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DOCTORAL THESIS

Pricing Pension, Life Insurance and Investment Products

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Contents

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I, Jin SUN, declare that this thesis titled, "Pricing Pension, Life Insurance and Investment Products" and the work presented in it are my own. I confirm that:

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Abstract

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Pricing Pension, Life Insurance and Investment Products

by Jin SUN

This thesis considers the pricing of pension, life insurance and investment products under a general framework. In particular, this thesis considers the pricing under the risk-neutral framework, the benchmark framework and the utility framework. This thesis demonstrates how long term contracts can be less expensively produced with higher returns on investments under the benchmark framework than under the risk-neutral framework. This thesis works under the minimal market model, a parsimonious model for well diversified equity market indexes that incorporates the well documented mean reversion of equity returns and the leverage effect. This thesis uses historical equity returns data for out-of-sample backtests to demonstrate the effective hedging of long-term financial and insurance contracts.

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Chapter 1

Introduction

Long dated securities (10 years plus) underpin major national and international investments funded by banks, insurance companies, pension funds and governments. Valuation and management of these investments based on classical asset and derivative pricing theories have been reasonably successful over the second half of the last century, due to relatively high interest rates, low debt levels and high productivity growth of world's major economies. However, since the turn of the century, developed economies have experienced ongoing low and even negative interest rates that can be expected to last. The lasting low interest rates create a foreseeable gap between long dated liabilities and accumulated assets for many pension funds and life insurance companies. The situation is worsened by increasing life expectancies due to improvement of overall welfare and quality of life over the majority part of the world; see OECD (2015a,b).

One of the classical asset pricing theories for determining the value of securities is the arbitrage pricing theory (APT) and the associated risk neutral pricing of financial derivatives, developed by Ross (1976); Merton (1973); Black & Scholes (1973). The valuation and management of long term pension and insurance products have relied on the classical APT and risk neutral pricing, where the locally risk-free savings account is taken as the numeraire, against which all future cash flows are priced. Lasting low interest rates make the valuation of long term financial claims expensive, and many pension schemes are likely to fail to meet the long term liabilities if valuation and management are continually considered under the classical approach and the interest rates remain low (Bateman et al., 2001). An alternative valuation approach is needed taking a numeraire that is more representative of the long term economic growth than the locally risk-free savings account is the solution, as is demonstrated in this thesis.

Classical APT and risk neutral pricing rely on a strong form of no-arbitrage condition of Delbaen & Schachermayer (1994), known as the no free lunch with vanishing risk condition, henceforth referred to as the NFLVR condition, on the dynamics of the market. Evidences of failure of this condition have been observed in reality, see Bakshi et al. (2000); Baldeaux et al. (2015). The benchmark approach (BA) and the associated real world pricing, developed by Eckhard Platen and coauthors in a series of articles and research monographs, see, e.g., Platen (1998, 2001a, 2002, 2006); Platen & Heath (2006); Platen & Bruti-Liberati (2010, Chapter 3), generalizes the APT and risk neutral pricing by weakening its strong no-arbitrage condition. The BA imposes only the weaker condition on the existence of the numeraire portfolio, or growth-optimal portfolio (GP), also known as the Kelly portfolio (Kelly, 1956). The GP is the best performing portfolio in several aspects. The GP maximizes the instantaneous growth rate, as well as the expected log-utility of terminal wealth. Furthermore, it minimizes the expected time to reach a certain level of wealth, see Kardaras & Platen (2010). The GP more closely tracks the growth of the overall economy growth than the locally risk-free savings account. Under the BA, the GP replaces the latter as numeraire for pricing of financial securities and contingent claims.

Since its introduction in the above-mentioned articles and monographs, the BA has drawn increasing attention in the literature; see, e.g., Karatzas & Kardaras (2007); Pal & Protter (2010); Ruf (2013) and essays in Chiarella & Novikov (2010). A significant body of research exists on the BA, including both theoretical foundations in, e.g., Filipović & Platen (2009) and empirical results and hedging applications in, e.g., Heath & Platen (2005); Platen & Rendek (2012); Barkhagen et al. (2016); Fergusson & Platen (2014); Baldeaux et al. (2015), among others.

The cornerstone of the BA is the existence of the GP in the investment universe where all primary financial securities and portfolios are traded. To model the GP, the fundamental observation is made that without making specific assumptions on the market dynamics, the GP follows a squared Bessel process of dimension four, after a suitable time transformation is made. The minimal market model (MMM), see Platen (2001b), captures this critical observation in a stylized manner and models the GP through the modeling of the growth time, which to the first approximation can be simply modeled as an exponential function of time with a constant growth rate. Such a parsimonious model is called a stylized MMM.

Pricing of financial claims and stochastic cash flows under the BA is called real-world pricing, where future cash flows are denominated in units of the GP, that is, benchmarked, and the real world expectation of the benchmarked future cash flows is taken as its current benchmarked price. In other words, the GP is taken as the stochastic discount factor for the pricing of cash flows, leading to the real world pricing. The concept avoids the specification of a risk neutral pricing measure, in contrast to the APT with its risk neutral pricing. The existence of an equivalent risk neutral pricing measure imposes the NFLVR condition on the market dynamics. The BA on the other hand weakens this condition, and requires only the existence of the GP. It can be viewed as a generalization of the APT with risk neutral pricing, to which it reduces when the strong condition on the existence of an equivalent risk neutral probability measure is indeed satisfied. For more details on the BA framework, see Platen & Heath (2006). For more general discussions on stochastic discount factor in the theory of asset pricing, see Cochrane (2001).

Similar to the Black-Scholes model (BSM) under risk neutral pricing, see Black & Scholes (1973), the stylized MMM under the real world pricing allows closed-form formulas for pricing a number of standard contingent claims including zero coupon bonds, plain vanilla options, etc.. As the log-normal distribution is underlying the BSM, the noncentral chi-squared (NCX^2) distribution underlies the MMM. Functionals of the NCX^2 distribution relevant to the real world pricing of many contingent claims are derived and implemented in this thesis, and closed form formulas of these contingent claims are presented.

The BA was established to work under the real world probability measure. This implies that historical market data can be used to estimate BA model parameters. In particular, parameters of the MMM can be estimated from historical returns of the GP. In reality, the GP of a given market depends on the instantaneous market prices of risk factors, which are generally hard to estimate, see, e.g., DeMiguel et al. (2009). As an alternative, the GP may be approximated by a well-diversified portfolio. For example, the S&P 500 index may be regarded as an approximation to the GP of the US domestic market, and its historical returns may be used to estimate the MMM parameters of this index, see Platen & Rendek (2012). On the other hand, the prices of options written on this index can also be used for estimation purposes, without measure transformation between the real world and the risk neutral probability measures, and regardless of the existence of the latter. In other words, the BA incorporates existing risk neutral option pricing theory as a special case under the general framework.

Target date funds (TDFs) are becoming increasingly popular investment choices for retirees seeking to save for retirement at given ages. TDFs provide risk-efficient exposure to a diversified range of asset classes that matches the evolving risk profile of the investment payoff as the investors age. This is often achieved by making increasingly conservative asset allocations over time as the retirement date approaches. Such dynamically evolving allocation strategies for the TDFs are often referred to as the glide paths. Chapter 3 propose under the BA a systematic approach to the design of TDF glide paths with different target dates and risk preferences. The TDF strategies proposed are dynamic portfolios consisting of units of the GP and the money market account. The design falls under the expected utility theory, where the optimal payoff of the TDF implied by the retirement date and the risk preference of the investor is maximized, and the corresponding dynamic asset allocation strategy that delivers the optimal payoff at a predetermined cost is constructed. Chapter 3 is based on Sun et al. (2020).

In Chapter 4, I consider the long-term investment optimization problem with the objective of maximizing the investor's utility function of the future payoff, given an initial budget constraint. I work under the benchmark approach, where the inverse of the (discounted) GP is taken as the pricing kernel, leading to the least expensive replication cost of any future contingent payoffs. I construct the optimal payoffs corresponding to a class of subjective utilities under the cumulative prospect theory (CPT) of Kahneman and Tversky (Kahneman & Tversky, 1979; Tversky & Kahneman, 1992), subject to initial budget constraints. The CPT goes beyond conventional economical utility theory by taking into consideration some well observed behaviors from real economical agents, including an S-shaped utility function, and a subjective belief. I further develop the corresponding replication strategies to deliver the optimal payoffs under these assumptions made by the CPT. Chapter 4 is based on Sun & Platen (2020).

In recent years, the pensions and life insurance market has seen a proliferation of variable annuities (VA) with some guarantees of living and death benefits to assist individuals in managing their pre-retirement and post-retirement financial plans. These products take the advantage of market growth while providing a protection of the wealth investment against the market downturns. Under the post-crisis market conditions, the VA guarantees have become more valuable, the fulfillment of the liabilities more demanding, and the effective and efficient hedging of risks associated with the VA guarantees increasingly important. In Chapters 5 and 6, I consider optimal withdrawal strategies of the VA policy that maximizes the pay-out of VA contracts, and the associated hedging strategies for these contracts. In particular, I consider the pricing of a VA with guaranteed minimum withdrawal benefits (GMWB) under the risk neutral framework and the BA framework. I consider both a passive policy holder who follows a static strategy that makes the contracted withdrawals, and an active policy holder who follows a dynamic strategy that makes optimal withdrawals to maximize the total value of the contract. I model the underlying equity index in which the VA is invested with the MMM and the BSM. I price a stylized VA contract linked to an equity index with GMWB protection, and construct a replicating portfolio to finance the withdrawals and the final payoff, under withdrawal and hedging strategies under both models, and compare the outcomes. Chapter 5 is based on Sun et al. (2018). Chapter 6 is based on Sun et al. (2019).

Traditional life insurance policies offer no equity investment opportunities for the premium paid, and suffer from low returns over the long insurance terms. Modern equity-linked insurance policies offer equity investment opportunities exposed to equity market risk. To combine the low-risk of traditional policies with the high returns offered by equity-linked policies, I consider in Chapter 7 insurance policies under the BA, where the policyholders' funds are invested in the GP and the locally risk-free savings account. Due to unhedgeable mortality risk, life insurance policies cannot be fully hedged. In this case benchmarked risk-minimization (BRM) can be applied to obtain hedging strategies with minimally fluctuating profit and loss processes, where the fluctuations can further be reduced through diversification. Chapter 7 is based on Sun & Platen (2019).

Chapter 2

A Brief Overview of the Benchmark Approach

In this chapter, we give a brief overview of the benchmark approach (BA) under a diffusive market model. Much of this section follows from Platen & Heath (2006), to which interested readers are referred for a more complete treatment.

We consider a general diffusive financial market model with uncertainties driven by a d-dimensional Brownian motion $\boldsymbol{W}$, with $\boldsymbol{W}(t) = \{(W(t)^1, ..., W(t)^d)^\top, t \in [0,T]\}$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where T is some fixed time horizon, and the filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$ satisfies the usual conditions of right continuity and completeness, and models the accumulation of information over time; see Karatzas & Shreve (1991). We assume that there exists a locally risk-free savings account $S^0(t)$ and m nonnegative risky primary security accounts $\boldsymbol{S}(t) = (S^1(t), ..., S^m(t))^\top$ satisfying the vector stochastic differential equation (SDE)

$$d\boldsymbol{S}(t) = \boldsymbol{S}(t)\left(a(t)dt + b(t) \cdot d\boldsymbol{W}(t)\right), \quad t \in [0,T], \tag{2.1}$$

where $a(t)$ is the instantaneous drift vector and $b(t)$ the instantaneous volatility matrix, which both are assumed to be predictable and such that a unique strong solution of the above system of SDEs exists. We assume that all dividends and interests are reinvested. Without loss of generality, we further assume that $S^0(t) \equiv 1$. This means that we denominate all security prices in units of the locally risk-free savings account. In practice, the locally risk-free savings account may be approximated by the money market account that invests in short-term T-bills in a rolling manner. Thus in our notation, all primary security accounts are discounted by the locally risk-free savings account.

We denote by $S^{\boldsymbol{\pi}}$ the value process of a strictly positive, self-financing portfolio with portfolio weights $\boldsymbol{\pi}(t) = (\pi^1(t), ..., \pi^m(t))^\top, t \in [0,T]$, which invests at time t a fraction $\pi^j(t)$ of the total wealth in the jth primary security account, and the remaining wealth in the locally risk-free savings account. The value process satisfies then the SDE

$$\frac{dS^{\boldsymbol{\pi}}(t)}{S^{\boldsymbol{\pi}}(t)} = \boldsymbol{\pi}(t)^\top(a(t)dt + b(t) \cdot d\boldsymbol{W}(t)), \quad t \in [0,T]. \tag{2.2}$$

By Ito's formula, the SDE for the log-price is of the form

$$d\log S^{\boldsymbol{\pi}}(t) = \boldsymbol{\pi}(t)^\top\left(\left(a(t) - \frac{1}{2}b(t)b(t)^\top\boldsymbol{\pi}(t)\right)dt + b(t) \cdot d\boldsymbol{W}(t)\right), \quad t \in [0,T]. \tag{2.3}$$

We consider the growth-optimal portfolio (GP) $S^{\boldsymbol{\pi}^*}$ of this investment universe for which the instantaneous expected growth rate, that is, the drift of (2.3), is maximized for all t. This is achieved by setting the optimal portfolio weights $\boldsymbol{\pi}^*(t)$ to

$$\boldsymbol{\pi}^*(t) = \arg\max_{\boldsymbol{\pi}} \boldsymbol{\pi}^\top \left(a(t) - \frac{1}{2} b(t) b(t)^\top \boldsymbol{\pi} \right), \quad t \in [0, T]. \tag{2.4}$$

We assume that a solution to (2.4) exists a.s. for all $t \in [0, T]$. One potential such solution is given by

$$\boldsymbol{\pi}^*(t) = \left(b(t) b(t)^\top \right)^+ a(t), \quad t \in [0, T], \tag{2.5}$$

where $\left(b(t)b(t)^\top\right)^+$ denotes the Moore-Penrose generalized inverse of the self-adjoint matrix $b(t)b(t)^\top$. Note that the value process of the GP is unique, however, the fractions may vary due to potential redundancies in the primary security accounts.

For the market model to be viable, we assume that the GP process, denoted as $S^*(t) := S^{\boldsymbol{\pi}^*}(t), t \in [0, T]$, with $\boldsymbol{\pi}^*(t)$ given by (2.5), exists and is strictly positive. By substituting (2.5) into (2.2), we obtain the SDE

$$\frac{dS^*(t)}{S^*(t)} = \|\boldsymbol{\theta}^*(t)\|^2 dt + \boldsymbol{\theta}^*(t) \cdot d\boldsymbol{W}(t), \quad t \in [0, T], \tag{2.6}$$

where $\boldsymbol{\theta}^*(t) = b(t)^\top \boldsymbol{\pi}^*(t)$ is the risk factor loading process of S^*. The above SDE can further be written as

$$dS^*(t) = \alpha(t) dt + \sqrt{\alpha(t) S^*(t)} dW^*(t), \quad t \in [0, T], \tag{2.7}$$

where the drift $\alpha(t) = \|\boldsymbol{\theta}^*(t)\|^2 S^*(t)$ is assumed to be strictly positive, and $W^*(t)$, defined by the SDE

$$dW^*(t) = \frac{\boldsymbol{\theta}^*(t)}{\|\boldsymbol{\theta}^*(t)\|} \cdot d\boldsymbol{W}(t) \quad t \in [0, T], \tag{2.8}$$

with $W^*(0) = 0$, forms a standard Brownian motion by Levy's characterization theorem. So far, we only re-parametrized the GP dynamics different to the common volatility modeling specification. Note that the above drift $\alpha(t)$ can be, at this stage, still very general. Later on, we will make this drift more specific, which yields then a proper model.

The GP is the unique portfolio which, when used as numeraire or benchmark, makes any benchmarked portfolio process $\hat{S}^{\boldsymbol{\pi}}$, defined as $\hat{S}^{\boldsymbol{\pi}}(t) = \frac{S^{\boldsymbol{\pi}}(t)}{S^*(t)}$, a local martingale. To see this, apply Ito's lemma to $\hat{S}^{\boldsymbol{\pi}}(t)$, making use of (2.2), (2.6) as well as the fact that $\left(b(t)b(t)^\top\right) \boldsymbol{\pi}^*(t) = a(t)$ from (2.5). After some algebra, the SDE satisfied by $\hat{S}^{\boldsymbol{\pi}}(t)$ is obtained as

$$d\hat{S}^{\boldsymbol{\pi}}(t) = \hat{S}^{\boldsymbol{\pi}}(t)(\boldsymbol{\theta}^{\boldsymbol{\pi}}(t) - \boldsymbol{\theta}^*(t)) \cdot d\boldsymbol{W}(t), \tag{2.9}$$

namely, $S^{\boldsymbol{\pi}}$ forms a local martingale. Here $\boldsymbol{\theta}^{\boldsymbol{\pi}}(t) = \boldsymbol{\pi}^\top(t) b(t)$ is the risk factor loading process of $S^{\boldsymbol{\pi}}$.

If we assume the portfolio process $S^{\boldsymbol{\pi}}$ to be nonnegative, the benchmarked portfolio process $\hat{S}^{\boldsymbol{\pi}}$ becomes a supermartingale by Fatou's lemma. Given an $\mathcal{F}_T$-measurable nonnegative contingent claim $V(T) \geq 0$ with maturity T, its, so called, fair price process under the BA is given by the real-world pricing formula as

$$V(t) = E_t \left(\frac{S^*(t)}{S^*(T)} V(T) \right), \quad t \in [0,T], \tag{2.10}$$

where $E_t(\cdot) = E(\cdot|\mathcal{F}_t)$ denotes the $\mathcal{F}_t$-conditional expectation under the real-world probability measure $\mathbb{P}$. The benchmarked fair price process, defined as $\hat{V}(t) = \frac{V(t)}{S^*(t)}$, forms then a nonnegative $(\mathbb{F}, \mathbb{P})$-martingale. The benchmarked fair price process $\hat{V}$, if replicable, represents the least expensive portfolio among all benchmarked nonnegative self-financing replication portfolios, which form supermartingales. To see this, consider an alternative replication portfolio $U = \{U(t) \geq 0, t \in [0,T]\}$ such that $U(T) = V(T)$ $\mathbb{P}$-a.s.. The corresponding benchmarked price process $\hat{U}(t) = \frac{U(t)}{S^*(t)}$ forms an $(\mathbb{F}, \mathbb{P})$-super-martingale, thus

$$\hat{U}(t) \geq E_t(\hat{U}(T)) = E_t(\hat{V}(T)) = \hat{V}(t), \qquad t \in [0,T], \tag{2.11}$$

which implies $U(t) \geq V(t)$ for all $t \in [0,T]$.

2.1 The Minimal Market Model

The optimality of the TDF strategy presented in Section 3.1 holds under a general diffusive model. To compute the optimal portfolio value (3.4) and the corresponding asset allocation (3.11), however, one needs to have a reasonable proxy for the GP, and a reasonable model to describe the dynamics of the GP. For the latter purpose, we propose the minimal market model (MMM) for the underlying (discounted) GP dynamics. In this section, we give a brief introduction to the MMM, for full details, the reader is referred to Platen & Heath (2006).

To motivate the MMM, we begin with the general framework outlined so far, where we see that under very general assumptions, the GP $S^*(t)$ is a well diversified portfolio with portfolio weights (2.4) drawn from all primary securities. This leads us to the first assumption, that the GP may be approximated by a well diversified market portfolio, such as the S&P 500 index for the US equity market, see Platen & Rendek (2012) for more details on this assumption. Furthermore, the GP satisfies the SDE (2.7), with a drift process $\alpha(t) = \|\boldsymbol{\theta}(t)\|^2 S^*(t)$, which is directly linked to the aggregate market price of risk process $\|\boldsymbol{\theta}(t)\|$, see (2.6). It is generally very difficult to specify the market price of risk process precisely. However, it is a reasonable assumption that this process should be stationary. On the other hand, we note that the GP generally follows a trend of exponential growth, due to the exponential nature of economical growth in general. Therefore, there are reasons to believe that the drift process $\alpha(t)$ should also assume a general exponential trend. Moreover, this drift can be linked to the fundamental economical growth underlying the market, which we thus assume to be relatively insensitive to short-term market fluctuations. Consequently, as a parsimonious approximation, we assume that the drift process $\alpha(t)$ follows a deterministic exponential function, given by $\alpha(t) = \alpha_0 e^{\eta t}$, where η is the long-term growth rate of the GP, and α_0 is a constant representing the initial scale of the GP. Under these assumptions, the GP follows by (2.7) the SDE

$$dS^*(t) = \left(\alpha_0 e^{\eta t}\right) dt + \sqrt{\alpha_0 e^{\eta t} S^*(t)} dW^*(t), \quad t \in [0,T] \tag{2.12}$$

under the real-world probability measure $\mathbb{P}$.

The GP dynamics (2.12) evidently follows a time-transformed squared Bessel process of dimension four, with a deterministic time transformation. To see this, define the market time $\tau(t) = \int_0^t \alpha_0 e^{\eta s} ds$, denominate S^* under τ as $S^*(\tau(t)) = S^*(t)$ with some abuse of notation, and substitute into (2.12). By Levy's characterization theorem, the GP dynamics in the market time satisfies the SDE

$$dS^*(\tau) = d\tau + \sqrt{S^*(\tau)} d\tilde{W}^*(\tau), \quad \tau \in [0, \tau(T)], \tag{2.13}$$

which forms a squared-Bessel process of dimension four in τ time. Here $\tilde{W}^*(\tau(t)) = \int_0^t \sqrt{\tau(s)} dW^*(s)$ is a Brownian motion in τ time. For further details on squared Bessel processes, see Revuz & Yor (1999).

We next define in the model the normalized GP as $Y(t) = \frac{\eta}{\alpha_0 e^{\eta t}} S^*(t), t \in [0, T]$, which satisfies the SDE

$$dY(t) = (1 - Y(t))\eta dt + \sqrt{Y(t)\eta} dW^*(t), \quad t \in [0, T]. \tag{2.14}$$

The normalized GP is mean-reverting around the level of 1. The mean reversion of the normalized index implies the mean reversion of the GP around the long-term growth trend. It is well documented that equity log-returns follow in the long run a mean-reverting pattern, see, e.g., Campbell & Viceira (2005); Shiller (2015). By decomposing the GP into the normalized GP index $Y(t)$ and a simple exponential fundamental value function $\frac{\alpha_0 e^{\eta t}}{\eta}$, the MMM captures this important stylized fact in a parsimonious structure. Furthermore, the instantaneous squared volatility of the normalized GP equals that of the GP and is inversely proportional to the value of the normalized GP, generating the so-called leverage effect, see Black (1976).

The key properties that the model reflects realistically are the mean-reversion of the normalized index, the exponential long-term growth of the fundamental value and the leverage effect modeled by the square root in its diffusion coefficient, making the inverse of the normalized GP its squared volatility. Additionally, the model generates leptokurtic log-returns with similar tail properties as the data. In contrast, many models under the risk-neutral framework, such as the Black-Scholes model, and the popular Heston model, do not exhibit the mean-reversion property and/or do not generate so easily log-returns with realistic tail properties. See Fergusson & Platen (2006).

The normalized GP described by (2.14) is a time transformed square-root process of dimension four, with the transition law of a noncentral Chi-squared (NCX^2) distribution, given by

$$Y(u) \overset{\mathcal{L}}{=} \frac{1 - e^{-\eta(u-t)}}{4} \chi_4^2 \left(\frac{4e^{-\eta(u-t)}}{1 - e^{-\eta(u-t)}} Y(t) \right), \quad 0 \le t < u \le T, \tag{2.15}$$

where $\chi_4^2(\zeta)$ denotes a NCX^2 random variable of dimension 4 and noncentrality parameter ζ; see, e.g., Broadie & Kaya (2006). The $\chi_4^2(\zeta)$ random variable has finite moments of all positive orders. The probability density function is given by

$$f(\zeta, x) = \frac{1}{2} e^{-\frac{\zeta + x}{2}} \left(\sqrt{\frac{x}{\zeta}} \right) I_1 \left(\sqrt{\zeta x} \right), \quad x > 0, \tag{2.16}$$

where $I_1(\cdot)$ is the first order modified Bessel function of the first kind, see, e.g., Revuz & Yor (1999).

It is worth mentioning that the normalized GP is dimensionless, and serves as a nontrivial state variable in that the transition density over the period (t, u) depends nonlinearly on the current state $Y(t)$. This is in marked contrast to the classical Black-Scholes model (BSM), where the current price serves as a scaling factor of a time-homogeneous geometric Brownian motion. This dependence implies that unlike the BSM, (2.12) is neither scale nor time-invariant. Note, however, the GP, as a time transformed squared Bessel process, has some self-similarity property.

Given the transition law (2.15), it is straightforward to exactly simulate the normalized GP under the MMM. In particular, an NCX^2 random variable can be generated by inverse transformation as

$$\chi_4^2(\zeta) = F^{-1}(\zeta, U), \tag{2.17}$$

where U is a standard uniform random variable, and $F^{-1}(\zeta, \cdot)$ is the inverse of the cumulative distribution function corresponding to the probability density (2.16).

The inverse transformation technique also allows for easy historical simulations of the normalized GP. In particular, given a realised normalized GP, $Y(t)$, sampled over a discrete set of time points, the corresponding NCX^2 random sample $\chi^2(\zeta)$, corresponding to a transition of $Y(t)$, can be extracted from (2.15). Applying the probability transformation to $\chi^2(\zeta)$, we obtain $U = F(\zeta, \chi^2(\zeta))$ as a uniformly distributed random sample. The collection of uniformly distributed samples thus obtained can be resampled to generate historical simulations of the realised GP, through the inverse transformation technique described above.

2.2 Estimation of the GP Dynamics Model

To implement the TDF strategy as outline in Section 3.1, in particular, to evaluate the optimal portfolio value (3.14), we need to specify the transition density for the GP. This can be obtained by specifying the dynamics of the GP, by, e.g., the minimal market model (MMM) introduced in 2.1, where the transition density of the GP is given in closed-form. In this section, we take the MMM as the underlying GP dynamics and consider the estimation of the MMM parameters from the historical data of the GP.

To do this we first need to identify the historical GP from financial market data. While the GP strategy under a diffusive market model is given explicitly by (2.4), to implement such a strategy even in retrospect requires knowledge of historical instantaneous drifts, volatilities, and correlations of all the primary securities traded in the market. It is well documented by the literature of sample-based mean-variance portfolio optimization that estimating these quantities with reasonable accuracy is extremely hard, and strategies based on these estimations often perform poorly, compared to simple strategies such as the $1/N$ strategy.

On the other hand, the GP strategy given by (2.4) is highly diversified and highly efficient, both properties resembling those of the market portfolio, see also, Platen & Rendek (2012). Consequently, as a first approximation, we propose to take the market portfolio as a proxy to the GP. In particular, we take the S&P500 total return index as a proxy to the GP. We take 80 years of historical monthly prices of the S&P500 index from 1938 to 2018, with all dividends reinvested, and discount by the (locally) risk-free asset. The S&P500 data, as well as the risk-free asset price data after 1963, are obtained from Datastream, and the earlier parts are reconstructed in Shiller (2015). The log-prices of the (discounted) S&P500 index are shown in Figure 2.1.

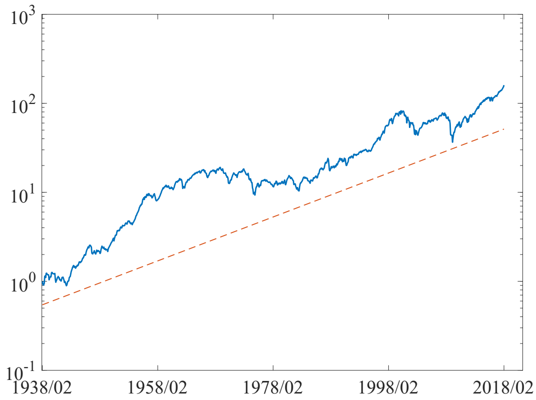


Figure 2.1 – Discounted S&P500 log-prices with a clear linear trend.

To estimate the MMM parameters, namely the initial scale parameter α_0 and long-term growth rate η, we consider several estimators: the maximum likelihood estimator (MLE), the Bayesian Markov chain Monte-Carlo estimator (BME) and the quadratic variation estimator (QVE). The MLE is based on maximizing the log-likelihood of the observed GP data under the MMM, which, thanks to its explicit transition law (2.15), allows the log-likelihood to be easily computed and optimized using standard optimization procedures. The BME is the weighted average of the prior beliefs on the parameters and the information contained by the likelihood (Koop, 2003).

We consider a realised GP time series $S^*_n := S^*_{t_n}$, where $\{t_n : n = 0, 1, \cdots, N\}$ are evenly distributed over $[0, T]$ inclusive. For the MLE, the log-likelihood function of these samples is given by

$$l(\eta, \alpha_0) = \sum_{n=1}^{N} \log\left(\frac{4}{1-e^{-\eta\Delta t}} f\left(\frac{4e^{-\eta\Delta t}}{1-e^{-\eta\Delta t}} Y_{n-1}, \frac{4}{1-e^{-\eta\Delta t}} Y_n\right)\right). \tag{2.18}$$

Here $\Delta t = \frac{T}{N}$ and $Y_n = Y_{t_n}$ is the normalized GP sample under (η, α_0), where the dependence on the parameters were suppressed. A simple optimisation procedure is used in estimating the MLE estimator of the model parameters.

For the QVE approach, we note that the quadratic variation of the square root of the GP is given by,

$$[\sqrt{S^*}]_t = \frac{1}{4}\int_0^t \alpha_0 e^{\eta s} ds = \frac{\alpha_0}{4\eta}(e^{\eta t} - 1), \quad t \in [0, T]. \tag{2.19}$$

Thus for $\eta\Delta t \ll 1$, we have

$$\frac{[\sqrt{S^*}]_{t+\Delta t} - [\sqrt{S^*}]_t}{\Delta t} \approx \frac{\alpha_0}{4} e^{\eta t}, \quad t \in [0, T], \tag{2.20}$$

and taking the logarithm,

$$\log\left(\frac{[\sqrt{S^*}]_{t+\Delta t} - [\sqrt{S^*}]_t}{\Delta t}\right) \approx \eta t + \log\left(\frac{\alpha_0}{4}\right), \quad t \in [0, T]. \tag{2.21}$$

Here the left-hand side may be estimated as the sum of squares of $\sqrt{S^*_t}$ variations over Δt containing multiple samples. This is followed by fitting a linear function of t through these estimates, where the slope corresponds to the η estimate, and the

Table 2.1 – Estimated MMM parameter values and standard deviations are obtained from different estimation schemes. The standard deviations are computed based on 1000 simulated GP paths

	η	α_0
MLE	0.0567 (0.0019)	0.0291 (0.0026)
QVE	0.0545 (0.0021)	0.0243 (0.0026)
BME	0.0564 (0.0020)	0.0292 (0.0028)

intercept can be used to estimate α_0.

For the Bayesian approach, the posterior distribution of the model parameters can be written as

$$f(\eta, \alpha_0 | S^*) \propto f_\eta(\eta) f_\alpha(\alpha_0) \exp\left(l(\eta, \alpha_0)\right)$$

where l is the log-likelihood function given by (2.18), f_η and f_α are the prior distributions of the parameters, η and α_0, respectively. In particular, we choose the following prior structure with

$$\eta \sim U(0, 1),$$

and

$$Y_0 = \frac{\eta}{\alpha_0} S_0^* \sim \Gamma(2, 2).$$

Here $\Gamma(2, 2)$ is the Gamma distribution with shape and scale parameters equal to 2. It is easily seen from (2.15) that $\Gamma(2, 2)$ is the long-term stationary distribution of the normalized GP.

