

Quantum Algorithm for Lexicographically Minimal String Rotation

Qisheng Wang *

Mingsheng Ying †

Abstract

Lexicographically minimal string rotation (LMSR) is a problem to find the minimal one among all rotations of a string in the lexicographical order, which is widely used in equality checking of graphs, polygons, automata and chemical structures.

In this paper, we propose an $O(n^{3/4})$ quantum query algorithm for LMSR. In particular, the algorithm has average-case query complexity $O(\sqrt{n} \log n)$, which is shown to be asymptotically optimal up to a polylogarithmic factor, compared with its $\Omega(\sqrt{n/\log n})$ lower bound. Furthermore, we claim that our quantum algorithm outperforms any (classical) randomized algorithms in both worst-case and average-case query complexities by showing that every (classical) randomized algorithm for LMSR has worst-case query complexity $\Omega(n)$ and average-case query complexity $\Omega(n/\log n)$.

Our quantum algorithm for LMSR is developed in a framework of nested quantum algorithms, based on two new results: (i) an $O(\sqrt{n})$ (optimal) quantum minimum finding on bounded-error quantum oracles; and (ii) its $O(\sqrt{n \log(1/\varepsilon)})$ (optimal) error reduction. As a byproduct, we obtain some better upper bounds of independent interest: (i) $O(\sqrt{N})$ (optimal) for constant-depth MIN-MAX trees on N variables; and (ii) $O(\sqrt{n \log m})$ for pattern matching which removes $\text{polylog}(n)$ factors.

*Department of Computer Science and Technology, Tsinghua University, China. Email: wqs17@mails.tsinghua.edu.cn

†Centre for Quantum Software and Information, University of Technology Sydney, Australia; State Key Laboratory of Computer Science, Institute of Software, Chinese Academy of Sciences, China; Department of Computer Science and Technology, Tsinghua University, China. Email: Mingsheng.Ying@uts.edu.au

1 Introduction

1.1 Lexicographically Minimal String Rotation

Lexicographically Minimal String Rotation (LMSR) is a problem to find the minimal one among all rotations of a string in the lexicographical order (see [Jeu93, CR94, CHL07]). It has been widely used in equality checking of graphs [CB80], polygons [IS89, Mae91], automata [Pup10] (and their minimizations [AZ08]) and chemical structures [Shi79], and in generating de Bruijn sequences [SWW16] (see also [GY03, vLWW01]). Booth [Boo80] first proposed an algorithm in linear time for LMSR based on the Knuth-Morris-Pratt string-matching algorithm [KMP77]. Shiloach [Shi81] later improved Booth’s algorithm in terms of performance. A more efficient algorithm was developed by Duval [Duv83] from a different point of view known as Lyndon Factorization. All these algorithms for LMSR are deterministic and have worst-case time complexity $\Theta(n)$. After that, several parallel algorithms for LMSR have also been proposed. Apostolico, Iliopoulos and Paige [AIP87] found an $O(\log n)$ time CRCW PRAM algorithm for LMSR using $O(n)$ processors, which was then improved by Iliopoulos and Smyth [IS92] to use only $O(n/\log n)$ processors.

The LMSR of a string s can also be computed by finding the lexicographically minimal suffix of $ss\$,$ i.e. the concatenation of two occurrences of s and an end-marker $\$,$ where $\$$ is considered as a character lexicographically larger than every character in s . The minimal suffix of a string can be solved in linear time with the help of data structures known as suffix trees [Wei73, AHU74, McC76] and suffix arrays [MM90, GBYS92, KSB06], and alternatively by some specific algorithms [Duv83, Cro92, Ryt03] based on Duval’s algorithm [Duv83] or the KMP algorithm [KMP77].

1.2 Quantum Solutions to Related Problems

Pattern matching problem is to determine whether a pattern p of length m occurs in a text t of length n . In classical computing, it is considered to be closely related to LMSR. The Knuth-Morris-Pratt algorithm [KMP77] used in Booth’s algorithm [Boo80] for LMSR mentioned above is one of the first few algorithms for pattern matching, with time complexity is $\Theta(n + m)$. Recently, several quantum algorithms have been developed for pattern matching; for example, Ramesh and Vinay [RV03] developed an $O(\sqrt{n} \log(n/m) \log m + \sqrt{m} \log^2 m)$ quantum pattern matching algorithm based on a useful technique for parallel pattern matching, namely deterministic sampling [Vis90], and Montanaro [Mon17] proposed an average-case $O\left((n/m)^d 2^{O(d^{3/2} \sqrt{\log m})}\right)$ quantum algorithm for d -dimensional pattern matching. However, it seems that these quantum algorithms for pattern matching cannot be directly generalized to solve LMSR.

1.3 Main Contributions of This Paper

A naive quantum algorithm for LMSR (see Definition 2.1 for its formal definition) is to find the LMSR of the string within $O(\sqrt{n})$ comparisons of rotations by quantum minimum finding [DH96, AK99] among all rotations. However, each comparison of two rotations in the lexicographical order costs $O(\sqrt{n})$ and is bounded-error. Combining the both, an $\tilde{O}(n)$ quantum algorithm for LMSR is obtained, which has no advantages compared to classical algorithms.

In this paper, however, we find a more efficient quantum algorithm for LMSR. Formally, we have:

Theorem 1.1 (Quantum Algorithm for LMSR). *There is a bounded-error quantum query algorithm for LMSR, of which the worst-case query complexity is $O(n^{3/4})$ and average-case query complexity is $O(\sqrt{n} \log n)$.*

In the top-level design of this algorithm, we are required to find the minimal value of a function, which is given by a bounded-error quantum oracle. To resolve this issue, we develop an efficient error reduction for nested quantum algorithms (see Section 1.4.1 for a outline). With this framework of nested quantum algorithms, we are able to solve problems in nested structures efficiently. The high level illustrations of the algorithm for the worst and average cases are given in Section 1.4.2 and Section 1.4.3, respectively. A detailed description of the algorithm is presented in Section 5.

We assume access to a quantum-read/classical-write random access memory (QRAM) and define time complexity as the number of elementary two-qubit quantum gates, input queries and QRAM operations (see Section 2.2.2 for more details). Our quantum algorithm uses only $O(\log^2 n)$ classical bits in QRAM and $O(\log n)$ “actual” qubits in the quantum computation. Thus, the time complexity of our quantum algorithm is just a polylogarithmic factor bigger than its query complexity (in both worst and average cases).

In order to show a separation between classical and quantum algorithms for LMSR, we settle the classical and quantum lower bounds for LMSR in both the worst and average cases. Let $R(LMSR)$ and $R^{unif}(LMSR)$ be the worst-case and average-case (classical) randomized query complexities for LMSR, and let $Q(LMSR)$ and $Q^{unif}(LMSR)$ be their quantum counterparts. Then we have:

Theorem 1.2 (Classical and Quantum Lower Bounds for LMSR). *1. For every bounded-error (classical) randomized algorithm for LMSR, it has worst-case query complexity $\Omega(n)$ and average-case query complexity $\Omega(n/\log n)$. That is, $R(LMSR) = \Omega(n)$ and $R^{unif}(LMSR) = \Omega(n/\log n)$.*

2. For every bounded-error quantum algorithm for LMSR, it has worst-case query complexity $\Omega(\sqrt{n})$ and average-case query complexity $\Omega(\sqrt{n/\log n})$. That is, $Q(LMSR) = \Omega(\sqrt{n})$ and $Q^{unif}(LMSR) = \Omega(\sqrt{n/\log n})$.

Remark 1.1. *It suffices to consider only bounded-error quantum algorithms for LMSR, as we can show that every exact (resp. zero-error) quantum algorithm for LMSR has worst-case query complexity $\Omega(n)$. This is achieved by reducing the searching problem to LMSR (see Appendix F), since the searching problem is known to have worst-case query complexity $\Omega(n)$ for exact and zero-error quantum algorithms [BBC⁺01].*

Theorem 1.2 is proved in Section 6. Our main proof technique is to reduce a total Boolean function to LMSR and to find a lower bound of that Boolean function based on the notion of block sensitivity. The key observation is that the block sensitivity of that Boolean function is related to the string sensitivity of input (Lemma 6.3, and see Section 1.4.3 for more discussions).

The results of Theorems 1.1 and 1.2 can be summarized as Table 1.

	Classical		Quantum	
	Lower bounds	Upper bounds	Lower bounds	Upper bounds
Worst-case	$\Omega(n)$	$O(n)$ [Boo80, Shi81, Duv83]	$\Omega(\sqrt{n})$	$O(n^{3/4})$
Average-case	$\Omega(n/\log n)$	$O(n)$ [IS94, BCN05]	$\Omega(\sqrt{n/\log n})$	$O(\sqrt{n} \log n)$

Table 1: Classical (randomized) and quantum query complexities of LMSR.

Note that

$$\Omega(\sqrt{n/\log n}) \leq Q^{unif}(LMSR) \leq O(\sqrt{n} \log n).$$

Therefore, our quantum algorithm is asymptotically optimal in the average case up to a logarithmic factor. Moreover, a quantum separation from (classical) randomized computation in both the worst-case and average-case query complexities is achieved:

1. Worst case: $Q(LMSR) = O(n^{3/4})$ but $R(LMSR) = \Omega(n)$; and
2. Average case: $Q^{unif}(LMSR) = O(\sqrt{n} \log n)$ but $R^{unif}(LMSR) = \Omega(n/\log n)$.

In other words, our quantum algorithm is faster than any classical randomized algorithms both in the worst case and the average case.

As an application, we show that our algorithm can be used in identifying benzenoids [Baš16] and minimizing disjoint-cycle automata [AZ08] (see Section 7). The quantum speedups for these problems are illustrated in Table 2.

	Classical	Quantum (this work)
LMSR	$O(n)$ [Boo80, Shi81, Duv83]	$O(n^{3/4})$
Canonical boundary-edges code	$O(n)$ [Baš16]	$O(n^{3/4})$
Disjoint-cycle automata minimization	$O(mn)$ [AZ08]	$\tilde{O}(m^{2/3}n^{3/4})$

Table 2: Classical and quantum query complexities of LMSR, canonical boundary-edges code and disjoint-cycle automata minimization. For disjoint-cycle automata minimization, m indicates the number of disjoint cycles and n indicates the length of each cycle. Here, $\tilde{O}(\cdot)$ suppresses logarithmic factors.

1.4 Overview of the Technical Ideas

Our results presented in the above subsection are achieved by introducing the following three new ideas:

1.4.1 Optimal Error Reduction for Nested Quantum Minimum Finding

Our main algorithm for LMSR is essentially a nest of quantum search and minimum finding. A major difficulty in its design is error reduction in nested quantum oracles, which has not been considered in the previous studies of nested quantum algorithms (e.g. nested quantum search analyzed by Cerf, Grover and Williams [CGW00] and nested quantum walks introduced by Jeffery, Kothari and Magniez [JKM13]).

A d -level nested classical algorithm need $O(\log^{d-1} n)$ repetitions to ensure a constant error probability by majority voting. For a d -level quantum algorithm composed of quantum minimum finding, it is known that only a small factor $O(\log n)$ of repetitions is required [CGYM08]. We show that this factor can be even better; that is, only $O(1)$ of repetitions are required as if there were no errors in quantum oracles:

- We extend quantum minimum finding algorithm [DH96, AK99] to the situation where the input is given by a bounded-error oracle so that it has query complexity $O(\sqrt{n})$ (see Lemma 3.4) rather than $O(\sqrt{n} \log n)$ straightforwardly by majority voting.
- We introduce a probability amplification method for quantum minimum finding on bounded-error oracles, which requires $O\left(\sqrt{n \log(1/\varepsilon)}\right)$ query complexity to obtain the minimum with error probability $\leq \varepsilon$ (see Lemma 3.5). In contrast, a straightforward solution by $O(\log(1/\varepsilon))$ repetitions of the previous $O(\sqrt{n})$ algorithm has query complexity $O(\sqrt{n} \log(1/\varepsilon))$.

These ideas are inspired by quantum searching on bounded-error oracles [HMdW03] and amplification of success of quantum search [BCdWZ99]. The above two algorithms will be used in the main algorithm for LMSR as a subroutine. Both of them are optimal because their simpler version (OR function) has lower bound $\Omega(\sqrt{n})$ [BBBV97, BBHT98, Zal99] for quantum bounded-error algorithm and $\Omega(\sqrt{n \log(1/\varepsilon)})$ [BCdWZ99] for its error reduction. To be clearer, we compare the results about quantum searching and quantum minimum finding in the previous literature and ours in Table 3.

Algorithm type	Oracle type	Search	Minimum finding
Bounded-error	exact	$O(\sqrt{n})$ [Grö96]	$O(\sqrt{n})$ [DH96, AK99]
	bounded-error	$O(\sqrt{n})$ [HMdW03]	$O(\sqrt{n})$
Error reduction	exact	$O(\sqrt{n \log(1/\varepsilon)})$ [BCdWZ99]	$O(\sqrt{n \log(1/\varepsilon)})$
	bounded-error	$O(\sqrt{n \log(1/\varepsilon)})$ [HMdW03]	$O(\sqrt{n \log(1/\varepsilon)})$

Table 3: Quantum query complexities of bounded-error quantum algorithms and their error reductions for search and minimum finding.

Based on the above results, we develop an $O(\sqrt{n \log^3 n \log \log n})$ quantum algorithm for deterministic sampling [Vis90], and furthermore obtain an $O(\sqrt{n \log m} + \sqrt{m \log^3 m \log \log m})$ quantum algorithm for pattern matching, which is better than the best known result [RV03] of $O(\sqrt{n} \log(n/m) \log m + \sqrt{m} \log^2 m)$. We also develop an optimal $O(\sqrt{n^d})$ quantum algorithm for evaluating d -level shallow MIN-MAX trees that matches the lower bound $\Omega(\sqrt{n^d})$ [Amb00, BS04] for AND-OR trees, and therefore it is optimal. The best known quantum query complexity of MIN-MAX trees is $O(W_d(n) \log n)$ [CGYM08], where $W_d(n)$ is the query complexity of d -level AND-OR trees and optimally $O(\sqrt{n^d})$ as known from [BCW98, HMdW03]. Our improvements on these problems are summarized in Table 4.

	Previous	Improved
Deterministic sampling	$O(\sqrt{n} \log^2 n)$ [RV03]	$O(\sqrt{n \log^3 n \log \log n})$
Pattern matching	$O(\sqrt{n} \log(n/m) \log m)$ [RV03]	$O(\sqrt{n \log m})$
d -level MIN-MAX tree	$O(\sqrt{n^d} \log n)$ [CGYM08]	$O(\sqrt{n^d})$

Table 4: Quantum query complexities improved by our nested quantum algorithm framework.

1.4.2 Exclusion Rule of LMSR

We find a useful property of LMSR, named *exclusion rule* (Lemma 5.1): for any two overlapped substrings which are prefixes of the canonical representation of a string, the LMSR of the string can not be the latter one. This property enables us to reduce the worst-case query complexity by splitting the string into blocks of suitable sizes, and in each block the exclusion rule can apply so that there is at most one candidate for LMSR. This kind of trick has been used in parallel algorithms, e.g. Lemma 1.1 of [IS92] and the Ricochet Property of [Vis90]. However, the exclusion rule of LMSR used here is not found in the literature (to the best of our knowledge).

We outline our algorithm as follows:

- 1 Let $B = \lfloor \sqrt{n} \rfloor$ and $L = \lfloor B/4 \rfloor$. We split s into $\lceil n/L \rceil$ blocks of size L (except the last block).
- 2 Find the prefix p of $SCR(s)$ of length B , where $SCR(s)$ is the canonical representation of s .
- 3 In each block, find the *leftmost* index that matches p as the candidate. Only the leftmost index is required because of the exclusion rule of LMSR.
- 4 Find the lexicographically minimal one among all candidates in blocks. In case of a tie, the minimal candidate is required.

A formal description and the analysis of this algorithm is given in Section 5.1. In order to find the leftmost index that matches p (Step 3 of this algorithm) efficiently, we adopt the deterministic sampling [Vis90] trick. That is, we preprocess a deterministic sample of p , with which whether an index matches p can be checked within $O(\log |p|)$ rather than $O(|p|)$ comparisons. Especially, we allow p to be periodic, and therefore extend the definition of deterministic samples for periodic strings (see Definition 4.1) and propose a quantum algorithm for finding a deterministic sample of a string (either periodic or aperiodic) (see Algorithm 4).

