Stability of Asynchronous Switched Systems with Sequence-based Average Dwell Time Approaches☆

Dianhao Zheng[a,b], Hongbin Zhang[b,*], J. Andrew Zhang[c], Weixing Zheng[d], Steven W. Su[a]

[a] *Faculty of Engineering and Information Technology, University of Technology Sydney, Ultimo, NSW 2007, Australia*

[b] *School of Information and Communication Engineering, University of Electronic Science and Technology of China, Chengdu, 611731, China*

[c] *Global Big Data Technologies Centre, University of Technology Sydney, Ultimo, NSW 2007, Australia*

[d] *School of Computing, Engineering and Mathematics, Western Sydney University, Sydney, NSW 2751, Australia*

Abstract

This paper studies the stability problem of asynchronous switched systems and proposes novel sequence-based average dwell time approaches. Both continuous-time and discrete-time systems are considered. The proposed approaches exploit the switching sequences of subsystems which were seldom utilized in the literature. More specifically, our approaches exploit the differences between different switching sequences, including the maximal asynchronous switching time, the energy changing degree at switching times, and the increasing speed of energy functions in asynchronous time intervals. As a result, the proposed approaches can reduce the threshold value of average dwell time significantly. We also propose an approach to counterbalance the increasing of energy functions in asynchronous time intervals by prolonging the preceding rather than subsequent subsystem. Numerical results demonstrate that the proposed approaches can improve the performance significantly in comparison with a well-known method.

Keywords: sequence-based average dwell time, asynchronous switched systems, sequence-based average subsequent dwell time, sequence-based average preceding dwell time

1. INTRODUCTION

A *switched system* is generally defined as a special dynamical system that includes a finite number of subsystems and a logical orchestrating rule, and these subsystems are often described by differential or difference equations[1]. Switched systems can be used in modeling many physical or man-made systems.When the switching between controllers and system modes is asynchronous, it is referred to as *asynchronous switching* [2, 3]. There have been strong research interests recently in

*Corresponding author
Email address: zhanghb@uestc.edu.cn (Hongbin Zhang)

applying switched systems for, e.g., fault detection, adaptive control, switched stochastic systems[4], sliding-mode control, robust control, multi-agent systems[5, 6] and 2-D switched systems[7].

Many results on switched systems are based on *dwell time*. The concept of dwell time was firstly introduced in [8] to study the stability of switched systems. A more flexible signal called average dwell time (ADT) switching was defined and used in [9]. In [10], the authors proposed a more practical approach, called mode-dependent average dwell time (MDADT) switching. These dwell time switching approaches have been applied to many switched systems, such as dynamic output feedback control [11], and 2-D systems[7]. On the basis of average dwell time switching, asynchronous switched systems have also been investigated. The stability and stabilization problems for asynchronous switched systems were investigated in [2, 12]. Asynchronous filtering of switched systems was studied in [13]. Asynchronous L_2 gain and H_∞ control were considered in [14, 15, 16, 17]. Asynchronous control for 2-D switched systems was also analyzed in [7]. Nonlinear systems under sampled-data control was discussed in [18, 19].

However, there are two main limitations in these studies on switched systems based on dwell time. Firstly, different sequences contain different asynchronous switching information, which is not well exploited in these studies. For example, the whole asynchronous switching has one maximum of asynchronous time and every sequence has its own maximum of asynchronous time. Almost all the past work did not consider the differences from sequences to sequences. Papers [20, 21] conducted stability analysis for switched systems using matrix sequences rather than mode switching sequences. Secondly, almost all studies try suppressing the increase of energy functions via prolonging the average dwell time of the subsequent subsystem. There are actually other options which can potentially lead to better performance as we are going to show in this paper.

This paper aims to overcome the aforementioned limitations by exploiting more asynchronous switching information and counterbalancing the energy functions via prolonging preceding subsystems other than the subsequent subsystem. The major contributions of this paper are as follows: (1) We propose two novel approaches for studying the stability problem of asynchronous switched systems. They can relax the restrictions on average dwell time, improve the computational accuracy and reduce the threshold value, compared to existing approaches; (2) We propose two novel parallel approaches for switched systems to improve the selectivity. These approaches provide a novel direction for solving the stability problem of asynchronous switched systems.

The remainder of this paper is structured as follows. In Section 2, we introduce some preliminaries on asynchronous switching. In Section 3, we first propose two novel approaches for studying the stability problem of general switched systems and then use the obtained results to solve linear systems problems. A numerical example is given to verify the effectiveness of these novel theorems in Section 4. Finally, Section 5 concludes this paper.

Notations: We use standard notations in this paper. The symbol $\times$ represents multiplication or Cartesian product of sets. For any given matrix P, the superscript "T" means the matrix transpose, and $P > (or \ <) \ 0$ denotes that matrix P is positive (or negative) definite and symmetric. If a function κ is said to be of a class $\mathcal{K}_\infty$ function, the function $\kappa : [0, \infty) \to [0, \infty)$, $\kappa(0) = 0$, must be strictly increasing, continuous, and unbounded. The symbols $t_1, \ t_2, \ t_3, \ \cdots$ or $k_1, \ k_2, \ k_3, \ \cdots$ denote the switching times for continuous-time asynchronous systems or discrete-time cases. The flag t_i^- stands for the moment just before the switching time t_i. For a switched system including s subsystems, the function $\sigma(t) \ or \ \sigma(k) : [0, +\infty) \to \mathcal{S} = \{1, 2, \cdots, s\}$ stands for the switching signal. The symbol $[p|q]$ denotes the situation when the p^{th} subsystem is aroused immediately after the q^{th} subsystem. Let $T_\downarrow(t_l, t_{l+1})$ and $T_\uparrow(t_l, t_{l+1})$ denote the unions (or the length of the unions) of the decreasing and increasing time (including unchanging time) of the Lyapunov function within the interval $[t_l, t_{l+1})$. For different switching sequences or subsystems, $T_\downarrow(t_l, t_{l+1})$ and $T_\uparrow(t_l, t_{l+1})$ have different subscripts.

2. PRELIMINARIES

Suppose that there is a time lag T between switched controllers and system modes. Consider the closed-loop asynchronous switched system in
(a) continuous-time case:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}K_{\sigma(t-T)}x(t); \tag{1}$$

and (b) discrete-time case:

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}K_{\sigma(k-T)}x(k). \tag{2}$$

In the above systems, $x(t)$ (or $x(k)$) $\in \mathcal{R}^n$ is the state vector; $A_{\sigma(t)}$ (or $A_{\sigma(k)}$) and $B_{\sigma(t)}$ (or $B_{\sigma(k)}$) are the system and input matrix; $u(t) = K_{\sigma(t-T)}x(t)$ (or $u(k) = K_{\sigma(k-T)}x(k)$) is the control input.

In order to analyze the stability of the above systems, [2] used the average dwell time (ADT) to develop effective switching strategy. Its definition is given below.

Definition 1. *[9] For any switching times $0 \le t_1 \le t_2$ and a switching signal $\sigma(t)$, let $N_\sigma(t_1, t_2)$ be the number of discontinuities of $\sigma(t)$ in the open interval (t_1, t_2). It is said that the signal $\sigma(t)$ has an average dwell time τ_a if there exist two scalar $N_0 \ge 0$ (N_0 is called the chatter bound) and τ_a such that $N_\sigma(t_1, t_2) \le N_0 + \frac{t_2 - t_1}{\tau_a}$.*

3. MAIN RESULTS

In this section, we propose two novel approaches for stabilizing asynchronous switched systems. We first present novel results for general situations and then apply them to the linear case.

3.1. General Switched Systems

We first consider general asynchronous switched systems. Two novel concepts are introduced below.