The estimated parameter values based on the historical data are summarized in Table 2.1, where the quantities shown in brackets represent the corresponding standard deviations, computed based on estimated values on 1000 simulated GP paths under the MMM dynamics, with parameters given by the MLE.

These estimated values exhibit high similarities. In particular, the MLE and BME are very close. The QVE for η is still fairly close, and more difference is found in its α_0 estimate, which nevertheless still lies within about two standard deviations from the other two.

Chapter 3

Dynamic Asset Allocation for Target Date Funds under the Benchmark Approach

Investing for the long-term has been an important aspect of both individual investors seeking to save for retirement and institutions such as pension funds and insurance companies seeking to manage long-term liabilities. It is generally advised that an investor holds more stocks and other risky assets when the investment horizon is in the far future, and slowly switches to the (locally) risk-free asset, e.g., money market account, as the investment horizon approaches. Such a strategy carries low risks in the late stages close to the terminal payoff, while in the meantime provides high returns by actively investing during the early stages. In the pension and life insurance industry, such strategies are sometimes offered in the form of TDF's. According to Mitchell & Utkus (2012), a TDF aims to simplify investment decision-making for investors and delegate important portfolio choices to TDF investment managers. These simplifications are three-fold: First, contributions to the TDFs are directed to multi-asset class funds designed to be "all-in-one" portfolios. Second, portfolio allocations are based on the investor's expected retirement age. Third, risk exposures are reduced over time through age-based rebalancing of the investor's portfolio.

TDFs have become an important component of pension plans. In the United States, the Pension Protection Act of 2006 created the need for safe-harbor type Qualifying Default Investment Alternatives including target-date funds for 401K savings plans, see Elton et al. (2015). From 2008 to 2012, the funds grew from 160 to 481 billion in total assets under management, with 91% of these assets in retirement plans, see VanDerhei et al. (2012, 2013); Mitchell & Utkus (2012). The trend has since continued. According to Morningstar (2018), the total assets under management in TDFs in the United States surpassed one trillion in 2017. TDFs play an increasingly important role in retirement funding for more and more investors.

One of the widely observed characteristics of TDFs is the gradual reduction over time of wealth invested in risky assets. However, according to Elton et al. (2015), such asset allocation patterns have limited and controversial theoretical and empirical support. Based on different assumptions, different authors have come up with rather different views. Due to this ambiguity, typical implementations of TDF strategies are often based on heuristics that are lacking consistency and optimality under a well-defined quantitative model. These often lead to inappropriate exposure to market risks for TDFs of different target dates and mixed investment outcomes, see Yoon (2010); Johnson & Yi (2017). To properly design the so-called "glide path" for the TDF, proper characterization of market risks and returns at different time horizons is essential, seeYoon (2010).

There is a limited amount of work on the quantitative design of TDF glide paths, partly due to limited and controversial theoretical and empirical support, as mentioned earlier. To provide a rigorous and consistent approach to the design of "optimal" TDF strategies that quantify the asset allocation pattern of generally decreasing stock investments, we propose to work under the framework of the benchmark approach (BA) proposed by Platen & Heath (2006). The BA offers investment strategies at minimum costs, given any investment targets in the form of future contingent claims. In particular, long-dated TDFs optimally allocate under the BA between risky and risk-free investment opportunities, leading to the least expensive benefits commensurate to the risk preferences and target dates of the investors.

In this chapter, we take the S&P500 total return index, a well-diversified stock index of the US domestic equity market, as a proxy for the GP of this market. We construct under the BA TDF products based on the real-world pricing formula, see (2.10) in Chapter 2, as well as dynamic asset allocation strategies associated with these products. Specifically, given a target date and the risk preference, we construct the dynamic asset allocation strategy that delivers optimally at the target date the terminal payoff corresponding to the investor's risk preference. A dynamic TDF strategy generates an optimal glide path of holdings between the GP and the (locally) risk-free asset over the investment period. We backtest our TDF strategies for a number of long-term target maturities over historical as well as simulated market scenarios to assess the performance of the TDF strategies over these scenarios and across different investment horizons and risk preferences. Chapter 3 is based on Sun et al. (2020).

3.1 Optimal Dynamic Asset Allocation under the Benchmark Approach

We consider a TDF investor with a utility function $u(\cdot)$ for final wealth $V(T)$ and an initial wealth $V(0) > 0$. This investor wants to manage a portfolio of the m risky securities $\boldsymbol{S}(t) = \{S^1(t), ..., S^m(t)\}$ and the locally risk-free asset to maximize his or her expected utility of the terminal wealth of his or her investment portfolio, at a given future date T, with an initial portfolio value $V(0) > 0$. The investor invests in a strategy

$$\boldsymbol{\Pi}(t) = \{\pi^0(t), \boldsymbol{\pi}(t)\} = \{\pi^0(t), \pi^1(t), ..., \pi^m(t)\}$$

where π_t^0 and π_t^i denote the number of units of S^0 and the ith risky asset held. Following Chapter 2, the (self-financing) trading strategy $\boldsymbol{\Pi}$ is admissible if the corresponding wealth process $V^{\boldsymbol{\Pi}}$ satisfies $V^{\boldsymbol{\Pi}}(t) \geq 0$ for all t. We further require that the terminal wealth

$$V^{\boldsymbol{\Pi}}(T) \in L_+^p(\Omega, \mathcal{F}_T, \mathbb{P})$$

for some fixed $p > 1$.

To make the optimal asset allocation problem well defined, we first make the following assumption about the utility function of the terminal wealth:

Assumption 1. *We assume that*

- $u(\cdot) : \mathbb{R}_+ \cup \infty \to \mathbb{R}_+$ *is twice differentiable with* $u'(x) > 0, u''(x) < 0$ *for all* $x \in \mathbb{R}_+$, *and* $u'(0) = \infty, u'(\infty) = 0$;

- *it is bounded from above*

$$u(x) \leq \beta_1 + \beta_2 x^{1-b} \text{ for all } x \in \mathbb{R}_+$$

for some constants $\beta_1 \geq 0$, $\beta_2 > 0$ *and* $b \in (0, 1)$.

Assumption 1 on the utility function is typically satisfied, e.g. for constant-relative-risk-aversion (CRRA) utility functions with a positive risk aversion γ,

$$\begin{cases} u(v) = \frac{v^{1-\gamma}}{1-\gamma}, & \gamma \neq 1, \\ u(v) = \log(v), & \gamma = 1. \end{cases} \tag{3.1}$$

The optimal portfolio selection question can be formulated as

$$\sup_{\mathbf{\Pi}} \mathbb{E}_0 \left(u(V^{\mathbf{\Pi}}(T)) \right)$$

with admissible $\mathbf{\Pi}$ of some given initial budget $V^{\mathbf{\Pi}}(0)$.

Here we shall follow the BA framework described in Chapter 2 that this optimal control problem can be formulated into a static variational problem

$$\sup_{V(T) \in L^p_+(\Omega, \mathcal{F}_T, \mathbb{P})} \mathbb{E}_0 \big(u(V(T)) \big) \tag{3.2}$$

subject to the budget constraint

$$V(0) = \mathbb{E}_0 \left(\frac{S^*(0)}{S^*(T)} V(T) \right), \tag{3.3}$$

under the real-word pricing measure where $\{S^*(t), t \in [0, T]\}$ is the GP, namely, the corresponding numeraire portfolio. Here we suppress the superscript $\mathbf{\Pi}$ in the wealth process V to simplify the notation. We next make the following assumption about the GP:

Assumption 2. *The GP,* $S^*(t), t \in [0, T]$, *is a strictly positive scalar Markov process such that,* $(S^*(T))^{-1} \in L^q_+(\Omega, \mathcal{F}_T, \mathbb{P})$, *where* $q = \frac{p}{p-1}$, *and* $\mathbb{E}_0((S^*(T))^k) < \infty$ *for all* $k > p$.

Assumption 2 is satisfied under MMM for all $p > 2$. Under Assimptions 1 and 2, the optimal terminal wealth $V^*(T)$ is well defined by (3.2) and (3.3) as a result of Cox and Huang (1991), Theorems 4.1 and 4.2.

Proposition 1. *Under Assumption 1, the optimal wealth process can be written as*

$$V^*(t) = \mathbb{E}_t \left(\frac{S^*(t)}{S^*(T)} u'^{-1} \left(\lambda^* \frac{S^*(0)}{S^*(T)} \right) \right), \quad t \in [0, T] \tag{3.4}$$

with λ^* *solved by setting* $V^*(0) = V(0)$.

Proof. For any $\xi \in L^p_+(\Omega, \mathcal{F}_T, \mathbb{P})$, and a nonnegative self-financing portfolio process $\{V(t) : t \in [0, T]\}$, such that $V(T) = \xi$ $\mathbb{P}$-a.s., and that the budget constraint (3.3) is satisfied, the benchmarked portfolio process $\tilde{V}(t) := \frac{V(t)}{S^*(t)}$, $t \in [0, T]$ is a supermartingale by the numeraire property of the GP (Platen & Heath, 2006). Thus for $t \in [0, T]$,

$$\tilde{V}(t) \geq \mathbb{E}_t(\tilde{V}(T)). \tag{3.5}$$

On the other hand, by the super-martingale property and (3.3),

$$\mathbb{E}_0(\tilde{V}(t)) \leq \tilde{V}(0) = \mathbb{E}_0(\tilde{V}(T)) = \mathbb{E}_0(\mathbb{E}_t(\tilde{V}(T))). \tag{3.6}$$

Combining (3.5) and (3.6), we conclude that $\tilde{V}(t) = \mathbb{E}_t(\tilde{V}(T))$ $\mathbb{P}$-a.s.. That is,

$$V(t) = \mathbb{E}_t\left(\frac{S^*(t)}{S^*(T)}\xi\right), \tag{3.7}$$

i.e., the budget constraint is satisfied for all $t \in [0, T]$. Note that the budget constraint (3.7) is the same as the real-world pricing formula (2.10). The constrained optimization problem defined by (3.2), (3.3) may thus be solved by following the standard Lagrangian technique. The Lagrangian $L_t(\xi, \lambda_t)$ for $t \in [0, T]$ is given by

$$\begin{aligned} L_t(\xi, \lambda_t) = &\quad \mathbb{E}_t\left(u(\xi)\right) + \lambda_t\left(V(t) - \mathbb{E}_t\left(\frac{S^*(t)}{S^*(T)}\xi\right)\right) \\ = &\quad \mathbb{E}_t\left(u(\xi) + \lambda_t\left(V(t) - \frac{S^*(0)}{S^*(t)}\xi\right)\right), \end{aligned} \tag{3.8}$$

where λ_t is the Lagrange multiplier corresponding to the budget constraint (3.7). The corresponding first order condition at the expiry is given by

$$u'(\xi^*) - \lambda_T^* \frac{S^*(0)}{S^*(T)} = 0, \tag{3.9}$$

where ξ^* is the potential optimal terminal wealth under the optimal wealth process V^*, and λ^* the corresponding Lagrange multiplier. By assuming invertibility of $u'(\cdot)$, the potential optimal terminal wealth $V^*(T)$ is given by

$$V^*(T) = \xi^* = u'^{-1}\left(\lambda_T^* \frac{S^*(0)}{S^*(T)}\right), \tag{3.10}$$

where $\lambda_T^* > 0$ is determined by substituting (3.10) into (3.3). The portfolio value process that delivers the optimal terminal wealth (3.10) is given by (3.7), or equivalently by (2.10), which gives the result. □

Under Assumption 2, the optimal portfolio value can be written as a function of the GP. If this function is suitably smooth, the number of GP units to be held by the investor's portfolio is then given by

$$\delta^*(t) = \frac{\partial V^*}{\partial S^*}(t). \tag{3.11}$$

That is, at $t \in [0, T]$, the portfolio strategy invests the fraction

$$\pi^*(t) = \delta^*(t)\frac{S^*(t)}{V^*(t)} \tag{3.12}$$

of the total portfolio value in the GP, and the rest in the (locally) risk-free asset.

The optimal dynamic allocation for the TDF investor is here seen to consist of two funds, the risk-free asset, and the GP. The GP evidently incorporates all investment opportunities in the growth-optimal sense, see (2.4), thus no further optimal allocations among the investment opportunities are necessary. This structure of two-fund separation largely simplifies the design of TDFs for different target dates and risk preferences, in that the only truly challenging part is to construct a good proxy for the GP of the investment universe and to model its dynamics. Once this is done, the glide paths for different TDF target dates and risk preferences are simply given by the "delta" with respect to the GP, of the price process of the corresponding optimal terminal portfolio value, as given by (3.11). By following this method, we overcome the current weaknesses of previously proposed TDF strategies, which left open whether the target payoff is produced in the least expensive way. In this sense, we make the optimal design of TDFs rigorous.

As a concrete example, we consider a TDF investor with a target date T and a CRRA utility function of the terminal wealth, with a risk-aversion coeficient $\gamma > 0$. By substituting (3.1) into (3.10), (3.3), (3.4) and (3.11), we readily obtain

$$V^*(T) = \bar{V}(0)\left(\frac{S^*(T)}{S^*(0)}\right)^{\frac{1}{\gamma}} \tag{3.13}$$

with the constant $\bar{V}(0) = \frac{V(0)}{E_0\left(\left(\frac{S^*(T)}{S^*(0)}\right)^{\frac{1}{\gamma}-1}\right)}$, and

$$V^*(t) = \bar{V}(0)E_t\left(\left(\frac{S^*(T)}{S^*(t)}\right)^{\frac{1}{\gamma}-1}\right)\left(\frac{S^*(t)}{S^*(0)}\right)^{\frac{1}{\gamma}}. \tag{3.14}$$

Once the optimal portfolio value is computed, the corresponding optimal GP allocation (3.11) is given by differentiating (3.14) with respect to $S^*(t)$. This can be done by specifying the real-world dynamics of $S^*(T)$. In particular, under the minimal market model to be introduced in Section 2.1, $S^*(T)$ conditional on the current GP value $S^*(t)$ assumes a closed-form transition density, with an explicit dependence on $S^*(t)$. This allows a semi-analytical evaluation of this differentiation, thus an efficient computation of the number of GP units to be held in the optimal portfolio.

3.2 Simulation Studies of the TDF Strategies

As a verification and validation procedure, we first conduct some simulation studies and gain some insight into the TDF strategies developed so far. In particular, we assume that the GP dynamic follows the MMM, with parameters estimated from historical data in Section 2.2. We take the MLE parameter estimates and simulate a number 1000 of sample paths of the GP at a monthly frequency. We show in Figure 3.1 a number of 20 simulated GP sample paths, out of 1000 paths simulated over a period of $T = 40$ years. We consider TDF strategies with maturities of 10, 20, and 40 years, which are typical life spans of a retirement investment portfolio, and the risk aversion coefficients $\gamma = \infty$, 5, and 0.8, representing different risk appetites from extremely risk-averse to less risk-averse. We implement these strategies over the simulated sample paths and assess the investment outcomes. To test the robustness of the strategies with transaction costs, we apply a charge of 0.5% to the total trading volume of the monthly rebalancing and deduct this charge from the portfolio value.

For each simulated sample-path, we apply the TDF strategies corresponding to the different risk aversions, with and without the transaction costs. The TDF strategies are based on the underlying true MMM model parameters, thus corresponding to an "in-sample" situation. To simulate an "out-of-sample" situation, for each sample-path where we apply the TDF strategies, we randomly select a different

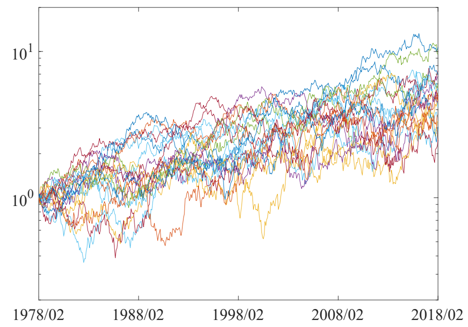


Figure 3.1 – Simulated sample paths of the GP prices at a monthly rate from the estimated MMM, over a period of 40 years (480 months).

sample-path and use it to estimate the model parameters used by the TDF strategies. We then apply the TDF strategies based on this out-of-sample-estimated model. The investment outcomes, e.g., annualized excess log-returns, for both in-sample and out-of-sample scenarios are summarized in Table 3.1, where the projected returns are based on portfolio values given by (3.14), and the realised returns without and with transaction costs (TC) are based on the corresponding monthly-rebalanced portfolio processes following the TDF strategies. The excess log-return distributions are shown in Figure 3.2.

The results show that the mean excess log-return increases with decreasing risk aversion, for obvious reasons. The standard deviation of the returns decreases with increasing time horizon, which reflects the effective reduction of risk over the long-term due to the mean reversion dynamics of the GP. For $\gamma = \infty$, the situation is a little different: the return standard deviation increases a little with the horizon. This is because at $\gamma = \infty$, the return standard deviation should be zero in theory, and the realised standard deviation is mainly due to accumulated model and hedging errors, which increase with the time horizon. The mean excess return, however, also increases significantly with the time horizon.

On the other hand, the negligible statistical difference between the projected and realised returns implies that the replication errors without or with transaction costs are much smaller than the realised excess returns in terms of mean and standard deviations, across all time horizons and risk preferences, for both in-sample and out-of-sample situations. This reflects the robustness of the strategies against discrete, monthly rebalancing and market frictions, as well as model estimation errors.

3.3 Backtesting with Historical Data

To assess the TDF strategy developed in the previous sections under the real-world scenarios, we conduct backtests of the strategy over historically realised data, with time to maturities $T = 10$, 20 and 40 years, and risk aversion coefficients $\gamma = \infty$, 5, and 0.8. We first consider in-sample and out-of-sample backtests of the TDF strategy on the historical S&P500 index prices starting from 02/1978 up to 02/2018. For in-sample backtests, we use the estimated MMM parameters by MLE as presented in Table 2.1. For out-of-sample backtests, we estimate the parameters based on 40-year historical S&P500 index prices from 02/1938 to 02/1978. The investment outcomes are summarised in Table 3.2.

These results review similar patterns as seen in the simulation studies in Section 3.2, e.g., increased annualized log-return and decreased hedging error, i.e., the difference between projected and realised log-returns, with increasing time horizon

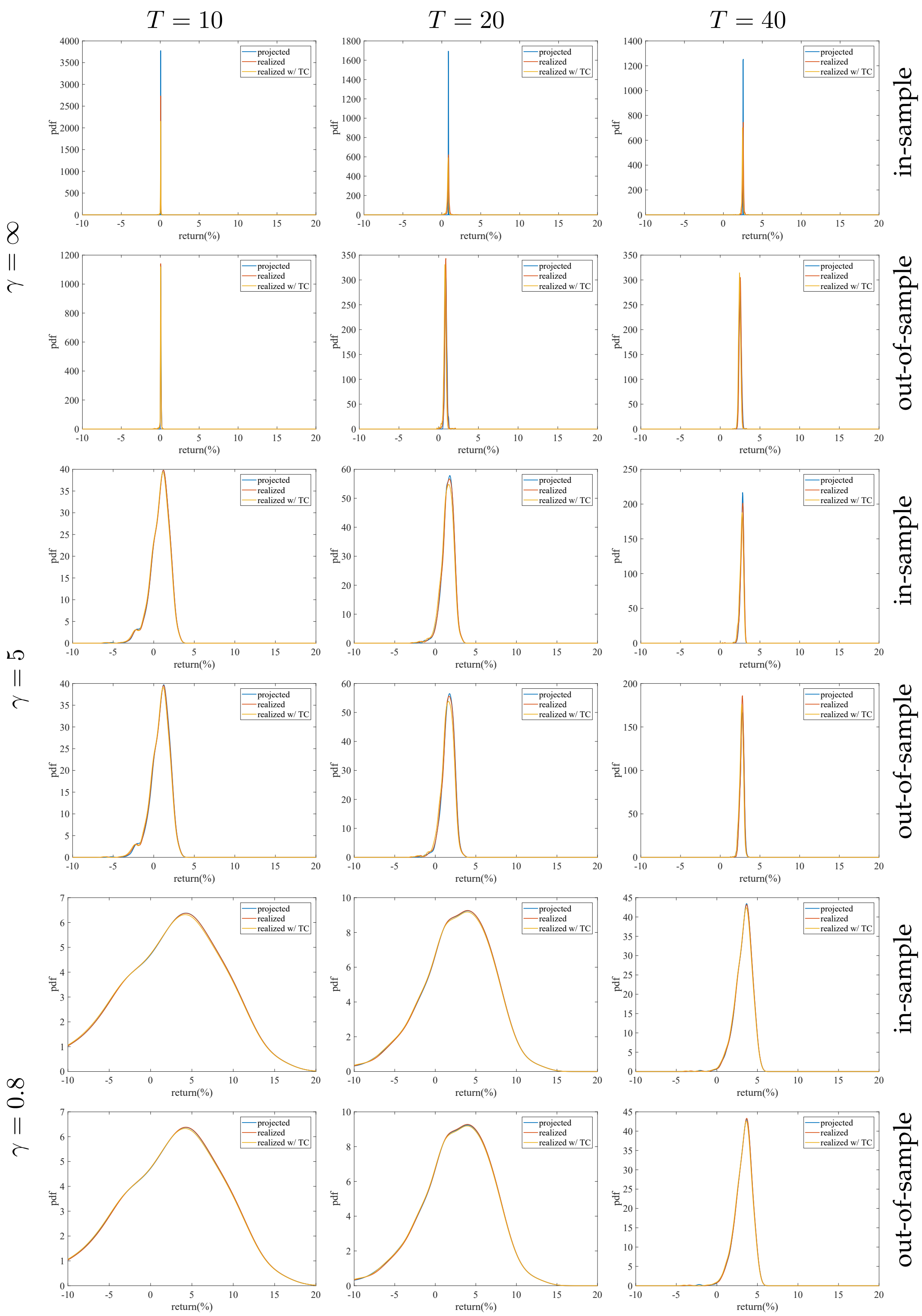


Figure 3.2 – The in-sample and out-of-sample projected and realised return distributions of the TDF strategies under simulated paths over 40 years.

Table 3.1 – Sample means and standard deviations of the pathwise annualized excess log-returns of the TDF strategies under the simulated scenarios. (All values are given in %.)

			$\gamma = \infty$		$\gamma = 5$		$\gamma = 0.8$	
$T = 10$	in-sample	projected	0.060	(0.000)	0.818	(1.128)	1.952	(7.047)
		realised	0.059	(0.041)	0.816	(1.138)	1.956	(7.031)
		realised (TC)	0.043	(0.045)	0.747	(1.156)	1.804	(7.163)
	out-of-sample	projected	0.083	(0.036)	0.855	(1.129)	1.922	(7.048)
		realised	0.069	(0.066)	0.826	(1.153)	1.959	(7.005)
		realised (TC)	0.050	(0.074)	0.755	(1.172)	1.807	(7.136)
$T = 20$	in-sample	projected	0.855	(0.000)	1.567	(0.716)	2.595	(4.474)
		realised	0.856	(0.129)	1.564	(0.741)	2.612	(4.444)
		realised (TC)	0.762	(0.137)	1.462	(0.773)	2.499	(4.585)
	out-of-sample	projected	0.908	(0.128)	1.614	(0.724)	2.576	(4.473)
		realised	0.878	(0.154)	1.581	(0.756)	2.609	(4.436)
		realised (TC)	0.783	(0.163)	1.479	(0.788)	2.497	(4.576)
$T = 40$	in-sample	projected	2.537	(0.000)	2.754	(0.205)	3.304	(1.026)
		realised	2.539	(0.087)	2.756	(0.225)	3.305	(1.031)
		realised (TC)	2.452	(0.093)	2.673	(0.242)	3.246	(1.058)
	out-of-sample	projected	2.518	(0.143)	2.742	(0.248)	3.291	(1.029)
		realised	2.522	(0.131)	2.747	(0.230)	3.301	(1.021)
		realised (TC)	2.434	(0.133)	2.665	(0.245)	3.242	(1.048)

and decreasing risk-aversion. These patterns again reflect the decreased market risk with a long time horizon as a result of the mean-reversion of the market index. The decreased hedging error with decreasing risk-aversion, which is also apparent in Figure 3.3 below, reflects the fact that hedging with a risk-aversion coefficient closer to 1 is generally easier, requiring less accurate estimates of the index dynamics. In particular, for $\gamma = 1$, the corresponding TDF strategy will be to simply follow the market index without active trading. Moreover, the small difference between in-sample and out-of-sample returns implies good robustness against estimation errors in the out-of-sample model estimation.

To illustrate these TDF strategies, we show in Figure 3.3 for each maturity and risk preference the out-of-sample projected and realised portfolios along with the GP over the same period. We also show in every second row in Figure 3.3 the corresponding risky positions of these strategies, that is, the fraction π^* of total portfolio value invested in the GP as given by (3.12), overlain by the normalized GP plotted against the right Y-axis.

A first glance at the realised portfolios of the TDF strategy confirms the common wisdom of risk-return compromise, where more risk-averse investors gain fewer positive returns over a bull market condition, and in the meantime suffer from fewer losses given a bearish market condition. We see more interesting patterns, however, from the risky positions of these strategies. In particular, for more risk-averse investors, i.e., investors with $\gamma > 1$, the risky position steadily reduces to the optimal short-term position corresponding to the investor's risk preference, from a more aggressive initial risky asset allocation. The longer the maturity, due to the mean reversion of the returns, the more aggressive the initial allocation: Since a long time horizon allows the market more time to recover from a possible drawdown, it pays to invest more aggressively to take advantage of the potential market excess returns

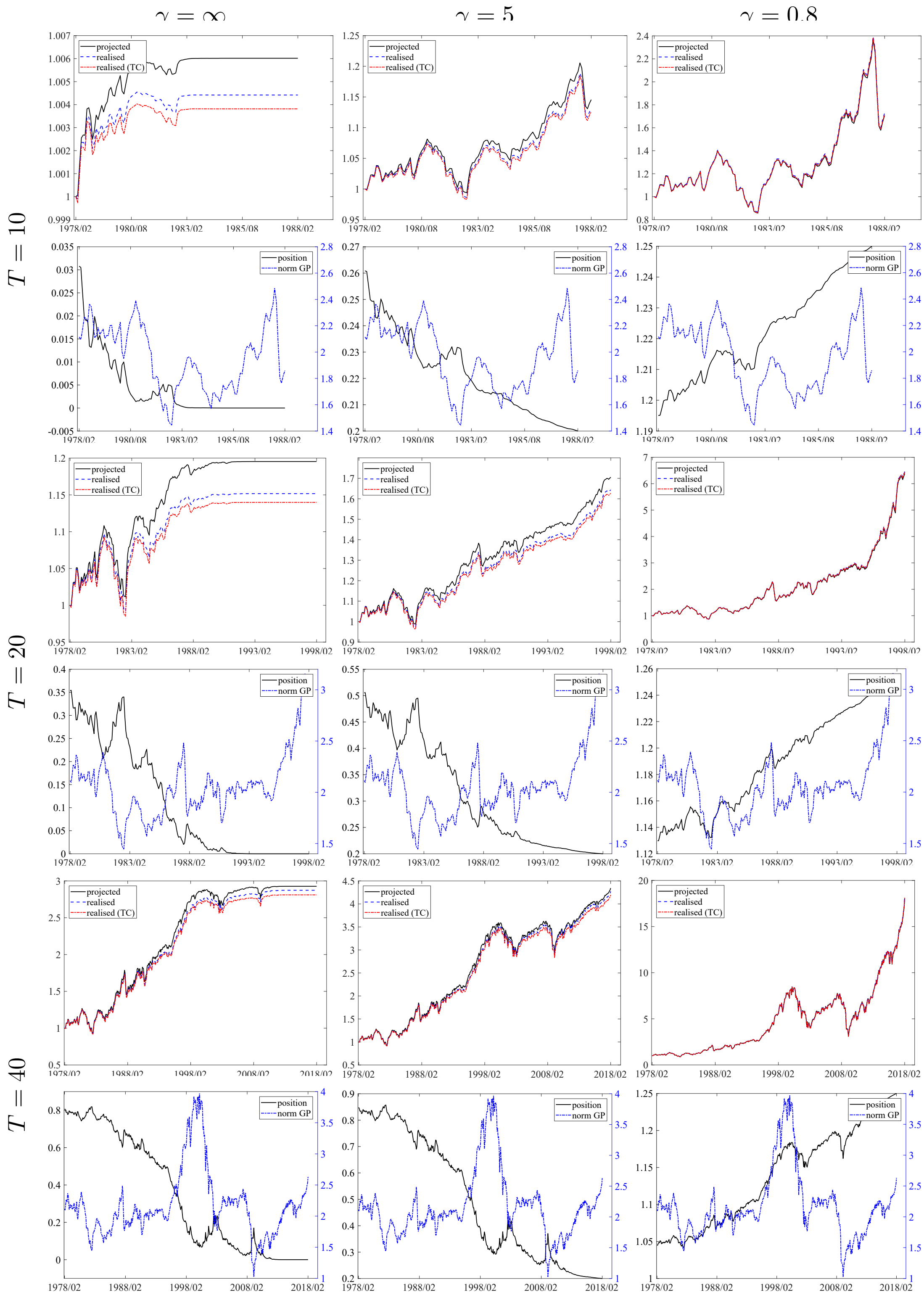


Figure 3.3 – The projected and realised portfolio processes of the TDF strategies, along with the corresponding risky-asset positions, over historically realised index prices, based on out-of-sample model estimations.

Table 3.2 – Realised TDF log-returns on historical S&P500 data starting from 02/1978. (All values are given in %.)

			$\gamma = \infty$	$\gamma = 5$	$\gamma = 0.8$
$T = 10$	in-sample	projected	0.060	1.360	5.340
		realised	0.044	1.232	5.472
		realised (TC)	0.038	1.186	5.397
	out-of-sample	projected	0.060	1.359	5.341
		realised	0.044	1.232	5.472
		realised (TC)	0.038	1.187	5.398
$T = 20$	in-sample	projected	0.855	2.637	9.280
		realised	0.672	2.457	9.342
		realised (TC)	0.622	2.405	9.323
	out-of-sample	projected	0.892	2.669	9.268
		realised	0.706	2.484	9.333
		realised (TC)	0.655	2.433	9.313
$T = 40$	in-sample	projected	2.537	3.553	7.269
		realised	2.490	3.523	7.287
		realised (TC)	2.437	3.478	7.273
	out-of-sample	projected	2.686	3.673	7.231
		realised	2.638	3.631	7.256
		realised (TC)	2.584	3.585	7.243

when the maturity is still in the far future. This is particularly obvious for the $\gamma = \infty$ investor, where risky positions for $T = 10$ are nearly zero throughout the investment period, while risky positions for $T = 40$ can be significant during the earlier stage.

Other than adjusting the risk exposure according to the remaining time horizon, the TDF strategies also adjust the risk exposure according to the normalized GP. It is clear from Figure 3.3 that for glide paths of more risk-averse investors, the risky positions are negatively correlated to the normalized GP. The normalized GP under the MMM represents the "over-valuation" of the market index. If the normalized GP decreases, the market index becomes under-valued relative to its previous value. Therefore, the strategies take more positions to make use of the under-valued GP at its current price level, and vice versa. However, the average mean-reversion time for the normalized GP is in the order of decades, this "buy low and sell high" behavior is, in general, not too aggressive for the maturities considered, and becomes progressively less noticeable as the maturity closes up.