1.4.3 String Sensitivity

We also observe the property of LMSR that almost all strings are of low string sensitivity (Lemma 5.3), which can be used to reduce the query complexity of our quantum algorithm significantly in the average case. Here, the string sensitivity of a string (see Definition 5.1) is a metric showing the difficulty to distinguish its substrings, and helpful to obtain lower bounds for LMSR (see Lemma 6.3).

We outline our improvements for better average-case query complexity as follows:

- 1 Let s_1 and s_2 be the minimal and the second minimal substrings of s of length $B = O(\log n)$, respectively.
- 2 If $s_1 < s_2$ lexicographically, then return the starting index of s_1 ; otherwise, run the basic quantum algorithm given in Section 1.4.2.

Intuitively, in the average case, we only need to consider the first $O(\log n)$ characters in order to compare two rotations. The correctness is straightforward but the average-case query complexity needs some analysis. See Section 5.2 for a formal description and the analysis of the improvements.

1.5 Organization of This Paper

We recall some basic definitions about strings and formally define the LMSR problem in Section 2. An efficient error reduction for nested quantum algorithms is developed in Section 3. An improved quantum algorithm for pattern matching based on the new error reduction technique is proposed in Section 4. The quantum algorithm for LMSR is proposed in Section 5. The classical and quantum lower bounds for LMSR are given in Section 6. The applications are discussed in Section 7.

2 Preliminaries

For convenience of the reader, in this section, we briefly review the lexicographically minimal string rotation (LMSR) problem, quantum query model and several notions of worst-case and average-case complexities used in the paper.

2.1 Lexicographically Minimal String Rotation

For any positive number n , let $[n] = \{0, 1, 2, \dots, n-1\}$. Let Σ be a finite alphabet with a total order $<$. A string $s \in \Sigma^n$ of length n is a function $s : [n] \rightarrow \Sigma$. The empty string is denoted ϵ . We use $s[i]$ to denote the i -th character of s . In case of $i \notin \mathbb{Z} \setminus [n]$, we define $s[i] \equiv s[i \bmod n]$. If $l < r$, $s[l \dots r] = s[l]s[l+1] \dots s[r]$ stands for the substring consisting of the l -th to the r -th character of s , and if $l > r$, we define $s[l \dots r] = \epsilon$. A prefix of s is a string of the form $s[0 \dots i]$ for $i \in [n] \cup \{-1\}$. The period of a string $s \in \Sigma^n$ is the minimal positive integer d such that $s[i] = s[i+d]$ for every $0 \leq i < n-d$. String s is called *periodic* if its period $\leq n/2$, and s is *aperiodic* if it is not periodic.

Let $s \in \Sigma^n$ and $t \in \Sigma^m$. The concatenation of s and t is string $st = s[0]s[1] \dots s[n-1]t[0]t[1] \dots t[m-1]$. We write $s = t$ if $n = m$ and $s[i] = t[i]$ for every $i \in [n]$. We say that s is smaller than t in the *lexicographical order*, denoted $s < t$, if either s is a prefix of t but $s \neq t$, or there exists an index $0 \leq k < \min\{n, m\}$ such that $s[i] = t[i]$ for $i \in [k]$ and $s[k] < t[k]$. For convenience, we write $s \leq t$ if $s < t$ or $s = t$.

Definition 2.1 (Lexicographically Minimal String Rotation). *For any string $s \in \Sigma^n$ of length n , we call $s^{(k)} = s[k \dots k+n-1]$ the rotation of s by offset k . The lexicographically minimal string rotation (LMSR) problem is to find an offset k such that $s^{(k)}$ is the minimal string among $s^{(0)}, s^{(1)}, \dots, s^{(n-1)}$ in the lexicographical order. The minimal $s^{(k)}$ is called the string canonical representation (SCR) of s , denoted $SCR(s)$; that is,*

$$SCR(s) = \min \left\{ s^{(0)}, s^{(1)}, \dots, s^{(n-1)} \right\}.$$

In case of a tie; that is, there are multiple offsets such that each of their corresponding strings equals to $SCR(s)$, then the minimal offset is desired, and the goal is to find

$$LMSR(s) = \min \left\{ k \in [n] : s^{(k)} = SCR(s) \right\}.$$

The LMSR problem has been well-studied in the literature [CB80, Boo80, Shi81, Duv83, Jeu93, CR94, CHL07], and several linear time (classical) algorithms for LMSR are known, namely Booth's, Shiloach's and Duval's Algorithms:

Theorem 2.1 ([Boo80, Shi81, Duv83]). *There is an $O(n)$ deterministic algorithm for LMSR.*

2.2 Quantum Query Algorithms

Our computational model is the quantum query model [Amb04, BdW02]. The goal is to compute an n -variable function $f(x) = f(x_0, x_1, \dots, x_{n-1})$, where x_0, x_1, \dots, x_{n-1} are input variables. For example, the LMSR problem can be viewed as function $f(x) = LMSR(x_0, x_1, \dots, x_{n-1})$, where $x_0x_1x_2 \dots x_{n-1}$ denotes the string of element x_0, x_1, \dots, x_{n-1} . The input variables x_i can be accessed by queries to a quantum oracle O_x (which is a quantum unitary operator) defined by $O_x |i, j\rangle = |i, j \oplus x_i\rangle$, where \oplus is the bitwise exclusive-OR operation. A quantum algorithm A with T queries is described by a sequence of quantum unitary operators

$$A : U_0 \rightarrow O_x \rightarrow U_1 \rightarrow O_x \rightarrow \dots \rightarrow O_x \rightarrow U_T.$$

The intermediate operators U_0, U_1, \dots, U_T can be arbitrary quantum unitary operators that are determined independent of O_x . The computation is performed in a Hilbert space $\mathcal{H} = \mathcal{H}_o \otimes \mathcal{H}_w$, where \mathcal{H}_o is the output space and \mathcal{H}_w is the work space. The computation starts from basis state $|0\rangle_o |0\rangle_w$, and then we apply $U_0, O_x, U_1, O_x, \dots, O_x, U_T$ on it in that order. The result state is

$$|\psi\rangle = U_T O_x \dots O_x U_1 O_x U_0 |0\rangle_o |0\rangle_w.$$

Measuring the output space, the outcome is then defined as the output $A(x)$ of algorithm A on input x . More precisely, $\Pr[A(x) = y] = \|M_y |\psi\rangle\|^2$, where $M_y = |y\rangle_o \langle y|$. Furthermore, A is said to be a bounded-error quantum algorithm that computes f , if $\Pr[A(x) = f(x)] \geq 2/3$ for every x .

2.2.1 Worst-case and Average-case Query Complexities

Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function. If A is a (either randomized or quantum) algorithm and $y \in \{0, 1\}$, we use $\Pr[A(x) = y]$ to denote the probability that A outputs y on input x . Let $T_A(x)$ denote the expected number of queries that A uses on input x . Note that $T_A(x)$ only depends on the algorithm A and its given input x (which is fixed rather than from some distribution). Let $\mathcal{R}(f)$ and $\mathcal{Q}(f)$ be the set of randomized and quantum bounded-error algorithms that compute f , respectively:

$$\begin{aligned}\mathcal{R}(f) &= \{\text{randomized algorithm } A : \forall x \in \{0, 1\}^n, \Pr[A(x) = f(x)] \geq 2/3\}, \\ \mathcal{Q}(f) &= \{\text{quantum algorithm } A : \forall x \in \{0, 1\}^n, \Pr[A(x) = f(x)] \geq 2/3\},\end{aligned}$$

Then the *worst-case* query complexities of f are:

$$\begin{aligned}R(f) &= \inf_{A \in \mathcal{R}(f)} \max_{x \in \{0, 1\}^n} T_A(x), \\ Q(f) &= \inf_{A \in \mathcal{Q}(f)} \max_{x \in \{0, 1\}^n} T_A(x).\end{aligned}$$

Let $\mu : \{0, 1\}^n \rightarrow [0, 1]$ be a probability distribution. We usually use $unif \equiv 2^{-n}$ to denote the uniform distribution. The *average-case* query complexity of an algorithm A with respect to μ is

$$T_A^\mu = \mathbb{E}_{x \sim \mu}[T_A(x)] = \sum_{x \in \{0, 1\}^n} \mu(x) T_A(x).$$

Thus, the randomized and quantum average-case query complexities of f with respect to μ are:

$$\begin{aligned}R^\mu(f) &= \inf_{A \in \mathcal{R}(f)} T_A^\mu, \\ Q^\mu(f) &= \inf_{A \in \mathcal{Q}(f)} T_A^\mu.\end{aligned}$$

Clearly, $Q^\mu(f) \leq R^\mu(f)$ for all f and μ .

2.2.2 Time and Space Efficiency

In order to talk about the “time” and “space” complexities of quantum algorithms, we assume access to a quantum-read/classical-write random access memory (QRAM), where it takes a single QRAM operation to either classically write a bit to the QRAM or make a quantum query to a bit stored in the QRAM. For simplicity, we assume the access to the QRAM is described by a quantum unitary operator U_{QRAM} that swaps the accumulator and a register indexed by another register:

$$U_{QRAM} |i, j\rangle |r_0, r_1, \dots, r_i, \dots, r_{M-1}\rangle = |i, r_i\rangle |r_0, r_1, \dots, j, \dots, r_{M-1}\rangle,$$

where r_0, r_1, \dots, r_{M-1} are bit registers that are only accessible through this QRAM operator.

Let A be a quantum query algorithm, and $t_A(x)$ denote the expected number of two-qubit quantum gates and QRAM operators U_{QRAM} composing intermediate operators, and the quantum input oracles O_x that A uses on input x . The space complexity of A measures the number of (qu)bits used in A . The worst-case and average-case time complexities of a Boolean function f are defined similarly to Section 2.2.1 by replacing $T_A(x)$ with $t_A(x)$.

3 Optimal Error Reduction for Nested Quantum Algorithms

Our quantum algorithm for LMSR (Theorem 1.1) is essentially a nested algorithm calling quantum search and quantum minimum finding. The error reduction is often crucial for nested quantum algorithms. Traditional probability amplification methods for randomized algorithms can obtain an $O(\log^d n)$ slowdown for d -level nested quantum algorithms by repeating the algorithm $O(\log n)$ times in each level. In this section, we introduce an efficient error reduction for nested quantum algorithms composed of quantum search and quantum minimum finding, which only costs a factor of $O(1)$. This improvement is obtained by finding an $O(\sqrt{n})$ quantum algorithm for minimum finding when the comparison oracle can have bounded errors (see Algorithm 1). Moreover, we also show how to amplify the success probability of quantum minimum finding with both exact and bounded-error oracles. In particular, we obtain an $O(\sqrt{n \log(1/\varepsilon)})$ quantum algorithm for minimum finding with success probability $\geq 1 - \varepsilon$ (see Algorithm 2). These two algorithms allow us to control the error produced by nested quantum oracles better than traditional (classical) methods. Both of them are optimal because their simpler version (OR function) has lower bound $\Omega(\sqrt{n})$ [BBBV97, BBHT98, Zal99] for quantum bounded-error algorithm and $\Omega(\sqrt{n \log(1/\varepsilon)})$ [BCdWZ99] for its error reduction. As an application, we develop a useful tool to find the first solution in the searching problem.

3.1 Quantum Search

Let us start from an $O(\sqrt{n})$ quantum algorithm to search on bounded-error inputs [HMdW03]. The searching problem is described by a function $f(x_0, x_1, \dots, x_{n-1})$ that finds an index $j \in [n]$ (if exists) such that $x_j = 1$, where $x_i \in \{0, 1\}$ for all $i \in [n]$. It was first shown by Gröver [Grö96] that the searching problem can be solved by an $O(\sqrt{n})$ quantum algorithm, which was later proved to be optimal [BBBV97, BBHT98, Zal99].

3.1.1 Quantum search on bounded-error oracles

A more robust approach for the searching problem on bounded-error oracles was proposed by Høyer, Mosca and de Wolf [HMdW03]. Rather than an exact quantum oracle $U_x |i, 0\rangle = |i, x_i\rangle$, they consider a bounded-error one introducing extra workspace $|0\rangle_w$:

$$U_x |i, 0\rangle |0\rangle_w = \sqrt{p_i} |i, x_i\rangle |\psi_i\rangle_w + \sqrt{1 - p_i} |i, \bar{x}_i\rangle |\phi_i\rangle_w,$$

where $p_i \geq 2/3$ for every $i \in [n]$, \bar{u} denotes the negation of u , and $|\psi_i\rangle_w$ and $|\phi_i\rangle_w$ are ignorable work qubits. This kind of bounded-error oracles is general in the sense that every bounded-error quantum algorithm and (classical) randomized algorithm can be described by it. A naive way to solve the searching problem on bounded-error oracles is to repeat $k = O(\log n)$ times and choose the majority value among the k outputs. This gives an $O(\sqrt{n} \log n)$ quantum algorithm. Surprisingly, it can be made better to $O(\sqrt{n})$ as shown in the following:

Theorem 3.1 (Quantum Search on Bounded-Error Oracles, [HMdW03]). *There is an $O(\sqrt{n})$ bounded-error quantum algorithm for the searching problem on bounded-error oracles. Moreover, if there are $t \geq 1$ solutions, the algorithm finds a solution in expected $O(\sqrt{n/t})$ queries (even if t is unknown).*

For convenience, we use $\mathbf{Search}(U_x)$ to denote the algorithm of Theorem 3.1 which, with probability $\geq 2/3$, returns an index $j \in [n]$ such that $x_j = 1$ or reports that no such j exists (we require the algorithm to return -1 in this case).

3.1.2 Amplification of the success of quantum search

Usually, we need to amplify the success probability of a quantum or (classical) randomized algorithm to make it sufficiently large. A common trick used in randomized algorithms is to repeat the bounded-error algorithm $O(\log(1/\varepsilon))$ times and choose the majority value among all outputs to ensure success probability $\geq 1 - \varepsilon$. Buhrman, Cleve, de Wolf and Zalka [BCdWZ99] showed that we can do better for quantum searching.

Theorem 3.2 (Amplification of the success of quantum search, [BCdWZ99]). *For every $\varepsilon > 0$, there is an $O\left(\sqrt{n \log(1/\varepsilon)}\right)$ bounded-error quantum algorithm for the searching problem with success probability $\geq 1 - \varepsilon$. Moreover, if there is a promise of $t \geq 1$ solutions, the algorithm finds a solution in $O\left(\sqrt{n}\left(\sqrt{t + \log(1/\varepsilon)} - \sqrt{t}\right)\right)$ queries.*

Theorem 3.2 also holds for bounded-error oracles. For convenience, we use $\mathbf{Search}(U_x, \varepsilon)$ to denote the algorithm of Theorem 3.2, which succeeds with probability $\geq 1 - \varepsilon$ (and error probability ε). Note that Theorem 3.2 does not cover the case that there can be $t \geq 2$ solutions without promise. In this case, we can obtain an $O\left(\sqrt{n/t} \log(1/\varepsilon)\right)$ bounded-error quantum algorithm with error probability $\leq \varepsilon$ by straightforward majority voting.

