Noname manuscript No.
(will be inserted by the editor)

Peak Oil: Testing Hubbert's curve via theoretical modeling

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Received: date / Accepted: date

Abstract A theoretical model of conventional oil production has been developed. In particular the model does not assume Hubbert's bell curve, an asymmetric bell curve or an R/P method is correct, and does not use oil production data as an input. The theoretical model is in close agreement with actual production data until the 1979 oil crisis with an R^2 value of greater than 0.98 in all three scenarios. Whilst the theoretical model indicates that an ideal production curve is slightly asymmetric, which contradicts Hubbert's curve, the ideal model compares well with the Hubbert model with R^2 values of greater than 0.95. Amending the theoretical model to take into account the 1979 oil crisis, and assuming a URR in the range of 2-3 trillion barrels, the amended model predicts conventional oil production to peak between 2010 and 2025. The amended model for the case when the URR is 2.2 trillion barrels indicates that oil production peaks in 2013.

Keywords Peak Oil · Modeling · Hubbert's Curve

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1 Introduction

There is considerable debate on when and how steeply oil production will peak, with a range of estimate from 2004 to 2047 e.g. ASPO (2004); Deffeyes (2002); Bakhtiari (2004); Mohr and Evans (2007); Wells (2005a,b); EIA (2004). The considerable range in peak oil estimates is due to two main reasons. The first problem is uncertainty in conventional oil URR, with Bauquis (2003) indicating that estimates range from 2 to 3 trillion barrels. The second reason is the different methods for modeling conventional oil production. It should be noted that oil production is model in three distinct ways. Wells (2005a,b); Mohr and Evans (2007); Deffeyes (2002) used a bell (or Hubbert) curve to model oil production. The second method, which was used by ASPO (2004); Bakhtiari (2004), was a graphical model with limited data as to how the model is created. The last method, which was used by EIA (2004) assumed oil production declines with a R/P ratio of 10. The different models create very different production profiles, and hence a wide range of predictions, which ultimately confuse the wider community. Rather than assume a production curve, and attempt to justify its use, this article will endevour to generate a model based on theory. With the theory explained, we will then determine what the oil production profile looks like.

2 Review of Literature

Before explaining how the current model works, it is important to look carefully at the theoretical models already developed by Reynolds (1999); Bardi (2005). Reynolds (1999) explains qualitatively how oil discoveries are comparable to the Mayflower problem. Bardi (2005) using this technique explains the model mathematically as:

$$p(t) = k(t)\frac{URR - C_d(t)}{URR}, \tag{1}$$

where $p(t)$ is the expected discovery percentage, URR is the Ultimate Recoverable Resources (TL), $C_d(t)$ is the cumulative discoveries (TL), and $k(t)$ is the technology function, which is quoted from Bardi (2005) as “a simple linear function of the amount of previously found [oil reserves] that starts at 1 and increases proportionally to the total amount of found [oil reserves]". The models of Reynolds (1999); Bardi (2005) are based on a simplified scenario with Robinson Crusoe digging for buried hardtacks (food).

The work done by Brandt (2007) is statistical. Brandt (2007) obtained production data for many places of various sizes. The result from Brandt (2007) research is that the rate difference, Δr, is slightly positive with a median of 0.05 year^{-1}, which implies that on average the rate of increase is slightly larger than the rate of decrease Brandt (2007), see Appendix A.

3 Model

The model of oil production is determined in several subsections. In the discovery subsection the amount of oil found in a given year will be determined. It will then be assumed that the amount of oil found each year is located in a single reservoir. The reservoir production subsection will model oil production in a reservoir by estimating the number of wells in operation and estimating the oil production production per well. The world production model is then determined by summing the oil production of all the reservoirs.

3.1 Discoveries:

We will assume that finding oil is equivalent to the mayflower problem, hence the expected discovery percentage function will be determined by Equation 1 (Bardi, 2005). Now the technology function $k(t)$ must be between 0 and 1, in order for the expected discovery percentage to remain bounded between 0 and 1. It is worth noting that some Optimists such as Linden (1998) believe that technology makes "marginal hydrocarbon resources" economic. It is also reasonable to assume that the technology function is non-decreasing. Given these constraints we will assume the technology function $k(t)$ is:

$$k(t) = \left[\tanh\left(b_t\left(t - t_t\right)\right) + 1\right]/2,$$

where b_t and t_t are constants with units (year^{-1}) and (year) respectively. Hence the expected discovery percentage function is:

$$p(t) = \left[\tanh\left(b_t\left(t - t_t\right)\right) + 1\right] \frac{URR - C_d(t)}{2URR}. \tag{2}$$