On the other hand, for less risk-averse investors ($\gamma < 1$), the glide paths are opposite to those of the more risk-averse ones: First, the risk exposure steadily *increases* towards the target short-term position corresponding to the risk preference, from a less aggressive initial position. The entire glide path takes on leverage, that is, the investor borrows money from the risk-free asset and invests in the GP. However, in the early stages, less leverage is taken to avoid premature bankruptcy, which is the main threat to more aggressive strategies. As maturity approaches, there is less chance of premature bankruptcy, and the risk exposure is adjusted to the target short-term level. Second, the glide path is now positively correlated to the normalized GP, corresponding in some sense to a momentum type strategy. This is sensible since the investor, being less risk-averse than the log-utility investor with $\gamma = 1$, would want to take on leverage to achieve the preferred risk-return balance. To avoid premature bankruptcy, it is important to reduce the leverage when the GP goes down and increases leverage only when the GP goes up. It should be noted that taking on leverage is highly risky in practice, in that the realised portfolio may hit bankruptcy, even though theoretically, the projected portfolio value is always positive.

It is interesting to see how the optimal strategies of more risk-averse investors differ from those of the less risk-averse ones: While the more risk-averse investors take a reversal strategy, the less risk-averse investors take a momentum strategy. This is in contrast to the common wisdom for short-term investments where reversal strategies are characterized by high risks and high returns, and most risk-averse investors follow the momentum type strategy. The key difference lies in the long-time horizons considered here, where the mean reversion of the normalized GP creates a decreasing term structure of risks with increasing time horizons. Moreover, it may be conjectured from the glide paths shown in Figure 3.3 that, as the maturity becomes very large, all strategies tend to invest fully in the GP without leverage. Since in theory, the GP almost surely outperforms any essentially different portfolios in the long run, it is the obvious optimal choice when the maturity is still in the far future, regardless of the investor's risk preference. Moreover, it is worth mentioning that the Sharpe ratio, when measured instantaneously, remains the same as that of the GP, regardless of the TDF strategies employed. In other words, all TDF strategies share essentially the same "optimal" Sharpe ratio as the GP.

To consider more historically realised scenarios, with limited historical data, we conduct historical simulations of the GP, over 40 years, and apply the TDF strategy to these simulated GP paths. Our approach to historical simulations is described in Section 2.1. We simulate 1000 GP paths, and 20 simulated GP paths are shown in Figure 3.4. Note that the historically simulated scenarios differ statistically from the simulated scenarios as shown in Figure 3.1, in that only historical realizations of the risk factor are sampled.

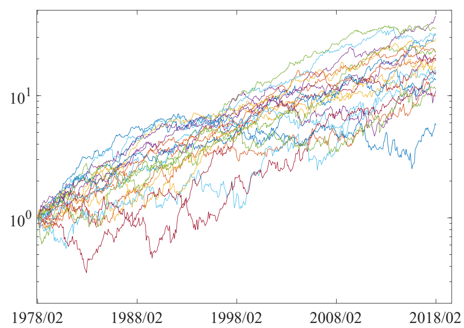


Figure 3.4 – Historically simulated sample paths of the GP prices at a monthly rate from the estimated MMM, over a period of 40 years (480 months).

We backtest the TDF strategies on the historical simulations in an out-of-sample way, under the same setting as in Section 3.2. In particular, for each simulated path, we randomly select another simulated path from which we estimate the model parameters using MLE, and apply the corresponding TDF strategy to the former path. As in Section 3.2, we consider all 1000 paths under all maturities and risk-aversion coefficients. The summary statistics of these backtest investment outcomes are given in Table 3.3.

From these statistics, the TDF strategies on average realised excess returns for all maturities, even for the most risk-averse investors. At a rather strong risk aversion of $\gamma = 2$, the TDF strategies realised on average excess returns of at least around 2% per annum. The excess return increases with maturity, where more risk-averse

Table 3.3 – Sample means and standard deviations of the pathwise annualized excess log-returns from out-of-sample TDF strategies, under the historically simulated scenarios. (All values are given in %.)

		$\gamma = \infty$		$\gamma = 5$		$\gamma = 0.8$	
$T = 10$	projected	0.113	(0.069)	2.050	(0.838)	9.141	(5.198)
	realised	0.107	(0.063)	2.037	(0.830)	9.148	(5.212)
	realised (TC)	0.093	(0.060)	1.982	(0.840)	9.075	(5.247)
$T = 20$	projected	1.007	(0.193)	2.684	(0.489)	8.702	(2.884)
	realised	0.992	(0.165)	2.670	(0.473)	8.704	(2.895)
	realised (TC)	0.936	(0.160)	2.608	(0.485)	8.667	(2.912)
$T = 40$	projected	2.621	(0.212)	3.724	(0.295)	7.909	(1.501)
	realised	2.613	(0.199)	3.721	(0.283)	7.907	(1.504)
	realised (TC)	2.561	(0.199)	3.677	(0.290)	7.889	(1.513)

investors see more significant improvements when maturity increases. More aggressive investors with $\gamma < 1$ realise significant returns, with the added risk of total loss (bankruptcy) due to errors in replicating the projected portfolio processes.

It is interesting to consider the case of risk aversion coefficient $\gamma = \infty$. In this case, the target terminal wealth (3.13) becomes a constant, i.e., proportional to the risk-free asset price at maturity. The fact that such a strategy realises excess returns without any risk seems to contradict the common belief in the risk-return compromise. In fact, under the classical theory of no-arbitrage, it is impossible to realise excess returns without taking risks. Under the MMM, this type of (theoretically) risk-free excess returns is allowed, which falls under the BA framework where the exclusion of classical arbitrage is not a necessary condition of market viability. However, this risk-free excess return can only be realised at longer time horizons, as shown in the left column in Figure 3.3, where the “risk-free" strategy with a 10-year maturity almost does not take any risk at all, for longer maturities, however, this strategy invests a fair amount of the total wealth in the GP in the earlier stage, and decreases the risk exposure over time rapidly as the maturity approaches. Hence for short maturities, classical wisdom still applies.

There are still risks associated with this seemingly “risk-free" strategy, including at least model errors and hedging errors. The standard deviations of the investment outcomes, as shown in Table 3.3 under the column $\gamma = \infty$, reflect these risks, where it is seen that although the standard deviation in log-return increases with maturity, relative to the mean log-return, it decreases. This relative decreasing of these “basis" risks at longer maturity again reflects the decreasing market risks associated with longer horizons.

Finally, we briefly extend our discussion to the case where regular contributions, instead of a lump-sum were made to an individual investor’s retirement account. In this case, the “fair value", namely, the real-world price of the sequence of contributions can be calculated under the BA, and the TDF strategy can be carried out with this fair value as its initial budget constraint. In the case of pre-defined contributions, the initial budget corresponds to the sum of zero-coupon bonds with maturities on the contribution dates. This approach requires shorting the zero-coupon bonds at the inception and maintaining the short positions throughout to the maturity. For a pension fund manager with a large portfolio of individual retirement accounts, individual TDFs may be aggregated to a large overall portfolio, again consists of the GP and the locally risk-free security, to achieve further diversification effects across different maturities, contribution patterns and risk preferences. Alternatively, the fund manager may also estimate "average" risk preferences and contribution patterns of individual retirees, and invest optimally with respect to these estimates. Hopefully the simplified strategy is close enough to the more individualized approach through mass diversification effects.

3.4 Summary

We have developed under the benchmark approach (BA) a systematic set of procedures for designing optimal glide paths for TDF investors with different investment horizons and risk preferences. By modelling the GP under the MMM with mean reversion and leverage effect, we construct TDF strategies that automatically adjust risk exposures according to both the current time to maturity and the current market condition. The strategies are optimal under the MMM by maximizing expected utilities of terminal wealth. No ad-hoc, subjective control of the glide paths are necessary. We demonstrated these strategies by running backtests over long histories of the S&P500 index, which serves as the GP proxy, across a wide range of risk preferences and investment horizons.

The out-of-sample backtests based on the historical simulation of the S&P500 data demonstrate that these TDF strategies generate robust and efficient returns over a range of maturities and risk aversions. In particular, the TDF strategy corresponding to infinite risk aversion is essentially free of market risk and offers excess returns over longer maturities. The proposed TDF strategies make the common financial planning idea rigorous. For example, by the Law of One Price, classical no-arbitrage assumptions would have requested the "risk-free" strategy to invest in the risk-free asset all the time and this would have been the only portfolio that would have produced the targeted payout of units of the risk-free asset. The BA relaxes the classical assumptions by only requesting the GP to exist. This extra freedom allows having several self-financing portfolio processes that generate the same payoff. The real-world pricing formula identifies the portfolio process that generates the payoff least expensively. As we have seen, for long-term investments it is the highest possible long-term growth of the GP that leads to high fractions invested in the GP. Most important is that the respective strategies are similarly rigorous and feasible as the strategies that have guided successful financial option pricing in recent decades. In contrast, current practices from TDF managers rely mostly on ad-hoc methods which cannot, in general, achieve good alignments of the strategy with the maturity, risk preference and market condition.

Chapter 4

Long-term Optimal Investments with Subjective Preferences

4.1 Introduction

Investing for the long term has been an important aspect of both invidual investor seeking to save for retirement, and institutions such as pension funds and insurance companies seeking to manage long-term liabilities. One of the earliest works on optimal investment has been Markowitz's portfolio theory based on mean-variance analysis (Markowitz, 1952), which was often considered as the starting point of modern finance. Based on Markowitz's theory, Tobin (1958) came up with the mutual fund theorem that states the remarkable result that all investors should hold the same relative proportions of risky assets in their mean-variance efficient portfolios.

However, financial planners in practice have largely resisted and rejected the simple myopic investment strategies as outlined by the Markowitz's mean variance analysis, partly due to its short-term nature. Financial planners have generally advised that an investor should hold more stocks and other risky assets when the investment horizon is in the far future, and slowly switch to the risk-free assets as the investment horizon approaches. Moreover, they have argued that more conservative investors with long-term horizons should hold more bonds than more aggressive ones. To reconcile these inconsistencies, many authors have studied optimal portfolio choices by long-term investors, taking into consideration changing investment opportunities over time, labor income, risk of inflation, and mean-reversion of excess stock returns over the long term, etc.; see Campbell et al. (2002) for an overview of the existing literature on this subject.

One of the key concepts that links short-term and long-term optimal investment strategies is the growth optimal portfolio (GP). The GP was first proposed by Kelly (1956), and has since received considerable attention. See MacLean et al. (2011) for a comprehensive overview. The GP at any moment seeks to maximize the instantaneous expected growth rate. From this point of view, the GP qualifies as a short-term myoptic investment strategy. On the other hand, the GP maximizes the expected log utilities of arbitrary long-term investment horizons. In fact, it has been pointed out by Samuelson (1969); Merton (1969, 1971) that the expected log utiliy investor makes the myopic portfolio choice of the GP, regardless of the investment horizon. The GP, as an investment strategy, has received considerable debate and criticism, most notably by Samuelson (1979), due to its significant down-side risk over the long term. Admittedly, the GP is not the "one size fits all" solution for all investors. However, the GP is an important theoretical tool that may be used to construct optimal investment strategies for a wide class of investors with different investment horizons, risk preferences and subjective views, as we will show in this chapter, under the benchmark approach (BA) described by Platen & Heath (2006).

Under the BA, following the real-world pricing formula, the price process of any nonnegative, replicable future payoff, when denominated in units of the GP, forms a martingale. The real-world pricing formula gives the minimal possible replication cost of the payoff. As a consequence, given an initial budget, an investor seeking to maximize the expected utility of the final payoff of the investment portfolio may do so by minimizing the cost of such an optimal payoff, and then follow the corresponding replication strategy of the optimal payoff. This provides an alternative to ad-hoc long-term investment strategies such as the 60/40 rule of thumb. If a tradeable portfolio can be constructed as a proxy to the GP, only the return dynamics of the GP proxy, rather than each individual risky assets, has to be quantitatively modelled, and optimal investment strategies for a wide range of objective functions and preferences can easily be implemented based on this parsimonious model.

It has been shown by DeMiguel et al. (2009) that there is very little evidence that short-term optimal portfolios based on Markowitz's theory outperform simple, model-independent strategies, such as the equal-weighted portfolio, largely due to estimation errors of model parameters. Moreover, it is commonly believed in the finance literature that the financial market is efficient, in the sense that no abnormal returns that outperform the market portfolio (MP) can be consistently generated based on publicly available information, see Fama (1970). Based on these observations, the MP may be regarded as a first approximation to the GP.

In this chapter we take the S&P 500 index as the proxy to the GP of the US equity market. We work under the BA and utilize the minimal market model (MMM) to model the index dynamics. The MMM models the mean-reversion of stock returns(Samuelson, 1991; Kim & Omberg, 1996; Campbell & Viceira, 1999), as well as the leverage effect(Black, 1976; Christie, 1982; Hamao et al., 1990), both of which are well documented empirical facts, in a quantitative and parsimonious manner. Based on this model, we consider under the BA the optimal investment problem for an investor who seeks to maximize a utility function of the type outlined under the cumulative prospect theory (CPT) of Kahneman and Tversky (Kahneman & Tversky, 1979; Tversky & Kahneman, 1992), with a long-term investment horizon and a fixed initial budget.

The rest of this chapter is organized as follows: In Section 4.2 we consider under the BA the optimal investment problem of an investor seeking to maximize the utility of the investment outcome. In Section 4.3, we consider the class of utility functions under the CPT and given detailed solutions of the optimal terminal wealth as a function of the underlying GP. In Section 4.4 we demonstrate the optimal strategies by conducting backtests with the historical returns of the S&P 500 total return index serving as the GP approximation. We summarize our results in Section 4.5. Chapter 4 is based on Sun & Platen (2020).

4.2 Optimal Terminal Wealth under the BA

We consider the long-term optimal consumption and savings problem for a price-taking investor, who seeks to maximize a utility function of terminal wealth $V(T)$ at a finite investment horizon T. The terminal wealth is funded from an initial wealth $V(0)$ through a self-financing investment portfolio $V = \{V(t), t \in [0,T]\}$.

The investor realizes from the terminal wealth $V(T)$ a utility $\mathcal{U}(V(T))$, which the investor seeks to maximize by appropriate choices of the consumption. That is,

$$V^*(T) = \underset{\xi \in L^1_+(\Omega, \mathcal{F}_T, \mathbb{P})}{\arg\max} \quad \mathcal{U}(\xi), \tag{4.1}$$

subject to the budget constraint $V(0)$. Here the terminal wealth ξ is assumed to be nonnegative and $(\Omega, \mathcal{F}_T, \mathbb{P})$ integrable, where $(\Omega, \mathcal{F}_T, \mathbb{P})$ is the usual stochastic basis with information set $\mathcal{F}_T$ at maturity T, and $\mathcal{U} : L^1_+(\Omega, \mathcal{F}_T, \mathbb{P}) \to \mathbb{R}$ is the terminal wealth utility function depending only on the distribution of the terminal wealth.

Following similar arguments as in Section 3.1, the budget constraint is given by

$$E_0\left(\frac{S^*(0)}{S^*(T)}V(T)\right) = V(0). \tag{4.2}$$

By solving the constrained optimization problem (4.1), (4.2) for the optimal terminal wealth $V^*(T)$, the optimal portfolio value process V^* is obtained from the real-world pricing formula (2.10). The optimal investor's portfolio strategy needs only to hold a number of

$$\delta^*(t) = \frac{\partial V^*(t)}{\partial S^*(t)} \tag{4.3}$$

shares of the GP, and invest the rest of the wealth in the risk-free security.

4.3 Utility of the Terminal Savings under the Cumulative Prospect Theory

In this section we consider a class of utilities of the terminal wealth in the form of a rank-dependent expected utility with a convex-concave utility function, given by

$$\mathcal{U}(V(T)) = E_0^w(U(V(T))) = \int U(v)dw(F_{V(T)}(v)), \tag{4.4}$$

where $F_{V(T)}(\cdot)$ denotes the cumulative distribution function (CDF) of the terminal wealth $V(T)$, and the utility function $U(\cdot)$ is assumed to be increasing and convex below a reference level $M >= 0$ and concave above the same level, and the rank-dependent expectation $E_0^w(\cdot)$ is weighted by a weight function $w(\cdot)$ taking values in $[0, 1]$. The weight function models the investor's subjective distortion of the distribution of the savings. The weight function assumes two parts, $w_+(\cdot)$ and $w_-(\cdot)$, which apply to the distribution function of the terminal wealth above and below the reference level M. In other words,

$$w(p) = \begin{cases} w_+(p), & 0 \le p < F_{V(T)}(M), \\ w_-(p), & F_{V(T)}(M) \le p \le 1. \end{cases} \tag{4.5}$$

The weight functions $w_\pm(\cdot)$ are monotonic on $[0, 1]$ and satisfy $w_\pm(0) = 0$, $w_\pm(1) = 1$. Note that $w(\cdot)$ need not be monotonic by definition. If $w_\pm$ represents the uniform distribution, the investor is objective and (4.4) reduces to the expected utility. The utility (4.4) falls in the paradigm of the cumulative prospect theory (CPT), see, Kahneman & Tversky (1979); Tversky & Kahneman (1992).

Under the BA, the optimal investment problem under the afore-mentioned utility of terminal wealth reduces to

$$\begin{cases} \max\limits_{V(T)} E_0^w(U(V(T))), \\ E_0\left(\frac{S^*(0)}{S^*(T)}V(T)\right) = V(0), \end{cases} \tag{4.6}$$

with the stochastic discount factor (SDF) given by $\frac{S^*(0)}{S^*(T)}$. The problem thus reduces to the construction of the optimal terminal wealth $V^*(T)$ subject to the budget constraint. To solve (4.6) we follow Rüschendorf & Vanduffel (2019) and find the solution to be given by

$$V^*(T) = \mathbb{1}_{\{K_0 \le S^*(T) < K_1\}} M + \mathbb{1}_{\{S^*(T) \ge K_1\}} {V_+}^*, \tag{4.7}$$

where $K_1 \ge K_0 \ge 0$ are constants, and ${V_+}^* \ge M$ is the conditional optimal savings on $\{S^*(T) \ge K_1\}$ solving the following restricted problem,

$$\begin{cases} \max\limits_{V_+ \ge M} E_0^w(\mathbb{1}_{\{S^*(T)\ge K_1\}} U(V_+)), \\ E_0\left(\mathbb{1}_{\{S^*(T)\ge K_1\}} \frac{S^*(0)}{S^*(T)} V_+\right) = V_+(0), \end{cases} \tag{4.8}$$

where $V_+(0)$ is a constant representing the budget assigned to $\{S^*(T) \ge K_1\}$ for the conditional optimal wealth ${V_+}^*$. The constants K_1, K_0 and $V_+(0)$ are chosen such that $V^*(T)$ optmizes (4.6). Evidently, the optimal terminal wealth $V^*(T)$ is 0 when $S^*(T) < K_0$, jumps to M when $K_0 \le S^*(T) < K_1$ and takes the higher conditional optimal wealth ${V_+}^* \ge M$ when $S^*(T) \ge K_1$. There is no distribution of the optimal terminal wealth in $(0, M)$ due to the assumed convexity of the utility $U(\cdot)$ over this region.

Following Rüschendorf & Vanduffel (2019), a solution to the restricted problem (4.8) may be given by

$${V_+}^*(\lambda, S^*(T)) = U_+'^{-1}\left(\lambda H_\downarrow\left(F_{S^*(T)}(S^*(T))\right)\right) \tag{4.9}$$

for $S^*(T) \ge K_1$, where $U_+(\cdot)$ is the restriction of $U(\cdot)$ over (M, ∞), i.e., above the reference level. Here $F_{S^*(T)}(\cdot)$ is the CDF of $S^*(T)$, and $H_\downarrow(\cdot)$ is the L^2 projection of the distorted quantile function $H(\cdot)$, given by

$$H(t) = \frac{S^*(0)}{Q_{S^*(T)}(w_+^{-1}(t)) w_+'(w_+^{-1}(t))}, \tag{4.10}$$

where $Q_{S^*(T)}(\cdot)$ is the quantile function of $S^*(T)$, onto the convex subset of non-increasing functions in $L^2([0,1])$, the space of L^2-integrable functions over $[0,1]$. In the special case when $w^+(\cdot)$ is convex, $H_\downarrow(\cdot) = H(\cdot)$. The constant $\lambda > 0$ is chosen such that (4.9) is well-defined, and the budget constraint of (4.8) is satisfied. Alternatively, it can be regarded that λ defines $V_+(0)$ through the budget constraint. The optimal terminal wealth (4.7) is thus obtained by choosing the parameters (K_0, K_1, λ) according to the following system:

$$\begin{cases} \max\limits_{K_0, K_1, \lambda} E_0^w\left(\mathbb{1}_{\{K_0 \le S^*(T) < K_1\}} M + \mathbb{1}_{\{S^*(T)\ge K_1\}} {V_+}^*\right), \\ K_1 \ge K_0 \ge 0, \\ 0 < \lambda \le \frac{U_+'(M)}{H_\downarrow(\mathbb{P}(S^*(T) \le K_1))}, \\ E_0\left(\frac{S^*(0)}{S^*(T)}\left(\mathbb{1}_{\{K_0 \le S^*(T) < K_1\}} M + \mathbb{1}_{\{S^*(T)\ge K_1\}} {V_+}^*\right)\right) = V(0), \end{cases} \tag{4.11}$$

where ${V_+}^*$ is given by (4.9). Here we assume $U_+'^{-1}(\cdot)$ is well defined over $(0, U_+'(M))$.

From (4.7) and (4.9), it is evident that the optimal terminal wealth $V^*(T)$ is monotonically increasing with $S^*(T)$. That is, the terminal wealth is higher when the GP value is higher, or equivalently, the SDF (state price) is lower. This is intuitively consistent with the state-independence of the CPT utility, where payoffs in different

Table 4.1 – The CPT investors' preference parameters.

Investor	γ_+	γ_-	α_+	α_-	β
#1	1	2	0.5	1.2	0.3
#2	1	2	0.5	1.2	0.5
#3	1	2	0.5	1.2	0.7
#4	1	2	0.5	1.2	0.9

states contribute equally to the overall utility, thus more payoffs in states of better return or cheaper state price is optimal.

4.4 Numerical Experiments

To illustrate the methods presented so far, we conduct in this section numerical experiments of the optimal investment strategies, based on the realised historical returns of the S&P 500 total return index of the US domestic equity market. We model the market index under the BA and consider backtests of optimal investment strategies for a number of different investors' subjective preferences. We propose the minimal market model (MMM), as described in 2.1, for modelling the underlying discounted GP. We consider backtesting of optimal investment strategies implied by the MMM under the BA. We fit the MMM with the historical data of the S&P 500 index returns as described in 2.2.

To assess the implications of the estimated model for investment purposes, we consider an investor making long-term investment decisions based on the predictions of the model, over the 30-year period from 1988.02 to 2018.01. The long-term horizon of 30 years resembles the typical life-time investments for retirement of individual retirees, and pension fund investments on behalf of the individual retirees. The investor starts from an initial wealth of unity, and seeks to maximize a utility of the terminal wealth following the general CPT framework of Section 4.3.

We consider a CPT investor #1 with a convex-concave utility of wealth function $u(\cdot)$, along with a probability distortion function $w(\cdot)$, as described in Section 4.3. We assume the investor adopts an exponential utility of wealth function as in Reige and Hens (2006), given by

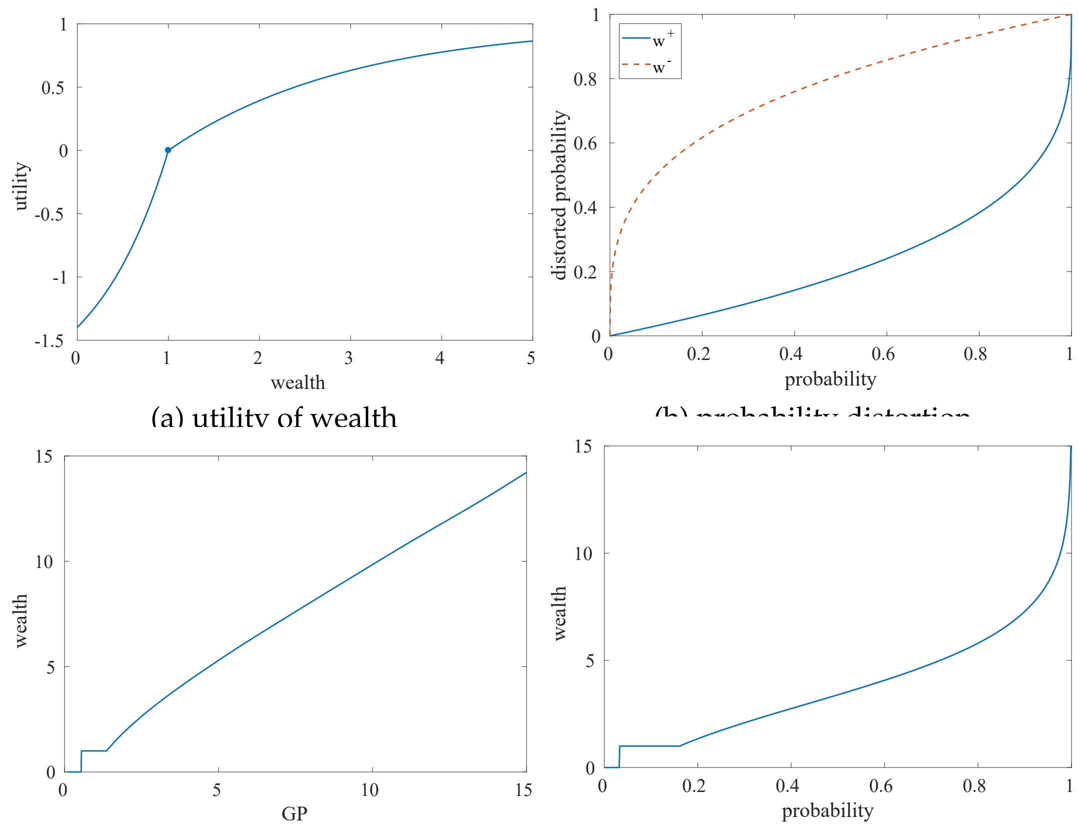
$$u(c) = \begin{cases} \gamma_+ \left(1 - e^{-\alpha_+(c-M)}\right), & c \geq M, \\ -\gamma_- \left(e^{\alpha_-(c-M)} - 1\right), & c \leq M, \end{cases} \tag{4.12}$$

where γ_+ and $-\gamma_-$ are upper and lower bounds of the utility of wealth, M is the reference level, and $\alpha_\pm$ model respectively the investor's degrees of risk aversion and risk seeking above and below the reference. The investor's subjective probability distortions $w_\pm(\cdot)$ are modeled by power functions as

$$\begin{cases} w_+(t) = 1 - (1-t)^\beta, \\ w_-(t) = t^\beta, \end{cases} \tag{4.13}$$

where, for simplicity, a single parameter $0 < \beta < 1$ models the nonlinear preference for small probabilities at both the upper and lower tails. These parameters are given in Table 4.1, and the function plots are shown in Fig. 4.1(a), (b).

To compute the optimal terminal wealth, the terminal GP $S^*(T)$ was simulated from the estimated model and an empirical distribution for $S^*(T)$ was empirically



(a) utility of wealth

(b) probability distortion

(c) optimal wealth vs terminal GP

(d) quantiles of the optimal wealth

Figure 4.1 – The CPT investor's utility of wealth, probability distortion for the gains, and the optimal terminal wealth as a function of the terminal GP as well as the quantile function.

Table 4.2 – The CPT investors' investment objectives and outcomes.

Investor	K_0	K_1	$\mathbb{P}(V^*(T)=0)$	$\mathbb{P}(V^*(T)=1)$	pred. ret.	real. ret.	corr. with GP
#1	0.5406	1.3766	0.0323	0.1309	0.0717	0.0716	-0.58
#2	0.4795	1.2589	0.0258	0.1154	0.0679	0.0676	-0.60
#3	0.4259	1.1527	0.0206	0.1017	0.0635	0.0630	-0.60
#4	0.3785	1.0522	0.0163	0.0886	0.0584	0.0577	-0.60

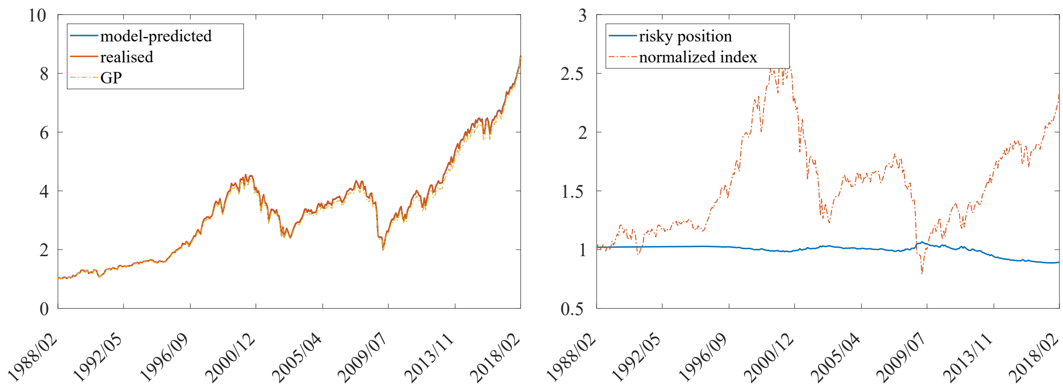
constructed using a nonparametric kernel density estimation. Alternaltively, the MMM assumes a closed form transition law as described in Section 2.1. Using the terminal GP distribution, the optimal terminal wealth for the CPT investor was subsequently constructed following Section 4.3 using standard numerical optimization techniques. The optimal terminal wealth as a function of the terminal GP is shown in Fig. 4.1(c), and the quantile function is shown in Fig. 4.1(d). It is seen that the optimal terminal wealth stays zero when the GP is very low, i.e., when $S^*(T) \leq K_0 = 0.5406$. The optimal wealth increases to the reference level 1 at K_0 and starts growing when $S^*(T) > K_1 = 1.3766$. Here we normalized the GP by the initial value so that $S^*(0) = 1$. The probability for $S^*(T) \leq K_0$, that is, for zero terminal wealth, is in this case valued at 3.23%. The probability of retaining exactly the initial wealth of unity, that is, $K_0 < S^*(T) \leq K_1$, is 13.09%.

Given the optimal terminal wealth, the investor follows a replicating strategy implied by the estimated model, and realizes a portfolio process based on the realised index, as shown in Fig. 4.2(a). The realised portfolio follows closely the model-predicted values throughout the investment period. The realised log return is valued at 7.16%, slightly less than the model predicted value of 7.17%, and the log return of the GP is valued at 7.18%.

It can be seen that CPT investor, even though highly risk-averse above the reference level with an exponential utility, ends up performing nearly as good as the GP. In other words, the seemingly exponentially risk-averse CPT investor invests almost as agressively as the expected log utility investor, who invests fully in the GP. This is partly due to the investor's distorted subjective view toward extremely favorable market returns, as is evident from the nearly linear optimal payoff as a function of the GP toward the upper end. On the other hand, the risk-seeking of the CPT investor below the reference level also plays a role, leading to more budget allocation to the scenarios of more returns, i.e., $S^*(T) > K_1$. The cost of maintaining even the minimal reference level of wealth under the scenario of a bear market (e.g., $S^*(T) \leq K_0$) is very high, thus giving up such a scenario leads to more chances of realising higher returns under a bull market scenario, and more utility for the investor.

The CPT investor's risky asset allocations, as shown in Fig .4.2(b), can be seen to be slowly fluctuating around 1, with an apparent negative correlation with the normalized index Y. The overall correlation is measured at -0.58. This negative correlation is attributed partly to the mean-reverting nature of the normalized index under the MMM. Toward the end of the investment period when the portfolio value is well above the reference level, the risk-aversion for positive returns seems to have a strong effect in reducing the risk exposure, which contribute to the negative correlation in this realised scenario.