3.2 Quantum Minimum Finding

We now turn to consider the minimum-finding problem. Given x_0, x_1, \dots, x_{n-1} , the problem is to find an index $j \in [n]$ such that x_j is the minimal element. Let $cmp(i, j)$ be the comparator to determine whether $x_i < x_j$:

$$cmp(i, j) = \begin{cases} 1 & x_i < x_j, \\ 0 & \text{otherwise.} \end{cases}$$

The comparison oracle U_{cmp} simulating cmp is defined by

$$U_{cmp} |i, j, k\rangle = |i, j, k \oplus cmp(i, j)\rangle.$$

We measure the query complexity by counting the number of queries to this oracle U_{cmp} . A quantum algorithm was proposed by Dürr and Høyer [DH96] and Ahuja and Kapoor [AK99] for finding the minimum:

Theorem 3.3 (Minimum finding, [DH96, AK99]). *There is an $O(\sqrt{n})$ bounded-error quantum algorithm for the minimum-finding problem.*

3.2.1 Optimal quantum minimum finding on bounded-error oracles

For the purpose of this paper, we need to generalize the above algorithm to one with a bounded-error version of U_{cmp} . For simplicity, we abuse a little bit of notation and define:

$$U_{cmp} |i, j, 0\rangle |0\rangle_w = \sqrt{p_{ij}} |i, j, cmp(i, j)\rangle |\psi_{ij}\rangle_w + \sqrt{1 - p_{ij}} |i, j, \overline{cmp(i, j)}\rangle |\phi_{ij}\rangle_w,$$

where $p_{ij} \geq 2/3$ for all $i, j \in [n]$, and $|\psi_{ij}\rangle_w$ and $|\phi_{ij}\rangle_w$ are ignorable work qubits. Moreover, for every index $j \in [n]$, we can obtain a bounded-error oracle U_{cmp}^j :

$$U_{cmp}^j |i, 0\rangle |0\rangle_w = \sqrt{p_{ij}} |i, cmp(i, j)\rangle |\psi_{ij}\rangle_w + \sqrt{1 - p_{ij}} |i, \overline{cmp(i, j)}\rangle |\phi_{ij}\rangle_w$$

Algorithm 1 $\text{Minimum}(U_{cmp})$: An algorithm for minimum finding on bounded-error oracles.

Input: A bounded-error oracle U_x for x_0, x_1, \dots, x_{n-1} .

Output: An index $j \in [n]$ such that $x_j \leq x_i$ for every $i \in [n]$ with probability $\geq 2/3$.

```

1: if  $n = 1$  then
2:   return 0.
3: end if
4:  $m \leftarrow \lceil 12 \ln n \rceil$ ,  $q \leftarrow \lceil 36 \ln m \rceil$ .
5: Choose  $j \in [n]$  uniformly at random.
6: for  $t = 1 \rightarrow m$  do
7:   if the total number of queries to  $U_{cmp} > 30C\sqrt{n} + mq$  then
8:     break.
9:   end if
10:   $i \leftarrow \text{Search}(U_{cmp}^j)$ .
11:   $b \leftarrow i \neq -1 \wedge x_i < x_j$  with probability  $\geq 1 - 1/m^2$  by  $q$  queries to  $U_{cmp}$  directly.
12:  if  $b$  then
13:     $j \leftarrow i$ .
14:  end if
15: end for
16: return  $j$ .

```

with only one query to U_{cmp} . Then we can provide a quantum algorithm for minimum finding on bounded-error oracles as Algorithm 1.

The constant $C > 0$ in Algorithm 1 is given so that $\text{Search}(U_x)$ in Theorem 3.2 takes at most $C\sqrt{n/\max\{t, 1\}}$ queries to U_x if there are t solutions.

Lemma 3.4. *Algorithm 1 is a bounded-error quantum algorithm for minimum finding on bounded-error oracles in expected $O(\sqrt{n})$ queries.*

Proof. The query complexity is trivially $O(\sqrt{n})$ due to the guard (Line 7) of Algorithm 1.

The correctness is proved as follows. Let $m = \lceil 12 \ln n \rceil$ and $q = \lceil 36 \ln m \rceil$. We assume that $n \geq 2$ and therefore $m \geq 12$. In each of the m iterations, Line 11 of Algorithm 1 calls U_{cmp} for q times and b obtains the desired Boolean value with probability $\geq 1 - 1/m^2$ (This is a straightforward majority voting. For completeness, its probability analysis is provided in Appendix A).

We only consider the case that the values of b in all iterations are desired. This case happens with probability $\geq (1 - 1/m^2)^m \geq 1 - 1/m \geq 11/12$. In each iteration, i finds a candidate index with probability $\geq 2/3$ such that $x_i < x_j$ if exists (and if there are many, any of them is obtained with equal probability). It is shown in [DH96] that: if i finds a candidate index with certainty, then the expected number of queries before j holds the index of the minimal is $\leq \frac{5}{2}C\sqrt{n}$; moreover, the expected number of iterations is $\leq \ln n$. In our case, i finds a candidate index in expected $3/2$ iterations. Therefore, the expected number of queries to U_{cmp} is $\leq \frac{15}{4}C\sqrt{n}$ and that of iterations is $\leq \frac{3}{2} \ln n$. When Algorithm 1 makes queries to the oracle $\geq 30C\sqrt{n}$ times (except those negligible queries in Line 11) or iterations $\geq m$ times (that is, more than 8 times their expectations), the error probability is $\leq 1/8 + 1/8 = 1/4$ by Markov's inequality. Therefore, the overall success probability is $\geq \frac{11}{12} \cdot \frac{3}{4} \geq 2/3$. \square

3.2.2 Amplifying the success probability of quantum minimum finding

We can amplify the success probability for quantum minimum finding better than a naive method, as shown in Algorithm 2

Algorithm 2 **Minimum**(U_{cmp}, ε): Amplification of the success of minimum finding.

Input: A bounded-error oracle U_{cmp} for x_0, x_1, \dots, x_{n-1} and $0 < \varepsilon < 1/2$.

Output: An index $j \in [n]$ such that $x_j \leq x_i$ for every $i \in [n]$ with probability $\geq 1 - \varepsilon$.

```

1: while true do
2:    $j \leftarrow \mathbf{Minimum}(U_{cmp})$ .
3:   if  $\mathbf{Search}(U_{cmp}^j, \varepsilon) \neq -1$  then
4:     break.
5:   end if
6: end while
7: return  $j$ .

```

Lemma 3.5. *Algorithm 2 runs in expected $O\left(\sqrt{n \log(1/\varepsilon)}\right)$ queries with error probability $\leq \varepsilon$.*

Proof. Algorithm 2 terminates with a guard by $\mathbf{Search}(U_{cmp}^j, \varepsilon)$. Therefore, it has error probability ε as the guard. Let p be the probability that j holds the index of the minimal with a single query to $\mathbf{Minimum}(U_{cmp})$. Then it holds that $p \geq 2/3$. Let q be the probability that Algorithm 2 breaks the “while” loop at each iteration. Then

$$q = p(1 - \varepsilon) + (1 - p)\varepsilon \geq p(1 - \varepsilon) \geq 1/3,$$

which is greater than a constant. So, the expected number of iterations is $O(1)$. In a single iteration, $\mathbf{Minimum}(U_{cmp})$ takes $O(\sqrt{n})$ queries (by Lemma 3.4) and $\mathbf{Search}(U_{cmp}^j, \varepsilon)$ takes $O\left(\sqrt{n \log(1/\varepsilon)}\right)$ queries (by Theorem 3.2). Therefore, the expected query complexity of Algorithm 2 is $O\left(\sqrt{n \log(1/\varepsilon)}\right)$. \square

3.3 An Application: Searching for the First Solution

In this subsection, we develop a tool needed in our quantum algorithm for LMSR as an application of the above two subsections. It solves the problem of finding the first solution (i.e. leftmost solution, or solution with the minimal index rather than an arbitrary solution) and thus can be seen as a generalization of quantum searching, but the solution is based on quantum minimum finding.

Formally, the query oracle U_x of x_0, x_1, \dots, x_{n-1} is given. The searching-first problem is to find the minimal index $j \in [n]$ such that $x_j = 1$ or report that no solution exists. This problem can be solved by minimum-finding with the comparator

$$cmp(i, j) = \begin{cases} 1 & x_i = 1 \wedge x_j = 0 \\ 1 & x_i = x_j \wedge i < j \\ 0 & \text{otherwise} \end{cases},$$

which immediately yields an $O(\sqrt{n})$ solution if the query oracle U_x is exact.

In the case that the query oracle U_x is bounded-error, a bounded-error comparison oracle U_{cmp} corresponding to cmp can be implemented with a constant number of queries to U_x . Therefore,

the results in Lemma 3.4 and Lemma 3.5 also hold for the searching-first problem. For convenience in the following discussions, we write $\mathbf{SearchFirst}(U_x)$ and $\mathbf{SearchFirst}(U_x, \varepsilon)$ to denote the algorithm for the searching-first problem based on the two algorithm $\mathbf{Minimum}(U_{cmp})$ and $\mathbf{Minimum}(U_{cmp}, \varepsilon)$, respectively. Symmetrically, we have $\mathbf{SearchLast}(U_x)$ and $\mathbf{SearchLast}(U_x, \varepsilon)$ for searching the last solution.

Recently, an $O(\sqrt{n})$ quantum algorithm for searching the first was proposed in [KKM⁺20]. Their approach is quite different from our presented above. It is specifically designed for this particular problem, but our approach is based on a more general framework of quantum minimum finding.

We believe that the techniques presented in this section can be applied in solving other problems. For this reason, we present a description of them in a general framework of nested quantum algorithm in Appendix B.

4 Quantum Deterministic Sampling

In this section, we prepare another tool to be used in our quantum algorithm for LMSR, namely an efficient quantum algorithm for deterministic sampling. It is based on our nested quantum algorithm composed of quantum search and quantum minimum finding given in the last section. Deterministic sampling is also a useful trick in parallel pattern matching [Vis90]. We provide a simple quantum lexicographical comparator in Section 4.1, and a quantum algorithm for deterministic sampling in Section 4.2. As an application, we obtain quantum algorithms for string periodicity and pattern matching in Section 4.3.

4.1 Lexicographical Comparator

Suppose there are two strings $s, t \in \Sigma^n$ of length n over a finite alphabet $\Sigma = [\alpha]$. Let U_s and U_t be their query oracles, respectively. That is,

$$U_s |i, j\rangle = |i, j \oplus s[i]\rangle, \quad U_t |i, j\rangle = |i, j \oplus t[i]\rangle.$$

In order to compare the two strings in the lexicographical order, we need to find the leftmost index $k \in [n]$ that $s[k] \neq t[k]$. If no such k exists, then $s = t$. A straightforward algorithm for lexicographical comparison based on the searching-first problem is described in Algorithm 3.

Algorithm 3 $\mathbf{LexicographicalComparator}(U_s, U_t)$: Lexicographical Comparator.

Input: Two query oracles U_s and U_t for two strings s and t , respectively.

Output: If $s < t$, then return -1 ; if $s > t$, then return 1 ; if $s = t$, then return 0 , with probability $\geq 2/3$.

```

1:  $k \leftarrow \mathbf{SearchFirst}(U_x)$ .
2: if  $k = -1$  then
3:   return  $0$ .
4: end if
5: if  $s[k] < t[k]$  then
6:   return  $-1$ .
7: else
8:   return  $1$ .
9: end if

```

Lemma 4.1. *Algorithm 3 is an $O(\sqrt{n})$ bounded-error quantum algorithm that compares two strings by their oracles in the lexicographical order.*

Remark 4.1. *We usually need to compare two strings in the lexicographical order as a subroutine nested as low-level quantum oracles in string algorithms. However, the lexicographical comparator (Algorithm 3) brings errors. Therefore, the error reduction trick for nested quantum oracles proposed in Section 3 is required here.*

4.2 Deterministic Sampling

Deterministic sampling [Vis90] is a useful technique for pattern matching in pattern analysis. In this subsection, we provide a quantum solution to deterministic sampling in the form of a nested quantum algorithm.

For our purpose, we extend the definition of deterministic samples to the periodic case. The following is a generalised definition of deterministic samples.

Definition 4.1 (Deterministic samples). *Let $s \in \Sigma^n$ and d be its period. A deterministic sample of s consists of an offset $0 \leq \delta < \lfloor n/2 \rfloor$ and a sequence of indices i_0, i_1, \dots, i_{l-1} (called checkpoints) such that*

1. $i_k - \delta \in [n]$ for $k \in [l]$;
2. For every $0 \leq j < \lfloor n/2 \rfloor$ but $j \not\equiv \delta \pmod{d}$, there exists $k \in [l]$ such that $i_k - j \in [n]$ and $s[i_k - j] \neq s[i_k - \delta]$. We denote $c_k = s[i_k - \delta]$ regardless of exact i_k and δ .

If s is aperiodic (i.e. $d > n/2$), the second condition degenerates into “for every $0 \leq j < \lfloor n/2 \rfloor$ but $j \neq \delta$ ”, which is consistent with the definition for aperiodic strings in [Vis90].

The following theorem shows that the size of the deterministic sample can be every small.

Theorem 4.2 (Deterministic sampling [Vis90]). *Let $s \in \Sigma^n$. There is a deterministic sample $(\delta; i_0, i_1, \dots, i_{l-1})$ of s with $l \leq \lfloor \log n \rfloor$.*

Proof. It was shown in [Vis90] for aperiodic strings. For periodic strings, the proof is similar if we limit the string s to its prefix $s[0 \dots d-1]$, where d is the period of s . \square

Now let us consider how can a quantum algorithm do deterministic sampling. We start from the case where $s \in \Sigma^n$ is aperiodic. Let U_s be the query oracle of s , that is, $U|i, j\rangle = |i, j \oplus s[i]\rangle$. Suppose the sequence of indices i_0, i_1, \dots, i_{l-1} is known as well as $c_k = s[i_k - \delta]$ for $k \in [l]$ (we need not know δ explicitly). For $0 \leq j < \lfloor n/2 \rfloor$, let x_j denote whether candidate j agrees with δ at all checkpoints, that is,

$$x_j = \begin{cases} 0 & \exists k \in [l], j \leq i_k < j + n \wedge s[i_k - j] \neq c_k \\ 1 & \text{otherwise} \end{cases}. \quad (1)$$

Based on the searching problem, there is a bounded-error oracle U_x for computing x_j with $O(\sqrt{l}) = O(\sqrt{\log n})$ queries to U_s .

A quantum algorithm for deterministic sampling is described in Algorithm 4. Initially, all offsets $0 \leq \xi < \lfloor n/2 \rfloor$ are candidates of δ . The idea of the algorithm is to repeatedly find two remaining candidates p and q that differ at an index j (if there is only one remaining candidate, the algorithm has already found a deterministic sample and terminates), randomly choose a character c being $s[j - p]$ or $s[j - q]$, and delete either p or q according to c . It is sufficient to select p and q to be

the first and the last solution of x_j defined in Eq. (1). To explicitly describe how to find an index j where p and q differ, we note that $q \leq j < p + n$ and j must exist because of the aperiodicity of s , and let

$$y_j = \begin{cases} 1 & q \leq j < p + n \wedge s[j - p] \neq s[j - q] \\ 0 & \text{otherwise} \end{cases}.$$

It is trivial that there is an exact oracle U_y for computing y_j with $O(1)$ queries to U_s .

For the case of periodic $s \in \Sigma^n$, the algorithm requires some careful modifications. We need a variable Q to denote the upper bound of current available candidates. Initially, $Q = \lfloor n/2 \rfloor - 1$. We modify the definition of x_j in Eq. (1) to make sure $0 \leq j \leq Q$ by

$$x_j = \begin{cases} 0 & j > Q \vee (\exists k \in [l], j \leq i_k < j + n \wedge s[i_k - j] \neq c_k) \\ 1 & \text{otherwise} \end{cases}. \quad (2)$$

For an aperiodic string s , the algorithm will reach Line 17-25 during its execution with small probability $\leq 1/m$, where $m = O(\log n)$. But for periodic string s , let d be its period, if $q - p$ is divisible by d , then the algorithm once reaches Line 17-25 with high probability $\geq 1 - 1/6m^2$. In this case, there does not exist $q \leq j < p + n$ such that $y_j = 1$. We set $Q = q - 1$ to eliminate all candidates $\geq q$. In fact, even for a periodic string s , the algorithm is intended to reach Line 17-25 only once (with high probability). If the algorithm reaches Line 17-25 the second (or more) time, it is clear that current $(\delta; i_0, i_1, \dots, i_{l-1})$ is a deterministic sample of s (with high probability $\geq 1 - 1/m$), and therefore consequent computation does not influence the correctness and can be ignored.

Lemma 4.3. *Algorithm 4 is an $O\left(\sqrt{n \log^3 n \log \log n}\right)$ bounded-error quantum algorithm for deterministic sampling.*

Proof. Assume $n \geq 2$ and let $m = \lceil 8 \log_2 n \rceil$ and $\varepsilon = 1/6m^2$. There are m iterations in Algorithm 4. In each iteration, there are less than 6 calls to **Search**, **SearchFirst** or **SearchLast**, which may bring errors. It is clear that each call to **Search**, **SearchFirst** or **SearchLast** has error probability $\leq \varepsilon$. Therefore, Algorithm 4 runs with no errors from **Search**, **SearchFirst** or **SearchLast** with probability $\geq (1 - \varepsilon)^{6m} \geq 1 - 1/m$.