Definition 2. *For the switching signal $\sigma(t)$ and any switching time $0 \leq t_1 < t_2$, let $N_{\sigma[p|q]}(t_1, t_2)$ and $T_{p,[p|q]}(t_1, t_2)$ represent the number of sequences and the total running time of the p^{th} subsystem, respectively, when the p^{th} subsystem is aroused immediately after the q^{th} subsystem over the time interval $[t_1, t_2)$. If there exist two nonnegative scalars $\tau_{a(p,[p|q])}$ and $N_{0(p,[p|q])}$ which is called the subsequent sequenced-based chatter bound, such that*

$$N_{\sigma[p|q]}(t_1, t_2) \leq N_{0(p,[p|q])} + \frac{T_{p,[p|q]}(t_1, t_2)}{\tau_{a(p,[p|q])}}, \tag{3}$$

then it is said that $\sigma(t)$ has a sequence-based average subsequent dwell time (SBASDT) $\tau_{a(p,[p|q])}$.

Definition 3. *For the switching signal $\sigma(t)$ and any switching time $0 \leq t_1 < t_2$, let $N_{\sigma[p|q]}(t_1, t_2)$ and $T_{q,[p|q]}(t_1, t_2)$ denote the number of the sequences and the total running time of the q^{th} subsystem, when the p^{th} subsystem is aroused immediately after the q^{th} subsystem over the time interval $[t_1, t_2)$. If there exist two nonnegative scalars $\tau_{a(q,[p|q])}$ and $N_{0(q,[p|q])}$ which is called the preceding sequenced-based chatter bound, such that*

$$N_{\sigma[p|q]}(t_1, t_2) \leq N_{0(q,[p|q])} + \frac{T_{q,[p|q]}(t_1, t_2)}{\tau_{a(q,[p|q])}}, \tag{4}$$

then it is said that $\sigma(t)$ has a sequence-based average preceding dwell time (SBAPDT) $\tau_{a(q,[p|q])}$.

Definition 2 and 3 provide two new sequence-based average dwell time (SBADT) concepts.

Remark 1. *Definition 2 and 3 take into consideration the influence of sequences. They are different from the definition of conventional average dwell time in [2, 9].*

Remark 2. *We note that paper [22] also considers sequence-based switching. It mainly studied a general class of switching signals with which the resulting switched system was input-to-state stable, while this paper considers the stability of asynchronous switched systems. Paper [22] recast generalized switching signal for input-to-state stability and did not address the concrete concept $T_{p,[p|q]}(t_1, t_2)$, especially for asynchronous switching, as in this paper.*

To simplify presentation, we assume that a proper decreasing (increasing) Lyapunov function can be found in the synchronous (asynchronous) switching time section. We can then obtain the following four theorems for general asynchronous switched systems.

Theorem 1. *For a switched system*

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \tag{5}$$

let $\alpha_{[p|q]} > 0$*,* $\lambda_q > 0$*,* $\lambda_p > 0$*,* $\mu_{[p|q]} > 1$ *be given constants. For every* $s \in \mathcal{S}$*, suppose there exist functions* k_{1s}*,* k_{2s} *of class* $\mathcal{K}_\infty$ *and* $\mathcal{C}^1$ *fuctions* $V_s : \mathcal{R}^n \to \mathcal{R}$*, such that*

$$k_{1s}(\|x(t)\|) \leq V_s(x(t)) \leq k_{2s}(\|x(t)\|), \tag{6}$$

$\forall i = 1,\ 2,\ \cdots,\ \forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S},\ p \neq q,$

$$\dot{V}_p(x(t)) \leq \begin{cases} -\lambda_p V_p(x(t)), \forall t \in T_\downarrow[t_i, t_{i+1}), \\ \alpha_{[p|q]} V_p(x(t)), \forall t \in T_\uparrow[t_i, t_{i+1}), \end{cases} \tag{7}$$

and

$$V_p(x(t_i)) \leq \mu_{[p|q]} V_q(x(t_i)). \tag{8}$$

Then the system is globally uniformly asymptotically stable for any switching signals with SBASDT

$$\tau_{a(p,[p|q])} \geq \tau^*_{a(p,[p|q])} = \frac{T_{[p|q]\_\max\uparrow}(\lambda_p + \alpha_{[p|q]}) + \ln \mu_{[p|q]}}{\lambda_p}, \tag{9}$$

where $T_{[p|q]\_\max\uparrow} \triangleq \max_{l, \sigma(t_l)=p,\ \sigma(t_l^-)=q} T_\uparrow(t_l, t_{l+1}),\ \forall l \in Z_+.$

Proof: Let $t_0 = 0$. For any $t > 0$, $t \in [t_l, t_{l+1})$, $l \in Z_+$, we define two novel sets: $\mathcal{S}' \triangleq \{(p,q); p \in \mathcal{S}, q \in \mathcal{S}, p \neq q\}$; $\mathcal{S}'' \triangleq \{(\sigma(t_i), \sigma(t_i^-)) \in \mathcal{S} \times \mathcal{S}, i = 1,\ 2,\ 3,\ \cdots,\ l\}$. The total numbers of the set $\mathcal{S}'$ and $\mathcal{S}''$ are s' and s''. Therefore, $S'' \subseteq S'$, $s' = s(s-1)$, $s'' \leq l$ and $s'' \leq s'$.

For all the elements in the set $\mathcal{S}''$, we order and list them. We use the symbol $[p|q]_{(k)}$ to represent the k^{th} element. The total aroused numbers of the k^{th} element is represented by $N_{\sigma[p|q]_{(k)}}(0,t)$. The sum of dwell time of the p^{th} subsystems for the sequence $[p|q]$ is denoted by $T_{p,[p|q]_{(k)}}(0,t)$.

When $\sigma(t_j) = p$, $\sigma(t_j^-) = q$, $\forall j = 1,\ 2,\ \cdots,\ l$, and the element (p,q) is the k^{th} element, we rewrite $\mu_{\sigma(t_j)}$ as $\mu_{[p|q]_{(k)}}$ to denote the switching sequences. In this situation, we also use $\alpha_{[p|q]_{(k)}}$ to denote $\alpha_{\sigma(t_j)}$.

According to (7) and (8), it holds that

$$\begin{aligned} & V_{\sigma(t)}(x(t)) \\ \leq & \exp\{-\lambda_{\sigma(t_l)} T_\downarrow(t_l, t) + \alpha_{\sigma(t_l)} T_\uparrow(t_l, t)\} V_{\sigma(t_l)}(x(t_l)) \\ \leq & \exp\{-\lambda_{\sigma(t_l)} T_\downarrow(t_l, t) + \alpha_{\sigma(t_l)} T_\uparrow(t_l, t)\}\, \mu_{\sigma(t_l)|\sigma(t_l^-)} V_{\sigma(t_l^-)}(x(t_l{}^-)). \end{aligned} \tag{10}$$

Recursively,

$$
\begin{aligned}
&V_{\sigma(t)}(x(t)) \\
&\leq \left\{\prod_{j=1}^{l} \mu_{\sigma(t_j)|\sigma(t_j^-)}\right\} \exp\left\{-\lambda_{\sigma(t_l)} T_{\downarrow}(t_l, t) - \cdots - \lambda_{\sigma(t_1)} T_{\downarrow}(t_1, t_2)\right\} \\
&\times \exp\left\{\alpha_{\sigma(t_l)} T_{\uparrow}(t_l, t) + \cdots + \alpha_{\sigma(t_1)} T_{\uparrow}(t_1, t_2)\right\} \exp\left\{-\lambda_{\sigma(0)}(t_1 - 0)\right\} V_{\sigma(0)}(x(0)).
\end{aligned}
\tag{11}
$$

We can also obtain the following formal transformation:

$$
\prod_{j=1}^{l} \mu_{\sigma(t_j)|\sigma(t_j^-)} = \prod_{k=1}^{s''} \mu_{[p|q]_{(k)}}^{N_{[p|q]_{(k)}}(0,t)}. \tag{12}
$$

Therefore,

$$
\begin{aligned}
&V_{\sigma(t)}(x(t)) \\
&\leq \left\{\prod_{k=1}^{s''} \mu_{[p|q]_{(k)}}^{N_{[p|q]_{(k)}}(0,t)}\right\} \exp\left\{\sum_{k=1}^{s''} -\lambda_p T_{p,[p|q]_{(k)}\downarrow}(0,t)\right\} \exp\left\{\sum_{k=1}^{s''} \alpha_{[p|q]_{(k)}} T_{p,[p|q]_{(k)}\uparrow}(0,t)\right\} \\
&\times \exp\left\{-\lambda_{\sigma(0)}(t_1 - 0)\right\} V_{\sigma(0)}(x(0)).
\end{aligned}
\tag{13}
$$

Without affecting the conclusion, we assume that there is no asynchronous switching between the first switching time t_1 and the initial time 0. Hence,

$$
\sum_{k=1}^{s''} \alpha_{[p|q]_{(k)}} T_{p,[p|q]_{(k)}\uparrow}(0,t) = \sum_{k=1}^{s''} \alpha_{[p|q]_{(k)}} T_{[p|q]_{(k)}\uparrow}(0,t),
$$

where $T_{[p|q]_{(k)}\uparrow}(0,t)$ stands for the total asynchronous switching time (or the increasing/unchanging time of Lyapunov functions) in the time internal $[0,t)$.

