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Cooperation of Multi-agent Systems

by

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in Partial Fulfillment of the
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Certificate of Authorship/Originality

I declare that this thesis is submitted in satisfaction of the requirements for the degree of Doctor of Philosophy, in the Faculty of Engineering and Information Technology at the University of Technology Sydney.

I certify that this thesis has been written by me. In addition, I also certify that all information sources and literature used are quoted in the thesis. This document has not been submitted for qualifications at any other academic institution.

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ABSTRACT

Cooperation of Multi-agent Systems

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The cooperation of multi-agent systems represents that a group of agents complete the common tasks that are difficult or impossible for an individual agent or a single system to finish.

The cooperation of multi-agent systems has received considerable attention and has been widely studied over the past few years. The consensus is the basis of the cooperation of multi-agent systems.

Researching the cooperation of multi-agent systems probably involves a large number of theories, such as graph theories and stability theories of switched systems.

In this thesis, we study consensus, tracking control, and containment control under a fixed graph. For a time-invariant network of multi-agent systems, its topological characteristic can be described by a fixed graph.

For the time-variant network, we model the multi-agent systems by switched systems in this thesis and get some new conclusions. We transform the consensus problems into stability problems first and then get some new conclusions by the aid of conventional methods.

For traditional approaches, there are many limitations. Many of the involved theories also need to be further improved. For example, the stability problems of switched systems have not yet been thoroughly resolved. Therefore, we also do much research work on stability analysis of switched systems.

For switched systems, we propose some novel theories and methods to reduce the limitations of conventional stability analysis methods. We propose some sequence-

based methods to resolve the stability problems and get some new results. It is proved that the switched systems are globally uniformly asymptotically stable when the sequence-based mode-dependence average dwell time satisfies the conditions deduced by this thesis.

We use the proposed stability theories and methods to analyse the consensus of multi-agent systems under switched systems and get some new results. A proposed model transformation can transform consensus problems into stability problems. We get some novel results.

In this thesis, we first introduce the background and then give a brief literature review on the stability analysis of switched systems and the cooperation of multi-agent systems. After that, Chapter 3 addresses multi-agent systems under a fixed topology. The research on the stability of switched systems is presented in Chapter 4. A consensus of second-order multi-agent systems under switched topologies based on the sequence is studied in Chapter 5.

The research plan, progress, and our publications on this research topic are also shown in this report. Finally, we summarize my past work and give a conclusion.

Dedication

To my parents

To my wife

To my child

Thank you for your love and support.

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List of Publications

Journal Papers

J-1. **D. Zheng**, H. Zhang and J.A. Zhang, W. Zheng, S.W. Su. "Stability of asynchronous switched systems with sequence-based average dwell time approaches," *Journal of the Franklin Institute*, 2020, 357(4): 2149-2166.

J-2. H. Gao, H. Zhang, **D. Zheng**, L. Zhang, Y. Li. " Finite-time event-triggered extended dissipative control for discrete time switched linear systems," *International Journal of General Systems*, 2019, 48(5):476–491.

J-3. **D. Zheng**, H. Zhang, J.A. Zhang, Y. Li. " Consensus of the second-order multi-agent systems under asynchronous switching with a controller fault," *International Journal of Control, Automation and Systems*, 2019, 17(1):136–144.

J-4. **D. Zheng**, H. Zhang, J.A. Zhang, G. Wang. " Consensus of multi-agent systems with faults and mismatches under switched topologies using a delta operator method," *Neurocomputing*, 2018, 315:198–209.

J-5. Y. Li, H. Zhang, **D. Zheng**, Y. Wang. " Asynchronous H_∞ control of discrete-time switched TS fuzzy systems with dwell time," *International Journal of Fuzzy Systems*, 2018, 20(4):1098–1114.

J-6. **D. Zheng**, H. Zhang, Q. Zheng. " Consensus analysis of multi-agent systems under switching topologies by a topology-dependent average dwell time approach," *IET Control Theory and Applications*, 2017, 11(3):429–438.

J-7. Q. Zheng, H. Zhang, **D. Zheng**. " Stability and asynchronous stabilization for a class of discrete-time switched nonlinear systems with stable and unstable subsystems," *International Journal of Control, Automation and Systems*, 2017, 15(3):986–994.

Conference Papers

C-1. **D. Zheng**, H. Zhang. " Research on the transformation of control protocols among three kinds of cooperative control for multi-agent systems," *International Conference on Intelligent Human-Machine Systems and Cybernetics*, chengdu, 2016, 301–304

Abbreviation

ADT: Average Dwell Time

CQLF: common quadratic Lyapunov function

DT: Dwell Time

GUES: Globally Uniformly Exponential Stability

MDADT: Mode-dependent Average Dwell Time

MLF: multiple Lyapunov functions

SBASDT: sequence-based average subsequence dwell time

SBAPDT: sequence-based average preceding dwell time

UAVs: Unmanned Air Vehicles

Nomenclature and Notation

Capital letters denote matrices.

Lower-case alphabets denote column vectors.

$.^T$ denotes the transpose operation.

$\rightarrow$ denotes going to.

I_n is the identity matrix of dimension $n \times n$.

0_n is the zero matrix of dimension $n \times n$.

$0_{c \times d}$ is the zero matrix of dimension $c \times d$.

$\mathbb{R}$, $\mathbb{R}^+$ denote the field of real numbers, and the set of positive reals, respectively.

$\mathbb{Z}, \mathbb{Z}^+$ denote the field of integers, and the set of positive integers, respectively.

$\mathbb{N}$ denotes the field of natural numbers, $\mathbb{N}= \{0, 1, 2, 3, \cdots\}$.

t is time.

x is the state.

f stands for functions.

$\sigma(t)$ stands for a switching signal.

$\sigma(k)$ stands for a switching signal.

m is the number of subsystems or topologies.

$\mathcal{M}$ is the set of subsystems or topologies.

$\mathcal{S}$ is the set of stable subsystems or of cases that agents may reach a consensus.

$\mathcal{U}$ is the set of unstable subsystems or of cases that agents cannot reach a consensus.

C^1 is the space of continuously differentiable functions.

$\mathbb{R}^N$ denotes N-dimensional Euclidean space.

$t_1, t_2, t_3, \cdots, t_i, t_{i+1}, \cdots$ or k_1, k_2, k_3, $\cdots$ are the switching times.

t_i^- represents the time approaches time t_i from the left side.

$\otimes$ represents the Kronecker product of two matrices.

$T_\uparrow$ is the time set when the Lyapunov-like function increases in certain time section.

$T_\downarrow$ is the set of time which is not in $T_\uparrow$ in certain time section.

$\mathcal{I}$ is the set of agent indexes.

$\mathcal{V}$ is the set of nodes.

$\mathcal{E}$ is the set of edges.

$\mathcal{A}$ is the corresponding adjacency matrix.

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is a directed graph describing the topology of multi-agent systems.

$\mathcal{N}_i$ stands for the set of neighboring node v_i.

L stands for the Laplacian matrix.

H stands for a Laplacian-like matrix.

$\mathcal{R}(u)$ denotes the real part of a complex number u.

$\mathcal{I}(u)$ denotes the imaginary part of a complex number u.

$Dist(x, \mathcal{C})$ denotes the distance from $x \in \mathbb{R}^N$ to the set $C \subseteq \mathbb{R}^N$.

$\mathcal{F}$ denotes the set of the followers.

$\mathcal{R}$ denotes the set of the leaders.

$\times$ represents multiplication or Cartesian product of sets.

$> (or\ <)\ 0$ represents that a matrix is positive (or negative) definite and symmetric, or that the value of a scalar is more (or less) than 0.

$[p|q]$ denotes the situation when the p^{th} subsystem is aroused immediately after the q^{th} subsystem.

$T_{\downarrow}(t_l, t_{l+1})$ and $T_{\uparrow}(t_l, t_{l+1})$ denote the unions (or the length of the unions) of the decreasing and increasing time (including unchanging time) of the Lyapunov function within the interval $[t_l, t_{l+1})$.

$\Delta_{\sigma(t_l)}(t_l, t_{l+1})$ represents the asynchronous time between the time slots $[t_l, t_{l+1})$.

$\nabla_{\sigma(t_l)}(t_l, t_{l+1})$ represents the synchronous time between the time slots $[t_l, t_{l+1})$.

$T_p(0, t)$ denotes the running time of the p^{th} topology between the time slots $[0, t)$.

$\sigma'(t)$ denotes the feedback controller switching signal for asynchronous switching.

Δ denotes the time lag between $\sigma(t)$ and $\sigma'(t)$ for asynchronous switching.

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Chapter 1

Introduction

In this chapter, we introduce the background, the main scope, contribution, and organization of this thesis.

1.1 Background

In the past decades, the development of society has meant that more and more difficult problems are proposed in the academic and engineering fields. In order to make more powerful technical tools, people are trying to resolve more challenging problems. Persons are studying more flexible means to solve these difficult and challenging problems. Many methods are based on the cooperation of multi-agent systems.

It was reported that multi-agent systems are useful for building large computer systems. As hardware and software become increasingly powerful, the computer systems are becoming correspondingly larger and more complex. However, the problems encountered in building such systems are not simply scaled-up versions of small computer programs. This requires consideration of multi-agent systems. Individual problem-solving entities are called agents and are grouped together to form communities. It is assumed that each agent is capable of a range of useful problem-solving activities in its own right, has its own goals and objectives, and can communicate with others. [1]

Some articles have also discussed the application of multi-agent systems in electric power network systems. Due to an increase of distributed renewable resources

and a decrease of traditional centralized generators, the electrical network is facing challenging problems within distribution networks that are becoming ever more difficult for distribution network operators to solve under present operational practices. Multi-agent systems are used to resolve the coordination problems of distributed energy resources.

The formation is also one of the application areas of multi-agent systems. In Fig. 1.1, many random circle points can form some characters or pictures. In the past years, the condition of the popular methods is that every circle point obtains the information about its position in global coordinates in the expected formation [2]. Alternatively, the methods based on multi-agent systems do not use the precondition any more. For the approaches based on multi-agent systems, every circle point is modelled as an agent, and the final characters or pictures are seen as the cooperative aim of the multi-agent systems. Such a group of agents is a multi-agent system.

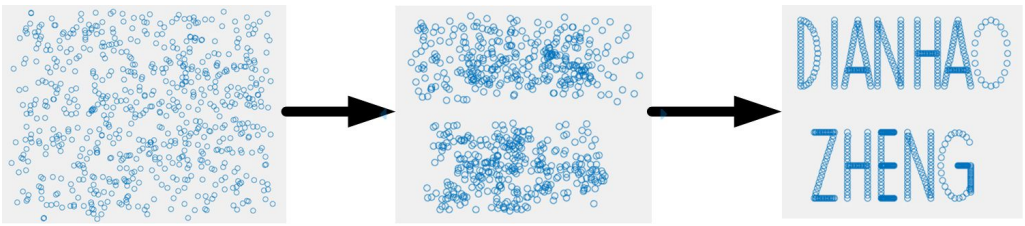


Figure 1.1 : A formation example

The conceptions of an agent are many and varied. Diverse definitions from different disciplines have been proposed over the past years [3]. In [4, 5], an agent is a software entity that functions continuously and autonomously in a particular environment, often inhabited by other agents and processes [3]. In [6], an agent is defined as anything that can be viewed as perceiving its environment through sensors and acting upon that environment through effectors. In [7], an agent refers to a computational system that inhabits a complex dynamic environment and can sense and act on its environment to achieve a set of goals or motivations that it tries

to achieve through these actions.

In this thesis, an agent is considered as anything that can change only according to the information of its neighbours and its own, and whose dynamic can be modelled by difference or differential equations. A group of agents comprise the multi-agent systems. The cooperation of multi-agent systems means that a group of agents complete the common tasks that are difficult or impossible for an individual agent or a single system to finish.

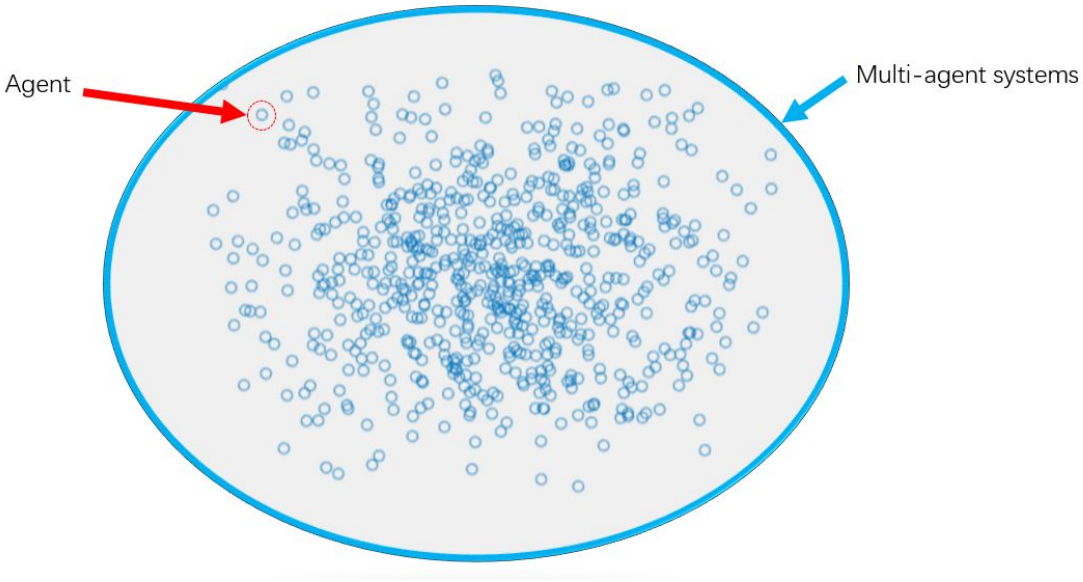


Figure 1.2 : Agent and multi-agent systems

In Figs. 1.1 and 1.2, every circle point can be called an agent. Every agent has its own dynamic system. The whole agent system can be called multi-agent systems.

Although the concept of multi-agent systems has been proposed for a few years and the methods based on them have been used in a large number of fields, there are many problems with multi-agent systems that have not yet been resolved. For example, the multi-agent systems can not form the regular characters shown in Fig. 1.3 because of improper coupling coefficients and lack of a spanning tree. In this thesis, we will discuss some of these problems.

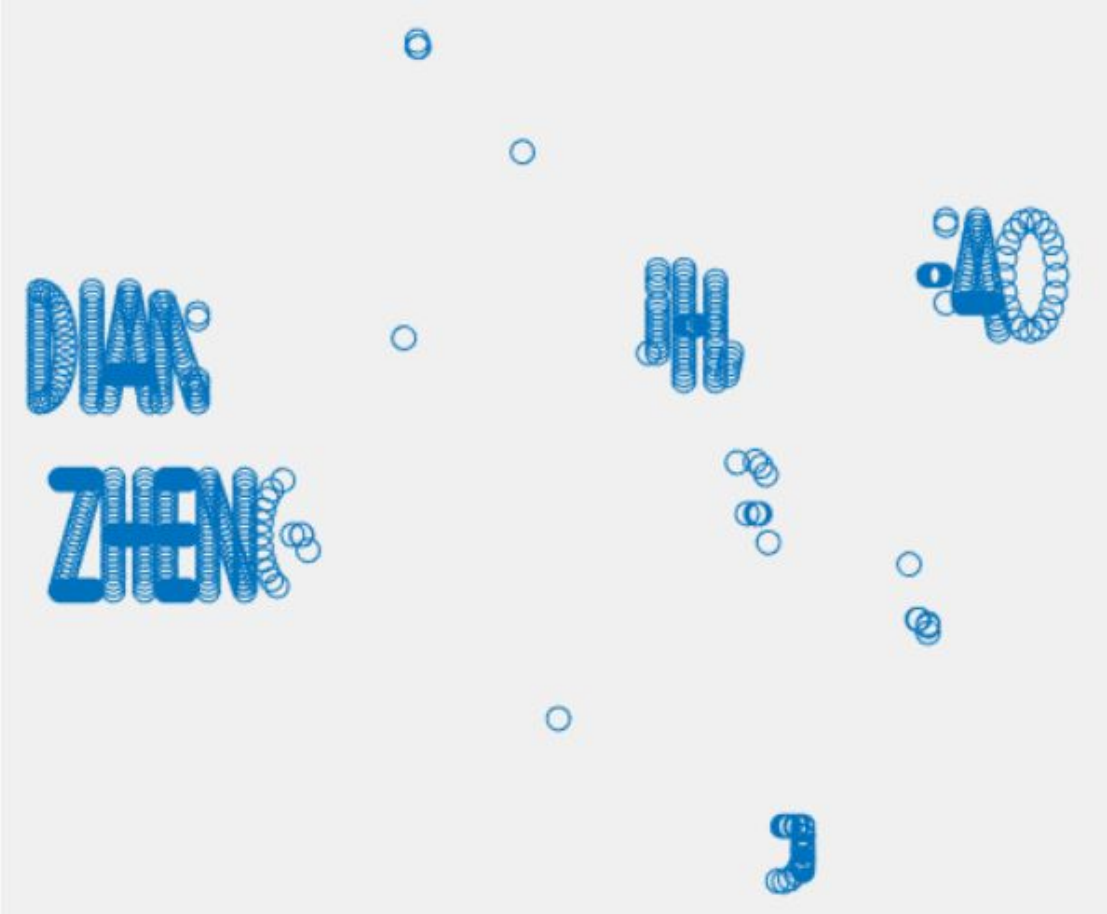


Figure 1.3 : Fail formation for multi-agent systems

1.2 Research Motivation

In this section, the research questions, objectives, scope, plus the contribution to knowledge are introduced. Ethics and risks are also considered.

Although the multi-agent systems are studied in a large number of fields, the problems in this thesis are limited to the control field. According to the topologies of networks, multi-agent systems can be divided into two categories: multi-agent systems under fixed topologies and those under switched topologies. The cooperation goals of multi-agent systems are many and varied, such as common consensus, tracking control, and contained control. For the multi-agent systems under fixed topologies, we study the consensus, tracking control and contained control prob-

lems. For the multi-agent systems under switched topologies, we take the second-order consensus problems as an example to analyze the cooperation of multi-agent systems.

In this thesis, objectives are to propose some methods to judge whether or not the multi-agent systems can reach a consensus, especially under switched topologies when (a) agents can reach a consensus under every separate topology; (b) agents can reach a consensus under part topologies; (c) agents cannot reach a consensus under any topology.

For these problems, there are several challenges that need to be tackled, such as: how to solve consensus problems for the agents that can not reach a consensus under any separate topology; how to solve the multi-agent problems when the systems have faults; what kinds of methods we can adopt. In the past few years, most existing research results are based on the topologies with a spanning tree; in this thesis, we hope to study some cases when the topologies have no spanning trees in this thesis.

The research methodology for the problems in this thesis is clear. For the multi-agent systems with a fixed topology, we try to solve cooperation problems using stability and matrix theories by transforming cooperation problems into stability problems. For the multi-agent systems with switched topologies, we first propose some novel methods to resolve the stability problems of switched systems and then apply the results to multi-agent systems by means of mode transformation. How to combine multi-agent systems and switched systems is another challenge for us.

The motivation to study the cooperation of multi-agent systems can be summarized as (1) enriching our knowledge about multi-agent systems; (2) enriching our knowledge about switched systems; (3) providing application cases for theories of switched systems.

In our research, we do not do any experiments on animals or humans. In our

experiments, the principal tool is computational software. All the copyrights of software have been bought by UTS. Therefore, no ethics problem or risk is likely to be involved.

1.3 Contributions of This Thesis

In this thesis, we complete the following tasks:

a): Judge whether the cooperation goals of multi-agent systems can be reached or not under a fixed topology;

b): Judge whether the cooperation goals of multi-agent systems can be reached or not under switched topologies, especially when

(b.1) agents can reach a consensus under every separate topology;

(b.2) agents can not reach a consensus under part topologies, because of

(b.2.1) improper coupling coefficients;

(b.2.2) lack of spanning trees;

(b.2.3) faults;

(b.3) agents cannot reach a consensus under any topology.

In order to reach the above goals, we propose the following model transactions:

a): translate the cooperation problems of multi-agent under a fixed topology into the stability problems of a dynamic system;

b): translate the consensus problems of multi-agent under switched topologies into the stability problems of switched systems.

For the existing methods of stability analysis of switched systems, there is conser-

vativeness to calculate some threshold values. That is to say, the existing results of switched systems need to be improved. Motivated by this, we propose the following new methods:

a): Sequence-based average dwell time methods for stability analysis of switched systems;

b): Sequence-based topology-dependence approaches for consensus analysis of multi-agent systems under switched topologies.

The novelty and main contribution of this thesis are summarized as follows:

a): Propose new methods to judge whether the cooperation goals of multi-agent systems can be reached or not under a fixed topology;

b): Propose new approaches to judge whether or not the consensus of multi-agent systems can be reached or not under switched topologies;

c): Design novel model transactions from cooperation problems of multi-agent systems under a fixed topology to the stability problems of a dynamic system;

d): Design novel approaches from consensus problems of multi-agent under switched topologies to stability problems of switched systems;

e): Exploit sequence-based average dwell time methods for stability analysis of switched systems;

f): Exploit sequence-based topology-dependence approaches for consensus analysis of multi-agent systems under switched topologies.

1.4 Thesis Organization

The flowchart of the thesis outline is shown in Fig. 1.4. This thesis is organized as follows:

- *Chapter 2*: A survey of switched systems and cooperation of multi-agent systems is presented in Chapter 2. Some basic and important concepts and dilemmas are addressed in the chapter. Some definitions are also given. They are used in the rest of this thesis.

- *Chapter 3*: Three kinds of cooperation problems of multi-agent systems under a fixed topology are analyzed in Chapter 3. They are consensus with no leader, consensus with one leader, and Containment Control. By transforming them into stability problems, we obtain the sufficient and necessary condition of achieving three cooperation goals.

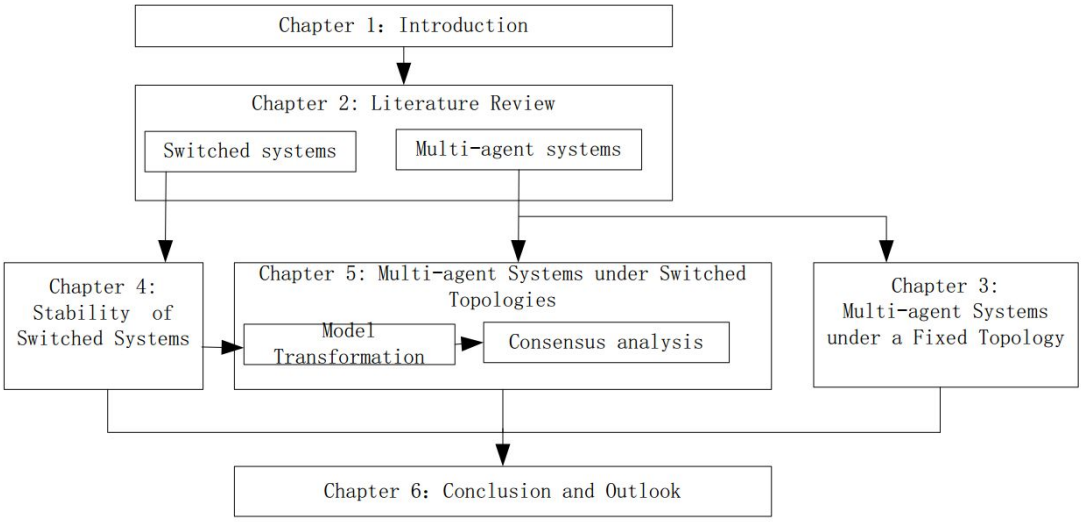


Figure 1.4 : Flowchart of the thesis outline

- *Chapter 4*: This chapter presents our proposed methods for the stability of switched systems under three situations. The novel methods are based on the switching sequence and have many advantages compared with the traditional methods. On the basis of the proposed methods, we obtain some new results which can apply to the scenarios in Chapter 5.

- *Chapter 5*: Consensus of second-order multi-agent systems under switched topologies are studied in Chapter 5 by proposed sequence-based methods. Not only the case that all agents can reach a consensus under the receptive fixed topology studied, but also the cases that all agents cannot reach a consensus under any receptive topology. Some special cases, such as asynchronous switching, and having faults, are also studied in this chapter.

- *Chapter 6*: Chapter 6 is the final chapter. A brief summary of the thesis contents and its contributions are given in Chapter 6. Recommendation for future works is given as well.

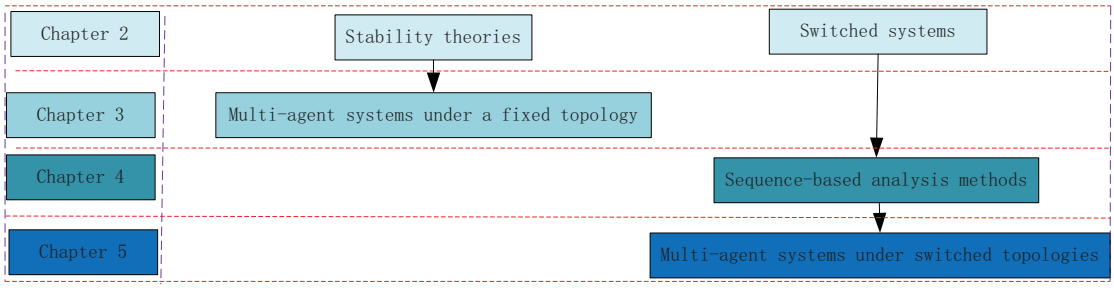


Figure 1.5 : Road-map and content structure

We show the technology road-map and content structure by Fig. 1.5.

Chapter 2

Literature Survey and Preliminaries

In this chapter, we present a survey of multi-agent systems and switched systems. The background of multi-agent systems, switched systems, and their combination is introduced. We also address some basic and important concepts and results.

2.1 Introduction

The cooperation of multi-agent systems has received much attention in the past decades. Two main reasons are the rapid developments of computer science and technology, and its wide potential application in a large number of areas, for example, cooperative control of unmanned air vehicles (UAVs) [8].

Some fundamental tasks of cooperation of multi-agent systems have been studied, such as consensus [9, 10, 11, 12], tracking control [13], containment control [14, 15].

Resolving the problem with multi-agent systems needs many theories, methods, and tools. Stability theories, matrix theories, and graph theories are often used in resolving problems around multi-agent systems [9].

Methods and results based on *Lyapunov* stability theories can be used to analyze the problems of multi-agent systems. They are useful whether or not the network of multi-agent systems changes. A Lyapunov-based method is introduced in [16] for convergence analysis of consensus problems.

Stability theories of switched systems can be used to resolve the cooperation problems of multi-agent systems under switched topologies. In [17, 18, 19], it was proved that some cooperation problems can be transformed into stability problems

when the systems are under switched topologies.

The rest of this chapter is organized as follows. In Section 2, we present the survey of stability analysis of switched systems. In this thesis, we use these stability theories to resolve the cooperation problems of multi-agent systems. In Section 3, a survey of the cooperation of multi-agent systems is shown. We introduce the basic and important concepts, methods, theories, and results. In the final section, a brief summary is given.

2.2 Stability Analysis of Switched Systems

2.2.1 Basic Concepts

In this section, we discuss the stability of switched systems. Consider a general non-autonomous system described by differential equations

$$\frac{dx}{dt} = f(t, y), \tag{2.1}$$

which can be rewritten in a compact matrix formation as

$$\dot{x}(t) = f(x(t)). \tag{2.2}$$

If the system (2.2) is linear, it can be rewritten as

$$\dot{x}(t) = Ax, \tag{2.3}$$

where A is often called as a system matrix.

First of all, we provide some definitions about stability.

Definition 1: [20, 21] If a function $\beta : [0, \infty) \to [0, \infty)$, $\beta(0) = 0$, is strictly increasing, continuous, unbounded, then it is said to be of class K_∞.

Definition 2: [20, 21]. For $\forall \varepsilon > 0$, $\forall t_0 \geq 0$, $\exists \delta(\varepsilon, t_0)$, $\forall x_0$, when $\|x_0\| < \delta(\varepsilon, t_0)$, if the condition $\|x(t, t_0, x_0)\| < \varepsilon$ holds for all $t > t_0$, then the trivial solution of the

formulation (2.1) is ***stable***. In the contrast, $\exists\varepsilon$, $\exists t_0 \geq 0$, for $\forall\delta > 0$, $\forall x_0$ or $\exists x_0$, $\exists t > t_0$ satisfying $\|x(t, t_0, x_0)\| > \varepsilon_0$,though $\|x_0\| < \delta$, then the trivial solution of the formulation (2.1) is ***unstable***.

Definition 3: [20,21]. For $\forall\varepsilon > 0$, $\exists\delta(\varepsilon)$, $\forall x_0$, when $\|x_0\| < \delta(\varepsilon)$, if the condition $\|x(t, t_0, x_0)\| < \varepsilon$ holds for all $t \geq t_0 \geq \tau$, then the trivial solution of the formulation (2.1) is ***uniformly stable***.

The word "uniformly" in Definition 3 means δ does not depend on t_0.

Definition 4: [20,21]. For $\forall t_0 > \tau$, $\forall\varepsilon > 0$, $\exists T(\varepsilon, t_0, x_0) > 0$, when $\|x_0\| < \delta(t_0)$, if the condition $\|x(t, t_0, x_0)\| < \varepsilon$ holds for all $t \geq t_0 + T(\varepsilon, t_0, x_0)$, that is to say, $x(t, t_0, x_0) \underset{t\to+\infty}{\to} 0$,then the trivial solution of the formulation (2.1) is ***attractive***. If $\delta(t_0)$ can take any value, the trivial solution of the formulation (2.1) is ***general attractive***.

Definition 5: [20, 21]. If the trivial solution of the formulation (2.1) is stable, and attractive, then it is ***asymptotically stable***.

Definition 6: [20, 21]. If the trivial solution of the formulation (2.1) is stable, uniformly stable and general attractive, then it is ***globally uniformly asymptotically stable***.

Definition 7: [20, 21]. For the formulation (2.1), if $\forall\varepsilon > 0$, $\exists\delta(\varepsilon)$, $\exists\lambda > 0$, $\forall t_0 \geq 0$, when $\|x_0\| < \delta$, the condition

$$\|x(t, t_0, x_0)\| \leq \varepsilon e^{-\lambda(t-t_0)},\ (t \geq t_0,) \tag{2.4}$$

always holds. In this case, the trivial solution of the formulation (2.1) is ***globally uniformly exponential stability***(GUES).

There are a large number of methods to check the stability of systems (2.2) and (2.3). Many of them are based on the *Lyapunov* stability theories.

Lemma 1: [22]. Let $V(x,t)$ be a non-negative function with derivative V' along the trajectories of the system (2.1). If $V(x,t)$ is positive definite and decrescent, and $V(x,t)$ is positive definite, then the origin of the system is globally uniformly asymptotically stable.

For the linear system (2.3), if we construct a Lyapunov function $V(x,t) = x^T P x$ where x is a column vector from the system (2.3), it can be proved that:

Lemma 2: The system (2.3) is globally uniformly asymptotically stable, if there exists a symmetrical positive matrix P satisfying

$$A^T P + PA < 0, \tag{2.5}$$

according to Lemma 1.

It has been proved that the condition in Lemma 2 is a sufficient and necessary condition according to [23]. In the past decades, many results have been obtained based on the Lyapunov theories. In the meantime, it has been proved the Lyapunov method is very useful.

In recent years, switched systems have attracted considerable attention and become active research topics for their wide range of applications, such as cooperation of multi-agent systems [17, 18, 19], modelling of electronic systems.

Switched systems consist of a finite number of subsystems and a logical rule that orchestrates switching between them [24].

For a switched system including m subsystems, it can be described by

$$\dot{x}(t) = f_{\sigma(t)}(x(t)). \tag{2.6}$$

The switching signal $\sigma(t)$ take its value from the set $\mathcal{M}$, $\mathcal{M} \triangleq \{1, 2, \cdots, m\}$.

If the system (2.6) is piecewise linear, it can be rewritten as

$$\dot{x}(t) = A_{\sigma(t)} x. \tag{2.7}$$

In order to analyze the stability of switched systems, we give the definition of globally uniformly exponential stability for switched systems.

Definition 8: [25] For a given system (2.6) and a switching signal $\sigma(t)$, an equilibrium $x^* = 0$ is ***globally uniformly exponential stable*** if there exist constants $\eta > 0$, $\lambda > 0$ for any initial conditions $x(t_0)$, such that the solution of the system satisfies

$$\| x(t) \| \leq \eta \| x(t_0) \| e^{-\lambda(t-t_0)}, t \geq t_0. \tag{2.8}$$

Compare with Definitions 7 and 8; we know that they give the same description of globally uniformly exponential stability.

In 1996, the authors of the paper [26] presented a definition on dwell time (DT) to resolve the problem of supervisory control of families of linear controllers. The dwell time was set as a constant.

In 1999, a definition on average dwell time (ADT) was proposed in [27]. This scheme sets that the *average* interval between consecutive discontinuities be not less than a constant. This method is considered as a more general and flexible one.

The concept "average dwell time" was given as:

Definition 9: [27] For any switching times $t_2 \geq t_1 \geq 0$ with a switching signal $\sigma(t)$, let $N_\sigma(t_2, t_1)$ stand for the number of discontinuities of switching signal $\sigma(t)$ over the time interval $[t_1, t_2]$. It is said that $\sigma(t)$ has a average dwell time τ_a if there exist two numbers N_0 (N_0 is called the chatter bound here), and τ_a such that

$$N_\sigma(t_2, t_1) \leq N_{0p} + \frac{t_2 - t_1}{\tau_a}. \tag{2.9}$$

In 2011, the authors of the paper [25] proposed a new method for the stability analysis of switched systems. It is named mode-dependent average dwell time (MDADT) method. It does not concern a minimum or maximum. It focuses on the average value. This method could reduce the restrictions on average dwell time.

The concept "mode-dependent average dwell time" was given as:

Definition 10: [25] For any switching times $t_2 \geq t_1 \geq 0$ with a switching signal $\sigma(t)$, let $N_{\sigma p}(t_2, t_1)$ stand for the number of the p^{th} subsystem that is activated over the time interval $[t_1, t_2]$, and $T_p(t_2, t_1)$ denote the total running time of the p^{th} subsystem over the time interval $[t_1, t_2]$, $p \in \mathcal{M}$. It is said that $\sigma(t)$ has a mode-dependent average dwell time τ_{ap} if there exist two numbers N_{0p} (N_{0p} is called the mode-dependent chatter bound here) and τ_{ap} such that

$$N_{\sigma p}(t_2, t_1) \leq N_{0p} + \frac{T_p(t_2, t_1)}{\tau_{ap}}. \tag{2.10}$$

In the meantime, the stability analysis has been pursued in the framework of multiple Lyapunov functions (MLF) [24]. Each Lyapunov or Lyapunov-like function corresponds to one single subsystem or a certain region in the state space. All of them are concatenated to produce a non-traditional Lyapunov function which may not be a monotonically decreasing function along the state trajectories. The constructed function may have discontinuities and be piecewise differentiable. The reason why they considered constructing multiple Lyapunov functions is that traditional ones may not exist for switched systems with restricted switching signals. [24]

On the basis of DT, ADT, MDADT, and MLF, a number of useful theories have been developed [28, 29, 30, 31].

2.2.2 System Switching Among Stable Subsystems

In this case, all the subsystems are stable. If we use the set $\mathcal{S}$ to denote the set of the stable subsystems, the first case means $\mathcal{M}$=$\mathcal{S}$.

The existence of a common quadratic Lyapunov function (CQLF) for all subsystems assures the stability of the switched system [24]. That is to say,

Lemma 3: the system (2.7) is globally uniformly asymptotically stable, if there

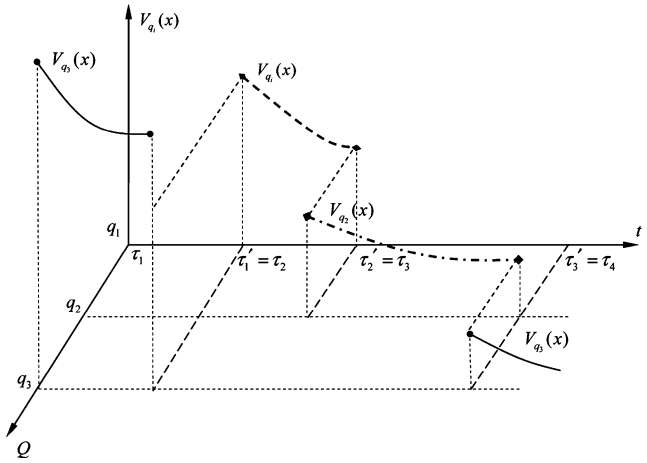


Figure 2.1 : The switched system is asymptotically stable if the values of MLF decrease at the switching instants [24]

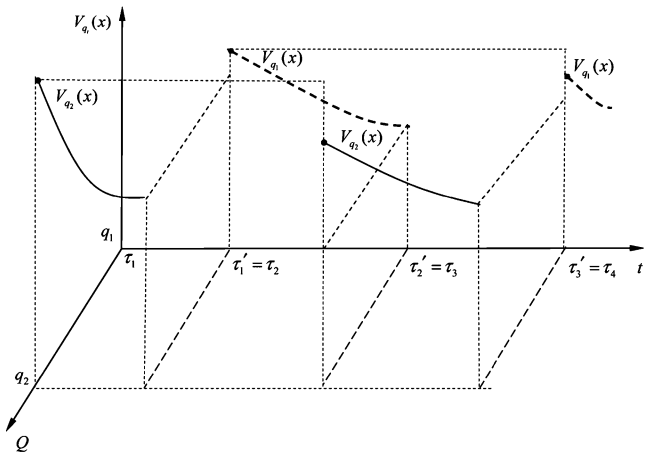


Figure 2.2 : The switched system can be asymptotically stable when the values of MLF increase at the switching instants [24]

exists a symmetrical positive matrix P satisfying

$$A_p^T P + P A_p < 0, \ \forall \ p \in \mathcal{M}. \tag{2.11}$$

As we discussed in this section, a common quadratic Lyapunov function may not exist for switched systems with restricted switching signals. For these systems, we can consider the multiple Lyapunov functions.

According to Lemma 1, there exists a monotonically decreasing Lyapunov function that corresponds to every single subsystem or a certain region in the state space. The constructed multiple Lyapunov functions or the Lyapunov-like function is shown in Figures 2.1 and 2.2.

Lemma 4: Consider the switched system (2.6), and give constants $0 < \mu_p < 1$, $p \in \mathcal{M}$. Suppose that there exist C^1 Lyapunov-like functions $V_{\sigma(t)}$, and class K_∞ functions k_{1p} and k_{2p}, such that $\forall p \in \mathcal{M}$,

$$k_{1p}(\|x(t)\|) < V_p(\|x(t)\|) < k_{2p}(\|x(t)\|), \tag{2.12}$$

$$\dot{V}_p(\|x(t)\|) < 0, \tag{2.13}$$

and $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in (\mathcal{M} \times \mathcal{M})$, $p \neq q$, one has

$$V_p(x(t_i)) < \mu_p V_q(x(t_i^-)). \tag{2.14}$$

Then the system (2.6) is globally uniformly asymptotically stable.

Lemma 5: [25] Consider the switched system (2.6), and give constants $\lambda_p > 0$, $\mu_p > 1$, $p \in \mathcal{M}$. Suppose that there exist C^1 Lyapunov-like functions $V_{\sigma(t)}$, and class K_∞ functions k_{1p} and k_{2p}, such that $\forall p \in \mathcal{M}$,

$$k_{1p}(\|x(t)\|) < V_p(\|x(t)\|) < k_{2p}(\|x(t)\|), \tag{2.15}$$

$$\dot{V}_p(\|x(t)\|) < -\lambda_p V_p(\|x(t)\|), \tag{2.16}$$

and $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in (\mathcal{M} \times \mathcal{M})$, $p \neq q$, one has

$$V_p(x(t_i)) < \mu_p V_q(x(t_i^-)). \tag{2.17}$$

Then the system (2.6) is globally uniformly asymptotically stable with MDADT

$$\tau_{ap} \geq \tau_{ap}^* = \frac{\ln \mu_p}{\lambda_p}. \tag{2.18}$$

Consider the switched linear systems given by (2.7) with input $u(t)$

$$\dot{x}(t) = A_{\sigma(t)}x + B_{\sigma(t)}u(t). \tag{2.19}$$

If we consider the control gain $K_{\sigma(t)}$, the system (2.19) can be rewritten as

$$\dot{x}(t) = A_{\sigma(t)}x + B_{\sigma(t)}K_{\sigma(t)}x(t), \tag{2.20}$$

because

$$u(t) = K_{\sigma(t)}x(t). \tag{2.21}$$

If there is time lag $\delta > 0$ between switched controllers and system modes, the control input will become

$$u(t) = K_{\sigma(t-\delta)}x(t). \tag{2.22}$$

This switching is called asynchronous switching in this situation [28]. The system (2.20) is changed into

$$\dot{x}(t) = A_{\sigma(t)}x + B_{\sigma(t)}K_{\sigma(t-\delta)}x(t). \tag{2.23}$$

Asynchronous switching probably results in the increase incidence of Lyapunov-like functions. We use $T_{\uparrow}(t_i, t_{i+1})$ to denote the time set when the Lyapunov-like function increases in a time section $[t_i, t_{i+1})$, $\forall i \in \mathbb{N}$. $T_{\downarrow}(t_i, t_{i+1})$ is the set of other time in $[t_i, t_{i+1})$.

For asynchronous switching, it is proved that:

Lemma 6: [28] Consider the continuous-time switched system (2.6), $\lambda > 0$, $\mu > 1$, $\alpha > 0$. Suppose that there exist C^1 Lyapunov functions $V_{\sigma(t)}$, and class K_∞ functions k_1 and k_2, such that $\forall p \in \mathcal{M}$,

$$k_1(\|x(t)\|) < V_p(\|x(t)\|) < k_2(\|x(t)\|), \tag{2.24}$$

$$\dot{V}_p(\|x(t)\|) < \begin{cases} -\lambda V_p(\|x(t)\|), & \forall t \in T_{\downarrow}(t_i, t_{i+1}), \forall i \in \mathbb{N}, \\ \alpha V_p(\|x(t)\|), & \forall t \in T_{\uparrow}(t_i, t_{i+1}), \forall i \in \mathbb{N}, \end{cases} \tag{2.25}$$

and $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in (\mathcal{M} \times \mathcal{M})$, $p \neq q$, one has

$$V_p(x(t_i)) < \mu V_q(x(t_i^-)). \tag{2.26}$$

Then the system (2.6) is globally uniformly asymptotically stable with ADT

$$\tau_a > \tau_a^* \triangleq \frac{\Delta_{\max}(\lambda + \alpha) + \ln \mu}{\lambda}, \tag{2.27}$$

where $\Delta_{\max} \triangleq max_l T_\uparrow([t_l, t_{l+1})$, for $\forall l \in \mathbb{N}$.

2.2.3 System Switching Among Stables and Unstable Subsystems

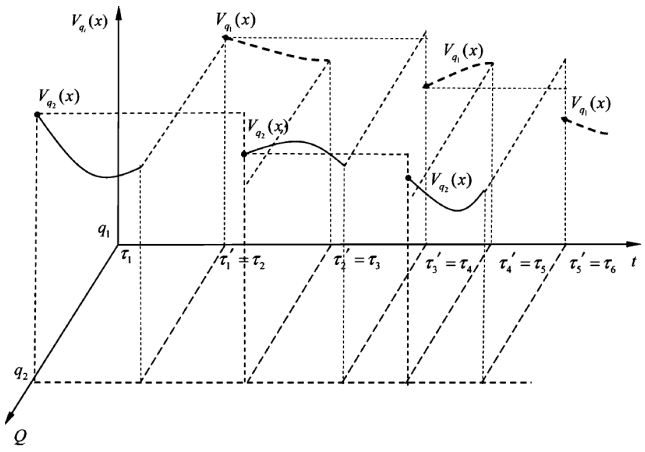


Figure 2.3 : The switched system can be asymptotically stable even when MLF increases during a certain period [24]

In this case, we use the sets $\mathcal{S}$, $\mathcal{U}$ to denote the sets of the stable, unstable subsystems respectively. It means $\mathcal{M}$=$\mathcal{S} \bigcup \mathcal{U}$.