Now suppose Algorithm 4 runs with no errors from **Search**, **SearchFirst** or **SearchLast**. To prove the correctness of Algorithm 4, we consider the following two cases:

1. **Case 1.** s is aperiodic. In this case, Algorithm 4 will never reach Line 17-25. In each iteration, the leftmost and the rightmost remaining candidates p and q are found. If $p = q$, then only one candidate remains, and thus a deterministic sample is found. If $p \neq q$, then there exists an index $q \leq j < p + n$ such that $s[j - p]$ and $s[j - q]$ differ. We set $i_l = j$ and set c_l randomly from $s[j - p]$ and $s[j - q]$ with equal probability. Then with probability $1/2$, half of the remaining candidates are eliminated. In other words, it is expected to find a deterministic sample in $2 \log_2 n$ iterations. The iteration limit to $m \geq 8 \log_2 n$ will make error probability $\geq 1/4$. That is, a deterministic sample is found with probability $\geq 3/4$.
2. **Case 2.** s is periodic and the period of s is $d \leq n/2$. In each iteration, the same argument for aperiodic s holds if Algorithm 4 does not reach Line 17-25. If Line 17-25 is reached for the first time, it means candidates between $q + 1$ and $\lfloor n/2 \rfloor - 1$ are eliminated and $q - p$ is divisible by d . If Line 17-25 is reached for the second time, all candidates $p \not\equiv q \pmod{d}$ are eliminated, and therefore a deterministic sample is found.

Algorithm 4 **DeterministicSampling**(U_s): Deterministic Sampling.

Input: The query oracles U_s for string $s \in \Sigma^n$.

Output: A deterministic sampling $(\delta; i_0, i_1, \dots, i_{l-1})$.

```
1: if  $n = 1$  then
2:   return  $(0; 0)$ .
3: end if
4:  $m \leftarrow \lceil 8 \log_2 n \rceil$ .
5:  $\varepsilon \leftarrow 1/6m^2$ .
6:  $l \leftarrow 0$ .
7:  $Q \leftarrow \lfloor n/2 \rfloor - 1$ .
8: for  $t = 1 \rightarrow m$  do
9:   Let  $U_x$  be the bounded-error oracle for computing  $x_j$  defined in Eq. (2).
10:   $p \leftarrow \mathbf{SearchFirst}(U_x, \varepsilon)$ .
11:   $q \leftarrow \mathbf{SearchLast}(U_x, \varepsilon)$ .
12:  if  $p = q$  then
13:    break.
14:  end if
15:   $i_l \leftarrow \mathbf{Search}(U_y, \varepsilon)$ .
16:  if  $i_l = -1$  then
17:     $Q \leftarrow q - 1$ .
18:     $q \leftarrow \mathbf{SearchLast}(U_x, \varepsilon)$ .
19:    if  $p = q$  then
20:      break.
21:    end if
22:     $i_l \leftarrow \mathbf{Search}(U_y, \varepsilon)$ .
23:    if  $i_l = -1$  then
24:      break.
25:    end if
26:  end if
27:   $c_l \leftarrow s[i_l - p]$  with probability  $1/2$  and  $s[i_l - q]$  with probability  $1/2$ .
28:   $l \leftarrow l + 1$ .
29: end for
30: Let  $U_x$  be the bounded-error oracle for computing  $x_j$  defined in Eq. (1).
31:  $\delta \leftarrow \mathbf{SearchFirst}(U_x, \varepsilon)$ .
32: return  $(\delta; i_0, i_1, \dots, i_{l-1})$ .
```

Combining the above two cases, a deterministic sample is found with probability $\geq 3/4(1 - 1/m) \geq 2/3$.

On the other hand, we note that a single call to **SearchFirst** and **SearchLast** in Algorithm 4 has $O(\sqrt{n \log(1/\varepsilon)}) = O(\sqrt{n \log \log n})$ queries to U_x (by Lemma 3.5), and **Seach** has query complexity $O(\sqrt{n \log(1/\varepsilon)}) = O(\sqrt{n \log \log n})$ (by Theorem 3.2). Hence, a single iteration has query complexity $O(\sqrt{nl \log \log n}) = O(\sqrt{n \log n \log \log n})$, and the total query complexity (m iterations) is $O(m\sqrt{n \log n \log \log n}) = O(\sqrt{n \log^3 n \log \log n})$. \square

Algorithm 4 is a 2-level nested quantum algorithm (see Appendix C for a more detailed discussion), and is a better solution for deterministic sampling in $O(\sqrt{n \log^3 n \log \log n})$ queries than the known $O(\sqrt{n} \log^2 n)$ solution in [RV03].

In order to make our quantum algorithm for deterministic sampling time and space efficient, we need to store and modify the current deterministic sample $(\delta; i_0, i_1, \dots, i_{l-1})$ during the execution in the QRAM, which needs $O(l \log n) = O(\log^2 n)$ bits of memory. Moreover, only $O(\log n)$ qubits are needed in the computation (for search and minimum finding). In this way, the time complexity of the quantum algorithm is $\tilde{O}(\sqrt{n})$, which is just a polylogarithmic factor bigger than its query complexity.

4.3 Applications

Based on our quantum algorithm for deterministic sampling, we provide applications for string periodicity and pattern matching.

4.3.1 String Periodicity

We can check whether a string is periodic (and if yes, find its period) with its deterministic sample. Formally, let $(\delta; i_0, i_1, \dots, i_{l-1})$ be a deterministic sample of s , and x_j defined in Eq. (1). Let j_1 denote the smallest index j such that $x_j = 1$, which can be computed by **SearchFirst** on x_j . After j_1 is obtained, define

$$x'_j = \begin{cases} 0 & j \leq j_1 \vee (\exists k \in [l], j \leq i_k < j + n \wedge s[i_k - j] \neq c_k) \\ 1 & \text{otherwise} \end{cases}.$$

Let j_2 denote the smallest index j such that $x'_j = 1$ (if not found then $j_2 = -1$), which can be computed by **SearchFirst** on x'_j . If $j_2 = -1$, then s is aperiodic; otherwise, s is periodic with period $d = j_2 - j_1$. This algorithm for checking periodicity is $O(\sqrt{n \log n})$. (See Appendix D for more details.)

4.3.2 Pattern Matching

As an application of deterministic sampling, we have a quantum algorithm for pattern matching with query complexity $O(\sqrt{n \log m} + \sqrt{m \log^3 m \log \log m})$, better than the best known solution in [RV03] with query complexity $O(\sqrt{n} \log(n/m) \log m + \sqrt{m} \log^2 m)$.

For readability, these algorithms are postponed to Appendix D.

Remark 4.2. *There are several mistakes in the details of the quantum pattern matching algorithm given in [RV03]. However, it can be fixed with some modifications and the same query complexity remains as stated.*

5 The Quantum Algorithm for LMSR

Now we are ready to present our quantum algorithm for LMSR and thus prove Theorem 1.1. This algorithm is designed in two steps:

1. Design a quantum algorithm with worst-case query complexity $O(n^{3/4})$ in Section 5.1; and
2. Improve the algorithm to average-case query complexity $O(\sqrt{n} \log n)$ in Section 5.2.

5.1 The Basic Algorithm

For convenience, we assume that the alphabet $\Sigma = [\alpha]$ for some $\alpha \geq 2$, where $[n] = \{0, 1, 2, \dots, n-1\}$ and the total order of Σ follows that of natural numbers. Suppose the input string $s \in \Sigma^n$ is given by an oracle U_{in} :

$$U_{in} |i, j\rangle = |i, j \oplus s[i]\rangle.$$

The overall idea of our algorithm is to split s into blocks of length B , and then in each block find a candidate with the help of the prefix of $SCR(s)$ of length B . These candidates are eliminated between blocks by the exclusion rule for LMSR (see Lemma 5.1). We describe it in detail in the next three subsections.

5.1.1 Find a prefix of SCR

Our first goal is to find the prefix $p = s[LM\text{SR}(s) \dots LM\text{SR}(s) + B - 1]$ of $SCR(s)$ of length B by finding an index $i^* \in [n]$ such that $s[i^* \dots i^* + B - 1]$ matches p , where $B = \lfloor \sqrt{n} \rfloor$ is chosen optimally (see later discussions). To achieve this, we need to compare two substrings of s of length B with the comparator cmp_B :

$$cmp_B(i, j) = \begin{cases} 1 & s[i \dots i + B - 1] < s[j \dots j + B - 1] \\ 0 & \text{otherwise} \end{cases}. \quad (3)$$

According to Algorithm 3, we can obtain a bounded-error comparison oracle U_{cmp_B} corresponding to cmp_B with $O(\sqrt{B})$ queries to U_{in} . After that, Let $i^* \in [n]$ be any index such that $s[i^* \dots i^* + B - 1] = p$ by calling **Minimum**(U_{cmp_B}), which needs $O(\sqrt{n})$ queries to U_{cmp_B} (by Lemma 3.4) and succeeds with a constant probability. In the following discussion, we use i^* to find possible candidates of $LM\text{SR}(s)$ and then find the solution among all candidates.

5.1.2 Candidate in each block

Being able to access contents of p by i^* , we can obtain a deterministic sample of p by Algorithm 4 in $O(\sqrt{B \log^3 B \log \log B})$ queries with a constant probability. Suppose a deterministic sample of p is known to be $(\delta; i_0, i_1, \dots, i_{l-1})$. We split s into blocks of length $L = \lfloor B/4 \rfloor$. In the i -th block (0-indexed, $0 \leq i < \lceil n/L \rceil$), with the index ranging from iL to $\min\{(i+1)L, n\} - 1$, a candidate h_i is computed by

$$h_i = \min\{iL \leq j < \min\{(i+1)L, n\} : s[j \dots j + B - 1] = p\}, \quad (4)$$

where $\min \emptyset = \infty$. Intuitively, for each $0 \leq i < \lceil n/L \rceil$, h_i defined by Eq. (4) denotes the leftmost possible candidate for $LM\text{SR}(s)$ such that $s[h_i \dots h_i + B - 1] = p$ in the i -th block. On the other hand, h_i denotes the first occurrence of p with starting index in the i -th block of s , and thus can be

computed by a procedure in quantum pattern matching (see Appendix D for more details), which needs $O(\sqrt{B \log B})$ queries to U_{in} with the help of the deterministic sample of p . We write U_h for the bounded-error oracle of h_i . Note that U_h is a 2-level nested quantum oracle.

5.1.3 Candidate elimination between blocks

After having computed h_i for $0 \leq i < \lceil n/L \rceil$, either h_i is a candidate or ∞ (indicating non-existence), we find $LMSR(s)$ among all h_i with the comparator

$$cmp(i, j) = \begin{cases} 1 & cmp_n(h_i, h_j) = 1, \\ 1 & cmp_n(h_i, h_j) = cmp_n(h_i, h_j) = 0 \wedge h_i < h_j, \\ 0 & \text{otherwise,} \end{cases} \quad (5)$$

where cmp_n is defined by Eq. (3), and ∞ can be regarded as n explicitly in the computation. Then we can obtain the bounded-error comparison oracle U_{cmp} corresponding to cmp with constant number of queries to U_{cmp_n} and U_h , with $O(\sqrt{B \log B} + \sqrt{n})$ queries to U_{in} . Here, U_{cmp} is a 2-level nested quantum oracle. At the end of the algorithm, the value of $LMSR(s)$ is chosen to be the minimal element among h_i by comparison oracle U_{cmp} according to comparator cmp . It can be seen that the algorithm is a 3-level nested quantum algorithm.

The Algorithm

We summarize the above design ideas in Algorithm 5. There are four main steps (Line 5, Line 6, Line 7 and Line 8) in the algorithm. Especially, Line 7 of Algorithm 5 involves a 3-level nested quantum algorithm. For convenience, we assume that each of these steps succeeds with a high enough constant probability, say ≥ 0.99 . To achieve this, each step just needs a constant number of repetitions to amplify the success probability from $2/3$ up to 0.99 .

Algorithm 5 BasicLMSR(U_{in}): Quantum algorithm for LMSR.

Input: The query oracle U_{in} for string s .

Output: $LMSR(s)$ with probability $\geq 2/3$.

- 1: **if** $n \leq 15$ **then**
 - 2: **return** $LMSR(s)$ by classical algorithms in Theorem 2.1.
 - 3: **end if**
 - 4: $B \leftarrow \lfloor \sqrt{n} \rfloor$.
 - 5: $i^* \leftarrow \mathbf{Minimum}(U_{cmp_B})$, where cmp_B is defined by Eq. (3).
 - 6: $(\delta; i_0, i_1, \dots, i_{l-1}) \leftarrow \mathbf{DeterministicSampling}(U_p)$, where $p = s[i^* \dots i^* + B - 1]$.
 - 7: $i \leftarrow \mathbf{Minimum}(U_{cmp})$, where cmp is defined by Eq. (5).
 - 8: **return** h_i , where h_i is defined by Eq. (4).
-

Complexity

The query complexity of Algorithm 5 comes for the following four parts:

1. One call to $\mathbf{Minimum}(U_{cmp_B})$, which needs $O(\sqrt{n})$ queries to U_{cmp_B} (by Lemma 3.4), i.e. $O(\sqrt{nB})$ queries to U_{in} .

2. One call to **DeterministicSampling**($U_{s[i^* \dots i^* + B - 1]}$), which needs $O\left(\sqrt{B \log^3 B \log \log B}\right)$ queries to U_{in} (by Lemma 4.3).

3. One call to **Minimum**(U_{cmp}), which needs $O\left(\sqrt{n/L}\right)$ queries to U_{cmp} (by Lemma 3.4), i.e.

$$O\left(\sqrt{n/L}\left(\sqrt{B \log B} + \sqrt{n}\right)\right) = O\left(n/\sqrt{B}\right)$$

queries to U_{in} .

4. Compute h_i , i.e. one query to U_h , which needs $O\left(\sqrt{B \log B}\right)$ queries to U_{in} .

Therefore, the total query complexity is

$$O\left(\sqrt{nB} + n/\sqrt{B}\right) = O\left(n^{3/4}\right)$$

by selecting $B = \Theta(\sqrt{n})$.

Correctness

The correctness of Algorithm 5 is not obvious due to the fact that we only deal with one candidate in each block, but there can be several candidates that matches p in a single block. This issue is resolved by the following exclusion rule:

- For every two equal substrings $s[i \dots i + B - 1]$ and $s[j \dots j + B - 1]$ ($0 \leq i < j < n, 1 \leq B \leq n/2$) of s that overlap each other, if both of them are prefixes of $SCR(s)$, then $LMSR(s)$ cannot be the larger index j .

More precisely, this exclusion rule can be stated as the following:

Lemma 5.1 (Exclusion Rule for LMSR). *Suppose $s \in \Sigma^n$ is a string of length n . Let $1 \leq B \leq n/2$, and two indices $i, j \in [n]$ with $i < j < i + B$. If $s[i \dots i + B - 1] = s[j \dots j + B - 1] = s[LMSR(s) \dots LMSR(s) + B - 1]$, then $LMSR(s) \neq j$.*

Proof. See Appendix E. □

Indeed, the above exclusion rule can be viewed as the Ricochet Property of LMSR. If there are two candidates in the same block, since each block is of length $L = \lfloor B/4 \rfloor < B$, then the two candidates must overlap each other. By this rule, the smaller candidate remains. Consequently, the correctness of Algorithm 5 is guaranteed because h_i defined by Eq. (4) always chooses the smallest candidate in each block.

After the above discussions, we obtain:

Theorem 5.2. *Algorithm 5 is an $O\left(n^{3/4}\right)$ bounded-error quantum query algorithm for LMSR.*

In order to make our quantum algorithm for LMSR time and space efficient, we only need to store the deterministic sample $(\delta; i_0, i_1, \dots, i_{l-1})$ of p in the QRAM, which needs $O(l \log n) = O(\log^2 n)$ bits of memory. Moreover, only $O(\log n)$ qubits are needed in the computation (for search and minimum finding). In this way, the time complexity of the quantum algorithm is $\tilde{O}(n^{3/4})$, which is just a polylogarithmic factor bigger than its query complexity.