According to (3), one has

$$
\begin{aligned}
&V_{\sigma(t)}(x(t)) \\
&\leq \left\{\prod_{k=1}^{s''} \mu_{[p|q]_{(k)}}^{N_{0[p|q]_{(k)}} + \frac{T_{p,[p|q]_{(k)}}(0,t)}{\tau_{a(p,[p|q]_{(k)})}}}\right\} \exp\left\{\sum_{k=1}^{s''} -\lambda_p T_{p,[p|q]_{(k)}\downarrow}(0,t) + \alpha_{[p|q]_{(k)}} T_{[p|q]_{(k)}\uparrow}(0,t)\right\} \\
&\times \exp\left\{-\lambda_{\sigma(0)}(t_1 - 0)\right\} V_{\sigma(0)}(x(0)).
\end{aligned}
\tag{14}
$$

Since $T_{p,[p|q]_{(k)}}(0,t) = T_{[p|q]_{(k)}\uparrow}(0,t) + T_{p,[p|q]_{(k)}\downarrow}(0,t)$, it is not difficult to obtain

$$
\begin{aligned}
&V_{\sigma(t)}(x(t)) \\
&\leq \exp\left\{\sum_{k=1}^{s''} N_{0[p|q]_{(k)}} \ln \mu_{[p|q]_{(k)}}\right\} \exp\left\{\sum_{k=1}^{s''} \left(\frac{\ln \mu_{[p|q]_{(k)}}}{\tau_{a(p,[p|q]_{(k)})}} - \lambda_p\right) T_{p,[p|q]_{(k)}}(0,t)\right\} \\
&\times \exp\left\{\sum_{k=1}^{s''} (\lambda_p + \alpha_{[p|q]_{(k)}}) T_{[p|q]_{(k)}\uparrow}(0,t)\right\} \exp\left\{-\lambda_{\sigma(0)}(t_1 - 0)\right\} V_{\sigma(0)}(x(0)).
\end{aligned}
\tag{15}
$$

According to the definition of $T_{[p|q]_{-}\max\uparrow}$, one can directly obtain

$$
T_{[p|q]_{(k)}\uparrow}(0,t) \leq N_{\sigma[p|q]_{(k)}}(0,t) T_{[p|q]_{(k)}\text{-}max\uparrow}. \tag{16}
$$

According to (3) and (16), one can arrive at

$$\begin{aligned}&V_{\sigma(t)}(x(t))\\&\leq \exp\left\{\sum_{k=1}^{s''} N_{0[p|q]_{(k)}}\ln\mu_{[p|q]_{(k)}}\right\}\exp\left\{\sum_{k=1}^{s''}\left(\frac{\ln\mu_{[p|q]_{(k)}}}{\tau_{a(p,[p|q]_{(k)})}}-\lambda_p\right)T_{p,[p|q]_{(k)}}(0,t)\right\}\\&\times\exp\left\{\sum_{k=1}^{s''}(\lambda_p+\alpha_{[p|q]_{(k)}})N_{0[p|q]_{(k)}}T_{[p|q]_{(k)}\text{-}max\uparrow}\right\}\exp\left\{\sum_{k=1}^{s''}(\lambda_p+\alpha_{[p|q]_{(k)}})\frac{T_{p,[p|q]_{(k)}}(0,t)}{\tau_{a(p,[p|q]_{(k)})}}T_{[p|q]_{(k)}\text{-}max\uparrow}\right\}\\&\times\exp\left\{-\lambda_{\sigma(0)}(t_1-0)\right\}V_{\sigma(0)}(x(0)).\end{aligned}\tag{17}$$

We define

$$C_1\triangleq\exp\left\{\sum_{k=1}^{s''}\left\{N_{0[p|q]_{(k)}}\ln\mu_{[p|q]_{(k)}}\right\}\right\}\exp\left\{\sum_{k=1}^{s''}\left\{(\lambda_p+\alpha_{[p|q]_{(k)}})N_{0[p|q]_{(k)}}T_{[p|q]_{(k)}\text{-}max\uparrow}\right\}\exp\left\{-\lambda_{\sigma(0)}(t_1-0)\right\}\right\}.$$

Inequality (17) can be rewritten as

$$\begin{aligned}&V_{\sigma(t)}(x(t))\\&\leq C_1\exp\left\{\sum_{k=1}^{s''}\left(\frac{\ln\mu_{[p|q]_{(k)}}}{\tau_{a(p,[p|q]_{(k)})}}-\lambda_p\right)T_{p,[p|q]_{(k)}}(0,t)\right\}\exp\left\{\sum_{k=1}^{s''}\left\{\frac{T_{[p|q]_{(k)}\text{-}max\uparrow}(\lambda_p+\alpha_{[p|q]_{(k)}})}{\tau_{a(p,[p|q]_{(k)})}}T_{p,[p|q]_{(k)}}(0,t)\right\}\right\}\\&\times V_{\sigma(0)}(x(0)).\end{aligned}\tag{18}$$

If there exist constants

$$\tau_{a(p,[p|q]_{(k)})}>\frac{(T_{[p|q]_{(k)}\text{-}max\uparrow}(\lambda_p+\alpha_{[p|q]_{(k)}})+\ln\mu_{[p|q]_{(k)}})}{\lambda_p},$$

we have

$$\begin{aligned}&V_{\sigma(t)}(x(t))\\&\leq C_1\exp\left\{value1\times(t-t_1)\right\}V_{\sigma(0)}(x(0)).\end{aligned}\tag{19}$$

where

$$value1\triangleq\max_k\left\{\frac{\ln\mu_{[p|q]_{(k)}}+T_{[p|q]_{(k)}\text{-}max\uparrow}(\lambda_p+\alpha_{[p|q]_{(k)}})}{\tau_{a(p,[p|q]_{(k)})}}-\lambda_p\right\}.$$

Therefore, it is concluded that $V_{\sigma(t)}(x(t))$ will converge to 0 as $t\to+\infty$. Then, the asymptotic stability can be deduced. □

Remark 3. *From Theorem 1, we can see that the proposed approaches exploit the differences of (a) the maximal asynchronous switching time $T_{[p|q]\text{-}\max\uparrow}$, (b) the energy changing degree $\mu_{[p|q]}$ at switching times, and (c) the increasing speed $\alpha_{[p|q]}$ of energy functions in asynchronous time intervals, for different switching sequences. Such sequence information is often ignored in the existing methods.*

In order to show the relationship between the proposed methods and the conventional ADT method, we derive the following corollary from Theorem 1.