When switching between stable and unstable subsystems, the switched system has some subsystems whose Lyapunov functions increase during a certain period, as shown in Fig. 2.3.

In this case, it has been proved that:

Lemma 7: [32] Consider switched system (2.6), and give constants α_p,$\mu_P > 1$, $p \in \mathcal{M}$. Suppose that there exist C^1 Lyapunov functions $V_{\sigma(t)}$, and class K_∞ functions k_{1p} and k_{2p}, such that $\forall p \in \mathcal{M}$,

$$k_{1p}(\|x(t)\|) < V_p(\|x(t)\|) < k_{2p}(\|x(t)\|), \tag{2.28}$$

$$\dot{V}_p(\|x(t)\|) < \alpha_p V_p(\|x(t)\|), \tag{2.29}$$

and $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in (\mathcal{M} \times \mathcal{M}), \mathrm{p} \neq \mathrm{q}$,

$$V_p(x(t_i)) < \mu_p V_q(x(t_i^-)), \tag{2.30}$$

then the system is globally uniformly asymptotically stable with marginal γ^* under any switching signal with the following conditions:

$$\begin{cases} \tau_{ap} \geq -\frac{\ln \mu_p}{\alpha_p}, & (\alpha_p < 0, p \in \mathcal{S}), \\ \tau_{ap} \geq \tau_{ap}^*, & (\alpha_p > 0, \forall \tau_{ap}^* > 0, p \in \mathcal{U}), \\ \frac{T^-}{T^+} > \frac{\gamma^+ + \gamma^*}{\gamma^- - \gamma^*}, & (0 < \gamma^* < \gamma^-), \end{cases} \tag{2.31}$$

where $T^- = \sum_{p=1}^{r} T_p(t,0)$, $T^+ = \sum_{p=r+1}^{M} T_p(t,0)$, $-\gamma^- = \max_{p\in\mathcal{S}}(\alpha_p + \frac{\ln \mu_p}{\tau_{ap}})$, $\gamma^+ = \max_{p\in\mathcal{U}}(\alpha_p + \frac{\ln \mu_p}{\tau_{ap}})$. $T_p(t,0)$ is defined in Definition 10.

2.2.4 System Switching Among Unstable Subsystems

In this case, all the subsystems are unstable. It means $\mathcal{M}$=$\mathcal{U}$.

When switching between unstable subsystems, all subsystems' Lyapunov functions increase during a certain period.

There are few conclusions about this case, one of them is

Lemma 8: [33] For switched system (2.6) with switching sequence $\{t_0, t_1, t_2, \cdots, t_i, \cdots\}$, and given constants α, $\mu < 1$, suppose that there exist C^1 Lyapunov functions $V_{\sigma(t)}$, and class K_∞ functions k_{1p} and k_{2p}, such that $\forall p \in \mathcal{M}$,

$$k_{1p}(\|x(t)\|) < V_p(\|x(t)\|) < k_{2p}(\|x(t)\|), \tag{2.32}$$

$$\dot{V}_p(\|x(t)\|) < \alpha V_p(\|x(t)\|), \tag{2.33}$$

and $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in (\mathcal{M} \times \mathcal{M}), \mathrm{p} \neq \mathrm{q}$,

$$V_p(x(t_i)) < \mu V_q(x(t_i^-)). \tag{2.34}$$

$$\ln \mu + \alpha\tau_n < 0, \ \forall n \in \mathbb{N}, \tag{2.35}$$

where $\tau_n = t_{n+1} - t_n$. Then the system is globally uniformly asymptotically stable with respect to switching law $\sigma(t)$.

Now we have finished the introduction with three kinds of switching for switched systems. Besides the lemmas discussed above, the following lemmas are also important to resolve the stability problems in this thesis.

Lemma 9: [34] Given matrices Q, R, E and H with proper dimensions, and $Q = Q^T$, $R = R^T > 0$, the inequality

$$Q + HFE + E^T F^T H^T < 0,$$

holds for all F satisfying $F^T F \leq R$, if and only if there exists some $\varepsilon > 0$, such that

$$Q + \varepsilon HH^T + \varepsilon^{-1} E^T RE < 0.$$

2.3 Cooperation of Multi-agent Systems

Many years of research in the field of agents and multi-agent systems has offered remarkable results from both theoretical and practical points of view. The cooperation of multi-agent systems has received considerable attention over the past few years due to the rapid developments of computer science and its wide application in many areas, such as industrial production [35], sensor networks [36], vehicle systems [37], and spacecraft flying [38].

However, after so many years of research, there is no general consensus on some basic notions. There is no strict definition of what an agent is. Literature offers a

variety of definitions ranging from the simple to the lengthy and demanding ones. All available definitions are strongly biased by the background field of those interested in agent technology.

In this thesis, we consider the agent whose dynamics can be described by the formulation (2.2). The multi-agent systems are a group of agents. The cooperation goals are given in different chapters by different definitions.

In the 1990s or 2000s, many persons researched on the flocks of mobile agents [39, 8], synchronization of coupled oscillators [40, 41], distributed formation control [42], distributed sensor fusion in sensor networks [43, 44], rendezvous in space [45], fast consensus in small-worlds [46].

Some preliminary results about the cooperation of multi-agent systems have been reported in the 2000s.

In [42], tools from algebraic graph theory proved useful in modelling the communication network and relating its topology to formation stability. Graph theories are often used to describe the network of multi-agent systems. For a multi-agent system with n agents, the set of agent indexes is $\mathcal{I} = \{1, 2, \ldots, n\}$. Let a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ describe the communication topology among the agents, where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and $\mathcal{A} = [a_{ij}]$. They are the set of nodes, a set of edges, and the corresponding adjacency matrix, respectively. We assume that there are no self loops in the networks. The set $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}, j \neq i, j \in \mathcal{I}, i \in \mathcal{I}\}$ stands for the set of neighboring node v_i. The matrix $L = [l_{ij}]$ is called as a *Laplacian matrix*, where

$$l_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{n} a_{ik} \ for \ i = j \ . \\ -a_{ij}, \ for \ i \neq j, \ and \ i, \ j \in \mathcal{I}. \end{cases} \tag{2.36}$$

A directed graph is called a directed tree when every node has exactly one parent except for the root. A directed tree is seen as a spanning tree of a directed graph

when the tree connects all the nodes of the graph. More details can be found in [47]. These concepts can be illustrated by Fig. 2.4.

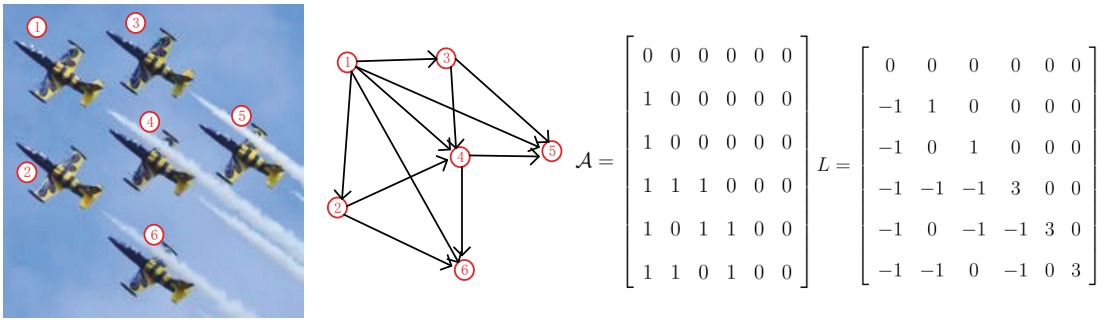


Figure 2.4 : Network, topology and matrix.

Some basic concepts and conclusions were introduced and given in [9]. The authors analyzed three topology cases and two protocols. In [9], consensus and related concepts are introduced. To make all agents converge to a common state was the major objective of consensus for multi-agent systems. The common state was called the group decision value [9]. The group decision value was not decided by centralized systems, but by each agent using the local information of its own and neighbours. The Fig. 2.5 presents the concept of consensus.

For the states x_1, x_2,x_3, $\cdots$, initial states of the system $a = x(0)$, the dynamics of each node or each dynamic agent,

$$\dot{x}_i = f(x_i, u_i), \tag{2.37}$$

and a distributed protocol

$$\dot{u}_i = k_i(x_{j1}, x_{j2}, \cdots), \tag{2.38}$$

where the set of $\{j1, j2, \cdots\}$ equals $\mathcal{N}_i$. Three kinds of consensus are defined in [9]. Average-consensus means the group decision value is the average value of initial states. For the dynamics

$$\dot{x}_i(t) = u_i(t), \tag{2.39}$$

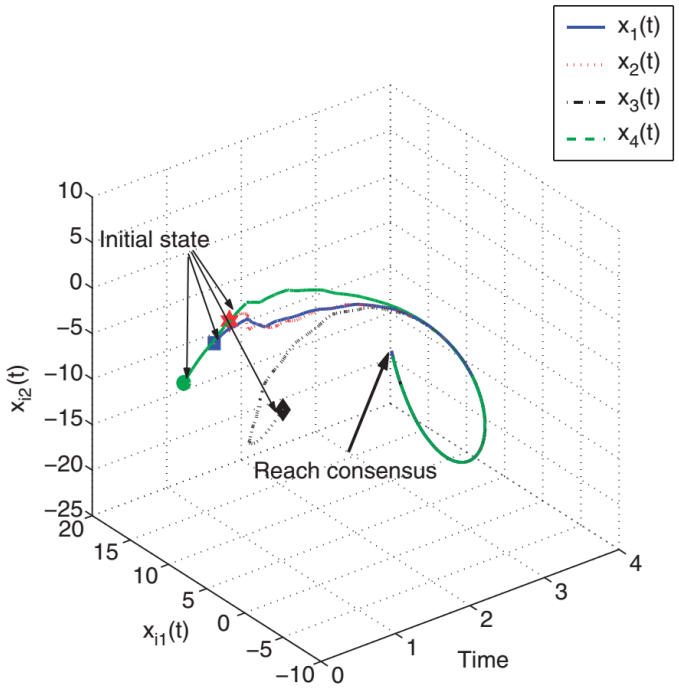


Figure 2.5 : State trajectories of four agents with the initial state x(0) [48]

and the distributed protocol

$$\dot{u}_i = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j - x_i), \tag{2.40}$$

the whole multi-agent systems can be rewritten in the matrix formation as

$$\dot{x} = -Lx, \tag{2.41}$$

where L is called graph Laplacian defined by (2.36). When the topologies changed, the protocol (2.40 and matrix formation changed into

$$\dot{u}_i = \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(x_j(t) - x_i(t)), \tag{2.42}$$

$$\dot{x} = -L_{\sigma(t)}x, \tag{2.43}$$

respectively.

Lemma 10: [9] Consider a network including n agents with dynamics (2.39) and the applied protocol (2.40). Assume it is a strongly connected digraph. Then, this system (2.41) globally asymptotically solves a consensus problem.

For a given directed graph (or digraph) with n agents. The in-degree and out-degree of node are defined as

$$deg_{in}(v_i) = \sum_{j=1}^{n} a_{ji},\ deg_{out}(v_i) = \sum_{j=1}^{n} a_{ij}, \tag{2.44}$$

respectively [9].

Definition 11: [9] A graph is called balanced if and only if all of its nodes are balanced, or

$$deg_{in}(v_i) = deg_{out}(v_i).$$

Lemma 11: [9] Consider a network of integrators with a fixed topology that is a strongly connected digraph. Then, protocol (2.40) globally asymptotically solves the average consensus problem if and only if the network is balanced.

In [10], it was shown that information consensus under dynamically changing interaction topologies can be achieved asymptotically if the union of the directed interaction graphs had a spanning tree frequently enough as the system evolved. An important lemma was also given in [10] as:

Lemma 12: [10] If the directed network has a directed spanning tree, the corresponding Laplacian matrix L would have a simple eigenvalue 0 and all the other eigenvalues would have positive real parts.

The first-order consensus under fixed and switching topologies was also studied in [9, 10, 49, 50, 51, 52, 53]. The first-order asynchronous consensus problem under directed switching topologies was shown to be solvable if the union of the topologies contained a spanning tree in [51]. First-order leader-following consensus under undirected switching topologies was researched in [52] [53], and consensus tracking under directed switching topologies in [54], showing that the joint connectivity was a key condition to ensure the leader-following consensus.

The authors in [55] considered a seconder-order control protocol

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i &= \beta_0 \sum_{j\in\mathcal{N}_i(t)} a_{ij}(x_j(t) - x_i(t)) + \beta_1 \sum_{j\in\mathcal{N}_i(t)} a_{ij}(v_j(t) - v_i(t)), i \in \mathcal{I}. \end{aligned} \tag{2.45}$$

For the system (2.45), $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{iN}(t))^T \in \mathbb{R}^N$ is the state vector of the ith agent.

Let

$$x = (x_1^T, x_2^T, \ldots, x_n^T)^T, \tag{2.46}$$

$$v = (v_1^T, v_2^T, \ldots, v_n^T)^T, \tag{2.47}$$

$$y = (x^T, v^T)^T. \tag{2.48}$$

the system (2.45) can be rewritten into a matrix form as

$$\dot{y}(t) = (\tilde{L} \otimes I_N)y(t), \tag{2.49}$$

where

$$\tilde{L} = \begin{bmatrix} O_n & I_n \\ -\beta_0 L & -\beta_1 L \end{bmatrix}, \tag{2.50}$$

and $\otimes$ is the Kronecker product.

The cooperation goal in [55] was given as

Definition 12: The cooperation goal of the system (2.45) is achieved if for any initial conditions,

$$\begin{aligned} &\lim_{t\to+\infty} \|x_i(t) - x_j(t)\| = 0, \\ &\lim_{t\to+\infty} \|v_i(t) - v_j(t)\| = 0, \end{aligned} \quad \forall i, j \in \mathcal{I}, i \neq j. \tag{2.51}$$

In [55], necessary and sufficient conditions for the second-order multi-agent systems consisting of (2.45) achieving their goal were given as

Lemma 13: the second-order multi-agent systems consisting of (2.45) can achieve their goal for any initial values if and only if

a) the network contains a directed spanning tree, and

b)

$$\frac{\beta_1^2}{\beta_0} > \max_{2\le i\le n} \frac{\mathcal{I}^2(\mu_i)}{\mathcal{R}(\mu_i)[\mathcal{R}^2(\mu_i)+\mathcal{I}^2(\mu_i)]}, \tag{2.52}$$

where μ_i is the ith nonzero eigenvalue of the Laplacian matrix L; $\mathcal{R}(\mu_i)$ and $\mathcal{I}(\mu_i)$ are the real and imaginary parts of the complex number μ_i.

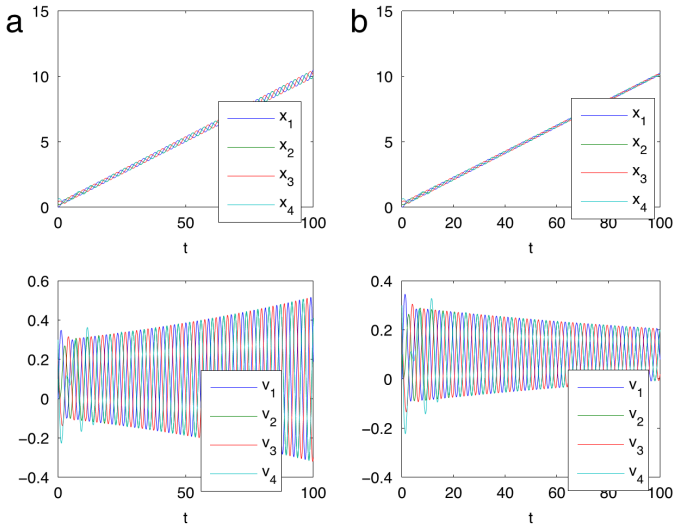


Figure 2.6 : Position and velocity states of agents in a network, when unsatisfying or satisfying Lemma 13. [55]

Both the example and the counterexample were given in [55] shown on the Fig. 2.6. The conclusion in [55] is very important for the second-order multi-agent systems.

In [56, 57, 58], a new Laplacian matrix H was given as

$$\begin{aligned} H &= [h_{ij}], \\ h_{ij} &= l_{ij} - l_{nj}. \end{aligned} \tag{2.53}$$

It is proved that

Lemma 14: [56, 58] Let $L = [l_{ij}]$ denote the Laplacian matrix associated with a strongly connected digraph. All the nonzero eigenvalues of L are the eigenvalues of the matrix $H = [h_{ij}]$, where $h_{ij} = l_{ij} - l_{nj}$.

Some other conclusions about the second-order consensus were also reported in [59, 55, 60, 61, 62, 63, 64]. In [59], some preliminary schemes for the second-order consensus have been studied, and it has been reported that the second-order consensus may fail to be reached even if the network topology had a directed spanning tree, i.e., consensus required other conditions besides having a spanning tree. The second-order consensus was affected by more factors than the first-order consensus. For the same agents and initial conditions, even when a consensus could be reached under some fixed topologies; the consensus could not be achieved under other topologies. In order to further study consensus conditions, the problem about second-order consensus in multi-agent dynamical systems under fixed topologies was discussed, and some conditions were given in [55, 60, 61]. In addition, second-order consensus under switching topologies with balanced networks was discussed in [62]. The second-order leader-following consensus was studied in [63], under the assumption that the leader was fixed for switching topologies.

The consensus of the three-order and the high-order multi-agent systems was also studied in [65, 66], where the network was fixed, and it was shown that all the gains must be chosen properly to ensure that consensus could be reached. Consensus regions were discussed for undirected switched topologies in [67]. The consensus of third-order nonlinear multi-agent systems was present in [68] recently, and a consensus protocol was proposed under the assumption that the topology was fixed. However, due to the loss or creation of the link, the topologies would be changed. The studies for consensus under switching topologies were more practical [69]. The consensus problem for high-order dynamic multi-agent systems with fixed and switching topologies was discussed in [70], and sufficient conditions of the consensus for switch-

ing topologies were provided under the assumption that interactions among agents were modelled by undirected graphs. A decentralized high gain control law was provided in [71] to solve the consensus problems with undirected switching topologies. But the consensus analysis for directed networks was more challenging than the case of undirected networks [9]. The consensus analysis for third-order multi-agent systems in directed networks was studied in [72] recently, where the topology was fixed. The relationship between high-order integral multi-agent systems and the first-order integral multi-agent systems was researched recently [69].

In [73], the authors proposed a kind of cooperation. That is called containment control. We first introduce some basic concepts.

Definition 13: [74] $Dist(x, \mathcal{C})$ denotes the distance from $x \in \mathbb{R}^N$ to the set $\mathcal{C} \subseteq \mathbb{R}^N$ in the Euclidean space, that is

$$dist(x, \mathcal{C}) = \inf_{y \in \mathcal{C}} \|x - y\|^2. \tag{2.54}$$

Definition 14: [75] A set $\mathcal{C}$ is said to be convex if $(1 - \lambda)x + \lambda y \in \mathcal{C}$, for any x, $y \in \mathcal{C}$ and any $\lambda \in [0, 1]$. The convex hull of a finite set of points $X = \{ x_1, x_2, \ldots, x_n\}$ is denoted by

$$co\{X\} = \{\sum_{i=1}^{n} \alpha_i x_i \mid \alpha_i \in \mathbb{R}, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1\}. \tag{2.55}$$

For containment control, the multi-agent systems are divided into two groups: the follower group denoted by the set $\mathcal{F}$ and leaders by $\mathcal{R}$. In [74], the authors assumed the followers had the dynamics

$$\dot{x}_i = A_i x_i + B_i u_i, \ i \in \mathcal{F}. \tag{2.56}$$

and the leaders were the dynamics

$$\dot{w}_k = S w_k. \tag{2.57}$$

The concept of containment is defined as

Definition 15: [74] The multi-agent system (2.56)(2.57) reached the containment control goal when the designed protocol can ensure that all followers converge to the convex hull spanned by the dynamic leaders as $t \to +\infty$, that is

$$\forall i \in \mathcal{F}, \lim_{t \to +\infty} dist(x_i(t), co(w_k)(t), k \in \mathcal{R}) = 0. \tag{2.58}$$

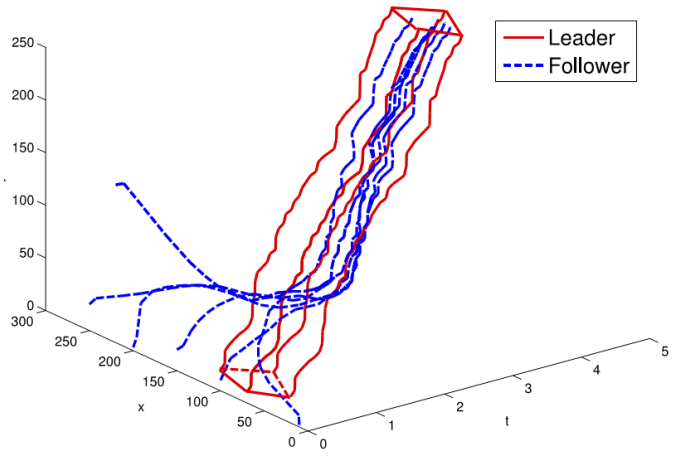


Figure 2.7 : State trajectories of agents for containment control in [14]

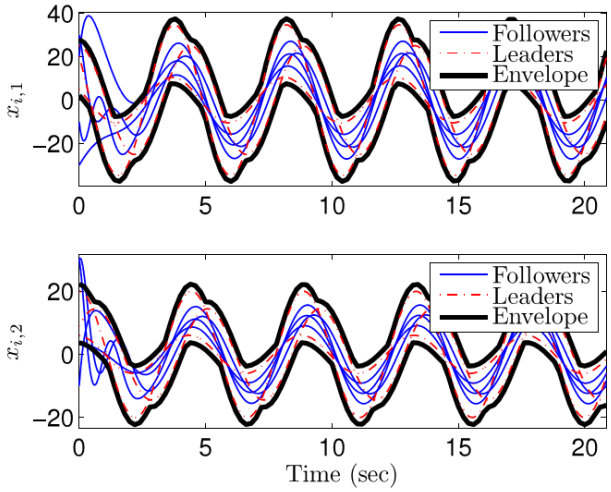


Figure 2.8 : State trajectories of agents for containment control in [74]

The containment control can be illustrated by Figs. 2.7 and 2.8.

In recent years, the containment control problems have been studied by [14, 74]. More conclusions can be found in these documents.

2.4 Summary

This chapter introduces some basic concepts and important conclusions for the cooperation of multi-agent systems and switched systems.

Chapter 3

Multi-agent Systems under a Fixed Topology

3.1 Introduction

In this chapter, we analyze the cooperation of multi-agent systems when the network of multi-agent systems is time-invariant.

The research in this chapter is limited to a fixed topology, which is different from the network in Chapter 5.

The network in this chapter has few limitations except a fixed topology. For example, the network can be balanced or not.

Three kinds of cooperation are studied. They are consensus with no leader, consensus with one leader (tracking control), and containment control.

Consensus control with no leader, tracking control with one leader and containment control with multiple leaders, as three particular fundamental problems of multi-agent systems, have been studied a lot [76, 77]. Although these three problems are with different numbers of leaders, they have a shared property that each agent communicates only with its neighbours to reach global behaviour. These three fundamental problems of multi-agent systems are considered as a common class of cooperative control with different numbers of leaders.

Many results about consensus and control protocols were designed in [76]. Some researches about tracking control with one leader, which also was called leader-following consensus, including control protocols have been done [52]. In order to analyze the tracking control problem, local neighbourhood synchronization error

was applied to denote the state error between the agent and the leader [78]. For containment control with multiple leaders, some study results and corresponding control protocols were reported [79].

3.2 Main Part

3.2.1 Consensus with No Leader

Consider a team of n agents' systems where each agent is modelled by the G^{th}-order dynamics as follows [65, 66]:

$$\begin{aligned} \dot{x}_i^{(0)}(t) &= x_i{}^{(1)}(t) \\ &\vdots \\ \dot{x}_i^{(G-2)}(t) &= x_i{}^{(G-1)}(t) \\ \dot{x}_i^{(G-1)}(t) &= u_i, i \in \mathcal{I}, \end{aligned} \tag{3.1}$$

where $x_i(t) = [x_{i1}(t), \ldots, x_{iN}(t)]^T \in \mathbb{R}^N$ is the state of vector of the i^{th} agent, $u_i \in \mathbb{R}^N$ is the control input, $x_i^{(k)}$ denotes the k^{th} derivative of x_i, and $x_i^{(0)} = x_i$.

The consensus algorithm is proposed as [55, 65]:

$$\begin{aligned} u_i(t) = \beta_0 &\sum_{j \in \mathcal{N}_i} a_{ij}(x_j^{(0)}(t) - x_i^{(0)}(t)) \\ &+ \cdots \\ &+ \beta_{G-2} \sum_{j \in \mathcal{N}_i} a_{ij}(x_j^{(G-2)}(t) - x_i^{(G-2)}(t)) \\ &+ \beta_{G-1} \sum_{j \in \mathcal{N}_i} a_{ij}(x_j^{(G-1)}(t) - x_i^{(G-1)}(t)),\ i \in \mathcal{I}, \end{aligned} \tag{3.2}$$

where $a_{ij},\ i,\ j \in \mathcal{I}$ are the entries of adjacency matrix $\mathcal{A}$, decided by the fixed topology $\mathcal{G}$. Positive constant $\beta_0, \beta_1, \beta_2, \ldots, \beta_{G-1}$ are the coupling strengths.

Definition 16: The G^{th}-order consensus is said to be reached among n agents for the multi-agent systems (3.1) if for all initial conditions, $x_i^{(k)}(t) \to x_j^{(k)}(t)$, as $t \to +\infty$, $k \in \mathcal{K}$, $\mathcal{K} \triangleq \{0, \ldots, \mathrm{G} - 1\}$.

Let $x^{(k)}(t) = [x_1^{(k)}(t)^T, \ldots, x_n^{(k)}(t)^T]^T$, and $y(t) = [x^{(0)}(t)^T, \ldots, x^{(G-1)}(t)^T]^T, k \in \mathcal{K}$. Then, network (3.1) with input (3.2) can be rewritten in a compact matrix as

$$\dot{y}(t) = (\tilde{L} \otimes I_N)y(t), \tag{3.3}$$

where

$$\tilde{L} = \begin{bmatrix} 0 & I \\ -\beta_0 L & \hat{L} \end{bmatrix}, \tag{3.4}$$

$$\hat{L} = [-\beta_1 L \quad -\beta_2 L \quad \cdots \quad -\beta_{G-1} L], \tag{3.5}$$

$y(t) \in \mathbb{R}^{GNn}$, and $L = [l_{ij}]$ is the Laplacian matrix.

The $x_i^{(k)} \in \mathbb{R}^N$, $x_i^{(k)}$ can be denoted as

$$x_i^{(k)}(t) = [x_{i,1}^{(k)}(t), x_{i,2}^{(k)}(t), \ldots, x_{i,N}^{(k)}(t)]^T, \tag{3.6}$$

where $x_{i,j}^{(k)} \in \mathbb{R}^1$, $i \in \mathcal{I}$, $j \in \{1, \ldots, N\}$.

Let $y_j(t) = [x_{1,j}^{(0)}(t), x_{2,j}^{(0)}(t), \cdots, x_{n,j}^{(0)}(t), x_{1,j}^{(1)}(t), \cdots, x_{n,j}^{(G-1)}(t)]^T$, $j \in \{1, \cdots N\}$, the system (3.3) could be rewritten as

$$\begin{cases} \dot{y}_1(t) = \tilde{L}y_1(t), y_1(0) = y_1(t)|_{t=0}, \\ \cdots \\ \dot{y}_N(t) = \tilde{L}y_N(t), y_N(0) = y_N(t)|_{t=0}, \end{cases}$$

with N independent subsystems having the same differential equation of motion as

$$\dot{c}(t) = \tilde{L}c(t) \tag{3.7}$$

where $c(t) \in \mathbb{R}^{Gn}$ stands for y_j, $j \in \{1, \cdots N\}$. It is defined by

$$c(t) = [c^{(0)}(t)^T, \ldots, c^{(G-1)}(t)^T]^T, \tag{3.8}$$

$$c^{(k)}(t) = [c_1^{(k)}(t), \ldots, c_n^{(k)}(t)]^T, k \in \mathcal{K}, \tag{3.9}$$

where $c(0) = c(t)|_{t=0}$ takes its values from $\{y_j(t)|_{t=0}\}$, $j \in \{1, \ldots, N\}$.

Now the consensus problem with (3.3) for all initial values is equivalent to that with equation (3.7) for all initial conditions. If we assume $N = 1$ for simplicity, (3.7) is (3.3).

By setting j as n for definition 16, and using(3.7), G^{th}-order consensus would be said to be reached if for all initial conditions, $c_i^{(k)}(t) \to c_n^{(k)}(t)$, as $t \to +\infty$, $k \in \mathcal{K}$, $\forall i \in \{1, \ldots, n-1\}$. That is, the consensus problem would be solved if

$$c_i^{(k)}(t) - c_n^{(k)}(t) \to 0,\ as\ t \to +\infty, \forall k \in \mathcal{K}, \forall \mathrm{i} \in \{1, \ldots, \mathrm{n}-1\}, \tag{3.10}$$

for all initial conditions.

$\forall k \in \mathcal{K}, \forall \mathrm{i} \in \{1, \ldots, \mathrm{n}-1\}$, let

$$\begin{cases} z_i^{(k)}(t) = c_i^{(k)}(t) - c_n^{(k)}(t), \\ z^{(k)}(t) = [z_1^{(k)}(t)^T, \ldots, z_{n-1}^{(k)}(t)^T]^T, \end{cases}$$

and $z(t) = [z^{(0)}(t)^T, \ldots, z^{(G-1)}(t)^T]^T \in \mathbb{R}^{G(n-1)}$.

Now, one can get Theorem 1.

Theorem 1: For the directed network $\mathcal{G}$, if the globally uniformly asymptotically stability problem of system

$$\dot{z}(t) = \tilde{H} z(t), z(t_0) = z(0). \tag{3.11}$$

is ensured, where

$$\tilde{H} = \begin{bmatrix} 0 & I \\ -\beta_0 H & \hat{H} \end{bmatrix} \tag{3.12}$$

$$\hat{H} = [-\beta_1 H \quad -\beta_2 H \quad \cdots \quad -\beta_{G-1} H], \tag{3.13}$$

then the consensus of system (3.1) is reached with a consensus algorithm (3.2), under all initial conditions.

Proof: The i^{th} row vector for the Laplacian matrix can be denoted as

$$L_i = [l_{i1}, l_{i2}, ..., l_{in}], i \in \mathcal{I}. \tag{3.14}$$

The i^{th} row vector for the H matrix can be denoted as

$$H_i = [h_{i1}, h_{i2}, ..., h_{in-1}], i \in \mathcal{I}, i \neq n. \tag{3.15}$$

1) When $\forall k \in \mathcal{K}$ and $k < G-1$,

according to (3.7) and (3.10),

$$\begin{aligned} \dot{z}_i^{(k)}(t) &= \dot{c}_i^{(k)}(t) - \dot{c}_n^{(k)}(t) \\ &= c_i^{(k+1)}(t) - c_n^{(k+1)}(t) \\ &= z_i^{(k+1)}(t), \end{aligned}$$

and

$$\begin{aligned} \dot{z}^{(k)}(t) &= (\dot{z}_1^{(k)}(t)^T, \ldots, \dot{z}_{n-1}^{(k)}(t)^T)^T \\ &= (z_1^{(k+1)}(t)^T, \ldots, z_{n-1}^{(k+1)}(t)^T)^T \\ &= z^{(k+1)}(t). \end{aligned} \tag{3.16}$$

2) For $\dot{z}_i^{(G-1)}(t)$,

$$\begin{aligned} \dot{z}_i^{(G-1)}(t) &= \dot{c}_i^{(G-1)}(t) - \dot{c}_n^{(G-1)}(t) \\ &= \beta_0 \sum_{j \in \mathcal{N}_i} a_{ij}(c_j^{(0)}(t) - c_i^{(0)}(t)) + \cdots \\ &+ \beta_{G-1} \sum_{j \in \mathcal{N}_i} a_{ij}(c_j^{(G-1)}(t) - c_i^{(G-1)}(t)) \\ &- \beta_0 \sum_{j \in \mathcal{N}_n} a_{ij}(c_j^{(0)}(t) - c_n^{(0)}(t)) - \cdots \\ &- \beta_{G-1} \sum_{j \in \mathcal{N}_n} a_{ij}(c_j^{(G-1)}(t) - c_n^{(G-1)}(t)), \end{aligned} \tag{3.17}$$

because of (3.7), (3.10), and properties of the Laplacian matrix.

By (3.14), one can get

$$\begin{aligned}\dot{z}_i^{(G-1)}(t) =& -\beta_0 L_i c^{(0)}(t) - \cdots - \beta_{G-1} L_i c^{(G-1)}(t) \\ &+ \beta_0 L_n c^{(0)}(t) + \cdots + \beta_{G-1} L_n c^{(G-1)}(t) \\ =& -\beta_0 L_i c^{(0)}(t) - \cdots - \beta_{G-1} L_i c^{(G-1)}(t) \\ &+ \beta_0 L_n c^{(0)}(t) + \cdots + \beta_{G-1} L_n c^{(G-1)}(t) \\ =& -\beta_0 (L_i - L_n) c^{(0)}(t) - \cdots \\ &- \beta_{G-1} (L_i - L_n) c^{(G-1)}(t).\end{aligned} \tag{3.18}$$

According to properties of the Laplacian matrix, and the definition of H matrix, one can get

$$(L_i - L_n) c^{(k)}(t) = H_i z^{(k)}(t), \forall k \in \mathcal{K}. \tag{3.19}$$

By (3.19),

$$\begin{aligned}\dot{z}_i^{(G-1)}(t) =& -\beta_0 (L_i - L_n) c^{(0)}(t) - \cdots - \beta_{G-1} (L_i - L_n) c^{(G-1)}(t) \\ =& -\beta_0 H_i z^{(0)}(t) - \cdots - \beta_{G-1} H_i z^{(G-1)}(t).\end{aligned} \tag{3.20}$$

By the definition of $z^{(k)}(t)$ and (3.15), then

$$\begin{aligned}\dot{z}^{(G-1)}(t) &= (\dot{z}_1^{(G-1)}(t)^T, \ldots, \dot{z}_{n-1}^{(G-1)}(t)^T)^T \\ &= -\beta_0 H z^{(0)}(t) - \cdots - \beta_{G-1} H z^{(G-1)}(t) \\ &= -\beta_0 H z^{(0)}(t) + \hat{H}[z^{(1)}(t)^T, \ldots, z^{(G-1)}(t)^T]^T.\end{aligned} \tag{3.21}$$

Now, by 1) and 2), the equation (3.11) has been proved.

If the globally uniformly asymptotically stability problem of system (3.11) is ensured, then

$$z_i^{(k)} = c_i^{(k)}(t) - c_n^{(k)}(t) \to 0 \quad as \quad t \to +\infty, \forall k \in \mathcal{K}, \forall \mathrm{i} \in \{1, \ldots, \mathrm{n}-1\}. \tag{3.22}$$

Compared with the consensus definition of equation (3.10), the consensus problem of (3.7) is solved. Based on the relationship, which has been analyzed, between (3.7) and (3.3), the consensus of the system (3.3) is reached. The consensus of the

system (3.1) with a consensus algorithm (3.2) is reached, because system (3.3) is the compact matrix form of system (3.1) with a consensus algorithm (3.2).

Now one can conclude the proof of this theorem. ■

Theorem 2: If the directed network $\mathcal{G}$ has a directed spanning tree with fixed topologies, the system (3.1) with a consensus algorithm (3.2) reaches a consensus exponentially if and only if all the eigenvalues of corresponding matrix $\tilde{H}$ have negative real parts.

Proof: Let $\tilde{J}$ be the Jordan form associated with $\tilde{H}$, then $\tilde{H} = S\tilde{J}S^{-1}$. By explicit calculation of (3.11), one has

$$z(t) = \exp(\tilde{H}t)z(0) = S\exp(\tilde{J}t)S^{-1}z(0) \tag{3.23}$$

$\exp(\tilde{J}t)$ converges to a zero matrix exponentially, as t$\rightarrow +\infty$, if and only if all the eigenvalues of corresponding matrix $\tilde{H}$ have negative real parts.

By the proof of Theorem 1, the systems (3.11) can be got from the system (3.1) with a consensus algorithm (3.2). System (3.1) with a consensus algorithm (3.2) reaches consensus exponentially if and only if the system (3.11) converges to origin exponentially, as t$\rightarrow +\infty$. The system (3.11) converges to origin exponentially only if all the eigenvalues of corresponding matrix $\tilde{H}$ have negative real parts, as t$\rightarrow +\infty$.

The proof is finished. ■

According to Lemma 2 in Chapter 2 and the conclusion in Theorem 1, one can get Theorem 3.

Theorem 3: For the multi-agent systems under a fixed network having a spanning tree, if there exists a symmetric matrix $P > 0$, such that

$$\tilde{H}^T P + P\tilde{H} < 0, \tag{3.24}$$

then the multi-agent systems (3.1) would reach a consensus with a consensus algorithm (3.2) under all initial values.

Proof: According to Lemma 2 in Chapter 2, the system(3.11) is globally uniformly asymptotically stable.

By Theorem 1, the multi-agent systems (3.1) with a consensus algorithm (3.2) would reach a consensus under all initial values.

This proof is completed. ■

3.2.2 Consensus with One leader

In this subsection, we consider the network having one leader. The consensus for this kind of network is also named tracking control.

For tracking control, the multi-agent systems are divided into two groups: the follower group denoted by the set $\mathcal{F}$ and leaders by $\mathcal{R}$. For agents' systems including n followers and one leader, the followers are labelled as $1,\ 2,\ 3, \cdots,\ n$. The set $\mathcal{F}$ and $\mathcal{R}$ are rewritten as $\mathcal{F} = \{1,\ 2,\ 3, \cdots,\ n\}$ and $\mathcal{R} = \{n+1\}$.

Consider the agent systems where each follower is modelled by the G^{th}-order dynamics as follows:

$$\begin{aligned} \dot{x}_i^{(0)}(t) &= {x_i}^{(1)}(t) \\ &\vdots \\ \dot{x}_i^{(G-2)}(t) &= {x_i}^{(G-1)}(t) \\ \dot{x}_i^{(G-1)}(t) &= u_i,\ i \in \mathcal{F}, \end{aligned} \tag{3.25}$$

the leader is modelled by

$$\begin{aligned} \dot{x}_{n+1}^{(0)}(t) &= 0 \\ &\vdots \\ \dot{x}_{n+1}^{(G-2)}(t) &= 0 \\ \dot{x}_{n+1}^{(G-1)}(t) &= 0, \end{aligned} \tag{3.26}$$

where $x_i(t) = [x_{i1}(t), \ldots, x_{iN}(t)]^T \in \mathbb{R}^N$ is the state of vector of the i^{th} agent, $u_i \in \mathbb{R}^N$ is the control input, $x_i^{(k)}$ denotes the k^{th} derivative of x_i, and $x_i^{(0)} = x_i$; $x_{n+1}(t) \in \mathbb{R}^N$ is the state of vector of the leader, $x_{n+1}^{(k)}$ denotes the k^{th} derivative of

x_{n+1}, and $x_{n+1}^{(0)} = x_{n+1}$

Definition 17: The G^{th}-order consensus is said to be reached for the multi-agent systems (3.25)-(3.26) if for all initial conditions, $x_i^{(k)}(t) \to x_{n+1}^{(k)}(t)$, $\forall i \in \mathcal{F}$, as $t \to +\infty$, $k \in \mathcal{K}$, $\mathcal{K} \triangleq \{0, \ldots, \mathrm{G}-1\}$.

An algorithm is considered as:

$$\begin{aligned} u_i(t) = \beta_0 & \sum_{j\in\mathcal{N}_i} a_{ij}(x_j^{(0)}(t) - x_i^{(0)}(t)) \\ & + \cdots \\ & + \beta_{G-2} \sum_{j\in\mathcal{N}_i} a_{ij}(x_j^{(G-2)}(t) - x_i^{(G-2)}(t)) \\ & + \beta_{G-1} \sum_{j\in\mathcal{N}_i} a_{ij}(x_j^{(G-1)}(t) - x_i^{(G-1)}(t)),\ i \in \mathcal{F}, \end{aligned} \tag{3.27}$$

where $a_{ij},\ i,\ j \in \mathcal{I}$ are the entries of adjacency matrix $\mathcal{A}$, decided by the fixed topology $\mathcal{G}$. Positive constant $\beta_0, \beta_1, \beta_2, \ldots, \beta_{G-1}$ are the coupling strengths.

Let $x^{(k)}(t) = [x_1^{(k)}(t)^T, \ldots, x_n^{(k)}(t)^T,\ x_{n+1}^{(k)}(t)^T]^T$, and $y(t) = [x^{(0)}(t)^T, \ldots, x^{(G-1)}(t)^T]^T, k \in \mathcal{K}$. Then, the systems (3.25)-(3.26) with input (3.27) can be rewritten in a compact matrix as

$$\dot{y}(t) = (\tilde{L} \otimes I_N)y(t), \tag{3.28}$$

where

$$\tilde{L} = \begin{bmatrix} 0 & I \\ -\beta_0 L & \hat{L} \end{bmatrix}, \tag{3.29}$$

$$\hat{L} = [-\beta_1 L \quad -\beta_2 L \quad \cdots \quad -\beta_{G-1} L], \tag{3.30}$$

$y(t) \in \mathbb{R}^{GN(n+1)}$, and $L = [l_{ij}]$ is the Laplacian matrix.

Let $y_j(t) = [x_{1,j}^{(0)}(t), x_{2,j}^{(0)}(t), \cdots, x_{n+1,j}^{(0)}(t), x_{1,j}^{(1)}(t), \cdots, x_{n+1,j}^{(G-1)}(t)]^T,\ j \in \{1, \cdots N\}$,

the system (3.28) could be rewritten as

$$\begin{cases} \dot{y}_1(t) = \tilde{L}y_1(t), y_1(0) = y_1(t)|_{t=0}, \\ \cdots \\ \dot{y}_N(t) = \tilde{L}y_N(t), y_N(0) = y_N(t)|_{t=0}, \end{cases}$$

with N independent subsystems having a same differential equation of motion as

$$\dot{c}(t) = \tilde{L}c(t) \tag{3.31}$$

where $c(t) \in \mathbb{R}^{G(n+1)}$ stands for y_j, $j \in \{1, \cdots N\}$. It is defined by

$$c(t) = [c^{(0)}(t)^T, \ldots, c^{(G-1)}(t)^T]^T, \tag{3.32}$$

$$c^{(k)}(t) = [c_1^{(k)}(t), \ldots, c_{n+1}^{(k)}(t)]^T, k \in \mathcal{K}, \tag{3.33}$$

where $c(0) = c(t)|_{t=0}$ takes its values from $\{y_j(t)|_{t=0}\}$, $j \in \{1, \ldots, N\}$.