5.2 An Improvement for Better Average-Case Query Complexity

In the previous subsection, we propose an efficient quantum algorithm for LMSR in terms of its worst-case query complexity. It is easy to see that its average-case query complexity remains the same as its worst-case query complexity. In this subsection, we improve this algorithm to achieve a better average-case query complexity.

The basic idea is to individually deal with several special cases, which cover almost all of the possibilities on average. Let $B = \lceil 3 \log_\alpha n \rceil$. Our strategy is to just consider substrings of length B . Let

$$k = \arg \min_{i \in [n]} s[i \dots i + B - 1]$$

denote the index of the minimal substring among all substrings of length B , and then let

$$k' = \arg \min_{i \in [n] \setminus \{k\}} s[i \dots i + B - 1]$$

denote the index of the second minimal substring among all substrings of length B . If $s[k \dots k + B - 1] \neq s[k' \dots k' + B - 1]$, then it immediately holds that $LMSR(s) = k$. To find the second minimal substring, the index k of the minimal substring should be excluded. For this, we need comparator $cmp_{B \setminus k}$:

$$cmp_{B \setminus k}(i, j) = \begin{cases} 0 & i = k \\ 1 & i \neq k \wedge j = k \\ cmp_B(i, j) & \text{otherwise} \end{cases} \quad (6)$$

The bounded-error quantum comparison oracle $U_{cmp_{B \setminus k}}$ corresponding to $cmp_{B \setminus k}(i, j)$ can be defined with at most one query to U_{cmp_B} .

The Algorithm

Our improved algorithm is presented as Algorithm 6. It has three main steps (Line 5, Line 6 and Line 10). For the same reason as in Algorithm 5, we assume that the third step (Line 10) succeeds with a high enough constant probability, say ≥ 0.99 .

Algorithm 6 ImprovedLMSR(U_{in}): Improved quantum algorithm for LMSR.

Input: The query oracle U_{in} for string s .
Output: $LMSR(s)$ with probability $\geq 2/3$.
1: **if** $n \leq 3$ **then**
2: **return** $LMSR(s)$ by classical algorithms.
3: **end if**
4: $B \leftarrow \lceil 3 \log_\alpha n \rceil$.
5: $k \leftarrow \mathbf{Minimum}(U_{cmp_B}, 1/2n)$, where cmp_B is defined by Eq. (3).
6: $k' \leftarrow \mathbf{Minimum}(U_{cmp_{B \setminus k}}, 1/2n)$, where $cmp_{B \setminus k}$ is defined by Eq. (6).
7: **if** $cmp_B(k, k') = 1$ **then**
8: **return** k .
9: **else**
10: **return** $\mathbf{BasicLMSR}(U_{in})$.
11: **end if**

Correctness

The correctness of Algorithm 6 is trivial. We only consider the case where $n \geq 4$, and all of the three main steps succeed with probability

$$\geq \left(1 - \frac{1}{2n}\right)^2 \cdot 0.99 > \frac{2}{3}.$$

If $\text{cmp}_B(k, k') = 1$, then $s[k \dots k + B - 1]$ is the minimal substring of length B , and it immediately holds that $\text{LMSR}(s) = k$. Otherwise, the correctness is based on that of **BasicLMSR**(U_{in}), which is guaranteed by Theorem 5.2.

Complexity

The worst-case query complexity of Algorithm 6 is $O(n^{3/4})$, obtained directly by Theorem 5.2. But settling the average-case query complexity of Algorithm 6 is a bit more subtle, which requires a better understanding of some properties of LMSR. To this end, we first introduce the notion of string sensitivity.

Definition 5.1 (String Sensitivity). *Let $s \in \Sigma^n$ be a string of length n over a finite alphabet Σ . The string sensitivity of s , denoted $C(s)$, is the smallest positive number l such that $s[i \dots i + l - 1] \neq s[j \dots j + l - 1]$ for all $0 \leq i < j < n$. In case that no such l exists, define $C(s) = \infty$.*

The string sensitivity of a string is a metric indicating the difficulty to distinguish its rotations by their prefixes. If we know the string sensitivity $C(s)$ of a string s , we can compute $\text{LMSR}(s)$ by finding the minimal string among all substrings of s of length $C(s)$, that is, $s[i \dots i + C(s) - 1]$ for all $i \in [n]$.

The following lemma shows that almost all strings have a low string sensitivity.

Lemma 5.3 (String Sensitivity Distribution). *Let s be a uniformly random string over Σ^n and $1 \leq B \leq n/2$. Then*

$$\Pr[C(s) \leq B] \geq 1 - \frac{1}{2}n(n-1)\alpha^{-B},$$

where $\alpha = |\Sigma|$. Especially,

$$\Pr[C(s) \leq \lceil 3 \log_\alpha n \rceil] \geq 1 - \frac{1}{n}.$$

Proof. Let $s \in \Sigma^n$ and $0 \leq i < j < n$. We claim that

$$\Pr[s[i \dots i + B - 1] = s[j \dots j + B - 1]] = \alpha^{-B}.$$

This can be seen as follows. Let d be the number of members that appear in both sequences $\{i \bmod n, (i+1) \bmod n, \dots, (i+B-1) \bmod n\}$ and $\{j \bmod n, (j+1) \bmod n, \dots, (j+B-1) \bmod n\}$. It is clear that $0 \leq d < B$. We note that $s[i \dots i + B - 1] = s[j \dots j + B - 1]$ implies the following system of B equations:

$$s[i + k] = s[j + k] \text{ for all } 0 \leq k < B.$$

On the other hand, these B equations involve $2B - d$ (random) characters. Therefore, there must be $(2B - d) - B = B - d$ independent characters, and the probability that the B equations hold is

$$\Pr[s[i \dots i + B - 1] = s[j \dots j + B - 1]] = \frac{\alpha^{B-d}}{\alpha^{2B-d}} = \alpha^{-B}.$$

Consequently, we have:

$$\begin{aligned}
\Pr[C(s) \leq B] &= 1 - \Pr[\exists 0 \leq i < j < n, s[i \dots i + B - 1] = s[j \dots j + B - 1]] \\
&\geq 1 - \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} \Pr[s[i \dots i + B - 1] = s[j \dots j + B - 1]] \\
&= 1 - \frac{1}{2}n(n-1)\alpha^{-B}.
\end{aligned}$$

In particular, in the case of $B = \lceil 3 \log_\alpha n \rceil$, it holds that $\alpha^B \geq n^3$ and we obtain:

$$\Pr[C(s) \leq B] \geq 1 - \frac{1}{2}n(n-1)n^{-3} \geq 1 - \frac{1}{n}.$$

□

With the above preparation, we now can analyze the average-case query complexity of Algorithm 6. Let $s \in \Sigma^n$ be a uniformly random string over Σ^n and $B = \lceil 3 \log_\alpha n \rceil$. Let k and k' denote the indices of the minimal and the second minimal substrings of length B of s . To compute k and k' , by Lemma 3.5, Algorithm 6 needs to make $O(\sqrt{n \log n})$ queries to U_{cmp_B} , which is equivalent to $O(\sqrt{n \log n} \sqrt{B}) = O(\sqrt{n} \log n)$ queries to U_{in} . On the other hand, it requires $O(B) = O(\log n)$ queries to U_{in} in order to check whether $cmp_B(k, k') = 1$, which is ignorable compared to other large complexities. Based on the result of $cmp_B(k, k')$, we only need to further consider the following two cases:

Case 1. $cmp_B(k, k') = 1$. Note that this case happens with probability

$$\Pr[cmp_B(k, k') = 1] \geq \Pr[C(s) \leq B] \geq 1 - \frac{1}{n}. \quad (7)$$

In this case, Algorithm 6 returns k immediately.

Case 2. $cmp_B(k, k') \neq 1$. According to Eq. (7), this case happens with probability $\leq 1/n$. In this case, Algorithm 6 makes one query to **BasicLMSR**(U_{in}), which needs $O(n^{3/4})$ queries to U_{in} (by Theorem 5.2).

Combining the above two cases yields the average-case query complexity of Algorithm 6:

$$\leq O(\sqrt{n} \log n) + \left(1 - \frac{1}{n}\right) \cdot O(1) + \frac{1}{n} \cdot O(n^{3/4}) = O(\sqrt{n} \log n).$$

After the above discussions, we obtain:

Theorem 5.4. *Algorithm 6 is an $O(n^{3/4})$ bounded-error quantum query algorithm for LMSR, whose average-case query complexity is $O(\sqrt{n} \log n)$.*

By an argument similar to that given in Section 5.1 for time and space efficiency, we can see that our quantum algorithm use $O(\log^2 n)$ bits of memory in the QRAM, and $O(\log n)$ qubits in the computation. Consequently, the time complexity is $\tilde{O}(n^{3/4})$ in the worst case and $\tilde{O}(\sqrt{n})$ in the average case, which is again a polylogarithmic factor bigger than its query complexity.

6 Lower Bounds of LMSR

In this section, we establish average-case and worst-case lower bounds of both classical and quantum algorithms for the LMSR problem and thus prove Theorem 1.2.

The notion of block sensitivity is the key tool we use to obtain lower bounds. Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function. If $x \in \{0, 1\}^n$ is a binary string and $S \subseteq [n]$, we use x^S to denote the binary string obtained by flipping the values of x_i for $i \in S$, where x_i is the i -th character of x :

$$(x^S)_i = \begin{cases} \bar{x}_i & i \in S \\ x_i & i \notin S \end{cases},$$

where \bar{u} denotes the negation of u , i.e. $\bar{0} = 1$ and $\bar{1} = 0$. The *block sensitivity* of f on input x , denoted $bs_x(f)$, is the maximal number m such that there are m disjoint sets $S_1, S_2, \dots, S_m \subseteq [n]$ for which $f(x) \neq f(x^{S_i})$ for $1 \leq i \leq m$.

6.1 Average-case Lower Bounds

For settling the average-case lower bound, we need the following useful result about block sensitivities given in [AdW99].

Theorem 6.1 ([AdW99]). *For every function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ and probability distribution $\mu : \{0, 1\}^n \rightarrow [0, 1]$, we have $R^\mu(f) = \Omega(\mathbb{E}_{x \sim \mu}[bs_x(f)])$ and $Q^\mu(f) = \Omega\left(\mathbb{E}_{x \sim \mu}\left[\sqrt{bs_x(f)}\right]\right)$.*

In order to give a lower bound for LMSR by using Theorem 6.1, we need a binary function that can be reduced $LMSR(x)$ to but is simpler than it. Here, we choose $LMSR_0(x) = LMSR(x) \bmod 2$. Obviously, if we can compute $LMSR(x)$, then $LMSR_0(x)$ is immediately obtained. Moreover, $LMSR_0$ enjoys the following basic property:

Lemma 6.2. *Let $x \in \{0, 1\}^n$ and $0 \leq r < n$. Then $bs_x(LMSR_0) = bs_{x^{(r)}}(LMSR_0)$.*

Proof. Let $m = bs_x(LMSR_0)$ and S_1, S_2, \dots, S_m be the m disjoint sets for which $LMSR_0(x) \neq LMSR_0(x^{S_i})$ for $1 \leq i \leq m$. We define $S'_i = \{(a+r) \bmod n : a \in S_i\}$ for $1 \leq i \leq m$. Then it can be verified that $LMSR_0(x^{(r)}) \neq LMSR_0\left(\left(x^{(r)}\right)^{S'_i}\right)$ for $1 \leq i \leq m$. Hence, $bs_{x^{(r)}}(LMSR_0) \geq m = bs_x(LMSR_0)$.

The same argument yields that $bs_x(LMSR_0) \geq bs_{x^{(r)}}(LMSR_0)$. Therefore, $bs_x(LMSR_0) = bs_{x^{(r)}}(LMSR_0)$. \square

Next, we establish a lower bound for $bs_x(LMSR_0)$.

Lemma 6.3. *Let $x \in \{0, 1\}^n$. Then*

$$bs_x(LMSR_0) \geq \left\lfloor \frac{n}{4C(x)} - \frac{1}{4} \right\rfloor. \quad (8)$$

Proof. We first note that inequality (8) is trivially true when $C(x) > n/5$ because the right hand side is equal to 0. For the case of $C(x) \leq n/5$, our proof is carried out in two steps:

Step 1. Let us start from the special case of $LMSR(x) = 0$. Note that $LMSR_0(x) = 0$. Let $B = C(x)$. We split x into $(k+1)$ substrings $x = y_1 y_2 \dots y_k y_{k+1}$, where $|y_i| = B$ for $1 \leq i \leq k$, $|y_{k+1}| = n \bmod B$, and $k = \lfloor n/B \rfloor$. By the assumption that $C(x) = B$, we have $y_1 < \min\{y_2, y_3, \dots, y_k\}$. It holds that $y_1 y_2 > 0^{2B}$; otherwise, $y_1 = y_2 = 0^B$, and then a contradiction $C(x) > B$ arises.

Let $m = \lfloor (k-1)/4 \rfloor$. We select some of y_i s and divide them into m groups (and the others are ignored). For every $1 \leq i \leq m$, the i -th group is $z_i = y_{4i-2}y_{4i-1}y_{4i}y_{4i+1}$. Let L_i be the number of characters in front of z_i . Then $L_i = (4i-3)B$. We claim that $bs_x(LMSR_0) \geq m$ by explicitly constructing m disjoint sets $S_1, S_2, \dots, S_m \subseteq [n]$ such that $LMSR_0(x) \neq LMSR_0(x^{S_i})$ for $1 \leq i \leq m$:

1. If L_i is even, then we define:

$$S_i = \{L_i \leq j \leq L_i + 2B : x_j \neq \delta_{j, L_i}\},$$

where $\delta_{x,y}$ is the Kronecker delta, that is, $\delta_{x,y} = 1$ if $x = y$ and 0 otherwise.

