

UNIVERSITY OF TECHNOLOGY SYDNEY

DOCTORAL THESIS

---

**Pricing Pension, Life Insurance and  
Investment Products**

---

*Author:*

Jin SUN

*Supervisor:*

Prof. Eckhard PLATEN

*Co-Supervisors:*

Prof. Pavel SHEVCHENKO

Prof. Erik SCHLOEGL

*Submitted for partial fulfillment of the degree of  
Doctor of Philosophy*

*in the*

School of Mathematical and Physical Sciences  
Faculty of Science

November 3, 2020



# Contents

<b>Declaration of Authorship</b>	<b>v</b>
<b>Abstract</b>	<b>vii</b>
<b>Acknowledgements</b>	<b>ix</b>
<b>List of Figures</b>	<b>xi</b>
<b>List of Tables</b>	<b>xiii</b>
<b>1 Introduction</b>	<b>1</b>
<b>2 A Brief Overview of the Benchmark Approach</b>	<b>5</b>
2.1 The Minimal Market Model . . . . .	7
2.2 Estimation of the GP Dynamics Model . . . . .	9
<b>3 Dynamic Asset Allocation for Target Date Funds</b>	<b>13</b>
3.1 Optimal Dynamic Asset Allocation under the Benchmark Approach . . . . .	14
3.2 Simulation Studies of the TDF Strategies . . . . .	17
3.3 Backtesting with Historical Data . . . . .	18
3.4 Summary . . . . .	25
<b>4 Long-term Optimal Investments with Subjective Preferences</b>	<b>27</b>
4.1 Introduction . . . . .	27
4.2 Optimal Terminal Wealth under the BA . . . . .	28
4.3 Utility of the Terminal Savings under the Cumulative Prospect Theory . . . . .	29
4.4 Numerical Experiments . . . . .	31
4.5 Summary . . . . .	36
<b>5 The Impact of Management Fees</b>	<b>37</b>
5.1 Introduction . . . . .	37
5.2 Formulation of the GMWDB Pricing Problem . . . . .	39
5.3 Calculating the Policyholder's Value Function . . . . .	42
5.4 Calculating the Insurer's Liability Function . . . . .	44
5.5 The Wealth Manager's Value Function and Optimal Withdrawals . . . . .	46
5.5.1 The wealth manager's value function . . . . .	46
5.5.2 Formulation of two optimization problems . . . . .	47
5.6 Numerical Examples . . . . .	48
5.6.1 Setup of the experiments . . . . .	48
5.6.2 Results and implications . . . . .	49
5.7 Conclusions . . . . .	50

<b>6</b>	<b>Optimal Hedging of Variable Annuities</b>	<b>57</b>
6.1	Introduction	57
6.2	Description of the VA Guarantee Product	59
6.3	Pricing of the VA with Guarantee	60
6.4	Modelling the Underlying Equity Index	62
6.4.1	The Black-Scholes model	62
6.4.2	The minimal market model	62
6.5	Empirical Studies on Historical Equity Index Data	62
6.6	Summary	66
<b>7</b>	<b>Benchmarked Risk Minimizing Hedging for Life Insurance</b>	<b>69</b>
7.1	The Financial Market Model	70
7.2	The Mortality Model	71
7.3	A Brief Overview on Benchmarked Risk-Minimization	73
7.4	BRM Hedging Strategies for Endowment Insurance Policies	75
7.5	Numerical Examples	77
7.6	Summary	81
<b>8</b>	<b>Conclusions and Future Works</b>	<b>83</b>

## Declaration of Authorship

I, Jin SUN, declare that this thesis titled, "Pricing Pension, Life Insurance and Investment Products" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.
- This research is supported by an Australian Government Research Training Program.

Production Note:

Signed: Signature removed prior to publication.

---

Date: 3 November 2020

---



UNIVERSITY OF TECHNOLOGY SYDNEY

# *Abstract*

Faculty of Science

Doctor of Philosophy

**Pricing Pension, Life Insurance and Investment Products**

by Jin SUN

This thesis considers the pricing of pension, life insurance and investment products under a general framework. In particular, this thesis considers the pricing under the risk-neutral framework, the benchmark framework and the utility framework. This thesis demonstrates how long term contracts can be less expensively produced with higher returns on investments under the benchmark framework than under the risk-neutral framework. This thesis works under the minimal market model, a parsimonious model for well diversified equity market indexes that incorporates the well documented mean reversion of equity returns and the leverage effect. This thesis uses historical equity returns data for out-of-sample backtests to demonstrate the effective hedging of long-term financial and insurance contracts.





## *Acknowledgements*

The author would like to thank Prof. Michael Sherris of UNSW, who carefully reviewed an earlier draft of this thesis with valuable and insightful suggestions.

The author would like to thank Dr. Dan Zhu of Monash University, who made significant contributions to a working paper that corresponds to one of the chapters in this thesis.

The author acknowledges inputs from Dr. Kevin Fergusson, Dr. Simon Fung, Prof. Eckhard Platen, Prof. Pavel Shevchenko to this thesis, as well as Erik Schloegl, who spent time reading an earlier draft of this thesis.

The author acknowledges funding from CSIRO in completing this thesis, which covered part of the travel costs required by the same agency.