Now the consensus problem with (3.28) for all initial values is equivalent to that with equation (3.31) for all initial conditions. If we assume $N = 1$ for simplicity, (3.31) is (3.28).

Let

$$\begin{cases} z_i^{(k)}(t) = c_i^{(k)}(t) - c_{n+1}^{(k)}(t), \\ z^{(k)}(t) = [z_1^{(k)}(t)^T, \ldots, z_n^{(k)}(t)^T]^T, \end{cases}, \ \forall k \in \mathcal{K}, \ \forall i \in \mathcal{F},$$

and $z(t) = [z^{(0)}(t)^T, \ldots, z^{(G-1)}(t)^T]^T \in \mathbb{R}^{Gn}$.

According to the above analysis, we can know the leader has some constraints.

Now, one can get Theorem 4.

Theorem 4: For the directed network $\mathcal{G}$, if the globally uniformly asymptotically stability problem of system

$$\dot{z}(t) = \tilde{H}z(t), z(t_0) = z(0). \tag{3.34}$$

is ensured, where

$$\tilde{H} = \begin{bmatrix} 0 & I \\ -\beta_0 H & \hat{H} \end{bmatrix} \tag{3.35}$$

$$\hat{H} = [-\beta_1 H \quad -\beta_2 H \quad \cdots \quad -\beta_{G-1} H], \tag{3.36}$$

then the consensus of systems (3.25)-(3.26) is reached with a consensus algorithm (3.27), under all initial conditions.

Proof: The i^{th} row vector for the Laplacian matrix can be denoted as

$$L_i = [l_{i_1}, l_{i_2}, \cdots, l_{i_{n+1}}],\ i \in \mathcal{I}. \tag{3.37}$$

The i^{th} row vector for the H matrix can be denoted as

$$H_i = [h_{i_1}, h_{i_2}, ..., h_{i_n}], i \in \mathcal{I}, i \neq n+1. \tag{3.38}$$

1) When $\forall k \in \mathcal{K}$ and $k < G-1$,

according to (3.31),

$$\begin{aligned} \dot{z}_i^{(k)}(t) &= \dot{c}_i^{(k)}(t) - \dot{c}_{n+1}^{(k)}(t) \\ &= c_i^{(k+1)}(t) - c_{n+1}^{(k+1)}(t) \\ &= z_i^{(k+1)}(t), \end{aligned}$$

and

$$\begin{aligned} \dot{z}^{(k)}(t) &= (\dot{z}_1^{(k)}(t)^T, \ldots, \dot{z}_n^{(k)}(t)^T)^T \\ &= (z_1^{(k+1)}(t)^T, \ldots, z_n^{(k+1)}(t)^T)^T \\ &= z^{(k+1)}(t). \end{aligned} \tag{3.39}$$

2) For $\dot{z}_i^{(G-1)}(t)$,

$$\begin{aligned} &\dot{z}_i^{(G-1)}(t) \\ &= \dot{c}_i^{(G-1)}(t) - \dot{c}_{n+1}^{(G-1)}(t) \\ &= \beta_0 \sum_{j \in \mathcal{N}_i} a_{ij}(c_j^{(0)}(t) - c_i^{(0)}(t)) + \cdots + \beta_{G-1} \sum_{j \in \mathcal{N}_i} a_{ij}(c_j^{(G-1)}(t) - c_i^{(G-1)}(t)) \\ &- \beta_0 \sum_{j \in \mathcal{N}_{n+1}(t)} a_{ij}(c_j^{(0)}(t) - c_{n+1}^{(0)}(t)) - \cdots - \beta_{G-1} \sum_{j \in \mathcal{N}_{n+1}(t)} a_{ij}(c_j^{(G-1)}(t) - c_{n+1}^{(G-1)}(t)), \end{aligned} \tag{3.40}$$

because of (3.31) and properties of the Laplacian matrix.

By (3.37), one can get

$$\begin{aligned}
&\dot{z}_i^{(G-1)}(t)\\
=&-\beta_0 L_i c^{(0)}(t)-\cdots-\beta_{G-1}L_i c^{(G-1)}(t)+\beta_0 L_{n+1}c^{(0)}(t)+\cdots+\beta_{G-1}L_{n+1}c^{(G-1)}(t)\\
=&-\beta_0 L_i c^{(0)}(t)-\cdots-\beta_{G-1}L_i c^{(G-1)}(t)+\beta_0 L_{n+1}c^{(0)}(t)+\cdots+\beta_{G-1}L_{n+1}c^{(G-1)}(t)\\
=&-\beta_0(L_i-L_{n+1})c^{(0)}(t)-\cdots-\beta_{G-1}(L_i-L_{n+1})c^{(G-1)}(t).
\end{aligned} \tag{3.41}$$

According to properties of the Laplacian matrix, and the definition of H matrix, one can get

$$(L_i-L_{n+1})c^{(k)}(t)=H_i z^{(k)}(t), \forall k\in\mathcal{K}. \tag{3.42}$$

By (3.42),

$$\begin{aligned}
\dot{z}_i^{(G-1)}(t)=&-\beta_0(L_i-L_{n+1})c^{(0)}(t)-\cdots-\beta_{G-1}(L_i-L_{n+1})c^{(G-1)}(t)\\
=&-\beta_0 H_i z^{(0)}(t)-\cdots-\beta_{G-1}H_i z^{(G-1)}(t).
\end{aligned} \tag{3.43}$$

By the definition of $z^{(k)}(t)$ and (3.38), then

$$\begin{aligned}
\dot{z}^{(G-1)}(t)=&(\dot{z}_1^{(G-1)}(t)^T,\ldots,\dot{z}_{n-1}^{(G-1)}(t)^T)^T\\
=&-\beta_0 H z^{(0)}(t)-\cdots-\beta_{G-1}H z^{(G-1)}(t)\\
=&-\beta_0 H z^{(0)}(t)+\hat{H}[z^{(1)}(t)^T,\ldots,z^{(G-1)}(t)^T]^T.
\end{aligned} \tag{3.44}$$

Now, by 1) and 2), the equation (3.34) has been proved.

If the globally uniformly asymptotically stability problem of system (3.34) is ensured, then

$$z_i^{(k)}=c_i^{(k)}(t)-c_{n+1}^{(k)}(t)\to 0,\ as\ t\to+\infty,\ \forall k\in\mathcal{K},\forall i\in\{1,\cdots,n\}. \tag{3.45}$$

Compared with consensus Definition (17), the consensus problem of (3.31)is solved. Based on the relationship, which has been analyzed, between (3.31) and (3.28), the consensus of the system (3.28) is reached. The consensus of system (3.25)-(3.26) with a consensus algorithm (3.27) is reached, because the system (3.28) is the compact matrix form of system (3.25)-(3.26) with a consensus algorithm (3.27).

Now one can conclude the proof of this theorem. ■

Theorem 5: If the directed network $\mathcal{G}$ has a directed spanning tree with fixed topologies, the system (3.25)-(3.26) with a consensus algorithm (3.27) reach a consensus exponentially if and only if all the eigenvalues of the corresponding matrix $\tilde{H}$ have negative real parts.

Proof: The proof of this theorem and Theorem 2 are the same. It is omitted here. ■

According to Lemma 2 in Chapter 2 and the conclusion in Theorem 4, one can get Theorem 6.

Theorem 6: For the multi-agent systems under a fixed network having a spanning tree, if there exists a symmetric matrix $P > 0$, such that

$$\tilde{H}^T P + P\tilde{H} < 0, \tag{3.46}$$

then the multi-agent systems (3.25)-(3.26) would reach a consensus with a consensus algorithm (3.27) under all initial values.

Proof: According to Lemma 2 in Chapter 2, the system(3.34) is globally uniformly asymptotically stable.

According to Theorem 4, the multi-agent systems (3.25)-(3.26) with a consensus algorithm (3.27) would reach a consensus under all initial values.

This proof is completed. ■

3.2.3 Containment Control

In this subsection, we consider the network owning more than one leader. One popular cooperation for this kind of network is named containment control.

For containment control, the multi-agent systems are divided into two groups: the follower group denoted by the set $\mathcal{F}$, and leaders by $\mathcal{R}$. $\mathcal{F} \cup \mathcal{R} = \mathcal{I}$. For

agent systems including n followers and m leaders, the followers are labelled as $1,\ 2,\ 3, \cdots,\ n$. The set $\mathcal{F}$ and $\mathcal{R}$ are rewritten as $\mathcal{F} = \{1,\ 2,\ 3, \cdots,\ n\}$ and $\mathcal{R} = \{n+1, \cdots, n+m\}$. It is assumed that the network has a directed forest that is a directed graph consisting of one or more directed trees, no two of which have a vertex in common.

The Laplacian matrix for the graph of the multi-agent systems can be partitioned as

$$L = \begin{bmatrix} L_{FF} & L_{FR} \\ 0_{m\times n} & 0_{m\times m} \end{bmatrix} \tag{3.47}$$

Where $L_{FF} \in \mathbb{R}^{n\times n}$, $L_{FR} \in \mathbb{R}^{n\times m}$.

Consider the agent systems where each follower is modelled by the following dynamics:

$$\dot{x}_i^{(0)}(t) = u_i,\ i \in \mathcal{F}. \tag{3.48}$$

And each leader is modelled by

$$\dot{w}_j^{(0)}(t) = 0,\ j \in \mathcal{R}. \tag{3.49}$$

where $x_i \in \mathbb{R}^1$ and u_i are the status and control protocol of the i^{th} follower. $w_j \in \mathbb{R}^1$ is the status of the j^{th} leader.

Definition 18: The first-order containment control is said to be reached for the multi-agent systems (3.48)-(3.49) if for all initial conditions,

$$\inf ||x_i - Co(w)||_2 \to 0,\ as\ \mathrm{t} \to +\infty,\ \forall i \in \mathcal{F},\ k \in \mathcal{R}, \tag{3.50}$$

where $Co(w)$ denotes the convex hull of leaders' states.

An algorithm is considered as:

$$u_i(t) = \sum_{j\in\mathcal{N}_i, j\in\mathcal{F}} a_{ij}(x_j(t) - x_i(t)) + \sum_{j\in\mathcal{N}_i, j\in\mathcal{R}} a_{ij}(w_j(t) - x_i(t)), \tag{3.51}$$

where $a_{ij},\ i,\ j \in \mathcal{I}$ are the entries of adjacency matrix $\mathcal{A}$, decided by the fixed topology $\mathcal{G}$.

Similar to the definition in [15], $\delta_i^k = 1,\ i \in \mathcal{F},\ k \in \mathcal{R}$, if the i^{th} follower agent can get the information from the k^{th} leader agent. $\delta_i^k = 0$, otherwise. A diagonal matrix Δ^k is composed of elements $\delta_1^k, \ldots, \delta_n^k$.

One can get

$$\Delta^k = \begin{bmatrix} \delta_1^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_n^k \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{3.52}$$

$$L_{FF} = L_F + \Delta^1 + \cdots + \Delta^m \tag{3.53}$$

$$L_{FR} = -\begin{bmatrix} \delta_1^1 & \cdots & \delta_1^m \\ \vdots & \ddots & \vdots \\ \delta_n^1 & \cdots & \delta_n^m \end{bmatrix} \in \mathbb{R}^{n\times m} \tag{3.54}$$

In this subsection, the extended local neighbourhood synchronization error is defined as

$$e_i(t) = \sum_{j\in\mathcal{N}_i, j\in\mathcal{F}} a_{ij}(x_j(t) - x_i(t)) + \sum_{j\in\mathcal{N}_i, j\in\mathcal{R}} a_{ij}(w_j(t) - x_i(t)), \tag{3.55}$$

where $i \in \mathcal{F}$, and $\mathcal{N}_i \subseteq (\mathcal{F} \cup \mathcal{R})$.

The compact formation is rewritten as

$$e(t) = L_{FF}x(t) + L_{FR}w(t). \tag{3.56}$$

Lemma 15: . For the multi-agent systems with multiple leaders, containment control will be reached for all initial values if the extended local neighbourhood synchronization error (3.55)(3.56) would satisfy

$$e_i(t) \to 0, i \in \mathcal{F}, as\ t \to +\infty \tag{3.57}$$

Proof: According to (3.57),

$$e(t) = L_{FF}x(t) + L_{FR}\mathrm{w(t)} \to 0, as\ t \to +\infty. \tag{3.58}$$

One can get

$$x(t) \to -L_{FF}^{-1} L_{FR} w(t) \; as \; t \to +\infty. \tag{3.59}$$

Lemma is verified for $-L_{FF}^{-1} L_{FR} \mathbf{1}_m = \mathbf{1}_n$ and all the elements of $-L_{FF}^{-1} L_{FR}$ are nonnegative [14]. ■

The extended local neighbourhood synchronization error (3.55) can be also used in the network with no leader or one leader. t For the first-order system with no leader, the extended local neighbourhood synchronization error (3.55) is rewritten as

$$e_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)). \tag{3.60}$$

The compact formation is rewritten as

$$e(t) = Lx(t). \tag{3.61}$$

Lemma 16: . For the first-order system with no leader, a consensus is reached for all initial values if the extended local neighbourhood synchronization error (3.60) (3.61) would satisfy

$$e_i(t) \to 0, as \; t \to +\infty. \tag{3.62}$$

Proof: Let $e = [e_1, \ldots, e_n]^T$, $\bar{e} = [e_1 - e_n, \ldots, e_{n-1} - e_n]^T$, and $\bar{x} = [x_1 - x_n, \ldots, x_{n-1} - x_n]^T$.

According to (3.62), one can get

$$e(t) = Lx(t) \to 0, \; as \; t \to +\infty. \tag{3.63}$$

According to the definitions of matrix H, one can get

$$\bar{e}(t) = H\bar{x}(t) \to 0, \; as \; t \to +\infty. \tag{3.64}$$

Therefore,

$$\bar{x}(t) = H^{-1}\bar{e}(t) \to 0, \; as \; t \to +\infty. \tag{3.65}$$

The proof is finished. ■

Lemma 17: For the tracking control with one leader, $m = 1$, this kind of cooperative control could be reached for all initial values if the extended local neighbourhood synchronization error (3.55)(3.56) would satisfy

$$e_i(t) \to 0, as\ t \to +\infty, i \in \mathcal{F}. \tag{3.66}$$

Proof: According to (3.53) and $m = 1$, then

$$\begin{aligned}
&(L_F + \Delta^1)(x - 1_n \otimes w_1) \\
=&(L_F + \Delta^1)x - (L_F + \Delta^1)(1_n \otimes w_1) \\
=&L_{FF}x - L_F(1_n \otimes w_1) - \Delta^1(1_n \otimes w_1) \\
=&L_{FF}x - \Delta^1(1_n \otimes w_1) \\
=&L_{FF}x - \begin{bmatrix} \delta_1^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_n^1 \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_1 \end{bmatrix} \\
=&L_{FF}x - \begin{bmatrix} \delta_1^1 \\ \vdots \\ \delta_n^1 \end{bmatrix} w_1 \\
=&L_{FF}x + L_{FR}w_1
\end{aligned} \tag{3.67}$$

One can get

$$\begin{aligned}
&(L_F + \Delta^1)(x(t) - 1_n \otimes w_1(t)) \\
=&L_{FF}x(t) + L_{FR}w(t) \\
=&e(t)
\end{aligned} \tag{3.68}$$

Combining (3.66) and (3.68), one can get

$$(L_F + \Delta^1)(x(t) - 1_n \otimes w_1(t)) = e(t) \to 0, \tag{3.69}$$

and

$$(x(t) - 1_n \otimes w_1(t)) = (L_F + \Delta^1)^{-1}e(t) \to 0. \tag{3.70}$$

The proof is finished.■

Now we can get the following theorem:

Theorem 7: For the multi-agent systems with no leader or one or more, the aim of cooperation control will be reached for all initial values if the extended local neighbourhood synchronization error (3.55)(3.56) would satisfy

$$e_i(t) \to 0, i \in \mathcal{F}, as\ t \to +\infty \tag{3.71}$$

Proof: The theorem can be directly derived because of Lemma (15)-(17).■

The containment control is with more than one leader. It can be used in geography systems for unmanned aircraft.

3.3 Summary

In this chapter, the multi-agent systems with none or one or more leaders under a fixed topology are studied. We analyzed consensus, tracking control, and containment control, respectively. We prove that reaching the aims of these three kinds of cooperation is equal to that making some traditional control systems be globally uniformly asymptotically stable.

Every leadership style of multi-agent systems has its own advantages and disadvantages. For the simple consensus, its structure is uncomplicated. Reaching a consensus is its single aim. When the systems have more leaders, the association becomes more complicated. The systems can complete multiple tasks.

In this chapter, the result is limited to a fixed topology. That is to say, it cannot be extended to different types of the complex network directly. If we also adapt to state-jump technology, the results probably are extended to different types of complex networks.

Chapter 4

A Basis for Cooperation Analysis under Switched Topologies: Sequence-based Stability Analysis Methods of Switched Systems

4.1 Introduction

In this chapter, we study the stability of switched systems. The conclusion in this chapter is the basis of chapter 5.

Many results on switched systems are based on dwell time. However, different sequences contain different switching information, which is not well exploited in these studies. The proposed sequence-based methods can overcome this weakness.

In this chapter, we use the proposed sequence-based methods to analyze the stability of switched systems when the subsystems are (a) all stable; (b) stable and unstable; (c) all unstable.

4.2 Switching among Stable Subsystems

According to whether or not the switching of the controllers has a lag to the switching of the system models, the switched systems can be divided into two classes: (a) synchronous switched systems and (b) asynchronous switched systems.

4.2.1 Synchronous Switched Systems

Consider a class of continuous-time or discrete-time switched linear systems:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \tag{4.1}$$

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{4.2}$$

where $x(t)$ $(x(k)) \in \mathbb{R}^n$ and $u(t)$ $(u(k))$ are the state vector and the control input.

For systems (4.1) and (4.2), some useful results can be found in recent literature.

The Definition 7 in Chapter 2 has given the definition of globally uniformly exponential stability for general systems. For systems (4.1) and (4.2), it can rewritten as Definition 19.

Definition 19: [25] For a given system and a switching signal $\sigma(t)$ (or $\sigma(k)$), an equilibrium $x^* = 0$ is globally uniformly exponentially stable if there exist constants $\eta > 0$, $\gamma > 0$ (or $0 < \varsigma < 1$) for any initial conditions $x(t_0)$ (or $x(k_0)$), such that the solution of the system satisfies $\| x(t) \| \leq \eta \| x(t_0) \| e^{-\gamma(t-t_0)}, \forall t \geq t_0$ for the continuous-time situation (or $\| x(k) \| \leq \eta \| x(k_0) \| \varsigma^{(k-k_0)}, \forall k \geq k_0$ for the discrete-time case).

In order to analyze the stability of switched systems, [25] and [27] proposed following Definition 10 and Lemmas 18-21 .

Lemma 18: [25](Continuous-time Situation) For the system (4.1) when $u(t) \equiv 0$ and given constants $\lambda_p > 0$, $\mu_p > 1$, $p \in \mathcal{S}$. Suppose that there exist matrices $P_p > 0$, such that $\forall p \in \mathcal{S}$, $A_p^T P_p + P_p A_p \leq -\lambda_p P_p$ and $\forall (p,q) \in \mathcal{S} \times \mathcal{S}, p \neq q$, it holds that $P_p \leq \mu_p P_q$. Then the system is globally uniformly asymptotically stable with the MDADT condition

$$\tau_{ap} \geq \tau_{ap}^* = \ln \mu_p / \lambda_p. \tag{4.3}$$

Lemma 19: [25](Discrete-time Situation) For the system (4.2), when $u(k) \equiv 0$, and the given constants $1 > \lambda_p > 0$, $\mu_p \geq 1$, $p \in \mathcal{S}$, suppose that there exist matrices $P_p > 0$ such that $\forall p \in \mathcal{S}$, $A_p^T P_p A_p - P_p \leq -\lambda_p P_p$ and $\forall (p,q) \in \mathcal{S} \times \mathcal{S}, p \neq q$, it holds that $P_p \leq \mu_p P_q$. Then the system is globally uniformly asymptotically stable with

the MDADT condition

$$\tau_{ap} \geq \tau_{ap}^* = -\ln \mu_p/(1-\lambda_p). \tag{4.4}$$

In the situation of $u(t) \not\equiv 0$, the work [25] considered a class of feedback $u(t) = K_{\sigma(t)}x(t)$ for the continuous-time case or $u(k) = K_{\sigma(k)}x(k)$ for the discrete-time case, where K_p, $p \in \mathcal{S}$, is the controller gain. The closed-loop system is given by

$$\dot{x}(t) = (A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)})x(t). \tag{4.5}$$

$$x(k+1) = (A_{\sigma(k)} + B_{\sigma(k)}K_{\sigma(k)})x(k). \tag{4.6}$$

For the systems (4.5) and (4.6), important results were present in some recent papers.

Lemma 20: [25](Continuous-time Situation) For the system (4.5) and the given constants $\lambda_p > 0$, $\mu_p > 1$, $p \in \mathcal{S}$, suppose that there exist matrices $U_p > 0$, and T_p such that $\forall p \in \mathcal{S}$, $A_pU_p + B_pT_p + U_pA_p^T + T_p^TB_p^T \leq -\lambda_pU_p$ and $\forall (p,q) \in \mathcal{S}\times\mathcal{S}, p \neq q$, it holds that $U_q \leq \mu_pU_p$. Then there exists a set of stabilizing controllers such that the system (4.5) is globally uniformly asymptotically stable with the MDADT condition (4.3). The controller gain can be given by

$$K_p = T_pU_p^{-1}. \tag{4.7}$$

Lemma 21: [25](Discrete-time Situation) For the system (4.6) and the given constants $1 > \lambda_p > 0$, $\mu_p \geq 1$, $p \in \mathcal{S}$, suppose that there exist matrices $U_p > 0$, and T_p such that $\forall p \in \mathcal{S}$,

$$\begin{bmatrix} -U_p & A_pU_p + B_pT_p \\ * & -(1-\lambda_p)U_p \end{bmatrix} \leq 0$$

and $\forall (p,q) \in \mathcal{S}\times\mathcal{S}, p \neq q$, it holds that $U_q \leq \mu_pU_p$. Then there exists a set of stabilizing controllers such that the system (4.6) is globally uniformly asymptotically stable with the MDADT condition (4.4). The controller gain can be given by (4.7).

More items can be found in recent works [28, 27, 25]. By reviewing the existing results concerning switched systems, we found that the influence of switching sequence on dwell time has not been explicitly considered so far. In this subsection, two switching sequence-based methods are proposed.

This subsection includes three divisions. In the first division, we propose a sequence-based average subsequence dwell time method. In the second division, we propose a sequence-based average preceding dwell time method. In the third part, we use the two proposed methods to analyze the stability of linear switched systems.

Sequence-based Average Subsequence Dwell Time Method

In this division, we propose a sequence-based average subsequence dwell time method. According to this method, the setting of the parameter μ_p or μ refers to the switching sequence. The relationships among sequences, the stability of switched systems, and average dwell times are reconsidered.

The definition of the sequence-based average subsequence dwell time is given as follows.

Definition 20: For any switching times $t_2 > t_1 \geq 0$ with a switching signal $\sigma(t)$, let $N_{\sigma[p|q]}(t_2, t_1)$ stand for the number of the sequence that the p^{th} subsystem is activated immediately after the q^{th} subsystem over the time interval $[t_1, t_2)$, and let $T_{p,[p|q]}(t_2, t_1)$ denote the total running time of the p^{th} subsystem activated immediately after the q^{th} subsystems over the time interval $[t_1, t_2)$, $p \in \mathcal{S}$. It is said that $\sigma(t)$ has a sequence-based average subsequence dwell time (SBASDT) $\tau_{a(p,[p|q])}$ if there exist two numbers $N_{0(p,[p|q])}$ ($N_{0(p,[p|q])}$ is called the sequenced-based subsequence chatter bound here) and $\tau_{a(p,[p|q])}$ such that

$$N_{\sigma[p|q]}(t_2, t_1) \leq N_{0(p,[p|q])} + \frac{T_{p,[p|q]}(t_2, t_1)}{\tau_{a(p,[p|q])}} \tag{4.8}$$

Remark 1: Definition 20 constructs a novel set of switching signals referring to the sequence-based average subsequence dwell time. It considers the switching sequence of the switched systems. Obviously, this definition is different from Definition 10 as the successor of each individual mode has been explicitly considered here.

On the basis of the new Definition 20, we present the following two theorems.

Theorem 8: (Continuous-time Situation) Consider a switched system described by

$$\dot{x}(t) = f_{\sigma(t)}(x(t)). \tag{4.9}$$

Let $\lambda_q > 0$, $\lambda_p > 0$, $\mu_{[p|q]} > 1$ be given constants. Suppose there exist $\mathcal{C}^1$ functions $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}$, and functions k_{1p}, k_{2p}, k_{1q} and k_{2q} of class $\mathcal{K}_\infty$, such that $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}$ and $p \neq q$,

$$k_{1q}(\|x(t)\|) \leq V_q(x(t)) \leq k_{2q}(\|x(t)\|), \tag{4.10}$$

$$\dot{V}_q(x(t)) \leq -\lambda_q V_q(x(t)), \tag{4.11}$$

$$k_{1p}(\|x(t)\|) \leq V_p(x(t)) \leq k_{2p}(\|x(t)\|), \tag{4.12}$$

$$\dot{V}_p(x(t)) \leq -\lambda_p V_p(x(t)), \tag{4.13}$$

and

$$V_p(x(t_i)) \leq \mu_{[p|q]} V_q(x(t_i)), \tag{4.14}$$

then the system is GUAS for any switching signals with SBASDT

$$\tau_{a(p,[p|q])} \geq \tau^*_{a(p,[p|q])} = \ln \mu_{[p|q]} / \lambda_p. \tag{4.15}$$

Proof: Let $t_0 = 0$ and use t_1, t_2, t_3, $\cdots$ to denote the switching moments. According

to (4.13) and (4.14), for any time $t > t_0$ and $t \in [t_l, t_{l+1})$, $l \in Z_+$, we have

$$\begin{aligned}
&V_{\sigma(t)}(x(t)) \\
&\leq \exp\left\{-\lambda_{\sigma(t_l)}(t-t_l)\right\} V_{\sigma(t_l)}(x(t_l)) \\
&\leq \exp\left\{-\lambda_{\sigma(t_l)}(t-t_l)\right\} \mu_{\sigma(t_l)|\sigma(t_l^-)} V_{\sigma(t_l{}^-)}(x(t_l{}^-)) \\
&\leq \exp\left\{-\lambda_{\sigma(t_l)}(t-t_l)\right\} \mu_{\sigma(t_l)|\sigma(t_l^-)} \\
&\quad \times \exp\left\{-\lambda_{\sigma(t_{l-1})}(t_l-t_{l-1})\right\} V_{\sigma(t_{l-1})}(x(t_{l-1})) \\
&= \mu_{\sigma(t_l)|\sigma(t_l^-)} \exp\left\{-\lambda_{\sigma(t_l)}(t-t_l)\right\} \\
&\quad \times \exp\left\{-\lambda_{\sigma(t_{l-1})}(t_l-t_{l-1})\right\} V_{\sigma(t_{l-1})}(x(t_{l-1}))
\end{aligned} \tag{4.16}$$

Recursively, we have

$$\begin{aligned}
&V_{\sigma(t)}(x(t)) \\
&\leq \left\{\prod_{k=1}^{l} \mu_{\sigma(t_k)|\sigma(t_k^-)}\right\} \exp\left\{-\lambda_{\sigma(t_l)}(t-t_l) \cdots -\lambda_{\sigma(t_1)}(t_2-t_1)\right\} \\
&\quad \times \exp\left\{-\lambda_{\sigma(0)}(t_1-0)\right\} V_{\sigma(0)}(x(0)).
\end{aligned} \tag{4.17}$$

On the basis of the set $\mathcal{S} = \{1, 2, \cdots, s\}$ (where s is the total amount of subsystems), we create a new set $\mathcal{S}'$ which is the Cartesian product of $\mathcal{S}$ and $\mathcal{S}$, i.e., $\mathcal{S}' = \mathcal{S} \times \mathcal{S} = \{(p, q) : p \in \mathcal{S}, and\, q \in \mathcal{S}\}$. It is obvious that the set $\mathcal{S}'$ includes all the possible ordered pairs $(\sigma(t_i), \sigma(t_i^-))$, $i \in Z_+$, and the total number of the elements in $\mathcal{S}'$ is $s' = s(s-1)$.

Consider the system switching up to l, and define a new related set $\mathcal{S}'' = \{(p, q) : p \in \sigma(t_i), q \in \sigma(t_i^-), i = 1, 2, 3, \cdots, l\}$. Obviously, $\mathcal{S}'' \subseteq \mathcal{S}'$.

If $\sigma(t_n) = p$ and $\sigma(t_n^-) = q$, we denote $\mu_{\sigma(t_n)|\sigma(t_n^-)}$ as $\mu_{[p|q]}$, $\forall n = 1, 2, \cdots, l$. The purpose of the formal transformation is to make it more readable under some situations. Furthermore, we list and number all the elements of the set $\mathcal{S}''$, i.e. we use $[p|q]_{(k)}$ to denote the pair (p, q) which is the k^{th} element of the set $\mathcal{S}''$. $N_{\sigma[p|q]_{(k)}}(t, 0)$ and $T_{p,[p|q]_{(k)}}(t, 0)$ denote the activated numbers and total subsequence dwell times of the k^{th} element in the time interval $[0, t)$ respectively. Assuming that the number of elements in $\mathcal{S}''$ is s'', we have $k \in \{1, 2, \cdots, s''\}$, $s'' \leq s'$, and $s'' \leq l$.

Hence,

$$\begin{aligned} & V_{\sigma(t)}(x(t)) \\ \leq & \left\{\prod_{k=1}^{s''} \mu_{[p|q]_{(k)}}^{N_{\sigma[p|q]_{(k)}}(t,0)}\right\} \exp\left\{\sum_{k=1}^{s''} -\lambda_{p,[p|q]_{(k)}} T_{p,[p|q]_{(k)}}(t,0)\right\} \\ & \times \exp\left\{-\lambda_{\sigma(0)}(t_1-0)\right\} V_{\sigma(0)}(x(0)). \end{aligned} \tag{4.18}$$

On the basis of (4.8), we can get

$$\begin{aligned} & V_{\sigma(t)}(x(t)) \\ \leq & \left\{\prod_{k=1}^{s''} \mu_{[p|q]_{(k)}}^{N_{0[p|q]_{(k)}}+\frac{T_{p,[p|q]_{(k)}}(t,0)}{\tau_{a(p,[p|q]_{(k)})}}}\right\} \exp\left\{\sum_{k=1}^{s''} -\lambda_{p,[p|q]_{(k)}} T_{p,[p|q]_{(k)}}(t,0)\right\} \\ & \times \exp\left\{-\lambda_{\sigma(0)}(t_1-0)\right\} V_{\sigma(0)}(x(0)) \\ \leq & \exp\left\{\sum_{k=1}^{s''} N_{0[p|q]_{(k)}} \ln \mu_{[p|q]_{(k)}}\right\} \exp\left\{\sum_{k=1}^{s''} \frac{T_{p,[p|q]_{(k)}}(t,0)}{\tau_{a(p,[p|q]_{(k)})}} \ln \mu_{[p|q]_{(k)}}\right\} \\ & \times \exp\left\{\sum_{k=1}^{s''} -\lambda_{p,[p|q]_{(k)}} T_{p,[p|q]_{(k)}}(t,0)\right\} \exp\left\{-\lambda_{\sigma(0)}(t_1-0)\right\} V_{\sigma(0)}(x(0)) \\ = & \exp\left\{\sum_{k=1}^{s''} N_{0[p|q]_{(k)}} \ln \mu_{[p|q]_{(k)}}\right\} \exp\left\{\sum_{k=1}^{s''} (\frac{\ln \mu_{[p|q]_{(k)}}}{\tau_{a(p,[p|q]_{(k)})}} - \lambda_{p,[p|q]_{(k)}}) T_{p,[p|q]_{(k)}}(t,0)\right\} \\ & \times \exp\left\{-\lambda_{\sigma(0)}(t_1-0)\right\} V_{\sigma(0)}(x(0)). \end{aligned} \tag{4.19}$$

If for all $k \in \{1, 2, \cdots, s''\}$

$$\tau_{a(p,[p|q]_{(k)})} \geq \ln \mu_{[p|q]_{(k)}} / \lambda_{p,[p|q]_{(k)}}, \tag{4.20}$$

then, we have

$$\begin{aligned} & V_{\sigma(t)}(x(t)) \\ \leq & \exp\left\{\sum_{k=1}^{s''} N_{0[p|q]_{(k)}} \ln \mu_{[p|q]_{(k)}}\right\} \exp\left\{\max\left\{\max_k\left\{\frac{\ln \mu_{[p|q]_{(k)}}}{\tau_{a(p,[p|q]_{(k)})}} - \lambda_{p,[p|q]_{(k)}}\right\}, -\lambda_{\sigma(0)}\right\} t\right\} \\ & \times V_{\sigma(0)}(x(0)). \end{aligned} \tag{4.21}$$

Therefore, we conclude that $V_{\sigma(t)}(x(t))$ convergences to zero as $t \to +\infty$ if the SBASDT satisfies the condition (4.15). Then, the asymptotic stability can be deduced according to Definition 19. ■

Theorem 9: (Discrete-time Situation) Consider a discrete switched system described by

$$x(k+1) = f_{\sigma(k)}(x(k)), \tag{4.22}$$

and let $1 > \lambda_q > 0$, $1 > \lambda_p > 0$, $\mu_{[p|q]} > 1$ be given constants. Assume there exist $\mathcal{C}^1$ functions $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$ and functions k_{1p}, k_{2p}, k_{1q} and k_{2q} of class $\mathcal{K}_\infty$, such that $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{S} \times \mathcal{S}$, $p \neq q$,

$$k_{1q}(\|x(k)\|) \leq V_q(x(k)) \leq k_{2q}(\|x(k)\|), \tag{4.23}$$

$$V_q(x(k+1)) - V_q(x(k)) \leq -\lambda_q V_q(x(k)), \tag{4.24}$$

$$k_{1p}(\|x(k)\|) \leq V_p(x(k)) \leq k_{2p}(\|x(k)\|), \tag{4.25}$$

$$V_p(x(k+1)) - V_p(x(k)) \leq -\lambda_p V_p(x(k)), \tag{4.26}$$

and

$$V_p(x(k_i)) \leq \mu_{[p|q]} V_q(x(k_i)), \tag{4.27}$$

then the system is GUAS for any switching signals with SBASDT

$$\tau_{a(p,[p|q])} \geq \tau^*_{a(p,[p|q])} = -\ln \mu_{[p|q]} / \ln(1 - \lambda_p). \tag{4.28}$$

Proof: This theorem can be proved by using a method similar to the proof of the continue-time situation. Due to the space limitation, the detailed proof is omitted here. ■

Remark 2: Actually, the conditions (4.10)-(4.13) (or (4.23)-(4.26)) are equivalent to the corresponding conditions of the MDADT method [25]. This remark is also valid for the sequence-based average preceding dwell time method.

Remark 3: Compared to the existing results, new methods allow a subsystem to have different switching restrictions $\mu_{[p|q]}$ for different sequences.

Remark 4: MDADT is a popular method for the analysis of the switched systems. For the p^{th} subsystem, if we set the values of $\mu_{[p|q]}$ as a fixed value μ_p, without considering the preceding different q^{th} subsystems, the SBASDT retrogresses to MDADT. Therefore, the method in this note has less conservativeness and restriction than MDADT.

Sequence-based Average Preceding Dwell Time Method

In this division, we propose another sequence-based method, the sequence-based average preceding dwell time method.

The definition of the sequence-based average preceding dwell time is given as follows.

Definition 21: For any switching times $t_2 > t_1 \geq 0$ with a switching signal $\sigma(t)$, let $N_{\sigma[p|q]}(t_2, t_1)$ stand for the number of the sequence that the p^{th} subsystem is activated immediately after the q^{th} subsystem over the time interval $[t_1, t_2)$, and let $T_{q,[p|q]}(t_2, t_1)$ denote the total running time of the q^{th} subsystem when the p^{th} subsystem is activated immediately after the q^{th} subsystems over the time interval $[t_1, t_2)$, $p \in \mathcal{S}$. It is said that $\sigma(t)$ has a sequence-based average preceding dwell time $\tau_{a(q,[p|q])}$ if there exist two numbers $N_{0(q,[p|q])}$ ($N_{0(q,[p|q])}$ is called the sequenced-based preceding chatter bound here) and $\tau_{a(q,[p|q])}$ such that

$$N_{\sigma[p|q]}(t_2, t_1) \leq N_{0(q,[p|q])} + \frac{T_{q,[p|q]}(t_2, t_1)}{\tau_{a(q,[p|q])}} \tag{4.29}$$

Remark 5: Definition 21 also constructs a novel set of switching signals referring to the sequence-based average preceding dwell time method (SBAPDT). Both Definition 20 and Definition 21 are based on switching sequence.

On the basis of Definition 21, we can get the following two theorems.

Theorem 10: (Continuous-time Situation) Consider a switched system depicted

by (4.9), and let $\lambda_q > 0$, $\lambda_p > 0$, $\mu_{[p|q]} > 1$ be given constants. Suppose there exist $\mathcal{C}^1$ functions $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}$, and functions k_{1p}, k_{2p}, k_{1q} and k_{2q} of class $\mathcal{K}_\infty$, such that $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}$, $p \neq q$,

$$k_{1q}(\|x(t)\|) \leq V_q(x(t)) \leq k_{2q}(\|x(t)\|), \tag{4.30}$$

$$\dot{V}_q(x(t)) \leq -\lambda_q V_q(x(t)), \tag{4.31}$$

$$k_{1p}(\|x(t)\|) \leq V_p(x(t)) \leq k_{2p}(\|x(t)\|), \tag{4.32}$$

$$\dot{V}_p(x(t)) \leq -\lambda_p V_p(x(t)), \tag{4.33}$$

and

$$V_p(x(t_i)) \leq \mu_{[p|q]} V_q(x(t_i)), \tag{4.34}$$

then the system is GUAS for any switching signals with SBAPDT

$$\tau_{a(q,[p|q])} \geq \ln \mu_{[p|q]} / \lambda_q. \tag{4.35}$$

Proof: Let $t_0 = 0$ and use t_1, t_2, t_3, $\cdots$ to denote the subsystems switching times. According to (4.33) and (4.34), for any time $t > t_0$ and $t \in [t_l, t_{l+1})$, $l \in Z_+$, we have systems (4.16)-(4.17).

We also use the definitions of $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{S}''$, and $[p|q]_{(k)}$ from the proof of Theorem 8. The symbol $T_{q,[p|q]_{(k)}}(t,0)$ denotes the total dwell times of the q^{th}subsystem for the k^{th} sequence union in the time interval $[0,t)$.

Hence,

$$\begin{aligned} & V_{\sigma(t)}(x(t)) \\ \leq & \left\{ \prod_{k=1}^{s''} \mu_{[p|q]_{(k)}}^{N_{\sigma[p|q]_{(k)}}(t,0)} \right\} \exp\left\{ \sum_{k=1}^{s''} -\lambda_{q,[p|q]_{(k)}} T_{q,[p|q]_{(k)}}(t,0) \right\} \\ & \times \exp\left\{ -\lambda_{\sigma(t_l)}(t - t_l) \right\} V_{\sigma(0)}(x(0)). \end{aligned} \tag{4.36}$$

According to (4.29), one can get

$$
\begin{aligned}
&V_{\sigma(t)}(x(t)) \\
&\leq \left\{ \prod_{k=1}^{s''} \mu_{[p|q]_{(k)}}^{N_{0(q,[p|q]_{(k)})}+\frac{T_{q,[p|q]_{(k)}}(t,0)}{\tau_{a(q,[p|q]_{(k)})}}} \right\} \exp\left\{ \sum_{k=1}^{s''} -\lambda_{q,[p|q]_{(k)}} T_{q,[p|q]_{(k)}}(t,0) \right\} \\
&\quad \times \exp\left\{-\lambda_{\sigma(t_l)}(t-t_l)\right\} V_{\sigma(0)}(x(0)) \\
&\leq \exp\left\{ \sum_{k=1}^{s''} N_{0(q,[p|q]_{(k)})} \ln \mu_{[p|q]_{(k)}} \right\} \exp\left\{ \sum_{k=1}^{s''} \frac{T_{q,[p|q]_{(k)}}(t,0)}{\tau_{a(q,[p|q]_{(k)})}} \ln \mu_{[p|q]_{(k)}} \right\} \\
&\quad \times \exp\left\{ \sum_{k=1}^{s''} -\lambda_{q,[p|q]_{(k)}} T_{q,[p|q]_{(k)}}(t,0) \right\} \exp\left\{-\lambda_{\sigma(t_l)}(t-t_l)\right\} V_{\sigma(0)}(x(0)) \\
&= \exp\left\{ \sum_{k=1}^{s''} N_{0(q,[p|q]_{(k)})} \ln \mu_{[p|q]_{(k)}} \right\} \exp\left\{ \sum_{k=1}^{s''} (\frac{\ln \mu_{[p|q]_{(k)}}}{\tau_{a(q,[p|q]_{(k)})}} - \lambda_{q,[p|q]_{(k)}}) T_{q,[p|q]_{(k)}}(t,0) \right\} \\
&\quad \times \exp\left\{-\lambda_{\sigma(t_l)}(t-t_l)\right\} V_{\sigma(0)}(x(0)).
\end{aligned}
\tag{4.37}
$$

If there exist constants

$$
\tau_{a(q,[p|q]_{(k)})} \geq \ln \mu_{[p|q]_{(k)}} / \lambda_{q,[p|q]_{(k)}},
$$

we have

$$
\begin{aligned}
&V_{\sigma(t)}(x(t)) \\
&\leq \exp\left\{ \sum_{k=1}^{s''} N_{0(q,[p|q]_{(k)})} \ln \mu_{[p|q]_{(k)}} \right\} \exp\left\{ \max\left\{ \max_{k} \left\{ \frac{\ln \mu_{[p|q]_{(k)}}}{\tau_{a(q,[p|q]_{(k)})}} - \lambda_{q,[p|q]_{(k)}} \right\}, -\lambda_{\sigma(t_l)} \right\} t \right\} \\
&\quad \times V_{\sigma(0)}(x(0))
\end{aligned}
\tag{4.38}
$$

Therefore, we conclude that $V_{\sigma(t)}(x(t))$ convergences to zero as $t \to +\infty$ if the SBAPDT satisfies the condition (4.35). Then, the asymptotic stability can be deduced according to Definition 19. ■

Theorem 11: (Discrete-time Situation) Consider a discrete switched system described by (4.22), and let $1 > \lambda_q > 0$, $1 > \lambda_p > 0$, $\mu_{[p|q]} > 1$ be given constants. Assume there exist $\mathcal{C}^1$ functions $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, and functions k_{1p}, k_{2p}, k_{1q} and k_{2q} of class $\mathcal{K}_\infty$, such that $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{S} \times \mathcal{S}$, $p \neq q$,

$$
k_{1q}(\|x(k)\|) \leq V_q(x(k)) \leq k_{2q}(\|x(k)\|), \tag{4.39}
$$

$$V_q(x(k+1)) - V_q(x(k)) \leq -\lambda_q V_q(x(k)), \tag{4.40}$$

$$k_{1p}(\|x(k)\|) \leq V_p(x(k)) \leq k_{2p}(\|x(k)\|), \tag{4.41}$$

$$V_p(x(k+1)) - V_p(x(k)) \leq -\lambda_p V_p(x(k)), \tag{4.42}$$

and

$$V_p(x(k_i)) \leq \mu_{[p|q]} V_q(x(k_i)), \tag{4.43}$$

then the system is GUAS for any switching signals with SBAPDT

$$\tau_{a(q,[p|q])} \geq -\ln \mu_{[p|q]} / \ln(1-\lambda_q). \tag{4.44}$$

Proof: This theorem can be proved by using a method similar to the proof of the continue-time situation. To save space, the proof is omitted here. ■

Linear Switched Systems with two Sequence-based Average Dwell Time Methods

In this division, we study the stability of linear switched systems using SBASDT and SBAPDT.