2. If L_i is odd, then we define:

$$S_i = \{L_i + 1 \leq j \leq L_i + 2B + 1 : x_j \neq \delta_{j, L_i+1}\}.$$

Note that $S_i \neq \emptyset$, and 0^{2B} is indeed the substring of x^{S_i} that starts at the index $L_i + 1$ if L_i is even and at $L_i + 2$ if L_i is odd. That is,

$$LMSR(x^{S_i}) = \begin{cases} L_i + 1 & L_i \equiv 0 \pmod{2} \\ L_i + 2 & L_i \equiv 1 \pmod{2} \end{cases}.$$

Then we conclude that $LMSR_0(x^{S_i}) = 1$ for $1 \leq i \leq m$. Consequently,

$$bs_x(LMSR_0) \geq m = \left\lfloor \frac{k-1}{4} \right\rfloor = \left\lfloor \frac{n}{4B} - \frac{1}{4} \right\rfloor.$$

Step 2. Now we remove the condition that $LMSR(x) = 0$ in Step 1. Let $r = LMSR_0(x)$ and we consider the binary string $x^{(r)}$. Note that $LMSR(x^{(r)}) = 0$. By Lemma 6.2, we have:

$$bs_x(LMSR_0) = bs_{x^{(r)}}(LMSR_0) \geq \left\lfloor \frac{n}{4B} - \frac{1}{4} \right\rfloor.$$

Therefore, inequality (8) holds for all $x \in \{0,1\}^n$ with $C(x) \leq n/5$. \square

We remark that inequality (8) can be slightly improved:

$$bs_x(LMSR_0) \geq \left\lfloor \frac{n-1}{2C(x)+2} \right\rfloor,$$

by splitting x more carefully. However, Lemma 6.3 is sufficient for our purpose. With it, we obtain a lower bound of the expected value of $bs_x(LMSR_0)$ when x is uniformly distributed:

Lemma 6.4. *Let $unif : \{0,1\}^n \rightarrow [0,1]$ be the uniform distribution that $unif(x) = 2^{-n}$ for every $x \in \{0,1\}^n$. Then*

$$\begin{aligned} \mathbb{E}_{x \sim unif} [bs_x(LMSR_0)] &= \Omega(n/\log n), \\ \mathbb{E}_{x \sim unif} \left[\sqrt{bs_x(LMSR_0)} \right] &= \Omega\left(\sqrt{n/\log n}\right). \end{aligned}$$

Proof. By Lemma 5.3 and Lemma 6.3, we have:

$$\begin{aligned}
\mathbb{E}_{x \sim \text{unif}}[bs_x(LMSR_0)] &= \sum_{x \in \{0,1\}^n} 2^{-n} bs_x(LMSR_0) \\
&\geq \sum_{x \in \{0,1\}^n} 2^{-n} \left| \frac{n}{4C(x)} - \frac{1}{4} \right| \\
&= \sum_{B=1}^n \sum_{x \in \{0,1\}^n: C(x)=B} 2^{-n} \left| \frac{n}{4B} - \frac{1}{4} \right| \\
&\geq \sum_{B=1}^{\lceil 3 \log_2 n \rceil} \sum_{x \in \{0,1\}^n: C(x)=B} 2^{-n} \left| \frac{n}{4B} - \frac{1}{4} \right| \\
&\geq \sum_{B=1}^{\lceil 3 \log_2 n \rceil} \sum_{x \in \{0,1\}^n: C(x)=B} 2^{-n} \left[\frac{n}{4 \lceil 3 \log_2 n \rceil} - \frac{1}{4} \right] \\
&= \left[\frac{n}{4 \lceil 3 \log_2 n \rceil} - \frac{1}{4} \right] \Pr_{x \sim \text{unif}}[C(x) \leq \lceil 3 \log_2 n \rceil] \\
&\geq \left[\frac{n}{4 \lceil 3 \log_2 n \rceil} - \frac{1}{4} \right] \left(1 - \frac{1}{n} \right) \\
&= \Omega\left(\frac{n}{\log n}\right).
\end{aligned}$$

A similar argument yields that $\mathbb{E}_{x \sim \text{unif}}[\sqrt{bs_x(LMSR_0)}] = \Omega(\sqrt{n/\log n})$. \square

By combining the above lemma with Theorem 6.1, we obtain lower bounds for randomized and quantum average-case bounded-error algorithms for LMSR:

$$R^{\text{unif}}(LMSR_0) = \Omega(n/\log n) \text{ and } Q^{\text{unif}}(LMSR_0) = \Omega(\sqrt{n/\log n}).$$

6.2 Worst-case Classical Lower Bound

Now we turn to consider the worst-case lower bounds. The idea is similar to the average-case. First, the following result similar to Theorem 6.1 was also proved in [AdW99].

Theorem 6.5 ([AdW99]). *Let A be a bounded-error algorithm for some function $f : \{0,1\}^n \rightarrow \{0,1\}$.*

1. *If A is classical, then $T_A(x) = \Omega(bs_x(f))$; and*
2. *If A is quantum, then $T_A(x) = \Omega(\sqrt{bs_x(f)})$.*

We still consider the function $LMSR_0$ in this subsection. The following lemma shows that its block sensitivity can be linear in the worst case.

Lemma 6.6. *There is a string $x \in \{0,1\}^n$ such that $bs_x(LMSR_0) \geq \lfloor n/2 \rfloor$.*

Proof. Let $x = 1^n$. Then $LMSR(x) = 0$ and $LMSR_0(x) = 0$. We can choose $m = \lfloor n/2 \rfloor$ disjoint sets S_1, S_2, \dots, S_m with $S_i = \{2i-1\}$. It may be easily verified that $LMSR_0(x^{S_i}) = 1$ for every $1 \leq i \leq m$. Thus, by the definition of block sensitivity, we have $bs_x(LMSR_0) \geq \lfloor n/2 \rfloor$. \square

Combining the above lemma with Theorem 6.5, we conclude that $R(LMSR_0) = \Omega(n)$ and $Q(LMSR_0) = \Omega(\sqrt{n})$, which give a lower bound for randomized and one for quantum worst-case bounded-error algorithms for LMSR, respectively. We have another more intuitive proof for the worst-case lower bound for quantum bounded-error algorithms and postpone into Appendix F.

7 Applications

In this section, we present some practical applications of our quantum algorithm for LMSR.

7.1 Benzenoid Identification

The first application of our algorithm is a quantum solution to a problem about chemical graphs. Benzenoid hydrocarbons are a very important class of compounds [Dia87, Dia88] and also popular as mimics of graphene (see [WSS⁺08, WMK08, PLB⁺16]). Several algorithmic solutions to the identification problem of benzenoids have been proposed in the previous literature; for example, Bašić [Baš16] identifies benzenoids by boundary-edges code [HLZ96] (see also [KPBF14]).

Formally, the boundary-edges code (BEC) of a benzenoid is a finite string over a finite alphabet $\Sigma_6 = \{1, 2, 3, 4, 5, 6\}$. The canonical BEC of a benzenoid is essentially the lexicographically maximal string among all rotations of any of its BECs and their reverses. Our quantum algorithm for LMSR can be used to find the canonical BEC of a benzenoid in $O(n^{3/4})$ queries, where n is the length of its BEC.

More precisely, it is equivalent to find the lexicographically minimal one if we assume that the lexicographical order is $6 < 5 < 4 < 3 < 2 < 1$. Suppose a benzenoid has a BEC s . Our quantum algorithm is described as follows:

- 1 Let $i = LMSR(s)$ and $i^R = LMSR(s^R)$, where s^R denotes the reverse of s . This is achieved by Algorithm 5 in $O(n^{3/4})$ query complexity.
- 2 Return the smaller one between $s[i \dots i + n - 1]$ and $s^R[i^R \dots i^R + n - 1]$. This is achieved by Algorithm 3 in $O(\sqrt{n})$ query complexity.

It is straightforward to see that the overall query complexity is $O(n^{3/4})$.

7.2 Disjoint-cycle Automata Minimization

Another application of our algorithm is a quantum solution to minimization of a special class of automata. Automata minimization is an important problem in automata theory [HMU00, BBCF10] and has many applications in various areas of computer science. The best known algorithm for minimizing deterministic automata is $O(n \log n)$ [Hop71], where n is the number of states. A few linear algorithms for minimizing some special automata are proposed in [Rev92, AZ08], which are important in practice, e.g. dictionaries in natural language processing.

We consider the minimization problem of disjoint-cycle automata discussed by Almeida and Zeitoun in [AZ08]. The key to this problem is a decision problem that checks whether there are two cycles that are equal to each other under rotations. Formally, suppose there are m cycles, which are described by strings s_1, s_2, \dots, s_m over a finite alphabet Σ . It is asked whether there are two strings s_i and s_j ($i \neq j$) such that $SCR(s_i) = SCR(s_j)$. For convenience, we assume that all strings are of equal length n , i.e. $|s_1| = |s_2| = \dots = |s_m| = n$.

A classical algorithm solving the above decision problem was developed in [AZ08] with time complexity $O(mn)$. With the help of our quantum algorithm for LMSR, this problem can be

solved more efficiently. We employ a quantum comparison oracle U_{cmp} that compares strings by their canonical representations in the lexicographical order, where the corresponding classical comparator is:

$$cmp(i, j) = \begin{cases} 1 & SCR(s_i) < SCR(s_j), \\ 0 & \text{otherwise,} \end{cases}$$

and can be computed by finding $r_i = LMSR(s_i)$ and $r_j = LMSR(s_j)$. In particular, it can be done by our quantum algorithm for LMSR in $O(n^{3/4})$ queries. Then the lexicographical comparator in Algorithm 3 can be used to compare $s_i[r_i \dots r_i + n - 1]$ and $s_j[r_j \dots r_j + n - 1]$ in query complexity $O(n^{3/4})$. Furthermore, the problem of checking whether there are two strings that are equal to each other under rotations among the m strings may be viewed as the element distinctness problem with quantum comparison oracle U_{cmp} , and thus can be solved by Ambainis's quantum algorithm [Amb07] with $\tilde{O}(m^{2/3})$ queries to U_{cmp} . In conclusion, the decision problem can be solved in quantum time complexity $\tilde{O}(m^{2/3}n^{3/4})$, which is better than the best known classical $O(mn)$ time.

References

- [AdW99] A. Ambainis and R. de Wolf. Average-case quantum query complexity. *Journal of Physics A General Physics*, 34(35):6741–6754, 1999.
- [AHU74] A. V. Aho, J. E. Hopcroft, and J. D. Ullman. *The Design and Analysis of Computer Algorithms*. Addison-Wesley, 1974.
- [AIP87] A. Apostolico, C. S. Iliopoulos, and R. Paige. An $O(n \log n)$ cost parallel algorithm for the one function partitioning problem. In *Proceedings of the International Workshop on Parallel Algorithms and Architectures*, pages 70–76, 1987.
- [AK99] A. Ahuja and S. Kapoor. A quantum algorithm for finding the maximum. ArXiv e-prints, 1999. arXiv:quant-ph/9911082.
- [Amb00] A. Ambainis. Quantum lower bounds by quantum arguments. In *Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing*, pages 636–643, 2000.
- [Amb04] A. Ambainis. Quantum query algorithms and lower bounds. In *Classical and New Paradigms of Computation and their Complexity Hierarchies*, pages 15–32, 2004.
- [Amb07] A. Ambainis. Quantum walk algorithm for element distinctness. *SIAM Journal on Computing*, 37(1):210–239, 2007.
- [AZ08] J. Almeida and M. Zeitoun. Description and analysis of a bottom-up DFA minimization algorithm. *Information Processing Letters*, 107(2):52–59, 2008.
- [Baš16] N. Bašić. *Algebraic approach to several families of chemical graphs*. PhD thesis, Faculty of Mathematics and Physics, University of Ljubljana, 2016.
- [BBBV97] C. H. Bennett, E. Bernstein, G. Brassard, and U. Vazirani. Strengths and weaknesses of quantum computing. *SIAM Journal on Computing*, 26(5):1524–1540, 1997.
- [BBC⁺01] R. Beals, H. Buhrman, R. Cleve, M. Mosca, and R. de Wolf. Quantum lower bounds by polynomials. *Journal of the ACM*, 48(4):778–797, 2001.

- [BBCF10] J. Berstel, L. Boasson, O. Carton, and I. Fagnot. Minimization of automata. ArXiv e-prints, 2010. arXiv:1010.5318.
- [BBHT98] M. Boyer, G. Brassard, P. Høyer, and A. Tapp. Tight bounds on quantum searching. *Fortschritte der Physik*, 46(4-5):493–505, 1998.
- [BCdWZ99] H. Buhrman, R. Cleve, R. de Wolf, and C. Zalka. Bounds for small-error and zero-error quantum algorithms. In *Proceedings of the Fortieth IEEE Annual Symposium on Foundations of Computer Science*, pages 358–368, 1999.
- [BCN05] F. Bassino, J. Clément, and C. Nicaud. The standard factorization of Lyndon words: an average point of view. *Discrete Mathematics*, 280(1):1–25, 2005.
- [BCW98] H. Buhrman, R. Cleve, and A. Wigderson. Quantum vs. classical communication and computation. In *Proceedings of the Thirtieth Annual ACM Symposium on Theory of Computing*, pages 63–68, 1998.
- [BdW02] H. Buhrman and R. de Wolf. Complexity measures and decision tree complexity: A survey. *Theoretical Computer Science*, 288(1):21–43, 2002.
- [Boo80] K. S. Booth. Lexicographically least circular substrings. *Information Processing Letters*, 10(4-5):240–242, 1980.
- [BS04] H. Barnum and M. Saks. A lower bound on the quantum query complexity of read-once functions. *Journal of Computer and System Sciences*, 69(2):244–258, 2004.
- [CB80] C. J. Colbourn and K. S. Booth. Linear time automorphism algorithms for trees, interval graphs, and planar graphs. *SIAM Journal on Computing*, 10(1):203–225, 1980.
- [CGW00] N. J. Cerf, L. K. Grover, and C. P. Williams. Nested quantum search and structured problems. *Physical Review A*, 61(3):032303, 2000.
- [CGYM08] R. Cleve, D. Gavinsky, and D. L. Yonge-Mallo. Quantum algorithms for evaluating MIN-MAX trees. In *Theory of Quantum Computation, Communication, and Cryptography: Third Workshop, TQC 2008*, pages 11–15, 2008.
- [CHL07] M. Crochemore, C. Hancart, and T. Lecroq. *Algorithms on Strings*. Cambridge University Press, 2007.
- [CR94] M. Crochemore and W. Rytter. *Text Algorithms*. Oxford University Press, 1994.
- [Cro92] M. Crochemore. String-matching on ordered alphabets. *Theoretical Computer Science*, 92(1):33–47, 1992.
- [DH96] C. Dürr and P. Høyer. A quantum algorithm for finding the minimum. ArXiv e-prints, 1996. arXiv:quant-ph/9607014.
- [Dia87] J. R. Dias. *Handbook of Polycyclic Hydrocarbons: Part A, Benzenoid Hydrocarbons*. Physical Sciences Data, Elsevier, 1987.
- [Dia88] J. R. Dias. *Handbook of Polycyclic Hydrocarbons: Part B, Polycyclic Isomers and Heteroatom Analogs of Benzenoid Hydrocarbons*. Physical Sciences Data, Elsevier, 1988.

- [Duv83] J. P. Duval. Factorizing words over an ordered alphabet. *Journal of Algorithms*, 8(8):363–381, 1983.
- [GBYS92] G. H. Gonnet, R. A. Baeza-Yates, and T. Snider. New indices for text: PAT trees and PAT arrays. In *Information Retrieval: Data Structures and Algorithms*, pages 66–82, 1992.
- [Grö96] L. K. Gröver. A fast quantum mechanical algorithm for database search. In *Proceedings of the Twenty-eighth Annual ACM Symposium on Theory of Computing*, pages 212–219, 1996.
- [GY03] J. L. Gross and J. Yellen. *Handbook of Graph Theory*. CRC Press, 2003.
- [HLZ96] P. Hansen, C. Lebatteux, and M. Zheng. The boundary-edges code for polyhexes. *Journal of Molecular Structure: THEOCHEM*, 363(2):237–247, 1996.
- [HMdW03] P. Høyer, M. Mosca, and R. de Wolf. Quantum search on bounded-error inputs. In *International Colloquium on Automata, Languages, and Programming*, pages 291–299, 2003.
- [HMU00] J. E. Hopcroft, R. Motwani, and J. D. Ullman. *Introduction to Automata Theory, Languages, and Computation*. Addison-Wesley, 2 edition, 2000.
- [Hoe63] W. Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, 1963.
- [Hop71] J. E. Hopcroft. An $n \log n$ algorithm for minimizing states in a finite automaton. In *Proceedings of an International Symposium on the Theory of Machines and Computations*, pages 189–196, 1971.
- [IS89] C. S. Iliopoulos and W. F. Smyth. PRAM algorithms for identifying polygon similarity. In *Proceedings of the International Symposium on Optimal Algorithms*, pages 25–32, 1989.
- [IS92] C. S. Iliopoulos and W. F. Smyth. Optimal algorithms for computing the canonical form of a circular string. *Theoretical Computer Science*, 92(1):87–105, 1992.
- [IS94] C. S. Iliopoulos and W. F. Smyth. A fast average case algorithm for Lyndon decomposition. *International Journal of Computer Mathematics*, 57(1-2):15–31, 1994.
- [Jeu93] J. T. Jeuring. *Theories for algorithm calculation*. Utrecht University, 1993.
- [JKM13] S. Jeffery, R. Kothari, and F. Magniez. Nested quantum walks with quantum data structures. In *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1474–1485, 2013.
- [KKM⁺20] R. Kapralov, K. Khadiev, J. Mokut, Y. Shen, and M. Yagafarov. Fast classical and quantum algorithms for online k -server problem on trees. ArXiv e-prints, 2020. arXiv:2008.00270.
- [KMP77] D. Knuth, J. H. Morris, and V. Pratt. Fast pattern matching in strings. *SIAM Journal on Computing*, 6(2):323–350, 1977.

