Recent development in nonuniformly spaced array synthesis methods

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***Abstract*—Synthesis of sparse arrays with reduced number of elements are significant for some applications where the available space, weight and the cost of the antenna system is very limited. In recent years, a variety of advanced techniques have been presented to deal with sparse array synthesis problems in either narrow and wideband cases. This paper presents a brief review of these techniques, and gives rough comparative study on some of sparse array synthesis methods.**

***Index Terms*—Sparse arrays, synthesis methods**

I. Introduction

Antenna arrays have been widely applied in many electronic systems such as radars, sonars and wireless communication systems. In general, uniformly spaced antenna arrays have been used and studied more extensively than nonuniformly spaced arrays due to the convenience of array configurations as well as the simplicity in mathematical processing. However, to avoid the presence of grating lobes, the element spacing for uniformly spaced arrays should be no large than one wavelength for a broadside beam and half a wavelength for a wide-angle scannable beam. Thus, a large number of elements would be required if a narrow beam pattern is required, for example, in high-resolution imaging systems and radio astronomy instruments. Since each antenna element is usually connected to one individual transmitting and/or receiving channel, and a large number of elements usually means a huge cost and weight of the whole hardware system.

To address this issue, nonuniformly spaced arrays with optimized element positions are designed to reduce the required number of elements for some applications where the space and weight of the hardware system are limited. In some applications, nonuniform spacing technique can be used as a kind of equivalent amplitude weighting to reduce the sidelobe level of array pattern. Another application of nonuniformly spaced arrays is in the ultra-wideband system where antenna elements may be much larger than one wavelength in high frequency. Thus using nonuniform spacing can significantly suppress the grating lobe level in high frequency band. Due to these advantages, design of nonuniformly spaced arrays have been received much attention over the past a few decades. The earliest research on the nonuniformly spaced array may be dated back to late 1950s when Unz theoretically analyze the stored energy and Q factor of a nonuniformly spaced linear array [1], [2]. Then from 1960s to 1980s, a few methods had been presented to design nonuniformly spaced linear arrays. These methods mainly include the orthogonal methods [3]–[5], dynamic programming technique [6], FIR filter concept-based thinning technique [7], and probabilistic approaches [8]–[10]. It is noted that probabilistic approaches are usually used to design statistic space-tapered arrays, and thus it is only suited for large-scale arrays.

In recent 30 years, huge progress has been made in many aspects including in numerical computation, optimization theory, signal processing and artificial smarm intelligence based on the factor that universal computers become available to everyone. Accordingly, in the area of antenna array theory, a number of advanced methods for synthesizing nonuniformly spaced arrays have been presented. In this paper, we try to briefly review these techniques, and give short discussion on some nonuniformly spaced array synthesis problems which remains to be challenging. Due to very limited space of this paper, we would not include all the contributions in this area over the past. We sincerely apologize for the authors who contribute intelligence to this area but we miss to mention in this paper.

II. Brief review of nonuniformly spaced array synthesis methods in recent decades

A. Problem description

Let us consider an array with N radiating elements placed at the positions $\vec{r}_n = (x_n, y_n, z_n), n = 1, 2, \ldots, N$. The array pattern can be described as follows

$$F(\theta, \phi) = \sum_{n=1}^{N} w_n E_n(\theta, \phi) e^{j\{\beta \vec{r}_n \cdot \vec{a}(\theta,\phi) + \varphi_n\}}, \tag{1}$$

where $j = \sqrt{-1}$, $\beta = 2\pi/\lambda$ denotes the wave number in freedom space, $\vec{a}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is the propagation direction vector, w_n and φ_n denotes the excitation amplitude and phase of the n-th element, respectively. $E_n(\theta, \phi)$ is the phase-adjusted pattern of the n-th element in

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the array. If mutual coupling effect is ignored or edge effect is ignored in uniformly spaced linear or planar arrays, all the phased-adjusted element patterns for different elements can be considered the same. In this situation, we need to synthesize only the array factor which is given by

$$AF(\theta,\phi)=\sum_{n=1}^{N} w_n e^{j(\beta \vec{r}_n\cdot \vec{a}(\theta,\phi)+\varphi_n)}. \quad (2)$$

The array pattern can be given by multiplying the array factor with the element pattern. In the case of planar array, for example, assumed that the planar array lies in xoy-plane, the array factor can be written as

$$AF(\theta,\phi)=\sum_{n=1}^{N} w_n e^{j\{\beta(x_n \sin\theta\cos\phi+y_n\sin\theta\sin\phi)+\varphi_n\}}. \quad (3)$$

In general, there are mainly two types of nonuniformly spaced synthesis problems: a) one kind of problem is optimizing the pattern performance at a given element number N. For example, for a given N, the element positions are optimized with aiming at minimizing the peak sidelobe level (PSLL) and/or minimizing the tolerance between the synthesized and the desired mainlobe shapes; b) the second kind is minimizing the element number N provided that the synthesized pattern meets the prescribed requirement in both mainlobe and sidelobe region. For different kinds of problems, the ways in the detail to solving the problem may be different to some extent. In recent decades, a number of advanced techniques have been presented, and most of them can be applicable to solve either Problem A or B. However, these two kinds of problems essentially differ not too much, and they can be sometime transformed to each other. In addition, some of techniques can be suited for both of the problems by slightly modifying the details of the algorithms.