First, we consider the general linear system (4.1) and (4.2). When $u(t) \equiv 0$, we can obtain Corollary 1 and 2 below.

Corollary 1: (Continuous-time Situation) Consider a switched system depicted by (4.1) with $u(t) \equiv 0$. Let $\lambda_q > 0$, $\lambda_p > 0$, $\mu_{[p|q]} > 1$ be given constants. Suppose that there exist matrices $P_q > 0$, $P_p > 0$, $p, q \in \mathcal{S}$, such that $\forall p \in \mathcal{S}$, $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}$, $p \neq q$,

$$A_q^T P_q + P_q A_q \leq -\lambda_q P_q, \tag{4.45}$$

$$A_p^T P_p + P_p A_p \leq -\lambda_p P_p, \tag{4.46}$$

and

$$P_p \leq \mu_{[p|q]} P_q, \tag{4.47}$$

then the system is GUAS for any switching signals with SBASDT

$$\tau_{a(p,[p|q])} \geq \tau^*_{a(p,[p|q])} = \ln \mu_{[p|q]}/\lambda_p. \tag{4.48}$$

or with SBAPDT

$$\tau_{a(q,[p|q])} \geq \tau^*_{a(q,[p|q])} = \ln \mu_{[p|q]}/\lambda_q. \tag{4.49}$$

Proof: This corollary can be proved according to Theorems 8 and 10 and the proof of Lemma 18. Detailed proof is omitted here. ■

Corollary 2: (Discrete-time Situation) Consider a discrete switched system described by (4.2) with $u(t) \equiv 0$. Let $1 > \lambda_q > 0$, $1 > \lambda_p > 0$, $\mu_{[p|q]} > 1$ be given constants. Suppose that there exist matrices $P_q > 0$, $P_p > 0$, $p, q \in \mathcal{S}$, such that $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{S} \times \mathcal{S}$, $p \neq q$,

$$A_q^T P_q A_q - P_q \leq -\lambda_q P_q \tag{4.50}$$

$$A_p^T P_p A_p - P_p \leq -\lambda_p P_p \tag{4.51}$$

and

$$P_p \leq \mu_{[p|q]} P_q, \tag{4.52}$$

then the system is GUAS for any switching signals with SBASDT

$$\tau_{a(p,[p|q])} \geq -\ln \mu_{[p|q]}/\ln(1 - \lambda_p). \tag{4.53}$$

or with SBAPDT

$$\tau_{a(q,[p|q])} \geq -\ln \mu_{[p|q]}/\ln(1 - \lambda_q). \tag{4.54}$$

Proof: This corollary can be proved by using a method similar to the proof of the continue-time situation. Detailed proof is omitted here. ■

For the case $u(t) \not\equiv 0$, we have Corollaries 3 and 4. Their proofs are very similar to those of Corollaries 1 and 2, respectively, and hence are omitted.

Corollary 3: (Continuous-time Situation) Consider a switched system depicted by (4.1) with $u(t) = K_{\sigma(t)}x(t)$. Let $\lambda_q > 0$, $\lambda_p > 0$, $\mu_{[p|q]} > 1$ be given constants. Suppose that there exist matrices $P_q > 0$, $P_p > 0$, $p, q \in \mathcal{S}$, such that $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}$, $p \neq q$,

$$(A_q + B_q K_q)^T P_q + P_q(A_q + B_q K_q) \leq -\lambda_q P_q, \tag{4.55}$$

$$(A_p + B_p K_p)^T P_p + P_p(A_p + B_p K_p) \leq -\lambda_p P_p, \tag{4.56}$$

and

$$P_p \leq \mu_{[p|q]} P_q, \tag{4.57}$$

then the system is GUAS for any switching signals with the SBASDT switching condition (4.48) or with the SBAPDT switching condition (4.49).

Corollary 4: (Discrete-time Situation) For a discrete switched system (4.2) and $u(k) = K_{\sigma(k)}x(k)$, let $1 > \lambda_q > 0$, $1 > \lambda_p > 0$, $\mu_{[p|q]} > 1$ be given constants. Suppose that there exist matrices $P_q > 0$, $P_p > 0$, $p, q \in \mathcal{S}$, such that $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{S} \times \mathcal{S}$, $p \neq q$

$$(A_q + B_q K_q)^T P_q (A_q + B_q K_q) - P_q \leq -\lambda_q P_q \tag{4.58}$$

$$(A_p + B_p K_p)^T P_p (A_p + B_p K_p) - P_p \leq -\lambda_p P_p \tag{4.59}$$

and

$$P_p \leq \mu_{[p|q]} P_q, \tag{4.60}$$

then the system is GUAS for any switching signals with the SBASDT switching condition (4.53) or with the SBAPDT switching condition. (4.54).

If we want to get a solution on controller gain $K_{\sigma(t)}$ and $K_{\sigma(k)}$, the following two corollaries can provide solutions.

Corollary 5: (Continuous-time Situation) Consider a switched system depicted by (4.1), and let $\lambda_q > 0$, $\lambda_p > 0$, $\mu_{[p|q]} > 1$ be given constants. Suppose that there

exist matrices $U_p > 0$, and T_p, $p \in \mathcal{S}$, such that $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}$, $p \neq q$,

$$A_p U_p + B_p T_p + U_p A_p^T + T_p^T B_p^T \leq -\lambda_p U_p \tag{4.61}$$

$$A_q U_q + B_q T_q + U_q A_q^T + T_q^T B_q^T \leq -\lambda_q U_q \tag{4.62}$$

and

$$U_q \leq \mu_{[p|q]} U_p. \tag{4.63}$$

Then there exists a set of stabilizing controllers such that the system is globally uniformly asymptotically stable with the with SBASDT (4.48) or with SBAPDT condition (4.49).

The controller gain can be calculated by

$$K_p = T_p U_p^{-1}. \tag{4.64}$$

$$K_q = T_q U_q^{-1}. \tag{4.65}$$

Proof: This corollary can be established according to Corollary 3 and the proof of lemma 20. Detailed proof is omitted here. ■

Corollary 6: (Discrete-time Situation) Consider a discrete switched system described by (4.2), and let $1 > \lambda_q > 0$, $1 > \lambda_p > 0$, $\mu_{[p|q]} > 1$ be given constants. Suppose that there exist matrices $U_q > 0$, $U_p > 0$, T_p, T_q, $p, q \in \mathcal{S}$, such that $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{S} \times \mathcal{S}$, $p \neq q$,

$$\begin{bmatrix} -U_p & A_p U_p + B_p T_p \\ * & -(1-\lambda_p) U_p \end{bmatrix} \leq 0 \tag{4.66}$$

$$\begin{bmatrix} -U_q & A_q U_q + B_q T_q \\ * & -(1-\lambda_q) U_q \end{bmatrix} \leq 0 \tag{4.67}$$

and

$$U_q \leq \mu_{[p|q]} U_p, \tag{4.68}$$

Then there exists a set of stabilizing controllers such that the system is globally uniformly asymptotically stable with the SBASDT condition (4.53) or with the SBAPDT condition (4.54).

The controller gain can be designed through (4.64)-(4.65).

Proof: This corollary can be derived according to Corollary 4 and the proof of lemma 21. Detailed proof is omitted here. ■

4.2.2 Asynchronous Systems

Suppose that there is a time lag T between switched controllers and system modes. Consider the closed-loop asynchronous switched system in

a) continuous-time case:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}K_{\sigma(t-T)}x(t); \tag{4.69}$$

b) discrete-time case:

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}K_{\sigma(k-T)}x(k). \tag{4.70}$$

In the above systems, $x(t)$ (or $x(k)$) $\in \mathbb{R}^n$ is the state vector; $A_{\sigma(t)}$ (or $A_{\sigma(k)}$) and $B_{\sigma(t)}$ (or $B_{\sigma(k)}$) are the system and input matrix; $u(t) = K_{\sigma(t-T)}x(t)$ (or $u(k) = K_{\sigma(k-T)}x(k)$) is the control input.

In order to analyze the stability of the above systems, [28] used the average dwell time (ADT) to develop an effective switching strategy. Its definition is given as Definition 9 in Chapter 2.

In this subsection, we propose two novel approaches for stabilizing asynchronous switched systems. We first present novel results for general situations and then apply them to the linear case.

General Switched Systems

We first consider general asynchronous switched systems. Two novel concepts are given in Definitions 20 and 21.

In this subsection, let $T_{\downarrow}(t_l, t_{l+1})$ and $T_{\uparrow}(t_l, t_{l+1})$ denote the unions (or the length of the unions) of the decreasing and increasing time (including unchanging time) of the Lyapunov function within the interval $[t_l, t_{l+1})$. For different switching sequences or subsystems, $T_{\downarrow}(t_l, t_{l+1})$ and $T_{\uparrow}(t_l, t_{l+1})$ have different subscripts.

To simplify the presentation, we assume that a proper decreasing (increasing) Lyapunov function can be found in the synchronous (asynchronous) switching time section. We can then obtain the following four theorems for general asynchronous switched systems.

Theorem 12: For a switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \tag{4.71}$$

let $\alpha_{[p|q]} > 0,\ \lambda_q > 0,\ \lambda_p > 0,\ \mu_{[p|q]} > 1$ be given constants. For $\forall s \in \mathcal{S}$, suppose there exist functions $k_{1s},\ k_{2s}$ of class $\mathcal{K}_\infty$ and $\mathcal{C}^1$ functions $V_s : \mathbb{R}^n \to \mathbb{R}$, such that

$$k_{1s}(\|x(t)\|) \leq V_s(x(t)) \leq k_{2s}(\|x(t)\|), \tag{4.72}$$

$\forall i = 1,\ 2,\ \cdots, \forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S},\ p \neq q,$

$$\dot{V}_p(x(t)) \leq \begin{cases} -\lambda_p V_p(x(t)), \forall t \in T_{\downarrow}[t_i, t_{i+1}), \\ \alpha_{[p|q]} V_p(x(t)), \forall t \in T_{\uparrow}[t_i, t_{i+1}), \end{cases} \tag{4.73}$$

and

$$V_p(x(t_i)) \leq \mu_{[p|q]} V_q(x(t_i)). \tag{4.74}$$

Then the system is globally uniformly asymptotically stable for any switching signals with SBASDT

$$\tau_{a(p,[p|q])} \geq \tau^*_{a(p,[p|q])} = \frac{T_{[p|q]_\max\uparrow}(\lambda_p + \alpha_{[p|q]}) + \ln \mu_{[p|q]}}{\lambda_p}, \tag{4.75}$$

where $T_{[p|q]_\max\uparrow} \triangleq \max\limits_{l,\sigma(t_l)=p,\ \sigma(t_l^-)=q} T_\uparrow(t_l, t_{l+1})$, $\forall l \in Z_+$.

Proof: Let $t_0 = 0$. For any $t > 0$, $t \in [t_l, t_{l+1})$, $l \in Z_+$, we define two novel sets: $\mathcal{S}' \triangleq \{(p,q); p \in \mathcal{S}, q \in \mathcal{S}, p \neq q\}$; $\mathcal{S}'' \triangleq \{(\sigma(t_i), \sigma(t_i^-)) \in \mathcal{S} \times \mathcal{S}, i = 1,\ 2,\ 3,\ \cdots,\ l\}$. The total numbers of the set $\mathcal{S}'$ and $\mathcal{S}''$ are s' and s''. Therefore, $S'' \subseteq S'$, $s' = s(s-1)$, $s'' \leq l$ and $s'' \leq s'$.

For all the elements in the set $\mathcal{S}''$, we order and list them. We use the symbol $[p|q]_{(k)}$ to represent the k^{th} element. The total aroused numbers of the k^{th} element is represented by $N_{\sigma[p|q]_{(k)}}(0,t)$. The sum of dwell time of the p^{th} subsystems for the sequence $[p|q]$ is denoted by $T_{p,[p|q]_{(k)}}(0,t)$.

When $\sigma(t_j) = p$, $\sigma(t_j^-) = q$, $\forall j = 1,\ 2,\ \cdots,\ l$, and the element (p,q) is the k^{th} element, we rewrite $\mu_{\sigma(t_j)}$ as $\mu_{[p|q]_{(k)}}$ to denote the switching sequence. In this situation, we also use $\alpha_{[p|q]_{(k)}}$ to denote $\alpha_{\sigma(t_j)}$.

According to (4.73) and (4.74), it holds that

$$\begin{aligned} & V_{\sigma(t)}(x(t)) \\ & \leq \exp\big\{-\lambda_{\sigma(t_l)}T_\downarrow(t_l,t) + \alpha_{\sigma(t_l)}T_\uparrow(t_l,t)\big\}V_{\sigma(t_l)}(x(t_l)) \\ & \leq \exp\left\{-\lambda_{\sigma(t_l)}T_\downarrow(t_l,t) + \alpha_{\sigma(t_l)}T_\uparrow(t_l,t)\right\}\mu_{\sigma(t_l)|\sigma(t_l^-)}V_{\sigma(t_l^-)}(x(t_l^-)). \end{aligned} \tag{4.76}$$

Recursively,

$$\begin{aligned} & V_{\sigma(t)}(x(t)) \\ & \leq \left\{\prod_{j=1}^{l}\mu_{\sigma(t_j)|\sigma(t_j^-)}\right\}\exp\left\{-\lambda_{\sigma(t_l)}T_\downarrow(t_l,t) - \cdots - \lambda_{\sigma(t_1)}T_\downarrow(t_1,t_2)\right\} \\ & \times \exp\left\{\alpha_{\sigma(t_l)}T_\uparrow(t_l,t) + \cdots + \alpha_{\sigma(t_1)}T_\uparrow(t_1,t_2)\right\}\exp\left\{-\lambda_{\sigma(0)}(t_1-0)\right\}V_{\sigma(0)}(x(0)). \end{aligned} \tag{4.77}$$

We can also obtain the following formal transformation:

$$\prod_{j=1}^{l}\mu_{\sigma(t_j)|\sigma(t_j^-)} = \prod_{k=1}^{s''}\mu_{[p|q]_{(k)}}^{N_{[p|q]_{(k)}}(0,t)}. \tag{4.78}$$

Therefore,

$$\begin{aligned}
&V_{\sigma(t)}(x(t)) \\
&\leq \left\{\prod_{k=1}^{s''} \mu_{[p|q]_{(k)}}^{N_{[p|q]_{(k)}}(0,t)}\right\} \exp\left\{\sum_{k=1}^{s''} -\lambda_p T_{p,[p|q]_{(k)}\downarrow}(0,t)\right\} \exp\left\{\sum_{k=1}^{s''} \alpha_{[p|q]_{(k)}} T_{p,[p|q]_{(k)}\uparrow}(0,t)\right\} \\
&\times \exp\left\{-\lambda_{\sigma(0)}(t_1-0)\right\} V_{\sigma(0)}(x(0)).
\end{aligned} \tag{4.79}$$

Without affecting the conclusion, we assume that there is no asynchronous switching between the first switching time t_1 and the initial time 0. Hence,

$$\sum_{k=1}^{s''} \alpha_{[p|q]_{(k)}} T_{p,[p|q]_{(k)}\uparrow}(0,t) = \sum_{k=1}^{s''} \alpha_{[p|q]_{(k)}} T_{[p|q]_{(k)}\uparrow}(0,t),$$

where $T_{[p|q]_{(k)}\uparrow}(0,t)$ stands for the total asynchronous switching time (or the increasing/unchanging time of Lyapunov functions) in the time internal $[0,t)$.

According to (4.8), one has

$$\begin{aligned}
&V_{\sigma(t)}(x(t)) \\
&\leq \left\{\prod_{k=1}^{s''} \mu_{[p|q]_{(k)}}^{N_{0[p|q]_{(k)}} + \frac{T_{p,[p|q]_{(k)}}(0,t)}{\tau_{a(p,[p|q]_{(k)})}}}\right\} \exp\left\{\sum_{k=1}^{s''} -\lambda_p T_{p,[p|q]_{(k)}\downarrow}(0,t) + \alpha_{[p|q]_{(k)}} T_{[p|q]_{(k)}\uparrow}(0,t)\right\} \\
&\times \exp\left\{-\lambda_{\sigma(0)}(t_1-0)\right\} V_{\sigma(0)}(x(0)).
\end{aligned} \tag{4.80}$$

Since $T_{p,[p|q]_{(k)}}(0,t) = T_{[p|q]_{(k)}\uparrow}(0,t) + T_{p,[p|q]_{(k)}\downarrow}(0,t)$, it is not difficult to obtain

$$\begin{aligned}
&V_{\sigma(t)}(x(t)) \\
&\leq \exp\left\{\sum_{k=1}^{s''} N_{0[p|q]_{(k)}} \ln \mu_{[p|q]_{(k)}}\right\} \exp\left\{\sum_{k=1}^{s''} \left(\frac{\ln \mu_{[p|q]_{(k)}}}{\tau_{a(p,[p|q]_{(k)})}} - \lambda_p\right) T_{p,[p|q]_{(k)}}(0,t)\right\} \\
&\times \exp\left\{\sum_{k=1}^{s''} (\lambda_p + \alpha_{[p|q]_{(k)}}) T_{[p|q]_{(k)}\uparrow}(0,t)\right\} \exp\left\{-\lambda_{\sigma(0)}(t_1-0)\right\} V_{\sigma(0)}(x(0)).
\end{aligned} \tag{4.81}$$

According to the definition of $T_{[p|q]_\max\uparrow}$, one can directly obtain

$$T_{[p|q]_{(k)}\uparrow}(0,t) \leq N_{\sigma[p|q]_{(k)}}(0,t) T_{[p|q]_{(k)}_max\uparrow}. \tag{4.82}$$

According to (4.8) and (4.82), one can arrive at

$$\begin{aligned}
&V_{\sigma(t)}(x(t)) \\
&\leq \exp\left\{\sum_{k=1}^{s''} N_{0[p|q]_{(k)}} \ln \mu_{[p|q]_{(k)}}\right\} \exp\left\{\sum_{k=1}^{s''} (\frac{\ln \mu_{[p|q]_{(k)}}}{\tau_{a(p,[p|q]_{(k)})}} - \lambda_p) T_{p,[p|q]_{(k)}}(0,t)\right\} \\
&\times \exp\left\{\sum_{k=1}^{s''} (\lambda_p + \alpha_{[p|q]_{(k)}}) N_{0[p|q]_{(k)}} T_{[p|q]_{(k)}_max\uparrow}\right\} \\
&\times \exp\left\{\sum_{k=1}^{s''} (\lambda_p + \alpha_{[p|q]_{(k)}}) \frac{T_{p,[p|q]_{(k)}}(0,t)}{\tau_{a(p,[p|q]_{(k)})}} T_{[p|q]_{(k)}_max\uparrow}\right\} \\
&\times \exp\left\{-\lambda_{\sigma(0)}(t_1 - 0)\right\} V_{\sigma(0)}(x(0)).
\end{aligned} \tag{4.83}$$

We define

$$C_1 \triangleq \exp\left\{\sum_{k=1}^{s''} \left\{N_{0[p|q]_{(k)}} \ln \mu_{[p|q]_{(k)}}\right\} + \sum_{k=1}^{s''} \left\{(\lambda_p + \alpha_{[p|q]_{(k)}}) N_{0[p|q]_{(k)}} T_{[p|q]_{(k)}_max\uparrow}\right\} \lambda_{\sigma(0)}(t_1)\right\}.$$

Inequality (4.83) can be rewritten as

$$\begin{aligned}
&V_{\sigma(t)}(x(t)) \\
&\leq C_1 \exp\left\{\sum_{k=1}^{s''} (\frac{\ln \mu_{[p|q]_{(k)}}}{\tau_{a(p,[p|q]_{(k)})}} - \lambda_p) T_{p,[p|q]_{(k)}}(0,t)\right\} \\
&\times \exp\left\{\sum_{k=1}^{s''} \{\frac{T_{[p|q]_{(k)}_max\uparrow}(\lambda_p + \alpha_{[p|q]_{(k)}})}{\tau_{a(p,[p|q]_{(k)})}} T_{p,[p|q]_{(k)}}(0,t)\}\right\} V_{\sigma(0)}(x(0)).
\end{aligned} \tag{4.84}$$

If there exist constants

$$\tau_{a(p,[p|q]_{(k)})} > \frac{(T_{[p|q]_{(k)}_max\uparrow}(\lambda_p + \alpha_{[p|q]_{(k)}}) + \ln \mu_{[p|q]_{(k)}})}{\lambda_p},$$

we have

$$V_{\sigma(t)}(x(t)) \leq C_1 \exp\left\{value1 \times (t - t_1)\right\} V_{\sigma(0)}(x(0)). \tag{4.85}$$

where

$$value1 \triangleq \max_k \left\{\frac{\ln \mu_{[p|q]_{(k)}} + T_{[p|q]_{(k)}_max\uparrow}(\lambda_p + \alpha_{[p|q]_{(k)}})}{\tau_{a(p,[p|q]_{(k)})}} - \lambda_p\right\}.$$

Therefore, it is concluded that $V_{\sigma(t)}(x(t))$ will converge to 0 as $t \to +\infty$. Then, the asymptotic stability can be deduced. ■

Remark 6: From Theorem 12, we can see that the proposed approaches exploit the differences of (a) the maximal asynchronous switching time $T_{[p|q]_\max\uparrow}$, (b) the energy changing degree $\mu_{[p|q]}$ at switching times, and (c) the increasing speed $\alpha_{[p|q]}$ of energy functions in asynchronous time intervals, for different switching sequences. Such sequence information is often ignored in the existing methods.

In order to show the relationship between the proposed methods and the conventional ADT method, we derive the following corollary from Theorem 12.

Corollary 7: For the switched system (4.71), let $\alpha > 0$, $\lambda > 0$, $\lambda > 0$, $\mu > 1$ be given constants. For $\forall s \in \mathcal{S}$, suppose there exist functions k_{1s}, k_{2s} of class $\mathcal{K}_\infty$ and $\mathcal{C}^1$ fuctions $V_s : \mathbb{R}^n \to \mathbb{R}$, such that

$$k_1(\|x(t)\|) \leq V_s(x(t)) \leq k_2(\|x(t)\|), \tag{4.86}$$

$\forall i = 1,\ 2,\ \cdots,$

$$\dot{V}_s(x(t)) \leq \begin{cases} -\lambda V_s(x(t)), \forall t \in T_\downarrow[t_i, t_{i+1}), \\ \alpha V_s(x(t)), \forall t \in T_\uparrow[t_i, t_{i+1}), \end{cases} \tag{4.87}$$

$\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}$,

$$V_p(x(t_i)) \leq \mu V_q(x(t_i)). \tag{4.88}$$

Then the system is globally uniformly asymptotically stable for any switching signals with average dwell time

$$\tau_a \geq \tau_a^* = \frac{T_{\max\uparrow}(\lambda + \alpha) + \ln \mu}{\lambda}, \tag{4.89}$$

where $T_{\max\uparrow} \triangleq \max_i T_\uparrow(t_i, t_{i+1})$, $\forall i \in Z_+$.

Proof: For Theorem 12, one gives the values that:

a) $\alpha_{[p|q]} = \alpha > 0$, for all $p,\ q \in \mathcal{S}$, $p \neq q$;

b) $\lambda_p = \lambda > 0$, for all $p \in \mathcal{S}$;

c) $\mu_{[p|q]} = \mu > 1$, for all $p,\ q \in \mathcal{S}$, $p \neq q$.

According to the above given values, the conditions (4.72), (4.73) and (4.74) are equal to (4.86), (4.87) and (4.88), respectively.

For Theorem 12, $T_{\max\uparrow}$, τ_a, τ_a^* are defined as follows

d) $T_{\max\uparrow} \triangleq \max_{p,q}\{T_{[p|q]_\max\uparrow}\}$, for all $p,\ q \in \mathcal{S}$, $p \neq q$;

e) $\tau_a \triangleq \max_{p,q}\{\tau_{a(p,[p|q])}\}$;

f) $\tau_a^* \triangleq \frac{T_{\max\uparrow}(\lambda+\alpha)+\ln\mu}{\lambda}$.

On the basis of (4.75) and above definitions, it can be seen

$$\begin{aligned}\tau_a \triangleq \max_{p,q}\{\tau_{a(p,[p|q])}\} &\geq \max_{p,q}\{\frac{T_{[p|q]_\max\uparrow}(\lambda_p+\alpha_{[p|q]})+\ln\mu_{[p|q]}}{\lambda_p}\}\\ &= \max_{p,q}\{\frac{T_{[p|q]_\max\uparrow}(\lambda+\alpha)+\ln\mu}{\lambda}\}\\ &= \frac{\max_{p,q}\{T_{[p|q]_\max\uparrow}\}(\lambda+\alpha)+\ln\mu}{\lambda}\\ &= \frac{T_{\max\uparrow}(\lambda+\alpha)+\ln\mu}{\lambda} \triangleq \tau_a^*.\end{aligned} \tag{4.90}$$

This completes the proof. ■

Remark 7: The relationship between Corollary 7 and Theorem 12 demonstrates that the existing ADT methods are special cases of the proposed sequential ADT. Corollary 7 is a conventional result based on conventional ADT methods in [28].

Theorem 13: For a discrete switched system

$$x(k+1) = f_{\sigma(k)}(x(k)), \tag{4.91}$$

let $1 > \lambda_p > 0$, $1 > \lambda_q > 0$, $\alpha_{[p|q]} > -1$, $\mu_{[p|q]} \geq 1$ be given constants. For $\forall s \in \mathcal{S}$, suppose there exist functions k_{1s}, k_{2s} of class $\mathcal{K}_\infty$ and $\mathcal{C}^1$ functions $V_s : \mathbb{R}^n \to \mathbb{R}$, such that

$$k_{1s}(\|x(k)\|) \leq V_s(x(k)) \leq k_{2s}(\|x(k)\|), \tag{4.92}$$

$\forall i = 1,\ 2,\ \cdots, \forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{S} \times \mathcal{S},\ p \neq q,$

$$V_p(x(k+1)) - V_p(x(k)) \leq \begin{cases} -\lambda_p V_p(x(k)), \forall t \in T_{\downarrow}[t_i, t_{i+1}), \\ \alpha_{[p|q]} V_p(x(k)), \forall t \in T_{\uparrow}[t_i, t_{i+1}), \end{cases} \tag{4.93}$$

and

$$V_p(x(t_i)) \leq \mu_{[p|q]} V_q(x(t_i)), \tag{4.94}$$

then the system is globally uniformly asymptotically stable for any switching signals with SBASDT

$$\tau_{a(p,[p|q])} \geq \tau^*_{a(p,[p|q])} = -\frac{(T_{[p|q]_\max\uparrow}(\ln(1 + \alpha_{[p|q]}) - \ln(1 - \lambda_p)) + \ln \mu_{[p|q]})}{\ln(1 - \lambda_p)}. \tag{4.95}$$

Proof: By using the technique similar to the proof of Theorem 12, this theorem can be proved. ■

Theorem 14: For the switched system (4.71), let $\lambda_q > 0$, $\lambda_p > 0$, $\mu_{[p|q]} > 1$ be given constants. For $\forall s \in \mathcal{S}$, suppose there exist functions k_{1s}, k_{2s} of class $\mathcal{K}_\infty$ and $\mathcal{C}^1$ functions $V_s : \mathbb{R}^n \to \mathbb{R}$, such that

$$k_{1s}(\|x(t)\|) \leq V_s \leq k_{2s}(\|x(t)\|), \tag{4.96}$$

$\forall i = 1,\ 2,\ \cdots, \forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S},\ p \neq q,$

$$\dot{V}_q(x(t)) \leq -\lambda_q V_q(x(t)), \forall t \in T_{\downarrow}[t_{i-1}, t_i), \tag{4.97}$$

$$\dot{V}_p(x(t)) \leq \alpha_{[p|q]} V_p(x(t)), \forall t \in T_{\uparrow}[t_i, t_{i+1}). \tag{4.98}$$

and

$$V_p(x(t_i)) \leq \mu_{[p|q]} V_q(x(t_i)). \tag{4.99}$$

Then the system is globally uniformly asymptotically stable for any switching signals with SBAPDT

$$\tau_{a(q,[p|q])} \geq \tau^*_{a(q,[p|q])} = \frac{T_{[p|q]_\max\uparrow}(\lambda_q + \alpha_{[p|q]}) + \ln \mu_{[p|q]}}{\lambda_q}. \tag{4.100}$$

Proof: Let $t_0 = 0$, and for any $t > 0$, $t \in [t_l, t_{l+1})$, $l \in Z_+$. We use $T_{q,[p|q]_{(k)}}(0,t)$ to represent the sum of dwell time of the q^{th} subsystems for the k^{th} union of the set $\mathcal{S}''$ in the time interval $[0,t)$.

According to (4.76) and (4.97)-(4.99), one can rewrite (4.77) as

$$\begin{aligned}
&V_{\sigma(t)}(x(t)) \\
&\leq \left\{\prod_{j=1}^{l} \mu_{\sigma(t_j)|\sigma(t_j^-)}\right\} \exp\left\{-\lambda_{\sigma(0)} T_{\downarrow}(0,t_1) + \alpha_{\sigma(t_1)} T_{\uparrow}(t_1,t_2)\right\} \\
&\times \exp\left\{-\lambda_{\sigma(t_1)} T_{\downarrow}(t_1,t_2) + \alpha_{\sigma(t_2)} T_{\uparrow}(t_2,t_3)\right\} \\
&\cdots \\
&\times \exp\left\{-\lambda_{\sigma(t_{l-1})} T_{\downarrow}(t_{l-1},t_l) + \alpha_{\sigma(t_l)} T_{\uparrow}(t_l,t)\right\} \\
&\times \exp\left\{-\lambda_{\sigma(t_1)} T_{\downarrow}(t_1,t_2)\right\} V_{\sigma(0)}(x(0)).
\end{aligned} \tag{4.101}$$

According to (4.78), it holds that

$$\begin{aligned}
&V_{\sigma(t)}(x(t)) \\
&\leq \left\{\prod_{k=1}^{s''} \mu_{[p|q]_{(k)}}^{N_{[p|q]_{(k)}}(0,t)}\right\} \exp\left\{\sum_{k=1}^{s''} -\lambda_q T_{q,[p|q]_{(k)}\downarrow}(0,t_l) + \alpha_{[p|q]_{(k)}} T_{[p|q]_{(k)}\uparrow}(0,t)\right\} \\
&\times \exp\left\{-\lambda_{\sigma(t_l)} T_{\downarrow}(t_l,t)\right\} V_{\sigma(0)}(x(0)).
\end{aligned} \tag{4.102}$$

For the second part in (4.102), one has

$$\begin{aligned}
&\sum_{k=1}^{s''} -\lambda_q T_{q,[p|q]_{(k)}\downarrow}(0,t_l) + \alpha_{[p|q]_{(k)}} T_{[p|q]_{(k)}\uparrow}(0,t) \\
=&\sum_{k=1}^{s''} -\lambda_q T_{q,[p|q]_{(k)}\downarrow}(0,t) + \alpha_{[p|q]_{(k)}} T_{[p|q]_{(k)}\uparrow}(0,t) \\
=&\sum_{k=1}^{s''} -\lambda_q T_{q,[p|q]_{(k)}\downarrow}(0,t) - \lambda_q T_{q,[p|q]_{(k)}\uparrow}(0,t) + \sum_{k=1}^{s''} \lambda_q T_{q,[p|q]_{(k)}\uparrow}(0,t) + \alpha_{[p|q]_{(k)}} T_{[p|q]_{(k)}\uparrow}(0,t) \\
=&\sum_{k=1}^{s''} -\lambda_q T_{q,[p|q]_{(k)}}(0,t) + \sum_{k=1}^{s''} \lambda_q T_{q,[p|q]_{(k)}\uparrow}(0,t) + \alpha_{[p|q]_{(k)}} T_{[p|q]_{(k)}\uparrow}(0,t).
\end{aligned} \tag{4.103}$$

It follows from (4.82) that

$$\begin{aligned}&\sum_{k=1}^{s''}-\lambda_q T_{q,[p|q]_{(k)}\downarrow}(0,t)+\alpha_{[p|q]_{(k)}}T_{[p|q]_{(k)}\uparrow}(0,t)\\&\leq\sum_{k=1}^{s''}-\lambda_q T_{q,[p|q]_{(k)}}(0,t)+\sum_{k=1}^{s''}(\lambda_q+\alpha_{[p|q]_{(k)}})N_{\sigma[p|q]_{(k)}}(0,t)T_{[p|q]_{(k)}_max\uparrow}.\end{aligned}\tag{4.104}$$

According to (4.29), one can obtain

$$\begin{aligned}&V_{\sigma(t)}(x(t))\\&\leq\left\{\prod_{k=1}^{s''}\mu_{[p|q]_{(k)}}^{N_{0[p|q]_{(k)}}+\frac{T_{q,[p|q]_{(k)}}(0,t)}{\tau_{a(q,[p|q]_{(k)})}}}\right\}\exp\left\{\sum_{k=1}^{s''}-\lambda_q T_{q,[p|q]_{(k)}}(0,t)\right\}\\&\times\exp\left\{\sum_{k=1}^{s''}(\lambda_q+\alpha_{[p|q]_{(k)}})(N_{0[p|q]_{(k)}}+\frac{T_{q,[p|q]_{(k)}}(0,t)}{\tau_{a(q,[p|q]_{(k)})}})T_{[p|q]_{(k)}_max\uparrow}-\lambda_{\sigma(t_l)}T_{\downarrow}(t_l,t)\right\}V_{\sigma(0)}(x(0))\\&=\exp\left\{\sum_{k=1}^{s''}N_{0[p|q]_{(k)}}\ln\mu_{[p|q]_{(k)}}\right\}\exp\left\{\sum_{k=1}^{s''}(\frac{\ln\mu_{[p|q]_{(k)}}}{\tau_{a(q,[p|q]_{(k)})}}-\lambda_q)T_{q,[p|q]_{(k)}}(0,t)\right\}\\&\times\exp\left\{\sum_{k=1}^{s''}(\lambda_q+\alpha_{[p|q]_{(k)}})N_{0[p|q]_{(k)}}T_{[p|q]_{(k)}_max\uparrow}\right\}\\&\times\exp\left\{\sum_{k=1}^{s''}(\lambda_q+\alpha_{[p|q]_{(k)}})\frac{T_{q,[p|q]_{(k)}}(0,t)}{\tau_{a(q,[p|q]_{(k)})}}T_{[p|q]_{(k)}_max\uparrow}-\lambda_{\sigma(t_l)}T_{\downarrow}(t_l,t)\right\}V_{\sigma(0)}(x(0)).\end{aligned}\tag{4.105}$$

It can be rewritten as

$$\begin{aligned}&V_{\sigma(t)}(x(t))\\&\leq\exp\left\{\sum_{k=1}^{s''}N_{0[p|q]_{(k)}}\ln\mu_{[p|q]_{(k)}}\right\}\exp\left\{\sum_{k=1}^{s''}(\lambda_p+\alpha_{[p|q]_{(k)}})N_{0[p|q]_{(k)}}T_{[p|q]_{(k)}_max\uparrow}\right\}\\&\times\exp\left\{\sum_{k=1}^{s''}\left\{(\frac{\ln\mu_{[p|q]_{(k)}}}{\tau_{a(q,[p|q]_{(k)})}}-\lambda_q)T_{q,[p|q]_{(k)}}(0,t)\right\}\right\}\\&\times\exp\left\{\sum_{k=1}^{s''}\{\frac{T_{[p|q]_{(k)}_max\uparrow}(\lambda_q+\alpha_{[p|q]_{(k)}})}{\tau_{a(q,[p|q]_{(k)})}}T_{q,[p|q]_{(k)}}(0,t)\}-\lambda_{\sigma(t_l)}T_{\downarrow}(t_l,t)\right\}V_{\sigma(0)}(x(0)).\end{aligned}\tag{4.106}$$

We define

$$C_2\triangleq\exp\left\{\sum_{k=1}^{s''}N_{0[p|q]_{(k)}}\ln\mu_{[p|q]_{(k)}}\right\}\exp\left\{\sum_{k=1}^{s''}(\lambda_p+\alpha_{[p|q]_{(k)}})N_{0[p|q]_{(k)}}T_{[p|q]_{(k)}_max\uparrow}\right\}.$$

If there exist constants

$$\tau_{a(q,[p|q]_{(k)})}>\frac{(T_{[p|q]_{(k)}_max\uparrow}(\lambda_q+\alpha_{[p|q]_{(k)}})+\ln\mu_{[p|q]_{(k)}})}{\lambda_q},$$

then one has

$$\begin{aligned} & V_{\sigma(t)}(x(t)) \\ & \leq C_2 \exp\{value2 \times (t_l\text{–}0)\text{–}\lambda_{\sigma(t_l)} T_{\downarrow}(t_l, t)\} V_{\sigma(0)}(x(0)) \\ & \leq C_2 \exp\left\{\max\{-\lambda_{\sigma(t_l)}, value2\}\, t\right\} V_{\sigma(0)}(x(0)), \end{aligned} \tag{4.107}$$

where

$$value2 \triangleq \max_k \left\{ \frac{\ln \mu_{[p|q]_{(k)}} + T_{[p|q]_{(k)}\text{-}max\uparrow}(\lambda_q + \alpha_{[p|q]_{(k)}})}{\tau_{a(q,[p|q]_{(k)})}} - \lambda_q \right\} < 0.$$

Therefore, it is concluded that $V_{\sigma(t)}(x(t))$ converges to 0 as $t \to +\infty$. Then, the asymptotic stability can be deduced. ■

Theorem 15: For the discrete switched system (4.91), let $1 > \lambda_p > 0$, $1 > \lambda_q > 0$, $\alpha_{[p|q]} > -1$, $\mu_{[p|q]} \geq 1$ be given constants. For $\forall s \in \mathcal{S}$, suppose there exist functions k_{1s}, k_{2s} of class $\mathcal{K}_\infty$ and $\mathcal{C}^1$ functions $V_s : \mathbb{R}^n \to \mathbb{R}$, such that

$$k_{1s}(\|x(k)\|) \leq V_s(x(k)) \leq k_{2s}(\|x(k)\|), \tag{4.108}$$

$\forall i = 1,\ 2,\ \cdots, \forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{S} \times \mathcal{S}$, $p \neq q$,

$$V_q(x(k+1))\text{–}V_q(x(k)) \leq \text{–}\lambda_q V_q(x(k)),\ \forall t \in T_{\downarrow}[t_{i-1}, t_i) \tag{4.109}$$

$$V_p(x(k+1))\text{–}V_p(x(k)) \leq \alpha_{[p|q]} V_p(x(t)), \forall t \in T_{\uparrow}[t_i, t_{i+1}), \tag{4.110}$$

and

$$V_p(x(t_i)) \leq \mu_{[p|q]} V_q(x(t_i)), \tag{4.111}$$

then the system is globally uniformly asymptotically stable for any switching signals with SBAPDT

$$\tau_{a(q,[p|q])} \geq \tau^*_{a(q,[p|q])} = -\frac{(T_{[p|q]\text{-}\max\uparrow}(\ln(1 + \alpha_{[p|q]}) - \ln(1 - \lambda_q)) + \ln \mu_{[p|q]})}{\ln(1 - \lambda_q)}. \tag{4.112}$$

Proof: By using the technique similar to the proof of Theorem 5, this theorem can be proved.■

Remark 8: Comparing Theorem 14 (or 15) with 12 (or 13) or with the existing methods, we can see that the proposed SBAPDT approach counterbalances the energy functions by prolonging the preceding rather than subsequent subsystems. This approach provides a novel direction for solving the stability problem of asynchronous switched systems.

Linear Systems

In this division, we extend the results in the last division to linear asynchronous switched systems and present the following theorems for their stability analysis.

Theorem 16: For system (4.69) and the given constants $\alpha_{[p|q]} > 0$, $\lambda_p > 0$, $\mu_{[p|q]} > 1$, if there exist matrices $P_p > 0$, $P_q > 0$, and $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}, p \neq q$, such that

$$(A_p + B_p K_p)^T P_p + P_p(A_p + B_p K_p) \leq -\lambda_p P_p, \tag{4.113}$$

$$(A_p + B_p K_q)^T P_p + P_p(A_p + B_p K_q) \leq \alpha_{[p|q]} P_p, \tag{4.114}$$

$$P_p \leq \mu_{[p|q]} P_q, \tag{4.115}$$

then the system is globally uniformly asymptotically stable for any switching signals under the SBASDT switching condition (4.75).

Proof: For system (4.69), the multiple Lyapunov functions are given by

$$V_s = x^T P_s x, \ \forall s \in \mathcal{S}. \tag{4.116}$$

According to (4.73), one has (4.113) and (4.114).

The condition for (4.115) can be given in terms of (4.74).

Now, the proof of Theorem 16 is completed. ■

Theorem 17: For the discrete-time system (4.70) and the given constants $1 > \lambda_p > 0$, $\alpha_{[p|q]} > -1$, $\mu_{[p|q]} \geq 1$, if there exist matrices $P_p > 0$, $P_q > 0$, $p, \ q \in \mathcal{S}$,

$\forall(\sigma(k_i)=p,\sigma(k_i-1)=q)\in\mathcal{S}\times\mathcal{S},p\neq q$ such that

$$\begin{bmatrix} -P_p & P_pA_p+P_pB_pK_p \\ * & -(1-\lambda_p)P_p \end{bmatrix}\leq 0, \tag{4.117}$$

$$\begin{bmatrix} -P_p & P_pA_p+P_pB_pK_q \\ * & -(1+\alpha_{[p|q]})P_p \end{bmatrix}\leq 0, \tag{4.118}$$

$$P_p\leq\mu_{[p|q]}P_q, \tag{4.119}$$

then the system is globally uniformly asymptotically stable for any switching signals under the SBASDT switching condition (4.95).

Proof: According to Theorem 13, Theorem 17 can be proved by using the technique similar to the proof of Theorem 16. ■

Theorem 18: For system (4.69) and the given constants $\alpha_{[p|q]}>0$, $\mu_{[p|q]}>1$ and $\lambda_q>0$, if there exist matrices $P_p>0$, $P_q>0$,and $\forall(\sigma(t_i)=p,\sigma(t_i^-)=q)\in\mathcal{S}\times\mathcal{S},p\neq q$, such that

$$(A_q+B_qK_q)^TP_q+P_q(A_q+B_qK_q)\leq-\lambda_qP_q, \tag{4.120}$$

$$(A_p+B_pK_q)^TP_p+P_p(A_p+B_pK_q)\leq\alpha_{[p|q]}P_p, \tag{4.121}$$

$$P_p\leq\mu_{[p|q]}P_q, \tag{4.122}$$

then the system is globally uniformly asymptotically stable for any switching signals under the SBAPDT switching condition (4.100).

Proof: According to Theorem 14, Theorem 18 can be proved by using the technique similar to the proof of Theorem 16. ■

Theorem 19: For the discrete-time system (4.70) and the given constants $1>\lambda_q>0$, $\alpha_{[p|q]}>-1$, $\mu_{[p|q]}\geq 1$, if there exist matrices $P_p>0$, $P_q>0$, $p,\ q\in\mathcal{S}$, $\forall(\sigma(k_i)=p,\sigma(k_i-1)=q)\in\mathcal{S}\times\mathcal{S},p\neq q$ satisfying

$$\begin{bmatrix} -P_q & P_qA_q+P_qB_qK_q \\ * & -(1-\lambda_q)P_q \end{bmatrix}\leq 0, \tag{4.123}$$

$$\begin{bmatrix} -P_p & P_pA_p + P_pB_pK_q \\ * & -(1+\alpha_{[p|q]})P_p \end{bmatrix} \leq 0, \tag{4.124}$$

$$P_p \leq \mu_{[p|q]}P_q, \tag{4.125}$$

then the system is globally uniformly asymptotically stable for any switching signals under the SBAPDT switching condition (4.112).