- [KPBF14] J. Kovič, T. Pisanski, A. T. Balaban, and P. W. Fowler. On symmetries of benzenoid systems. *MATCH Communications in Mathematical and in Computer Chemistry*, 72(1):3–26, 2014.
- [KSB06] J. Kärkkäinen, P. Sanders, and S. Burkhardt. Linear work suffix array construction. *Journal of the ACM*, 53(6):918–9360, 2006.
- [Mae91] M. Maes. Polygonal shape recognition using string-matching techniques. *Pattern Matching*, 24(5):433–440, 1991.
- [McC76] E. M. McCreight. A space-economical suffix tree construction algorithm. *Journal of the ACM*, 23(2):262–272, 1976.
- [MM90] U. Manber and G. Myers. Suffix arrays: a new method for on-line string searches. In *Proceedings of the First Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 319–327, 1990.
- [Mon17] A. Montanaro. Quantum pattern matching fast on average. *Algorithmica*, 77:16–39, 2017.
- [PLB⁺16] R. Papadakis, H. Li, J. Bergman, A. Lundstedt, K. Jorner, R. Ayub, S. Haldar, B. O. Jahn, A. Denisova, B. Zietz, R. Lindh, B. Sanyal, H. Grennberg, K. Leifer, and H. Ottosson. Metal-free photochemical silylations and transfer hydrogenations of benzenoid hydrocarbons and graphene. *Nature Communications*, 7:12962, 2016.
- [Pup10] G. Puppis. *Automata for branching and layered temporal structures: an investigation into regularities of infinite transition systems*. Springer Science & Business Media, 2010.
- [Rev92] D. Revuz. Minimisation of acyclic deterministic automata in linear time. *Theoretical Computer Science*, 92(1):181–189, 1992.
- [RV03] H. Ramesh and V. Vinay. String matching in $\tilde{O}(\sqrt{n} + \sqrt{m})$ quantum time. *Journal of Discrete Algorithms*, 1(1):103–110, 2003.
- [Ryt03] W. Rytter. On maximal suffixes and constant-space linear-time versions of KMP algorithm. *Theoretical Computer Science*, 299(1-3):763–774, 2003.
- [Shi79] Y. Shiloach. A fast equivalence-checking algorithm for circular lists. *Information Processing Letters*, 8(5):236–238, 1979.
- [Shi81] Y. Shiloach. Fast canonization of circular strings. *Journal of Algorithms*, 2(2):107–121, 1981.
- [SWW16] J. Sawada, A. Williams, and D. Wong. A surprisingly simple de Bruijn sequence construction. *Discrete Mathematics*, 339(1):127–131, 2016.
- [Vis90] U. Vishkin. Deterministic sampling - a new technique for fast pattern matching. In *Proceedings of the Twenty-Second Annual ACM Symposium on Theory of Computing*, pages 170–180, 1990.
- [vLWW01] J. H. van Lint, R. W. Wilson, and R. M. Wilson. *A Course in Combinatorics*. Cambridge University Press, 2001.

- [Wei73] P. Weiner. Linear pattern matching algorithms. In *Proceedings of the Fourteenth Annual Symposium on Switching and Automata Theory*, pages 1–11, 1973.
- [WMK08] W. L. Wang, S. Meng, and E. Kaxiras. Graphene NanoFlakes with large spin. *Nano Letters*, 8(1):241–245, 2008.
- [WSS⁺08] T. Wassmann, A. P. Seitsonen, A. M. Saitta, M. Lazzeri, and F. Mauri. Structure, stability, edge states, and aromaticity of graphene ribbons. *Physical Review Letters*, 101(9):096402, 2008.
- [Zal99] C. Zalka. Grover’s quantum searching algorithm is optimal. *Physical Review A*, 60(4):2746–2751, 1999.

A Probability Analysis of Majority Voting

For convenience of the reader, let us first recall the Hoeffding’s inequality [Hoe63].

Theorem A.1 (The Hoeffding’s inequality, [Hoe63]). *Let X_1, X_2, \dots, X_n be independent random variables with $0 \leq X_i \leq 1$ for every $1 \leq i \leq n$. We define the empirical mean of these variables by*

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i.$$

Then for every $\varepsilon > 0$, we have:

$$\Pr[\bar{X} \leq \mathbb{E}[\bar{X}] - \varepsilon] \leq \exp(-2n\varepsilon^2).$$

We only consider the case where b is true. In this case, we have $q = \lceil 36 \ln m \rceil$ independent random variables X_1, X_2, \dots, X_q , where for $1 \leq i \leq q$, $X_i = 0$ with probability $\leq 1/3$ and $X_i = 1$ with probability $\geq 2/3$ (Here, $X_i = 1$ means the i -th query is true and 0 otherwise). It holds that $\mathbb{E}[\bar{X}] \geq 2/3$. By Theorem A.1 and letting $\varepsilon = 1/6$, we have:

$$\Pr \left[\bar{X} \leq \frac{1}{2} \right] \leq \Pr[\bar{X} \leq \mathbb{E}[\bar{X}] - \varepsilon] \leq \exp(-2q\varepsilon^2) \leq \exp(-2 \ln m) = \frac{1}{m^2}.$$

By choosing \hat{b} to be true if $\bar{X} > 1/2$ and false otherwise, we obtain $\Pr [\hat{b} = b] = 1 - \Pr [\bar{X} \leq 1/2] \geq 1 - 1/m^2$ as required in Algorithm 1.

B A Framework of Nested Quantum Algorithms

In this appendix, we provide a general framework to explain how the improvement given in Section 3 can be achieved on nested quantum algorithms composed of quantum search and quantum minimum finding.

Suppose there is a d -level nested quantum algorithm composed of d bounded-error quantum algorithms A_1, A_2, \dots, A_d on a d -dimensional input $x(\theta_{d-1}, \dots, \theta_2, \theta_1, \theta_0)$ given by an (exact) quantum oracle U_0 :

$$U_0 |\theta_{d-1}, \dots, \theta_2, \theta_1, \theta_0, j\rangle = U_0 |\theta_{d-1}, \dots, \theta_2, \theta_1, \theta_0, j \oplus x(\theta_{d-1}, \dots, \theta_2, \theta_1, \theta_0)\rangle,$$

where $\theta_k \in [n_k]$ for $k \in [d]$ and $d \geq 2$. For $1 \leq k \leq d$, A_k is a bounded-error quantum algorithm given parameters $\theta_{d-1}, \dots, \theta_k$ that computes the function $f_k(\theta_{d-1}, \dots, \theta_k)$. In particular, $f_0 \equiv x$, and A_d computes a single value f_d , which is considered to be the output of the nested quantum algorithm A_1, A_2, \dots, A_d .

Let $t_k \in \{s, m\}$ denote the type of A_k , where s indicates the searching problem and m indicates the minimum finding problem. Let $S_m = \{1 \leq k \leq d : t_k = m\}$ denote the set of indices of the minimum finding algorithms. The behavior of algorithm A_k is defined as follows:

1. **Case 1.** $t_k = s$. Then A_k is associated with a checker $p_k(\theta_{d-1}, \dots, \theta_k, \xi)$ that determines whether ξ is a solution (which returns 1 for “yes” and 0 for “no”). A_k is to find a solution $i \in [n_{k-1}]$ such that

$$p_k(\theta_{d-1}, \dots, \theta_k, f_{k-1}(\theta_{d-1}, \dots, \theta_k, i)) = 1$$

(which returns -1 if no solution exists). Here, computing $p_k(\theta_{d-1}, \dots, \theta_k, f_{k-1}(\theta_{d-1}, \dots, \theta_k, i))$ requires the value of $f_{k-1}(\theta_{d-1}, \dots, \theta_k, i)$, which in turn requires a constant number of queries to A_{k-1} with parameters $\theta_{d-1}, \dots, \theta_k, i$.

2. **Case 2.** $t_k = m$. Then A_k is associated with a comparator $c_k(\theta_{d-1}, \dots, \theta_k, \alpha, \beta)$ that compares α and β . A_k is to find the index $i \in [n_{k-1}]$ of the minimal element such that

$$c_k(\theta_{d-1}, \dots, \theta_k, f_{k-1}(\theta_{d-1}, \dots, \theta_k, j), f_{k-1}(\theta_{d-1}, \dots, \theta_k, i)) = 0$$

for all $j \neq i$. Here, computing $c_k(\theta_{d-1}, \dots, \theta_k, f_{k-1}(\theta_{d-1}, \dots, \theta_k, j), f_{k-1}(\theta_{d-1}, \dots, \theta_k, i))$ requires the value of $f_{k-1}(\theta_{d-1}, \dots, \theta_k, j)$ and $f_{k-1}(\theta_{d-1}, \dots, \theta_k, i)$, which requires a constant number of queries to A_{k-1} with parameters $\theta_{d-1}, \dots, \theta_k, j$ and $\theta_{d-1}, \dots, \theta_k, i$, respectively.

The following lemma settles the query complexity of nested quantum algorithms, which shows that nested algorithms can do much better than naively expected.

Lemma B.1 (Nested quantum algorithms). *Given a d -level nested quantum algorithm A_1, A_2, \dots, A_d , where $d \geq 2$, and $n_0, n_1, \dots, n_{d-1}, S_m, f_d$ are defined as above. Then:*

1. *There is a bounded-error quantum algorithm that computes f_d with query complexity*

$$O\left(\sqrt{\prod_{k=0}^{d-1} n_k}\right).$$

2. *There is a quantum algorithm that computes f_d with error probability $\leq \varepsilon$ with query complexity*

$$O\left(\sqrt{\prod_{k=0}^{d-1} n_k \log \frac{1}{\varepsilon}}\right).$$

Proof. Immediately yields by Theorem 3.1, Theorem 3.2, Lemma 3.4 and Lemma 3.5. \square

Remark B.1. *Lemma B.1 can be seen as a combination of Theorem 3.1, Theorem 3.2, Lemma 3.4 and Lemma 3.5. For convenience, we assume $n_0 = n_1 = \dots = n_{d-1} = n$ in the following discussions. Note that traditional probability amplification methods for randomized algorithms usually introduce an $O(\log^{d-1} n)$ slowdown for d -level nested quantum algorithms by repeating the algorithm $O(\log n)$ times in each level. However, our method obtains an extremely better $O(1)$ factor as if there are no errors in oracles at all.*

Remark B.2. *Lemma B.1 only covers a special case of nested quantum algorithms. A more general form of nested quantum algorithms can be described as a tree rather than a sequence, which allows intermediate quantum algorithms to compute their results by queries to several low-level quantum algorithms. We call them **adaptively nested quantum algorithms**, and an example of this kind algorithm is presented in Appendix D (for pattern matching).*

C Remarks for Quantum Deterministic Sampling

Algorithm 4 uses several nested quantum algorithms as subroutines, but they are not described as nested quantum algorithms explicitly. Here, we provide an explicit description for Line 10 in Algorithm 4 as an example. Let $\theta_0 \in [l]$ and $\theta_1 \in [n]$. Then:

- The 0-level function is

$$f_0(\theta_1, \theta_0) = \begin{cases} 1 & \theta_1 \leq i_{\theta_0} < \theta_1 + n \wedge s[i_{\theta_0} - \theta_1] \neq s[i_{\theta_0} - \delta], \\ 0 & \text{otherwise,} \end{cases}$$

which checks whether candidate θ_1 does not match the current deterministic sample at the θ_0 -th checkpoint.

- The 1-level function is

$$f_1(\theta_1) = \begin{cases} 0 & \theta_1 > Q \vee \exists i, f_0(\theta_1, i) = 1, \\ 1 & \text{otherwise,} \end{cases}$$

which checks candidate θ_1 matches the current deterministic sample at all checkpoints.

- The 2-level function is

$$f_2 = \min \{i : f_1(i) = 1\},$$

which finds the first candidate of δ .

By Lemma B.1, f_2 can be computed with error probability $\leq \varepsilon = 1/6m^2$ in

$$O\left(\sqrt{mn \log(1/\varepsilon)}\right) = O\left(\sqrt{n \log n \log \log n}\right).$$

D Quantum Algorithm for Pattern Matching

In this appendix, we give a detailed description of our quantum algorithm for pattern matching.

D.1 Quantum Algorithm for String Periodicity

The algorithm will be presented in the form of a nested quantum algorithm. Suppose a string $s \in \Sigma^n$ is given by a quantum oracle U_s that $U_s |i, j\rangle = |i, j \oplus s[i]\rangle$. We are asked to check whether s is periodic, and if yes, find its period.

Let $(\delta; i_0, i_1, \dots, i_{l-1})$ be a deterministic sample of s . We need a 2-level nested quantum algorithm. Let $\theta_0 \in [l]$ and $\theta_1 \in [n]$. Then:

- The 0-level function is

$$f_0(\theta_1, \theta_0) = \begin{cases} 1 & \theta_1 \leq i_{\theta_0} < \theta_1 + n \wedge s[i_{\theta_0} - \theta_1] = s[i_{\theta_0} - \delta], \\ 0 & \text{otherwise,} \end{cases}$$

which checks whether offset θ_1 matches the deterministic sample at the θ_0 -th checkpoint. There is obviously an exact quantum oracle that computes $f_0(\theta_1, \theta_0)$ with a constant number of queries to U_s .

- The 1-level function is

$$f_1(\theta_1) = \begin{cases} 0 & \exists i \in [l], f_0(\theta_1, i) = 1, \\ 1 & \text{otherwise,} \end{cases}$$

where $f_1(\theta_1)$ means offset θ_1 matches the deterministic sample of s . The 2-level function is

$$f_2 = \min\{i \in [n] : f_1(i) = 1\},$$

which finds the minimal offset that matches the deterministic sample.

By Lemma B.1, we obtain an $O(\sqrt{nl})$ bounded-error quantum algorithm that finds the smallest possible offset δ_1 of the deterministic sample of s . According to δ_1 , we define another 2-level function

$$f'_2 = \min\{i \in [n] : i > \delta_1 \wedge f_1(i) = 1\},$$

which finds the second minimal offset that matches the deterministic sample of s , where $\min \emptyset = \infty$. Similarly, we can find the second smallest offset δ_2 of the deterministic sample of s in query complexity $O(\sqrt{nl})$ with bounded-error. If $\delta_2 = \infty$, then s is aperiodic; otherwise, s is periodic with period $d = \delta_2 - \delta_1$. Therefore, we obtain an $O(\sqrt{n \log n})$ bounded-error quantum algorithm that checks whether a string is periodic and, if yes, finds its period.

D.2 Quantum Algorithm for Pattern Matching

Suppose text $t \in \Sigma^n$ and pattern $p \in \Sigma^m$, and a deterministic sample of p is $(\delta; i_0, i_1, \dots, i_{l-1})$. The idea for pattern matching is to split the text t into blocks of length $L = \lfloor m/4 \rfloor$, the i -th (0-indexed) of which consists of indices ranged from iL to $\min\{(i+1)L, n\} - 1$. Our algorithm applies to the case of $m \geq 4$, but does not to the case $1 \leq m \leq 3$, where a straightforward quantum search is required (we omit it here).

The key step for pattern matching is to find a candidate h_i for $0 \leq i < \lceil n/L \rceil$, indicating the first occurrence in the i -th block with starting index $iL \leq j < \min\{(i+1)L, n\}$, where $j + m \leq n$ and $t[j \dots j + m - 1] = p$. Formally,

$$h_i = \min\{iL \leq j < \min\{(i+1)L, n\} : j + m \leq n \wedge t[j \dots j + m - 1] = p\}, \quad (9)$$

where $\min \emptyset = \infty$. Note that Eq. (4) is similar to Eq. (9) but without the condition of $j + m \leq n$, which can easily be removed from our algorithm given in the following discussion. In the previous subsection, we presented an efficient quantum algorithm that checks whether the string is periodic or not. Then we are able to design a quantum algorithm for pattern matching. Let us consider the cases of aperiodic patterns and periodic patterns, separately:

D.2.1 Aperiodic Patterns

The Algorithm. For an aperiodic pattern p , h_i can be computed in the following two steps:

1. Search j from $iL \leq j < \min\{(i+1)L, n\}$ such that $t[i_k - \delta + j] = p[i_k - \delta]$ for every $k \in [l]$ (we call such j a candidate of matching). For every j , there is an $O(\sqrt{l})$ bounded-error quantum algorithm to check whether $t[i_k - \delta + j] = p[i_k - \delta]$ for every $k \in [l]$. Therefore, finding any j requires $O(\sqrt{Ll}) = O(\sqrt{m \log m})$ queries (by Theorem 3.1).
2. Check whether the index j found in the previous step satisfies $j+m \leq n$ and $t[j \dots j+m-1] = p$. This can be computed by quantum search in $O(\sqrt{m})$ queries. If the found index j does not satisfy this condition, then $h_i = \infty$; otherwise, $h_i = j$.

