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RISING BINOMIAL COEFFICIENTS – TYPE 1: EXTENSIONS OF CARLITZ AND RIORDAN

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Abstract. Following some ideas initiated by Leonard Carlitz and John Riordan, this paper generalizes some properties of the ordinary binomial coefficient through the use of rising factorials. Properties include connections with Beta and Gamma functions.

1. Introduction

We explore here some extensions of Carlitz' definition of the q-series analogue of the binomial coefficients [1]

$$\left[\begin{array}{c} n \\ k \end{array}\right] = \frac{(q)_n}{(q)_k\,(q)_{n-k}} \tag{1}$$

in which

$$(q)_n = (1-q)\left(1-q^2\right)\dots(1-q^n)\,. \tag{2}$$

The latter suggests that we define a rising factorial analogue of the binomial coefficients in a similar fashion and then investigate some of its properties. We first recall rising and falling factorial coefficients. The rising factorial of n can be given by

$$n^{\overline{r}} = n\,(n+1)\dots(n+r-1) \tag{3}$$

which is an r permutation of $(n+r-1)$ things, and in contrast the falling factorial is then given by

$$n^{\underline{r}} = n\,(n-1)\dots(n-r+1) \tag{4}$$

which is equivalent to $P(n,r)$, an r permutation of n distinct things [7]. These two factorials occupy a central position in the finite difference calculus [8] because

$$\nabla x^{\overline{n}} = nx^{\overline{n-1}},$$

and

$$\nabla x^{\underline{n}} = nx^{\underline{n-1}},$$

for the shift operator

$$\nabla P\,(n,r) = P\,(n,r) - P\,(n-1,r) = rP\,(n-1,r-1)\,.$$

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2. Rising Binomial Coefficients – Type 1

We define

$$\left[\begin{array}{c} n \\ k \end{array}\right]^a = \frac{a^{\overline{n}}}{a^{\overline{k}} a^{\overline{n-k}}} \tag{5}$$

It is clear that

$$\left[\begin{array}{c} i \\ j \end{array}\right]^1 = \left(\begin{array}{c} i \\ j \end{array}\right)$$

because

$$\begin{aligned} \left[\begin{array}{c} i \\ j \end{array}\right]^1 &= \frac{(1+j)\,(2+j)\ldots(i)\,j!}{(1)\,(2)\ldots(i-j)\,j!} \\ &= \frac{i!}{j!\,(i-j)!} \\ &= \left(\begin{array}{c} i \\ j \end{array}\right). \end{aligned}$$

Some of the simplest properties of Type 1 follow immediately. Thus

$$\left[\begin{array}{c} n \\ k \end{array}\right]^{-a} = \frac{\left(\begin{array}{c} a-n+k \\ k \end{array}\right)}{\left(\begin{array}{c} a \\ k \end{array}\right)}. \tag{6}$$

Proof.

$$\begin{aligned} \left[\begin{array}{c} n \\ k \end{array}\right]^{-a} &= \frac{(-a)^{\overline{n}}}{(-a)^{\overline{k}}\,(-a)^{\overline{n-k}}} \\ &= \frac{(-1)^n\,a!\,(a-k)!\,(a-n+k)!}{(-1)^k\,a!\,(a-n)!\,(-1)^{n-k}\,a!} \\ &= \frac{\frac{(a-n+k)!}{(a-n)!}}{\frac{a!}{(a-k)!}} \\ &= \frac{\left(\begin{array}{c} a-n+k \\ k \end{array}\right)}{\left(\begin{array}{c} a \\ k \end{array}\right)} \end{aligned}$$

□

In this proof a result stated by Carlitz [6] has been used, namely

$$\left(\begin{array}{c} a \\ k \end{array}\right) = \frac{(-1)^k\,(-a)^{\overline{k}}}{k!}, \tag{7}$$

which is readily verified as follows:

$$\begin{aligned}\frac{(-1)^k(-a)^{\overline{k}}}{k!} &= \frac{(-1)^k(-a)(-a+1)\dots(-a+k-1)}{k!}\\ &= \frac{a(a-1)\dots(a-k+1)}{k!}\\ &= \binom{a}{k}.\end{aligned}$$