Proof: According to Theorem 15, Theorem 19 can be proved by using the technique similar to the proof of Theorem 16. ■

4.3 Switching among Stable and Unstable Subsystems

In this division, we consider the switched systems with both unstable and stable subsystems. In this case, it means $\mathcal{M} = \mathcal{S} \cup \mathcal{U}$, where $\mathcal{M}$, $\mathcal{S}$, and $\mathcal{U}$ denote the set of all subsystems, stable and unstable subsystems, respectively.

In this section, we consider the following nonlinear system models:
discrete-time switched systems:

$$x(k+1) = f_{\sigma(k)}\left(x(k)\right), \ x(k_0) = x_0; \tag{4.126}$$

continuous-time switched systems:

$$\dot{x}(t) = f_{\sigma(t)}\left(x(t)\right), \ x(t_0) = x_0. \tag{4.127}$$

The symbol $x(k)$ $(or\ x(t)) \in \mathbb{R}^n$ denotes a n-dimension state vector. We use $\sigma(k)$ (or $\sigma(t)$) to denote the switching signal which is a piecewise continuous function.

Correspondingly, the following linear switched system models are also considered:
discrete-time switched systems:

$$x(k+1) = A_{\sigma(k)}x(k), \ x(k_0) = x(0); \tag{4.128}$$

continuous-time switched systems:

$$\dot{x}(t) = A_{\sigma(t)}x(t), \ x(t_0) = x(0). \tag{4.129}$$

In order to obtain a new solution for the stability problem of switched systems (4.126)-(4.129), we first introduce some concepts about switched systems.

Definition 22: For a given linear or nonlinear discrete-time switched system with a switching signal $\sigma(k)$ and any switching time k_1, k_2 and $k_2 > k_1 \geq 0$, let $N_{\sigma[p|q]}(k_2, k_1)$ represent the number of the sequences that the p^{th} subsystem is activated immediately after the q^{th} subsystem over the time interval $[k_1, k_2)$. The symbol $T_{p,[p|q]}(k_2, k_1)$ $(or\ T_{q,[p|q]}(k_2, k_1))$ stands for the total running time of the $p^{th}(or\ the\ q^{th})$ subsystem under this circumstance. It is said that:
(1) $\sigma(k)$ has a slow sequence-based average subsequent dwell time (SBASDT) $\tau_{a(p,[p|q])}$ if there exist two numbers $\tau_{a(p,[p|q])}$ and $N_{0(p,[p|q])}$ which is called the sequenced-based subsequent slowly switching chatter bound here, such that

$$N_{\sigma[p|q]}(k_2, k_1) \leq N_{0(p,[p|q])} + \frac{T_{p,[p|q]}(k_2, k_1)}{\tau_{a(p,[p|q])}}; \tag{4.130}$$

(2) $\sigma(k)$ has a slow sequence-based average preceding dwell time (SBAPDT) $\tau_{a(q,[p|q])}$ if there exist two numbers $\tau_{a(q,[p|q])}$ and $N_{0(q,[p|q])}$ which is called the sequenced-based preceding slowly switching chatter bound here, such that

$$N_{\sigma[p|q]}(k_2, k_1) \leq N_{0(q,[p|q])} + \frac{T_{q,[p|q]}(k_2, k_1)}{\tau_{a(q,[p|q])}}. \tag{4.131}$$

A definition for continuous switched systems can be given similarly, we omit it here.

Definition 22 is known for the sequence-based slowly switching. Below, we introduce a new concept for fast switching.

Definition 23: A given linear or nonlinear discrete-time switched system with a switching signal $\sigma(k)$ and any switching time k_1, k_2 and $k_2 > k_1 \geq 0$, let $N^c_{\sigma[p|q]}(k_2, k_1)$ represent the number of the sequences that the p^{th} subsystem is activated immediately after the q^{th} subsystem over the time interval $[k_1, k_2)$. The symbol $T^c_{p,[p|q]}(k_2, k_1)$ $(or\ T^c_{q,[p|q]}(k_2, k_1))$ stands for the total running time of the $p^{th}(or\ q^{th})$ subsystem under this circumstance. It is said that

(1) $\sigma(k)$ has a fast SBASDT $\tau^c_{a(p,[p|q])}$ if there exist two numbers $\tau^c_{a(p,[p|q])}$ and $N^c_{0(p,[p|q])}$ which is called the sequence-based subsequent fast switching chatter bound here, such that

$$N^c_{\sigma[p|q]}(k_2,k_1) \geq N^c_{0(p,[p|q])} + \frac{T^c_{p,[p|q]}(k_2,k_1)}{\tau^c_{a(p,[p|q])}}; \tag{4.132}$$

(2) $\sigma(k)$ has a fast SBAPDT $\tau^c_{a(q,[p|q])}$ if there exist two numbers $\tau^c_{a(q,[p|q])}$ and $N^c_{0(q,[p|q])}$ which is called the sequence-based preceding fast switching chatter bound here, such that

$$N^c_{\sigma[p|q]}(k_2,k_1) \geq N^c_{0(q,[p|q])} + \frac{T^c_{q,[p|q]}(k_2,k_1)}{\tau^c_{a(q,[p|q])}}. \tag{4.133}$$

This definition for continuous switched systems can be given similarly.

For the nonlinear discrete-time switched system (4.126), if the unstable subsystems are fast switching and that the stable systems are slowly switching, we can get the following Lemma 22.

Lemma 22: Consider the system (4.126), and let ς_p, ς_q, $\mu_{[p|q]}$, be given constants. Suppose that there exist $\mathcal{C}^1$ functions $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, and class $\mathcal{K}_\infty$ functions κ_{q1}, κ_{q2}, κ_{p1}, κ_{p2}, such that $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{M} \times \mathcal{M}$, $p \neq q$,

$$\begin{cases} \kappa_{p1}(\| x(k) \|) \leq V_p(x(k),k) \leq \kappa_{p2}(\| x(k) \|), \\ \kappa_{q1}(\| x(k) \|) \leq V_q(x(k),k) \leq \kappa_{q2}(\| x(k) \|), \end{cases} \tag{4.134}$$

$$\begin{cases} V_p(x(k+1),k+1) - V_p(x(k),k) \leq \varsigma_p V_p(x(k),k), \\ V_q(x(k+1),k+1) - V_q(x(k),k) \leq \varsigma_q V_q(x(k),k), \end{cases} \tag{4.135}$$

$$V_p(x(k_i),k_i) \leq \mu_{[p|q]} V_q(x(k_i),k_i). \tag{4.136}$$

Then the system is globally uniformly asymptotically stable for any switching signals with SBASDT

$$\begin{cases} \tau_{a(p,[p|q])} > \tau^*_{a(p,[p|q])} \triangleq -\frac{\ln \mu_{[p|q]}}{\ln(1+\varsigma_p)}, (-1 < \varsigma_p < 0, \mu_{[p|q]} > 1, p \in \mathcal{S}), \\ \tau^c_{a(p,[p|q])} < \tau^*_{a(p,[p|q])} \triangleq -\frac{\ln \mu_{[p|q]}}{\ln(1+\varsigma_p)}, (\varsigma_p > 0, 0 < \mu_{[p|q]} \leq 1, p \in \mathcal{U}), \end{cases} \tag{4.137}$$

or with SBAPDT

$$\begin{cases} \tau_{a(q,[p|q])} > \tau^*_{a(q,[p|q])} \triangleq -\frac{\ln \mu_{[p|q]}}{\ln (1+\varsigma_q)}, (-1 < \varsigma_q < 0, \mu_{[p|q]} > 1, q \in \mathcal{S}). \\ \tau^c_{a(q,[p|q])} < \tau^*_{a(q,[p|q])} \triangleq -\frac{\ln \mu_{[p|q]}}{\ln (1+\varsigma_q)}, (\varsigma_q > 0, 0 < \mu_{[p|q]} \leq 1, q \in \mathcal{U}). \end{cases} \tag{4.138}$$

Proof: For any $k > 0$ and $\forall k \in [\, k_i, k_{i+1}\,)$, $i \in Z_+$, let $k_0 = 0$, the set $\mathcal{S}'' = \{(p,q) : p = \sigma(k_j), q = \sigma(k_j - 1), j = 1, 2, 3, \cdots, i\}$, $V_{\sigma(k)}(k)$ denote $V_{\sigma(k)}(x(k), k)$, and $\overline{\varsigma}$ stand for $1 + \varsigma$. According to (4.134)-(4.136), one can get

$$\begin{aligned} & V_{\sigma(k)}(k) \\ & \leq (1 + \varsigma_{\sigma(k_i)})^{(k-k_i)} V_{\sigma(k_i)}(k_i) \\ & \leq \mu_{\sigma(k_i)} (1 + \varsigma_{\sigma(k_i)})^{(k-k_i)} V_{\sigma(k_i - 1)}(k_i) \\ & \leq \mu_{\sigma(k_i)} \overline{\varsigma}^{(k-k_i)}_{\sigma(k_i)} \overline{\varsigma}^{(k_i - k_{i-1})}_{\sigma(k_{i-1})} V_{\sigma(k_{i-1})}(k_{i-1}). \end{aligned} \tag{4.139}$$

Finally, we can get

$$V_{\sigma(k)}(k) \leq \prod_{j=1}^{i} \mu_{\sigma(k_j)} \overline{\varsigma}^{(k-k_i)}_{\sigma(k_i)} \cdots \overline{\varsigma}^{(k_1 - 0)}_{\sigma(0)} V_{\sigma(0)}(0). \tag{4.140}$$

The symbols $N^c_{\sigma[p|q]_{(g)}}(k,0)$ (or $N_{\sigma[p|q]_{(g)}}(k,0)$), $T^c_{p,[p|q]_{(g)}}(k,0)$ (or $T_{p,[p|q]_{(g)}}(k,0)$) and $T^c_{q,[p|q]_{(g)}}(k,0)$ (or $T_{q,[p|q]_{(g)}}(k,0)$) denote the activated numbers, total subsequent dwell time and total preceding dwell time of the g^{th} element in the time interval $[0,k)$ for fast (slowly) switching respectively.

Next, we provide proof for the SBASDT and SBAPDT switching separately.

(a) The SBASDT switching:

According to (4.130) and (4.132), one can obtain

$$\begin{aligned} & V_{\sigma(k)}(k) \\ & \leq \prod_{g=1, p\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(p,[p|q]_{(g)})} + \frac{T_{p,[p|q]_{(g)}}(k,k_1)}{\tau_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}^{T_{p,[p|q]_{(g)}}(k,k_1)} \\ & \times \prod_{g=1, p\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N^c_{0(p,[p|q]_{(g)})} + \frac{T^c_{p,[p|q]_{(g)}}(k,k_1)}{\tau^c_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}^{T^c_{p,[p|q]_{(g)}}(k,k_1)} \overline{\varsigma}^{(k_1)}_{\sigma(0)} V_{\sigma(0)}(x(0)) \\ & = \prod_{g=1, p\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(p,[p|q]_{(g)})}} (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}})^{T_{p,[p|q]_{(g)}}(k,k_1)} \\ & \times \prod_{g=1, p\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N^c_{0(p,[p|q]_{(g)})}} (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}})^{T^c_{p,[p|q]_{(g)}}(k,k_1)} \overline{\varsigma}^{(k_1)}_{\sigma(0)} V_{\sigma(0)}(x(0)). \end{aligned} \tag{4.141}$$

We give the following definitions:

$$\gamma_3 \triangleq \max_g \left\{ \mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}, \mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}} \right\},$$

$$K_3 \triangleq \left\{ \prod_{g=1,p\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(p,[p|q]_{(g)})}} \right\} \left\{ \prod_{g=1,p\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N^c_{0(p,[p|q]_{(g)})}} \right\} \gamma_3^{-k_1} \overline{\varsigma}_{\sigma(0)}^{-k_1}.$$

According to the switching condition (4.137), we can know that $0 < \gamma_3 < 1$. It follows

$$V_{\sigma(k)}(k) \leq K_3 \gamma_3^{(k-k_0)} V_{\sigma(0)}(x(0)). \tag{4.142}$$

Therefore, the switched system is globally uniformly asymptotically stable.

(b) The SBAPDT switching:

According to (4.131) and (4.133), one can obtain

$$\begin{aligned}
&V_{\sigma(k)}(k) \\
&\leq \prod_{g=1,q\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(q,[p|q]_{(g)})}+\frac{T_{q,[p|q]_{(g)}}(k_i,0)}{\tau_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}}^{T_{q,[p|q]_{(g)}}(k_i,0)} \\
&\times \prod_{g=1,q\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N^c_{0(q,[p|q]_{(g)})}+\frac{T^c_{q,[p|q]_{(g)}}(k_i,0)}{\tau^c_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}}^{T^c_{q,[p|q]_{(g)}}(k_i,0)} \overline{\varsigma}_{\sigma(k-k_i)}^{(k-k_i)} V_{\sigma(0)}(x(0)) \\
&= \prod_{g=1,q\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(q,[p|q]_{(g)})}} (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}})^{T_{q,[p|q]_{(g)}}(k_i,0)} \\
&\times \prod_{g=1,q\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N^c_{0(q,[p|q]_{(g)})}} (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}})^{T^c_{q,[p|q]_{(g)}}(k_i,0)} \overline{\varsigma}_{\sigma(k-k_i)}^{(k-k_i)} V_{\sigma(0)}(x(0)).
\end{aligned} \tag{4.143}$$

We give the following definitions:

$$\gamma_4 \triangleq \max_g \left\{ \mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}}, \mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}} \right\}$$

$$K_4 \triangleq \left\{ \prod_{g=1,q\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(q,[p|q]_{(g)})}} \right\} \left\{ \prod_{g=1,q\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N^c_{0(q,[p|q]_{(g)})}} \right\} \gamma_4^{-(k-k_i)} \overline{\varsigma}_{\sigma(k_i)}^{(k-k_i)}$$

According to the switching condition (4.138), we can know that $0 < \gamma_4 < 1$. It follows

$$V_{\sigma(k)}(k) \leq K_4 \gamma_4^{(k-k_0)} V_{\sigma(0)}(x(0)). \tag{4.144}$$

Therefore, the switched system is globally uniformly asymptotically stable. Combining (a) and (b), this lemma is proved.■

If the considered system is linear, we have the following Theorem 20.

Theorem 20: Consider the system (4.128). Let ς_p, ς_q, $\mu_{[p|q]}$, $\tau_{min,p} = \min\limits_{\sigma(k_i)=p}(k_i - k_{i-1})$, $\tau_{min,q} = \min\limits_{\sigma(k_i)=q}(k_i - i_{i-1})$ be given constants. For $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{M} \times \mathcal{M}$, $p \neq q$, assume that there exist a set of positive definite matrices satisfying:

(a) if $p \in \mathcal{U}$, then $P_{p,\tau_{min,p}} > 0$, $P_{p,j_p} > 0$, $j_p = 0,\ 1, \cdots,\ \tau_{min,p} - 1$, and

$$\begin{cases} \begin{bmatrix} (\varsigma_p + 1)P_{p,j_p+1}, & A_p^T P_{p,j_p} \\ * & P_{p,j_p} \end{bmatrix} > 0 \\ \begin{bmatrix} (\varsigma_p + 1)P_{p,\tau_{min,p}}, & A_p^T P_{p,\tau_{min,p}} \\ * & P_{p,\tau_{min,p}} \end{bmatrix} > 0 \end{cases}; \tag{4.145}$$

(b) if $p \in \mathcal{S}$, then $P_p > 0$ and

$$\begin{bmatrix} (\varsigma_p + 1)P_p & A_p^T P_p \\ * & P_p \end{bmatrix} > 0; \tag{4.146}$$

(c) if $q \in \mathcal{U}$, then $P_{q,\tau_{min,q}} > 0$, $P_{q,j_q} > 0$, $j_q = 0,\ 1, \cdots,\ \tau_{min,q} - 1$, and

$$\begin{cases} \begin{bmatrix} (\varsigma_q + 1)P_{q,j_q+1}, & A_q^T P_{q,j_q} \\ * & P_{q,j_q} \end{bmatrix} > 0 \\ \begin{bmatrix} (\varsigma_q + 1)P_{q,\tau_{min,q}}, & A_q^T P_{q,\tau_{min,q}} \\ * & P_{q,\tau_{min,q}} \end{bmatrix} > 0 \end{cases}; \tag{4.147}$$

(d) if $q \in \mathcal{S}$, then $P_q > 0$ and

$$\begin{bmatrix} (\varsigma_q + 1)P_q & A_q^T P_q \\ * & P_q \end{bmatrix} > 0, \tag{4.148}$$

and

$$P_{p,0} \leq \mu_{[p|q]} P_{q,\tau_{min,p}}. \tag{4.149}$$

Then for any switching signals with SBASDT satisfying (4.137) or with SBAPDT satisfying (4.138), the system (4.128) is globally uniformly asymptotically stable.

Proof: The proof of this theorem is largely similar to that of Theorem 24. The only additional requirement is:

$$\begin{cases} P_p = P_{p,j_p} > 0,\ j_p = 0,\ 1, \cdots,\ \tau_{min,p},\ p \in \mathcal{S} \\ P_q = P_{q,j_q} > 0,\ j_q = 0,\ 1, \cdots,\ \tau_{min,q},\ q \in \mathcal{S} \end{cases}.$$

It means that a constant P_p (or P_q) replaces all P_{p,j_p} (or P_{q,j_q}) if $p \in \mathcal{S}$ ($q \in \mathcal{S}$).

■

For the continuous-time systems, we can get Lemma 23 and Theorem 21 below, and the proofs are similar to those for discrete-time systems.

Lemma 23: Consider the system (4.127). Let $\mu_{[p|q]}$, α_p and α_q be given constants. Suppose that there exist class $\mathcal{K}_\infty$ functions κ_{p1}, κ_{p2}, κ_{q1}, and κ_{q2} and $\mathbb{C}^1$ functions $V_{\sigma(t)}(x(t)) : \mathbb{R}^n \to \mathbb{R}$, such that $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M}$ and $p \neq q$,

$$\begin{cases} \kappa_{p1}(\| x(t) \|) \leq V_p(x(t), t) \leq \kappa_{p2}(\| x(t) \|), \\ \kappa_{q1}(\| x(t) \|) \leq V_q(x(t), t) \leq \kappa_{q2}(\| x(t) \|), \end{cases} \tag{4.150}$$

$$\begin{cases} \dot{V}_p(x(t), t) \leq \alpha_p V_p(x(t), t), \\ \dot{V}_q(x(t), t) \leq \alpha_q V_q(x(t), t), \end{cases} \tag{4.151}$$

$$V_p(x(t_i), t_i) \leq \mu_{[p|q]} V_q(x(t_i), t_i). \tag{4.152}$$

Then for any switching signals with SBASDT

$$\begin{cases} \tau_{a(p,[p|q])} > \tau^*_{a(p,[p|q])} \triangleq -ln\mu_{[p|q]}/\alpha_p, (\alpha_p < 0, \mu_{[p|q]} > 1, p \in \mathcal{S}), \\ \tau^c_{a(p,[p|q])} < \tau^*_{a(p,[p|q])} \triangleq -ln\mu_{[p|q]}/\alpha_p, (\alpha_p > 0, 0 < \mu_{[p|q]} < 1, p \in \mathcal{U}), \end{cases} \tag{4.153}$$

or with SBAPDT

$$\begin{cases} \tau_{a(q,[p|q])} > \tau^*_{a(q,[p|q])} \triangleq -ln\mu_{[p|q]}/\alpha_q, (\alpha_q < 0, \mu_{[p|q]} > 1, q \in \mathcal{S}), \\ \tau^c_{a(q,[p|q])} < \tau^*_{a(q,[p|q])} \triangleq -ln\mu_{[p|q]}/\alpha_q, (\alpha_q > 0, 0 < \mu_{[p|q]} < 1, q \in \mathcal{U}), \end{cases} \tag{4.154}$$

the system is globally uniformly asymptotically stable.

Theorem 21: Consider the system (4.129). Let $\mu_{[p|q]},\ \alpha_p,\ \alpha_q, \tau_{min,p} = \min_{\sigma(t_i)=p}(t_i - t_{i-1}),\ \tau_{min,q} = \min_{\sigma(t_i)=q}(t_i - t_{i-1})$ and $l_p,\ l_q$ be the given constants. For $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M},\ p \neq q$, assume that there exist a set of positive definite matrices satisfying:

(a) if $p \in \mathcal{U}$ then $P_{p,l_p} > 0,\ P_{p,j_p} > 0,\ j_p = 0,\ 1, \cdots,\ l_p - 1$, and

$$\begin{cases} A_p^T P_{p,j_p} + P_{p,j_p} A_p + \frac{l_p(P_{p,j_p+1} - P_{p,j_p})}{\tau_{min,p}} \leq \alpha_p P_{p,j_p} \\ A_p^T P_{p,j_p+1} + P_{p,j_p+1} A_p + \frac{l_p(P_{p,j_p+1} - P_{p,j_p})}{\tau_{min,p}} \leq \alpha_p P_{p,j_p+1} \\ A_p^T P_{p,l_p} + P_{p,l_p} A_p \leq \alpha_p P_{p,l_p} \end{cases} ; \tag{4.155}$$

(b) if $p \in \mathcal{S}$ then $P_p > 0$, and

$$A_p^T P_p + P_p A_p \leq \alpha_p P_p; \tag{4.156}$$

(c) if $q \in \mathcal{U}$ then $P_{q,l_q} > 0,\ P_{q,j_q} > 0,\ j_q = 0,\ 1, \cdots,\ l_q - 1$, and

$$\begin{cases} A_q^T P_{q,j_q} + P_{q,j_q} A_q + \frac{l_q(P_{q,j_q+1} - P_{q,j_q})}{\tau_{min,q}} \leq \alpha_q P_{q,j_q} \\ A_q^T P_{q,j_q+1} + P_{q,j_q+1} A_q + \frac{l_q(P_{q,j_q+1} - P_{q,j_q})}{\tau_{min,q}} \leq \alpha_q P_{q,j_q+1} \\ A_q^T P_{q,l_q} + P_{q,l_q} A_q \leq \alpha_q P_{q,l_q} \end{cases} ; \tag{4.157}$$

(d) if $p \in \mathcal{S}$ then $P_p > 0$, and

$$A_q^T P_q + P_q A_q \leq \alpha_q P_q;$$

and

$$P_{p,0} \leq \mu_{[p|q]} P_{q,l_p}. \tag{4.158}$$

Then the system (4.129) is globally uniformly asymptotically stable for any switching signals with SBASDT satisfying (4.153) or with SBAPDT satisfying (4.154).

The preceding theorems in this division require $\mu_{[p|q]} < 1$ when the subsequent or preceding subsystem is unstable. This is a relatively strong requirement in some cases. In the following, we relax this requirement.

Lemma 24: Consider the system (4.126). Let ς_p, ς_q, $\mu_{[p|q]} > 1$ be given constants. Suppose that there exist class $\mathcal{K}_\infty$ functions $\kappa_{p1}, \kappa_{p2}, \kappa_{q1}, \kappa_{q2}$ and $\mathcal{C}^1$ functions $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, such that $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{M} \times \mathcal{M}$, $p \neq q$,

$$\begin{cases} \kappa_{p1}(\| x(k) \|) \leq V_p(k) \leq \kappa_{p2}(\| x(k) \|), \\ \kappa_{q1}(\| x(k) \|) \leq V_q(k) \leq \kappa_{q2}(\| x(k) \|), \end{cases} \tag{4.159}$$

$$\begin{cases} V_p(k+1) - V_p(k) \leq \varsigma_p V_p(k), \\ V_q(k+1) - V_q(k) \leq \varsigma_q V_q(k), \end{cases} \tag{4.160}$$

$$V_p(k_i) \leq \mu_{[p|q]} V_q(k_i). \tag{4.161}$$

Then for any switching signals with SBASDT,

$$\begin{cases} \tau_{a(p,[p|q])} > -\frac{\ln \mu_{[p|q]}}{\ln (1+\varsigma_p)}, (-1 < \varsigma_p < 0, p \in \mathcal{S}), \\ \tau_{a(p,[p|q])} > \tau^*_{a(p,[p|q])}, (\forall \tau^*_{a(p,[p|q])} > 0, \varsigma_p > 0, p \in \mathcal{U}), \\ \frac{T^-}{T^+} > \frac{\ln \gamma_s^+ - \ln \gamma}{\ln \gamma - \ln \gamma_s^-}, (0 < \gamma_s^- < \gamma < 1), \end{cases} \tag{4.162}$$

or with SBAPDT

$$\begin{cases} \tau_{a(q,[p|q])} > -\frac{\ln \mu_{[p|q]}}{\ln (1+\varsigma_q)}, \ (-1 < \varsigma_q < 0, \ q \in \mathcal{S}), \\ \tau_{a(q,[p|q])} > \tau^*_{a(q,[p|q])}, (\forall \tau^*_{a(q,[p|q])} > 0, \varsigma_q > 0, q \in \mathcal{U}), \\ \frac{T^-}{T^+} > \frac{\ln \gamma_p^+ - \ln \gamma}{\ln \gamma - \ln \gamma_p^-}, (0 < \gamma_p^- < \gamma < 1), \end{cases} \tag{4.163}$$

the system is globally uniformly asymptotically stable with marginal γ, where T^- and T^+ stand for the total running time of stable and unstable subsystems. The definitions of γ_s^+,γ_p^+, γ_s^-,γ_p^- are given in (4.167)-(4.168) and (4.174)-(4.175).

Proof: For any $k > 0$ and $\forall k \in [\, k_i, k_{i+1}\,)$, $i \in Z_+$, let $k_0 = 0$, the set $\mathcal{S}'' \triangleq \{(p, q) : p = \sigma(k_j), q = \sigma(k_j - 1), j = 1, 2, 3, \cdots, i\}$, $V_{\sigma(k)}(k)$ denote $V_{\sigma(k)}(x(k))$, and $\overline{\varsigma}$ stand for $1 + \varsigma$. According to (4.159)-(4.161), one can get the system (4.140).

Let $N_{\sigma[p|q]_{(g)}}(k, 0)$, $T_{q,[p|q]_{(g)}}(k, 0)$ and $T_{p,[p|q]_{(g)}}(k, 0)$ denote the activated numbers, total proceeding dwell time and total subsequent dwell time of the g^{th} element in the time interval $[0, k)$ respectively.

Next, we provide proof for the SBASDT and SBAPDT switching separately.

(a) The SBASDT switching:

According to (4.130), one can obtain

$$\begin{aligned}
&V_{\sigma(k)}(k)\\
&\leq \prod_{g=1,p\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(p,[p|q]_{(g)})}+\frac{T_{p,[p|q]_{(g)}}(k,k_1)}{\tau_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}^{T_{p,[p|q]_{(g)}}(k,k_1)}\\
&\times \prod_{g=1,p\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(p,[p|q]_{(g)})}+\frac{T_{p,[p|q]_{(g)}}(k,k_1)}{\tau_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}^{T_{p,[p|q]_{(g)}}(k,k_1)} \overline{\varsigma}_{\sigma(0)}^{(k_1)} V_{\sigma(0)}(x(0))\\
&= \left\{\prod_{g=1}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(p,[p|q]_{(g)})}}\right\} \left\{\prod_{g=1,p\in\mathcal{S}}^{g=s''} (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}})^{T_{p,[p|q]_{(g)}}(k,k_1)}\right\}\\
&\times \left\{\prod_{g=1,p\in\mathcal{U}}^{g=s''} (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}})^{T_{p,[p|q]_{(g)}}(k,k_1)}\right\} \left\{\overline{\varsigma}_{\sigma(0)}^{(k_1)} V_{\sigma(0)}\right\}(x(0)).
\end{aligned} \tag{4.164}$$

We make the following definitions:

$$T^- \triangleq \begin{cases} \sum\limits_{p\in\mathcal{S}} T_{p,[p|q]}(k,k_1)+k_1, \sigma(0)\in\mathcal{S},\\ \sum\limits_{p\in\mathcal{S}} T_{p,[p|q]}(k,k_1), \sigma(0)\in\mathcal{U}, \end{cases} \triangleq \sum_{p\in\mathcal{S}} T_p(k,0), \tag{4.165}$$

$$T^+ \triangleq \begin{cases} \sum\limits_{p\in\mathcal{U}} T_{p,[p|q]}(k,k_1)+k_1, \sigma(0)\in\mathcal{U},\\ \sum\limits_{p\in\mathcal{U}} T_{p,[p|q]}(k,k_1), \sigma(0)\in\mathcal{S}, \end{cases} \triangleq \sum_{p\in\mathcal{U}} T_p(k,0), \tag{4.166}$$

$$\gamma_s^- \triangleq \max_{p\in\mathcal{S}} \left\{\mu_{[p|q]}^{\frac{1}{\tau_{a(p,[p|q])}}} \overline{\varsigma}_{p,[p|q]}\right\}, \tag{4.167}$$

$$\gamma_s^+ \triangleq \max_{p\in\mathcal{U}} \left\{\mu_{[p|q]}^{\frac{1}{\tau_{a(p,[p|q])}}} \overline{\varsigma}_{p,[p|q]}\right\}, \tag{4.168}$$

$$K_5 \triangleq \prod_{g=1}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(p,[p|q]_{(g)})}} \gamma^{-k_1} \overline{\varsigma}_{\sigma(0)}^{k_1}. \tag{4.169}$$

According to (4.164)-(4.168) and the SBASDT condition (4.162), one can have

$$\begin{aligned} & V_{\sigma(k)}(k) \\ & \leq K_5 \gamma_s^{-T^-} \gamma_s^{+T^+} V_{\sigma(0)}(0) \\ & \leq K_5 \gamma^{(k-k_0)} V_{\sigma(0)}(0), \end{aligned} \tag{4.170}$$

Therefore, the switched system is globally uniformly asymptotically stable.

(b) The SBAPDT switching:

According to (4.131), one can obtain

$$\begin{aligned} & V_{\sigma(k)}(k) \\ & \leq \prod_{g=1,q\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(q,[p|q]_{(g)})}+\frac{T_{q,[p|q]_{(g)}}(k_i,0)}{\tau_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}}^{T_{q,[p|q]_{(g)}}(k_i,0)} \\ & \times \prod_{g=1,q\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(q,[p|q]_{(g)})}+\frac{T_{q,[p|q]_{(g)}}(k_i,0)}{\tau_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}}^{T_{q,[p|q]_{(g)}}(k_i,0)} \overline{\varsigma}_{\sigma(k_i)}^{(k-k_i)} V_{\sigma(0)}(x(0)) \\ & = \left\{ \prod_{g=1}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(q,[p|q]_{(g)})}} \right\} \left\{ \prod_{g=1,q\in\mathcal{S}}^{g=s''} (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}})^{T_{q,[p|q]_{(g)}}(k_i,0)} \right\} \\ & \times \left\{ \prod_{g=1,p\in\mathcal{U}}^{g=s''} (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}})^{T_{q,[p|q]_{(g)}}(k_i,0)} \right\} \overline{\varsigma}_{\sigma(k_i)}^{(k-k_i)} V_{\sigma(0)}(x(0)). \end{aligned} \tag{4.171}$$

We make the following definitions:

$$T^- \triangleq \begin{cases} \sum\limits_{q\in\mathcal{S}} T_{q,[p|q]}(k_i,0) + k - k_i, \sigma(k_i) \in \mathcal{S}, \\ \sum\limits_{q\in\mathcal{S}} T_{q,[p|q]}(k_i,0), \sigma(k_i) \in \mathcal{U}, \end{cases} \triangleq \sum_{q\in\mathcal{S}} T_q(k,0), \tag{4.172}$$

$$T^+ \triangleq \begin{cases} \sum\limits_{q\in\mathcal{U}} T_{q,[p|q]}(k_i,0) + k - k_i, \sigma(k_i) \in \mathcal{U}, \\ \sum\limits_{q\in\mathcal{U}} T_{q,[p|q]}(k_i,0), \sigma(k_i) \in \mathcal{S}, \end{cases} \triangleq \sum_{q\in\mathcal{U}} T_q(k,0), \tag{4.173}$$

$$\gamma_p^- \triangleq \max_{q\in\mathcal{S}} \left\{ \mu_{[p|q]}^{\frac{1}{\tau_{a(q,[p|q])}}} \overline{\varsigma}_{q,[p|q]} \right\}, \tag{4.174}$$

$$\gamma_p^+ \triangleq \max_{q\in\mathcal{U}}\left\{\mu_{[p|q]}^{\frac{1}{\tau_{a(q,[p|q])}}}\bar{\varsigma}_{q,[p|q]}\right\}, \tag{4.175}$$

$$K_6 \triangleq \prod_{g=1}^{g=s''}\mu_{[p|q]_{(g)}}^{N_{0(q,[p|q]_{(g)})}}\gamma^{-(k-k_i)}\bar{\varsigma}_{\sigma(k_i)}^{(k-k_i)}. \tag{4.176}$$

According to (4.171)-(4.176) and the SBAPDT condition (4.163), one can get

$$\begin{aligned} & V_{\sigma(k)}(k) \\ \leq\ & K_6\gamma_p^{-T^-}\gamma_p^{+T^+}V_{\sigma(0)}(0) \\ \leq\ & K_6\gamma^{(k-k_0)}V_{\sigma(0)}(0), \end{aligned} \tag{4.177}$$

Therefore, the switched system is globally uniformly asymptotically stable.

Combining (a) and (b), this lemma is proved.■

If the considered system is linear, we have the following Theorem 22.

Theorem 22: Consider the system (4.128). Let $\varsigma_p > 0$, $\varsigma_q > 0$, $\mu_{[p|q]} > 1$ be given constants. Suppose that there exist matrices $P_p > 0$, $P_q > 0$, such that $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{M} \times \mathcal{M}$ and $p \neq q$,

$$\begin{bmatrix} (\varsigma_p+1)P_p & A_p^T P_p \\ * & P_p \end{bmatrix} > 0, \tag{4.178}$$

$$\begin{bmatrix} (\varsigma_q+1)P_q & A_q^T P_q \\ * & P_q \end{bmatrix} > 0, \tag{4.179}$$

$$V_p(k_i) \leq \mu_{[p|q]}V_q(k_i). \tag{4.180}$$

Then for any switching signals with SBASDT satisfying (4.162) or with SBAPDT satisfying (4.163), the system (4.128) is globally uniformly asymptotically stable.

For the continuous-time systems, We can obtain Lemma 25 and Theorem 23 below. The proofs are similar to those for the discrete systems.

Lemma 25: Consider the system (4.127). Let α_p, α_q and $\mu_{[p|q]} > 1$ be given constants. Suppose that there exist $\mathbb{C}^1$ functions $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}$, and class $\mathcal{K}_\infty$ functions $\kappa_{p1}, \kappa_{p2}, \kappa_{q1}, \kappa_{q2}$, such that $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M}$ and $p \neq q$,

$$\begin{cases} \kappa_{p1}(\| x(t) \|) \leq V_p(x(t)) \leq \kappa_{p2}(\| x(t) \|), \\ \kappa_{q1}(\| x(t) \|) \leq V_q(x(t)) \leq \kappa_{q2}(\| x(t) \|), \end{cases} \tag{4.181}$$

$$\begin{cases} \dot{V}_p(x(t)) \leq \alpha_p V_p(x(t)), \\ \dot{V}_q(x(t)) \leq \alpha_q V_q(x(t)), \end{cases} \tag{4.182}$$

$$V_p(x(t_i)) \leq \mu_{[p|q]} V_q(x(t_i)). \tag{4.183}$$

Then for any switching signals with SBASDT

$$\begin{cases} \tau_{a(p,[p|q])} > \tau^*_{a(p,[p|q])} \triangleq -\ln \mu_{[p|q]}/\alpha_p, (\alpha_p < 0, p \in \mathcal{S}), \\ \tau_{a(p,[p|q])} > \tau^*_{a(p,[p|q])}, (\forall \tau^*_{a(p,[p|q])} > 0, \alpha_p > 0, p \in \mathcal{U}), \\ \frac{T_s^-}{T_s^+} > \frac{\gamma_s^+ + \gamma^*}{\gamma_s^- - \gamma^*}, (0 < \gamma^* < \gamma_s^-), \end{cases} \tag{4.184}$$

or with SBAPDT

$$\begin{cases} \tau_{a(q,[p|q])} > \tau^*_{a(q,[p|q])} \triangleq -\ln \mu_{[p|q]}/\alpha_q, (\alpha_q < 0, q \in \mathcal{S}), \\ \tau_{a(q,[p|q])} > \tau^*_{a(q,[p|q])}, (\forall \tau^*_{a(q,[p|q])} > 0, \alpha_q > 0, q \in \mathcal{U}), \\ \frac{T_p^-}{T_p^+} > \frac{\gamma_p^+ + \gamma^*}{\gamma_p^- - \gamma^*}, (0 < \gamma^* < \gamma_p^-), \end{cases} \tag{4.185}$$

the system (4.127) is globally uniformly asymptotically stable with marginal γ^*, where T^- and T^+ stand for the total running time of stable and unstable subsystems. The parameters of γ_s^+,γ_p^+, γ_s^-,γ_p^- are given as:

$$\gamma_s^- \triangleq \max_{p \in \mathcal{S}} \left\{ \alpha_p + \frac{ln\mu_{[p|q]}}{\tau_{a(p,[p|q])}} \right\}, \tag{4.186}$$

$$\gamma_s^+ \triangleq \max_{p\in\mathcal{U}}\left\{\alpha_p + \frac{ln\mu_{[p|q]}}{\tau_{a(p,[p|q])}}\right\}, \tag{4.187}$$

$$\gamma_p^- \triangleq \max_{q\in\mathcal{S}}\left\{\alpha_q + \frac{ln\mu_{[p|q]}}{\tau_{a(q,[p|q])}}\right\}, \tag{4.188}$$

$$\gamma_p^+ \triangleq \max_{q\in\mathcal{U}}\left\{\alpha_q + \frac{ln\mu_{[p|q]}}{\tau_{a(q,[p|q])}}\right\}. \tag{4.189}$$

Theorem 23: Consider the system (4.129). Let α_p, α_q and $\mu_{[p|q]} > 1$ be given constants. Suppose that there exist matrices $P_p > 0$, $P_q > 0$, such that $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M}\times\mathcal{M}$ and $p \neq q$,

$$A_p^T P_p + P_p A_p \leq \alpha_p P_p, \tag{4.190}$$

$$A_q^T P_q + P_q A_q \leq \alpha_q P_q, \tag{4.191}$$

$$P_p \leq \mu_{[p|q]} P_q. \tag{4.192}$$

Then the system (4.129) is globally uniformly asymptotically stable for any switching signals with SBASDT satisfying (4.184) or with SBAPDT satisfying (4.185).

4.4 Switching among Unstable Subsystems

In this section, we consider the switched systems, which are all unstable.

We consider the following nonlinear system models:

a) discrete-time switched systems,

$$x(k+1) = f_{\sigma(k)}\left(x(k)\right),\ x(k_0) = x_0; \tag{4.193}$$

b) continuous-time switched systems:

$$\dot{x}(t) = f_{\sigma(t)}\left(x(t)\right),\ x(t_0) = x_0. \tag{4.194}$$

The symbol $x(k)$ $(or\ x(t)) \in \mathbb{R}^n$ denotes a n-dimension state vector.

Correspondingly, the considered linear switched system models are:
discrete-time switched systems

$$x(k+1) = A_{\sigma(k)}x(k),\ x(k_0) = x(0); \tag{4.195}$$

continuous-time switched systems

$$\dot{x}(t) = A_{\sigma(t)}x(t),\ x(t_0) = x(0). \tag{4.196}$$

For the nonlinear discrete-time switched system (4.193), we can get the following Lemma 26 using the fast sequence-based average dwell time approach.

Lemma 26: Consider the nonlinear discrete-time switched system (4.193) with the given constants $\varsigma_p > 0$, $\varsigma_q > 0$, $0 < \mu_{[p|q]} < 1$. Suppose that there exist $\mathcal{C}^1$ functions $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, and class $\mathcal{K}_\infty$ functions $\kappa_{p1}, \kappa_{p2}, \kappa_{q1}, \kappa_{q2}$, such that $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{M} \times \mathcal{M}$, $p \neq q$,

$$\begin{cases} \kappa_{p1}(\| x(k) \|) \leq V_p(x(k), k) \leq \kappa_{p2}(\| x(k) \|), \\ \kappa_{q1}(\| x(k) \|) \leq V_q(x(k), k) \leq \kappa_{q2}(\| x(k) \|), \end{cases} \tag{4.197}$$

$$\begin{cases} V_p(x(k+1), k+1) - V_p(x(k), k) \leq \varsigma_p V_p(x(k), k), \\ V_q(x(k+1), k+1) - V_q(x(k), k) \leq \varsigma_q V_q(x(k), k), \end{cases} \tag{4.198}$$

$$V_p(x(k_i), k_i) \leq \mu_{[p|q]} V_q(x(k_i), k_i), \tag{4.199}$$

then the system is globally uniformly asymptotically stable for any switching signals with fast SBASDT

$$\tau^c_{a(p,[p|q])} < -\frac{\ln \mu_{[p|q]}}{\ln (1 + \varsigma_p)}, \tag{4.200}$$

or with fast SBAPDT

$$\tau^c_{a(q,[p|q])} < -\frac{\ln \mu_{[p|q]}}{\ln (1 + \varsigma_q)}. \tag{4.201}$$

Proof: For any $k > 0$ and $\forall k \in [\, k_i, k_{i+1}\,)$, $i \in Z_+$, let $k_0 = 0$, the set $\mathcal{S}'' \triangleq \{(p, q) : p = \sigma(k_j), q = \sigma(k_j - 1), j = 1, 2, 3, \cdots, i\}$, $V_{\sigma(k)}(k)$ denote $V_{\sigma(k)}(x(k), k)$, and $\overline{\varsigma}$

stand for $1+\varsigma$. According to (4.197)-(4.199), one can get

$$\begin{aligned} & V_{\sigma(k)}(k) \\ \leq & (1+\varsigma_{\sigma(k_i)})^{(k-k_i)} V_{\sigma(k_i)}(k_i) \\ \leq & \mu_{\sigma(k_i)}(1+\varsigma_{\sigma(k_i)})^{(k-k_i)} V_{\sigma(k_i-1)}(k_i) \\ \leq & \mu_{\sigma(k_i)} \overline{\varsigma}_{\sigma(k_i)}^{(k-k_i)} \overline{\varsigma}_{\sigma(k_{i-1})}^{(k_i-k_{i-1})} V_{\sigma(k_{i-1})}(k_{i-1}). \end{aligned} \tag{4.202}$$

At last, we can get

$$V_{\sigma(k)}(k) \leq \prod_{j=1}^{i} \mu_{\sigma(k_j)} \overline{\varsigma}_{\sigma(k_i)}^{(k-k_i)} \cdots \overline{\varsigma}_{\sigma(0)}^{(k_1-0)} V_{\sigma(0)}(0). \tag{4.203}$$

If $\sigma(k_j)=p$ and $\sigma(k_j-1)=q$, $\mu_{\sigma(k_j)}$ is represented by $\mu_{[p|q]}$ to reveal the switching sequence.

We order all the elements in the set $\mathcal{S}''$, and represent the ordered elements as $[p|q]_{(g)}$ which means the element (p,q) is the g^{th} element of the set $\mathcal{S}''$. The amount of all its elements is s''.