Corollary 1. *For the switched system (5), let $\alpha > 0$, $\lambda > 0$, $\lambda > 0$, $\mu > 1$ be given constants. For every $s \in \mathcal{S}$, suppose there exist functions k_{1s}, k_{2s} of class $\mathcal{K}_\infty$ and $\mathcal{C}^1$ fuctions $V_s : \mathcal{R}^n \to \mathcal{R}$, such that*

$$k_1(\|x(t)\|) \leq V_s(x(t)) \leq k_2(\|x(t)\|), \tag{20}$$

$\forall i = 1,\ 2,\ \cdots,$

$$\dot{V}_s(x(t)) \leq \begin{cases} -\lambda V_s(x(t)), \forall t \in T_{\downarrow}[t_i, t_{i+1}), \\ \alpha V_s(x(t)), \forall t \in T_{\uparrow}[t_i, t_{i+1}), \end{cases} \tag{21}$$

$\forall (\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S},$

$$V_p(x(t_i)) \leq \mu V_q(x(t_i)). \tag{22}$$

Then the system is globally uniformly asymptotically stable for any switching signals with average dwell time

$$\tau_a \geq \tau_a^* = \frac{T_{\max\uparrow}(\lambda + \alpha) + \ln \mu}{\lambda}, \tag{23}$$

where $T_{\max\uparrow} \triangleq \max_i T_{\uparrow}(t_i, t_{i+1})$, $\forall i \in Z_+$.

Proof:

For Theorem 1, one gives the values that :

(a) $\alpha_{[p|q]} = \alpha > 0$, for all $p,\ q \in \mathcal{S}$, $p \neq q$;

(b) $\lambda_p = \lambda > 0$, for all $p \in \mathcal{S}$;

(c) $\mu_{[p|q]} = \mu > 1$, for all $p,\ q \in \mathcal{S}$, $p \neq q$.

According to the above given values, the conditions (6), (7) and (8) are equal to (20), (21) and (22), respectively.

For Theorem 1, $T_{\max\uparrow}$, τ_a, τ_a^* are defined as follows

(d)$T_{\max\uparrow} \triangleq \max_{p,q}\{T_{[p|q]\_\max\uparrow}\}$, for all $p,\ q \in \mathcal{S}$, $p \neq q$;

(e)$\tau_a \triangleq \max_{p,q}\{\tau_{a(p,[p|q])}\}$;

(f)$\tau_a^* \triangleq \frac{T_{\max\uparrow}(\lambda+\alpha)+\ln\mu}{\lambda}$.

On the basis of (9) and above definitions, it can be seen

$$\begin{aligned} \tau_a \triangleq \max_{p,q}\{\tau_{a(p,[p|q])}\} &\geq \max_{p,q}\{\frac{T_{[p|q]\_\max\uparrow}(\lambda_p + \alpha_{[p|q]}) + \ln \mu_{[p|q]}}{\lambda_p}\} \\ &= \max_{p,q}\{\frac{T_{[p|q]\_\max\uparrow}(\lambda + \alpha) + \ln \mu}{\lambda}\} \\ &= \frac{\max_{p,q}\{T_{[p|q]\_\max\uparrow}\}(\lambda + \alpha) + \ln \mu}{\lambda} \\ &= \frac{T_{\max\uparrow}(\lambda + \alpha) + \ln \mu}{\lambda} \triangleq \tau_a^*. \end{aligned} \tag{24}$$

This completes the proof. □

Remark 4. *The relationship between Corollary 1 and Theorem 1 demonstrates that the existing ADT methods are special cases of the sequential ADT. Corollary 1 is a conventional result based on conventional ADT methods[2].*

Theorem 2. *For a discrete switched system*

$$x(k+1) = f_{\sigma(k)}(x(k)), \tag{25}$$

let $1 > \lambda_p > 0$, $1 > \lambda_q > 0$, $\alpha_{[p|q]} > -1$, $\mu_{[p|q]} \geq 1$ *be given constants. For every* $s \in \mathcal{S}$, *suppose there exist functions* k_{1s}, k_{2s} *of class* $\mathcal{K}_\infty$ *and* $\mathcal{C}^1$ *fuctions* $V_s : \mathcal{R}^n \to \mathcal{R}$, *such that*

$$k_{1s}(\|x(k)\|) \leq V_s(x(k)) \leq k_{2s}(\|x(k)\|), \tag{26}$$

$\forall i = 1,\ 2,\ \cdots, \forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{S} \times \mathcal{S},\ p \neq q,$

$$V_p(x(k+1)) - V_p(x(k)) \leq \begin{cases} -\lambda_p V_p(x(k)), \forall t \in T_{\downarrow}[t_i, t_{i+1}), \\ \alpha_{[p|q]} V_p(x(k)), \forall t \in T_{\uparrow}[t_i, t_{i+1}), \end{cases} \tag{27}$$

and

$$V_p(x(t_i)) \leq \mu_{[p|q]} V_q(x(t_i)), \tag{28}$$

then the system is globally uniformly asymptotically stable for any switching signals with SBASDT

$$\tau_{a(p,[p|q])} \geq \tau^*_{a(p,[p|q])} = -\frac{(T_{[p|q]\text{-}\max\uparrow}(\ln(1+\alpha_{[p|q]}) - \ln(1-\lambda_p)) + \ln \mu_{[p|q]})}{\ln(1-\lambda_p)}. \tag{29}$$

Proof: By using the technique similar to the proof of Theorem 1, this theorem can be proved. □

Theorem 3. *For the switched system (5), let* $\lambda_q > 0$, $\lambda_p > 0$, $\mu_{[p|q]} > 1$ *be given constants. For every* $s \in \mathcal{S}$, *suppose there exist functions* k_{1s}, k_{2s} *of class* $\mathcal{K}_\infty$ *and* $\mathcal{C}^1$ *fuctions* $V_s : \mathcal{R}^n \to \mathcal{R}$, *such that*

$$k_{1s}(\|x(t)\|) \leq V_s(x(t)) \leq k_{2s}(\|x(t)\|), \tag{30}$$

$\forall i = 1,\ 2,\ \cdots, \forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S},\ p \neq q,$

$$\dot{V}_q(x(t)) \leq -\lambda_q V_q(x(t)), \forall t \in T_{\downarrow}[t_{i-1}, t_i), \tag{31}$$

$$\dot{V}_p(x(t)) \leq \alpha_{[p|q]} V_p(x(t)), \forall t \in T_{\uparrow}[t_i, t_{i+1}). \tag{32}$$

and

$$V_p(x(t_i)) \leq \mu_{[p|q]} V_q(x(t_i)). \tag{33}$$

Then the system is globally uniformly asymptotically stable for any switching signals with SBAPDT

$$\tau_{a(q,[p|q])}\geq\tau^*_{a(q,[p|q])}=\frac{T_{[p|q]\_\max\uparrow}(\lambda_q+\alpha_{[p|q]})+\ln\mu_{[p|q]}}{\lambda_q}. \tag{34}$$

Proof: Let $t_0=0$, and for any $t>0$, $t\in[t_l,t_{l+1})$, $l\in Z_+$. We use $T_{q,[p|q]_{(k)}}(0,t)$ to represent the sum of dwell time of the q^{th} subsystems for the k^{th} union of the set $\mathcal{S}''$ in the time interval $[0,t)$.

According to (10) and (31)-(33), one can rewrite (11) as

$$\begin{aligned}
&V_{\sigma(t)}(x(t))\\
&\leq\left\{\prod_{j=1}^{l}\mu_{\sigma(t_j)|\sigma(t_j^-)}\right\}\exp\left\{-\lambda_{\sigma(0)}T_\downarrow(0,t_1)+\alpha_{\sigma(t_1)}T_\uparrow(t_1,t_2)\right\}\\
&\times\exp\left\{-\lambda_{\sigma(t_1)}T_\downarrow(t_1,t_2)+\alpha_{\sigma(t_2)}T_\uparrow(t_2,t_3)\right\}\\
&\cdots\\
&\times\exp\left\{-\lambda_{\sigma(t_{l-1})}T_\downarrow(t_{l-1},t_l)+\alpha_{\sigma(t_l)}T_\uparrow(t_l,t)\right\}\\
&\times\exp\left\{-\lambda_{\sigma(t_1)}T_\downarrow(t_1,t_2)\right\}V_{\sigma(0)}(x(0)).
\end{aligned}\tag{35}$$