By analogy with

$$\binom{n-1}{k}+\binom{n-1}{k-1}=\binom{n}{k}$$

we then have that

$$\left[\begin{array}{c}n-1\\k\end{array}\right]^a+\left[\begin{array}{c}n-1\\k-1\end{array}\right]^a=\frac{2a+n-2}{a+n-1}\left[\begin{array}{c}n\\k\end{array}\right]^a \tag{8}$$

which reduces to the former when $a=n=1$. To prove (8) we first need

$$\left[\begin{array}{c}n\\k\end{array}\right]^a=\frac{(a+k)(a+k+1)\dots(a+n-1)}{a(a+1)\dots(a+n-k-1)}$$

which follows immediately from the expansion of the terms in the definition and the cancellation of common terms in the numerator and denominator. The proof of (8) now follows as:

$$\begin{aligned}\left[\begin{array}{c}n-1\\k\end{array}\right]^a+\left[\begin{array}{c}n-1\\k-1\end{array}\right]^a &= \frac{(a+k)(a+k+1)\dots(a+n-2)}{a(a+1)\dots(a+n-k-2)}\\ &\quad+\frac{(a+k-1)(a+k)\dots(a+n-2)}{a(a+1)\dots(a+n-k-1)}\\ &= \frac{(a+k)(a+k+1)\dots(a+n-2)}{a(a+1)\dots(a+n-k-2)}\left(1+\frac{a+k-1}{a+n-k-1}\right)\end{aligned}$$

which yields the desired result on multiplying numerator and denominator by $(a+n-1)$.

3. Rising binomial coefficients and Beta and Gamma functions

We start with the connection in (9):

$$\left[\begin{array}{c}n\\k\end{array}\right]^a=\frac{\Gamma(a)\Gamma(a+n)}{\Gamma(a+n-k)\Gamma(a+k)} \tag{9}$$

where $\Gamma(a)$ is the Gamma Function defined by

$$\begin{aligned}\Gamma(a) &= \int_0^\infty e^{-x}x^{a-1}\\ &= \lim_{n\to\infty}\left(n!\frac{n^a}{a^{\overline{n+1}}}\right)\end{aligned}$$

which curiously has unexpected applications in medicine [3]. The proof of (9) follows: From the definition of the Gamma function, it follows that if a is a positive integer, then [4]

$$a!=\Gamma(a+1),$$

from which we get

$$\begin{aligned} \frac{\Gamma(a+1)}{\Gamma(a)} &= a\,(a+1)\ldots(a+n-1) \\ &= a^{\overline{n}}. \end{aligned}$$

Thus,

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}^a &= \frac{a^{\overline{n}}}{a^{\overline{k}}a^{\overline{n-k}}} \\ &= \frac{\Gamma\,(a+n)\,\Gamma\,(a)\,\Gamma\,(a)}{\Gamma\,(a)\,\Gamma\,(a+k)\,\Gamma\,(a+n-k)} \\ &= \frac{\Gamma\,(a)\,\Gamma\,(a+n)}{\Gamma\,(a+n-k)\,\Gamma\,(a+k)} \end{aligned}$$

as required.

We next show that

$$\begin{bmatrix} n \\ k \end{bmatrix}^a = \frac{B_{a,k}}{B_{a+n-k,k}} \tag{10}$$

for the Beta function defined by

$$B_{a,k} = \int_0^1 u^{a-1}\,(1-u)^{k-1}\,du.$$

We use the result proved in Gillespie [3]:

$$B_{a,k} = \frac{\Gamma\,(a)\,\Gamma\,(k)}{\Gamma\,(a+k)}.$$

From (9) and (10)

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}^a &= \frac{\Gamma\,(a)\,\Gamma\,(a+n)}{\Gamma\,(a+n-k)\,\Gamma\,(a+k)} \\ &= \frac{\Gamma\,(a+n)}{\Gamma\,(a+n-k)\,\Gamma\,(a+k)} \times \frac{\Gamma\,(a)\,\Gamma\,(k)}{\Gamma\,(a+k)} \\ &= \frac{\Gamma\,(a+n)}{\Gamma\,(a+n-k)\,\Gamma\,(a+k)} \times B_{a,k} \\ &= \frac{B_{a,k}}{B_{a+n-k,k}}. \end{aligned}$$

The following matrix reduces to the well-known Pascal array (or associated triangle) when $a = 1$.