Let $N^c_{\sigma[p|q]_{(g)}}(k,0)$, $T^c_{p,[p|q]_{(g)}}(k,0)$ and $T^c_{q,[p|q]_{(g)}}(k,0)$ denote the activated numbers, total subsequent dwell time and total proceeding dwell time of the g^{th} element in the time interval $[0,k)$ for fast switching respectively.

Therefore, it follows

$$V_{\sigma(k)}(k) \leq \left\{ \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{\sigma[p|q]_{(k)}}(k,0)} \overline{\varsigma}_{p,[p|q]_{(g)}}^{T^c_{p,[p|q]_{(g)}}(k,0)} \right\} \overline{\varsigma}_{\sigma(0)}^{(k_1)} V_{\sigma(0)}(x(0)). \tag{4.204}$$

Next, we provide proof separately for the SBASDT and SBAPDT switching.

(a) The SBASDT switching:

According to (4.132) and $T^c_{p,[p|q]_{(g)}}(k,0) = T^c_{p,[p|q]_{(g)}}(k,k_1)$, one can obtain

$$\begin{aligned}
&V_{\sigma(k)}(k)\\
&\leq \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(p,[p|q]_{(g)})}+\frac{T^c_{p,[p|q]_{(g)}}(k,0)}{\tau^c_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}^{T^c_{p,[p|q]_{(g)}}(k,k_1)} \overline{\varsigma}^{(k_1)}_{\sigma(0)} V_{\sigma(0)}(x(0))\\
&= \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(p,[p|q]_{(g)})}} \mu_{[p|q]_{(g)}}^{\frac{T^c_{p,[p|q]_{(g)}}(k,0)}{\tau^c_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}^{T^c_{p,[p|q]_{(g)}}(k,k_1)} \overline{\varsigma}^{(k_1)}_{\sigma(0)} V_{\sigma(0)}(x(0))\\
&= \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(p,[p|q]_{(g)})}} (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}})^{T^c_{p,[p|q]_{(g)}}(k,k_1)} \overline{\varsigma}^{(k_1)}_{\sigma(0)} V_{\sigma(0)}(x(0)).
\end{aligned} \tag{4.205}$$

We make the following definitions:

$$\gamma_1 \triangleq \max_g \left\{ (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}) \right\},$$

$$K_1 \triangleq \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(p,[p|q]_{(g)})}} \gamma_1^{-k_1} \overline{\varsigma}^{k_1}_{\sigma(0)}.$$

According to the switching condition (4.200) and $\varsigma_p > 0$, we can know that $0 < \gamma_1 < 1$.

Therefore, we have

$$V_{\sigma(k)}(k) \leq K_1 \gamma_1^{(k-k_0)} V_{\sigma(0)}(x(0)). \tag{4.206}$$

Therefore, the switched system is globally uniformly asymptotically stable.

(b) The SBAPDT switching:

According to (4.133), (4.204) and $T^c_{q,[p|q]_{(g)}}(k,0) = T^c_{q,[p|q]_{(g)}}(k_i,0)$, one can obtain

$$\begin{aligned}
&V_{\sigma(k)}(k)\\
&\leq \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(q,[p|q]_{(g)})}+\frac{T^c_{q,[p|q]_{(g)}}(k,0)}{\tau^c_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}}^{T^c_{q,[p|q]_{(g)}}(k_i,0)} \overline{\varsigma}^{(k-k_i)}_{\sigma(k_i)} V_{\sigma(0)}(x(0))\\
&= \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(q,[p|q]_{(g)})}} \mu_{[p|q]_{(g)}}^{\frac{T^c_{q,[p|q]_{(g)}}(k,0)}{\tau^c_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}}^{T^c_{q,[p|q]_{(g)}}(k_i,0)} \overline{\varsigma}^{(k-k_i)}_{\sigma(k_i)} V_{\sigma(0)}(x(0))\\
&= \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(q,[p|q]_{(g)})}} (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}})^{T^c_{q,[p|q]_{(g)}}(k_i,0)} \overline{\varsigma}^{(k-k_i)}_{\sigma(k_i)} V_{\sigma(0)}(x(0)).
\end{aligned} \tag{4.207}$$

We make the following definitions:

$$\gamma_2 \triangleq \max_g \left\{ (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}}) \right\},$$

$$K_2 \triangleq \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(q,[p|q]_{(g)})}} \gamma_2^{-(k-k_i)} \overline{\varsigma}_{\sigma(k_i)}^{(k-k_i)}.$$

From the switching condition (4.201) and $\varsigma_p > 0$, we can know that $0 < \gamma_2 < 1$.

We can then get

$$V_{\sigma(k)}(k) \leq K_2 \gamma_2^{(k-k_0)} V_{\sigma(0)}(x(0)). \tag{4.208}$$

Therefore, the switched system is globally uniformly asymptotically stable.

Combining (a) and (b), this lemma is proved. ■

If the considered system is linear, we have the following theorem.

Theorem 24: Consider the system (4.195), let $\varsigma_p > 0$, $\varsigma_q > 0$, $0 < \mu_{[p|q]} < 1$, $\tau_{min,p} = \min_{\sigma(k_i)=p}(k_i - k_{i-1})$, $\tau_{min,q} = \min_{\sigma(k_i)=q}(k_i - i_{i-1})$ be given constants. If there exist a set of positive definite matrices $P_{p,j_p} > 0$, $P_{q,j_q} > 0$, $j_p = 0,\ 1, \cdots,\ \tau_{min,p}$, $j_q = 0,\ 1, \cdots,\ \tau_{min,q}$, such that $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{M} \times \mathcal{M}$, $p \neq q$, $\forall j_p = 0,\ 1, \cdots,\ \tau_{min,p} - 1$, $\forall j_q = 0,\ 1, \cdots,\ \tau_{min,q} - 1$,

$$\begin{cases} \begin{bmatrix} (\varsigma_p + 1)P_{p,j_p+1}, & A_p^T P_{p,j_p} \\ * & P_{p,j_p} \end{bmatrix} > 0, \\ \begin{bmatrix} (\varsigma_p + 1)P_{p,\tau_{min,p}}, & A_p^T P_{p,\tau_{min,p}} \\ * & P_{p,\tau_{min,p}} \end{bmatrix} > 0, \end{cases} \tag{4.209}$$

$$\begin{cases} \begin{bmatrix} (\varsigma_q + 1)P_{q,j_q+1}, & A_q^T P_{q,j_q} \\ * & P_{q,j_q} \end{bmatrix} > 0, \\ \begin{bmatrix} (\varsigma_q + 1)P_{q,\tau_{min,q}}, & A_q^T P_{q,\tau_{min,q}} \\ * & P_{q,\tau_{min,q}} \end{bmatrix} > 0, \end{cases} \tag{4.210}$$

$$P_{p,0} \leq \mu_{[p|q]} P_{q,\tau_{min,p}}, \tag{4.211}$$

then the system is globally uniformly asymptotically stable for any switching signals with fast SBASDT satisfying (4.200) or with fast SBAPDT satisfying (4.201).

Proof: For the multiple Lyapunov functions for linear discrete-time switched systems, (4.195) can be rewritten as

$$V_{\sigma(k)}(k) = x^T(k) P_{\sigma(k)}(k) x(k), \sigma(k) \in \mathcal{M}. \tag{4.212}$$

For $\forall k \in [k_i, k_{i+1})$, the matrix $P_{\sigma(k)}(k)$ is defined as

$$P_{\sigma(k)}(k) = \begin{cases} P_{\sigma(t_i)}(k - k_i), \ k \in [k_i, k_i + \tau_{min,\sigma(k_i)}) \\ P_{\sigma(t_i)}(\tau_{min,\sigma(k_i)}), \ k \in [k_i + \tau_{min,\sigma(k_i)}, k_{i+1}) \end{cases}. \tag{4.213}$$

We rewrite $P_{\sigma(k_i)}(k - k_i)$ and $P_{\sigma(k_i)}(\tau_{min,\sigma(k_i)})$ as $P_{\sigma(k_i),k-k_i}$ and $P_{\sigma(k_i),\tau_{min,\sigma(k_i)}}$.

The condition (4.209) implies (4.197). The condition (4.210) means (4.198). Because of (4.211), we can get (4.199). ■

For the continuous-time systems, we can get Lemma 27 and Theorem 25 below, while the proofs are similar to those for Lemma 1 and Theorem1.

Lemma 27: Consider the system (4.194), let $0 < \mu_{[p|q]} < 1, \alpha_p > 0, \alpha_q > 0$ be the given constants. Suppose that there exist $\mathbb{C}^1$ functions $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}$, and class $\mathcal{K}_\infty$ functions $\kappa_{p1}, \ \kappa_{p2}, \ \kappa_{q1}$ *and* κ_{q2}, such that $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M}$ and $p \neq q$,

$$\begin{cases} \kappa_{p1}(\| x(t) \|) \leq V_p(x(t), t) \leq \kappa_{p2}(\| x(t) \|), \\ \kappa_{q1}(\| x(t) \|) \leq V_q(x(t), t) \leq \kappa_{q2}(\| x(t) \|), \end{cases} \tag{4.214}$$

$$\begin{cases} \dot{V}_p(x(t), t) \leq \alpha_p V_p(x(t), t), \\ \dot{V}_q(x(t), t) \leq \alpha_q V_q(x(t), t), \end{cases} \tag{4.215}$$

and

$$V_p(x(t_i), t_i) \leq \mu_{[p|q]} V_q(x(t_i), t_i), \tag{4.216}$$

then the system is globally uniformly asymptotically stable for any switching signals with fast SBASDT

$$\tau_{a(p,[p|q])} < \tau^*_{a(p,[p|q])} = -ln\mu_{[p|q]}/\alpha_p, \tag{4.217}$$

or with fast SBAPDT

$$\tau_{a(q,[p|q])} < \tau^*_{a(q,[p|q])} = -ln\mu_{[p|q]}/\alpha_q. \tag{4.218}$$

Theorem 25: Consider the system (4.196), let $0 < \mu_{[p|q]} < 1,\ \alpha_p > 0,\ \alpha_q > 0,$ $\tau_{min,p} = \min_{\sigma(t_i)=p}(t_i - t_{i-1})$, $\tau_{min,q} = \min_{\sigma(t_i)=q}(t_i - t_{i-1})$ and l_p, l_q be the given constants. If there exist a set of positive definite matrices $P_{p,j_p} > 0,\ P_{q,j_q} > 0,\ j_p = 0,\ 1, \cdots,\ l_p,$ $j_q = 0,\ 1, \cdots,\ l_q,$ such that $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M},\ p \neq q,\ \forall j_p = 0,\ 1, \cdots,\ l_p - 1, \forall j_q = 0,\ 1, \cdots,\ l_q - 1,$

$$\begin{cases} A_p^T P_{p,j_p} + P_{p,j_p} A_p + \frac{l_p(P_{p,j_p+1} - P_{p,j_p})}{\tau_{min,p}} \leq \alpha_p P_{p,j_p}, \\ A_p^T P_{p,j_p+1} + P_{p,j_p+1} A_p + \frac{l_p(P_{p,j_p+1} - P_{p,j_p})}{\tau_{min,p}} \leq \alpha_p P_{p,j_p+1}, \\ A_p^T P_{p,l_p} + P_{p,l_p} A_p \leq \alpha_p P_{p,l_p}, \end{cases} \tag{4.219}$$

$$\begin{cases} A_q^T P_{q,j_q} + P_{q,j_q} A_q + \frac{l_q(P_{q,j_q+1} - P_{q,j_q})}{\tau_{min,q}} \leq \alpha_q P_{q,j_q}, \\ A_q^T P_{q,j_q+1} + P_{q,j_q+1} A_q + \frac{l_q(P_{q,j_q+1} - P_{q,j_q})}{\tau_{min,q}} \leq \alpha_q P_{q,j_q+1}, \\ A_q^T P_{q,l_q} + P_{q,l_q} A_q \leq \alpha_q P_{q,l_q}, \end{cases} \tag{4.220}$$

$$P_{p,0} \leq \mu_{[p|q]} P_{q,l_p}, \tag{4.221}$$

then the system is globally uniformly asymptotically stable for any switching signals with SBASDT satisfying (4.217) or with SBAPDT satisfying (4.218).

Remark 9: For the sequence-based approach, it just requires $0 < \mu_{[p|q]} < 1$, when $\sigma(t_i) = p, \sigma(t_i^-) = q,\ i \in Z_+$. If some sequences do not appear, there is no constraint on their μ values. For example, Fig. 4.1 shows the periodically switched systems. For this kind of switched systems, we do not need to check whether $\mu_{[3|1]}$,$\mu_{[2|3]}$ $\mu_{[1|2]}$ are less than 1 or not.

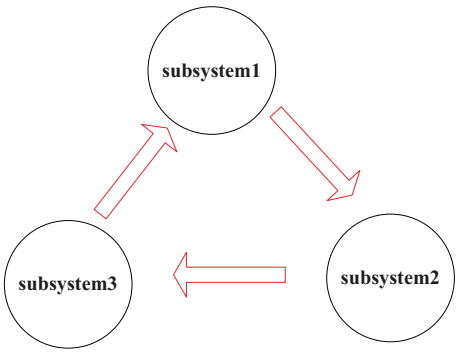


Figure 4.1 : A subsystem running sequence graph.

4.5 Numerical Examples

In this subsection, we use numerical examples to show the effectiveness of the proposed methods and the derived results. Because of the space limitation and the similarity between continuous-time and discrete-time systems, we only verify Corollary 5, Theorem 16, Theorem 18 and Theorem 20. We get four examples, respectively. Other obtained results can be verified in the same way.

4.5.1 Example 1

In this subsection, we use a numerical example to show the effectiveness of the proposed methods and the derived results in Corollary 5. Other obtained results can be verified in the same way.

Consider switched linear systems including three subsystems described as

$$A_1 = \begin{bmatrix} 100.3 & -20.1 \\ -10.1 & -10.2 \end{bmatrix}, B_1 = \begin{bmatrix} -0.5 & -0.8 \end{bmatrix}^T,$$

$$A_2 = \begin{bmatrix} 10.8 & -10.2 \\ 2 & 10.5 \end{bmatrix}, B_2 = \begin{bmatrix} -10.1 & 10 \end{bmatrix}^T,$$

$$A_3 = \begin{bmatrix} 0.2 & -3.59 \\ 12 & 10.4 \end{bmatrix}, B_3 = \begin{bmatrix} 5.1 & -10.1 \end{bmatrix}^T,$$

Table 4.1 : compution for the MDADT and SBASDT switching

Schemes	MDADT switching	SBASDT switching
Criteria	Theorem 1 in [28]	Corollary 5 in this note
Parameters	λ_1=3; $\lambda_2 = 1.5$; $\lambda_3 = 2.5$; $\mu_1 = 18$; $\mu_2 = 2.3$; $\mu_3 = 41$.	λ_1=3; $\lambda_2 = 1.5$; $\lambda_3 = 2.5$; $\mu_{1\|2} = 18$; $\mu_{2\|1} = 2.3$; $\mu_{3\|1} = 41$; $\mu_{1\|3} = 13$; $\mu_{2\|3} \to 1$; $\mu_{3\|2} = 17$.
Controller	$K1 = [403.6393, -107.2597]$; $K2 = [2.8646, -0.6579]$; $K3 = [-9.4055, -2.6831]$.	K1 = [371.7662, -100.0154]; K2 = [2.8330, -0.4917]; K3 = [-7.9922, -2.0744].
Average Dwell Time Thresholds	τ^*_{a1}=0.96; τ^*_{a2}=0.56; τ^*_{a3}=1.5.	$\tau^*_{a(1,1\|2)} = 0.96$; $\tau^*_{a(1,1\|3)} = 0.86$; $\tau^*_{a(2,2\|1)} = 0.56$; $\tau^*_{a(2,2\|3)} \to 0$; $\tau^*_{a(3,3\|2)} = 1.1$; $\tau^*_{a(3,3\|1)} = 1.5$.

In this example, we want to get a set of proper sequence-based stabilizing controller gains and search the admissible minimal SBADT to make the closed-loop system stable.

In order to show the advantage of the sequence-based average dwell time, we use Table 4.1 to show the numerical results for the mode-dependent average dwell time switching [25] and the sequence-based average subsequence dwell time switching.

It can be seen that the dwell time thresholds are reduced notably, according to Table 4.1. For the first dwell time set, 0.96 and 0.86 replace 0.96. Two values with one being 0.56 and the other approaching 0 replace 0.56 for the second set. The values 1.1 and 1.5 replace 1.5 for the third set. The control gains are also shown in Table 4.1.

4.5.2 Example 2

In this subsection, we verify the validity of Theorem 16, whose switching situation is shown in Fig. 4.2. Other theorems and cases can be verified similarly.

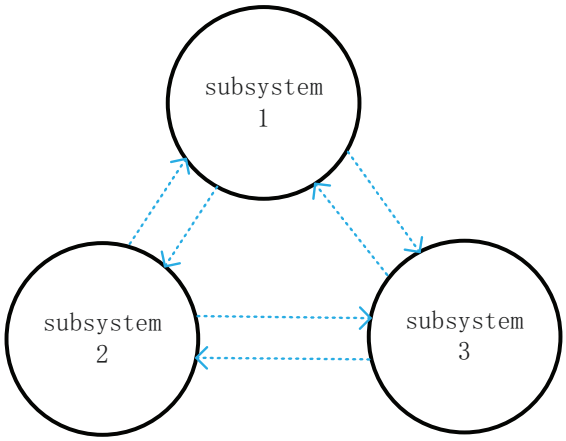


Figure 4.2 : Subsystems and switching in example 2.

In this example, the asynchronous switched system (4.69) is with

$$A_1 = \begin{bmatrix} 0.3 & -0.6 \\ 0.6 & -0.4 \end{bmatrix}, B_1 = \begin{bmatrix} -0.5 & -1.9 \end{bmatrix}^T, K_1 = [-1.4316 \quad 0.7373],$$

$$A_2 = \begin{bmatrix} 0.3 & 0.3 \\ -1.2 & 0.4 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & -1.5 \end{bmatrix}^T, K_2 = [-0.1760 \quad 0.5674],$$

$$A_3 = \begin{bmatrix} 0.3 & 0.3 \\ -1 & 0.4 \end{bmatrix}, B_3 = \begin{bmatrix} 0.2 & 1.5 \end{bmatrix}^T, K_3 = [-1.6987 \quad -0.6626].$$

In this example, each subsystem can switch to the other two subsystems randomly, as shown in Fig. 4.2.

We assume that asynchronous switching time is not more than 0.5000, 0, 0.7000, 0.8000, 0.9000, and 1.000 when $[p|q] = [2|1]$, $[p|q] = [1|2]$, $[p|q] = [1|3]$, $[p|q] = [3|1]$, $[p|q] = [3|2]$, and $[p|q] = [2|3]$, respectively. For the conventional ADT methods [28], the Matlab LMIs toolbox can find a feasible solution only if $\lambda \leq 0.3000$ or $\mu \geq 3.969$

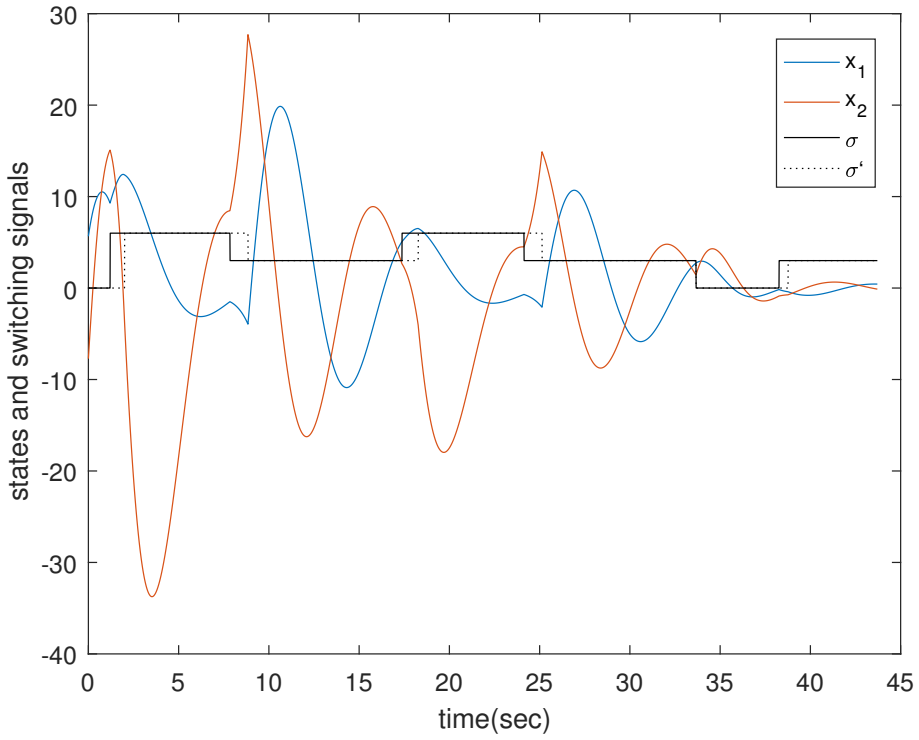


Figure 4.3 : States and switching signals

or $\alpha \geq 4.456$. When $\lambda = 0.3000$, $\mu = 3.969$, and $\alpha = 4.456$, we obtain $\tau_a^* = 20.45$. But for our proposed SBASDT method, the Matlab LMIs toolbox can find a solution even if we set $\lambda_1 = \lambda_2 = 0.3000$, $\lambda_3 = 0.6107$, $\alpha_{[2|3]} = 2.560$, $\alpha_{[1|2]} = 0.2678$, $\mu_{[2|1]} = \mu_{[1|2]} = 3.969$, $\mu_{[1|3]} = \mu_{[2|3]} = \mu_{[3|1]} = \mu_{[3|2]} = 1.001$, $\alpha = 4.456$, $\alpha_{[1|3]} = 4.456$, $\alpha_{[2|1]} = 0.2159$, $\alpha_{[3|1]} = 4.455$, and $\alpha_{[3|2]} = 3.975$. The conditions and results for this example are shown in Table 4.2.

It can be seen that our proposed sequence-based approach makes the threshold of average dwell time decrease significantly. Therefore, the proposed SBASDT method has considerable advantages compared with traditional ADT methods.

In order to show the globally asymptotic stability, the states x1 and x2 of the second-order switched system are shown in Fig. 4.3, and its Lyapunov function value is given in Fig. 4.4, where σ' is the asynchronous switching signal.

Table 4.2 : Comparison between ADT and SBASDT asynchronous switching for example 2

	in [28]	SBASDT results in this thesis
para-	$\lambda = 0.3000$;	$\lambda_1 = \lambda_2 = 0.3000; \lambda_3 = 0.6107; \alpha_{[2\|3]} = 2.560; \alpha_{[1\|2]} = 0.2678;$
me-	$\mu = 3.969$;	$\mu_{[2\|1]} = \mu_{[1\|2]} = 3.969$; $\mu_{[1\|3]} = \mu_{[2\|3]} = \mu_{[3\|1]} = \mu_{[3\|2]} = 1.001$;
ters	$\alpha = 4.456$.	$\alpha_{[1\|3]} = 4.456$; $\alpha_{[2\|1]} = 0.2159$; $\alpha_{[3\|1]} = 4.455$; $\alpha_{[3\|2]} = 3.975$.
switch-		$\tau^*_{a(1,[1\|2])} = 4.595$; $\tau^*_{a(2,[2\|1])} = 5.455$;
ing	τ^*_a=20.45.	$\tau^*_{a(3,[3\|1])} = 6.636$; $\tau^*_{a(1,[1\|3])} = 11.09$;
signals		$\tau^*_{a(2,[2\|3])} = 9.533$; $\tau^*_{a(3,[3\|2])} = 6.758$.

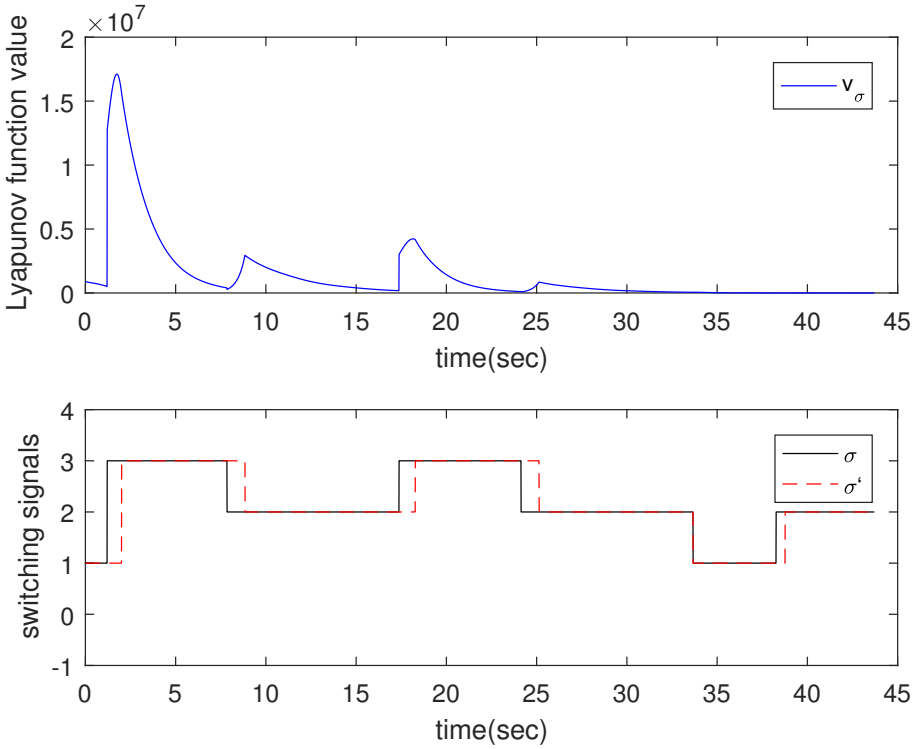


Figure 4.4 : Lyapunov function value and switching signals

4.5.3 Example 3

In this subsection, we verify the validity of Theorem 18, whose switching situation is shown in Fig. 4.5. Other theorems and cases can be verified similarly.

In this example, the asynchronous switched system (4.69) is with

$$A_1 = \begin{bmatrix} 0.3 & -0.6 \\ 0.6 & -0.4 \end{bmatrix}, B_1 = \begin{bmatrix} -0.5 & -1.9 \end{bmatrix}^T, K_1 = [-1.4316 \quad 0.7373],$$

$$A_2 = \begin{bmatrix} 0.3 & 0.3 \\ -1.2 & 0.4 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & -1.5 \end{bmatrix}^T, K_2 = [-0.1760 \quad 0.5674],$$

$$A_3 = \begin{bmatrix} 0.3 & 0.3 \\ -1 & 0.4 \end{bmatrix}, B_3 = \begin{bmatrix} 0.2 & 1.5 \end{bmatrix}^T, K_3 = [-1.6987 \quad -0.6626].$$

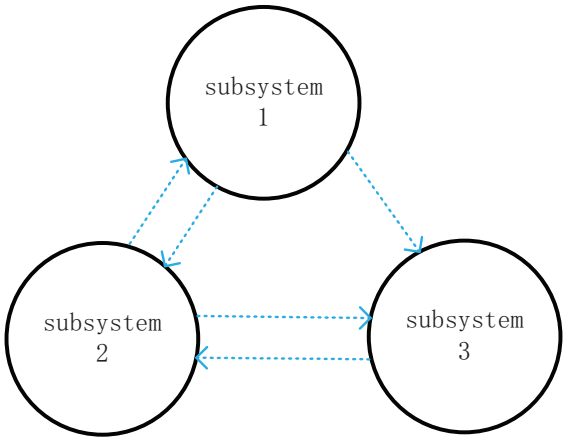


Figure 4.5 : Subsystems and switching in example 3

In this example, the subsystems 1 and 2 can switch to the other two subsystems randomly. The subsystem 3 can switch only to subsystem 1.

It is hard to know $T_{[p|q]_max}$ in practice. A common way is to estimate its value, e.g., as proposed in [28]. $T_{[p|q]_max}$ is not more than the asynchronous time "$T_{[p|q]}$" in every sequence (p,q). As it comes from the time "T" in system (2). Therefore, the maximal asynchronous time "$T_{[p|q]}$" is often taken as $T_{[p|q]_max}$. We assume that all the asynchronous switching time is 0.1.

For the conventional ADT methods, it is not feasible for the Matlab LMIs toolbox to find a solution according to the results in [28] when we set $\lambda = 0.3200$, $\mu = 3.800$,

Table 4.3 : Comparison between ADT and SBAPDT asynchronous switching for example 1

	theorems in [28]	SBAPDT results in this thesis
para-	$\lambda = 0.3200;$	$\lambda_1 = \lambda_2 = \lambda_3 = 0.3200;$
meters	$\mu = 3.800;$	$\mu_{[3\|1]} = \mu_{[2\|3]} = \mu_{[1\|2]} = \mu_{[2\|1]} = \mu_{[2\|1]} = 3.800;$
	$\alpha = 4.400.$	$\alpha_{[3\|1]} = \alpha_{[2\|3]} = \alpha_{[1\|2]} = \alpha_{[2\|1]} = \alpha_{[2\|1]} = 4.400;$
switching		$\tau^*_{a(2,[1\|2])} = 4.172;\ \tau^*_{a(1,[2\|1])} = 5.647;$
signals	Not feasible.	$\tau^*_{a(1,[3\|1])} = 5.647\ ;\ \tau^*_{a(2,[3\|2])} = 5.647;$
		$\tau^*_{a(3,[2\|3])} = 5.647.$

$\alpha = 4.400$. But for our proposed SBAPDT method, the Matlab LMIs toolbox can find a solution if we set $\lambda_1 = \lambda_2 = \lambda_3 = 0.3200$, $\mu_{[3|1]} = \mu_{[2|3]} = \mu_{[1|2]} = \mu_{[2|1]} = \mu_{[2|1]} = 3.800$, $\alpha_{[3|1]} = \alpha_{[2|3]} = \alpha_{[1|2]} = \alpha_{[2|1]} = \alpha_{[2|1]} = 4.400$. Therefore, the SBAPDT method is superior for the discussed problems. The conditions and numerical results for this example are shown in Table 4.3.

The design parameters are selected based on the desired system performance and parameters' physical meaning discussed in [24, 57, 80].

Remark 10: The sequences in this thesis are mainly referred to the set $\{(p,q) : (\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}, p \neq q, i = 1,\ 2, \cdots, \}$ and its elements. As $i \to +\infty$, the sequence set is not related to the time t_i anymore. It can be obtained directly from the switching population rather than being computed in real time. That is, it can be known in advance. For Example 3, all possible switching events are shown in Fig. 4.5. According to the set $\{(p,q) : (\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}, p \neq q, i = 1,\ 2, \cdots, \}$ and Fig. 4.5., one can obtain the sequences set as $\{(1,2),(2,1),(2,3),(3,1),(3,2)\}$. A sequence is one of its elements.

Remark 11: In some switched systems, certain sequences, which belong to $\mathcal{S}' \triangleq \{(p,q); p \in \mathcal{S}, q \in \mathcal{S}, p \neq q\}$, may not exist in practice. For these sequences, we do not need to check whether they satisfy the conditions (4.113)-(4.115), (4.117)-(4.119), (4.120)-(4.122), (4.123)-(4.125). That is, only the sequences in the set $\{(p,q) : (\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}, p \neq q, i = 1,\ 2, \cdots, \}$ need to be checked to determine whether they satisfy these conditions. In Example 1, the set $\{(p,q) : (\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{S} \times \mathcal{S}, p \neq q, i = 1,\ 2, \cdots, \}$=$\{(1,2),\ (2,1),\ (2,3),\ (3,1),\ (3,2)\}$. The sequence $\{1,3\}$ does not exist. Therefore, we do not need to check any condition in terms of $\mu_{[1|3]}$ or $\alpha_{[1|3]}$.

4.5.4 Example 4

In order to verify our proposed approach, we present an illustrative numerical example here. A comparison with the existing method in [81] is also provided. Considering the space limitation, here we only verify Theorem 20. Fast SBASDT is studied in that theorem.

Let us consider the linear discrete-time switched systems with

$$A_1 = \begin{bmatrix} -0.012 & -0.022 \\ -0.11 & 0.012 \end{bmatrix},\ A_2 = \begin{bmatrix} -0.5 & -0.9 \\ -0.7 & 0.2 \end{bmatrix},\ A_3 = \begin{bmatrix} 0.011 & -0.013 \\ -0.23 & 0.031 \end{bmatrix}.$$

The first and third subsystems are Schur stable. The second subsystem is Schur unstable.

Parameters and the results are provided in Table 4.4 for results in [81] and Theorem 20 in this thesis for these three subsystems. From Table 4.4, we can get the following observations.

a) For stable subsystems:

Table 4.4 : Parameters and results for the numerical example for verifying Theorem 20 in this thesis and results in [81].

	Results in [81]	Theorem 20 in this thesis
Given constants	$\lambda_1 = -0.98;$	$\lambda_1 = -0.98;$
	$\lambda_2 = 0.036;$	$\lambda_2 = 0.036;$
	$\lambda_3 = -0.95;$	$\lambda_3 = -0.95;$
	$\mu_1 = 2.5;$	$\mu_{[1\|2]} = 2.5;\ \mu_{[1\|3]} \to 1;$
	$\mu_2 = 0.41;$	$\mu_{[2\|1]} = 0.41; \mu_{[2\|3]} = 0.40;$
	$\mu_3 = 3.0.$	$\mu_{[3\|1]} = 1.2; \mu_{[3\|2]} = 3.0.$
Time thresholds (stable subsystems)	$\tau^*_{a1} = 0.9350;$ $\tau^*_{a3} = 1.1564.$	$\tau^*_{a(1,[1\|2])} = 0.9350;$ $\tau^*_{a(1,[1\|3])} \to 0;$ $\tau^*_{a(3,[3\|2])} = 1.1564;$ $\tau^*_{a(3,[3\|1])} = 0.1919.$
Time thresholds (unstable subsystems)	$\tau^*_{a2} = 24.7666.$	$\tau^*_{a(2,[2\|1])} = 24.7666;$ $\tau^*_{a(2,[2\|3])} = 25.4525.$

(a.1) According to results in [81]

$$\begin{cases} \tau_{a1} > 0.9350 \\ \tau_{a3} > 1.1564 \end{cases};$$

(a.2) According to Theorem 20 in this thesis

$$\begin{cases} \tau_{a(1,[1|2])} > 0.9350; \tau_{a(1,[1|3])} > 0 \\ \tau_{a(3,[3|2])} > 1.1564; \tau_{a(3,[3|1])} > 0.1919 \end{cases}.$$

b) For unstable subsystems:

(b.1) According to results in [81]

$$\tau_{a2}^{c} < 24.7666,$$

(b.2) According to Theorem 20 in this thesis

$$\tau_{a(2,[2|1])}^{c} < 24.7666; \tau_{a(2,[2|3])}^{c} < 25.4525.$$

Combining (a) and (b), we can know that our proposed sequence-based approach provides a less conservative condition for ensuring the stability of switched systems with unstable subsystems.

4.6 Summary

In the chapter, the stability of switched systems is analyzed based on proposed methods. The validation is shown by numerical examples.

Chapter 5

Consensus of the Second-order Multi-agent Systems under Switched Topologies Based on the Sequence

5.1 Introduction

In this chapter, the consensus of second-order systems is taken as an example to illustrate the cooperation of multi-agent systems. In this chapter, the context is limited to the second-order systems. The research object is restricted to the consensus.

The results in Chapter 4 will be applied to the analysis of the consensus of the multi-agent systems. We build model transactions from switched topologies to switched systems first, and then we apply the results in Chapter 4 to multi-agent systems under switched topologies. Finally, we get the new results about the consensus when the multi-agent systems change topologies in three cases: (a) a consensus can be reached under each topology; (b) a consensus cannot be reached under part topologies; (c) a consensus cannot be reached under any topology.

Various situations are analyzed in the chapter. We study the common second-order multi-agent systems, multi-agent systems with a fixed fault, multi-agent systems with uncertain faults, and asynchronous switching.

5.2 Model Transformation: from Switched Topologies to Switched systems

5.2.1 Common Second-order Multi-agent Systems

Consider a team of n agent systems where each agent is modelled with the second-order dynamics as follows [65, 66]:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t), \end{aligned} \tag{5.1}$$

where $x_i(t) = [x_{i1}(t), \ldots, x_{iN}(t)]^T \in \mathbb{R}^N$ is the state of the i^{th} agent, u_i is the control input.

The consensus algorithm is proposed as:

$$u_i(t)=\beta_0 \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(x_j(t) - x_i(t)) + \beta_1 \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(v_j(t) - v_i(t)), \tag{5.2}$$

where $a_{ij}, i, j \in \mathcal{I}$ are the entries of adjacency matrix $\mathcal{A}(\mathcal{G})$, given by the interaction topology $\mathcal{G}(t)$. Positive constant β_0, β_1 are the coupling strengths, and $\mathcal{K} = \{0, 1\}$.

Definition 24: The second-order consensus is said to be reached among n agents for the multi-agent systems if for all initial conditions, $x_i(t) \to x_j(t)$, $v_i(t) \to v_j(t)$, as $t \to +\infty$.

Let $x(t)=[x_1(t)^T, \ldots, x_n(t)^T]^T$, $v(t)=[\dot{x}_1(t)^T, \ldots, \dot{x}_n(t)^T]^T$ and $y(t)=[x(t)^T, v(t)^T]^T$. Then, network (5.1) with input (5.2) can be rewritten in a compact matrix as

$$\dot{y}(t) = (\tilde{L}(t) \otimes I_N)y(t) \tag{5.3}$$

where

$$\tilde{L}(t)= \begin{bmatrix} 0 & I \\ -\beta_0 L(t) & -\beta_1 L(t) \end{bmatrix}. \tag{5.4}$$

and $y(t) \in \mathbb{R}^{2Nn}$, $L = [l_{ij}]$ is the Laplacian matrix.

The $x_i, v_i \in \mathbb{R}^N$, x_i and v_i can be denoted as

$$x_i(t) = [x_{i,1}(t), x_{i,2}(t), \ldots, x_{i,N}(t)]^T, \tag{5.5}$$

$$v_i(t) = [v_{i,1}(t), v_{i,2}(t), \ldots, v_{i,N}(t)]^T. \tag{5.6}$$

where $x_{i,j},\ v_{i,j} \in \mathbb{R}^1$, $i \in \mathcal{I}$, $j \in \{1, \ldots, N\}$.

Let $y_j(t) = [x_{1,j}(t), \cdots, x_{n,j}(t), v_{1,j}(t), \cdots, v_{n,j}(t)]^T$, $j \in \{1, \cdots N\}$, the system (5.3) could be rewritten as

$$\begin{cases} \dot{y}_1(t) = \tilde{L}y_1(t), y_1(0) = y_1(t)|_{t=0}, \\ \cdots \\ \dot{y}_N(t) = \tilde{L}y_N(t), y_N(0) = y_N(t)|_{t=0}, \end{cases}$$

with N independent subsystems. These N independent subsystems have the same differential equation of motion as

$$\dot{c}(t) = \tilde{L}(t)c(t), \tag{5.7}$$

where $c(t) \in \mathbb{R}^{2n}$ stands for y_j, $j \in \{1, \cdots N\}$, defined as

$$c(t) = [\bar{x}(t)^T, \bar{v}(t)^T]^T, \tag{5.8}$$

$$\bar{x}(t) = [\bar{x}_1(t), \ldots, \bar{x}_n(t)]^T, \tag{5.9}$$

$$\bar{v}(t) = [\bar{v}_1(t), \ldots, \bar{v}_n(t)]^T. \tag{5.10}$$

And, $c(0) = c(t)|_{t=0}$ takes its values from $\{y_j(t)|_{t=0}\}$, $j \in \{1, \ldots, N\}$.

Now the consensus problem with (5.3) for all initial values is equivalent to that with equation (5.7) for all initial conditions. If we assume $N = 1$ for simplicity, (5.7) is (5.3).

By now, the second-order consensus would be said to be reached if for all initial conditions, $\bar{x}_i(t) \to \bar{x}_j(t)$, $\bar{v}_i(t) \to \bar{v}_j(t)$, as $t \to +\infty$, $\forall i,\ j \in \{1, \ldots, n\}$.

That is, for a definite n, the consensus problem would be solved if

$$\bar{x}_i(t) - \bar{x}_n(t) \to 0, \bar{v}_i(t) - \bar{v}_n(t) \to 0, \ as\ t \to +\infty, \ \forall i \in \{1, \ldots, n-1\} \tag{5.11}$$

for all initial conditions.

$\forall k \in \mathcal{K}, \forall i \in \{1, \ldots, n-1\}$, let

$$\begin{cases} z_i^{(0)} = \bar{x}_i - \bar{x}_n, \ z_i^{(1)} = \bar{v}_i - \bar{v}_n, \\ z^{(0)} = [z_1^{(0)T}, \ldots, z_{n-1}^{(0)T}]^T, z^{(1)} = [z_1^{(1)T}, \ldots, z_{n-1}^{(1)T}]^T, \end{cases}$$

and $z(t) = [z^{(0)}(t)^T, z^{(1)}(t)^T]^T \in \mathbb{R}^{2(n-1)}$.

Now, one can get the following Theorem 26.

Theorem 26: For the directed network $\mathcal{G}(t)$, if the globally uniformly asymptotic stability problem of system

$$\dot{z}(t) = \tilde{H}(t)z(t), z(t_0) = z(0). \tag{5.12}$$

is ensured, where

$$\tilde{H}(t) = \begin{bmatrix} \mathbf{0} & I \\ -\beta_0 H(t) & -\beta_1 H(t) \end{bmatrix} \tag{5.13}$$

then the consensus of the system (5.1) is reached with the consensus algorithm (5.2), under all initial conditions.