Finally, we explore the effect of the CPT investor's probability distortion. We consider four CPT investors (numbered from 1 to 4, with 1 discussed earlier) of different levels of probability distortions from very little ($\beta = 0.9$) to very deep ($\beta = 0.3$). The



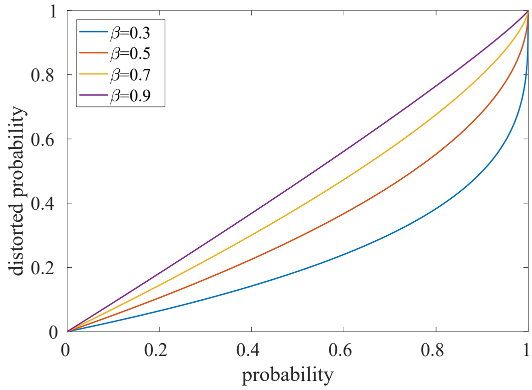
(a) portfolio value processes (b) risky asset allocations

Figure 4.2 – The CPT investor's model-predicated and realised portfolio processes, as well as risky asset allocations in the GP.

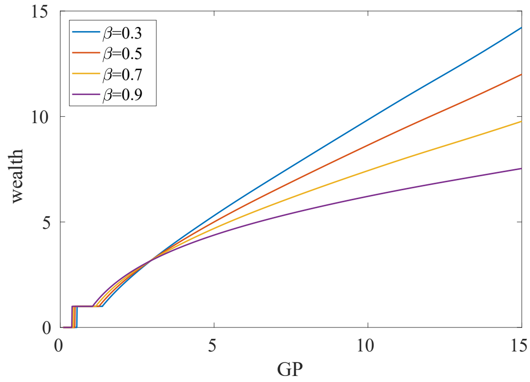
rest of the preference parameters remain the same, as shown in Table 4.1. Plots of the distortion function $w_+(\cdot)$ are shown in Fig. 4.3 (a). The corresponding optimal terminal wealths are shown in Fig. 4.3 (b). Evidently the CPT investor is increasingly optimistic with a more distorted subjective view toward high market returns, leading to less risk aversion under such scenarios, as is seen by the increasingly linear optimal payoff as a function of the GP under high market returns. On the other hand, though to a lesser extent, the investor seems to be increasingly willing to give up the minimal reference wealth, in exchange for more chances of high market returns. The symmetric distortion $w^-(t) = 1 - w^+(1-t)$ does not seem to discourage the CPT investor from giving up the minimal reference wealth enough to offset the attraction from increased prospects at the other end, possibly due to the asymmetric distribution of the GP around the reference level.

The CPT investor's portfolios for different levels of distortions are realised as shown in Fig. 4.3 (c), with the corresponding model-predicted processes shown in Fig. 4.3 (d). The risky asset allocations of the CPT investor are shown in Fig. 4.3(e). It can be seen that with increasing probability distortions, the CPT investor becomes increasingly risk-taking, and invests with increasing leverage. The investment strategies under the realised scenario resulted in increasingly spectacular returns, with the highest (excess) return of 7.17% under the strongest probability distortion. Of course the increasingly high returns come at the cost of increased uncertainty, acceptable due to the probability distortion, as well as increased chances of winding up losing all wealths under adverse market conditions. Such investment strategies are not unseen in reality among market speculators driven by subjective views.

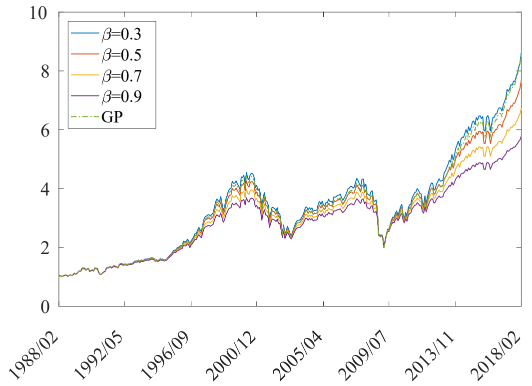
With monthly rebalancing, the realised returns are very close to the model predicted returns, as shown in Table 4.2, as well as from a visual comparison of Fig. 4.3 (c) and Fig. 4.3 (d). The risky asset allocations are relatively stable over the time, and vary slowly against the normalized index, as seen from Fig. 4.3(e). This makes the strategies relatively practical with less exposure to transaction costs. The negative correlation of the risky asset allocations with the normalized index remains a robust feature regardless of the probability distortions. This means mean-reversion of the normalized index is robust under probability distortions of the CPT investor.



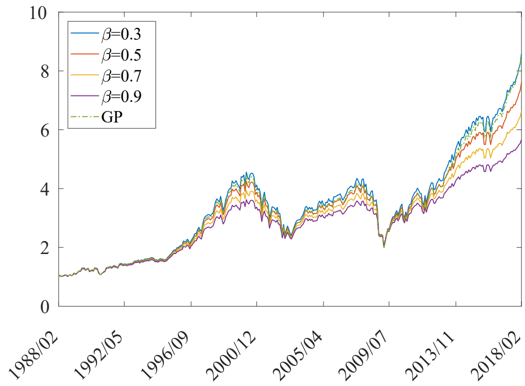
(a) weight functions



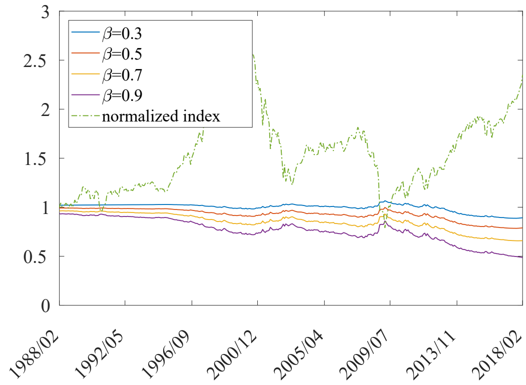
(b) optimal wealth vs terminal GP



(c) model-predicted portfolio processes



(d) realised portfolio processes



(e) portfolio risky asset allocations

Figure 4.3 – The CPT investor's probability distortions and investment outcomes.

4.5 Summary

We have proposed a novel approach to the long-term financial planning problem under the BA framework, where the benchmarked savings account is taken as the least expensive SDF, leading to the minimal pricing of any consumption-savings strategies. The BA framework generalizes the classical risk-neutral pricing framework, leading to potentially more realistic and robust modelling of the financial market under the real-world probability measure, where long-term contingent claims may be priced less expensively and long-term investments may generate higher returns.

To illustrate the key ideas of the BA, We take the MMM as a stylized BA model, where an equivalent risk-neutral probability measure cannot exist. We take the S&P 500 index as a proxy to the GP of the US equity market, and conduct investment backtestings on the historical prices of the index under the MMM. We consider a highly risk-averse investor as well as CPT investors of different subjective views, and apply the general solution to the optimal investment problem of Rüschendorf & Vanduffel (2019) under the BA. We perform out-of-sample backtests of these optimal strategies and obtain the investment outcomes. Our findings illustrate how typical CPT investors may construct long-term optimal investment strategies and realise different investment outcomes according to different preferences. Our stylized model with few parameters is simple and robust to estimate, and results in optimal investment strategies that are stable, requiring relatively less dramatic monthly rebalancing, which further produces robust investment outcomes that well replicate the desired objectives. Our results may help individual investors as well as fund managers investing for the long term, who would want to incorporate subjective preferences in their investment strategies.

Finally, it is worth mentioning that the method outlined in this chapter also applies if one adopts a risk-neutral pricing model, instead of the MMM. In particular, the former is a special case under the BA framework. The key difference is, perhaps, that a risk-neutral model generally under estimates the growth potential of the GP, thus leading to more conservative strategies even for the more subjective CPT investors. Whether to choose a risk-neutral model or one that is incompatible with the risk-neutral framework under the BA framework is, itself, a subjective matter.

Chapter 5

The Impact of Management Fees on the Pricing of Variable Annuity Guarantees

5.1 Introduction

In this chapter, we discuss pricing of variable annuities (VA) with guarantees under the risk-neutral pricing framework. As this subject is widely discussed in the literature, see e.g., references later in this section, we focus our attention to the problem of a nonlinear management fee typically charged by the mananger of the underlying mutual fund on which the VA contract is written. We will discuss similar contracts under the BA pricing framework in Chapter 6, where implications of the two pricing frameworks are compared and contrasted.

Variable annuities (VA) with guarantees of living and death benefits are offered by wealth management and insurance companies worldwide to assist individuals in managing their pre-retirement and post-retirement financial plans. These products take advantages of market growth while providing a protection of the savings against market downturns. Similar guarantees are also available for life insurance policies (Bacinello & Ortu (1996)). The VA contract cash flows received by the policyholder are linked to the investment portfolio choice and performance (e.g. the choice of a mutual fund and its strategy) while traditional annuities provide a predefined income stream in exchange for a lump sum payment. Holders of VA policies are required to pay management fees regularly during the term of the contract for the wealth management services.

A variety of VA guarantees, also known as VA riders, can be elected by policyholders at the cost of additional insurance fees. Common examples of VA guarantees include guaranteed minimum accumulation benefit (GMAB), guaranteed minimum withdrawal benefit (GMWB), guaranteed minimum income benefit (GMIB) and guaranteed minimum death benefit (GMDB), as well as a combination of them, e.g., guaranteed minimum withdrawal and death benefit (GMWDB), among others. These guarantees, generically denoted as GMxB, provide different types of protection against market downturns, shortfall of savings due to longevity risk or assurance of stability of income streams. Precise specifications of these products can vary across categories and issuers. See Bauer et al. (2008); Ledlie et al. (2008); Kalberer & Ravindran (2009) for an overview of these products.

The Global Financial Crisis during 2007-08 led to lasting adverse market conditions such as low interest rates and asset returns as well as high volatilities for VA providers. Under these conditions, the VA guarantees become more valuable, and the fulfillment of the corresponding required liabilities become more demanding. The post-crisis market conditions have called for effective hedging of risks associated with the VA guarantees (Sun et al. (2016)). As a consequence, the need for accurate estimation of hedging costs of VA guarantees has become increasingly important. Under the classical risk-neutral approach, such estimations consist of risk-neutral pricing of future cash flows that must be paid by the insurer to the policyholder in order to fulfill the liabilities of the VA guarantees.

There have been a number of contributions in the academic literature considering the pricing of VA guarantees. A range of numerical methods are considered, including standard and regression-based Monte Carlo (Huang & Kwok (2016)), partial differential equation (PDE) and direct integration methods (Milevsky & Salisbury (2006); Dai et al. (2008); Chen & Forsyth (2008); Bauer et al. (2008); Luo & Shevchenko (2015a,b); Forsyth & Vetzal (2014); Shevchenko & Luo (2017)). A comprehensive overview of numerical methods for the pricing of VA guarantees is provided in Shevchenko & Luo (2016).

In this chapter we focus on a GMWDB VA policy which provides a guaranteed withdrawal amount per year until the maturity of the contract regardless of the investment performance, as well as a lump-sum of death benefit in case the policyholder dies over the contract period. The guaranteed withdrawal amount is determined such that the initial investment is returned over the life of the contract. The death benefit may assume different forms depending on the details of the contract. When pricing GMWDB, one typically assumes either a pre-determined (static) policyholder behavior in withdrawal and surrender, or an active (dynamic) strategy where the policyholder "optimally" decides the amount of withdrawal at each withdrawal date depending on the information available at that date.

One of the most debated aspects in the pricing of GMWDB with active withdrawal strategies is the policyholders' withdrawal behavior (Cramer et al. (2007); Chen & Forsyth (2008); Moenig & Bauer (2015); Forsyth & Vetzal (2014)). It is often customary to refer to the withdrawal strategy that maximizes the expected liability, or the hedging cost, of the VA guarantee as the "optimal" strategy. Even though such a strategy underlies the worst case scenario for the VA provider with the highest hedging cost, it may not coincide with the real-world behavior of the policyholder. Nevertheless, the price of the guarantee under this strategy provides an upper bound of hedging cost from the insurer's perspective, which is often referred to as the "value" of the guarantee. The real-world behaviors of policyholders often deviate from this "optimal" strategy, as is noted in Moenig & Bauer (2015). Different models have been proposed to account for the real-world behavior of policyholders, including the subjective risk neutral valuation approach taken by Moenig & Bauer (2015). In particular, it is concluded by Moenig & Bauer (2015) that a subjective risk-neutral valuation methodology that takes different tax structures into consideration is in line with the corresponding findings from empirical observations.

Similar to the tax consideration in Moenig & Bauer (2015), the management fee is a form of market friction that would affect policyholders' rational behaviors. However, such effects of management fees have not been considered in the VA pricing literature. When the management fee is zero and deterministic withdrawal behavior is assumed, Hyndman & Wenger (2014) and Fung et al. (2014) show that risk-neutral pricing of guaranteed withdrawal benefits in both a policyholder's and an insurer's perspectives will result in the same fair insurance fee. Few studies that take management fees into account in the pricing of VA guarantees include Bélanger et al. (2009), Chen et al. (2008) and Kling et al. (2011). In these studies, fair insurance fees are considered from the insurer's perspective with the management fees as given. The important question of how the management fees as a form of market friction will impact withdrawal behaviors of the policyholder, and hence the hedging cost for the insurer, is yet to be examined in a dynamic withdrawal setting. The main goal of the chapter is to address this question under the classical risk-neutral approach. We will address the VA problem in the next chapter under the generalized framework.

The chapter contributes to the literature in three aspects. First, we consider two pricing approaches based on the policyholder's and the insurer's perspective. In the literature it is most often the case that only an insurer's perspective is considered, which might result in mis-characterisation of the policyholder's withdrawal strategies. Second, we characterize the impact of management fees on the pricing of GMWDB, and demonstrate that the two afore-mentioned pricing perspectives lead to different fair insurance fees due to the presence of management fees. In particular, the fair insurance fees from the policyholder's perspective are lower than those from the insurer's perspective. This provides a possible justification of lower insurance fees observed in the market. Third, the sensitivity of the fair insurance fees to management fees under different market conditions and contract parameters are investigated and quantified through numerical examples.

The chapter is organized as follows. In Section 5.2 we present the contract details of the GMWDB guarantee together with its pricing formulation under a stochastic optimal control framework. Section 5.3 derives the policyholder's value function under the risk-neutral pricing approach, followed by the insurer's net liability function in Section 5.4. In Section 5.5 we compare the two withdrawal strategies that maximize the policyholder's value and the insurer's liability, respectively, and discuss the role of the management fees in their relations. Section 5.6 demonstrates the approaches via numerical examples. Section 5.7 concludes with remarks and discussion. Chapter 5 is based on Sun et al. (2018).

5.2 Formulation of the GMWDB Pricing Problem

We begin with the setup of the framework for the pricing of the GMWDB and describe the features of this type of guarantees. The problem is formulated under a general setting so that the resulting pricing formulation can be applied to different GMWDB contract specifications. Besides the general setting, we also consider a very specific simple GMWDB contract, which will be subsequently used for illustration purposes in numerical experiments presented in Section 5.6.

The VA policyholder's retirement fund is usually invested in a managed wealth account that is exposed to financial market risks. A management fee is usually charged for this investment service. In addition, if the GMWDB is elected, extra insurance fees will be charged for the protection offered by the guarantee provider (insurer). We assume the wealth account guaranteed by the GMWDB is subject to continuously charged proportional management fees independent of any fees charged for the guarantee insurance. The purpose of these management fees is to compensate for the wealth management services provided, or perhapse merely for the access to the guarantee insurance on the investment. This fee should not be confused with other forms of market frictions, e.g., transaction costs, if any, that must incur when tracking a given equity index. Given the proliferation of index-tracking exchange-traded funds in recent years, with much desired liquidity at a fraction of the costs of the conventional index mutual funds, see, e.g., Agapova (2011); Kostovetsky (2003); Poterba & Shoven (2002), regarding these management fees as additional costs to policyholders beyond the normal market frictions seems to be a reasonable assumption.

The hedging cost of the guarantee, on the other hand, is paid by proportional insurance fees continuously charged to the wealth account. The fair insurance fee rate, or the fair fee in short, refers to the minimal insurance fee rate required to fund the hedging portfolio, so that the guarantee provider can eliminate the market risk associated with the selling of the guarantees.

We consider the situation where a policyholder purchases the GMWDB rider in order to protect his wealth account that tracks an equity index $S(t)$ at time $t \in [0, T]$, where 0 and T correspond to the inception and expiry dates. The equity index account is modelled under the assumed risk-neutral probability measure $\mathbb{Q}$ following the stochastic differential equation (SDE)

$$dS(t) = S(t) \left(r(t)dt + \sigma(t)dB(t) \right), \quad t \in [0, T], \tag{5.1}$$

where $r(t)$ is the risk-free short-term interest rate, $\sigma(t)$ is the volatility of the index, which are time-dependent and can be stochastic, and $B(t)$ is a standard $\mathbb{Q}$-Brownian motion modelling the uncertainty of the index. Here, we follow standard practices in the literature of VA guarantee pricing by modelling under the risk-neutral probability measure $\mathbb{Q}$, which allows the pricing of stochastic cash flows to be given as the risk-neutral expectation of the discounted cash flows. The risk-neutral probability measure $\mathbb{Q}$ exists if the underlying financial market satisfies the NFLVR condition mentioned in Chapter 1. Adopting risk-neutral pricing here assumes that stochastic cash flows in the future can be replicated by dynamic hedging without transaction fees. For details on risk-neutral pricing and the underlying assumptions, see, e.g., Delbaen & Schachermayer (2006) for an account under very general settings.

The wealth account $W(t), t \in [0, T]$ over the lifetime of the GMWDB contract is invested into the index S, subject to management fees charged by a wealth manager at the rate $\alpha_{\mathrm{m}}(t)$. An additional charge of insurance fees at rate $\alpha_{\mathrm{ins}}(t)$ for the GMWDB rider is collected by the insurer to pay for the hedging cost of the guarantee. We assume that both fees are deterministic, time-dependent and continuously charged to the wealth account. (Sometimes the insurance fees are charged to the guarantee account mentioned shortly.) Discrete fees may be modelled similarly without any difficulty. The wealth account in turn evolves as

$$dW(t) = W(t) \left((r(t) - \alpha_{\mathrm{tot}}(t))dt + \sigma(t)dB(t) \right), \tag{5.2}$$

for any $t \in [0, T]$ at which no withdrawal of wealth is made. Here, $\alpha_{\mathrm{tot}}(t) = \alpha_{\mathrm{ins}}(t) + \alpha_{\mathrm{m}}(t)$ is the total fee rate. The GMWDB contract allows the policyholder to withdraw from a guarantee account $A(t), t \in [0, T]$ on a sequence of pre-determined contract event dates, $0 = t_0 < t_1 < \cdots < t_N = T$. The initial guarantee $A(0)$ usually matches the initial wealth $W(0)$. The guarantee account stays constant unless a withdrawal is made on one of the event dates, which changes the guarantee account balance. If the policyholder dies on or before the maturity T, the death benefit will be paid at the next event date immediately following the death of the policyholder. Additional features such as early surrender can be included straightforwardly but will not be considered in this article to avoid unnecessary complexities.

To simplify notations, we denote by $\mathbf{Y}(t)$ the vector of state variables at t, given by

$$\mathbf{Y}(t) = (r(t), \sigma(t), S(t), W(t), A(t)), \quad t \in [0, T]. \tag{5.3}$$

Here, we assume that all state variables follow Markov processes under the risk-neutral probability measure $\mathbb{Q}$, so that $\mathbf{Y}(t)$ contains all the market and account balances information available at t. For simplicity, we assume the state variables $r(t)$, $\sigma(t)$ and $S(t)$ are continuous, and $W(t)$ and $A(t)$ are left continuous with right limit (LCRL). We include the index value $S(t)$ in $\mathbf{Y}(t)$, which under the current model may seem redundant, due to the scale-invariance of the geometric Brownian motion type model (5.1). In general, however, $S(t)$ may determine the future dynamics of S in a nonlinear fashion, as is the case under, e.g., the minimal market model described in Chapter 1.

We define $I(t), t \in [0, T]$ as the life indicator function of an individual policyholder as the following: $I(t) = 1$ if the policyholder was alive on the last event date on or before t; $I(t) = 0$ if the policyholder was alive on the second-to-the-last event date prior to t but died on or before the last event date; $I(t) = -1$ if the policyholder died on or before the second-to-the-last event date prior to t. We assume the policyholder is alive at t_0. The life indicator function $I(t)$ therefore starts at $I(t_0) = 1$, is right continuous with left limit (RCLL), and remains constant between two consecutive event dates. Note that mortality information contained in $I(t)$ (RCLL) comes before any jumps of the LCRL account balances $W(t)$ and $A(t)$ on the event dates, reflecting the situation that any jumps in these account balances may depend on the mortality information. We denote the vector of state variables including $I(t)$ as $\boldsymbol{X}(t) = (\mathbf{Y}(t)^\top, I(t))^\top$, and we denote by $\mathrm{E}_t^{\mathbb{Q}}[\cdot]$ the risk-neutral expectation conditional on the state variables $\boldsymbol{X}(t)$ at t, i.e., $\mathrm{E}_t^{\mathbb{Q}}[\cdot] := \mathrm{E}^{\mathbb{Q}}[\cdot|\boldsymbol{X}(t)]$. Note that the risk-neutral measure $\mathbb{Q}$ is assumed to extend to the mortality risk represented by the life indicator $I(t)$.

On event dates $t_n, n = 1, \ldots, N$, a nominal withdrawal γ_n from the guarantee account is made. The policyholder, if alive, may choose γ_n on $t_n < T$. Otherwise, a liquidation withdrawal of $\max(W(t_n), A(t_n))$ is made. That is,

$$\gamma_n = \Gamma(t_n, \mathbf{Y}(t_n))\mathbb{1}_{\{I(t_n)=1, n<N\}} + \max\left(W(t_n), A(t_n)\right)\mathbb{1}_{\{I(t_n)=0 \text{ or } n=N\}}, \tag{5.4}$$

where $\mathbb{1}_{\{\}}$ denotes the indicator function of an event, and $\Gamma(\cdot,\cdot)$ is referred to as the *withdrawal strategy* of the policyholder. The real cash flow received by the policyholder, which may differ from the nominal amount, is denoted by $C_n(\gamma_n, \boldsymbol{X}(t_n))$. This is given by

$$C_n(\gamma_n, \boldsymbol{X}(t_n)) = P_n(\gamma_n, \mathbf{Y}(t_n))\mathbb{1}_{\{I(t_n)=1\}} + D_n(\mathbf{Y}(t_n))\mathbb{1}_{\{I(t_n)=0\}}, \tag{5.5}$$

where $P_n(\gamma_n, \mathbf{Y}(t_n))$ is the payment received if the policyholder is still alive, and $D_n(\mathbf{Y}(t_n))$ denotes the death benefit if the policyholder died during the last period. As a specific example, $P_n(\gamma_n, \mathbf{Y}(t_n))$ may be given by

$$P_n(\gamma_n, \mathbf{Y}(t_n)) = \gamma_n - \beta \max(\min(\gamma_n, A(t_n)) - G_n, 0). \tag{5.6}$$

Here the contractual withdrawal G_n is a pre-determined withdrawal amount specified in the GMWDB contract, and β is the penalty rate applied to the part of the withdrawal from the guarantee account exceeding the contractual withdrawal G_n. Note that the $\min(\gamma_n, A(t_n))$ term in (5.6) accommodates the situation that at expiration of the contract, both accounts are liquidated, but only the *guarantee account withdrawals* exceeding the contractual rate G_n will be penalized. Excess balance on the wealth account after the guaranteed withdrawal is not subject to this penalty. An example of the death benefit may simply be taken as the total withdrawal without penalty, i.e.,

$$D_n(\mathbf{Y}(t_n)) = \max\left(W(t_n), A(t_n)\right). \tag{5.7}$$

Upon withdrawal by the policyholder, the guarantee account is changed by the amount $J_n(\gamma_n, \mathbf{Y}(t_n))$, that is,

$$A(t_n^+) = A(t_n) - J_n(\gamma_n, \mathbf{Y}(t_n)), \tag{5.8}$$

where $A(t_n^+)$ denotes the guarantee account balance "immediately after" the withdrawal. For example, $J_n(\gamma_n, \mathbf{Y}(t_n))$ may be given by

$$J_n(\gamma_n, \mathbf{Y}(t_n)) = \gamma_n \mathbb{1}_{\{I(t_n)=1\}} + A(t_n)\mathbb{1}_{\{I(t_n)\leq 0\}}, \tag{5.9}$$

i.e., the guarantee account balance is reduced by the withdrawal amount if the policyholder is alive and the policy has not expired. Otherwise, the account is liquidated. The guarantee account stays nonnegative, that is, γ_n if chosen by the policyholder must be such that $J_n(\gamma_n, \mathbf{Y}(t_n)) \leq A(t_n)$. The wealth account is reduced by the amount γ_n upon withdrawal and remains nonnegative. That is,

$$W(t_n^+) = \max(W(t_n) - \gamma_n, 0), \tag{5.10}$$

where $W(t_n^+)$ is the wealth account balance immediately after the withdrawal. It is assumed that $\gamma_0 = 0$, i.e., no withdrawals at the start of the contract. Both the wealth and the guarantee account balance are 0 after contract expiration. That is

$$W(T^+) = A(T^+) = 0. \tag{5.11}$$

The policyholder's value function at time t is denoted by $V(t, \boldsymbol{X}(t)), t \in [0, T]$, which corresponds to the risk-neutral expected value of all cash flows to the policyholder on or after time t. The remaining policy value after the final cash flow is thus 0, i.e.,

$$V(T^+, \boldsymbol{X}(T^+)) = 0. \tag{5.12}$$

5.3 Calculating the Policyholder's Value Function

We now calculate the policyholder's value function $V(t, \boldsymbol{X}(t))$ as the risk-neutral expected value of policyholder's future cash flows at time $t \in [0, T]$. The risk-neutral valuation of the policyholder's future cash flows can be regarded as the value of the remaining term of the VA contract from the poliyholder's perspective. As mentioned in the beginning of Section 5.2, valuation under the risk-neutral pricing approach assumes that the cash flows may be replicated by hedging portfolios without market frictions. This may be carried out by a third-party independent agent, if not directly by the individual policyholder.

Following Section 5.2, the policyholder's value function on an event date t_n can be written as

$$V(t_n, \boldsymbol{X}(t_n)) = C_n(\gamma_n, \boldsymbol{X}(t_n)) + V(t_n^+, \boldsymbol{X}(t_n^+)), \tag{5.13}$$

which by (5.5) can be further written as

$$V(t_n, \boldsymbol{X}(t_n)) = (P_n(\gamma_n, \mathbf{Y}(t_n)) + V_n(t_n^+, \boldsymbol{X}(t_n^+)))\mathbb{1}_{\{I(t_n)=1\}} + D_n(\mathbf{Y}(t_n))\mathbb{1}_{\{I(t_n)=0\}}, \tag{5.14}$$

since if the policyholder died during the last period, the death benefit is the only cash flow to receive. Taking the risk-neutral expectation $\mathrm{E}^{\mathbb{Q}}_{t_n^-}[\cdot]$, we obtain the jump condition

$$V(t_n^-, \boldsymbol{X}(t_n^-)) = \Big((1-q_n)\Big(P_n(\gamma_n, \mathbf{Y}(t_n))+V(t_n^+, \boldsymbol{X}(t_n^+))\Big)+q_nD_n(\mathbf{Y}(t_n))\Big)\mathbb{1}_{\{I(t_n^-)=1\}}, \tag{5.15}$$

where q_n is the risk-neutral probability that the policyholder died over (t_{n-1}, t_n), given that he is alive on the last withdrawal date t_{n-1}. That is,

$$q_n = \mathbb{Q}[I(t_n) = 0 | I(t_n^-) = 1]. \tag{5.16}$$

Here, we assume that the mortality risk is independent of the market risk under the risk-neutral probability measure. Under the assumption that the mortality risk is completely diversifiable, the risk-neutral mortality rate may be indentified with that under the real-world probability measure and inferred from a historical life table. (Since an individual policyholder cannot hedge the mortality risk through diversification, risk-neutral pricing of the policyholder's value function essentially assumes that the policyholder is risk-neutral toward the mortality risk.) Here the mortality information over $(t_{n-1}, t_n]$ is revealed at t_n, thus at t_n^- such information is not yet available. This assumption is not a model constraint since all decisions are made only on event dates.

The policy value at $t \in (t_{n-1}, t_n)$ is given by the expected discounted future policy value under the risk-neutral probability measure, given by

$$V(t, \boldsymbol{X}(t)) = \mathrm{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{t_n} r(s)ds} V(t_n^-, \boldsymbol{X}(t_n^-))\right], \tag{5.17}$$

where $e^{-\int_t^{t_n} r(s)ds}$ is the discount factor. The initial policy value, given by $V(0, \boldsymbol{X}(0))$, can be calculated backward in time starting from the terminal condtion (5.12), using (5.15) and (5.17), as described in Algorithm 1.

As an illustrative example, we assume $r(t) \equiv r$, $\sigma(t) \equiv \sigma$ and $\alpha_{\text{tot}}(t) \equiv \alpha_{\text{tot}}$ as constants. Since $A(t)$ and $I(t)$ are constants between the withdrawals, the only effective state variable between the withdrawal dates is $W(t)$. Thus we can simply write $V(t, \boldsymbol{X}(t)) = V(t, W(t))$ for $t \in (t_{n-1}, t_n)$ without confusion. We now derive the partial differential equation (PDE) satisfied by the value function V through a hedging argument. We consider a delta hedging portfolio that, at time $t \in (t_{n-1}, t_n)$, takes a long position of the value function V and a short position of $\frac{W(t)\partial_W V(t,W(t))}{S(t)}$ shares of the index S. Here $\partial_W V(t, W(t)) \equiv \frac{\partial V(t,W)}{\partial W}|_{W=W(t)}$ is the partial derivative of $V(t, W)$ with respect to the second argument, evaluated at $W(t)$. Denoting the total value of this portfolio at $t \in (t_{n-1}, t_n)$ as $\Pi^V(t)$, the value of the delta hedging portfolio is given by

$$\Pi^V(t) = V(t, W(t)) - W(t)\partial_W V(t, W(t)). \tag{5.18}$$

By Ito's formula and (5.1), the SDE for Π^V can be obtained as

$$ld\Pi^V(t) = \left(\partial_t V(t, W(t)) - \alpha_{\text{tot}} W(t)\partial_W V(t, W(t)) + \frac{1}{2}\sigma^2 W(t)^2 \partial_{WW} V(t, W(t))\right) dt, \tag{5.19}$$

for $t \in (t_{n-1}, t_n)$. Since the hedging portfolio Π^V is locally riskless, it must grow at the risk-free rate r, that is $d\Pi^V(t) = r\Pi^V(t)dt$. This along with (5.18) implies that the

PDE satisfied by the value function $V(t, W)$ is given by

$$\partial_t V - rV + (r - \alpha_{\text{tot}})W\partial_W V + \frac{1}{2}\sigma^2 W^2 \partial_{WW} V = 0, \tag{5.20}$$

for $t \in (t_{n-1}, t_n)$ and $n = 1, \ldots, N$. The boundary conditions at t_n are specified by (5.12) and (5.15). The valuation formula (5.17) or the PDE (5.20) may be solved recursively by following Algorithm 1 to compute the initial policy value $V(0, \mathbf{Y}(0))$. It should be noted that (5.17) is general, and does not depend on the simplifying assumptions made in the PDE derivation.