$$
\begin{bmatrix}
\frac{2a-1}{a}\begin{bmatrix} 1 \\ 1 \end{bmatrix}^a & \frac{2a-1}{a}\begin{bmatrix} 1 \\ 1 \end{bmatrix}^a & 0 & 0 & \cdots & 0 \\
\frac{2a-1}{a}\begin{bmatrix} 2 \\ 1 \end{bmatrix}^a & \frac{2a-1}{a}\begin{bmatrix} 2 \\ 1 \end{bmatrix}^a + \frac{2a}{a+1}\begin{bmatrix} 2 \\ 2 \end{bmatrix}^a & \frac{2a}{a+1}\begin{bmatrix} 2 \\ 2 \end{bmatrix}^a & 0 & \cdots & 0 \\
\frac{2a-1}{a}\begin{bmatrix} 3 \\ 1 \end{bmatrix}^a & \frac{2a-1}{a}\begin{bmatrix} 3 \\ 1 \end{bmatrix}^a + \frac{2a}{a+1}\begin{bmatrix} 3 \\ 2 \end{bmatrix}^a & \frac{2a}{a+1}\begin{bmatrix} 3 \\ 2 \end{bmatrix}^a + \frac{2a+1}{a+2}\begin{bmatrix} 3 \\ 3 \end{bmatrix}^a & \frac{2a+1}{a+2}\begin{bmatrix} 3 \\ 3 \end{bmatrix}^a & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
\frac{2a-1}{a}\begin{bmatrix} n-2 \\ 1 \end{bmatrix}^a & \frac{2a-1}{a}\begin{bmatrix} n-2 \\ 1 \end{bmatrix}^a + \frac{2a}{a+1}\begin{bmatrix} n-2 \\ 2 \end{bmatrix}^a & \frac{2a}{a+1}\begin{bmatrix} n-2 \\ 2 \end{bmatrix}^a + \frac{2a+1}{a+2}\begin{bmatrix} n-2 \\ 3 \end{bmatrix}^a & \frac{2a+1}{a+2}\begin{bmatrix} n-2 \\ 3 \end{bmatrix}^a + \frac{2a+2}{a+3}\begin{bmatrix} n-2 \\ 4 \end{bmatrix}^a & \cdots & \frac{2a+n-5}{a+n-4}\begin{bmatrix} n-2 \\ a+n-2 \end{bmatrix}^a + \frac{2a+n-4}{a+n-3}\begin{bmatrix} n-2 \\ a+n-1 \end{bmatrix}^a
\end{bmatrix}.
$$

(12)

Two numerical examples now follow:

$a = 1,\ n = 6$

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 3 & 1 & 0 & 0 & 0 \\
3 & 6 & 4 & 1 & 0 & 0 \\
4 & 10 & 10 & 5 & 1+\begin{bmatrix} 4 \\ 5 \end{bmatrix}^1 & \begin{bmatrix} 4 \\ 5 \end{bmatrix}^1 + \begin{bmatrix} 4 \\ 6 \end{bmatrix}^1
\end{bmatrix}_{4\times 6}
$$

$a = 1,\ n = 8$

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\
3 & 6 & 4 & 1 & 0 & 0 & 0 & 0 \\
4 & 10 & 10 & 5 & 1 & 0 & 0 & 0 \\
5 & 15 & 20 & 15 & 6 & 1 & 0 & 0 \\
6 & 21 & 35 & 35 & 21 & 7 & 1+\begin{bmatrix} 6 \\ 7 \end{bmatrix}^1 & \begin{bmatrix} 6 \\ 7 \end{bmatrix}^1 + \begin{bmatrix} 6 \\ 8 \end{bmatrix}^1
\end{bmatrix}_{6\times 8}
$$

5. Conclusion

One could continue generating similar results which have a certain elegance based on the fundamental structure of their similar fractional format [4]. Likewise, rising binomial coefficients – Type 2 were defined in [10] as

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^a = \frac{n^{\overline{a}}}{k^{\overline{a}}\,(n-k)^{\overline{a}}}$$

and surprisingly they give rise to quite different properties. Some of these have been investigated more recently [5]. The notation used in this paper for these two factorials apparently arose from D.E. Knuth in the discussion which follows a paper by Riordan at the University of North Carolina in 1967 [9].

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