In the following, we choose to separately describe the nonuniformly spaced array synthesis methods in narrow-band and wideband cases. Although such a classification may also be not strict due to the factor that many of narrow-band methods can be further extended to deal with wideband nonuniformly spaced array synthesis, this would be meaningful and clear in view of practical applications.

B. Narrow-band Sparse Array Synthesis Methods

Most of antenna arrays work in narrow-band. Accordingly, a number of nonuniformly spaced array synthesis techniques are mainly designed to deal with narrow-band problems. These techniques can be roughly summarized as five categories: analytical methods [11], [12], stochastic optimization methods [13]–[21], sparse array reconstruction methods [22]–[29], iterative convex optimization techniques [30]–[32], and iterative FFT-based array thinning techniques [33], [34]. Each has its own merits and demerits. For example, the analytical methods in [11], [12] can give a solution to element position distribution in a very efficient manner but they cannot control the obtained sidelobe distribution very well.

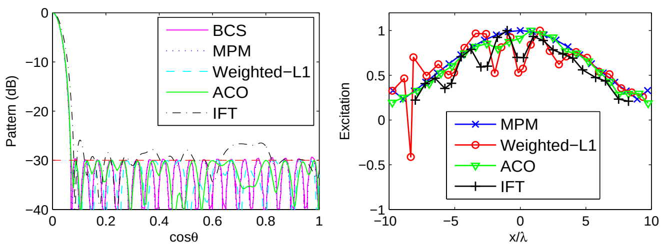
Due to the ability of refraining from locally optimal solutions, stochastic optimization (SO) methods such as genetic algorithm (GA) in [13], simulated annealing (SA) in [14], modified tabu search (TS) in [15], differential evolution (DE) in [16], particle swarm optimization (PSO) in [17], and others [18], [19] are very suitable to solve the nonlinear and nonconvex problem like as the synthesis of sparse arrays. However, all these SO methods are time-consuming especially for the array with large number of elements. Besides, when applied to optimize the element positions for planar arrays, these methods are hard to control the minimum element spacing constraint (MESC). In [20], a modified GA is introduced to transform the MESC into Chebyshev distance constraint in which the element spacing can be easy to control. In [21], an asymmetric mapping method is presented to guarantee that the synthesized array always meets the MESC, but this mapping method leads to greatly increase in the dimension of search space.

Sparse array reconstruction methods typically includes the matrix pencil method (MPM) and Bayesian compressive sensing (BCS). With respect to the synthesis of sparse linear arrays, the MPM in [22] is introduced to rearrange the element positions and excitation distribution after performing singular value decomposition (SVD) on a Hankel matrix constructed by using the desired pattern samples to obtain a lower-rank approximation, which corresponds to fewer elements. In [23], the forward-backward MPM (FBMPM) is presented to achieve an accurate pattern shape by giving a restriction on the poles which correspond to element positions. In [24], the extended MPM is introduced to synthesize multiple pattern on the base of the MPM or FBMPM synthesis methods. In [25], a unitary matrix pencil (UMP) method is presented to improve the matching accuracy of shaped patterns by establishing a new relationship between the element positions and the generalized eigenvalues by means of a unitary transformation. Besides, two-dimensional UMP in [26] is also proposed to synthesize sparse planar array. Importantly, the MPM and its variants are efficient and solve the problem that traditional sparse array synthesis methods cannot prior estimate the best number of elements. However, they are hard to control the sidelobe distribution. Recently, the BCS techniques are applied in the synthesis of sparse linear array [27], and planar array [28] by matching the obtained pattern and a reference one in both excitation amplitudes and phases, which can achieve ultra-sparse position distribution, and a review on this technique is presented in [29]. However, prescribing a realizable reference pattern is a problem itself to be solved. In addition, the minimum element spacing is difficult to control in both the MPM and BCS algorithm frameworks.