Algorithm 1 Recursive computation of $V(0, \boldsymbol{X}(0))$

choose a withdrawal strategy Γ
initialize $V(T^+, \boldsymbol{X}(T^+)) = 0$
set $n = N$
while $n > 0$ **do**
  compute the withdrawal amount γ_n by by (5.4)
  compute $V(t_n^-, \boldsymbol{X}(t_n^-))$ by applying jump condition (5.15) with appropriate cash flows
  compute $V(t_{n-1}^+, \boldsymbol{X}(t_{n-1}^+))$ by solving (5.17) or (5.20) with terminal condition $V(t_n^-, \boldsymbol{X}(t_n^-))$
  $n = n - 1$
end while
return $V(0, \boldsymbol{X}(0)) = V(0^+, \boldsymbol{X}(0^+))$

5.4 Calculating the Insurer's Liability Function

The GMWDB contract may be considered from the insurer's perspective by examining the insurer's liabilities, given by the risk-neutral value of the cash flows that must be paid by the insurer in order to fulfill the GMWDB contract. We define the *net* liability function $L(t, \boldsymbol{X}(t)), t \in [0, T]$ as the time-t risk-neutral value of all future payments on or after t made to the policyholder by the insurer, less that of all insurance fee incomes over the same period.

The insurance fees, charged at the rate $\alpha_{\text{ins}}(t)$, $t \in [0, T]$, is called fair if the total fees exactly compensate for the insurer's total liability, such that the net liability is zero at time $t = 0$. That is,

$$L(0, \boldsymbol{X}(0)) = 0. \tag{5.21}$$

If $\alpha_{\text{ins}}(t) \equiv \alpha_{\text{ins}}$ is a constant, its value can be found by solving (5.21). The fair insurance fees represent the hedging cost for the insurer to deliver the GMWDB guarantee to the policyholder, which is often regarded as the value of the GMWDB rider, at least from the insurer's perspective. We emphasize here that this value may not be equal to the added value of the GMWDB rider to the policyholder's wealth account, as we will show in Section 5.5.

On an event date t_n, the actual cash flow received by the policyholder is given by (5.5). This cash flow is first paid out of the policyholder's real withdrawal from the wealth account, which is equal to $\min(W(t_n), \gamma_n)$, the smaller of the nominal withdrawal and the available wealth. If the wealth account has an insufficient balance, the rest must be paid by the insurer. If the real withdrawal exceeds the cash flow entitled to the policyholder, the insurer keeps the surplus. The payment made by the insurer at t_n is thus given by

$$c_n(\gamma_n, \boldsymbol{X}(t_n)) = C_n(\gamma_n, \boldsymbol{X}(t_n)) - \min(W(t_n), \gamma_n). \tag{5.22}$$

To compute $L(t, \boldsymbol{X}(t))$ for all $t \in [0, T]$ we first note that at maturity T, the terminal condition on L is given by

$$L(T^+, \boldsymbol{X}(T^+)) = 0, \tag{5.23}$$

i.e., no further liability or insurance fee income after maturity. Analogous to (5.13) and (5.15), the jump condition of L on event date t_n is given by

$$L(t_n, \boldsymbol{X}(t_n)) = c_n(\gamma_n, \boldsymbol{X}(t_n)) + L(t_n^+, \boldsymbol{X}(t_n^+)) \tag{5.24}$$

and

$$L(t_n^-, \boldsymbol{X}(t_n^-)) = \Big((1-q_n)\Big(p_n(\gamma_n, \mathbf{Y}(t_n)) + L(t_n^+, \mathbf{Y}(t_n^+))\Big) + q_n d_n(\mathbf{Y}(t_n))\Big)\mathbb{1}_{\{I(t_n^-)=1\}}, \tag{5.25}$$

where the insurance payments under $I(t_n) = 1$ and $I(t_n) = 0$ are given by, respectively, $p_n(\gamma_n, \mathbf{Y}(t_n)) = P_n(\gamma_n, \mathbf{Y}(t_n)) - \min(W(t_n), \gamma_n)$ and $d_n(\mathbf{Y}(t_n)) = D_n(\mathbf{Y}(t_n)) - \min(W(t_n), \gamma_n)$. See (5.22), (5.5) and (5.4).

At $t \in (t_{n-1}, t_n)$, the net liability function is given by the risk-neutral value of the remaining liabilities at t_n^- less any insurance fee incomes over the period (t, t_n), discounted at the risk-free rate. Specifically, we have

$$L(t, \boldsymbol{X}(t)) = \mathrm{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^{t_n} r(s)ds} L(t_n^-, \boldsymbol{X}(t_n^-))\Big] - \mathrm{E}_t^{\mathbb{Q}}\Big[\int_t^{t_n} e^{-\int_t^s r(u)du} \alpha_{\text{ins}}(s) W(s) ds\Big]. \tag{5.26}$$

Note that the net liability at t is reduced by expecting to receive insurance fees over time. Since this reduction decreases with time, the net liability *increases* with time over (t_{n-1}, t_n).

To give an example, we again assume constant $r(t) \equiv r$, $\sigma(t) \equiv \sigma$, $\alpha_{\text{ins}}(t) \equiv \alpha_{\text{ins}}$, $\alpha_{\text{tot}}(t) \equiv \alpha_{\text{tot}}$. Under these simplifying assumptions we have $L(t, \boldsymbol{X}(t)) = L(t, W(t))$ for $t \in (t_{n-1}, t_n)$. To derive the PDE satisfied by $L(t, W)$, consider a delta hedging portfolio that, at time $t \in (t_{n-1}, t_n)$, consists of a long position in the net liability function L and a short position of $\frac{W(t)\partial_W L(t, W(t))}{S(t)}$ shares of the index S. The value of the delta hedging portfolio, denoted as $\Pi^L(t)$, is given by

$$\Pi^L(t) = L(t, W(t)) - W(t)\partial_W L(t, W(t)). \tag{5.27}$$

By Ito's formula and (5.1), we obtain the SDE for Π^L as

$$d\Pi^L(t) = \Big(\partial_t L(t, W(t)) - \alpha_{\text{tot}} W(t)\partial_W L(t, W(t)) + \frac{1}{2}\sigma^2 W(t)^2 \partial_{WW} L(t, W(t))\Big)\, dt, \tag{5.28}$$

where $t \in (t_{n-1}, t_n)$. Since the hedging portfolio Π^L is locally riskless and must grow at the risk-free rate r, as well as increase with the insurance fee income at rate $\alpha_{\text{ins}} W(t)$ (see remarks after (5.26)), we must also have $d\Pi^L(t) = \big(r\Pi^L(t) + \alpha_{\text{ins}} W(t)\big)\, dt$. This along with (5.27) implies that the PDE satisfied by the value function $L(t, W)$ is given by

$$\partial_t L - \alpha_{\text{ins}} W - rL + (r - \alpha_{\text{tot}}) W \partial_W L + \frac{1}{2}\sigma^2 W^2 \partial_{WW} L = 0, \tag{5.29}$$

for $t \in (t_{n-1}, t_n)$. The initial net liability can thus be computed by recursively solving (5.26) or (5.29) from terminal and jump conditions (5.23) and (5.25), as described in Algorithm 2.

Algorithm 2 Recursive computation of $L(0, \boldsymbol{X}(0))$

choose a withdrawal strategy Γ
initialize $L(T^+, \boldsymbol{X}(T^+)) = 0$
set $n = N$
while $n > 0$ **do**
    compute the withdrawal amount γ_n by (5.4)
    compute $L(t_{n-1}^-, \boldsymbol{X}(t_{n-1}^-))$ by applying jump condition (5.25) with appropriate cash flows
    compute $L(t_{n-1}^+, \boldsymbol{X}(t_{n-1}^+))$ by solving (5.26) or (5.29) with terminal condition $L(t_n^-, \boldsymbol{X}(t_n^-))$
    $n = n - 1$
end while
return $L(0, \boldsymbol{X}(0)) = L(0^+, \boldsymbol{X}(0^+))$

5.5 The Wealth Manager's Value Function and Optimal Withdrawals

In the previous sections, the withdrawal strategy Γ has been assumed to be given. The withdrawal strategy serves as a control sequence affecting the policyholder's value function and the insurer's liability function. These withdrawals may thus be chosen to maximize either of these functions, leading to two distinct withdrawal strategies. In this section we formulate these two strategies and discuss their relations. In particular, we identify the wealth manager's value function that connects the two perspectives.

5.5.1 The wealth manager's value function

We establish the relationship between the policy value V and the net liability L by defining the process

$$M(t, \boldsymbol{X}(t)) := L(t, \boldsymbol{X}(t)) + W(t) - V(t, \boldsymbol{X}(t)), \tag{5.30}$$

for $t \in [0, T]$. From (5.11) and (5.12) we obtain

$$M(T^+, \boldsymbol{X}(T^+)) = 0, \tag{5.31}$$

as the terminal condition for M. The jump condition for M can be obtained from (5.10), (5.13), (5.24) and (5.22) as

$$M(t_n, X(t_n)) = M(t_n^+, \boldsymbol{X}(t_n^+))), \tag{5.32}$$

and further more,

$$M(t_n^-, X(t_n^-)) = (1 - q_{n-1}) M(t_n^+, \boldsymbol{X}(t_n^+))) \mathbb{1}_{\{I(t_n^-)=1\}} \tag{5.33}$$

due to the possible death occurance of the policy holder over the last period. Note that the death information is not revealed until t_n.

From (5.17) and (5.26) we find the recursive relation for M as,

$$\begin{aligned} lM(t, \boldsymbol{X}(t)) = \mathrm{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^{t_n} r(s)ds} M(t_n^-, \boldsymbol{X}(t_n^-))\Big] \\ +W(t) - \mathrm{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^{t_n} r(s)ds} W(t_n)\Big] \\ -\mathrm{E}_t^{\mathbb{Q}}\Big[\int_t^{t_n} e^{-\int_t^s r(u)du} \alpha_{\text{ins}}(s) W(s) ds\Big]. \end{aligned} \tag{5.34}$$

Note that the second and third lines in (5.34) can be identified with the time-t risk-neutral value of management fees over (t, t_n). To see this, we first note that the difference of the two terms in the second line is the time-t risk-neutral value of the total fees charged on the wealth account over (t, t_n), and the expectation in the third line is the time-t risk-neutral value of the insurance fees over the same period.

In lights of (5.31), (5.33) and (5.34), the process $M(t, \boldsymbol{X}(t)), t \in [0, T]$ defined by (5.30) is precisely the time-t risk-neutral value of future management fees, or *wealth manager's value function*. From (5.30), the policy value may be written as

$$V(t, \boldsymbol{X}(t)) = W(t) + L(t, \boldsymbol{X}(t)) - M(t, \boldsymbol{X}(t)), \tag{5.35}$$

i.e., the sum of the wealth and the value of the GMWDB rider, less the wealth manager's value. At $t = 0$, this gives

$$V(0, \boldsymbol{X}(0)) = W(0) + L(0, \boldsymbol{X}(0)) - M(0, \boldsymbol{X}(0)). \tag{5.36}$$

Hence, a withdrawal strategy that maximizes the total net liability $L(0, \boldsymbol{X}(0))$, in general, is sub-optimal in maximizing the policyholder's total value $V(0, \boldsymbol{X}(0))$, since the wealth manager's total value $M(0, \boldsymbol{X}(0))$ depends on the withdrawals. The fair fee condition (5.21) becomes

$$V(0, \boldsymbol{X}(0)) = W(0) - M(0, \boldsymbol{X}(0)), \tag{5.37}$$

as in contrast to the $V(0) = W(0)$ condition often seen in the literature, when no management fees are charged.

5.5.2 Formulation of two optimization problems

We first formulate the policyholder's value maximization problem, i.e., maximizing the initial policy value $V(0, \boldsymbol{X}(0))$ by optimally choosing the sequence γ_n under $I(t_n) = 1$ for $n = 1, \ldots, N-1$. Following the principle of dynamic programming, this is accomplished by choosing the withdrawal γ_n as

$$\gamma_n = \Gamma^V(t_n, \mathbf{Y}(t_n)) = \arg\max_{\gamma \in \mathcal{A}} \big\{P_n(\gamma, \mathbf{Y}(t_n)) + V\left(t_n^+, \boldsymbol{X}(t_n^+ | \boldsymbol{X}(t_n), \gamma)\right)\big\} \tag{5.38}$$

in the admissible set $\mathcal{A} = \{\gamma : \gamma \geq 0, A(t_n^+ | \boldsymbol{X}(t_n), \gamma) \geq 0\}$. Here, we used the expressions $\boldsymbol{X}(t_n^+ | \boldsymbol{X}(t_n), \gamma)$ and $A(t_n^+ | \boldsymbol{X}(t_n), \gamma)$ to denote the state variables $\boldsymbol{X}$ and guarantee account balance A after withdrawal γ is made, given the value of the state variables $\boldsymbol{X}$ before the withdrawal. At any withdrawal time t_n the policyholder chooses the withdrawal $\gamma \in \mathcal{A}$ to maximize the sum of the payment $P_n(\gamma, \mathbf{Y}(t_n))$ received and the present value of the remaining term of the policy $V(t_n^+, \boldsymbol{X}(t_n^+))$. The strategy Γ^V given by (5.38) is called the *value maximization strategy*.

On the other hand, the optimization problem from the insurer's perspective considers the most unfavourable situation for the insurer. That is, by making suitable choices of γ_n's, the policyholder attempts to maximize the net initial liability function $L(0, \boldsymbol{X}(0))$. Even though a policyholder has little reason to pursue such a strategy, the fair fee rate under this strategy is guaranteed to cover the hedging cost of the GMWDB rider regardless of the withdrawal strategy of the policyholder (assuming the insurer can perfectly hedge the market risk). The withdrawal γ_n under $I(t_n) = 1$ for this strategy is given by

$$\gamma_n = \Gamma^L(t_n, \mathbf{Y}(t_n)) = \arg\max_{\gamma \in \mathcal{A}} \{p_n(\gamma, \mathbf{Y}(t_n)) + L\left(t_n^+, \boldsymbol{X}(t_n^+ | \boldsymbol{X}(t_n), \gamma)\right)\}, \quad (5.39)$$

i.e., the sum of the payment made by the insurer and the net liability of the remaining term of the contract is maximized. The strategy Γ^L given by (5.39) is called the *liability maximization strategy*.

5.6 Numerical Examples

To demonstrate the effect of management fees on the fair fees of GMWDB contracts under different withdrawal strategies, we carry out in this section several numerical experiments. We investigate how the presence of management fees will lead to different fair fees for the two withdrawal strategies studied in previous sections under different market conditions and contract parameters.

5.6.1 Setup of the experiments

For illustration purposes, we assume a simple GMWDB contract as specified by (5.6), (5.9) as well as constant r, σ, α_{m} and α_{ins} so that the PDEs (5.20) and (5.29) hold, and set all mortality rates to zero for simplicity. We consider different contractual scenarios and calculate the fair fees implied by (5.21) under the withdrawal strategies given in Section 5.5.

It is assumed that the wealth and the guarantee accounts start at $W(0) = A(0) = 1$. The maturities of the contracts range from 5 to 20 years, with annual contractual withdrawals evenly distributed over the lifetime of the contracts. The first withdrawal occurs at the end of the first year and the last at the maturity. The management fee rate ranges from 0% up to 2%.

We consider several investment environments with the risk-free rate r at levels 1% and 5%, and the volatility of the index σ at 10% and 30%, to represent different market conditions such as low/high growth and low/high volatility scenarios. In addition, the penalty rate β may take values at 10% or 20%.

We compute the initial policy value $V(0, \mathbf{Y}(0))$ as well as the initial net liability $L(0, \boldsymbol{X}(0))$ at time 0 numerically by following Algorithms 1 and 2 simultaneously. The withdrawal strategies Γ^L and Γ^V are considered separately. The PDEs (5.20) and (5.29) are solved using Crank-Nicholson finite difference method (Crank & Nicolson (1947); Hirsa (2012)) with appropriate terminal and jump conditions for both functions under both strategies. This leads to the initial values and liabilities $V(0, \boldsymbol{X}(0); \Gamma^L)$, $L(0, \boldsymbol{X}(0); \Gamma^L)$, $V(0, \boldsymbol{X}(0); \Gamma^V)$ and $L(0, \boldsymbol{X}(0); \Gamma^V)$ under both strategies. Here, we made the dependence of these functions on the strategies explicit. The fair fee rates under both strategies were obtained by solving (5.21) using a standard root-finding numerical scheme.

5.6.2 Results and implications

The fair fees and corresponding total policy values are shown in Figures 5.1 and 5.2 for two market conditions: a low interest rate market with high volatility ($r = 1\%, \sigma = 30\%$) and a high interest rate market with low volatility ($r = 5\%, \sigma = 10\%$), respectively. Fair fee rates obtained for all market conditions and contract parameters and the corresponding policy values can be found in Tables 5.1 through 5.4.

We first observe from these numerical results that the fair fee rate implied by the liability maximization strategy is always higher, and the corresponding policyholder's total value always lower, than those implied by the value maximization strategy, unless management fees are absent, in which case these quantities are equal. These are to be expected from the definitions of the two strategies. We also note that under the market condition of low interest rate with high volatility, a much higher insurance fee rate is required than under the market condition of high interest rate with low volatility, for the obvious reason that under adverse market conditions, the guarantee is more valuable. Moreover, a higher penalty rate results in a lower insurance fee since a higher penalty rate discourages the policyholder from making more desirable withdrawals that exceed the contracted values.

Furthermore, the results show that under most market conditions or contract specifications, the fair insurance fee rate obtained is highly sensitive to the management fee rate regardless of the withdrawal strategies, as seen from Figures 5.1 and 5.2. In particular, the fair fee rate implied by the liability maximization strategy always increases with the management fee rate, since the management fees cause the wealth account to decrease, leading to higher liability for the insurer to fulfill. On the other hand, the fair fee rate implied by the value maximization strategy first increases then decreases with the management fee rate, since at high management fee rates, a rational policyholder tends to withdraw more and early to avoid the management fees, which in turn reduces the liability and generates more penalty incomes for the insurer.

A major insight from the numerical results is that with increasing management fees, the value maximization withdrawals of a rational policyholder deviates more from the liability maximization withdrawals assumed by the insurer. In particular, it is seen by examining Figures 5.1 and 5.2 that the fair fee rates implied by the two strategies differ more significantly under the following conditions:

- longer maturity T,
- lower penalty rate β,
- higher interest rate r, and
- higher management fee rate α_{m}.

Moreover, careful examination of results listed in Tables 5.1 and 5.2 reveals that the index volatility σ does not contribute significantly to this discrepancy. These observations are intuitively reasonable: The contributors listed above all imply that the wealth manager's total value $M(0)$ will be higher. There are more incentives to withdraw early to achieve higher policy value in the form of reduced management fees. The corresponding differences between the policyholder's values follow similar patterns. Of particular interest is that in some cases, as shown in Figure 5.2, the fair fee rate implied by maximizing policyholder's value can become negative. This implies that the policyholder would want to withdraw more and early due to high management fees to such an extent, that the penalties incurred exceed the total value of the GMWDB rider. On the other hand, the fair fee rate implied by maximizing the liability is always positive.

5.7 Conclusions

Determining accurate hedging costs of VA guarantees is a significant issue for VA providers. While the effect of management fees on policyholder's withdrawal behaviors is typically ignored in the VA literature, it was demonstrated in this chapter, assuming the risk-neutral framework, that this effect on the pricing of GMWDB contract can be significant. As a form of market friction, management fees can affect policyholders' withdrawal behaviors, causing large deviations from the "optimal" (liability maximization) withdrawal behaviors often assumed in the literature.

Two different policyholder's withdrawal strategies were considered: liability maximization and value maximization when management fees are present. We demonstrated that these two withdrawal strategies imply different fair insurance fee rates, where maximizing policy value implies lower fair fees than those implied by maximizing liability, or equivalently, maximizing the "total value" of the contract, which represents the maximal hedging costs from the insurer's perspective.

We identify the difference between the initial investment plus the value of the guarantee and the total value of the policyholder as the wealth manager's total value, which causes the discrepancy between the two withdrawal strategies. We further identify a number of factors that contribute to this discrepancy through a series of illustrating numerical experiments. Our findings identify the management fees as a potential cause of discrepancy between the fair fee rates implied by the liability maximization strategy, often assumed from the insurer's perspective for VA pricing, and the prevailing market rates for VA contracts with GMWDB or similar riders.

Figures and Tables

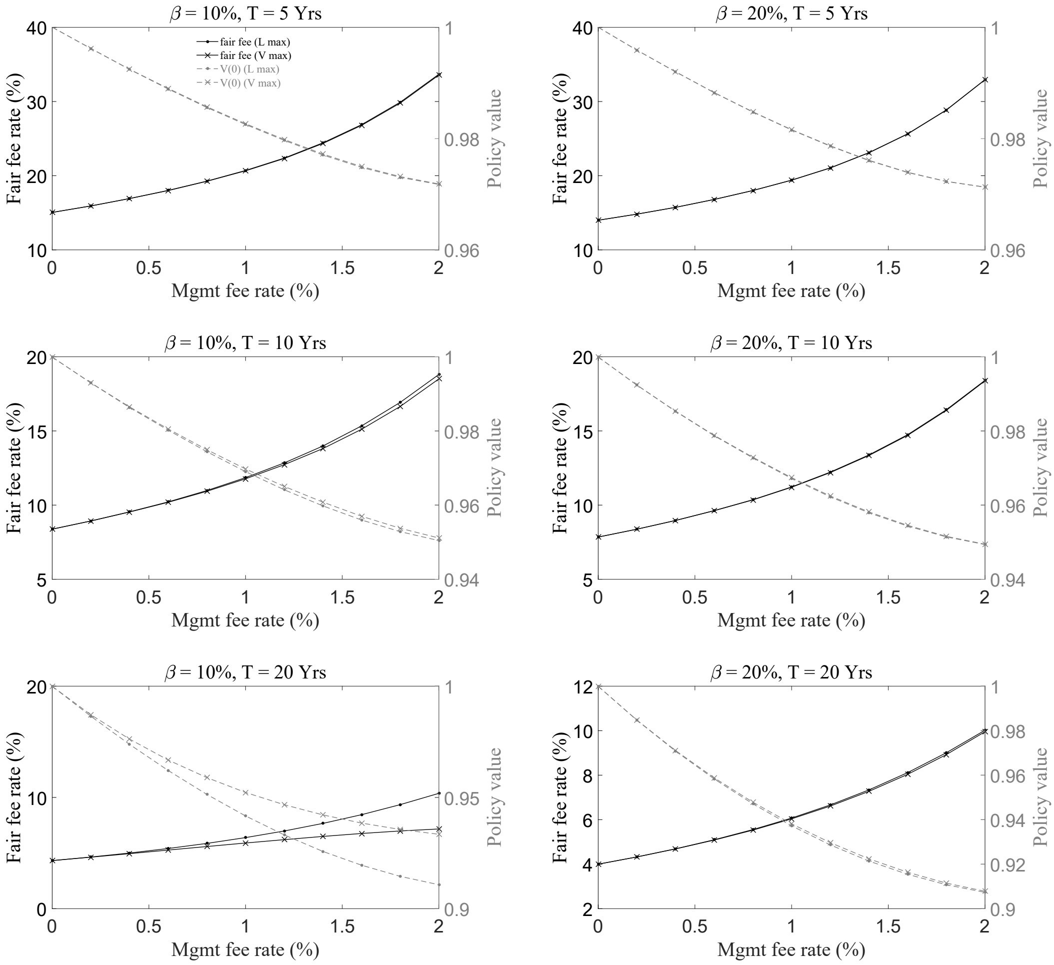


Figure 5.1 – Fair insurance fee rates and policy values as a function of management fee rates α_{m} for risk-free rate $r = 1\%$ and volatility $\sigma = 30\%$, for penalty rates $\beta = 10\%\,,20\%$ and maturities $T = 5\,,10\,,20$ years. The left axis and dark plots refer to the fair fees in percentage; The right axis and gray plots refer to the policy values. Legends across all plots are shown in the upper left panel.

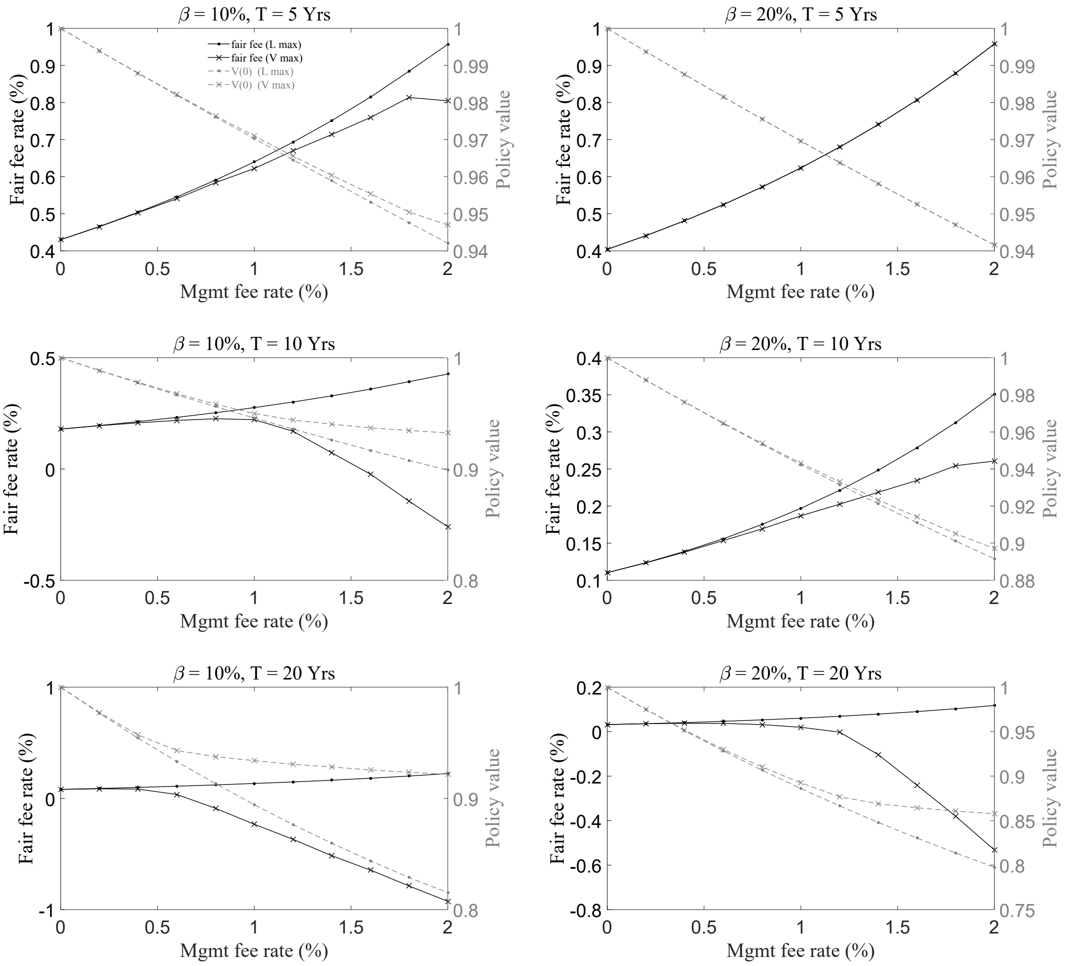


Figure 5.2 – Fair insurance fee rates and policy values as a function of management fee rates α_{m} for risk-free rate $r = 5\%$ and volatility $\sigma = 10\%$, for penalty rates $\beta = 10\%, 20\%$ and maturities $T = 5, 10, 20$ years. The left axis and dark plots refer to the fair fees in percentage; The right axis and gray plots refer to the policy values. Legends across all plots are shown in the upper left panel.

Table 5.1 – Fair fee rate α_{ins} (%) based on the liability maximization strategy Γ^L.

Parameters				α_{m}										
$r(\%)$	$\sigma(\%)$	$\beta(\%)$	T	0%	0.2%	0.4%	0.6%	0.8%	1%	1.2%	1.4%	1.6%	1.8%	2%
1	10	10	5	3.08	3.48	3.96	4.59	5.41	6.65	8.66	12.18	17.67	23.53	29.92
			10	1.66	1.92	2.25	2.66	3.21	3.97	5.08	6.77	9.26	12.17	15.31
			20	0.82	0.97	1.17	1.43	1.77	2.22	2.83	3.68	4.81	6.19	7.71
		20	5	3.08	3.47	3.95	4.55	5.34	6.48	8.39	12.01	17.66	23.55	29.92
			10	1.66	1.91	2.23	2.62	3.13	3.83	4.88	6.60	9.20	12.17	15.31
			20	0.81	0.96	1.16	1.40	1.71	2.13	2.72	3.57	4.74	6.17	7.71
	30	10	5	15.05	15.92	16.92	18.02	19.27	20.70	22.37	24.43	26.88	29.91	33.69
			10	8.38	8.93	9.55	10.22	10.98	11.85	12.84	13.98	15.34	16.93	18.81
			20	4.32	4.64	5.00	5.41	5.87	6.39	6.98	7.66	8.45	9.35	10.37
		20	5	13.99	14.82	15.73	16.80	18.01	19.40	21.06	23.11	25.64	28.86	32.95
			10	7.85	8.38	8.97	9.63	10.36	11.22	12.22	13.38	14.74	16.43	18.41
			20	4.00	4.33	4.69	5.10	5.55	6.07	6.66	7.34	8.11	9.01	10.02
5	10	10	5	0.43	0.47	0.50	0.55	0.59	0.64	0.69	0.75	0.81	0.88	0.96
			10	0.18	0.20	0.21	0.23	0.25	0.28	0.30	0.33	0.36	0.39	0.43
			20	0.08	0.09	0.10	0.11	0.12	0.13	0.15	0.16	0.18	0.20	0.22
		20	5	0.40	0.44	0.48	0.52	0.57	0.62	0.68	0.74	0.81	0.88	0.96
			10	0.11	0.12	0.14	0.16	0.18	0.20	0.22	0.25	0.28	0.31	0.35
			20	0.03	0.04	0.04	0.05	0.05	0.06	0.07	0.08	0.09	0.10	0.12
	30	10	5	5.33	5.48	5.65	5.81	5.99	6.17	6.35	6.55	6.75	6.96	7.17
			10	2.91	3.02	3.12	3.23	3.35	3.47	3.60	3.73	3.86	4.01	4.16
			20	1.58	1.65	1.74	1.82	1.91	2.00	2.10	2.21	2.32	2.43	2.55
		20	5	4.97	5.13	5.28	5.44	5.61	5.79	5.97	6.15	6.35	6.55	6.76
			10	2.27	2.35	2.43	2.52	2.61	2.71	2.81	2.91	3.02	3.13	3.25
			20	1.08	1.13	1.19	1.24	1.31	1.37	1.43	1.50	1.58	1.65	1.73

Table 5.2 – Fair fee rate α_{ins} (%) based on the policy value maximization strategy Γ^V.