According to (12), it holds that

$$\begin{aligned}
&V_{\sigma(t)}(x(t))\\
&\leq\left\{\prod_{k=1}^{s''}\mu_{[p|q]_{(k)}}^{N_{[p|q]_{(k)}}(0,t)}\right\}\exp\left\{\sum_{k=1}^{s''}-\lambda_qT_{q,[p|q]_{(k)}\downarrow}(0,t_l)+\alpha_{[p|q]_{(k)}}T_{[p|q]_{(k)}\uparrow}(0,t)\right\}\\
&\times\exp\left\{-\lambda_{\sigma(t_l)}T_\downarrow(t_l,t)\right\}V_{\sigma(0)}(x(0)).
\end{aligned}\tag{36}$$

For the second part in (36), one has

$$\begin{aligned}
&\sum_{k=1}^{s''}-\lambda_qT_{q,[p|q]_{(k)}\downarrow}(0,t_l)+\alpha_{[p|q]_{(k)}}T_{[p|q]_{(k)}\uparrow}(0,t)\\
&=\sum_{k=1}^{s''}-\lambda_qT_{q,[p|q]_{(k)}\downarrow}(0,t)+\alpha_{[p|q]_{(k)}}T_{[p|q]_{(k)}\uparrow}(0,t)\\
&=\sum_{k=1}^{s''}-\lambda_qT_{q,[p|q]_{(k)}\downarrow}(0,t)-\lambda_qT_{q,[p|q]_{(k)}\uparrow}(0,t)+\sum_{k=1}^{s''}\lambda_qT_{q,[p|q]_{(k)}\uparrow}(0,t)+\alpha_{[p|q]_{(k)}}T_{[p|q]_{(k)}\uparrow}(0,t)\\
&=\sum_{k=1}^{s''}-\lambda_qT_{q,[p|q]_{(k)}}(0,t)+\sum_{k=1}^{s''}\lambda_qT_{q,[p|q]_{(k)}\uparrow}(0,t)+\alpha_{[p|q]_{(k)}}T_{[p|q]_{(k)}\uparrow}(0,t).
\end{aligned}\tag{37}$$

It follows from (16) that

$$\begin{aligned}&\sum_{k=1}^{s''}-\lambda_q T_{q,[p|q]_{(k)}\downarrow}(0,t)+\alpha_{[p|q]_{(k)}}T_{[p|q]_{(k)}\uparrow}(0,t)\\&\leq\sum_{k=1}^{s''}-\lambda_q T_{q,[p|q]_{(k)}}(0,t)+\sum_{k=1}^{s''}(\lambda_q+\alpha_{[p|q]_{(k)}})N_{\sigma[p|q]_{(k)}}(0,t)T_{[p|q]_{(k)}\text{-}max\uparrow}.\end{aligned}\tag{38}$$

According to (4), one can obtain

$$\begin{aligned}&V_{\sigma(t)}(x(t))\\&\leq\left\{\prod_{k=1}^{s''}\mu_{[p|q]_{(k)}}^{N_{0[p|q]_{(k)}}+\frac{T_{q,[p|q]_{(k)}}(0,t)}{\tau_{a(q,[p|q]_{(k)})}}}\right\}\exp\left\{\sum_{k=1}^{s''}-\lambda_q T_{q,[p|q]_{(k)}}(0,t)\right\}\\&\times\exp\left\{\sum_{k=1}^{s''}(\lambda_q+\alpha_{[p|q]_{(k)}})(N_{0[p|q]_{(k)}}+\frac{T_{q,[p|q]_{(k)}}(0,t)}{\tau_{a(q,[p|q]_{(k)})}})T_{[p|q]_{(k)}\text{-}max\uparrow}\right\}\exp\left\{-\lambda_{\sigma(t_l)}T_{\downarrow}(t_l,t)\right\}V_{\sigma(0)}(x(0))\\&=\exp\left\{\sum_{k=1}^{s''}N_{0[p|q]_{(k)}}\ln\mu_{[p|q]_{(k)}}\right\}\exp\left\{\sum_{k=1}^{s''}(\frac{\ln\mu_{[p|q]_{(k)}}}{\tau_{a(q,[p|q]_{(k)})}}-\lambda_q)T_{q,[p|q]_{(k)}}(0,t)\right\}\\&\times\exp\left\{\sum_{k=1}^{s''}(\lambda_q+\alpha_{[p|q]_{(k)}})N_{0[p|q]_{(k)}}T_{[p|q]_{(k)}\text{-}max\uparrow}\right\}\exp\left\{\sum_{k=1}^{s''}(\lambda_q+\alpha_{[p|q]_{(k)}})\frac{T_{q,[p|q]_{(k)}}(0,t)}{\tau_{a(q,[p|q]_{(k)})}}T_{[p|q]_{(k)}\text{-}max\uparrow}\right\}\\&\times\exp\left\{-\lambda_{\sigma(t_l)}T_{\downarrow}(t_l,t)\right\}V_{\sigma(0)}(x(0)).\end{aligned}\tag{39}$$

It can be rewritten as

$$\begin{aligned}&V_{\sigma(t)}(x(t))\\&\leq\exp\left\{\sum_{k=1}^{s''}N_{0[p|q]_{(k)}}\ln\mu_{[p|q]_{(k)}}\right\}\exp\left\{\sum_{k=1}^{s''}(\lambda_p+\alpha_{[p|q]_{(k)}})N_{0[p|q]_{(k)}}T_{[p|q]_{(k)}\text{-}max\uparrow}\right\}\\&\times\exp\left\{\sum_{k=1}^{s''}\left\{(\frac{\ln\mu_{[p|q]_{(k)}}}{\tau_{a(q,[p|q]_{(k)})}}-\lambda_q)T_{q,[p|q]_{(k)}}(0,t)\right\}\right\}\exp\left\{\sum_{k=1}^{s''}\{\frac{T_{[p|q]_{(k)}\text{-}max\uparrow}(\lambda_q+\alpha_{[p|q]_{(k)}})}{\tau_{a(q,[p|q]_{(k)})}}T_{q,[p|q]_{(k)}}(0,t)\}\right\}\\&\times\exp\left\{-\lambda_{\sigma(t_l)}T_{\downarrow}(t_l,t)\right\}V_{\sigma(0)}(x(0)).\end{aligned}\tag{40}$$

We define

$$C_2\triangleq\exp\left\{\sum_{k=1}^{s''}N_{0[p|q]_{(k)}}\ln\mu_{[p|q]_{(k)}}\right\}\exp\left\{\sum_{k=1}^{s''}(\lambda_p+\alpha_{[p|q]_{(k)}})N_{0[p|q]_{(k)}}T_{[p|q]_{(k)}\text{-}max\uparrow}\right\}.$$

If there exist constants

$$\tau_{a(q,[p|q]_{(k)})}>\frac{(T_{[p|q]_{(k)}\text{-}max\uparrow}(\lambda_q+\alpha_{[p|q]_{(k)}})+\ln\mu_{[p|q]_{(k)}})}{\lambda_q},$$

then one has

$$\begin{aligned}&V_{\sigma(t)}(x(t))\\&\leq C_2\exp\left\{value2\times(t_l-0)-\lambda_{\sigma(t_l)}T_{\downarrow}(t_l,t)\right\}V_{\sigma(0)}(x(0))\\&\leq C_2\exp\left\{\max\left\{-\lambda_{\sigma(t_l)},value2\right\}t\right\}V_{\sigma(0)}(x(0)),\end{aligned}\tag{41}$$

where

$$value2 \triangleq \max_k \left\{ \frac{\ln \mu_{[p|q]_{(k)}} + T_{[p|q]_{(k)}\text{-}max\uparrow}(\lambda_q + \alpha_{[p|q]_{(k)}})}{\tau_{a(q,[p|q]_{(k)})}} - \lambda_q \right\} < 0.$$

Therefore, it is concluded that $V_{\sigma(t)}(x(t))$ converges to 0 as $t \to +\infty$. Then, the asymptotic stability can be deduced. □