It is clear that h_i can be computed with bounded-error in $O(\sqrt{m \log m})$ queries according to the above discussion. Recall that p appears in t at least once, if and only if there is at least a $0 \leq i < \lceil n/L \rceil$ such that $h_i = 1$. By Theorem 3.1, this can be checked in $O(\sqrt{n/L} \sqrt{m \log m}) = O(\sqrt{n \log m})$ queries.

Correctness. Note that in the i -th block, if there is no such j or there are more than two values of j such that $t[i_k - \delta + j] = p[i_k - \delta]$ for every $k \in [l]$, then there is no matching in the i -th block. More precisely, we have:

Lemma D.1 (The Ricochet Property [Vis90]). *Let $p \in \Sigma^m$ be aperiodic, $(\delta; i_0, i_1, \dots, i_{l-1})$ be a deterministic sample, and $t \in \Sigma^n$ be a string of length $n \geq m$, and let $j \in [n - m + 1]$. If $t[i_k + j - \delta] = p[i_k - \delta]$ for every $k \in [l]$, then $t[j' \dots j' + m - 1] \neq p$ for every $j' \in [n - m + 1]$ with $j - \delta \leq j' < j - \delta + \lfloor m/2 \rfloor$.*

Proof. Assume that $t[j' \dots j' + m - 1] = p$ for some $j - \delta \leq j' < j - \delta + \lfloor m/2 \rfloor$. Let $x = \delta - j + j'$. Note that $0 \leq x < \lfloor m/2 \rfloor$. We further assume that $x \neq \delta$. Then by the definition of a deterministic sample, there exists $k \in [l]$ such that $x \leq i_k < x + m$ and $p[i_k - x] \neq p[i_k - \delta]$. On the other hand, $p[i_k - x] = t[j' + i_k - x] = t[i_k + j - \delta] = p[i_k - \delta]$. A contradiction arises, which implies $x = \delta$ and therefore $j = j'$. \square

We can see that if there are two different indices j_1, j_2 in the i -th block such that $t[i_k - \delta + j_r] = p[i_k - \delta]$ for every $k \in [l]$ and $r \in \{1, 2\}$, it must hold that $|j_1 - j_2| \leq m/4$, since each block has length $L = \lfloor m/4 \rfloor$. If we apply Lemma D.1 on j_1 , then $t[j_2 \dots j_2 + m - 1] \neq p$; and if we apply it on j_2 , then $t[j_1 \dots j_1 + m - 1] \neq p$. Thus, we conclude that neither j_1 nor j_2 can be a starting index of occurrence p in t . As a result, there is at most one candidate of matching in each block.

A Description in the Form of a Nested Quantum Algorithm. The above algorithm can be more clearly described as a 3-level nested quantum algorithm.

- The level-0 function is defined by

$$f_0(\theta_2, \theta_1, \theta_0) = \begin{cases} 1 & i_{\theta_0} - \delta + \theta_2 L + \theta_1 \in [n] \wedge t[i_{\theta_0} - \delta + \theta_2 L + \theta_1] = p[i_{\theta_0} - \delta] \\ 0 & \text{otherwise} \end{cases},$$

which checks whether t matches p at the θ_0 -th checkpoint at the θ_1 -th index in the θ_2 -th block, where $\theta_0 \in [l]$, $\theta_1 \in [L]$ and $\theta_2 \in [\lceil n/L \rceil]$.

- The level-1 function is defined by

$$f_1(\theta_2, \theta_1) = \begin{cases} 0 & \exists i \in [l], f_0(\theta_2, \theta_1, i) = 0, \\ 1 & \text{otherwise,} \end{cases}$$

which checks whether t matches the deterministic sample at the θ_1 -th index in the θ_2 -th block.

- The level-2 function $f_2(\theta_2)$ finds a solution $i \in [L]$ such that $f_1(\theta_2, i) = 1$, which indicates the (only) candidate in the θ_2 -th block.
- The level-3 function f_3 finds a matching among $f_2(i)$ over all $i \in \lceil n/L \rceil$ by checking whether $iL + f_2(i) + m \leq n$ and $t[iL + f_2(i) \dots iL + f_2(i) + m - 1] = p$, where the latter condition can be checked by a quantum searching algorithm, which can be formulated as a 1-level nested quantum algorithm:

- The level-0 function

$$g_0(\xi_2, \xi_1, \xi_0) = \begin{cases} 1 & t[\xi_2 L + \xi_1 + \xi_0] = p[\xi_0], \\ 0 & \text{otherwise,} \end{cases}$$

which checks whether the ξ_0 -th character of substring of t starting at offset ξ_1 in the ξ_2 -th block matches the ξ_0 -th character of p , where $\xi_0 \in [m]$, $\xi_1 \in [L]$ and $\xi_2 \in \lceil n/L \rceil$.

- The 1-level function

$$g_1(\xi_2, \xi_1) = \begin{cases} 0 & \exists i \in [m], g_0(\xi_2, \xi_1, i) = 0, \\ 1 & \text{otherwise,} \end{cases}$$

which checks whether the substring of t starting at offset ξ_1 in the ξ_2 -th block matches p . Finally, we have that f_3 finds a solution $i \in \lceil n/L \rceil$ such that $iL + f_2(i) + m \leq n$ and $g_1(i, f_2(i)) = 1$.

The structure of our algorithm can be visualized as the tree in Figure 1. It is worth noting that f_3 calls both f_2 and g_1 .

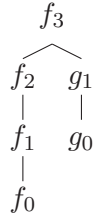


Figure 1: Quantum pattern matching algorithm for aperiodic strings.

By a careful analysis, we see that the query complexity of the above algorithm is $O(\sqrt{n \log m})$.

D.2.2 Periodic Patterns

For a periodic pattern p , a similar result can be achieved with some minor modifications to the algorithm for an aperiodic pattern. First, we have:

Lemma D.2 (The Ricochet Property for periodic strings). *Let $p \in \Sigma^m$ be periodic with period $d \leq m/2$, $(\delta; i_0, i_1, \dots, i_{l-1})$ be a deterministic sample, and $t \in \Sigma^n$ be a string of length $n \geq m$, and let $j \in [n - m + 1]$. If $t[i_k + j - \delta] = p[i_k - \delta]$ for every $k \in [l]$, then $t[j' \dots j' + m - 1] \neq p$ for every $j' \in [n - m + 1]$ with $j - \delta \leq j' < j - \delta + \lfloor m/2 \rfloor$ and $j' \not\equiv j \pmod{d}$.*

Proof. Similar to proof of Lemma D.1. □

Now in order deal with periodic pattern p , the algorithm for an aperiodic pattern can be modified as follows. For the i -th block, in order to compute h_i , we need to find (by minimum finding) the leftmost and the rightmost candidates j_l and j_r , which requires $O(\sqrt{Ll}) = O(\sqrt{m \log m})$ queries (by Algorithm 1). Let us consider two possible cases:

1. If $j_l \not\equiv j_r \pmod{d}$, then by Lemma D.2, there is no matching in the i -th block and thus $h_i = \infty$;
2. If $j_l \equiv j_r \pmod{d}$, find the smallest $j_l \leq q \leq R_i$ such that $t[q \dots R_i] = p[q - j_l \dots q - j_l + R_i - q]$, where $R_i = \min\{(i + 1)L, n\} - 1$ denotes the right endpoints of the i -th block, by minimum finding in $O(\sqrt{L}) = O(\sqrt{m})$ queries (by Algorithm 1). Then the leftmost candidate will be

$$j = j_l + \left\lceil \frac{q - j_l}{d} \right\rceil d.$$

If $j \leq \min\{n - m, j_r\}$ and $t[j \dots j + m - 1] = p$ (which can be checked by quantum search in $O(\sqrt{m})$ queries), then $h_i = j$; otherwise, $h_i = \infty$.

Correctness. It is not straightforward to see that the leftmost occurrence in the i -th block is found in the case $j_l \equiv j_r \pmod{d}$ if there does exist an occurrence in that block. By Lemma D.2, if there exists an occurrence starting at index $j_l \leq j \leq j_r$ in the i -th block, then $j_l \equiv j \equiv j_r \pmod{d}$. Let the leftmost and the rightmost occurrences of p in the i -th block be j'_l and j'_r , respectively. Then $j_l \leq j'_l \leq j'_r \leq j_r$ and $j_l \equiv j'_l \equiv j'_r \equiv j_r \pmod{d}$. By the minimality of q with $j_l \leq q \leq R_i$ and $t[q \dots R_i] = p[q - j_l \dots q - j_l + R_i - q]$, we have $q \leq j'_l$, and therefore the candidate determined by q is

$$j_q = j_l + \left\lceil \frac{q - j_l}{d} \right\rceil d \leq j'_l.$$

On the other hand, if $j_q \neq j'_l$, the existence of j'_l leads immediately to that $t[j_q \dots j_q + m - 1]$ matches p , i.e. $j_q < j'_l$ is also an occurrence, which contradicts with the minimality of j'_l . As a result, we have $j_q = j'_l$. That is, our algorithm finds the leftmost occurrence in the block, if exists.

Complexity. According to the above discussion, it is clear that h_i can be computed with bounded-error in $O(\sqrt{m \log m})$ queries. Thus, the entire problem can be solved by searching on bounded-error oracles (by Theorem 3.1) and the query complexity is

$$O\left(\sqrt{n/L} \sqrt{m \log m}\right) = O\left(\sqrt{n \log m}\right).$$

Combining the above two cases, we conclude that there is a bounded-error quantum algorithm for pattern matching in $O\left(\sqrt{n \log m} + \sqrt{m \log^3 m \log \log m}\right)$ queries.

E Proof of Exclusion Rule for LMSR

In this Appendix, we present a proof of Lemma 5.1. To this end, we first observe:

Proposition E.1. *Suppose $s \in \Sigma^n$ and $s[LMSR(s) \dots LMSR(s) + B - 1] = aba$, where $|a| \geq 1$, $|b| \geq 0$, and $B = 2|a| + |b| \leq n/2$. For every $m > 0$ and $i \in [n]$, if $s[i \dots i + |b| + |a| - 1] = ba$, then*

$$s[i \dots i + m - 1] \leq s[i + |b| + |a| \dots i + |b| + |a| + m - 1].$$

Proof. We prove it by induction on m .

Basis. For every $i \in [n]$ with $s[i \dots i + |b| + |a| - 1] = ba$, we note that $s[i + |b| \dots i + |b| + B - 1] = as[i + |b| + |a| \dots i + |b| - B - 1]$. On the other hand, by the definition of LMSR, we have

$$s[i + |b| \dots i + |b| + B - 1] \geq s[LMSR(s) \dots LMSR(s) + B - 1] = aba.$$

Therefore, it holds that $s[i + |b| + |a| \dots i + |b| - B - 1] \geq ba = s[i \dots i + |b| + |a| - 1]$. Immediately, we see that the proposition holds for $1 \leq m \leq |b| + |a|$.

Induction. Assume that the proposition holds for $m' = k(|b| + |a|)$ and $k \geq 1$, and we are going to prove it for the case $m' < m \leq (k + 1)(|b| + |a|)$. According to the induction hypothesis, we have

$$s[i \dots i + m' - 1] \leq s[i + |b| + |a| \dots i + |b| + |a| + m' - 1]$$

for every $0 \leq i < n$. Let us consider the following two cases:

1. $s[i \dots i + m' - 1] < s[i + |b| + |a| \dots i + |b| + |a| + m' - 1]$. In this case, it is trivial that $s[i \dots i + m - 1] < s[i + |b| + |a| \dots i + |b| + |a| + m - 1]$ for every $m > m'$.
2. $s[i \dots i + m' - 1] = s[i + |b| + |a| \dots i + |b| + |a| + m' - 1]$. In this case, we have $s[i \dots i + (k + 1)(|b| + |a|) - 1] = (ba)^{k+1}$, and

$$s[i + |b| + |a| \dots i + (k + 2)(|b| + |a|) - 1] = (ba)^k s[i + (k + 1)(|b| + |a|) \dots i + (k + 2)(|b| + |a|) - 1].$$

According to the induction hypothesis for $i' = i + m' = i + k(|b| + |a|)$ (this can be derived from the index $i'' = i' \bmod n \in [n]$), we have:

$$s[i + k(|b| + |a|) \dots i + (k + 1)(|b| + |a|) - 1] \leq s[i + (k + 1)(|b| + |a|) \dots i + (k + 2)(|b| + |a|) - 1].$$

Therefore, we obtain $s[i \dots i + (k + 1)(|b| + |a|) - 1] \leq s[i + |b| + |a| \dots i + (k + 2)(|b| + |a|) - 1]$, which means the proposition holds for $m = m' + |b| + |a|$. Immediately, we see that the proposition also holds for $m' < m \leq (k + 1)(|b| + |a|)$.

□

Now we are ready to prove Lemma 5.1. Let $\delta = j - i$, then we have $1 \leq \delta \leq B - 1$. We consider the following two cases:

- **Case 1.** $\delta > B/2$. In this case, $s[LMSR(s) \dots LMSR(s) + B - 1] = aba$ for some strings a and b , where $|a| = B - \delta$ and $|b| = 2\delta - B$. In order to prove that $LMSR(s) \neq j$, it is sufficient to show that $s[i \dots i + n - 1] \leq s[j \dots j + n - 1]$. Note that

$$\begin{aligned} s[i \dots i + n - 1] &= ababas[j + B \dots j + n - \delta - 1], \\ s[j \dots j + n - 1] &= abas[j + B \dots j + n - 1]. \end{aligned}$$

We only need to show that $bas[j + B \dots j + n - \delta - 1] \leq s[j + B \dots j + n - 1]$, that is, $s[i + B \dots i + n - 1] \leq s[i + B + \delta \dots i + \delta + n - 1]$, which can be immediately obtained from Proposition E.1 by letting $m \equiv n - B$ and $i \equiv i + B$.

- **Case 2.** $\delta \leq B/2$. Let $t = s[\text{LMSR}(s) \dots \text{LMSR}(s) + B - 1]$. Note that $t[i + \delta] = t[i]$ for every $i \in [B]$. We immediately see that t has period $d = \gcd(B, \delta)$, that is, $t = a^k$, where $|a| = d$ and $B = kd$ with $k \geq 2$. For convenience, we denote $\delta = ld$ for some $1 \leq l \leq k - 1$. In order to prove that $\text{LMSR}(s) \neq j$, it is sufficient to show that $s[i \dots i + n - 1] \leq s[j \dots j + n - 1]$. Note that

$$\begin{aligned} s[i \dots i + n - 1] &= a^{k+m} s[j + B \dots j + n - \delta - 1], \\ s[j \dots j + n - 1] &= a^k s[j + B \dots j + n - 1]. \end{aligned}$$

We only need to show that $a^m s[j + B \dots j + n - \delta - 1] \leq s[j + B \dots j + n - 1]$, i.e. $s[i + B \dots i + n - 1] \leq s[i + B + \delta \dots i + \delta + n - 1]$, which can be immediately obtained from Proposition E.1 by noting that $s[\text{LMSR}(s) \dots \text{LMSR}(s) + B - 1] = a^l a^{k-2l} a^l$.

F Worst-case Quantum Lower Bound

The worst-case quantum lower bound can be examined in a way different from that in Section 6.2. Let us consider a special case where all strings are binary, that is, the alphabet is $\Sigma = \{0, 1\}$. An solution to the LMSR problem implies the existence of a 0 character. That is, $s[i] = 0$ for some $i \in [n]$ if and only if $s[\text{LMSR}(s)] = 0$. Therefore, the searching problem can be reduced to LMSR. It is known that the searching problem has a worst-case query complexity lower bound $\Omega(\sqrt{n})$ for bounded-error quantum algorithms [BBBV97, BBHT98, Zal99], and $\Omega(n)$ for exact and zero-error quantum algorithms [BBC⁺01]. Consequently, we assert that the LMSR problem also has a worst-case quantum query complexity lower bound $\Omega(\sqrt{n})$ for bounded-error quantum algorithms, and $\Omega(n)$ for exact and zero-error quantum algorithms.