Parameters				α_{m}										
$r(\%)$	$\sigma(\%)$	$\beta(\%)$	T	0%	0.2%	0.4%	0.6%	0.8%	1%	1.2%	1.4%	1.6%	1.8%	2%
1	10	10	5	3.08	3.47	3.96	4.57	5.36	6.57	8.55	12.12	17.66	23.55	29.93
			10	1.66	1.92	2.23	2.61	3.13	3.81	4.86	6.44	8.99	12.11	15.30
			20	0.82	0.96	1.09	1.21	1.33	1.44	1.50	1.59	1.65	1.70	1.81
		20	5	3.08	3.47	3.95	4.55	5.34	6.48	8.38	12.01	17.66	23.55	29.92
			10	1.66	1.91	2.23	2.62	3.12	3.80	4.84	6.56	9.19	12.17	15.31
			20	0.81	0.96	1.16	1.39	1.67	2.05	2.60	3.43	4.65	6.14	7.70
	30	10	5	15.05	15.92	16.91	18.00	19.25	20.66	22.32	24.34	26.79	29.80	33.58
			10	8.38	8.93	9.53	10.19	10.93	11.77	12.72	13.81	15.11	16.65	18.51
			20	4.32	4.63	4.94	5.27	5.59	5.90	6.20	6.49	6.76	6.99	7.16
		20	5	13.99	14.82	15.73	16.80	18.00	19.39	21.04	23.09	25.62	28.84	32.93
			10	7.85	8.38	8.96	9.62	10.35	11.20	12.19	13.34	14.70	16.38	18.37
			20	4.00	4.32	4.68	5.08	5.53	6.04	6.61	7.27	8.04	8.92	9.95
5	10	10	5	0.43	0.47	0.50	0.54	0.58	0.62	0.67	0.71	0.76	0.81	0.80
			10	0.18	0.19	0.21	0.22	0.23	0.22	0.17	0.07	-0.02	-0.14	-0.26
			20	0.08	0.09	0.08	0.04	-0.09	-0.23	-0.37	-0.51	-0.64	-0.79	-0.93
		20	5	0.40	0.44	0.48	0.52	0.57	0.62	0.68	0.74	0.81	0.88	0.96
			10	0.11	0.12	0.14	0.15	0.17	0.19	0.20	0.22	0.23	0.25	0.26
			20	0.03	0.04	0.04	0.04	0.03	0.02	-0.00	-0.10	-0.24	-0.38	-0.53
	30	10	5	5.33	5.48	5.64	5.79	5.95	6.11	6.27	6.43	6.60	6.75	6.94
			10	2.91	3.01	3.10	3.18	3.26	3.34	3.41	3.47	3.53	3.58	3.60
			20	1.58	1.64	1.68	1.72	1.74	1.75	1.75	1.74	1.74	1.73	1.71
		20	5	4.97	5.13	5.28	5.44	5.61	5.78	5.96	6.14	6.32	6.51	6.70
			10	2.27	2.35	2.43	2.51	2.59	2.67	2.74	2.82	2.88	2.94	3.00
			20	1.08	1.13	1.18	1.22	1.23	1.24	1.23	1.21	1.18	1.14	1.09

Table 5.3 – Total policy value $V(0, \boldsymbol{X}(0); \Gamma^L)$ based on the liability maximization strategy Γ^L.

Parameters				α_{m}										
$r(\%)$	$\sigma(\%)$	$\beta(\%)$	T	0%	0.2%	0.4%	0.6%	0.8%	1%	1.2%	1.4%	1.6%	1.8%	2%
1	10	10	5	1.00	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97	0.97	0.97
			10	1.00	0.99	0.98	0.97	0.97	0.96	0.95	0.95	0.95	0.95	0.95
			20	1.00	0.98	0.96	0.95	0.94	0.92	0.92	0.91	0.90	0.90	0.90
		20	5	1.00	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97	0.97	0.97
			10	1.00	0.99	0.98	0.97	0.96	0.96	0.95	0.95	0.95	0.95	0.95
			20	1.00	0.98	0.96	0.95	0.93	0.92	0.91	0.91	0.90	0.90	0.90
	30	10	5	1.00	1.00	0.99	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97
			10	1.00	0.99	0.99	0.98	0.97	0.97	0.96	0.96	0.96	0.95	0.95
			20	1.00	0.99	0.97	0.96	0.95	0.94	0.93	0.93	0.92	0.91	0.91
		20	5	1.00	1.00	0.99	0.99	0.98	0.98	0.98	0.98	0.97	0.97	0.97
			10	1.00	0.99	0.99	0.98	0.97	0.97	0.96	0.96	0.95	0.95	0.95
			20	1.00	0.98	0.97	0.96	0.95	0.94	0.93	0.92	0.92	0.91	0.91
5	10	10	5	1.00	0.99	0.99	0.98	0.98	0.97	0.96	0.96	0.95	0.95	0.94
			10	1.00	0.99	0.98	0.97	0.96	0.95	0.94	0.93	0.92	0.91	0.90
			20	1.00	0.98	0.95	0.93	0.91	0.89	0.88	0.86	0.84	0.83	0.82
		20	5	1.00	0.99	0.99	0.98	0.98	0.97	0.96	0.96	0.95	0.95	0.94
			10	1.00	0.99	0.98	0.96	0.95	0.94	0.93	0.92	0.91	0.90	0.89
			20	1.00	0.97	0.95	0.93	0.91	0.89	0.87	0.85	0.83	0.81	0.80
	30	10	5	1.00	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.96	0.96	0.96
			10	1.00	0.99	0.98	0.98	0.97	0.96	0.95	0.95	0.94	0.93	0.93
			20	1.00	0.98	0.97	0.96	0.94	0.93	0.92	0.91	0.90	0.89	0.88
		20	5	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.96	0.96	0.95	0.95
			10	1.00	0.99	0.98	0.97	0.96	0.95	0.94	0.93	0.92	0.91	0.91
			20	1.00	0.98	0.96	0.94	0.92	0.90	0.89	0.87	0.86	0.84	0.83

Table 5.4 – Total policy value $V(0, \boldsymbol{X}(0); \Gamma^V)$ based on the policy value maximization strategy Γ^V.

Parameters				α_{m}										
$r(\%)$	$\sigma(\%)$	$\beta(\%)$	T	0%	0.2%	0.4%	0.6%	0.8%	1%	1.2%	1.4%	1.6%	1.8%	2%
1	10	10	5	1.00	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97	0.97	0.97
			10	1.00	0.99	0.98	0.97	0.97	0.96	0.95	0.95	0.95	0.95	0.95
			20	1.00	0.98	0.97	0.96	0.95	0.95	0.94	0.94	0.94	0.93	0.93
		20	5	1.00	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97	0.97	0.97
			10	1.00	0.99	0.98	0.97	0.96	0.96	0.95	0.95	0.95	0.95	0.95
			20	1.00	0.98	0.96	0.95	0.93	0.92	0.91	0.91	0.90	0.90	0.90
	30	10	5	1.00	1.00	0.99	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97
			10	1.00	0.99	0.99	0.98	0.97	0.97	0.97	0.96	0.96	0.95	0.95
			20	1.00	0.99	0.98	0.97	0.96	0.95	0.95	0.94	0.94	0.94	0.93
		20	5	1.00	1.00	0.99	0.99	0.98	0.98	0.98	0.98	0.97	0.97	0.97
			10	1.00	0.99	0.99	0.98	0.97	0.97	0.96	0.96	0.95	0.95	0.95
			20	1.00	0.98	0.97	0.96	0.95	0.94	0.93	0.92	0.92	0.91	0.91
5	10	10	5	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.96	0.96	0.95	0.95
			10	1.00	0.99	0.98	0.97	0.96	0.95	0.94	0.94	0.94	0.93	0.93
			20	1.00	0.98	0.96	0.94	0.94	0.93	0.93	0.93	0.93	0.92	0.92
		20	5	1.00	0.99	0.99	0.98	0.98	0.97	0.96	0.96	0.95	0.95	0.94
			10	1.00	0.99	0.98	0.96	0.95	0.94	0.93	0.92	0.91	0.91	0.90
			20	1.00	0.97	0.95	0.93	0.91	0.89	0.88	0.87	0.86	0.86	0.86
	30	10	5	1.00	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.97	0.96	0.96
			10	1.00	0.99	0.98	0.98	0.97	0.97	0.96	0.96	0.95	0.95	0.95
			20	1.00	0.99	0.97	0.97	0.96	0.95	0.94	0.94	0.94	0.93	0.93
		20	5	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.96	0.96	0.95	0.95
			10	1.00	0.99	0.98	0.97	0.96	0.95	0.94	0.94	0.93	0.92	0.92
			20	1.00	0.98	0.96	0.94	0.93	0.92	0.91	0.90	0.89	0.88	0.88

Chapter 6

Optimal Hedging of Variable Annuities with Guarantees under the Benchmark Approach

6.1 Introduction

In the previous chapter we studied Variable annuities (VA) under the classical risk-neutral framework. Most of the conclusions there are most likely valid when pricing under the BA because short-term contracts yield under both approaches similar prices. However, since long-term contracts can be less expensively produced under the BA, it is interesting to see in the current chapter what effect that has. In this chapter we continue the study of VA guarantee pricing, under the BA framework, hoping to shed some light on the implications on the hedging of these products under the classical vs BA approaches.

VAs with guarantees of living and death benefits are offered by wealth management and insurance companies worldwide to assist individuals in managing their pre-retirement and post-retirement financial plans. These products take advantages of market growth while provide a protection of the savings against market downturns. The VA contract cash flows received by the policyholder are linked to the choice of investment portfolio (e.g. the choice of mutual fund and its strategy) and its performance while traditional annuities provide a pre-defined income stream in exchange for a lump sum payment.

As introduced in Chapter 5, a variety of VA guarantees can be added by policyholders at the cost of additional fees. These guarantees, generically denoted as GMxB, provide different types of protection against market downturns, shortfall of savings due to longevity risk or assurance of stability of income streams. Pricing of these products under the risk-neutral framework was introduced in Chapter 5.

The benchmark approach (BA) offers an alternative pricing theory. Under very general conditions, there exists in a given investment universe a unique growth-optimal portfolio (GP). The BA takes the GP as the numeraire, or benchmark, such that any benchmarked nonnegative portfolio price process assumes zero instantaneous expected returns. The GP assumes the highest expected instantaneous growth rate among all nonnegative portfolios in the investment universe, and maximizes the expected log-utility of the terminal wealth. It is a well-diversified portfolio that draws on all tradable risk factors and the corresponding risk premia to achieve the growth optimality. Following the real-world pricing formula under the BA, the real-world pricing of any given future cash flows represents the minimal possible replication cost of these cash flows. The BA is considered a generalization of the risk-neutral pricing theory, in that an equivalent risk-neutral pricing measure need not exist, and includes the latter as a special case under additional conditions that ensures existence of the equivalent measure.

In this chapter we consider a standard VA product with GMWB, which provides a guaranteed withdrawal amount per year until the maturity of the contract regardless of the investment performance. The total withdrawal amount is such that the initial investment is guaranteed to be returned over the life of the contract. In this chapter, we do not consider additional features such as death benefits, which adds to the complexity of the discussion and especially on the numerical simulations, without shedding additional lights on the comparison of the two approaches. Pricing of death benefits under the BA framework will be discussed in Chapter 7 under the context of life insurance policies.

Two classes of withdrawal strategies of the policyholder are often considered in the literature: a static withdrawal strategy under which the policyholder withdraw a predetermined amount on each withdrawal date; or a dynamic strategy where the policyholder "optimally" decides the amount of withdrawal at each withdrawal date depending on the information available at that date, where the optimality usually refers to the maximization of the value of the current and future cash flows. By assuming an optimal policyholder's withdrawal behavior, the pricing of the VA product corresponds to the hedging cost of the worst case scenario faced by the VA provider. In other words, the price of the VA product under the dynamic strategy provides an upper bound of hedging cost from the VA provider's perspective. It should be noted that the actual policyholders' withdrawal strategies could be far from optimal. See, e.g., Moenig & Bauer (2015).

Assuming that the policyholder takes the dynamic withdrawal strategy, that is, the optimal strategy that maximizes the present value of current and future cash flows of the VA product, the actual withdrawals still depend on the pricing method adopted by the policyholder. On the other hand, the VA provider, who maintains a hedging portfolio to deliver the liability cash flows of the VA product also faces the same choices for pricing and the hedging strategies for the portfolio. In this chapter, we consider two pricing methods, the risk-neutral pricing approach and the BA. We investigate the outcomes and implications of different choices of withdrawal and hedging strategies by the policyholder and the VA provider. In particular, we study empirically the cases when the two parties take different pricing approaches. In the VA pricing literature, it is most often the case that the same pricing model is adopted by both the policyholder and the VA provider. The important situation where the policyholder and the VA provider hold fundamentally different valuation perspectives, as is described in this chapter, has not been investigated. This chapter attempts to fill this vacancy and hopefully initiate more interests in this direction.

The chapter is organized as follows. In Section 6.2 we present the contract details of the GMWB guarantee together with its pricing formulation under a stochastic optimal control framework. Section 6.3 formulates the VA pricing problem with optimal policyholder's withdrawals under a general framework, and describes the numerical algorithm to solve the problem. In Section 6.4 we introduce the pricing models under the risk-neutral approach and the BA. In Section 6.5 we empirically test these pricing and corresponding withdrawl and hedging strategies respectively for the policyholder and the VA provider, when the two parties use the same and different pricing approaches. Section 6.6 concludes with remarks and discussion. Chapter 6 is based on Sun et al. (2019).

6.2 Description of the VA Guarantee Product

In this section we describe the VA product with GMWB in detail. The product under consideration is similar to that described in Section 5.2, with some different details. To avoid any potential confusion, we give a complete description here at the cost of some repetition over Section 5.2.

We consider the VA product where a policy holder invests at time $t = 0$ a lump-sum of $W(0)$ into a wealth account $W(t), t \in [0, T]$ that tracks an equity index $S(t), t \in [0, T]$, where $t = T$ corresponds to the expiry date of the VA contract. We assume both $W(t)$ and $S(t)$ are discounted by the risk-free savings account, as are all other values of wealth encountered in the sequel. The (discounted) equity index evolves under the real-world probability measure $\mathbb{P}$ the SDE

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)dB(t), \qquad t \in [0, T], \tag{6.1}$$

where $\mu(t)$ and $\sigma(t)$ are the instantaneous market risk premium and volatility of the index, and $B(t), t \in [0, T]$ is a standard $\mathbb{P}$-Brownian motion driving the traded market uncertainties.

The policy holder selects a GMWB rider in order to protect his wealth account $W(t), t \in [0, T]$ over the lifetime of the VA contract. The GMWB contract allows the policyholder to withdraw from a guarantee account $A(t), t \in [0, T]$, on a sequence of pre-determined contract event dates, $0 = t_0 < t_1 < \cdots < t_N = T$. The initial guarantee $A(0)$ matches the initial wealth $W(0)$. We assume here that the guarantee account stays constant over time, unless a withdrawal is made on one of the event dates, which reduces the guarantee account balance. Other forms of guaranteed returns can be modelled similarly. For simplicity, we do not include in our discussion features such as death or early surrender benefits. Under a more realistic setting, these additional features can be included straightforwardly within the framework described in this chapter.

To simplify notations, we denote by $\boldsymbol{X}(t)$ the vector of state variables at t, given by

$$\boldsymbol{X}(t) = (\mu(t), \sigma(t), S(t), W(t), A(t)), \quad t \in [0, T]. \tag{6.2}$$

Here, we assume that all state variables follow Markov processes, so that $\boldsymbol{X}(t)$ contains all the market and account balances information available at t. For simplicity, we assume the state variables $\mu(t)$, $\sigma(t)$ and $S(t)$ are continuous, and $W(t)$ and $A(t)$ are left continuous with right limit (LCRL).

On event dates $t_n, n = 1, \ldots, N$, a nominal withdrawal γ_n from the guarantee account is made. The policyholder may choose $\gamma_n \leq A(t_n)$ on $t_n < T$. Otherwise a liquidation withdrawal of $\max(W(t_n), A(t_n))$ is made. That is,

$$\gamma_n = \begin{cases} \Gamma(t_n, \boldsymbol{X}(t_n)) & n < N, \\ \max\left(W(t_n), A(t_n)\right) & n = N, \end{cases} \tag{6.3}$$

where $\Gamma(\cdot, \cdot)$ is referred to as the *withdrawal strategy* of the policyholder. The real cash flow received by the policyholder, which may differ from the nominal amount, is denoted by $C_n(\gamma_n, \boldsymbol{X}(t_n))$. This is given by

$$C_n(\gamma_n, \boldsymbol{X}(t_n)) = \begin{cases} \gamma_n - \beta \max(\gamma_n - G_n, 0), & n < N, \\ \max\left(W(t_n), A(t_n)\right) & n = N, \end{cases} \tag{6.4}$$

where G_n is a pre-determined withdrawal amount specified in the GMWB contract, and β is the penalty rate applied to the part of the withdrawal from the guarantee account exceeding the contractual withdrawal G_n.

Upon withdrawal by the policyholder, the guarantee account is reduced by the nominal withdrawal γ_n, that is,

$$A(t_n^+) = A(t_n) - \gamma_n, \tag{6.5}$$

where $A(t_n^+)$ denotes the guarantee account balance "immediately after" the withdrawal. Note that $A(t_n^+) \geq 0$. The wealth account is thus reduced by the amount $\min(\gamma_n, W(t_n))$, and remains nonnegative. That is,

$$W(t_n^+) = \max(W(t_n) - \gamma_n, 0), \tag{6.6}$$

where $W(t_n^+)$ is the wealth account balance immediately after the withdrawal. It is assumed that $\gamma_0 = 0$, i.e., no withdrawals at the start of the contract. Both the wealth and the guarantee account balance are 0 after contract expiration. That is

$$W(T^+) = A(T^+) = 0. \tag{6.7}$$

Throughout the VA contract, the wealth account is charged an insurance fees continuously at rate α for the GMWB rider by the insurer to pay for the hedging cost of the guarantee. Discrete fees may be modelled similarly without any difficulty. The wealth account in turn evolves as

$$\frac{dW(t)}{W(t)} = (\mu(t) - \alpha(t))dt + \sigma(t)dB(t), \tag{6.8}$$

for any $t \in [0, T]$ at which no withdrawal of wealth is made. Here, $\alpha_{\text{tot}}(t) = \alpha_{\text{ins}}(t) + \alpha_{\text{m}}(t)$ is the total fee rate.

We denote the VA *plus* guarantee value function at time t by $V(t, \boldsymbol{X}(t)), t \in [0, T]$, which corresponds to the present value of all future cash flows entitled to the policyholder on or after the current time t. The remaining value after the final cash flow is obviously 0, i.e.,

$$V(T^+, \boldsymbol{X}(T^+)) = 0. \tag{6.9}$$

6.3 Pricing of the VA with Guarantee

In this section we consider the pricing of the VA product described in Section 6.2. For the pricing of a given set of cash flows, we adopt the concept of a stochastic discount factor (SDF), where the present value of the cash flows are given by the sum of their expected values, after discounting by the SDF, conditional on all current information. Both risk neutral pricing and pricing under BA can be formulated in terms of an appropriate SDF. For more information on risk-neutral pricing theory, see Delbaen & Schachermayer (2006) for an account.

To price the VA with guarantee value function $V(0, \boldsymbol{X}(0))$, we note that no withdrawal is made between any withdrawal dates (t_{n-1}, t_n), and that the wealth account is self-financing within this period. This leads to the following recurrence relation for the guarantee value function,

$$V(t_{n-1}^+, \boldsymbol{X}(t_{n-1}^+)) = E_{t_{n-1}^+}\left(D(t_{n-1}, t_n)V(t_n, \boldsymbol{X}(t_n))\right), \tag{6.10}$$

where $E_t(\cdot) = E(\cdot|\boldsymbol{X}(t))$ is the expectation under the real-world measure $\mathbb{P}$, conditional on the current information represented by $\boldsymbol{X}(t)$, and $D(t,u), 0 \leq t < u \leq T$ is the SDF over (t,u). Under the BA, the SDF is given by $D(t,u) = \frac{S(t)}{S(u)}$, i.e., the ratio of the inverse GP. Under the risk-neutral pricing framework, the SDF $D(t,u) = \frac{Z(u)}{Z(t)}$, with $Z(t) := E_t\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)$ being the Radon-Nikodym derivative of the measure change from the real-world probability measure $\mathbb{P}$ to the equivalent risk-neutral measure $\mathbb{Q}$, conditional on all available information at t.

Upon withdrawal at time t_n, $0 < n < N$, the value function satisfies the following jump condition

$$V(t_n, \boldsymbol{X}(t_n)) = V(t_n^+, \boldsymbol{X}(t_n^+)) + C(\gamma_n, \boldsymbol{X}(t_n)). \tag{6.11}$$

In other words, the guarantee value immediately before the withdrawal is the sum of the value immediately after the withdrawal and the cash flow of the withdrawal. The active policy holder follows a dynamic strategy. That is, for $0 < n < N$, the withdrawal amount γ_n is chosen according to the following total value maximizing strategy,

$$\gamma_n = \Gamma(t_n, \boldsymbol{X}(t_n)) = \underset{0 \leq \gamma \leq A(t_n)}{\arg\max} \left\{ V(t_n^+, \boldsymbol{X}(t_n) \setminus \gamma) + C(\gamma, \boldsymbol{X}(t_n)) \right\}, \tag{6.12}$$

where $\boldsymbol{X}(t_n) \setminus \gamma$ denotes the state variables $\boldsymbol{X}(t_n^+)$ after withdrawal γ is made, given the value of the state variables $\boldsymbol{X}(t_n)$ before the withdrawal. On the other hand, the passive policy holder follows a static strategy of pre-determined withdrawal values. The contract value $V(0, \boldsymbol{X}(0))$ can thus be computed recursively from (6.9), (6.5), (6.6) (6.11) and (6.10), along with the chosen strategy. These procedures are summarized in Algorithm 3, which is replicated from Algorithm 1, with references to relevant equations replaced by those from the current chapter. Alternatively, for the static strategy, cash flows of each withdrawals may be computed individually and the contract value $V(t, \boldsymbol{X}(t))$ is given by the sum of values of all future cash flows.

Algorithm 3 Recursive computation of $V(0, \boldsymbol{X}(0))$

initialize $V(T^+, \boldsymbol{X}(T^+)) = 0$
set $n = N$
while $n > 0$ **do**
  compute the withdrawal amount γ_n with the optimal strategy (6.12) or a pre-determined static strategy
  compute $V(t_n, \boldsymbol{X}(t_n))$ by applying jump condition (6.11) with appropriate cash flows
  compute $V(t_{n-1}^+, \boldsymbol{X}(t_{n-1}^+))$ by computing (6.10) with terminal value $V(t_n, \boldsymbol{X}(t_n))$ and the appropriate SDF $D(t_{n-1}, t_n)$
  set $n = n - 1$
end while
return $V(0, \boldsymbol{X}(0)) = V(0^+, \boldsymbol{X}(0^+))$

The contract value under the dynamic strategy is bounded from below by the corresponding value from any simple strategy such as the static one. If a closed-form pricing formula for the simple strategy value is available, Algorithm 3 can be easily modified to compute the optimal withdrawal premium on top of the suboptimal value of the simple strategy.

6.4 Modelling the Underlying Equity Index

In this section we specify the model parameters of the SDE governing the underlying equity index (6.1). The model parameters are described respectively under the risk-neutral pricing framework and the BA. For simplicity, we assume that the risk-free interest rate is identically zero. This is equivalent to taking the risk-free security account as the numeraire, and considering all prices denominated in units of the risk-free security account.

6.4.1 The Black-Scholes model

The classical BSM is probably the most widely known model to describe the price dyanmics of a risky security within the framework of risk-neutral pricing theory. Under the BSM, the equity index follows under the real-world probability measure $\mathbb{P}$ the geometric Brownian motion,

$$S(t) = S(0)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B(t)\right), \quad t \in [0,T], \tag{6.13}$$

where $S(0) > 0$ and μ, σ are constant risk premium and volatility parameters, and B is a standard $\mathbb{P}$-Brownian motion. The BSM admits a unique equivalent risk neutral pricing measure $\mathbb{Q}$ with the Radon-Nikodym derivative given by

$$Z(t) = e^{-\frac{\mu}{\sigma}B(t) - \frac{\mu^2}{2\sigma^2}t} = \left(\frac{S_0}{S_t}\right)^{\frac{\mu}{\sigma^2}} e^{\left(\frac{\mu^2}{2\sigma^2} - \frac{\mu}{2}\right)t}, \quad t \in [0,T]. \tag{6.14}$$

Under the risk-neutral measure $\mathbb{Q}$, the index S is driftless and satisfies

$$dS(t) = \sigma S(t) dB^{\mathbb{Q}}(t), \quad t \in [0,T], \tag{6.15}$$

where $B^{\mathbb{Q}}(t) = B(t) + \frac{\mu}{\sigma}t$ is a standard $\mathbb{Q}$-Brownian motion.

6.4.2 The minimal market model

We propose the minimal market model (MMM) as described in Section 2.1 for the dynamics of the underlying equity index. See Sections 2.1, 2.2 for more details.

6.5 Empirical Studies on Historical Equity Index Data

In this section, we conduct backtests of the proposed pricing models for VA products. We consider both from the policy holder's perspective, where he or she decides the optimal withdrawal amount based on the pricing model chosen by the policy holder, and from the VA provider's perspective, where the strategy of the hedging portfolio for the VA product is based on the pricing model chosen by the provider.

We take the S&P 500 index as the equity index underlying the VA product, and run simulated withdrawals and hedging strategies on the historical data of the index. In particular, we take the historically observed monthly prices of the S&P500

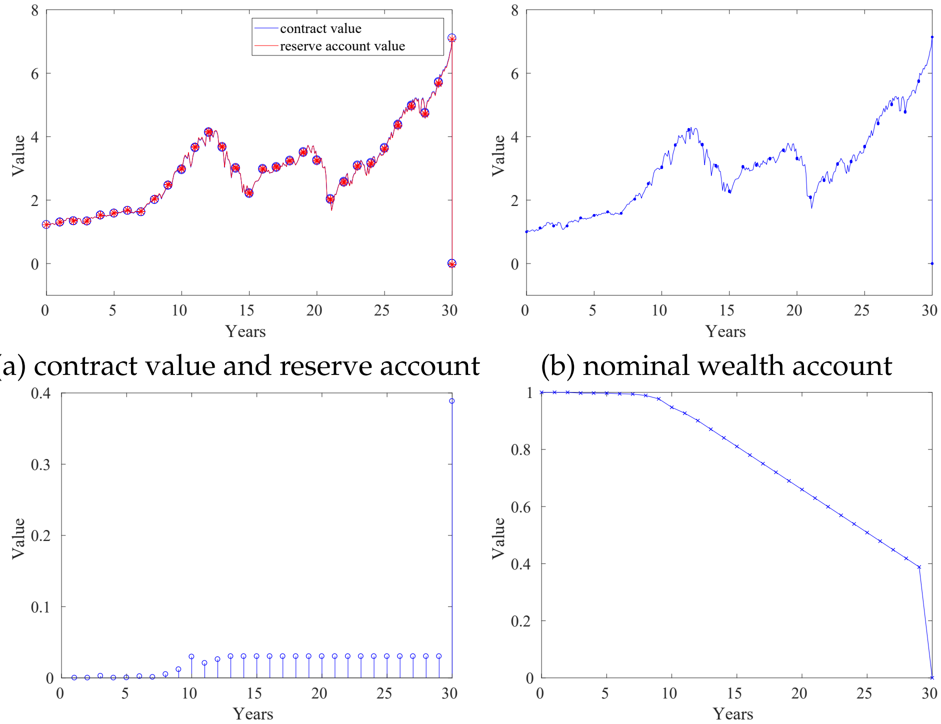
index from 1871 to 2018, with all dividends reinvested, and discounted by the locally risk-free savings account, see Section 2.2 for details. We consider the pricing of a GMWB contract written on a VA account tracking the index over the 30-year period from Feb. 1988 to Feb. 2018, based on the underlying models estimated from the historical index prices prior to this period. We consider both the MMM and the BSM as the underlying dynamics, and compare the evolution of the guarantee values under both models. The BSM parameters were estimated using standard sample mean and variance of the index log returns. The MMM parameters were estimated as described in Section 2.2.

We consider a stylized VA contract with GMWB where the policy holder invests \$1 Million in a mutual fund that tracks the S&P 500 total return index. For simplicity, we assume there are no mutual fund management fees. The policy holder purchases a GMWB rider that guarantees a return of the initial investment of \$1 Million over a period of 30 years. The contracted withdrawals are therefore rated at \$33,333 per annum. If the policy holder decided to withdraw more than the contracted amount, a penalty charge of 10% should apply to the excess part of the withdrawal. We assume the penalty charge also applies to the last withdrawal. That is, if the balance of the guarantee account exceeds the contracted withdrawal amount at maturity, withdrawal of this balance is mandatory and the same penalty rate applies to the excess part.

We first consider the situation where the VA provider prices the product under the BSM with the risk-neutral pricing approach, assuming that the policy holder makes optimal withdrawals under the same pricing model, which the policy holder actually does. The VA provider maintains a nominal wealth account $W(t)$ of the policy holder's wealth, and a nominal guarantee account $A(t)$ to keep track of the remaining guaranteed withdrawal allowance. The VA provider maintains an actual hedging portfolio $V(t)$ consisting of shares of the index-tracking mutual fund, or the index for short, and the risk-free security account. We refer to the hedging portfolio as the reserve account of the VA product, which is the only real investment account involved. The reserve account starts at value $V(0)$, the initial price of the VA product, and maintains a self-financing hedging strategy until a withdrawal is made on one of the withdrawal dates t_n, when the actual cash flow C_n is paid out of this account to the policy holder. The strategy maintained by the reserve account between withdrawal dates is the delta-hedging strategy.

Following Algorithm 3 described in Section 6.3, we compute recursively the price process of the VA product, based on the historical index prices. The price process is shown in Figure 6.1 (a), with an initial value of 1.2223 M, and a terminal value of 7.1051 M before the final liquidation. The reserve account process, realized through delta-hedging, is shown in the same plot for comparison. The initial value of the reserve account is the same as the price process, and the terminal value is 7.0707 M. After liquidation, the reserve account ended up with a small deficit of -0.0344 M, possibly due to hedging errors from discrete hedging.

The nominal wealth account $W(t)$ is shown in Figure 6.1 (b), the optimal withdrawals made by the policy holder are shown in Figure 6.1 (c), and the guarantee account balance is shown in Figure 6.1 (d). Note that both the wealth account and the guarantee account are nominal only, used for keeping track of the status of the policy holder's VA contract. No actual trading happens to these accounts. The optimal withdrawals are relatively uniform, except for no withdrawals in the beginning periods, and a large withdrawal in the last period. The relatively uniform withdrawal behavior is typical for the BSM, where the equity index dynamics is time-homogeneous. The only motivations to change withdrawal behaviors are changing date to maturity and wealth / guarantee account raio.



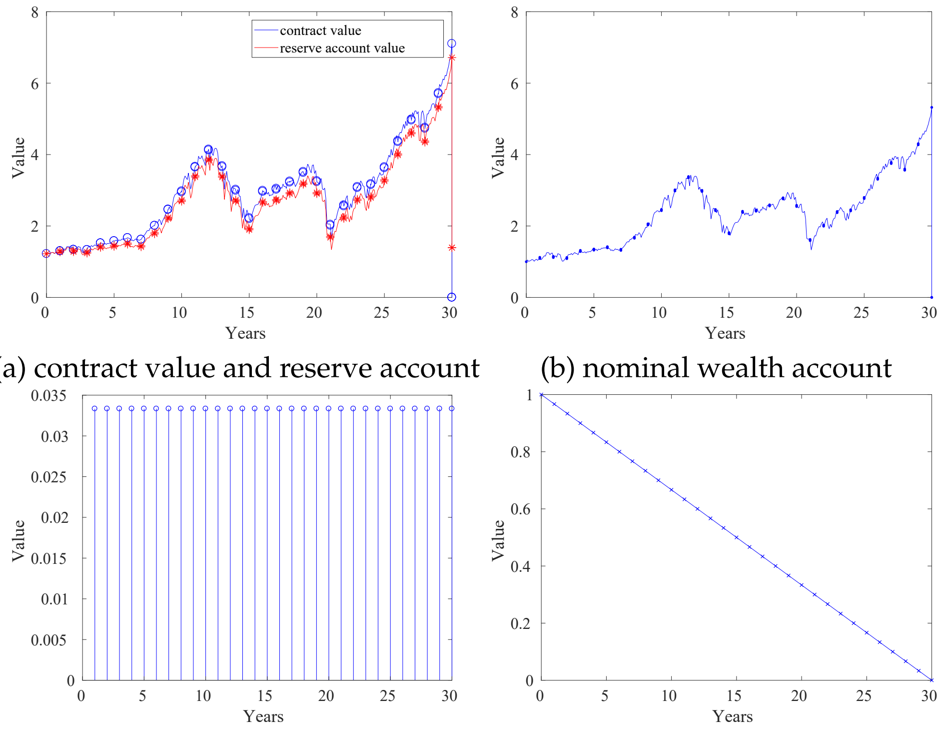
(a) contract value and reserve account (b) nominal wealth account

(c) withdrawals (d) nominal guarantee account

Figure 6.1 – Value processes associated with the VA product when the pricing and hedging as well as optimal withdrawals are performed based on the BSM and risk-neutral pricing.