Theorem 4. *For the discrete switched system (25), let* $1 > \lambda_p > 0$, $1 > \lambda_q > 0$, $\alpha_{[p|q]} > -1$, $\mu_{[p|q]} \geq 1$ *be given constants. For every* $s \in \mathcal{S}$, *suppose there exist functions* k_{1s}, k_{2s} *of class* $\mathcal{K}_\infty$ *and* $\mathcal{C}^1$ *fuctions* $V_s : \mathcal{R}^n \to \mathcal{R}$, *such that*

$$k_{1s}(\|x(k)\|) \leq V_s(x(k)) \leq k_{2s}(\|x(k)\|), \tag{42}$$

$\forall i = 1,\ 2,\ \cdots, \forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{S} \times \mathcal{S},\ p \neq q,$

$$V_q(x(k+1)) - V_q(x(k)) \leq -\lambda_q V_q(x(k)), \forall t \in T_\downarrow[t_{i-1}, t_i) \tag{43}$$

$$V_p(x(k+1)) - V_p(x(k)) \leq \alpha_{[p|q]} V_p(x(t)), \forall t \in T_\uparrow[t_i, t_{i+1}), \tag{44}$$

and

$$V_p(x(t_i)) \leq \mu_{[p|q]} V_q(x(t_i)), \tag{45}$$

then the system is globally uniformly asymptotically stable for any switching signals with SBAPDT

$$\tau_{a(q,[p|q])} \geq \tau^*_{a(q,[p|q])} = -\frac{(T_{[p|q]\text{-}\max\uparrow}(\ln(1 + \alpha_{[p|q]}) - \ln(1 - \lambda_q)) + \ln \mu_{[p|q]})}{\ln(1 - \lambda_q)}. \tag{46}$$

Proof: By using the technique similar to the proof of Theorem 5, this theorem can be proved.□

Remark 5. *Comparing Theorem 3 (or 4) with 1 (or 2) or with the existing methods, we can see that the SBAPDT approach counterbalances the energy functions by prolonging the preceding rather than subsequent subsystems. This approach provides a novel direction for solving the stability problem of asynchronous switched systems.*

3.2. Linear Systems

In this subsection, we extend the results in the last subsection to linear asynchronous switched systems and present the following theorems for their stability analysis.

Theorem 5. *For system (1) and the given constants* $\alpha_{[p|q]} > 0$, $\lambda_p > 0$, $\mu_{[p|q]} > 1$, *if there exist matrices* $P_p > 0$, $P_q > 0$, *and* $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}, p \neq q$, *such that*

$$(A_p + B_p K_p)^T P_p + P_p(A_p + B_p K_p) \leq -\lambda_p P_p, \tag{47}$$

$$(A_p + B_pK_q)^T P_p + P_p(A_p + B_pK_q) \le \alpha_{[p|q]} P_p, \tag{48}$$

$$P_p \le \mu_{[p|q]} P_q, \tag{49}$$

then the system is globally uniformly asymptotically stable for any switching signals under the SBASDT switching condition (9).

Proof: For system (1), the multiple Lyapunov functions are given by

$$V_s = x^T P_s x,\ \forall s \in \mathcal{S}. \tag{50}$$

According to (7), one has (47) and (48).

The condition for (49) can be given in terms of (8).

Now, the proof of Theorem 5 is completed. □

Theorem 6. *For the discrete-time system (2) and the given constants $1 > \lambda_p > 0$, $\alpha_{[p|q]} > -1$, $\mu_{[p|q]} \ge 1$, if there exist matrices $P_p > 0$, $P_q > 0$, $p,\ q \in \mathcal{S}$, $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{S}\times\mathcal{S}, p \ne q$ such that*

$$\begin{bmatrix} -P_p & P_pA_p + P_pB_pK_p \\ * & -(1-\lambda_p)P_p \end{bmatrix} \le 0, \tag{51}$$

$$\begin{bmatrix} -P_p & P_pA_p + P_pB_pK_q \\ * & -(1+\alpha_{[p|q]})P_p \end{bmatrix} \le 0, \tag{52}$$

$$P_p \le \mu_{[p|q]} P_q, \tag{53}$$

then the system is globally uniformly asymptotically stable for any switching signals under the SBASDT switching condition (29).

Proof: According to Theorem 2, Theorem 6 can be proved by using the technique similar to the proof of Theorem 5. □

Theorem 7. *For system (1) and the given constants $\alpha_{[p|q]} > 0$, $\mu_{[p|q]} > 1$ and $\lambda_q > 0$, if there exist matrices $P_p > 0$, $P_q > 0$, and $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S}\times\mathcal{S}, p \ne q$, such that*

$$(A_q + B_qK_q)^T P_q + P_q(A_q + B_qK_q) \le -\lambda_q P_q, \tag{54}$$

$$(A_p + B_pK_q)^T P_p + P_p(A_p + B_pK_q) \le \alpha_{[p|q]} P_p, \tag{55}$$

$$P_p \le \mu_{[p|q]} P_q, \tag{56}$$

then the system is globally uniformly asymptotically stable for any switching signals under the SBAPDT switching condition (34).

Proof: According to Theorem 3, Theorem 7 can be proved by using the technique similar to the proof of Theorem 5. □

Theorem 8. *For the discrete-time system (2) and the given constants* $1 > \lambda_q > 0$, $\alpha_{[p|q]} > -1$, $\mu_{[p|q]} \ge 1$, *if there exist matrices* $P_p > 0$, $P_q > 0$, $p,\ q \in \mathcal{S}$, $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{S} \times \mathcal{S}, p \ne q$ *satisfying*

$$\begin{bmatrix} -P_q & P_qA_q + P_qB_qK_q \\ * & -(1-\lambda_q)P_q \end{bmatrix} \le 0, \tag{57}$$

$$\begin{bmatrix} -P_p & P_pA_p + P_pB_pK_q \\ * & -(1+\alpha_{[p|q]})P_p \end{bmatrix} \le 0, \tag{58}$$

$$P_p \le \mu_{[p|q]} P_q, \tag{59}$$

then the system is globally uniformly asymptotically stable for any switching signals under the SBAPDT switching condition (46).

Proof: According to Theorem 4, Theorem 8 can be proved by using the technique similar to the proof of Theorem 5. □

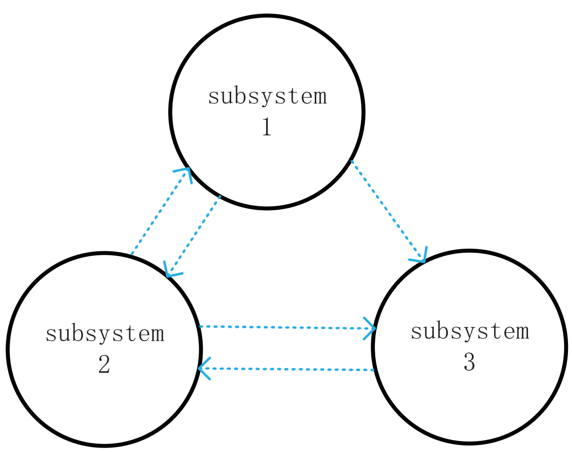

Figure 1: Subsystems and switching in example 1

Table 1: Comparison between ADT and SBAPDT asynchronous switching for example 1

	theorems in [2]	SBAPDT results in this paper
parameters	λ=0.3200;	$\lambda_1 = \lambda_2 = \lambda_3$=0.3200;
	$\mu = 3.800$;	$\mu_{[3\|1]} = \mu_{[2\|3]} = \mu_{[1\|2]} = \mu_{[2\|1]} = \mu_{[2\|1]}$=3.800;
	$\alpha = 4.400$.	$\alpha_{[3\|1]} = \alpha_{[2\|3]} = \alpha_{[1\|2]} = \alpha_{[2\|1]} = \alpha_{[2\|1]}$=4.400;
switching signals	Not feasible.	$\tau^*_{a(2,[1\|2])} = 4.172$; $\tau^*_{a(1,[2\|1])} = 5.647$; $\tau^*_{a(1,[3\|1])} = 5.647$; $\tau^*_{a(2,[3\|2])} = 5.647$; $\tau^*_{a(3,[2\|3])} = 5.647$.

For the linear or nonlinear switched asynchronous systems, the stability conditions presented in this paper are sufficient conditions which can guarantee the stability of switched asynchronous systems.