# List of Figures

2.1	Discounted S&P500 log-prices with a clear linear trend. . . . .	10
3.1	Simulated sample paths of the GP prices at a monthly rate from the estimated MMM, over a period of 40 years (480 months). . . . .	18
3.2	The in-sample and out-of-sample projected and realised return distributions of the TDF strategies under simulated paths over 40 years. . .	19
3.3	The projected and realised portfolio processes of the TDF strategies, along with the corresponding risky-asset positions, over historically realised index prices, based on out-of-sample model estimations. . . .	21
3.4	Historically simulated sample paths of the GP prices at a monthly rate from the estimated MMM, over a period of 40 years (480 months). . . .	23
4.1	The CPT investor's utility of wealth, probability distortion for the gains, and the optimal terminal wealth as a function of the terminal GP as well as the quantile function. . . . .	32
4.2	The CPT investor's model-predicated and realised portfolio processes, as well as risky asset allocations in the GP. . . . .	34
4.3	The CPT investor's probability distortions and investment outcomes. . .	35
5.1	Fair insurance fee rates and policy values as a function of management fee rates $\alpha_m$ for risk-free rate $r = 1\%$ and volatility $\sigma = 30\%$ , for penalty rates $\beta = 10\%, 20\%$ and maturities $T = 5, 10, 20$ years. The left axis and dark plots refer to the fair fees in percentage; The right axis and gray plots refer to the policy values. Legends across all plots are shown in the upper left panel. . . . .	51
5.2	Fair insurance fee rates and policy values as a function of management fee rates $\alpha_m$ for risk-free rate $r = 5\%$ and volatility $\sigma = 10\%$ , for penalty rates $\beta = 10\%, 20\%$ and maturities $T = 5, 10, 20$ years. The left axis and dark plots refer to the fair fees in percentage; The right axis and gray plots refer to the policy values. Legends across all plots are shown in the upper left panel. . . . .	52
6.1	Value processes associated with the VA product when the pricing and hedging as well as optimal withdrawals are performed based on the BSM and risk-neutral pricing. . . . .	64
6.2	Value processes associated with the VA product when the pricing and hedging are performed based on the BSM and risk-neutral pricing assuming an optimal withdrawal behavior, while the actual withdrawal behavior follows the static strategy. . . . .	65
6.3	Value processes associated with the VA product when the pricing and hedging are performed based on the BSM and risk-neutral pricing assuming an optimal withdrawal behavior, while the actual withdrawal behavior follows the MMM under the BA. . . . .	65

6.4	Value processes associated with the VA product when the pricing and hedging are performed based on the MMM and BA assuming an optimal withdrawal behavior, while the actual withdrawal behavior follows the BSM under risk-neutral pricing. . . . .	67
6.5	Value processes associated with the VA product when the pricing and hedging as well as the optimal withdrawals are performed based on the MMM and the BA. . . . .	67
7.1	Force of mortality over 1987-2017 for US females aged 30 at the beginning of 1987. . . . .	78
7.2	Simulated GP sample trajectories from the estimated MMM and the number of deaths within the portfolio from the historical force of mortality model. The thick trajectory in panel (a) is the realized S&P 500 index prices process; The thick trajectory in panel (b) is a randomly chosen trajectory that, we assume, corresponds to the realized GP in the subsequent simulations. . . . .	78
7.3	The sample trajectories of the BRM price process of the portfolio of life insurance policies and the benchmarked price process for the first 20 simulated scenarios, where the thick trajectories represent the realized scenario. . . . .	79
7.4	The trajectories of the realized price process and the hedging errors. Panel (a) shows the realized self-financing part of the BRM price process $\tilde{V}^{\vartheta^*}$ ; Panel (c) shows the corresponding hedging error process $V^{\nu^*} - \tilde{V}^{\vartheta^*}$ ; Panel (b) shows the realized delta-hedging implementation of the BRM price process $\tilde{V}^{\delta^*}$ ; and Panel (d) shows the corresponding hedging error process $V^{\nu^*} - \tilde{V}^{\delta^*}$ . . . . .	80
8.1	Histograms of uniform samples backed out from the S&P 500 historical index prices: the MMM backed-out "uniform" samples result in a p-value of $1.4 \times 10^{-9}$ for the KS statistic; the BSM backed-out "uniform" samples result in a p-value of $5.68 \times 10^{-65}$ for the same statistic. . . . .	84

# List of Tables

2.1	Estimated MMM parameter values and standard deviations are obtained from different estimation schemes. The standard deviations are computed based on 1000 simulated GP paths . . . . .	11
3.1	Sample means and standard deviations of the pathwise annualized excess log-returns of the TDF strategies under the simulated scenarios. ( All values are given in %.) . . . . .	20
3.2	Realised TDF log-returns on historical S&P500 data starting from 02/1978. (All values are given in %.) . . . . .	22
3.3	Sample means and standard deviations of the pathwise annualized excess log-returns from out-of-sample TDF strategies, under the historically simulated scenarios. (All values are given in %.) . . . . .	24
4.1	The CPT investors' preference parameters. . . . .	31
4.2	The CPT investors' investment objectives and outcomes. . . . .	33
5.1	Fair fee rate $\alpha_{\text{ins}}$ (%) based on the liability maximization strategy $\Gamma^L$ . . . . .	53
5.2	Fair fee rate $\alpha_{\text{ins}}$ (%) based on the policy value maximization strategy $\Gamma^V$ . . . . .	54
5.3	Total policy value $V(0, \mathbf{X}(0); \Gamma^L)$ based on the liability maximization strategy $\Gamma^L$ . . . . .	55
5.4	Total policy value $V(0, \mathbf{X}(0); \Gamma^V)$ based on the policy value maximization strategy $\Gamma^V$ . . . . .	56
7.1	Sample variance reductions of terminal portfolio values, P&Ls and hedging error processes . . . . .	81