Some iterative convex optimizations such as those in [30], [31] can be applied to efficiently synthesize the focused or shaped patterns by formulating the unequally spaced array synthesis problem as a re-weighted ℓ_1-norm problem, and these methods are not required for the reference pattern. However, they are hard to control the minimum element spacing so that the obtained array are sometime non-realizable. Recently, a novel method called alternating convex optimization (ACO)

TABLE I
THE PERFORMANCE COMPARISON AMONG VARIOUS SPARSE SYNTHESIS METHODS

Algorithms	Performance			Ref.
	Time Cost	SLL Control	MESC	
Analytical methods	extremely low	hard	possible	[11], [12]
Reweighted ℓ_1	low	easy	hard	[30], [31]
MPMs	low	hard	hard	[22]–[26]
BCS	low	hard	hard	[27]–[29]
ACO	low	easy	easy	[32]
Stochastic Alg.	high	easy	possible	[13]–[21]
IFT	extremely low	hard	easy	[33], [34]



(a) (b)

Fig. 1. The result comparison obtained by different algorithms in a sparse linear array with length $L = 19.5\lambda$. (a) the focused pattern, and (b) the obtained excitation distribution.

in [32] is proposed to control the minimum element spacing by formulating the unequally spaced array synthesis problem as a sequence of alternating convex optimization problems in which the MESC is incorporated.

In [33], [34], iterative fast Fourier transform (IFT) technique is exploited to efficiently synthesize very large thinned arrays based on the factor that the relationship between excitation distribution and the array pattern in uniformly spaced arrays is the discrete Fourier transform (DFT) pair. In general, the IFT is very computationally efficient and is particularly suitable for synthesizing large-scale arrays. However, due to the use of DFT, the IFT is used only for thinning uniformly spaced arrays. The obtained thinned arrays are easier to fabricate but having less degrees of synthesis freedoms compared with truly nonuniformly spaced arrays. Thus, the synthesized pattern performance such as the obtained sidelobe level may be not as good as that of a nonuniformly spaced array with the same number of elements. One possible improvement would be picking up the active elements from a densely uniformly spaced array. This provide more degrees of freedoms to achieve better performance, but it requires an additional minimum element spacing control. Besides, one recent research is extending the IFT to efficiently synthesize the patterns of nonuniformly spaced arrays including mutual coupling and platform effect by introducing a virtual active element pattern expansion method [35].

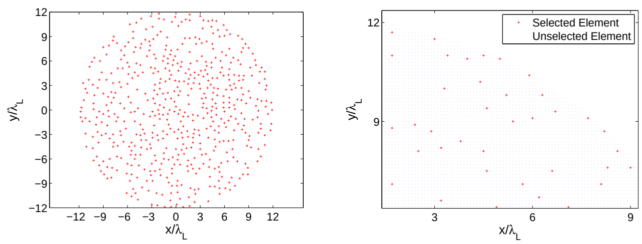
A brief summary of the mentioned sparse array synthesis methods in narrow-band case is given in Table I. For a representation case, a focused beam pattern in Fig. 4(c) of [27] is reconstructed by some representative methods including the MPM, weighted L_1-norm optimization, ACO and IFT. In this example, the array length is restricted to 19.5λ, and the desired sidelobe level is -30 dB. Compared with 26 elements using in BCS [27], these methods including the MPM, reweighted ℓ_1-norm optimization, ACO and IFT use 23, 28, 24 and 23 elements to reconstruct this pattern, respectively. Fig. 1(a) and (b) show the patterns and excitation distributions obtained by these methods, respectively. From the Fig. 1, we can see that the BCS, MPM, reweighted ℓ_1-norm optimization and ACO achieve the desired results, but the IFT obtains a bad result whose beamwidth is broadened and the SLL is only -25.86 dB, which is worse than others methods. The reason is mainly that the IFT only selects the elements in a uniformly gridded array, which leads to the lack of the degrees of synthesis freedom. It is worthy noting that, the MPM can reduce more elements than other methods, and the corresponding excitation distribution is symmetry about the array center. In addition, compared with reweighted ℓ_1-norm optimization, ACO can not only reduce more elements but also control the minimum element spacing constraint.

C. Wideband Sparse Array Synthesis Methods

Different from the narrow-band arrays, the design of wideband arrays need to solve such a problem: if the element spacing is too large, there exists a number of grating lobes at the highest frequency; if the element spacing is too small, mutual coupling effects between adjacent elements are very strong and this is not convenient for the design of the antenna element. Due to the requirement for wideband or ultra-wide band arrays in some applications, this naturally promotes the development of various wideband array synthesis methods [36]–[42], [48]–[50]. It is worthy that most of these synthesis methods come from prof. D. H. Werner and his group. For example, in [36], an analytical method based on raised power series (RPS) representation was introduced, which can obtain a stable grating lobe level across the whole ultra-wide frequency band.