4. NUMERICAL EXAMPLES

In this section, we verify the validity of the two sequence-based approaches for switched systems. Owing to the similarities between continuous-time and discrete-time asynchronous switched systems, here we only verify the results for the continuous-time cases. Other theorems and cases can be verified similarly.

In this section, we give two examples. The first one is to verify the results on SBAPDT, and its switching situation is shown in Fig. 1. The results on SBASDT is verified in example 2, whose switching situation is shown in Fig. 2.

In both examples, the asynchronous switched system (1) is with $A_1 = \begin{bmatrix} 0.3 & -0.6 \\ 0.6 & -0.4 \end{bmatrix}$, $B_1 = \begin{bmatrix} -0.5 & -1.9 \end{bmatrix}^T$, $K_1 = [-1.4316 \quad 0.7373]$, $A_2 = \begin{bmatrix} 0.3 & 0.3 \\ -1.2 & 0.4 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 & -1.5 \end{bmatrix}^T$, $K_2 = [-0.1760 \quad 0.5674]$, $A_3 = \begin{bmatrix} 0.3 & 0.3 \\ -1 & 0.4 \end{bmatrix}$, $B_3 = \begin{bmatrix} 0.2 & 1.5 \end{bmatrix}^T$, $K_3 = [-1.6987 \quad -0.6626]$.

Example 1. In this example, the subsystems 1 and 2 can switch to other two subsystems randomly. The subsystems 3 can only switch to subsystem 1.

It is hard to know $T_{[p|q]\_max}$ in practice. A common way is to estimate its value, e.g., as proposed in [2]. $T_{[p|q]\_max}$ is not more than the asynchronous time "$T_{[p|q]}$" in every sequence (p,q). As it comes from the time "T" in system (2). Therefore, the maximal asynchronous time "$T_{[p|q]}$" is often taken as $T_{[p|q]\_max}$. We assume that all the asynchronous switching time is 0.1.

Table 2: Comparison between ADT and SBASDT asynchronous switching for example 2

	theorems in [2]	SBASDT results in this paper
parameters	λ=0.3000;	$\lambda_1 = \lambda_2 = 0.3000$;$\lambda_3 = 0.6107$;$\alpha_{[2\|3]}$=2.560;$\alpha_{[1\|2]}$=0.2678;
	$\mu = 3.969$;	$\mu_{[2\|1]} = \mu_{[1\|2]} = 3.969$;$\mu_{[1\|3]} = \mu_{[2\|3]} = \mu_{[3\|1]} = \mu_{[3\|2]}$=1.001;
	$\alpha = 4.456$.	$\alpha_{[1\|3]}$=4.456;$\alpha_{[2\|1]}$=0.2159;$\alpha_{[3\|1]}$=4.455; $\alpha_{[3\|2]}$=3.975.
switching		$\tau^*_{a(1,[1\|2])} = 4.595$; $\tau^*_{a(2,[2\|1])} = 5.455$;
signals	τ^*_a=20.45.	$\tau^*_{a(3,[3\|1])} = 6.636$; $\tau^*_{a(1,[1\|3])} = 11.09$;
		$\tau^*_{a(2,[2\|3])} = 9.533$; $\tau^*_{a(3,[3\|2])} = 6.758$.

For the conventional ADT methods, it is not feasible for the Matlab LMIs toolbox to find a solution according to the results in [2] when we set λ=0.3200, $\mu = 3.800$, $\alpha = 4.400$. But for our SBAPDT method, the Matlab LMIs toolbox can find a solution if we set $\lambda_1 = \lambda_2 = \lambda_3$=0.3200, $\mu_{[3|1]} = \mu_{[2|3]} = \mu_{[1|2]} = \mu_{[2|1]} = \mu_{[2|1]}$=3.800, $\alpha_{[3|1]} = \alpha_{[2|3]} = \alpha_{[1|2]} = \alpha_{[2|1]} = \alpha_{[2|1]} =$ 4.400. Therefore, the SBAPDT method is superior for the discussed problems. The conditions and numerical results for this example are shown in Table 1.

The design parameters are selected based on the desired system performance and parameters' physical meaning discussed in [1, 5, 6, 23].

Remark 6. *The sequences in this papers are mainly referred to the set* $\{(p,q) : (\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}, p \neq q, i = 1,\ 2, \cdots,\}$ *and its elements. As* $i \to +\infty$*, the sequence set is not related to the time* t_i *anymore. It can be obtained directly from the switching population rather than being computed in real time. That is, it can be known in advance. For Example 1, all possible switching events are shown in Fig. 1. According to the set* $\{(p,q) : (\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}, p \neq q, i = 1,\ 2, \cdots,\}$ *and Fig. 1., one can obtain the sequences set as* $\{(1,2),(2,1),(2,3),(3,1),(3,2)\}$*. A sequence is one of its elements.*

Remark 7. *In some switched systems, certain sequences, which belong to* $\mathcal{S}' \triangleq \{(p,q); p \in \mathcal{S}, q \in \mathcal{S}, p \neq q\}$*, may not exist in practice. For these sequences, we do not need to check whether they satisfy the conditions (47)-(49), (51)-(53), (54)-(56), (57)-(59). That is, only the sequences in the set* $\{(p,q) : (\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}, p \neq q, i = 1,\ 2, \cdots,\}$ *need to be checked to determine whether they satisfy these conditions. In example 1, the set* $\{(p,q) : (\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}, p \neq q, i = 1,\ 2, \cdots,\}$=$\{(1,2),\ (2,1),\ (2,3),\ (3,1),\ (3,2)\}$*. The sequence* $\{1,3\}$ *does not exist. Therefore, we do not need to check any condition in terms of* $\mu_{[1|3]}$ *or* $\alpha_{[1|3]}$*.*

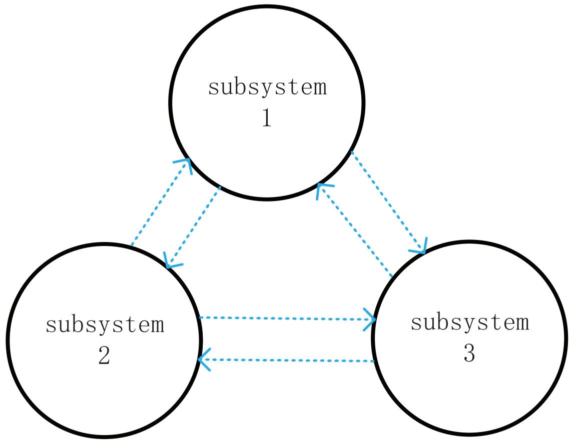

Figure 2: Subsystems and switching in example 2

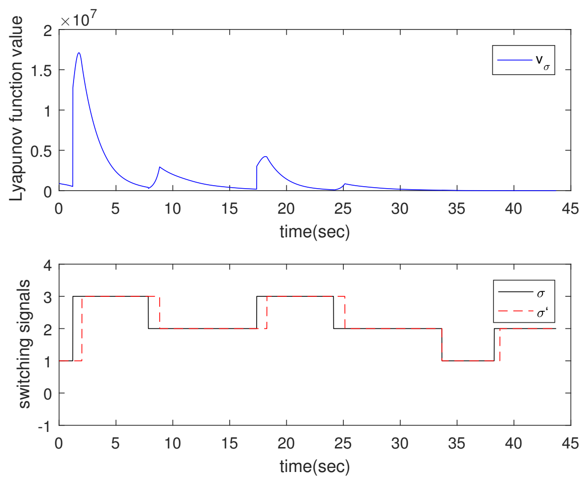

Figure 3: Lyapunov function value and switching signals

Although the sequence-based methods and LMIs algorithms lead to much more LMIs and parameters, they do not dramatically increase the computational complexity for calculating LMIs' feasible solutions. Meanwhile, as the stability analysis is off-line, the computing time is not a major issue.

Example 2. In this example, each subsystem can switch to the other two subsystems randomly as shown in Fig. 2.