For verification purposes we consider the alternative static withdrawal behavior from the policy holder. That is, we assume the policy holder makes a uniform withdrawal equal to $1/N$ on any one of the N withdrawal dates. On the other hand, the VA provider manages the reserve account in the same way as in the optimal case irrespective of the policy holder's withdrawal behavior, which is considered suboptimal in this case. The results are shown in Figure 6.2. The contract value process in this case is the same as the case with optimal withdrawals. The reserve account process as well as the nominal wealth account process differ from the previous case due to different (suboptimal) withdrawals. The reserve account ended up with a rather significant surplus of 1.3945 M. This is due to the loss made by the policy holder for withdrawing suboptimally. In particular, premature withdrawals led to less wealth accumulations in the nominal wealth account, leading to significantly less liquidation cash flow entitled to the policy holder. Since the reserve account maintained the same hedging strategy as in the previous case, it ended up having a surplus after paying the reduced liabilities.

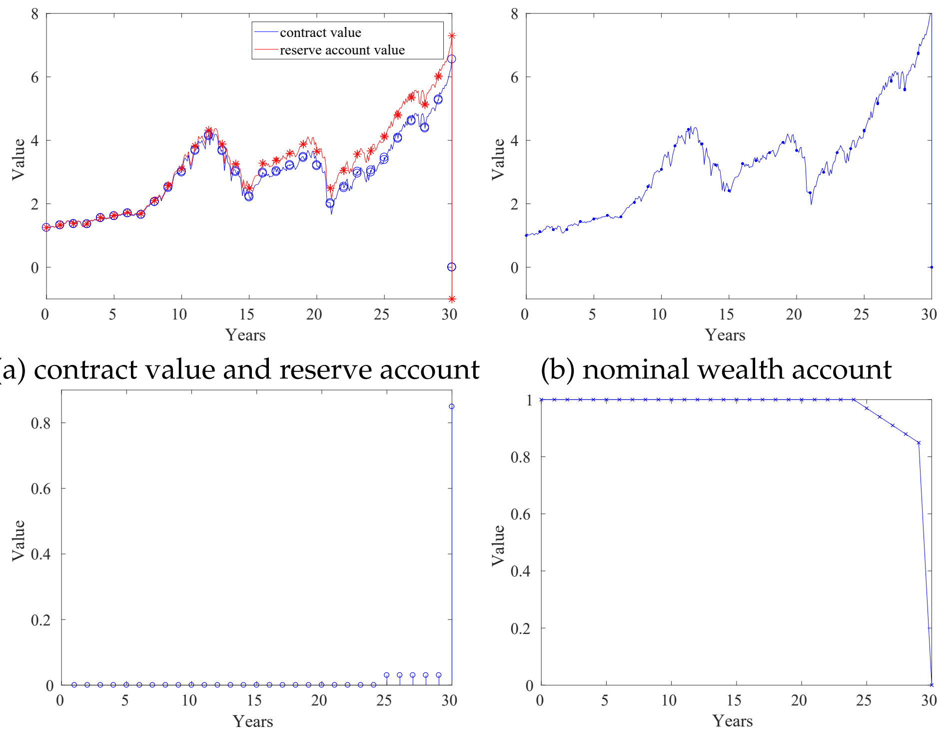
We next consider the situation where the policy holder makes withdrawals based on the MMM under the BA. That is, the policy holder's withdrawals maximize the value of the VA contract as priced by the MMM under the BA. The VA provider, believing in the BSM under the risk-neutral pricing framework, manages the reserve account in the same way as the previous cases, and views the policy holder's withdrawals as being suboptimal. The VA provider thus expects to receive a surplus in the reserve account after maturity of the VA contract. The outcomes of this scenario are shown in Figure 6.3.



(a) contract value and reserve account (b) nominal wealth account

(c) withdrawals (d) nominal guarantee account

Figure 6.2 – Value processes associated with the VA product when the pricing and hedging are performed based on the BSM and risk-neutral pricing assuming an optimal withdrawal behavior, while the actual withdrawal behavior follows the static strategy.



(a) contract value and reserve account (b) nominal wealth account

(c) withdrawals (d) nominal guarantee account

Figure 6.3 – Value processes associated with the VA product when the pricing and hedging are performed based on the BSM and risk-neutral pricing assuming an optimal withdrawal behavior, while the actual withdrawal behavior follows the MMM under the BA.

To the surprise of the VA provider, instead of having a surplus, the reserve account in this case ended up with a deficit of 1 M, as indicated in Figure 6.3 (a). The withdrawal behavior of the policy holder is such that there are no withdrawals until the very end of the contract term, where a number of small withdrawals were followed by a large withdrawal on the maturity date. The nominal wealth account accumulated to a high level of wealth due to no withdrawals in the early stages. The liquidation of this large wealth led to the deficit of the reserve account, which followed a hedging strategy assuming more early withdrawals.

The failure of the VA provider in hedging the VA product, when the policy holder behaved optimally under a different pricing framework indicates the potential inappropriateness of the BSM and risk-neutral pricing adopted by the VA provider. In particular, the policy holder believed in the long-term growth of the market and invested for this growth according to the MMM. The VA provider, from a risk-neutral perspective, did not recognize the long-term growth, and managed the reserve account with a short-term vision, leading to the failure of matching the performance of the policy holder's wealth account. Note that the MMM and the associated long-term growth rate were estimated from prior returns of the index. Thus no "looking into the future" is associated with the policy holder's withdrawal behavior.

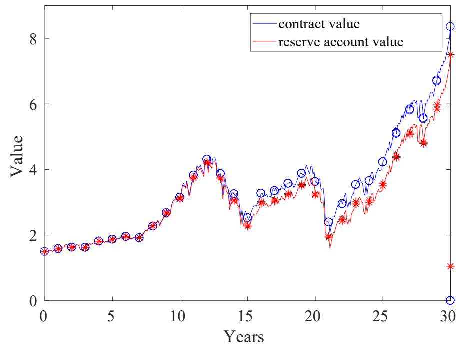
It is interesting to see what happens in a reversed scenario, where the VA provider prices and hedges under the MMM and BA, and the policy holder makes optimal withdrawals according to the BSM and risk-neutral pricing. Without repeating the detailed description of this scenario, the outcomes are shown in Figure 6.4, where the reserve account was managed recognizing the long-term growth under the MMM and BA, leading to a higher level of wealth accumulation than the nominal wealth account, and a surplus of 1.05 M. The withdrawal behavior of the policy holder is similar to the first case considered, with a rather uniform withdrawal a few years into the contract term.

Finally, to complete the empirical study, we consider the scenario when both the VA provider and the policy holder follow the MMM under the BA. The outcomes are shown in Figure 6.5. It can be seen that the VA provider in this case successfuly hedged the VA product, ending up with a small deficit of 0.0319 M in the reserve account.

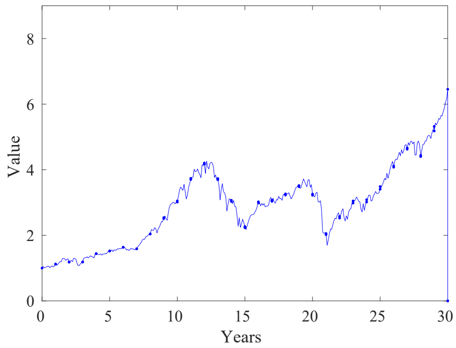
6.6 Summary

We considered the pricing of VA with GMWB under the BA, where a classical risk-neutral pricing measure may not exist. We rely on the real-world pricing formula to compute the value of the VA contract with GMWB under the MMM, and the associated hedging strategy for the VA product. We compare our results with those under the classical BSM under the risk-neutral pricing framework, through empirical backtests on the historical prices of the S&P 500 total return index, which were taken as the underlying of the VA product.

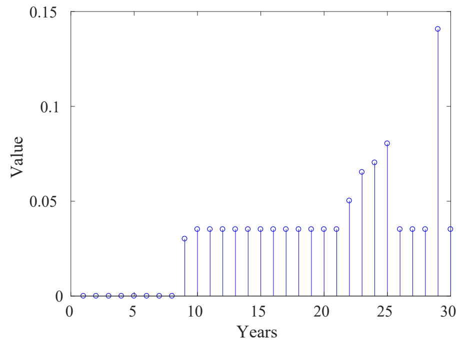
From the empirical studies, we found that the VA provider can successfully hedge the VA product when the VA provider and the policy holder rely on the same pricing model to make hedging and withdrawal decisions. When the policy holder takes a static withdrawal strategy without optimizing, the VA provider ended up with a surplus. When the VA provider relies on the MMM and the BA to make hedging decisions and the policy holder takes the BSM and risk-neutral approach, the VA provider ended up with a surplus. However, if the VA provider takes the BSM and risk-neutral approach, and the policy holder relies on MMM and the BA,



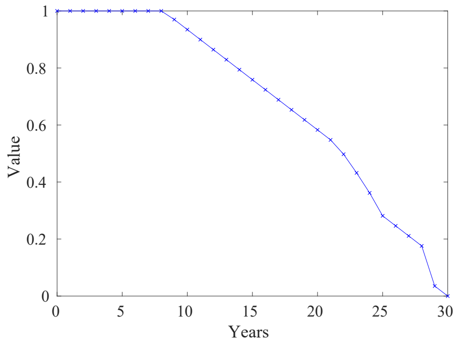
(a) contract value and reserve account



(b) nominal wealth account

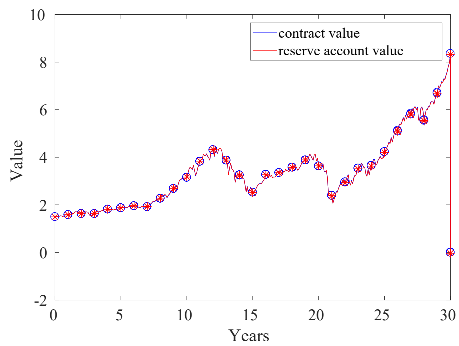


(c) withdrawals

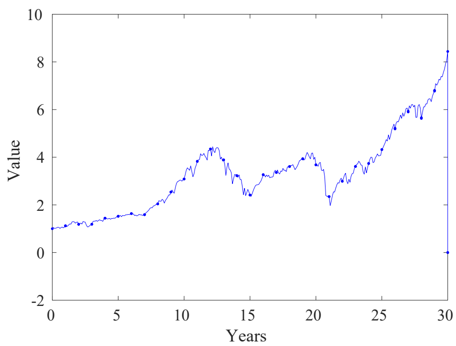


(d) nominal guarantee account

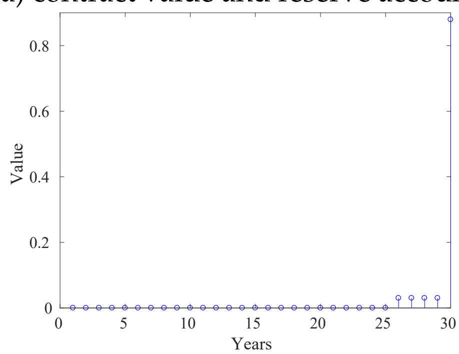
Figure 6.4 – Value processes associated with the VA product when the pricing and hedging are performed based on the MMM and BA assuming an optimal withdrawal behavior, while the actual withdrawal behavior follows the BSM under risk-neutral pricing.



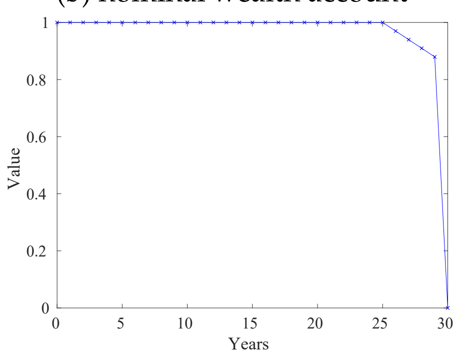
(a) contract value and reserve account



(b) nominal wealth account



(c) withdrawals



(d) nominal guarantee account

Figure 6.5 – Value processes associated with the VA product when the pricing and hedging as well as the optimal withdrawals are performed based on the MMM and the BA.

the VA provider ended up with a deficit. Our empirical studies show that the BSM and risk-neutral approach to the VA pricing problem may not be appropriate, in that when a sophisticated policy holder armed with a model under the more general BA framework, the VA provider risks having a significant deficit in the hedging of the VA product, at least in the historical scenario considered in this chapter. In general, risk-neutral models are more restrictive due to the additional requirement of admitting a risk-neutral pricing measure, thus generally less "faithful" agreement with the empirical observations should be expected, given that the two models are of comparable complexity. Here again we remind that the existence of the risk-neutral measure is not a necessary condition for market viability, as discussed earlier in Chapters 1-3.

Chapter 7

Benchmarked Risk Minimizing Hedging Strategies for Life Insurance Policies

Standard life insurance policies, such as term insurance policies, which pay a lump sum benefit upon death of the policyholder during a specified term; and pure endowment policies, which pay a lump sum upon survival of the policyholder at the end of a specified term; or the combination of the two, are traditionally priced under the actuarial principle of equivalence. That is, the benefits paid and the premiums received by the insurer are matched on average, after discounting by the (locally) risk-free savings account. See, e.g., Dickson et al. (2013), for a comprehensive introduction of classical actuarial pricing theory for life contingencies.

As policyholders become more interested in investment returns on their premiums, combined insurance and savings plans are becoming more popular, such as equity-linked insurance policies. These policies carry both mortality and market risks due to the explicit dependence of the benefit on some specified equity index. To alleviate market risk exposures, guaranteed benefits are often added to these contracts, such as guaranteed minimal maturity benefit (GMMB), guaranteed minimal death benefit (GMDB), etc., and combinations of these guarantees. See, e.g., Bauer et al. (2008) for an overview of such guarantee products. The addition of these guaranteed benefits partly removes the downside market risk at the expense of additional costs to the policyholder. The additional guaranteed benefits can be viewed as embedded options. The market risk, including any additional guaranteed benefits, are usually priced under an equivalent risk-neutral probability measure following the classical risk-neutral pricing theory. On the other hand, mortality risk is often considered diversifiable and priced under the real-world probability measure, following the actuarial principle of equivalence. In the pricing of insurance policies involving both market and mortality risks, the two types of risks are often assumed to be independent and separately priced under their respective pricing rules. We refer to Møller (1998) for a general approach to the risk minimization of insurance policies under independent market and mortality risks in a risk-neutral pricing framework.

As interest rates have been low across developed economies for an extended period of time since the recent financial crisis, and are expected to stay low for longer, the low interest rates imply low returns on the insurance premiums for traditional life insurance policies, or equivalently, high premiums for the same level of benefit. On the other hand, equity-linked policies are either exposed to market risk or subject to additional costs of the guarantees. We demonstrate how to overcome these difficulties by following the benchmark approach (BA) proposed by Platen & Heath (2006), and consider the minimal producing costs of the life insurance policies. In particular, the standard term insurance and pure endowment policies may be produced less expensively than the classical theory implies. Additionally, the aim of benefitting from the superior expected long-term growth of the equity market is naturally implemented by the strategies we derive.

The BA assumes that in a given investment universe, there exists a unique growth-optimal portfolio (GP), which we call the benchmark. The BA takes the GP as numeraire, or benchmark, such that any benchmarked (in units of the GP denominated) nonnegative portfolio price process assumes zero expected instantaneous returns. The GP achieves the highest expected instantaneous growth rate among all nonnegative portfolios in the investment universe by maximizing the expected log-utility of terminal wealth. The GP is a well-diversified portfolio that draws on all tradable risk factors and the corresponding risk premiums to achieve the growth-optimality. Under the BA, the real-world pricing formula of any nonnegative, replicable contingent claim, which makes the price denominated in units of the GP a martingale, represents the minimal possible replication cost of the claim.

The purpose of this chapter is to propose the pricing of standard life insurance policies under the BA, where the GP of the financial market may be approximated by a well-diversified equity index. Our chapter extends the approach described by Møller (1998), where the formulation is under the risk-neutral pricing framework, to that under the BA, where less restrictive conditions are assumed on the financial market model. Our approach extends straightforwardly to more complicated life insurance policies such as equity-linked insurance policies, and uses the real-world probability measure as pricing measure. Due to the non-hedgeable mortality risk inherent to the policies, the insurance claims cannot be fully hedged. Under this type of market incompleteness, we resort to benchmarked risk-minimization (BRM), where the mean-squared benchmarked hedging error is minimized under the real-world probability measure. The minimal-cost hedging portfolio minimizing this error is obtained as a dynamic portfolio consisting of the GP and the locally risk-free savings account. The optimal hedging strategy depends on current information about the well-diversified equity index, as well as, mortality information of the insured.

The chapter is organized as follows: In Sections 7.1 and 7.2 we present the financial market model and the mortality model for the insurance policies under consideration. In Section 7.3 we briefly review the BRM. Section 7.4 analyzes the insurance policies under the framework presented in the previous sections. We present numerical simulation results based on historical data in Section 7.5 before we summarize our findings in Section 7.6. Chapter 7 is based on Sun & Platen (2019).

7.1 The Financial Market Model

We work under the general BA framework of Chapter 2 for the financial market model. Under the general framework, the GP is the unique portfolio which, when used as numeraire or benchmark, makes any benchmarked portfolio process $\hat{S}^{\pi}$, defined as $\hat{S}^{\pi}(t) = \frac{S^{\pi}(t)}{S^{*}(t)}$, a local martingale. Given an $\mathcal{F}_T$-measurable nonnegative contingent claim $H \geq 0$ with maturity T, its fair price process under the BA is given by the real-world pricing formula (2.10). For example, consider a zero-coupon bond that pays one unit of the locally risk-free savings account at maturity T. Following

(2.10), the fair price process of this bond is given by

$$P^*(t,T) = E_t\left(\frac{S^*(t)}{S^*(T)}\right), \quad t \in [0,T]. \tag{7.1}$$

Assuming that the GP is a scalar Markov process, and $P^*(t,T)$ is a suitably smooth function of $S^*(t)$ and t, the fair zero-coupon bond (7.1) can be synthesized by a replication portfolio of value $P^*(t,T)$ at $t \in [0,T]$, following the delta hedging strategy given by (3.11). By applying Ito's formula to the benchmarked fair zero-coupon bond price $\hat{P}^*(t,T) = \frac{P^*(t,T)}{S^*(t)}$, and considering (2.7) as well as the fact that $\hat{P}^*(t,T)$ forms a martingale with zero drift, we obtain the SDE for $\hat{P}^*(t,T)$ as

$$d\hat{P}^*(t,T) = \sqrt{\frac{\alpha(t)}{S^*(t)}}\left(\delta^*(t) - \hat{P}^*(t,T)\right) dW^*(t), \quad t \in [0,T], \tag{7.2}$$

with $\hat{P}^*(T,T) = S^*(T)^{-1}$. Note that by applying Ito's formula to $P^*(t,T) = S^*(t)\hat{P}^*(t,T)$, we obtain

$$dP^*(t,T) = \delta^*(t)dS^*(t), \quad t \in [0,T], \tag{7.3}$$

where $\delta^*(t)$ is give by (3.11), verifying the hedge ratio for replicating the fair zero-coupon bond.

7.2 The Mortality Model

In this section we turn our attention to the mortality model, and consider pricing of insurance policies under the combined model. By incorporating mortality risk associated with insurance policies, the financial market becomes incomplete, and insurance policies cannot be fully replicated by trading the financial securities alone.

We consider a portfolio of life insurance policies sharing the same policy details and maturity T, written for a homogeneous group of individual policyholders with independent and identically distributed remaining life times, denoted by $T_i > 0, i = 1, \ldots, L$. Here L is the total number of policyholders in the portfolio. We assume that the remaining life times are independent of the financial market, and share the common survival function given in terms of the hazard rate as

$$s(t) := \mathbb{P}(T_i > t) = \exp\left(-\int_0^t \mu(\tau)d\tau\right), \quad t \in [0,T], \tag{7.4}$$

where $\mu(t) \geq 0, t \in [0,T]$ is the left-continuous force of mortality process common to all policyholders. For simplicity, we assume the hazard rate process to be deterministic. We define the conditional survival function $s(t,u)$ as

$$s(t,u) := \mathbb{P}(T_i > u | T_i > t) = \exp\left(-\int_t^u \mu(\tau)d\tau\right), \quad 0 \leq t \leq u, \tag{7.5}$$

i.e., the conditional probability that a policyholder survives through $u \geq t$, given that he or she survived through t. In terms of standard actuarial notation, we have $s(t,u) =_{u-t} p_{x+t}$ for cohorts aged x at $t = 0$.

The counting process $N(t), t \in [0,T]$, of the total number of deaths within the portfolio is given by

$$N(t) := \sum_{i=1}^{L} \mathbb{1}_{\{T_i \le t\}}, \quad t \in [0,T], \tag{7.6}$$

with $N(0) = 0$. Here $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function for an event. Evidently, N is right continuous with left-hand limits. We assume the current information $\mathcal{F}_t$ contains all information available at time t regarding the financial market, the hazard rate and the survival of individual policyholders. Under this assumption, N is adapted to $\mathbb{F}$. The stochastic intensity process $\lambda(t), t \in [0,T]$, of the mortality counting process N is evidently given by

$$\lambda(t) := \lim_{h \downarrow 0} \frac{1}{h} E_t \left(N(t+h) - N(t) \right) = (L - N(t))\mu(t), \quad t \in [0,T), \tag{7.7}$$

and the compensated counting process

$$M(t) := N(t) - \int_0^t \lambda(u_-)du = N(t) - \int_0^t (L - N(u_-))\mu(u)du, \quad t \in [0,T] \tag{7.8}$$

is an $(\mathbb{F}, \mathbb{P})$-martingale.

We assume that the i-th individual life insurance policy pays a lump-sum benefit H_i at $T_i \wedge T := \min(T_i, T)$, where H_i is $\mathcal{F}_{T_i \wedge T}$-measurable, and does not depend on other policyholders' survival information. Examples of such policies include standard term insurance, pure endowment, endowment insurance and equity-linked insurance policies. The insurance premium is assumed to be paid as a lump-sum at $t = 0$. The price process of an individual policy is given by (2.10) as

$$V_{H_i}(t) = E_t \left(\frac{S^*(t)}{S^*(T_i \wedge T)} H_i \right), \quad t \in [0,T]. \tag{7.9}$$

Note, to simplify notations, we assume that if $T_i < T$, then during the period $[T_i, T]$, V_{H_i} accumulates as the GP does over this period, so that V_{H_i} is defined over $[0,T]$ even for $T_i < T$. The price process of the insurance portfolio is given by

$$V_{\boldsymbol{H}}(t) = \sum_{i=1}^{L} V_{H_i}(t) = S^*(t) \sum_{i=1}^{L} E_t \left(\frac{H_i}{S^*(T_i \wedge T)} \right), \quad t \in [0,T]. \tag{7.10}$$

For example, we consider a standard endowment insurance policy with a deterministic benefit of one unit of the locally risk-free savings account, payable upon the policyholder's death on or before the maturity T, or at T if the policyholder survives. The terminal value of the i-th policy's fair price process is given by

$$V_{H_i}(T) = \frac{S^*(T)}{S^*(T_i \wedge T)}. \tag{7.11}$$

The insurance portfolio's terminal value is given by

$$V_{\boldsymbol{H}}(T) = \sum_{i=1}^{L} V_{H_i}(T) = \int_0^T \frac{S^*(T)}{S^*(\tau)} dN(\tau) + (L - N(T)), \tag{7.12}$$

where the first term after the second equality sign represent the total death benefits accumulated until maturity, and the second term represent the total survival benefits. Following (2.10), the fair portfolio value process is given by

$$V_{\boldsymbol{H}}(t) = E_t\left(\frac{S^*(t)}{S^*(T)} V_{\boldsymbol{H}}(T)\right) = \int_0^t \frac{S^*(t)}{S^*(u)} dN(u) + (L - N(t))R(t), \quad t \in [0, T]. \tag{7.13}$$

Here the first term after the second equality sign represents the fair value of the benefits paid so far, and the second term corresponds to the fair value of future benefits to be paid, where

$$R(t) = \int_t^T s(t,u)\mu(u)P^*(t,u)du + s(t,T)P^*(t,T), \quad t \in [0,T] \tag{7.14}$$

may be seen as the current fair value of the "reserve account" for an individual policyholder who is still alive. This reserve account consists of a collection of fair zero-coupon bonds of different maturities. Note that the weights of zero-coupon bonds change over time, and this reserve account is not self-financing.

7.3 A Brief Overview on Benchmarked Risk-Minimization

In the model described in Sections 7.1 and 7.2, insurance claims are contingent upon mortality information of policyholders, making these claims not fully hedgeable by trading in the financial market. For pricing in such incomplete market under the BA, we resort to the concept of benchmarked risk-minimization (BRM). In this section, we give a brief overview of the BRM. For a detailed presentation, we refer to Du & Platen (2016).

A dynamic trading strategy to deliver a contingent claim which is not perfectly hedgeable is, in general, not self-financing. We define under BRM a dynamic trading strategy as the $m+1$-dimensional stochastic process $\boldsymbol{\nu} = \{\boldsymbol{\nu}(t) = (\eta(t), \boldsymbol{\vartheta}(t)), t \in [0,T]\}$, where the vector-valued predictable process $\boldsymbol{\vartheta} = \{\boldsymbol{\vartheta}(t) = (\vartheta_1(t), \ldots, \vartheta_m(t))^\top, t \in [0,T]\}$ denotes the number of respective shares of the risky primary security accounts $\boldsymbol{S}$ held at time t. Here $\boldsymbol{\vartheta}$ represents the self-financing part of the trading strategy $\boldsymbol{\nu}$, where the remaining wealth of the self-financing portfolio is invested in the locally risk-free savings account; and $\eta = \{\eta(t), t \in [0,T]\}$ with $\eta(0) = 0$ represents the adapted cumulative P&L process of the strategy. Note that the P&L process η represents the non-self-financing part of the strategy $\boldsymbol{\nu}$, which appears in addition to the gains and losses of the self-financing part represented by the vector stochastic integral

$$\int_0^t \boldsymbol{\vartheta}(u)^\top d\boldsymbol{S}(u), \quad t \in [0,T]. \tag{7.15}$$

Here the predictable process $\boldsymbol{\vartheta}$ is assumed to be such that (7.15) is well-defined. The portfolio value or price process of the strategy $\boldsymbol{\nu}$ is given by

$$V^{\boldsymbol{\nu}}(t) = V^{\boldsymbol{\nu}}(0) + \int_0^t \boldsymbol{\vartheta}(u)^\top d\boldsymbol{S}(u) + \eta(t), \quad t \in [0,T]. \tag{7.16}$$

A trading strategy $\boldsymbol{\nu}$ is said to deliver an $\mathcal{F}_T$-measurable contingent claim $H_T \geq 0$ if $V^{\boldsymbol{\nu}}(T) = H_T$ $\mathbb{P}$-almost surely. We call such a (potentially non-self-financing) trading strategy a *hedging strategy delivering the contingent claim* H_T. The price process $V^{\boldsymbol{\nu}}(t)$ can be regarded as the value of the current self-financing portfolio $V^{\boldsymbol{\nu}}(0) + \int_0^t \boldsymbol{\vartheta}(u)^\top d\boldsymbol{S}(u)$ plus the accumulated P&L $\eta(t)$. Intuitively, the P&L process can be thought of as the hedging error of the self-financing strategy with respect to the contingent claim H_T, or equivalently the additional cost process of the strategy. The hedging strategy $\boldsymbol{\nu}$ is called replicating if $\eta(t) = 0$ for all $t \in [0, T]$. The hedging strategy is called mean self-financing if the benchmarked P&L process $\hat{\eta} = \{\hat{\eta}(t) = \frac{\eta(t)}{S^*(t)}, t \in [0, T]\}$ forms a local martingale. The hedging strategy is said to have an orthogonal benchmarked P&L process $\hat{\eta}$ if the process $\hat{\eta}\hat{\boldsymbol{S}} = \{\hat{\eta}(t)\hat{\boldsymbol{S}}(t), t \in [0, T]\}$ is a (vector) local martingale. Intuitively, $\hat{\eta}$ captures in a generalized least-squares sense the minimally fluctuating benchmarked P&L process.

Conceptually, there may exist more than one hedging strategy delivering a given contingent claim H_T. In fact, almost any self-financing strategy $\boldsymbol{\vartheta}$ paired with an appropriate P&L process η may be a valid hedging strategy delivering the claim. BRM seeks to find an optimal hedging strategy $\boldsymbol{\nu}^*$, which is mean self-financing, has an orthogonal benchmarked P&L process, and satisfies

$$V^{\boldsymbol{\nu}^*}(t) \leq V^{\boldsymbol{\nu}}(t) \tag{7.17}$$

for all $t \in [0, T]$ and every mean self-financing hedging strategy $\boldsymbol{\nu}$ delivering the contingent claim with an orthogonal benchmarked P&L process. In other words, the BRM hedging strategy, or BRM strategy in short, is the least expensive mean self-financing hedging strategy delivering the contingent claim, with an orthogonal benchmarked P&L process.

The optimality of the BRM strategy as described above is twofold: The strategy minimizes a generalized form of the hedging error, in that the benchmarked, unhedged part of the contingent claim forms a local martingale, orthogonal in a generalized sense to the benchmarked primary security accounts. Moreover, the initial cost of the BRM strategy is minimal among all such hedging strategies.

Not surprisingly, the price process $V^{\boldsymbol{\nu}^*}$ of the BRM strategy $\boldsymbol{\nu}^*$, assuming the latter exists, is given by the real-world pricing formula

$$V^{\boldsymbol{\nu}^*}(t) = E_t\left(\frac{S^*(t)}{S^*(T)} H_T\right) \quad t \in [0, T], \tag{7.18}$$

see Du & Platen (2016), Corollary 5.4. For (7.18) to be useful, one needs the existence of the corresponding BRM strategy $\boldsymbol{\nu}^*$ for the contingent claim. For this purpose we introduce the notion of a regular contingent claim. A contingent claim is called regular if a well-defined BRM strategy $\boldsymbol{\nu}^* = (\eta^*, \boldsymbol{\vartheta}^*)$ exists for this claim. In other words, an $\mathcal{F}_T$-measurable contingent claim $H_T \geq 0$ is called regular if the benchmarked contingent claim $\hat{H}_T = \frac{H_T}{S^*(T)}$ assumes the representation

$$\hat{H}_T = E_t\left(\hat{H}_T\right) + \int_t^T \boldsymbol{\vartheta}(u)^{*\top} d\hat{\boldsymbol{S}}(u) + \hat{\eta}^*(T) - \hat{\eta}^*(t) \quad t \in [0, T], \tag{7.19}$$

for some predictable self-financing trading strategy process $\boldsymbol{\vartheta}^*$, and some local martingale $\hat{\eta}^*$ with $\hat{\eta}^*(0) = 0$ as benchmarked P&L process, which is orthogonal to $\hat{\boldsymbol{S}}$ in the sense that $\hat{\eta}^*\hat{\boldsymbol{S}} = \{\hat{\eta}^*(t)\hat{\boldsymbol{S}}(t), t \in [0, T]\}$ forms a vector local martingale.