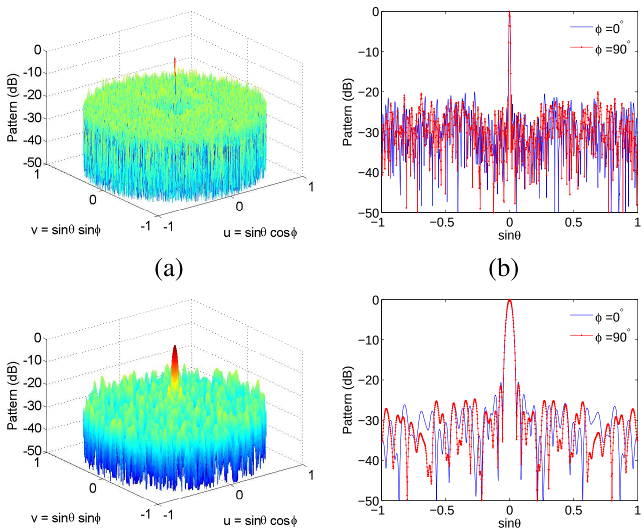
Stochastic optimization methods are suitable for the optimization of the complexity problem, and a review on the application of these methods in the synthesis of ultra-wideband aperiodic antenna arrays is presented in [37]. Typically, in [38], [39], they apply the GA on the base of the polyfractal array representation to obtain a wideband linear array with a sparse distribution. In [40], a self-similar structure called aperiodic tilings is exploited to synthesize large ultra-wideband sparse array layout by combing with GA perturbation. However, the radiation performance obtained by this manner is limited by the selected array representation. Covariance matrix adaptation evolutionary strategy (CMAES) is also applied in the synthesis of the linear array [41], and planar array with the rotational symmetry structure [42], which shows the effect on avoiding the generation of grating lobe.

Recently, we extend the IFT to synthesize wideband sparse arrays by gradually reducing the number of selected elements with the minimum element spacing control. For comparison



(a) (b)

Fig. 2. The position optimization of 551 elements in a circular aperture of radius $R = 12\lambda_L$. (a) the obtained layout, and (b) detail view.



(a) (b)

Fig. 3. The array pattern obtained by the modified IFT at the highest frequency $f_{\rm H}$ and the lowest frequency $f_{\rm L}$ over the whole 5:1 frequency bandwidth. (a) 3D pattern in (u, v)-plane at $f_{\rm H}$, (b) two orthogonal cuts of the array pattern at $f_{\rm H}$, (c) 3D pattern in (u, v)-plane at $f_{\rm L}$, and (d) two orthogonal cuts of the array pattern at $f_{\rm L}$.

with a Penrose tiling array combing GA perturbation in [40], we design a sparse array with 551 elements and $5:1$ frequency bandwidth in a circular aperture of $R = 12\lambda_L = 60\lambda_H$ where λ_L and λ_H denote the wavelength at the lowest frequency $f_{\rm L}$ and highest frequency $f_{\rm H} = 5f_{\rm L}$, respectively. The minimum element spacing is required to be $2.5\lambda_H$. The obtained layout and its detail view are shown in Fig. 2(a) and (b), respectively. It can be checked that the obtained layout meets the minimum element spacing constraint and the elements are located in the required spacing grid. Furthermore, the SLLs obtained by the modified IFT are -17.98 dB and -17.64 dB at $f_{\rm L}$ and $f_{\rm H}$, respectively. They are much lower than those obtained in [40]. Fig. 3(a)-(b) show the corresponding patterns at $f_{\rm L}$ and $f_{\rm H}$, respectively.

In some applications such as acoustic hearing aid arrays where wideband signals need to be transmitted or received without any waveform distortion, the array should have frequency-invariant beam response over a wide frequency band. In [43], the asymptotic theory of unequally spaced arrays is employed to design an ultra-wideband pattern with invariant mainlobe shape. However, this method has less control on the sidelobe level. In [44] , a hybrid method in which the positions are obtained by the simulated annealing optimization and the filter coefficients are analytically computed are introduced to achieve robust synthesis of sparse super-directive arrays. Another hybrid strategy by combining the genetic algorithm and a gradient-based method can be found in [45]. The compressive sensing (CS)-based technique can be also generalized to synthesize the broadband frequency-invariant sparse arrays [46], [47]. Other sparse array synthesis methods such as the iterative convex optimization and the matrix pencil method have been also generalized to synthesize sparse linear array with frequency-invariant single and even multiple beam patterns [48]- [50].

III. Final Remarks

Many of advanced methods have been developed in recent decades to deal with sparse array synthesis problems in either narrow-band or wideband. These methods have their own advantages and disadvantages. Short comments and comparison on these synthesis methods have been presented. This provides a rough guideline for someone who need to seek a method to solve sparse array synthesis problems.

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