We assume that asynchronous switching time is not more than 0.5000, 0, 0.7000, 0.8000, 0.9000, and 1.000 when $[p|q] = [2|1]$, $[p|q] = [1|2]$, $[p|q] = [1|3]$, $[p|q] = [3|1]$, $[p|q] = [3|2]$, and $[p|q] = [2|3]$, respectively. For the conventional ADT methods [2], the Matlab LMIs toolbox can find a feasible solution only if $\lambda \leq$0.3000 or $\mu \geq 3.969$ or $\alpha \geq 4.456$. When $\lambda =$0.3000, $\mu = 3.969$, and $\alpha = 4.456$,

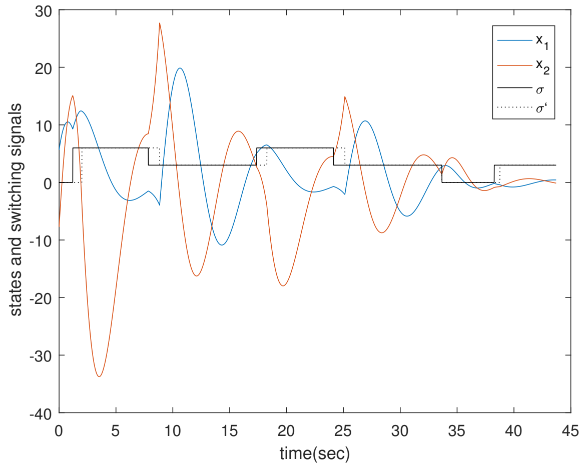

Figure 4: States and switching signals

we obtain τ_a^*=20.45. But for the SBASDT method, the Matlab LMIs toolbox can find a solution even if we set $\lambda_1 = \lambda_2 = 0.3000$, $\lambda_3 = 0.6107$, $\alpha_{[2|3]}$=2.560, $\alpha_{[1|2]}$=0.2678, $\mu_{[2|1]} = \mu_{[1|2]} = 3.969$, $\mu_{[1|3]} = \mu_{[2|3]} = \mu_{[3|1]} = \mu_{[3|2]}$=1.001, $\alpha = 4.456$, $\alpha_{[1|3]}$=4.456, $\alpha_{[2|1]}$=0.2159, $\alpha_{[3|1]}$=4.455, and $\alpha_{[3|2]}$=3.975. The conditions and results for this example are shown in Table 2.

It can be seen that the sequence-based approach makes the threshold of average dwell time decrease significantly. Therefore, the SBASDT method has considerable advantage compared with traditional ADT methods.

In order to show the global asymptotic stability, the states x1 and x2 of the second order switched system is shown in Fig. 4, and its Lyapunov function value is given in Fig. 3, where σ' is the asynchronous switching signal.

5. CONCLUSIONS

We have presented two novel methods for the stability analysis of asynchronous switching by discriminating the switching sequences of subsystems. Several less conservative conditions are derived. Unlike existing methods, the proposed approaches consider and exploit the differences among different switching sequences of subsystems, such as the maximal asynchronous switching time, the energy changing degree at switching times, and the increasing speed of energy functions in asynchronous time intervals. The SBAPDT approach counterbalances the increasing of energy functions in asynchronous time intervals by prolonging the preceding rather than subsequent subsystems. Numerical results verify the effectiveness of the proposed approaches, especially when sequences have small asynchronous switching time.

References

[1] H. Lin, P. J. Antsaklis, Stability and stabilizability of switched linear systems: a survey of recent results, IEEE Trans. Autom. Control 54 (2) (2009) 308–322.

[2] L. Zhang, H. Gao, Asynchronously switched control of switched linear systems with average dwell time, Automatica 46 (5) (2010) 953–958.

[3] Y. Wang, B. Niu, B. Wu, C. Wu, X. Xie, Asynchronous switching for switched nonlinear input delay systems with unstable subsystems, J. Frankl. Inst. 355 (5) (2018) 2912–2931.

[4] X. Su, X. Liu, P. Shi, R. Yang, Sliding mode control of discrete-time switched systems with repeated scalar nonlinearities, IEEE Trans. Autom. Control 62 (9) (2017) 4604–4610.

[5] D. Zheng, H. Zhang, Q. Zheng, Consensus analysis of multi-agent systems under switching topologies by a topology-dependent average dwell time approach, IET Control Theory Appl. 11 (3) (2017) 429–438.

[6] D. Zheng, H. Zhang, J. A. Zhang, G. Wang, Consensus of multi-agent systems with faults and mismatches under switched topologies using a delta operator method, Neurocomputing 315 (2018) 198–209.

[7] Z. Fei, S. Shi, C. Zhao, L. Wu, Asynchronous control for 2-D switched systems with mode-dependent average dwell time, Automatica 79 (2017) 198–206.

[8] A. S. Morse, Supervisory control of families of linear set-point controllers-part i. exact matching, IEEE Trans. Autom. Control 41 (10) (1996) 1413–1431.

[9] J. P. Hespanha, A. S. Morse, Stability of switched systems with average dwell-time, in: Proc. 38th IEEE Conf. Decision Control, Vol. 3, 1999, pp. 2655–2660.

[10] X. Zhao, L. Zhang, P. Shi, M. Liu, Stability and stabilization of switched linear systems with mode-dependent average dwell time, IEEE Trans. Autom. Control 57 (7) (2012) 1809–1815.

[11] Z. Fei, S. Shi, Z. Wang, L. Wu, Quasi-time-dependent output control for discrete-time switched system with mode-dependent average dwell time, IEEE Trans. Autom. Control 63 (8) (2018) 2647–2653.

[12] L. Zhang, P. Shi, Stability, l_2-gain and asynchronous h_∞ control of discrete-time switched systems with average dwell time, IEEE Trans. Autom. Control 54 (9) (2009) 2192–2199.

[13] L. Zhang, N. Cui, M. Liu, Y. Zhao, Asynchronous filtering of discrete-time switched linear systems with average dwell time, IEEE Trans. Circuits Syst. I 58 (5) (2011) 1109–1118.

[14] Q. Lu, L. Zhang, H. R. Karimi, Y. Shi, h_∞ control for asynchronously switched linear parameter-varying systems with mode-dependent average dwell time, IET Control Theory Appl. 7 (5) (2013) 673–683.

[15] G. Zong, R. Wang, W. Zheng, L. Hou, Finite-time h_∞ control for discrete-time switched nonlinear systems with time delay, Int. J. Robust Nonlinear Control 25 (6) (2015) 914–936.

[16] S. Yuan, L. Zhang, B. De Schutter, S. Baldi, A novel Lyapunov function for a non-weighted l_2 gain of asynchronously switched linear systems, Automatica 87 (2018) 310–317.

[17] H. Sang, H. Nie, Asynchronous h_∞ control for discrete-time switched systems under state-dependent switching with dwell time constraint, Nonlinear Anal. Hybrid Syst 29 (2018) 187–202.

[18] S. Li, C. K. Ahn, Z. Xiang, Sampled-data adaptive output feedback fuzzy stabilization for switched nonlinear systems with asynchronous switching, IEEE Trans. Fuzzy Syst. 27 (1) (2019) 200–205.

[19] S. Li, J. Guo, Z. Xiang, Global stabilization of a class of switched nonlinear systems under sampled-data control, IEEE Trans. Syst., Man, Cyber. 49 (9) (2019) 1912–1919.

[20] C. Briat, Convex lifted conditions for robust l_2-stability analysis and l_2-stabilization of linear discrete-time switched systems with minimum dwell-time constraint, Automatica 50 (3) (2014) 976–983.

[21] W. Xiang, D. Tran, T. T. Johnson, Nonconservative lifted convex conditions for stability of discrete-time switched systems under minimum dwell-time constraint, IEEE Trans. Autom. Control 64 (8) (2019) 3407–3414.

[22] A. Kundu, D. Chatterjee, D. Liberzon, Generalized switching signals for input-to-state stability of switched systems, Automatica 64 (2016) 270–277.

[23] D. Zheng, H. Zhang, J. A. Zhang, Y. Li, Consensus of the second-order multi-agent systems under asynchronous switching with a controller fault, International Journal of Control, Automation and Systems 17 (1) (2019) 136–144.