It should be noted that BRM makes the same minimal assumptions on the underlying financial market model as the BA. The only requirement is the existence of the GP, where all nonnegative, self-financing portfolio processes form local martingales when denominated in units of the GP. The concept of BRM is in the same spirit as classical risk-minimization as developed by Föllmer & Sondermann (1986); Schweizer (1994); Møller (1998). However, BRM allows to deliver the contingent claim with minimal cost, with less regularity conditions imposed on the market model.

7.4 BRM Hedging Strategies for Endowment Insurance Policies

Classical actuarial theory considers the risk associated with standard life insurance policies as diversifiable under the assumption that individual deaths are independent, and that the policy benefits are not subject to common factors such as market risk. Under the BA, the latter assumption does not hold even for standard life insurance policies. In fact, benefits of the standard policies, when priced under the BA, are subject to market risk associated with the GP, c.f. (7.1). Fortunately, this "systematic" risk associated with the GP can be hedged by following the BRM hedging strategy, leaving only the independent mortality risk which, in principle, can be diversified away.

As an illustrative example, we consider the pricing and hedging of a homogeneous portfolio of standard endowment insurance policies as discussed in Section 7.1, using the BRM strategy. We assume the insurance premiums to be paid upfront at the policies' inception at $t = 0$. Results obtained herein are easily modified to cover cases of different types of policies and/or premium payment schemes.

Following (7.18), the price process V^{ν^*} of the BRM strategy for the life insurance policies is the same as the one for the fair price process $V_{\boldsymbol{H}}$ given by (7.13), with benchmarked price process

$$\hat{V}^{\boldsymbol{\nu}^*}(t) = \int_0^t \frac{dN(u)}{S^*(u)} + (L - N(t))\hat{R}(t), \quad t \in [0, T], \tag{7.20}$$

and the benchmarked individual reserve account value

$$\hat{R}(t) = \int_t^T s(t,u)\mu(u)\hat{P}^*(t,u)du + s(t,T)\hat{P}^*(t,T), \quad t \in [0,T]. \tag{7.21}$$

To obtain the BRM representation (7.18), we apply Ito's formula to (7.20) and exploit the fact that $\hat{V}^{\boldsymbol{\nu}^*}$ forms a martingale. This leads to the following representation for the benchmarked price process of the BRM strategy:

$$\hat{V}^{\boldsymbol{\nu}^*}(t) = \hat{V}^{\boldsymbol{\vartheta}^*}(t) + \hat{\eta}^{\boldsymbol{\nu}^*}(t), \quad t \in [0, T], \tag{7.22}$$

with the self-financing part

$$\hat{V}^{\boldsymbol{\vartheta}^*}(t) = \hat{V}^{\boldsymbol{\nu}^*}(0) + \int_0^t \gamma^{\boldsymbol{\vartheta}^*}(\tau)\sqrt{\frac{\alpha(\tau)}{S^*(\tau)}}dW^*(\tau), \tag{7.23}$$

where

$$\gamma^{\boldsymbol{\vartheta}^*}(t) = (L - N(t))S^*(t)\frac{\partial \hat{R}(t)}{\partial S^*(t)}. \tag{7.24}$$

The benchmarked P&L process is given by

$$\hat{\eta}^{\boldsymbol{\nu}^*}(t) = \int_0^t \left(S^*(\tau)^{-1} - \hat{R}(\tau)\right) dM(\tau). \tag{7.25}$$

The latter forms a local martingale, orthogonal to the benchmarked primary security accounts, since the mortality counting process N and its compensated process M are independent of the GP. To verify whether (7.22) is in the correct form of a BRM representation, we construct the self-financing strategy $\boldsymbol{\vartheta}^*$ that replicates the hedgeable part $V^{\boldsymbol{\vartheta}^*}(t)$ of the BRM price process. By applying Ito's formula to $V^{\boldsymbol{\vartheta}^*}(t) = \hat{V}^{\boldsymbol{\vartheta}^*}(t)S^*(t)$ and considering (2.7), we obtain

$$V^{\boldsymbol{\vartheta}^*}(t) = V^{\boldsymbol{\nu}^*}(0) + \int_0^t \left(\gamma^{\boldsymbol{\vartheta}^*}(\tau) + \hat{V}^{\boldsymbol{\vartheta}^*}(\tau)\right) dS^*(\tau) \tag{7.26}$$

for $t \in [0,T]$. Evidently, the self-financing part of the strategy $\boldsymbol{\vartheta}^*$ holds the number of

$$\delta^{\boldsymbol{\vartheta}^*}(t) = \gamma^{\boldsymbol{\vartheta}^*}(t) + \hat{V}^{\boldsymbol{\vartheta}^*}(t) \tag{7.27}$$

units of the GP, and invest the remaining wealth of the self-financing part of the strategy in the locally risk-free savings account. The BRM strategy $\boldsymbol{\nu}$ for the life insurance policies is, thus, completely characterized by (7.20) through (7.27).

To gain more insights, notice that both the self-financing part of the strategy $\boldsymbol{\vartheta}^*$ and the P&L process $\eta^{\boldsymbol{\nu}^*}$ invest in the GP. In particular, the P&L process invests at $t \in [0,T]$ a number of $\hat{\eta}^{\boldsymbol{\nu}^*}(t)$ units in the GP. The total number of GP units held at $t \in [0,T]$ is, thus, given by

$$\delta^{\boldsymbol{\nu}^*}(t) = \delta^{\boldsymbol{\vartheta}^*}(t) + \hat{\eta}^{\boldsymbol{\nu}^*}(t) = \hat{V}^{\boldsymbol{\nu}^*}(t) + \gamma^{\boldsymbol{\vartheta}^*}(t) = \int_0^t \frac{dN(u)}{S^*(u)} + (L - N(t))\frac{\partial R(t)}{\partial S^*(t)}, \tag{7.28}$$

where the identity $S^*(t)\frac{\partial \hat{R}(t)}{\partial S^*(t)} = \frac{\partial R(t)}{\partial S^*(t)} - \hat{R}(t)$ is used. It is seen from (7.13) and (7.28) that $\delta^{\boldsymbol{\nu}^*}(t) = \frac{\partial V^{\boldsymbol{\nu}^*}(t)}{\partial S^*(t)}$. That is, (7.28) corresponds to the hedge ratio, or the "delta" of the BRM price process with respect to the GP. Evidently, the BRM strategy, effectively, resembles a delta-hedging strategy, with the delta given by the sensitivity of the fair price process with respect to the GP. However, an important difference between the BRM strategy and the corresponding delta-hedging strategy is that the BRM strategy is, in general, not self-financing. It is mean self-financing in terms of the benchmarked price process under the real-world probability measure. Moreover, the real-world pricing formula (2.10), generally, differs from the classical risk-neutral pricing formula, yielding prices and deltas different from classical theory. We refer to the delta-hedging strategy $\delta^{\boldsymbol{\nu}^*}$ that effectively mimics the BRM strategy $\boldsymbol{\nu}^*$ as the delta-hedging implementation of the BRM strategy $\boldsymbol{\nu}^*$. The delta-hedging implementation of the BRM strategy differs from the exact BRM strategy only in that the exact BRM strategy compensates for any hedging errors of the delta-hedging implementation by injecting and extracting additional funds into and from the locally risk-free savings account, so that the exact BRM price, as given by (7.13), is maintained throughout the entire hedging process. The injection and extraction of additional funds reveal the non-self-financing nature of the BRM strategy. On the other hand, we assume that the delta-hedging implementation is self-financing. That is, while maintaining exactly the same positions in the GP, including those of the P&L process, the delta-hedging implementation does NOT inject into or extract from the locally risk-free savings account additional funds to maintain the BRM price process (7.13). However, it should be noted that from a regulatory perspective, such injection and/or extraction of funds may be formally required in order to maintain the overview on the reserve account balance at or above a requested level.

7.5 Numerical Examples

To illustrate the methodology presented, we conduct some numerical experiments. In particular, we consider the pricing of a portfolio of homogeneous endowment insurance policies and the associated BRM strategy, based on the minimal market model (MMM), see Section 2.1, for the underlying discounted GP, estimated from the discounted historical prices of the S&P500 total return index, as described in Section 2.2. For the mortality modelling, we take the historical mortality table for the US and estimate the historical force of mortality. We conduct Monte-Carlo simulations to assess the distribution of the P&Ls from the BRM strategy. We further backtest the strategies on historical data and generate the P&L process corresponding to the historically realized scenario.

Using the closed-form transition density (2.16), the fair zero-coupon bond price (7.1) may be obtained by direct integration as

$$P^*(t,T) = Q_0\left(\sqrt{\frac{4e^{-\rho T}}{1-e^{-\rho(T-t)}}\frac{\rho}{\alpha_0}S^*(t)}\right), \quad t \in [0,T], \tag{7.29}$$

where the function $Q_0(a)$, given by

$$Q_0(a) = \int_0^\infty aI_1(ax)e^{-\frac{x^2+a^2}{2}}dx, \quad a > 0 \tag{7.30}$$

is a variant of the Marcum Q function; see Marcum (1960). From (**??**), at $t \in [0,T]$, the replication portfolio of the fair zero-coupon bond invests

$$\delta^*(t,T) = \sqrt{\frac{e^{-\rho T}}{1-e^{-\rho(T-t)}}\frac{\rho}{\alpha_0}\frac{1}{S^*(t)}}Q_0'\left(\sqrt{\frac{4e^{-\rho T}}{1-e^{-\rho(T-t)}}\frac{\rho}{\alpha_0}S^*(t)}\right) \tag{7.31}$$

units in the GP, and its remaining value in the locally risk-free savings account.

To model the portfolio of life insurance policies, we assume the group of policyholders to belong to the cohort of US females aged $x = 30$ at the beginning of 1987. The portfolio consists of standard endowment insurance policies with a term of $T = 30$ years. The mortality rates were obtained from The Human Mortality Database (Shkolnikov et al., 2018). We assume a yearly piecewise-constant force of mortality process for the group. From (7.5), the force of mortality is given by

$$\mu(t) = -\log(1 - q_{x+\lfloor t\rfloor}), \quad t \in [0,T], \tag{7.32}$$

where $\lfloor t\rfloor$ is the integer part of t, with mortality rate $q_{x+\lfloor t\rfloor} := 1 - s(\lfloor t\rfloor, \lfloor t\rfloor + 1)$. The obtained deterministic force of mortality trajectory is shown in Figure 7.1.

To gain some insights to the dynamics of the MMM, we first simulate from the fitted model a number of realized GP sample trajectories. We show the first 20 simulated sample trajectories in Figure 7.2 (a). We also simulate an equal number of sample trajectories of the counting process N, assuming a total number of $L = 1000$ policies in the portfolio. We show the first 20 sample trajectories for N in Figure 7.2 (b). It can be seen qualitatively that the market risk associated with the GP is dominant due to the considerable non-diversifiable uncertainty of the GP, as compared to the mortality counts for a moderately diversified portfolio of 1000 lives. As the number of life insurance policies increases, the mortality risk can be foreseen to be completely

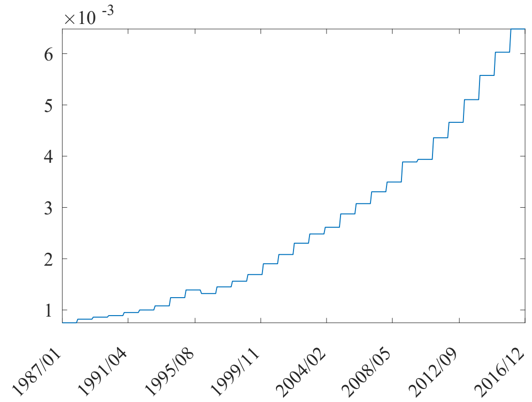


Figure 7.1 – Force of mortality over 1987-2017 for US females aged 30 at the beginning of 1987.

diversifiable (assuming the deterministic force of mortality model is correctly specified), leaving only the market risk to be hedged.

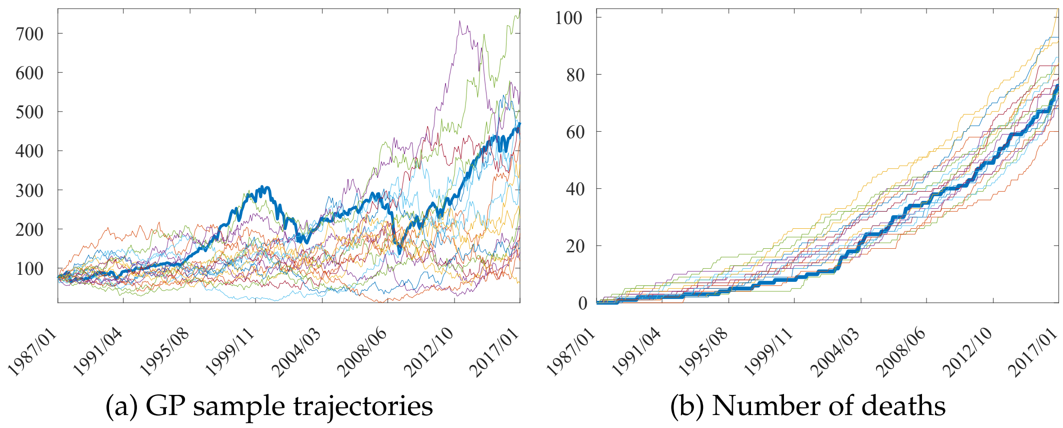
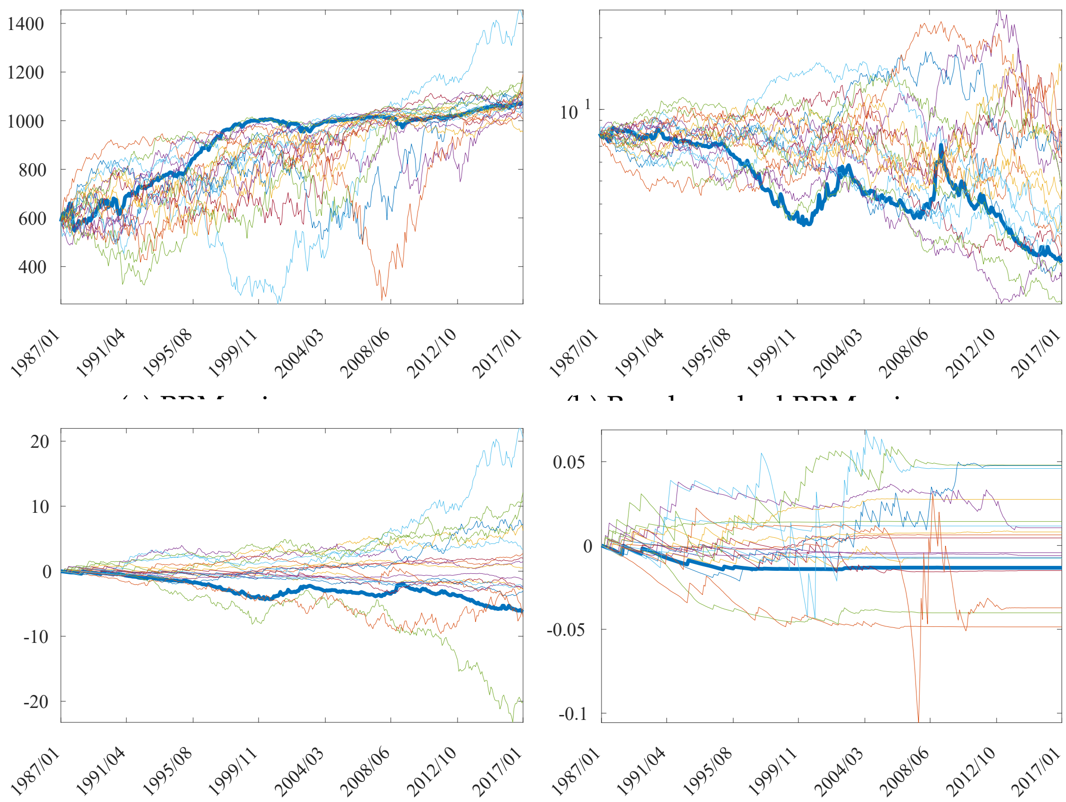


Figure 7.2 – Simulated GP sample trajectories from the estimated MMM and the number of deaths within the portfolio from the historical force of mortality model. The thick trajectory in panel (a) is the realized S&P 500 index prices process; The thick trajectory in panel (b) is a randomly chosen trajectory that, we assume, corresponds to the realized GP in the subsequent simulations.

We next proceed to consider the BRM hedging strategy for the portfolio of insurance policies. We first observe from (7.13) that the price process of the BRM strategy for the portfolio of insurance policies consists of a combination of prices of fair zero-coupon bonds with different maturities. As a result, the BRM strategy, essentially, consists of replication strategies of fair zero-coupon bonds, given by (7.29) and (7.31) under the MMM outlined earlier in this section. It should be pointed out that the fair zero-coupon bond under the MMM is priced less expensively than under the classical risk-neutral pricing theory, where a zero-coupon bond is priced precisely at its face value. Consequently, the BRM strategy under the MMM should be priced less expensively than under the classical theory.

Following (7.13), we compute the BRM price trajectories along the simulated scenarios. We show for the first 20 scenarios the realized BRM price trajectories in Figure 7.3 (a), and in Figure 7.3 (b) the corresponding benchmarked price trajectories. We see that the benchmarked price trajectories seem to be trendless, while the price trajectories exhibit a visible positive trend. In fact, the sample mean of the benchmarked price trajectories stays roughly constant over time, while the sample mean of the price trajectories exhibits on average a 2% annual growth. Recall that we take the locally risk-free savings account as denominator, so that the locally risk-free interest rate is effectively zero. Therefore, under the risk-neutral pricing theory, regardless of the equivalent risk-neutral probability measure adopted, the price process of any zero-coupon bond is a constant equal to its face value, thus no growth is possible. By following the BA and employing the MMM, significant savings on the hedging costs for long-term insurance policies should be expected.

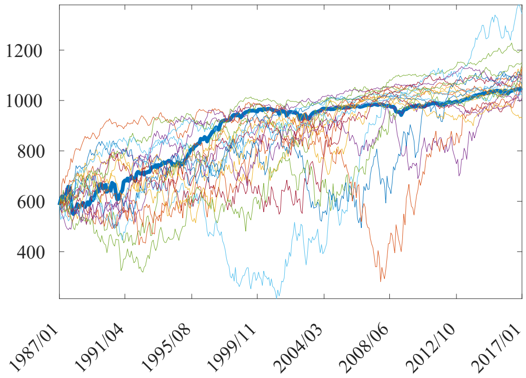


(c) P&L processes (d) Benchmarked P&L processes

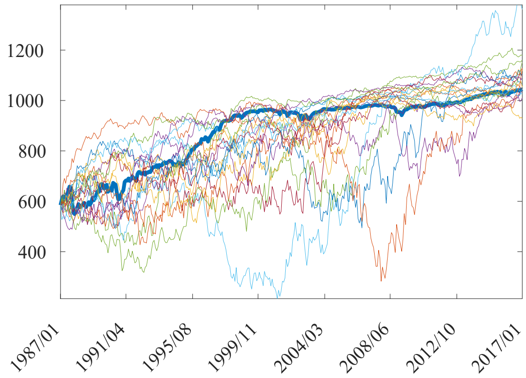
Figure 7.3 – The sample trajectories of the BRM price process of the portfolio of life insurance policies and the benchmarked price process for the first 20 simulated scenarios, where the thick trajectories represent the realized scenario.

To assess the effectiveness of the BRM strategy, we show in Figure 7.3 (c) and (d) the P&L and benchmarked P&L trajectories, respectively. It can be seen that the benchmarked P&L processes show very little variances for most sample trajectories. Comparing with Figure 7.3 (a) and (b), the P&L trajectories evidently show much less variances than the portfolio value trajectories. The small variance in the P&L trajectories indicates that the risk associated with the portfolio of insurance policies can be effectively hedged by following the BRM strategy, where the hedgeable part of the price process dominates the unhedgeable part, even with the rather modest diversification effects from only 1000 life insurance policies.

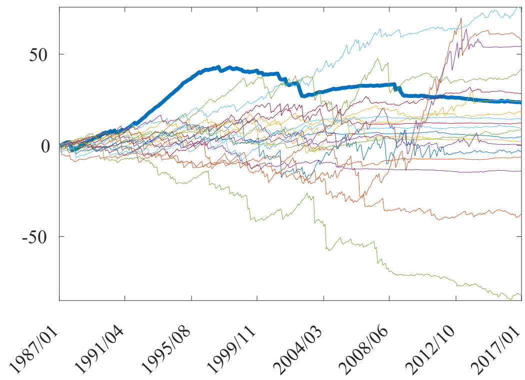
From (7.28) we compute the number of GP units held by the self-financing strategy δ^{ϑ^*} and simulate the realized price process $\tilde{V}^{\vartheta^*}$ by following the strategy ϑ^*. We show these realized price trajectories in Figure 7.4 (a), and the corresponding total hedging error trajectories $V^{\nu^*} - \tilde{V}^{\vartheta^*}$ in Figure 7.4 (c), for selected scenarios. It should be noted that the total hedging error process is given by the sum of the BRM P&L process η^{ν^*} and the hedging error process of the self-financial part. On the other hand, the "pure" hedging errors due to imperfect, discrete hedging, as well



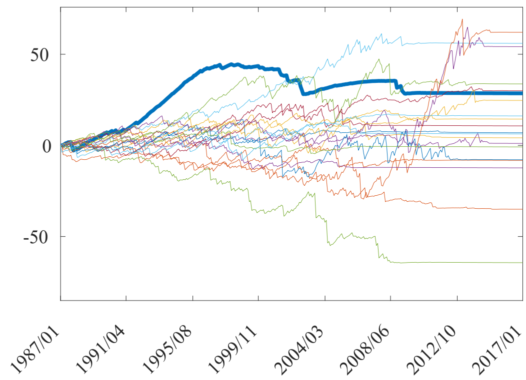
(a) Realized self-financing portfolio prices



(b) Realized delta-hedging portfolio prices



(c) Hedging errors of the self-financing portfolio



(d) Hedging errors of the delta-hedging portfolio

Figure 7.4 – The trajectories of the realized price process and the hedging errors. Panel (a) shows the realized self-financing part of the BRM price process $\tilde{V}^{\boldsymbol{\vartheta}^*}$; Panel (c) shows the corresponding hedging error process $V^{\boldsymbol{\nu}^*} - \tilde{V}^{\boldsymbol{\vartheta}^*}$; Panel (b) shows the realized delta-hedging implementation of the BRM price process $\tilde{V}^{\delta^*}$; and Panel (d) shows the corresponding hedging error process $V^{\boldsymbol{\nu}^*} - \tilde{V}^{\delta^*}$.

Table 7.1 – Sample variance reductions of terminal portfolio values, P&Ls and hedging error processes

term. val.	$V^{\boldsymbol{\vartheta}^*}(T)$	$\eta^{\boldsymbol{\nu}^*}(T)$	$\tilde{V}^{\boldsymbol{\vartheta}^*}(T)$	$\tilde{V}^{\delta^{\boldsymbol{\nu}^*}}(T)$	$V^{\boldsymbol{\nu}^*}(T) - \tilde{V}^{\boldsymbol{\vartheta}^*}(T)$	$V^{\boldsymbol{\nu}^*}(T) - \tilde{V}^{\delta^{\boldsymbol{\nu}^*}}(T)$
var.	0.8846	0.0324	1.0435	1.1474	0.1606	0.1282
benchmarked var.	0.9958	0.0005	1.0299	1.0345	0.0098	0.0092

as potential inaccuracies of the MMM, can be assessed by examining the hedging errors of the delta-hedging implementation of the BRM strategy. Figure 7.4 (b) shows the realized delta-hedging price processes $\tilde{V}^{\delta^*}$, and Figure 7.4 (d) shows the corresponding "pure" hedging errors $V^{\boldsymbol{\nu}^*} - \tilde{V}^{\delta^*}$, under the selected scenarios. It is seen from these simulations that the total hedging error of the self-financing part is dominated by the model and discretization errors. This should not be surprising since the market risk in this case is the dominant risk factor after the mortality risk has been largely diversified.

As a final step, we show in Table 7.1 the sample variances of the terminal values of various price trajectories and their hedging errors, denominated both in the locally risk-free savings account and in the GP, respectively, where the variances are normalized by sample variances of the BRM price trajectories.

Two observations can be made from these numerical scenario simulation results: First, the BRM framework effectively allows the investment in risky assets for the hedging of standard life insurance policies, which is generally not the case under risk-neutral approach, where zero-coupon bonds are not hedged by investing in risky assets. In these simulations we see a 2% growth rate in the BRM-based portfolio, while this growth rate should be zero if a risk-neutral approach is adopted. Second, we see that the BRM strategy effectively reduces the risk of the portfolio, as visualized in Figures 7.3, 7.4, as well as in the significant variance reduction ratios shown in Table 7.1.

7.6 Summary

In this chapter, we considered pricing and hedging of standard life insurance policies under the BA. Under our setting, the classical risk-neutral pricing measure may not exist, and standard life insurance policies are subject to both mortality risk and market risk. The mortality risk cannot be hedged by trading in the financial market. We employ benchmarked risk minimization (BRM) as a generalized concept of risk minimization, and developed the BRM hedging strategy for a homogeneous portfolio of standard endowment insurance policies subject to a deterministic force of mortality. We conducted numerical simulation studies under the MMM, based on historical equity market and mortality data, and demonstrated a BRM hedging strategy for a fictional portfolio of life insurance policies subject to historical market and mortality risks. Our studies revealed good hedging outcomes over simulated, as well as, historical scenarios with small hedging errors. Moreover, the BRM hedging strategy realized significant growth over the period of the insurance policies beyond that of the risk-free interest rate. This implies significant savings on the hedging costs of the insurance policies, which may be passed on to the insurance company as additional profits and/or to the policyholders as reduced premiums. The approach developed in this chapter can be generalized to more complicated insurance policies, including equity-linked contracts, etc., and more realistic market and mortality models.

Chapter 8

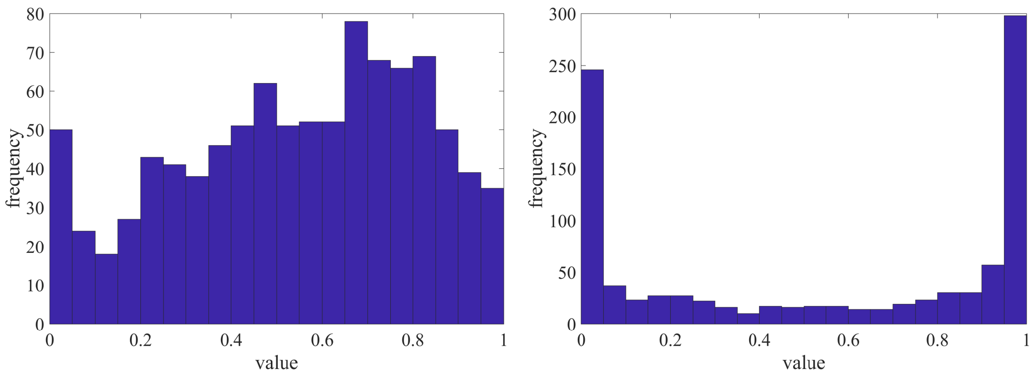
Conclusions and Future Works

In this thesis I have explored the application of benchmark approach (BA) to long-term investments and life insurance pricing and hedging problems. The BA and the real-world pricing formula can be considered as a generalization of the risk neutral pricing theory widely accepted in the academic literature of quantitative finance. Under incomplete markets, benchmarked risk minimization (BRM) is a generalization of classical risk minimization under the risk-neutral pricing theory. For modelling purposes, I make use of the minimal market model (MMM) for modelling the dynamics of the growth optimal portfolio (GP), the central concept of BA and an essential requirement for market viablility.

We first make a short summary of contributions from this thesis: In Chapter 3 we developed under the BA a systematic set of procedures for designing optimal glide paths for TDF investors with different investment horizons and risk preferences, that automatically adjust risk exposures according to both the current time to maturity and the current market condition. In Chapter 4 we illustrate under the BA framework how typical CPT investors may construct long-term optimal investment strategies and realise different investment outcomes according to different preferences. In Chapters 5, 6 we considered the pricing of VA with GMW(D)B under the risk-neutral framework and the BA framework, respectively. We compare results from the two through empirical backtests on the historical prices of the S&P 500 total return index and show how the MMM under the BA framework may be advantageous. In Chapter 7, we considered pricing and hedging of standard life insurance policies using BRM, a generalized concept of risk minimization under the BA, and construct hedging strategies that realize significant returns beyond those implied the risk-free interest rate. In short, we have shown empirically for a number of long-term investment situations, strategies following the MMM generate good excess returns compared with classical approaches, under historically realized and simulated scenarios.

Our investigations of BA applications in long-term investments, pension and life insurance products reveal much potential of realizing excess returns with controlled risk exposures, at costs cheaper than the classical theory based primarily on short-term views would predict. The long-term return characteristics of the equity market has been known to differ from those of the short term, which can be well understood and potentially well modelled under BA. Compared with the classical risk-neutral framework, the BA is less restrictive, in that it allows certain types of arbitrage to exist, thus offering more freedom in modelling more accurately the market dynamics under the real-world probability measure. On the other hand, such a relaxation does still ensure market viability, in that under these settings, an agent who prefers more to less and who has limited access to credit may have an optimum.

Unfortunately, empirical studies show that the over-simplified MMM, though qualitatively reflects some stylized observations, cannot explain quantitatively and



(a) uniform samples based on MMM (b) uniform samples based on BSM

Figure 8.1 – Histograms of uniform samples backed out from the S&P 500 historical index prices: the MMM backed-out "uniform" samples result in a p-value of 1.4×10^{-9} for the KS statistic; the BSM backed-out "uniform" samples result in a p-value of 5.68×10^{-65} for the same statistic.

statistically the real-world dynamics of the market. On the other hand, the risk-neutral models, such as the Black-Scholes model and its variants, hiding behind the unknown and arbitrary equivalent "risk-neutral" probability measure, by definition, are largely exempt from real-world statistical tests. (This partly explains the unjustified proliferation of the field of "derivative pricing", which largely ignores empirical facts under the real-world statistics, and focus on fancy mathematical models and equations that essentially come from nowhere.) To see how inappropriate the MMM is in terms of describing the real-world dynamics, consider the "uniform" samples backed out of the S&P 500 index data considered in the previous chapters, following the inverse function of (2.17) for the MMM, and generate the histogram of these theoretically uniformly distributed samples, which is shown in Figure 8.1 (a). It is obvious that these samples do not follow the supposed uniform distribution. The Kolmogorov-Smirnov (KS) test fails for all significance levels up to $1 - 1.4 \times 10^{-9}$. On the other hand, the classical BSM, following similar procedures, produces a histogram of "uniform" samples shown in Figure 8.1 (b), with a p-value of 5.68×10^{-65} for the KS statistic. Clearly the BSM severely underestimates extreme moves of the index prices, while the MMM has a skewed estimate of the price moves, with an over-estimation of the down-side moves. Even though both models are rejected with overwhelming confidence, the MMM is obviously more appropriate for long-term investments compared with the BSM, as extreme market moves are better much estimated by the MMM. On the other hand, while over-estimate of moderate downside risk by the MMM is undesirable, it leads to more conservative strategies, which do less harm than those guided by an under-estimating model.

Regardless of real-world test results, we argue that risk-neutral models are generally unsuitable for the pricing of long-term pension and life insurance products. They serve at most a summary of the current market information without any insights to the long-term market behavior that is critical in the successful management of long-term risks associated with pension and life insurance products. As a consequence, more sophisticated models that extends the MMM, unrestricted from the "risk-neutral" requirement, and that explain well the real-world dynamics of the market, are crucial in the construction of optimal investment strategies developed in this thesis. Such a major challenge has been largely avoided by the quantitative finance literature under the protection umbrella provided by the risk-neutral pricing measure, and has to be the subject of future investigations as the risk-neutral pricing cliche abates.

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