Proof: The i^{th} row vector for the Laplacian matrix can be denoted as

$$L_i(t) = [l_{i1}(t), l_{i2}(t), ..., l_{in}(t)], i \in \mathcal{I}. \tag{5.14}$$

The i^{th} row vector for the H matrix can be denoted as

$$H_i(t) = [h_{i1}(t), h_{i2}(t), ..., h_{in-1}(t)], i \in \mathcal{I}, \ i \neq n. \tag{5.15}$$

1) According to (5.7) and (5.11),

$$\begin{aligned} \dot{z}_i^{(0)}(t) &= \dot{\bar{x}}_i(t) - \dot{\bar{x}}_n(t) \\ &= \bar{v}_i(t) - \bar{v}_n(t), \\ &= z_i^{(1)}(t) \end{aligned}$$

and

$$\begin{aligned}\dot{z}^{(0)}(t) &= (\dot{z}_1^{(0)}(t)^T, \ldots, \dot{z}_{n-1}^{(0)}(t)^T)^T \\ &= (z_1^{(1)}(t)^T, \ldots, z_{n-1}^{(1)}(t)^T)^T \\ &= z^{(1)}(t)\end{aligned} \tag{5.16}$$

2) According to (5.7) and (5.11), and properties of the Laplacian matrix

$$\begin{aligned}\dot{z}_i^{(1)}(t) &= \dot{\bar{v}}_i(t) - \dot{\bar{v}}_n(t) \\ &= \beta_0 \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(\bar{x}_j(t) - \bar{x}_i(t)) + \beta_1 \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(\bar{v}_j(t) - \bar{v}_i(t)) \\ &- \beta_0 \sum_{j\in\mathcal{N}_n(t)} a_{ij}(t)(\bar{x}_j(t) - \bar{x}_n(t)) - \beta_1 \sum_{j\in\mathcal{N}_n(t)} a_{ij}(t)(\bar{v}_j(t) - \bar{v}_n(t))\end{aligned} \tag{5.17}$$

By (5.14), one can get

$$\begin{aligned}\dot{z}_i^{(1)}(t) =& -\beta_0 L_i(t)\bar{x}(t) - \beta_1 L_i(t)\bar{v}^{(1)}(t) + \beta_0 L_n(t)\bar{x}(t) + \beta_1 L_n(t)\bar{v}(t) \\ =& -\beta_0 L_i(t)\bar{x}(t) - \beta_1 L_i(t)\bar{v}(t) + \beta_0 L_n(t)\bar{x}(t) + \cdots + \beta_1 L_n(t)\bar{v}(t) \\ =& -\beta_0(L_i(t) - L_n(t))\bar{x}(t) - \beta_1(L_i(t) - L_n(t))\bar{v}(t)\end{aligned} \tag{5.18}$$

with properties of the Laplacian matrix, and the definition of H matrix. One can get

$$(L_i(t) - L_n(t))\bar{x}(t) = H_i(t)z^{(0)}(t), \tag{5.19}$$

$$(L_i(t) - L_n(t))\bar{v}(t) = H_i(t)z^{(1)}(t), \tag{5.20}$$

Therefore,

$$\begin{aligned}\dot{z}_i^{(1)}(t) =& -\beta_0(L_i(t) - L_n(t))\bar{x}(t) - \beta_1(L_i(t) - L_n(t))\bar{v}(t) \\ =& -\beta_0 H_i(t)z^{(0)}(t) - \beta_1 H_i(t)z^{(1)}(t)\end{aligned} \tag{5.21}$$

By the definition of $z^{(0)}(t)$, $z^{(1)}(t)$ and (5.15), then

$$\begin{aligned}\dot{z}^{(1)}(t) &= (\dot{z}_1^{(1)}(t)^T, \ldots, \dot{z}_{n-1}^{(1)}(t)^T)^T \\ &= -\beta_0 H(t)z^{(0)}(t) - \beta_1 H(t)z^{(1)}(t).\end{aligned} \tag{5.22}$$

Now, by 1) and 2), the equation (5.12) has been proved.

If the globally uniformly asymptotic stability problem of system (5.12) is ensured, then

$$z_i^{(0)} \to 0, z_i^{(1)} \to 0, \ as\ t \to +\infty, \forall i \in \{1, \ldots, n-1\}. \tag{5.23}$$

Compared with the consensus definition of equation (5.11), the consensus problem of (5.7) is solved. Based on the relationship, which has been analyzed, between (5.7) and (5.3), the consensus of the system (5.3) is reached. The consensus of the system (5.1) with the consensus algorithm (5.2) is reached, because the system (5.3) is the compact matrix form of system (5.1) with the consensus algorithm (5.2).

Now one can conclude the proof of this theorem. ■

5.2.2 Multi-agent Systems with a Fixed Fault

Consensus with mismatching coefficients is of practical importance. Some preliminary results on the consensus are reported in the literature [9,10,59,82,65,83,84,85]. According to the survey [9], consensus can be reached if the undirected network digraph is strongly connected. Because the undirected networks can be treated as special cases of directed networks where the adjacency matrices are symmetric, the conditions of directed networks are stricter than those of undirected networks. The article [10] reports that the first-order consensus can be achieved asymptotically if the union of the directed interaction graphs has a spanning tree sufficiently frequently as the system evolves. It is shown in [59] that the second-order consensus requires other conditions in addition to having a spanning tree. The work [84] states that a suitable Laplacian matrix and proper coupling strengths are indispensable to achieving consensus for the second-order systems under fixed topologies. That is, the second-order systems cannot reach consensus with improper strength coefficients even for fixed topologies with a spanning tree. These studies and results indicate that achieving consensus with mismatching coefficients is important and useful.

Mismatches resulting in failure of the consensus under fixed topologies do not

necessarily mean a failure under switched topologies. There is limited research on the consensus for topologies with mismatches and faults, considering varying topologies. In the literature [65, 83, 85], conditions for the consensus are analyzed with reference to the choice of coefficients under fixed topologies. But these papers do not explain when and how these topologies switch with the proper or improper coefficients.

With the development of cooperative control, multi-agent systems become increasingly complex, and faults cause more severe impacts on system performance [86]. The sources of faults are multifarious, such as external measurements, internal settings, and weighted deviations. These faults can cause system performance deterioration and lead to instability, which can further result in catastrophic accidents. Thus, many effective fault-tolerant control approaches have been proposed to improve system reliability [86, 58, 87, 88, 89]. The authors in [86] studied cooperative adaptive fuzzy tracking control for a network with unknown actuator faults, which are limited to a weighted directed graph with a fixed topology. The paper [58] presents a cooperative actuator fault accommodation strategy for a team of multi-agent systems with a switching topology on the assumption that the digraph is always strongly connected. The paper [87] presents an adaptive fault-tolerant control scheme for leader-follower consensus control of uncertain mobile agents with actuator faults under a fixed topology. The study in [88] addresses the cooperative fault-tolerant tracking control problem for a class of multi-agent systems subject to mismatched parameter uncertainties, external disturbances, and actuator faults under an undirected connected graph with a fixed topology. The authors in [89] propose an H_∞ cooperative fault recovery control scheme confined to the fixed topology with a spanning tree. In actuality, a consensus is likely reached under switched topologies with unknown faults and a lack of the spanning tree. However, the detailed realization is unknown as yet.

For a group of n agents systems, every agent is modelled by the second-order

dynamics as

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t), \end{aligned} \tag{5.24}$$

where $x_i(t)$ is the state of the i^{th} agent, u_i is the control input.

A consensus algorithm is represented as

$$u_i(t) = f\beta_0 \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(x_j(t) - x_i(t)) - f\beta_1 v_i(t), \tag{5.25}$$

where $a_{ij}, i, j \in \mathcal{I}$ are the elements in the adjacency matrix $\mathcal{A}(\mathcal{G})$. The topology $\mathcal{G}$ changes at the switching times $t_1, t_2, t_3, \cdots, t_l, t_{l+1}, \cdots$. The positive parameters β_0, β_1 are the coefficients, and f is defined as

$$f \triangleq \frac{measured \quad value}{actual \quad value}.$$

When the known positive constant $f \neq 1$, it is implied that the consensus algorithm (5.25) has a fault. For example, $f > 1$ means the measured value is more than the actual value.

For all initial conditions, the multi-agent systems (5.24) are said to reach a consensus if $\lim_{t \to +\infty} [x_i - x_j] = 0$, $\lim_{t \to +\infty} [v_i - v_j] = 0, \forall i, j \in \mathcal{I}, i \neq j$.

By the following Theorem 27, the problem of the consensus with a fixed fault is converted into that of the stability of switched systems.

Theorem 27: For the multi-agent systems with directed network $\mathcal{G}(t)$, if the system

$$\dot{z}(t) = FKB_{a\sigma(t)} z(t), z(t_0) = z(0), \tag{5.26}$$

is globally uniformly asymptotically stable, then the multi-agent systems with the dynamic system (5.24) will reach a consensus with the consensus algorithm (5.25),

where

$$F = \begin{bmatrix} I_{n-1} & 0_{n-1} \\ 0_{n-1} & f_{diag} \end{bmatrix}, \tag{5.27}$$

$$K = \begin{bmatrix} 0_{n-1} & I_{n-1} \\ -\beta_0 I_{n-1} & -\beta_1 I_{n-1} \end{bmatrix}, \tag{5.28}$$

$$B_{a\sigma(t)} = \begin{bmatrix} H & 0_{n-1} \\ 0_{n-1} & I_{n-1} \end{bmatrix}_{\sigma(t)}, \tag{5.29}$$

$$f_{diag} = diag\{\underbrace{f, \cdots, f}_{n-1}\}. \tag{5.30}$$

Proof: Firstly, we define $z(t) \triangleq [x_1 - x_n, \ldots, x_{n-1} - x_n, v_1 - v_n, \ldots, v_{n-1} - v_n]^T(t)$.

According to the definition of the consensus, if $\lim_{t \to +\infty} z(t) = 0$, the multi-agent systems will reach a consensus.

From (5.25), one can get

$$\begin{aligned} & u_i - u_n \\ =& f\beta_0 \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(x_j(t) - x_i(t)) - f\beta_1 v_i(t) - f\beta_0 \sum_{k \in \mathcal{N}_n(t)} a_{nk}(t)(x_k(t) - x_n(t)) + f\beta_1 v_n(t) \\ =& - f\beta_0 L_i(t)x(t) - \beta_1 f v_i(t) + f\beta_0 L_n(t)x(t) + \beta_1 f v_n(t) \\ =& - f\beta_{0_{\sigma(t-\Delta)}}(L_i(t) - L_n(t))x(t) - f\beta_{1_{\sigma(t-\Delta)}}(v_i(t) - v_n(t)) \\ =& - f\beta_0 H_i(t)[x_1 - x_n, \ldots, x_{n-1} - x_n]^T - f\beta_1(v_i(t) - v_n(t)), \end{aligned} \tag{5.31}$$

and

$$\begin{aligned} & [\dot{v}_1 - \dot{v}_n, \ldots, \dot{v}_{n-1} - \dot{v}_n]^T(t) \\ =& [u_1 - u_n, \ldots, u_{n-1} - u_n]^T \\ =& - f\beta_0 H(t)[x_1 - x_n, \ldots, x_{n-1} - x_n]^T(t) - f\beta_1 [v_1 - v_n, \ldots, v_{n-1} - v_n]^T(t) \\ =& [-f\beta_0 H(t), -f\beta_1 I] z(t), \end{aligned} \tag{5.32}$$

where L_i (or H_i) is the i^{th} row vector of the Laplacian (or H) matrix.

From (5.24), one can get

$$
\begin{aligned}
&[\dot{x}_1 - \dot{x}_n, \ldots, \dot{x}_{n-1} - \dot{x}_n]^T \\
=&[v_1 - v_n, \ldots, v_{n-1} - v_n]^T \\
=&[0_{n-1}, I_{n-1}]z.
\end{aligned} \tag{5.33}
$$

Now, combining (5.32) and (5.33), we can get

$$
\dot{z}(t) = \begin{bmatrix} 0_{n-1} & I_{n-1} \\ -f\beta_0 H(t) & -f\beta_1 I_{n-1} \end{bmatrix} z(t), z(t_0) = z(0). \tag{5.34}
$$

We further have

$$
\begin{aligned}
&\begin{bmatrix} 0_{n-1} & I_{n-1} \\ -f\beta_0 H(t) & -f\beta_1 I_{n-1} \end{bmatrix} \\
=&\begin{bmatrix} I_{n-1} & 0_{n-1} \\ 0_{n-1} & f\ I_{n-1} \end{bmatrix} \begin{bmatrix} 0_{n-1} & I_{n-1} \\ -\beta_0 I_{n-1} & -\beta_1 I_{n-1} \end{bmatrix} \begin{bmatrix} H & 0_{n-1} \\ 0_{n-1} & I_{n-1} \end{bmatrix}_{\sigma(t)} \\
\triangleq& FKB_{a\sigma(t)},
\end{aligned} \tag{5.35}
$$

where f is a scalar, and f_{diag} denotes the diagonal matrix with diagonal elements. The equation (5.26) can now be proved.

According to the definition of $z(t)$, the consensus problem can be solved if the system (5.26) is globally uniformly asymptotically stable. This completes the proof of this theorem. ■

For the case without a controller fault or an uncertain parameter, it can be denoted by $f = 1$. From the proof of Theorem 27, we can see that Theorem 27 is also applicable to the situations without a controller fault.

5.2.3 Multi-agent Systems with Uncertain Faults

In this subsection, we also consider the multi-agent systems (5.24) with the algorithm (5.25).

The fault considered in this subsection is uncertain and

$$\begin{cases} 0 < f_d \leq f \leq f_u, \\ f_d \leq 1 \leq f_u, \end{cases} \tag{5.36}$$

where f_d and f_u are known constants. In this situation, we only assume the range of the deviation is known.

Remark 12: When the positive constants $f_d \neq 1$ or $f_u \neq 1$ are known, it is implied that consensus feedback has an uncertain parameter f. It may originate from an inaccurate coefficient β or other factors and results in a controller fault.

For all initial conditions, the multi-agent systems (5.24) are said to reach a consensus if $\lim_{t\to+\infty}[x_i - x_j] = 0$, $\lim_{t\to+\infty}[v_i - v_j] = 0, \forall i, j \in \mathcal{I}, i \neq j$.

Similarly, we can also get Theorem 27.

We define

$$\begin{cases} F_u \triangleq diag(1, \cdots, 1, f_u, \cdots f_u) \\ F_d \triangleq diag(1, \cdots, 1, f_d, \cdots f_d) \\ F_0 \triangleq \frac{1}{2}(F_u + F_d) \\ F_1 \triangleq \frac{1}{2}(F_u - F_d) \end{cases},$$

then

$$F = F_0 + EF_1, \tag{5.37}$$

where $E = diag\,(1 \cdots 1, e, \cdots, e)$, and $-1 \leqslant e \leqslant 1$.

Systems (5.26) can be replaced with

$$\begin{aligned} \dot{z}(t) &= FKB_{a\sigma(t)}z(t) \\ &= (F_0 + EF_1)KB_{a\sigma(t)}z(t) \\ &= F_0KB_{a\sigma(t)}z(t) + EF_1KB_{a\sigma(t)}z(t) \\ &= \bar{A}_{\sigma(t)}z(t), \end{aligned} \tag{5.38}$$

where $\bar{A}_{\sigma(t)} = F_0KB_{a\sigma(t)}z(t) + EF_1KB_{a\sigma(t)}z(t)$.

5.2.4 Asynchronous Switching

For a group of n agents' systems, every agent is modelled by the second-order dynamics as

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t), \end{aligned} \tag{5.39}$$

where $x_i(t)$ is the state of the i^{th} agent, u_i is the control input.

For asynchronous switching, there exists a time lag Δ between the topologies switching signal $\sigma(t)$ and the feedback controller coefficients switching $\sigma'(t)$. When the lag is small enough, the models tend to be synchronous. Using one of the many algorithms for synchronous switching [66,72] as an example, the corresponding consensus algorithm with asynchronous switching can be represented as

$$u_i(t) = f\beta_{0_{\sigma(t-\Delta)}} \sum_{j\in\mathcal{N}_i(t)} a_{ij_{\sigma(t)}}(x_j(t)-x_i(t)) - f\beta_{1_{\sigma(t-\Delta)}} v_i(t), \tag{5.40}$$

where $a_{ij_{\sigma(t)}}, i, j \in \mathcal{I}$ are the elements in the adjacency matrix $\mathcal{A}(\mathcal{G})$. The topology $\mathcal{G}$ and $\sigma(t)$ change at the switching times $t_1, t_2, t_3, \cdots, t_l, t_{l+1}, \cdots$. The positive parameters β_0, β_1 are the coefficients, and f considered in this model is uncertain and

For all initial conditions, the multi-agent systems (5.39) are said to reach a consensus if $\lim_{t\to+\infty}[x_i - x_j] = 0$, $\lim_{t\to+\infty}[v_i - v_j] = 0, \forall i, j \in \mathcal{M}, i \neq j$.

By the following Theorem 28, the problem of the consensus under asynchronous switching is converted into that of the asynchronous stability of switched systems.

Theorem 28: For the multi-agent systems with directed network $\mathcal{G}(t)$, if the system

$$\dot{z}(t) = FK_{\sigma(t-\Delta)}B_{a\sigma(t)}z(t), z(t_0) = z(0), \tag{5.41}$$

is globally uniformly asymptotically stable, then the multi-agent systems with the dynamic system (5.39) will reach a consensus with the consensus algorithm (5.40),

where

$$F = \begin{bmatrix} I_{n-1} & 0_{n-1} \\ 0_{n-1} & f_{diag} \end{bmatrix}, \tag{5.42}$$

$$K_{\sigma(t-\Delta)} = \begin{bmatrix} 0_{n-1} & I_{n-1} \\ -\beta_0 I_{n-1} & -\beta_1 I_{n-1} \end{bmatrix}_{\sigma(t-\Delta)}, \tag{5.43}$$

$$B_{a\sigma(t)} = \begin{bmatrix} H & 0_{n-1} \\ 0_{n-1} & I_{n-1} \end{bmatrix}_{\sigma(t)}, \tag{5.44}$$

$$f_{diag} = diag\{\underbrace{f, \cdots, f}_{n-1}\}. \tag{5.45}$$

Proof: Firstly, we define $z(t) \triangleq [x_1 - x_n, \ldots, x_{n-1} - x_n, v_1 - v_n, \ldots, v_{n-1} - v_n]^T(t)$.

According to the definition of the consensus, if $\lim_{t\to+\infty} z(t) = 0$, the multi-agent systems will reach a consensus.

From (5.40), one can get

$$\begin{aligned}
&u_i - u_n \\
=&f\beta_{0_{\sigma(t-\Delta)}} \sum_{j\in\mathcal{N}_i(t)} a_{ij_{\sigma(t)}}(x_j(t) - x_i(t)) - f\beta_{1_{\sigma(t-\Delta)}} v_i(t) \\
&- f\beta_{0_{\sigma(t-\Delta)}} \sum_{k\in\mathcal{N}_n(t)} a_{nk_{\sigma(t)}}(x_k(t) - x_n(t)) + f\beta_{1_{\sigma(t-\Delta)}} v_n(t) \\
=&- f\beta_{0_{\sigma(t-\Delta)}} L_i(t)x(t) - \beta_{1_{\sigma(t-\Delta)}} f v_i(t) + f\beta_{0_{\sigma(t-\Delta)}} L_n(t)x(t) + \beta_{1_{\sigma(t-\Delta)}} f v_n(t) \\
=&-f\beta_{0_{\sigma(t-\Delta)}}(L_i(t) - L_n(t))x(t) - f\beta_{1_{\sigma(t-\Delta)}}(v_i(t) - v_n(t)) \\
=&- f\beta_{0_{\sigma(t-\Delta)}} H_i(t)[x_1 - x_n, \ldots, x_{n-1} - x_n]^T - f\beta_{1_{\sigma(t-\Delta)}}(v_i(t) - v_n(t)),
\end{aligned} \tag{5.46}$$

and

$$\begin{aligned}
&[\dot{v}_1 - \dot{v}_n, \ldots, \dot{v}_{n-1} - \dot{v}_n]^T(t) \\
=&[u_1 - u_n, \ldots, u_{n-1} - u_n]^T \\
=&-f\beta_{0_{\sigma(t-\Delta)}} H(t)[x_1 - x_n, \ldots, x_{n-1} - x_n]^T(t) - f\beta_{1_{\sigma(t-\Delta)}}[v_1 - v_n, \ldots, v_{n-1} - v_n]^T(t) \\
=&[-f\beta_{0_{\sigma(t-\Delta)}} H(t), -f\beta_{1_{\sigma(t-\Delta)}} I]z(t),
\end{aligned} \tag{5.47}$$

where L_i (or H_i) is the i^{th} row vector of the Laplacian (or H) matrix.

From (5.39), one can get

$$\begin{aligned}&[\dot{x}_1-\dot{x}_n,\ldots,\dot{x}_{n-1}-\dot{x}_n]^T\\=&[v_1-v_n,\ldots,v_{n-1}-v_n]^T\\=&[0_{n-1},I_{n-1}]z.\end{aligned}\tag{5.48}$$

Now, combining (5.47) and (5.48), we can get

$$\dot{z}(t)=\begin{bmatrix}0_{n-1} & I_{n-1}\\ -f\beta_{0_{\sigma(t-\Delta)}}H(t) & -f\beta_{1_{\sigma(t-\Delta)}}I_{n-1}\end{bmatrix}z(t), z(t_0)=z(0).\tag{5.49}$$

We further have

$$\begin{aligned}&\begin{bmatrix}0_{n-1} & I_{n-1}\\ -f\beta_{0_{\sigma(t-\Delta)}}H(t) & -f\beta_{1_{\sigma(t-\Delta)}}I_{n-1}\end{bmatrix}\\=&\begin{bmatrix}I_{n-1} & 0_{n-1}\\ 0_{n-1} & f\ I_{n-1}\end{bmatrix}\begin{bmatrix}0_{n-1} & I_{n-1}\\ -\beta_0 I_{n-1} & -\beta_1 I_{n-1}\end{bmatrix}_{\sigma(t-\Delta)}\begin{bmatrix}H & 0_{n-1}\\ 0_{n-1} & I_{n-1}\end{bmatrix}_{\sigma(t)}\\ \triangleq&FK_{\sigma(t-\Delta)}B_{a\sigma(t)},\end{aligned}\tag{5.50}$$

where f is a scalar, and f_{diag} denotes the diagonal matrix with diagonal elements. The equation (5.41) can now be proved.

According to the definition of $z(t)$, the consensus problem can be solved if the system (5.41) is globally uniformly asymptotically stable. This completes the proof of this theorem. ■

Remark 13: For the case without a controller fault or an uncertain parameter, it can be denoted by $f = 1$. From the proof of Theorem 28, we can see that Theorem 28 is also applicable to the situations without a controller fault or with a fixed controller fault.

5.3 Consensus Analysis When Agents Can Reach a Consensus under All Topologies

5.3.1 Usual Switching

In [55] [65], some certain relations to coupling strengths with a topology must be satisfied in order to reach a consensus. Any agent could not know whether or not the topologies have changed, for each agent only using the information from its neighbours. Suppose that the relationships, such as coupling strengths, would not change under switching topologies. The multi-agent systems would undergo m kinds of topologies, including $m - r$ kinds of topologies that would not make (5.12) asymptotically stable (the consensus could not be reached). The m kinds of topologies are denoted by $\mathcal{M}$. For brevity, the topologies that can make (5.12) asymptotically stable (that is to say, the consensus could be reached) are labelled as $1, \cdots, r$ denoted by $\mathcal{S}$, while others are denoted by $\mathcal{U}$. These assumptions are similar to the assumptions with switched systems.

Corresponding to the Definitions 20 and 21, we give two new definitions.

Definition 25: For any switching times $t_2 > t_1 \geq 0$ with a topology switching signal $\sigma(t)$, let $N_{\sigma[p|q]}(t_2, t_1)$ stand for the number of the topology switching sequence that the p^{th} topology is activated immediately after the q^{th} topology over the time interval $[t_1, t_2)$, and let $T_{p,[p|q]}(t_2, t_1)$ denote the total running time of the p^{th} topology activated immediately after the q^{th} topology over the time interval $[t_1, t_2)$, $p \in \mathcal{S}$. It is said that $\sigma(t)$ has a topology sequence-based average subsequence dwell time $\tau_{a(p,[p|q])}$ if there exist two numbers $N_{0(p,[p|q])}$ (nonnegative $N_{0(p,[p|q])}$ is called the topology sequenced-based subsequence chatter bound here) and $\tau_{a(p,[p|q])}$ such that

$$N_{\sigma[p|q]}(t_2, t_1) \leq N_{0(p,[p|q])} + \frac{T_{p,[p|q]}(t_2, t_1)}{\tau_{a(p,[p|q])}} \tag{5.51}$$

Definition 26: For any switching times $t_2 > t_1 \geq 0$ with a topology switching

signal $\sigma(t)$, let $N_{\sigma[p|q]}(t_2, t_1)$ stand for the number of the topology switching sequence that the p^{th} topology is activated immediately after the q^{th} topology over the time interval $[t_1, t_2)$, and let $T_{q,[p|q]}(t_2, t_1)$ denote the total running time of the q^{th} topology when the p^{th} topology is activated immediately after the q^{th} topology over the time interval $[t_1, t_2)$, $p \in \mathcal{S}$. It is said that $\sigma(t)$ has a topology sequence-based average preceding dwell time $\tau_{a(q,[p|q])}$ if there exist two numbers $N_{0(q,[p|q])}$ (nonnegative $N_{0(q,[p|q])}$ is called the topology sequenced-based preceding chatter bound here) and $\tau_{a(q,[p|q])}$ such that

$$N_{\sigma[p|q]}(t_2, t_1) \leq N_{0(q,[p|q])} + \frac{T_{q,[p|q]}(t_2, t_1)}{\tau_{a(q,[p|q])}} \tag{5.52}$$

Definitions 25-26 are Definitions 20 and 21 by transforming subsystems and switched systems into topologies and multi-agent systems under switched topologies.

On the basis of Theorem 26, we can get the following Theorem 29.

Theorem 29: For the multi-agent systems under switching topologies network $\mathcal{G}_p$, $p \in \mathcal{M}$, if there exist symmetric matrices $P > 0$, $\forall p \in \mathcal{M}$ such that

$$\tilde{H}_p^T P + P\tilde{H}_p < 0. \tag{5.53}$$

Then the multi-agent systems (5.1) would reach a consensus with the consensus algorithm (5.2) under all initial values.

Proof: A common Lyapunov function is constructed as

$$V(z(t)) = z^T(t)Pz(t) \tag{5.54}$$

and $\forall t \in [t_i, t_{i+1})$, with (5.53)

$$\begin{aligned} & \dot{V}(z(t)) \\ &= \dot{z}^T(t)Pz(t) + z^T(t)P\dot{z}(t) \\ &= z^T(t)\tilde{H}_p^T Pz(t) + z^T(t)P\tilde{H}_p z(t) \\ &= z^T(t)(\tilde{H}_p^T P + P\tilde{H}_p)z(t) \\ &< 0 \end{aligned} \tag{5.55}$$

the system(5.12) is globally uniformly asymptotically stable.

By Theorem 26, the multi-agent systems (5.1) with the consensus algorithm (5.2) would reach a consensus under all initial values.

This proof is completed. ■

With condition (5.53), the multi-agent systems under switching topologies are stable under arbitrary switching.

In some situations, finding a common $P > 0$ under the condition (5.53) is difficult, compared to finding an individual $P_p > 0$, $\forall p \in \mathcal{M}$, for every topology. The following Theorem 30 is an alternative method.

Theorem 30: For the multi-agent systems under switching topologies network $\mathcal{G}_p$, $p \in \mathcal{M}$, and giving constants $\lambda_p > 0$, $\lambda_q > 0$, $\mu_{[p|q]} > 1$, $\forall p \in \mathcal{M}$, if there exist symmetric matrices $P_p > 0$, $\forall p \in \mathcal{M}$, such that

$$\tilde{H}_p^T P_p + P_p \tilde{H}_p < -\lambda_p P_p \tag{5.56}$$

And $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M}, p \neq q$, such that

$$P_p \leq \mu_{[p|q]} P_q \tag{5.57}$$

Then the multi-agent systems (5.1) with the consensus algorithm (5.2) and all initial values reach a consensus with

$$\tau_{a(p,[p|q])} \geq \tau^*_{a(p,[p|q])} = \ln \mu_{[p|q]} / \lambda_p. \tag{5.58}$$

or

$$\tau_{a(q,[p|q])} \geq \tau^*_{a(q,[p|q])} = \ln \mu_{[p|q]}/\lambda_q. \tag{5.59}$$

Proof: A multiple Lyapunov function is constructed as

$$V_p(z(t)) = z^T(t)P_p z(t) \tag{5.60}$$

and $\forall t \in [t_i, t_{i+1})$, with (5.56) and (5.60), one can get

$$\begin{aligned}
&\dot{V}_p(z(t)) \\
&= \dot{z}^T(t)P_p z(t) + z^T(t)P_p \dot{z}(t) \\
&= z^T(t)\tilde{H}_p^T P_p z(t) + z^T(t)P_p \tilde{H}_p z(t) \\
&= z^T(t)(\tilde{H}_p^T P_p + P_p \tilde{H}_p)z(t) \\
&< -\lambda_p z^T(t)P_p z(t) \\
&= -\lambda_p V_p(z(t))
\end{aligned} \tag{5.61}$$

From (5.57) and (5.60), then

$$\begin{aligned}
&V_p(z(t_i)) \\
=&z^T(t_i)P_p z(t_i) \\
<&\mu_{[p|q]} z^T(t_i^-)P_q z(t_i^-) \\
=&\mu_{[p|q]} V_q(z(t_i^-))
\end{aligned} \tag{5.62}$$

According to systems (5.60)-(5.62), Theorems 8 and 10 and Corollary 1 in Chapter 4, the system (5.12) is globally uniformly asymptotically stable with (5.58) or (5.59).

By Theorem 26, the multi-agent systems (5.1) with the consensus algorithm (5.2) would reach a consensus with (5.58) or (5.59).

This proof is completed. ■

It can be seen that the results in Chapter 5 are based on some theorems of Chapter 4.

5.3.2 Asynchronous Switching with Faults

For the multi-agent systems with a controller fault represented by the uncertain parameter, we only know the range of f.

From (5.37) and Theorem 28, we now have the following Theorem 31:

Theorem 31: For the given constants $\lambda_p > 0$, $\lambda_q > 0$, $\mu_{[p|q]} > 1$, and the multi-agent systems with switched topologies $\mathcal{G}_p$, $p \in \mathcal{M}$, if there exist symmetric matrices $P_p > 0$, positive constants ε_p, and ε_{pq}, $\forall p \in \mathcal{M}$, such that

$$\begin{bmatrix} D_p & \varepsilon_p F_1 & P_p(K_p B_{ap})^T \\ \varepsilon_p {F_1}^T & -\varepsilon_p I_{n-1} & 0_{n-1} \\ K_p B_{ap} P_p & 0_{n-1} & -\varepsilon_p I_{n-1} \end{bmatrix} < 0, \tag{5.63}$$

where $D_p = (F_0 K_p B_{ap}) P_p + P_p (F_0 K_p B_{ap})^T + \lambda_p P_p$, and $\forall (\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M}$, $p \neq q$,

$$P_p^{-1} \leq \mu_{[p|q]} P_q^{-1}, \tag{5.64}$$

$$\begin{bmatrix} \bar{D}_p & \varepsilon_{pq} F_1 & P_p(K_q B_{ap})^T \\ \varepsilon_{pq} {F_1}^T & -\varepsilon_{pq} I_{n-1} & 0_{n-1} \\ K_q B_{ap} P_p & 0_{n-1} & -\varepsilon_{pq} I_{n-1} \end{bmatrix} < 0, \tag{5.65}$$

where $\bar{D}_p = (F_0 K_q B_{ap}) P_p + P_p (F_0 K_q B_{ap})^T - \alpha_{[p|q]} P_p$, then the multi-agent systems (5.39) under consensus algorithm (5.40) with fault-tolerance (5.36) will reach a consensus with

$$\tau_{a(p,[p|q])} \geq \tau^*_{a(p,[p|q])} = \frac{T_{[p|q]_\max\uparrow}(\lambda_p + \alpha_{[p|q]}) + \ln \mu_{[p|q]}}{\lambda_p}, \tag{5.66}$$

or

$$\tau_{a(q,[p|q])} \geq \tau^*_{a(q,[p|q])} = \frac{T_{[p|q]_\max\uparrow}(\lambda_q + \alpha_{[p|q]}) + \ln \mu_{[p|q]}}{\lambda_q}. \tag{5.67}$$

Proof: Based on the Schur complement lemma, one can see that the system

(5.63) is equivalent to

$$D_p - S_p^T \begin{bmatrix} -\varepsilon_p I_{n-1} & 0_{n-1} \\ 0_{n-1} & -\varepsilon_p I_{n-1} \end{bmatrix}^{-1} S_p < 0, \tag{5.68}$$

where $S_p = \begin{bmatrix} \varepsilon_p {F_1}^T \\ (K_p B_{ap}) P_p \end{bmatrix}$. This can be further written as

$$\begin{aligned} & D_p - S_p^T \begin{bmatrix} -\varepsilon_p I_{n-1} & 0_{n-1} \\ 0_{n-1} & -\varepsilon_p I_{n-1} \end{bmatrix}^{-1} S_p \\ & = D_p + S_p^T \begin{bmatrix} {\varepsilon_p}^{-1} I_{n-1} & 0_{n-1} \\ 0_{n-1} & {\varepsilon_p}^{-1} I_{n-1} \end{bmatrix} S_p \\ & = D_p + \begin{bmatrix} F_1 & {\varepsilon_p}^{-1} P_p (K_p B_{ap})^T \end{bmatrix} S_p \\ & = D_p + \varepsilon_p F_1 {F_1}^T + {\varepsilon_p}^{-1} P_p (K_p B_{ap})^T (K_p B_{ap}) P_p \\ & < 0. \end{aligned} \tag{5.69}$$

According to Lemma 9 in Chapter 2, one can get

$$D_p + F_1 E (K_p B_{ap}) P_p + P_p (K_p B_{ap})^T E^T {F_1}^T < 0. \tag{5.70}$$

This can also be further represented as

$$\begin{aligned} & D_p + F_1 E (K_p B_{ap}) P_p + P_p (K_p B_{ap})^T E^T {F_1}^T \\ =& (F_0 K_p B_{ap} + F_1 E K_p B_{ap}) P_p + P_p (F_0 K_p B_{ap} + F_1 E (K_p B_{ap}))^T + \lambda_p P_p \\ <& 0. \end{aligned} \tag{5.71}$$

Using $diag\{{P_p}^{-1}\}$ to pre- and post-multiply both sides of systems (5.71), one has

$${P_p}^{-1}(F_0 K_p B_{ap} + F_1 E K_p B_{ap}) + (F_0 K_p B_{ap} + F_1 E K_p B_{ap})^T {P_p}^{-1} + \lambda_p {P_p}^{-1} < 0. \tag{5.72}$$

Denote $\bar{P}_p = {P_p}^{-1}$. We can also see that $\bar{P}_p > 0$ and systems (5.72) are equivalent to

$$\bar{P}_p (F_0 K_p B_{ap} + F_1 E K_p B_{ap}) + (F_0 K_p B_{ap} + F_1 E K_p B_{ap}) \bar{P}_p + \lambda_p \bar{P}_p < 0. \tag{5.73}$$

Based on systems (5.41) and (5.73), one has

$$\bar{P}_p(FK_pB_{ap}) + (FK_pB_{ap})^T\bar{P}_p < -\lambda_p\bar{P}_p. \tag{5.74}$$

Similarly, we can have

$$\bar{P}_p(FK_qB_{ap}) + (FK_qB_{ap})^T\bar{P}_p < \alpha_{[p|q]}\bar{P}_p. \tag{5.75}$$

From (5.64), one has

$$\bar{P}_p \leq \mu_{[p|q]}\bar{P}_q. \tag{5.76}$$

According to systems (5.74)-(5.76), Theorems 12, 14, 16 and 18 in Chapter 4, the system (5.41) is globally uniformly asymptotically stable with (5.66) or (5.67).

By Theorem 28, the multi-agent systems (5.39) under consensus algorithm (5.40) with fault-tolerance (5.36) will reach a consensus with (5.58) or (5.59).

The proof is completed. ■

Theorem 31 is based on Theorems 12 and 14 and on the idea that the topologies changing replaces subsystem switching. Hence the results on SBASDT, SBAPDT of switched systems are also applicable to the multi-agent systems under switched topologies.

Using a similar process, we can establish and prove Corollary 8 below when using a common P to replace P_p.

Corollary 8: For the multi-agent systems with switched topologies $\mathcal{G}_p$, $p \in \mathcal{M}$, if there exist a symmetric matrix $P > 0$, positive constants ε_p, $\forall p \in \mathcal{M}$, such that

$$\begin{bmatrix} D_p & \varepsilon_p F_1 & P(K_pB_{ap})^T \\ \varepsilon_p {F_1}^T & -\varepsilon_p I_{n-1} & 0_{n-1} \\ (K_pB_{ap})P & 0_{n-1} & -\varepsilon_p I_{n-1} \end{bmatrix} < 0, \tag{5.77}$$

where $D_p = (F_0 K_p B_{ap})P + P(F_0 K_p B_{ap})^T + \lambda_p P$, and $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M}, p \neq q$,

$$\begin{bmatrix} \bar{D}_p & \varepsilon_{pq} F_1 & P(K_q B_{ap})^T \\ \varepsilon_{pq} F_1{}^T & -\varepsilon_{pq} I & 0_{n-1} \\ K_q B_{ap} P & 0_{n-1} & -\varepsilon_{pq} I_{n-1} \end{bmatrix} < 0, \tag{5.78}$$

where $\bar{D}_p = (F_0 K_q B_{ap})P + P(F_0 K_q B_{ap})^T - \alpha_{[p|q]} P$, then the multi-agent systems (5.39) under consensus algorithm (5.40) with fault-tolerance (5.36) will reach the consensus finally with

$$\tau_{a(p,[p|q])} \geq \tau^*_{a(p,[p|q])} = \frac{T_{[p|q]_\max\uparrow}(\lambda_p + \alpha_{[p|q]})}{\lambda_p}, \tag{5.79}$$

or

$$\tau_{a(q,[p|q])} \geq \tau^*_{a(q,[p|q])} = \frac{T_{[p|q]_\max\uparrow}(\lambda_q + \alpha_{[p|q]})}{\lambda_q}. \tag{5.80}$$

SBADT is used in the proof for reaching a consensus when the multi-agent systems are switching. In order to guarantee the condition of SBADT, the statistical information on the dwell time under each topology is required to confirm that the average dwell time is longer than the offset.

One can see that if the time lag $\Delta(t) = 0$, the asynchronous switching is replaced with synchronous switching in the algorithm (5.40).

In the synchronous switching situation, systems (5.40) change to

$$u_i(t) = f\beta_{0_{\sigma(t)}} \sum_{j \in \mathcal{N}_i(t)} a_{ij_{\sigma(t)}}(x_j(t) - x_i(t)) - f\beta_{1_{\sigma(t)}} v_i(t). \tag{5.81}$$

Likewise, the following corollaries can be obtained for the case of synchronous switching multi-agent systems.

Corollary 9: For the given constants $\lambda_p > 0$, $\lambda_q > 0$, $\mu_{[p|q]} > 1$, and the switched topologies $\mathcal{G}_p$ of the multi-agent systems, $p \in \mathcal{M}$, if there exist symmetric matrices

$P_p > 0$, positive constants ε_p, $\forall p \in \mathcal{M}$, such that

$$\begin{bmatrix} D_p & \varepsilon_p F_1 & P_p(K_p B_{ap})^T \\ \varepsilon_p {F_1}^T & -\varepsilon_p I_{n-1} & 0_{n-1} \\ K_p B_{ap} P_p & 0_{n-1} & -\varepsilon_p I_{n-1} \end{bmatrix} < 0, \tag{5.82}$$

where $D_p = (F_0 K_p B_{ap}) P_p + P_p (F_0 K_p B_{ap})^T + \lambda_p P_p$, $\forall (\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M}, p \neq q$,

$$P_p^{-1} \leq \mu_{[p|q]} P_q^{-1}, \tag{5.83}$$

then the multi-agent systems (5.39) under consensus algorithm (5.81) with fault-tolerance (5.36) reach the consensus when

$$\tau_{a(p,[p|q])} \geq \tau^*_{a(p,[p|q])} = \ln \mu_{[p|q]} / \lambda_p. \tag{5.84}$$

or

$$\tau_{a(q,[p|q])} \geq \tau^*_{a(q,[p|q])} = \ln \mu_{[p|q]} / \lambda_q. \tag{5.85}$$

Corollary 10: For the switched topologies $\mathcal{G}_p$ of the multi-agent systems, $p \in \mathcal{M}$, if there exist a symmetric matrix $P > 0$, positive constants ε_p, $\forall p \in \mathcal{M}$, such that

$$\begin{bmatrix} D_p & \varepsilon_p F_1 & P(K_p B_{ap})^T \\ \varepsilon_p {F_1}^T & -\varepsilon_p I_{n-1} & 0_{n-1} \\ (K_p B_{ap}) P & 0_{n-1} & -\varepsilon_p I_{n-1} \end{bmatrix} < 0, \tag{5.86}$$

where $D_p = (F_0 K_p B_{ap}) P + P (F_0 K_p B_{ap})^T + \lambda_p P$, then the multi-agent systems (5.39) under consensus algorithm (5.81) with fault-tolerance (5.36) will reach the consensus finally.

5.4 Consensus Analysis When Agents Cannot Reach a Consensus under Part Topologies

5.4.1 Usual Circumstances

If there is an unstable subsystem in the system 5.12, finding an individual $P_p > 0$, satisfying (5.56), $\forall p \in \mathcal{M}$ for every topology, is impossible. The following Theorem

32 is used for the consensus problem in this situation.

Theorem 32: For the multi-agent systems under switching topologies network $\mathcal{G}_p$, $p \in \mathcal{M}$, and giving constants $\alpha_{[p|q]}$, $\mu_{[p|q]} > 1$, $\forall p \in \mathcal{M}$, if there exist symmetric matrices $P_p > 0$, $\forall p \in \mathcal{M}$, such that

$$\tilde{H}_p^T P_p + P_p \tilde{H}_p < \alpha_{[p|q]} P_p, \tag{5.87}$$

And $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M}, p \neq q$, such that

$$P_p \leq \mu_{[p|q]} P_q, \tag{5.88}$$

and if

$$\begin{cases} \tau_{a(p,[p|q])} > \tau^*_{a(p,[p|q])} \triangleq -\ln \mu_{[p|q]}/\alpha_p, (\alpha_p < 0, p \in \mathcal{S}), \\ \tau_{a(p,[p|q])} > \tau^*_{a(p,[p|q])}, (\forall \tau^*_{a(p,[p|q])} > 0, \alpha_p > 0, p \in \mathcal{U}), \\ \frac{T_s^-}{T_s^+} > \frac{\gamma_s^+ + \gamma^*}{\gamma_s^- - \gamma^*}, (0 < \gamma^* < \gamma_s^-), \end{cases} \tag{5.89}$$

or

$$\begin{cases} \tau_{a(q,[p|q])} > \tau^*_{a(q,[p|q])} \triangleq -\ln \mu_{[p|q]}/\alpha_q, (\alpha_q < 0, q \in \mathcal{S}), \\ \tau_{a(q,[p|q])} > \tau^*_{a(q,[p|q])}, (\forall \tau^*_{a(q,[p|q])} > 0, \alpha_q > 0, q \in \mathcal{U}), \\ \frac{T_p^-}{T_p^+} > \frac{\gamma_p^+ + \gamma^*}{\gamma_p^- - \gamma^*}, (0 < \gamma^* < \gamma_p^-), \end{cases} \tag{5.90}$$

then the multi-agent systems (5.1) with the consensus algorithm (5.2) and all initial values reach a consensus with marginal γ^*,

where T^- and T^+ stand for the total running time of stable and unstable subsystems in (5.12). The parameters of γ_s^+,γ_p^+, γ_s^-,γ_p^- are given as:

$$\gamma_s^- \triangleq \max_{p \in \mathcal{S}} \left\{ \alpha_p + \frac{\ln \mu_{[p|q]}}{\tau_{a(p,[p|q])}} \right\}, \tag{5.91}$$

$$\gamma_s^+ \triangleq \max_{p \in \mathcal{U}} \left\{ \alpha_p + \frac{\ln \mu_{[p|q]}}{\tau_{a(p,[p|q])}} \right\}, \tag{5.92}$$

$$\gamma_p^- \triangleq \max_{q \in \mathcal{S}} \left\{ \alpha_q + \frac{\ln \mu_{[p|q]}}{\tau_{a(q,[p|q])}} \right\}, \tag{5.93}$$

$$\gamma_p^+ \triangleq \max_{q\in\mathcal{U}} \left\{ \alpha_q + \frac{\ln \mu_{[p|q]}}{\tau_{a(q,[p|q])}} \right\}. \tag{5.94}$$

Proof: A multiple Lyapunov function is constructed as

$$V_p(z(t)) = z^T(t) P_p z(t) \tag{5.95}$$

and $\forall t \in [t_i, t_{i+1})$, with (5.87) and (5.95), one can get

$$\begin{aligned}
&\dot{V}_p(z(t)) \\
&= \dot{z}^T(t) P_p z(t) + z^T(t) P_p \dot{z}(t) \\
&= z^T(t) \tilde{H}_p^T P_p z(t) + z^T(t) P_p \tilde{H}_p z(t) \\
&= z^T(t) (\tilde{H}_p^T P_p + P_p \tilde{H}_p) z(t) \\
&< \alpha_{[p|q]} z^T(t) P_p z(t) \\
&= \alpha_{[p|q]} V_p(z(t))
\end{aligned} \tag{5.96}$$

From (5.88) and (5.95), then

$$\begin{aligned}
&V_p(z(t_i)) \\
=&z^T(t_i) P_p z(t_i) \\
<&\mu_{[p|q]} z^T(t_i^-) P_q z(t_i^-) \\
=&\mu_{[p|q]} V_q(z(t_i^-))
\end{aligned} \tag{5.97}$$

Because of systems (5.95)-(5.97), Lemma 25 and Theorem 23, the system(5.12) is globally uniformly asymptotically stable with (5.89) or (5.90).

Because of Theorem 26, the multi-agent systems (5.1) with the consensus algorithm (5.2) would reach a consensus with (5.89) or (5.90).

This proof is completed. ■

5.4.2 Systems with a Fixed Fault

The following Theorem 33 is used for the consensus problem when there are unstable subsystems in the system (5.26) with a fixed fault.

Theorem 33: For the multi-agent systems under switching topologies network $\mathcal{G}_p$, $p \in \mathcal{M}$, and giving constants $\alpha_{[p|q]}$, $\mu_{[p|q]} > 1$, $\forall p \in \mathcal{M}$, if there exist symmetric matrices$P_p > 0$, $\forall p \in \mathcal{M}$, such that

$$FKB_p{}^T P_p + P_p FKB_p < \alpha_{[p|q]} P_p, \tag{5.98}$$

and $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M}, p \neq q$, such that

$$P_p \leq \mu_{[p|q]} P_q, \tag{5.99}$$

then the multi-agent systems (5.24) with the consensus algorithm (5.25) and all initial values reach a consensus with (5.89) or (5.90).

Proof: A multiple Lyapunov function is constructed as

$$V_p(z(t)) = z^T(t) P_p z(t) \tag{5.100}$$

and $\forall t \in [t_i, t_{i+1})$, with (5.98) and (5.100), one can get

$$\begin{aligned}
&\dot{V}_p(z(t)) \\
&= \dot{z}^T(t) P_p z(t) + z^T(t) P_p \dot{z}(t) \\
&= z^T(t) FKB_p{}^T P_p z(t) + z^T(t) P_p FKB_p z(t) \\
&= z^T(t)(FKB_p{}^T P_p + P_p FKB_p) z(t) \\
&< \alpha_{[p|q]} z^T(t) P_p z(t) \\
&= \alpha_{[p|q]} V_p(z(t))
\end{aligned} \tag{5.101}$$

From (5.99) and (5.100), then

$$\begin{aligned}
&V_p(z(t_i)) \\
&= z^T(t_i) P_p z(t_i) \\
&< \mu_{[p|q]} z^T(t_i^-) P_q z(t_i^-) \\
&= \mu_{[p|q]} V_q(z(t_i^-)).
\end{aligned} \tag{5.102}$$

Because of systems (5.100)-(5.102), Lemma 25 and Theorem 23, the system(5.26) is globally uniformly asymptotically stable with (5.89) or (5.90).

According to Theorem 27, the multi-agent systems (5.24) with the consensus algorithm (5.25) would reach a consensus with (5.89) or (5.90).

This proof is completed. ■

5.4.3 Systems with Uncertain Faults

Systems (5.26) can be rewritten as

$$\dot{z}(t) = Az(t) + FB(t)z(t), z(t_0) = z(0). \tag{5.103}$$

where

$$\hat{A} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \tag{5.104}$$

$$\hat{B} = \begin{bmatrix} 0 & 0 \\ -\beta_0 H & -\beta_1 I \end{bmatrix}. \tag{5.105}$$

According to

$$\begin{cases} F_u \triangleq diag(1, \cdots, 1, f_u, \cdots f_u) \\ F_d \triangleq diag(1, \cdots, 1, f_d, \cdots f_d) \\ F_0 \triangleq \frac{1}{2}(F_u + F_d) \\ F_1 \triangleq \frac{1}{2}(F_u - F_d) \end{cases}, \tag{5.106}$$

then

$$F = F_0 + EF_1, \tag{5.107}$$

where $E = diag\,(1 \cdots 1, e, \cdots, e)$, and $-1 \leqslant e \leqslant 1$.

Based on (5.107), systems (5.103) can be revised as

$$\begin{aligned} &\dot{z}(t) \\ &= Az(t) + FB_\sigma z(t) \\ &= Az(t) + (F_0 + EF_1)B_\sigma z(t) \\ &= Az(t) + F_0 B_\sigma z(t) + EF_1 B_\sigma z(t) \\ &= \bar{A}_\sigma z(t), \end{aligned} \tag{5.108}$$

where $\bar{A}_\sigma = A + F_0 B_\sigma + EF_1 B_\sigma$.

Now we can get Theorem 34 for the robust unknown fault case.

Theorem 34: For given constants $\alpha_{[p|q]}$, $\mu_{[p|q]} > 1$, and the switched topologies $\mathcal{G}_p$ of the multi-agent systems, $p \in \mathcal{J}$, if there exist symmetric matrices $P_p > 0$, positive constants ε_p, $\forall p \in \mathcal{J}$, such that

$$\begin{bmatrix} D_p & \varepsilon_p F_1 & P_p B_p^T \\ \varepsilon_p {F_1}^T & -\varepsilon_p I & 0 \\ B_p P_p & 0 & -\varepsilon_p I \end{bmatrix} < 0, \tag{5.109}$$

where $D_p = (A + F_0 B_p)P_p + P_p(A + F_0 B_p)^T - \alpha_{[p|q]} P_p$,

and for $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{J} \times \mathcal{J}, p \neq q$, such that

$$P_p^{-1} \leq \mu_{[p|q]} P_q^{-1}, \tag{5.110}$$

then the multi-agent systems (5.24) under consensus algorithm (5.25) with fault tolerance (5.36) will reach consensus with marginal γ^* under the condition (5.89) or (5.90).

Proof: According to the Schur Complement lemma, one can get that systems (5.109) are equivalent to

$$D_p - \begin{bmatrix} \varepsilon_p F_1 & P_p B_p^T \end{bmatrix} \begin{bmatrix} -\varepsilon_p I & 0 \\ 0 & -\varepsilon_p I \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon_p {F_1}^T \\ B_p P_p \end{bmatrix} < 0. \tag{5.111}$$

This can be further written as

$$\begin{aligned}
&D_p - \begin{bmatrix} \varepsilon_p F_1 & P_p B_p^T \end{bmatrix} \begin{bmatrix} -\varepsilon_p I & 0 \\ 0 & -\varepsilon_p I \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon_p {F_1}^T \\ B_p P_p \end{bmatrix} \\
&= D_p - \begin{bmatrix} \varepsilon_p F_1 & P_p B_p^T \end{bmatrix} \begin{bmatrix} {\varepsilon_p}^{-1} I & 0 \\ 0 & {\varepsilon_p}^{-1} I \end{bmatrix} \begin{bmatrix} \varepsilon_p {F_1}^T \\ B_p P_p \end{bmatrix} \\
&= D_p + \begin{bmatrix} F_1 & {\varepsilon_p}^{-1} P_p B_p^T \end{bmatrix} \begin{bmatrix} \varepsilon_p {F_1}^T \\ B_p P_p \end{bmatrix} \\
&= D_p + \varepsilon_p F_1 {F_1}^T + {\varepsilon_p}^{-1} P_p B_p^T B_p P_p \\
&= D_p + \varepsilon_p F_1 + {\varepsilon_p}^{-1} P_p B_p^T B_p P_p \\
&< 0.
\end{aligned} \tag{5.112}$$

According to Lemma 9 in Chapter 2, one can get

$$D_p + F_1 E B_p P_p + P_p B_p^T E^T {F_1}^T < 0. \tag{5.113}$$

This can also be further represented as

$$D_p + F_1 E B_p P_p + P_p B_p^T E^T {F_1}^T = \hat{D}_p < 0, \tag{5.114}$$

where $\hat{D}_p = (A + F_0 B_p + F_1 E B_p) P_p + P_p (A + F_0 B_p + F_1 E B_p)^T - \alpha_{[p|q]} P_p$.

Using ${P_p}^{-1}$ to pre- and post-multiply both sides of systems (5.114), respectively, one has

$${P_p}^{-1}(A + F_0 B_p + F_1 E B_p) + (A + F_0 B_p + F_1 E B_p)^T {P_p}^{-1} + \lambda_p {P_p}^{-1} < 0. \tag{5.115}$$

Denoting $\bar{P}_p = {P_p}^{-1}$, one can get $\bar{P}_p > 0$ and systems (5.115) are equivalent to

$$\bar{P}_p(A + F_0 B_p + F_1 E B_p) + (A + F_0 B_p + F_1 E B_p)^T \bar{P}_p + \lambda_p \bar{P}_p < 0. \tag{5.116}$$

Based on systems (5.108) and (5.116), one has

$$\bar{P}_p(A + F B_p) + (A + F B_p)^T \bar{P}_p + \lambda_p \bar{P}_p < 0. \tag{5.117}$$

That is,

$$\bar{P}_p(A + FB_p) + (A + FB_p)^T \bar{P}_p < -\lambda_p \bar{P}_p. \tag{5.118}$$

A multiple Lyapunov function is constructed as

$$V_p(z(t)) = z(t)^T \bar{P}_p z(t). \tag{5.119}$$

Based on systems (5.110), one has

$$\bar{P}_p \leq \mu_{[p|q]} \bar{P}_q. \tag{5.120}$$

Because of systems (5.118)-(5.120), Lemma 25 and Theorem 23, the system(5.26) is globally uniformly asymptotically stable with (5.89) or (5.90).

According to Theorem 27, the multi-agent systems (5.24) with the consensus algorithm (5.25) with fault-tolerance (5.36) would reach a consensus with (5.89) or (5.90).■

5.5 Consensus Analysis When Agents Cannot Reach a Consensus under Any Topology

In this section, we propose a novel method for determining whether the multi-agent system with a specific protocol can reach a consensus when all topologies have no spanning tree.

Corresponding to the Definitions 20 and 21 in Chapter 4 and 25 and 26 in Chapter 5, we give a new simplified definition.

Definition 27: For the multi-agent systems network switching among m topologies, we use $T_{[p|q]}(t_0, t)$ to denote the total running time of the p^{th} topology when it is instantly activated after the q^{th} topology in time section $[t_0, t)$, and $N_{[p|q]}(t_0, t)$ to indicate the total switching times in this case. Its sequence-based topology-dependent

average dwell time is

$$\tau_{a([p|q])} \triangleq \frac{T_{[p|q]}(t_0,t)}{N_{[p|q]}(t_0,t)}. \tag{5.121}$$

Remark 14: The Definition 27 can be seen as a simplified version of 25 and 26.

Theorem 35: For the multi-agent systems including m subsystem (5.1) with the control protocol (5.2), assume that the p^{th} topology was instantly activated after the q^{th} topology. Given the constants $l_p \in \mathbb{N}^+$, $l_q \in \mathbb{N}^+$, $0 < \mu_{[p|q]} < 1,\ \alpha_p > 0,\ \alpha_q > 0$, $\tau_{min,p} = \min\limits_{\sigma(t_i)=p}(t_i - t_{i-1})$, $\tau_{min,q} = \min\limits_{\sigma(t_i)=q}(t_i - t_{i-1})$, if there exist a set of positive definite matrices $P_{p,j_p} > 0,\ P_{q,j_q} > 0$, $j_p = 0,\ 1, \cdots,\ l_p$, $j_q = 0,\ 1, \cdots,\ l_q$, such that $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M}$, $p \neq q$, $\forall j_p = 0,\ 1, \cdots,\ l_p - 1$,

$$\begin{cases} \tilde{H}_p^T P_{p,j_p} + P_{p,j_p}\tilde{H}_p + \frac{l_p(P_{p,j_p+1} - P_{p,j_p})}{\tau_{min,p}} \leq \alpha_p P_{p,j_p}, \\ \tilde{H}_p^T P_{p,j_p+1} + P_{p,j_p+1}\tilde{H}_p + \frac{l_p(P_{p,j_p+1} - P_{p,j_p})}{\tau_{min,p}} \leq \alpha_p P_{p,j_p+1}, \\ \tilde{H}_p^T P_{p,l_p} + P_{p,l_p}\tilde{H}_p \leq \alpha_p P_{p,l_p}, \end{cases} \tag{5.122}$$

$$P_{p,0} \leq \mu_{[p|q]} P_{q,l_q}, \tag{5.123}$$

$$\tau_{a([p|q])} < -\ln \mu_{[p|q]}/\alpha_p, \tag{5.124}$$

then the multi-agent systems including m subsystems (5.1) with the protocol (5.2) will reach a consensus.

Proof: Firstly, we construct the state vector $z(t)$ as

$$z(t) = \begin{bmatrix} x_1 - x_n \\ \cdots \\ x_{n-1} - x_n \\ v_1 - v_n \\ \cdots \\ v_{n-1} - v_n \end{bmatrix}(t). \tag{5.125}$$

According to the proof for Theorem 26, we can get the system (5.12).

Secondly, we construct the multiple Lyapunov functions

$$V_p(t) = z^T(t)P_p(t)z(t),\ p \in \mathcal{M}. \tag{5.126}$$

For $\forall t \in [t_i, t_{i+1})$, $\sigma(t_i) = p$, $i > 0$, we divide the time section $[t_i, t_i + \tau_{min,p})$ into l_p equal length segments. Every segment is described by $S_{i,q} = [t_i + q * \tau_{min,p}/l_p, t_i + (q+1) * \tau_{min,p}/l_p)$, $q = 0, \cdots, l_p - 1$.

We first design a matrix $P_p(t)$ with piecewise-linear elements over time. Let $P_{p,q}$ denote $P_p(t_i + q * \tau_{min,p}/l_p)$. For any segment, q takes its value from $\{0, \cdots, l_p - 1\}$, and $P_p(t) = (1-\alpha)P_{p,q} + \alpha P_{p,q+1} = P_p^{(q)}(\alpha)$, $\alpha = (t - t_i - q * \tau_{min,p}/l_p) * l_p/\tau_{min,p}$, $0 \le \alpha \le 1$. In the time section $[t_i + \tau_{min,p}, t_{i+1})$, we use the time invariant matrix P_{p,l_p}.

Now, we can get

$$P_p(t) = \begin{cases} P_p^{(q)}(\alpha), t \in S_{i,q}, \\ P_{p,l_p}, t \in [t_i + \tau_{min,p}, t_{i+1}), \end{cases} \tag{5.127}$$

$$V_p(t) = \begin{cases} z^T(t)P_p^{(q)}(\alpha)z(t), t \in S_{i,q} \\ z^T(t)P_{p,l_p}z(t), t \in [t_i + \tau_{min,p}, t_{i+1}) \end{cases}. \tag{5.128}$$

We use $\lambda_{min} > 0$ (or $\lambda_{max} > 0$) to denote the minimal (or maximal) eigenvalue of all matrices $P_{p,j}$, $p \in \mathcal{M}$, $j \in \{1, \cdots, l_p\}$, i.e.,

$$\lambda_{min} z^T(t)z(t) \le V(t) \le \lambda_{max} z^T(t)z(t). \tag{5.129}$$

According to (5.122) and (5.128), we have

$$\dot{V}_p(t) \le \alpha_p V_p(t). \tag{5.130}$$

According to (5.123), we obtain

$$V_p(t) \le \mu_{[p|q]} V_q(t), t \in t_0, t_1, \cdots \tag{5.131}$$

when the network is switching from the q^{th} topology to the p^{th} topology. That is, the Lyapunov function drops at all topologies switching time $t_0, t_1, \cdots$.

Thirdly, according to (5.129)-(5.131), we have

$$\begin{aligned} & V(z(t)) \\ & \leq \exp\left\{\alpha_{\sigma(t_i)}(t-t_i)\right\} V_{\sigma(t_i)}(z(t_i)) \\ & \leq \exp\left\{\alpha_{\sigma(t_i)}(t-t_i)\right\} \mu_{\sigma(t_i)} V_{\sigma(t_i{}^-)}(z(t_i{}^-)) \\ & \leq \exp\left\{\alpha_{\sigma(t_i)}(t-t_i)\mu_{\sigma(t_i)} + \alpha_{\sigma(t_{i-1})}(t_i - t_{i-1})\right\} V_{\sigma(t_{i-1})}(z(t_{i-1})) \\ & = \mu_{\sigma(t_i)} \exp\left\{\alpha_{\sigma(t_i)}(t-t_i) + \alpha_{\sigma(t_{i-1})}(t_i - t_{i-1})\right\} V_{\sigma(t_{i-1})}(z(t_{i-1})). \end{aligned} \tag{5.132}$$

Recursively, we have

$$V(z(t)) \leq \mu_{\sigma(t_i)} \cdots \mu_{\sigma(t_1)} \exp\{\alpha_{\sigma(t_i)}(t-t_i) + \cdots + \alpha_{\sigma(0)}(t_1 - t_0)\} V_{\sigma(t_0)}(z(t_0)). \tag{5.133}$$

We assume there are s'' different sequences in the time interval $[t_0, t)$. If the sequence $[p|q]$ corresponds to the g^{th} sequence, it is written as $[p|q]_{(g)}$, $g = 1,\ 2, \ldots, s''$.

Therefore,

$$\begin{aligned} & \mu_{\sigma(t_i)} \cdots \mu_{\sigma(t_1)} \\ & = \mu_{[\sigma(t_i)|\sigma(t_{i-1})]} \cdots \mu_{[\sigma(t_1),\sigma(t_0)]} \\ & = \prod_{g=1}^{s''} \mu_{[p|q](g)}{}^{N_{[p|q](g)}(t_0,t)} \end{aligned} \tag{5.134}$$

and

$$\begin{aligned} & \alpha_{\sigma(t_i)}(t-t_i) + \cdots + \alpha_{\sigma(0)}(t_1 - t_0) \\ & = \sum_{g=1}^{s''} \alpha_p T_{[p|q](g)}(t_0, t) + t_1 - t_0. \end{aligned} \tag{5.135}$$

Let $t_0 = 0$ and we obtain

$$\begin{aligned} & V(z(t)) \\ & \leq \left\{ \prod_{g=1}^{s''} \mu_{[p|q](g)}{}^{N_{[p|q](g)}(0,t)} \right\} \exp\left\{ \sum_{g=1}^{s''} \alpha_p T_{[p|q](g)}(0,t) + t_1 \right\} V_{\sigma(0)}(z(0)) \\ & = \exp\left\{ \sum_{g=1}^{s''} \{\alpha_p T_{[p|q](g)}(0,t) + N_{[p|q](g)}(0,t) ln \mu_{[p|q](g)}\} + t_1 \right\} V_{\sigma(0)}(z(0)). \end{aligned} \tag{5.136}$$

Let $K_1 = e^{t_1}$ and one can get

$$\begin{aligned} & V(z(t)) \\ & \leq K_1 \exp\left\{ \sum_{g=1}^{s''} \{\alpha_p T_{[p|q](g)}(0,t) + N_{[p|q](g)}(0,t) \ln \mu_{[p|q](g)}\} \right\} V_{\sigma(0)}(z(0)) \\ & = K_1 \exp\left\{ \sum_{g=1}^{s''} T_{[p|q](g)}(0,t) \{\alpha_p + \tfrac{1}{\tau_{a([p|q])}} \ln \mu_{[p|q](g)}\} \right\} V_{\sigma(0)}(z(0)). \end{aligned} \tag{5.137}$$

According to (5.121) and (5.124), we have

$$\alpha_p + \frac{1}{\tau_{a([p|q])}} \ln \mu_{[p|q](g)} = -\gamma^*, \tag{5.138}$$

where $-\gamma^* = \alpha_p + \ln \mu_p / \tau_{a(p)}$ is a negative value.

Therefore,

$$V(z(t)) < K_1 \exp\{-\gamma^* t\} V_{\sigma(0)}(z(0)). \tag{5.139}$$

According to (5.139), $V(z(t)) \to 0$ as $t \to +\infty$. According to formulation (5.129), we can know $z(t) \to 0$ as $t \to +\infty$, which means a consensus can be reached according to (5.125). ■

In Theorem 35, l_p denotes how many segments we divide the time section $[t_i, t_i + \tau_{min,p})$ into. The parameter $\mu_{[p|q]}$ stands for the variation degree of Lyapunov functions at the topology changing instance. The parameter α_p represents the increasing degree of Lyapunov functions.

Remark 15: From the proof process, we can know Theorem 35 can be also got by the Theorem25 and Lemma 27 in Chapter 4 and Theorem26 if the Definition 27 replaces 25 and 26.

If we do not consider the differences of sequences, and let

$$T_p = \sum_q T_{[p|q]}, \tag{5.140}$$

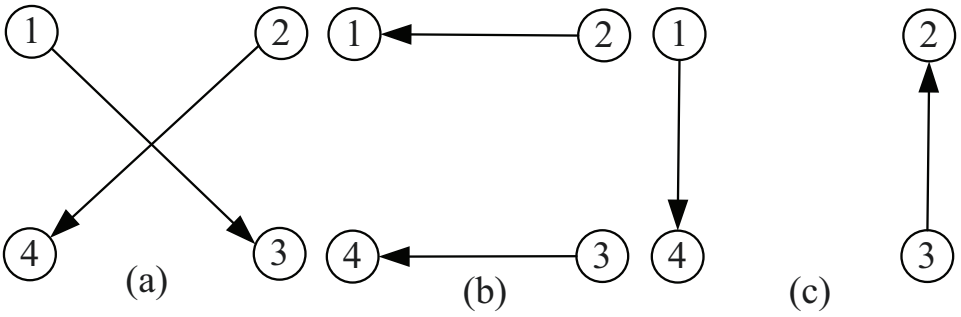


Figure 5.1 : Topologies of multi-agent systems in Examples 1 and 2.

$$N_p = \sum_q N_{[p|q]}, \tag{5.141}$$

we can get the topology-dependent average dwell time

$$\tau_{a(p)} \triangleq \frac{T_p(t_1,t)}{N_p(t_1,t)},$$

and the following corollary.

Corollary 11: For the system (5.12), let $0 < \mu_p < 1,\ \alpha_p > 0$, $\tau_{min,p} = \min_{\sigma(t_i)=p}(t_i - t_{i-1})$ and l_p, $p \in \mathcal{M}$, be the given constants. If there exist a set of positive definite matrices $P_{p,j} > 0$, $j = 0,\ 1, \cdots,\ l_p$, such that $\forall j = 0,\ 1, \cdots,\ l_p - 1$,

$$\begin{cases} \tilde{H}_p^T P_{p,j} + P_{p,j}\tilde{H}_p + \frac{l_p(P_{p,j+1}-P_{p,j})}{\tau_{min,p}} \le \alpha_p P_{p,j}, \\ \tilde{H}_p^T P_{p,j+1} + P_{p,j+1}\tilde{H}_p + \frac{l_p(P_{p,j+1}-P_{p,j})}{\tau_{min,p}} \le \alpha_p P_{p,j+1}, \\ \tilde{H}_p^T P_{p,l_p} + P_{p,l_p}\tilde{H}_p \le \alpha_p P_{p,l_p}, \end{cases} \tag{5.142}$$

$$P_{p,0} \le \mu_p P_{q,l_q}, p \ne q, \tag{5.143}$$

$$\tau_{a(p)} < -\ln \mu_p / \alpha_p, \tag{5.144}$$

then the multi-agent system (5.1) with the protocol (5.2) will reach a consensus.

5.6 Numerical Examples

For multi-agent systems, there are two classes of consensus problems in general: (a) there is no leader; (b) there is a leader. In this section, we simulate both cases to

verify the validity of our results. We consider multi-agent systems with four agents switching among three topologies. Every topology has no spanning tree. We use the values of the three-level signal $\sigma(t)$ to represent these topologies.

5.6.1 Example 1

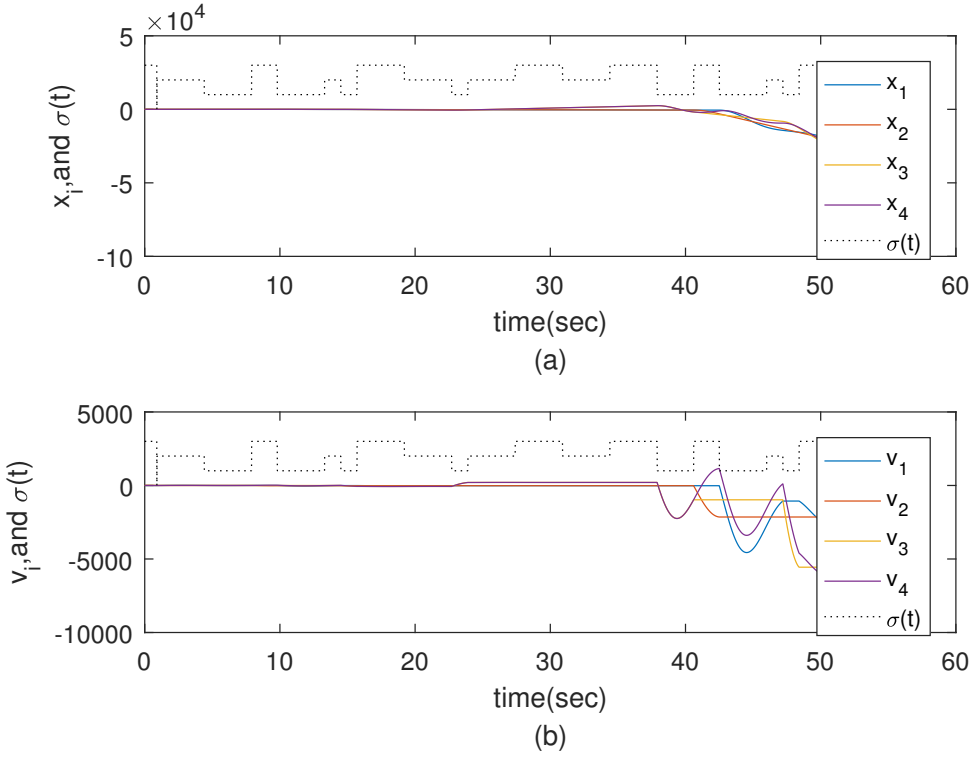


Figure 5.2 : Positions and speed trajectories of agents in Example 1 when conditions in Theorem 35 and Corollary 11 are not satisfied.

We consider the topologies in Fig. 5.1 in this example. We consider these three topologies with time parameters $\tau_{min,1} = 1.2$, $\tau_{min,2} = 1.2$, $\tau_{min,3} = 0.90$, $\tau_{a([1|3])} = 2.7$, $\tau_{a([2|3])} = 3.5$, $\tau_{a([3|1])} = 1.9$, $\tau_{a([1|2])} = 1.2$, $\tau_{a([2|1])} = 1.2$, and $\tau_{a([3|2])} = 0.90$. We set the coupling strength parameters as $\beta_{0_{\sigma(t)}} = 1.0$ and $\beta_{1_{\sigma(t)}} = 0.40$. If we set $l_1 = 3$, $l_2 = 3$, $l_3 = 3$, $\mu_{[3|1]} = 0.041$, $\mu_{[1|3]} = 0.045$, $\mu_{[2|1]} = 0.047$, $\mu_{[1|2]} = 0.048$, $\mu_{[3|2]} = 0.049$, $\mu_{[2|3]} = 0.057$, $\alpha_1 = 0.75$, $\alpha_2 = 0.85$, and $\alpha_3 = 0.90$, Matlab LMI tool boxes cannot find a proper set of positive definite matrices satisfying conditions

(5.122)-(5.123) except for the condition (5.124).

The positions and speed trajectories of all agents are shown in Fig.5.2. As time goes by, agents are getting farther and farther apart from each other. We can see that a consensus cannot be reached.

5.6.2 Example 2

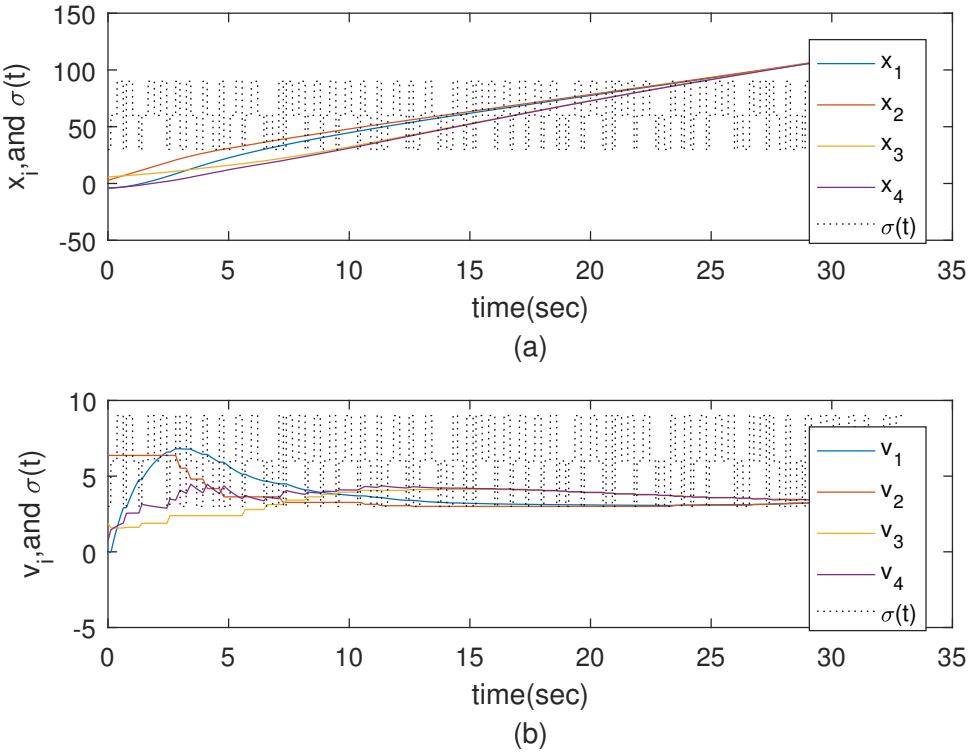


Figure 5.3 : Positions and speed trajectories of agents in Example 2 when all conditions in Theorem 35 and Corollary 11 are satisfied.

For the topologies in Fig. 5.1, we consider the dwell time of three topologies with time parameters $\tau_{min,1} = 0.120$, $\tau_{min,2} = 0.240$, $\tau_{min,3} = 0.180$, $\tau_{a([1|3])} = 0.124$, $\tau_{a([2|3])} = 0.260$, $\tau_{a([3|1])} = 0.184$, $\tau_{a([1|2])} = 0.120$, $\tau_{a([2|1])} = 0.240$, and $\tau_{a([3|2])} = 0.180$. We set the coupling strength parameters as $\beta_{0_{\sigma(t)}} = 0.10$ and $\beta_{1_{\sigma(t)}} = 1.0$. If we set $l_1 = 3$, $l_2 = 3$, $l_3 = 3$, $\mu_{[3|1]} = 0.790$, $\mu_{[1|3]} = 0.794$, $\mu_{[2|1]} = 0.780$, $\mu_{[1|2]} = 0.800$, $\mu_{[3|2]} = 0.800$, $\mu_{[2|3]} = 0.760$, $\alpha_1 = 1.748$, $\alpha_2 = 1.030$, and $\alpha_3 = 1.200$, Matlab LMI

tool boxes can find a proper set of positive definite matrices satisfying conditions (5.122)-(5.124). Now all conditions in Theorem 35 are satisfied.

The positions and speed trajectories of all agents are shown in Fig. 5.3. As time goes by, agents get closer and closer to each other. We can see that they reach a consensus.

5.6.3 Example 3

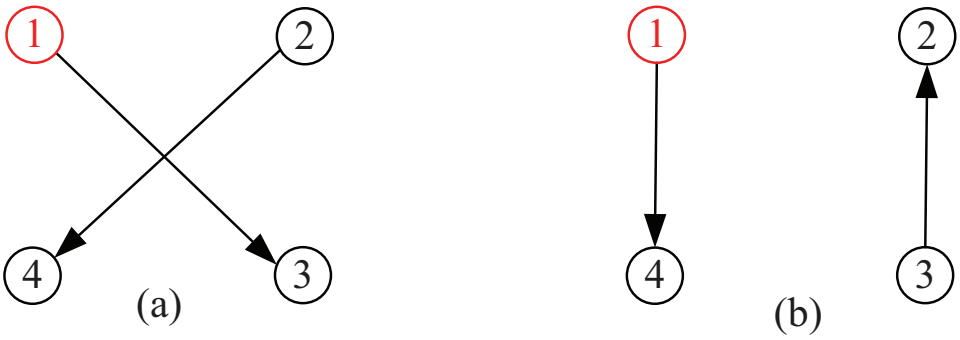


Figure 5.4 : Topologies of multi-agent systems in Examples 3 and 4.

We consider the topologies as shown in Fig. 5.4 in this example. Obviously, the first agent is a fixed leader. We set $\beta_{0_{\sigma(t)}} = 1.0$, $\beta_{1_{\sigma(t)}} = 0.40$, $l_1 = 3$, $l_2 = 3$, $\mu_1 = 0.89$, $\mu_2 = 0.89$, $\alpha_1 = 0.89$ and $\alpha_2 = 1.1$.

Remark 16: According to (5.121), (5.140), (5.141) and the relationship between Theorem 35 and Corollary 11, when the switching topologies are shown in Fig. 5.4, we have

$$\begin{cases} \mu_{[1|2]} = \mu_1, \\ \mu_{[2|1]} = \mu_2, \\ \tau_{a([1|2])} = \tau_{a(1)}, \\ \tau_{a([2|1])} = \tau_{a(2)}. \end{cases} \tag{5.145}$$

We consider the dwell time of two topologies with time parameters $\tau_{min,1} = \tau_{a(1)} = 12$ and $\tau_{min,2} = \tau_{a(2)} = 9.0$(here, although we fixed the dwell time, it is also

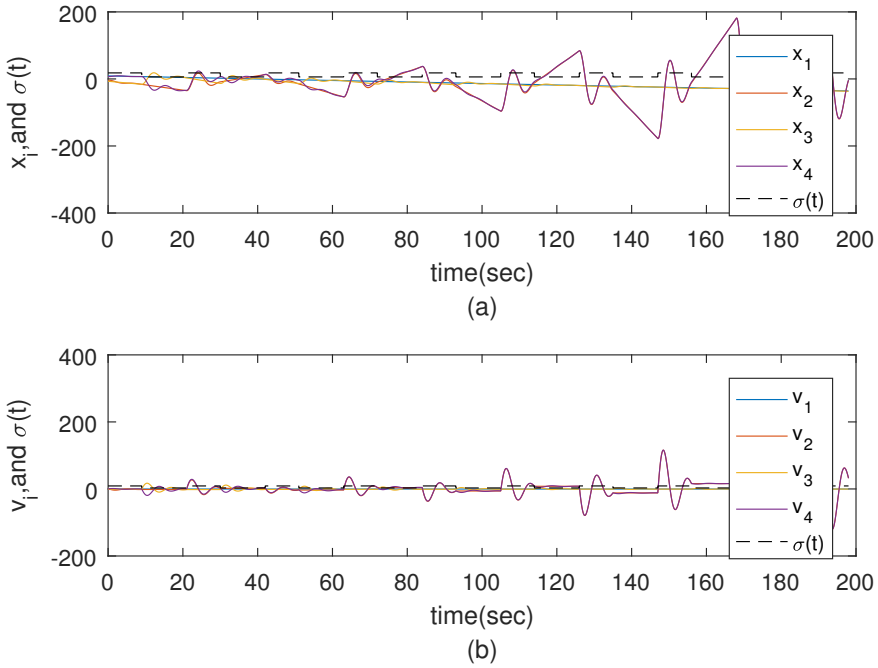


Figure 5.5 : Positions and speed trajectories of agents in Example 3 when conditions in Theorem 35 and Corollary 11 are not satisfied.

valid for *average* dwell time). We can find a proper set of positive definite matrices satisfying conditions (5.142)-(5.143) but not (5.144).

The positions and speed trajectories of all agents are shown in Fig. 5.5. As time goes by, some agents are getting farther and farther apart from others. We can see that they cannot reach a consensus.

5.6.4 Example 4

For the topologies shown in Fig. 5.4, we consider the dwell time of two topologies with time parameters $\tau_{min,1} = \tau_{a(1)} = 0.12$ and $\tau_{min,2} = \tau_{a(2)} = 0.090$. We can find a proper set of positive definite matrices satisfying conditions (5.142)-(5.144). Now all conditions in Corollary 11 are satisfied. The positions and speed trajectories of all agents are shown in Fig. 5.6. We can see that a consensus can be reached.

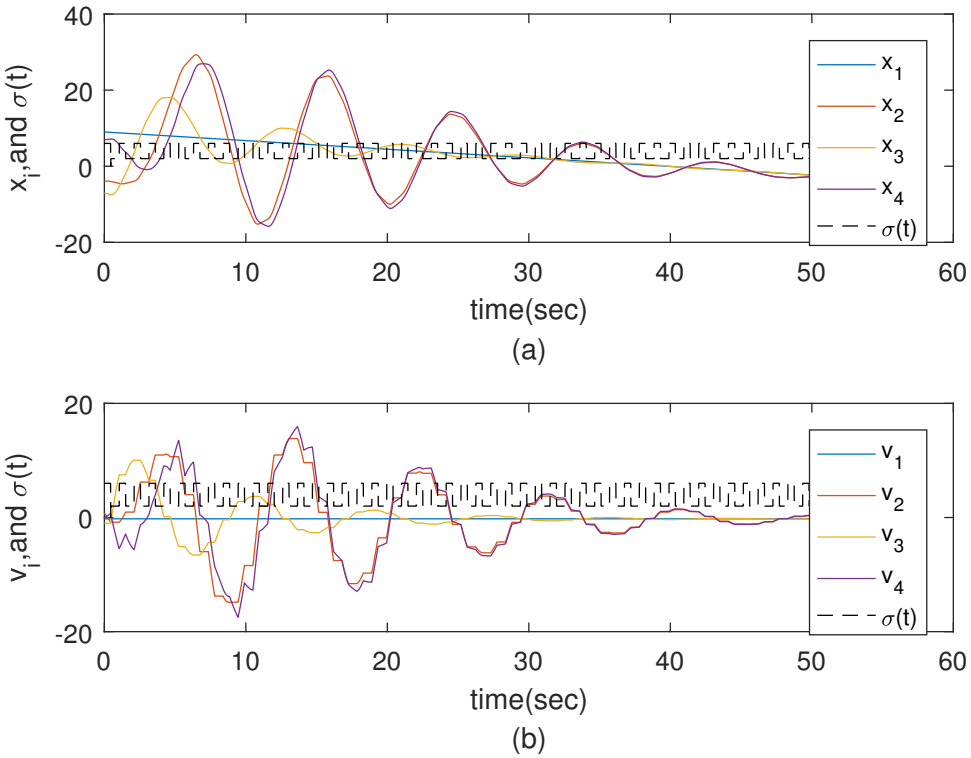


Figure 5.6 : Positions and speed trajectories of agents in Example 4 when all conditions in Theorem 35 and Corollary 11 are satisfied.

In Figs. 5.2 and 5.5 (Examples 1 and 3), it can be seen that the second-order multi-agent systems 5.1 with control protocols 5.2 can reach scattering status rather than a consensus when all switching topologies have no spanning tree even if they have jointly connected topologies.

In this section, we verify the validity of Theorem 35 and Corollary 11, whose switching situation is that a consensus cannot be reached under any separate topology. Other theorems and situations can be verified similarly.

5.7 Summary

In this chapter, we analyze the consensus problem of multi-agent systems under switched topologies by switching topologies sequence-based methods.

The results in Chapter 4 are applied to the cases in this chapter by the model transaction. We get new results about multi-agents' systems. Numerical examples are given.

Chapter 6

Conclusion and Outlook

In this thesis, we study the cooperation of multi-agent systems. The content of this thesis can be summarized as follows:

- We introduce the background of the multi-agent systems. It has been reported that many distributed systems can be modelled as multi-agent systems. In Chapter 1, the research background of this thesis is introduced.

- We summarize the research on the cooperation of multi-agent systems and the stability of switched systems. The cooperation of multi-agent systems and stability analysis of switched systems has received much attention in the past decades. Some fundamental tasks have been studied. The research methods and obtained results are summarized in Chapter 2. Literature Survey and Preliminaries of switched systems and multi-agent systems are introduced in Sections 2.2 and 2.3, relatively.

- We investigate the cooperation of multi-agent systems under a fixed topology. According to the changing of topologies, the running environment of multi-agent systems can be divided into two classes, the fixed topology and the switched topologies. The cooperation of multi-agent systems under a fixed topology is investigated in Chapter 3.

- We propose sequence-based methods to study the stability of switched systems. Many theorems and methods about switched systems have been presented in the past few years, but their conditions are conservative. Using our

proposed sequence-based methods can obtain less conservative conditions. The sequence-based methods and responding conclusions are addressed in Chapter 4.

- We propose sequence-based topology-dependent approaches to analyze the consensus of multi-agent systems under switched topologies. The switching of topologies includes much sequence information that has been explored in more detail in the past years. Our proposed sequence-based topology-dependent approaches utilize this sequence information for deriving consensus conditions. The sequence-based topology-dependent approaches and conclusions are declared in Chapter 5.

The most important proposed methods and approaches in this thesis are:

- Our proposed sequence-based methods to study the stability of switched systems. In existing approaches, the sequence information of switching subsystems is seldom exploited. By exploiting the sequence information, the new methods can release some preconditions and obtain more appropriate threshold values for the stability of switched systems.

- Our proposed sequence-based topology-dependent approaches to analyze the consensus of multi-agent systems. By utilizing the sequence information of switching topologies, our approaches can be used to derive less conservative consensus conditions.

Based on these methods and approaches, many new results are obtained for switched systems and multi-agent systems in this thesis. These new results can be used to

- Analyze the stability of switched systems when every subsystem is stable, in Section 4.2. It has been proved that the switched systems can be unstable even if all subsystems are stable. The proposed methods can help us analyze the stability of switched systems when every subsystem is stable.

- Study the stability of switched systems when part subsystems are unstable in Section 4.3. As some subsystems are unstable, the switched systems comprising these subsystems will probably be unstable. The proposed methods can help us to guarantee the stability of switched systems.

- Research the stabilization of switched systems when every subsystem is unstable in Section 4.4. How to make the switched system stable is an important question. The proposed methods can help us to resolve this problem.

- Design controllers for the switched systems in Sections 4.2-4.4. The proposed methods can help us to design feedback controllers to make the systems stable.

- Investigate the multi-agent systems under switched topologies when the systems can not reach a consensus under part topologies in Section 5.4. In this situation, the multi-agent systems can scatter. The proposed approaches can guarantee a consensus in this situation.

- Judge whether a consensus can be reached for the multi-agent systems under switched topologies when the systems can not reach a consensus under every separate topology in Section 5.5. A touchstone is obtained in light of the proposed approaches.

The possible extension and application of this thesis are fit for:

- Stability analysis of more complex control systems. Because the proposed methods for switched systems can be seen as an update for some popular

methods and they have been applied to many other complex control systems, such time-delayed systems, fuzzy systems, and positive systems, our proposed methods also have the potential for resolving problems in these complex control systems.

- Cooperation problems of real complex distributed systems. As many real distributed systems can be modelled as multi-agent systems, the proposed approaches and obtained results in this thesis have the potential for resolving the cooperation problems of real complex distributed systems.

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