Initially the expected discovery percentage is low as our knowledge is limited, as time continues the expected discovery percentage increases as our knowledge grows, whilst the amount of oil discovered is still small (relative to the URR). Eventually we have good knowledge of where the oil is to be found, but the amount of oil left to be discovered is small (relative to the URR) hence the expected discovery percentage is low. Let $C_d(t)$ denote the cumulative discoveries of oil made to the beginning of year t (TL). Now, the amount of oil found in year t equals the expected discovery percentage times the amount of oil left to be found in year t, which mathematically is

$$C_d(t+1) - C_d(t) = p(t)(URR - C_d(t)). \tag{3}$$

Now since the expected discovery percentage function $p(t)$ is continuous, we can express Equation 3 in the continuous form as

$$\frac{dC_d(t)}{dt} = p(t)(URR - C_d(t)).$$

Substituting Equation 2 for the expected discovery percentage function $p(t)$ we obtain

$$\frac{dC_d(t)}{dt} = \left[\tanh\left(b_t\left(t - t_t\right)\right) + 1\right] \frac{(URR - C_d(t))^2}{2URR}. \tag{4}$$

With the trivial assumption that initially $C_d(0) = 0$, Equation 4 is solved to get Equation 5

$$C_d(t) = URR - \frac{2b_t URR}{2b_t + tb_t + \ln\left(\frac{\cosh(b_t(t-t_t))}{\cosh(b_t t_t)}\right)}. \tag{5}$$

Let $y_d(t)$ denote the yearly discoveries (TL/year), $dC_d(t)/dt$, then by differentiating Equation 5 we obtain,

$$y_d(t) = \frac{2b_t^2 URR\left(1+\tanh(b_t(t-t_t))\right)}{\left(2b_t + tb_t + \ln\left(\frac{\cosh(b_t(t-t_t))}{\cosh(b_t t_t)}\right)\right)^2}. \tag{6}$$

Let URR_l denote the size of the l-th reservoir (TL), which is assumed to be found in the year t_l. Now since we assume that the amount of oil found each year is found in a single reservoir, we have

$$URR_l = y_d(t_l).$$

3.2 Reservoir Production:

To determine the production curve from a reservoir, we will assume that oil production is related to the number of wells drilled, and the production per well. Let $C_{p_l}(t)$ denote the cumulative production from the l-th reservoir (TL). Let $w_l(t)$ denote the number of wells in operation at time t. The function $w_l(t)$, will be defined by Equation 7

$$w_l(t) = w_{l_T} + (1-w_{l_T})e^{-k_{w_l}\left(\frac{C_{p_l}(t)}{URR_l}\right)},\, t \geq t_l \tag{7}$$

Where k_{w_l} is a proportionality constant and w_{l_T} is the total number of wells in operation assuming $C_{p_l}(t)$ increases to infinity. Note the boundary condition $C_{p_l}(t_l) = 0$ which implies $w_l(t_l) = 1$, hence initially there is only one well built. As cumulative production increases the number of wells exponentially decays upwards from 1 well to w_{l_T} wells. Note the total number of wells built is not w_{l_T} but $w_{l_{T_{act}}}$ which is defined as

$$w_{l_{T_{act}}} = \left\lceil w_{l_T} - (w_{l_T}-1)e^{-k_{w_l}} \right\rceil.$$

Lets assume that every well in the l-th reservoir extracts a total of $URR_l/w_{l_{T_{act}}}$ (TL) of oil. Let the i-th well start production in the t_{l_i}-th year, where t_{l_i} is the year such that $\lceil w_l(t_{l_i}-1)\rceil < i \leq \lceil w_l(t_{l_i})\rceil$ (initially $t_{l_1} = t_l$). Let $C_{p_{l_i}}$ denote the cumulative production from well i. Production for an individual well is assumed to be the idealized well explained in Arps (1945). In this case, there is no water injection, and oil production in the i-th well, P_{l_i}, is proportional to the pressure in the i-th well, Pr_{l_i}. Further the pressure in the well is proportional to the remaining amount of oil in the i-th well, $(URR_l/w_{l_{T_{act}}} - C_{p_{l_i}}(t - t_{l_i}))$, as shown in Equations 8 and 9 (Arps, 1945):

$$P_{l_i}(t) = k_{1_{l_i}} Pr_{l_i}(t), \tag{8}$$

$$Pr_{l_i}(t) = k_{2_{l_i}} \left(URR_l/w_{l_{T_{act}}} - C_{p_{l_i}}(t) \right). \tag{9}$$

Note $k_{1_{l_i}}$ and $k_{2_{l_i}}$ are proportionality constants. Equations 8 and 9 can be combined to obtain

$$P_{l_i}(t) = k_{1_{l_i}} k_{2_{l_i}} \left(URR_l/w_{l_{T_{act}}} - C_{p_{l_i}}(t) \right).$$

Now, $dC_{p_{l_i}}(t)/dt = P_{l_i}(t)$, hence

$$\frac{dC_{p_{l_i}}(t)}{dt} = k_{p_{l_i}} \left(URR_l/w_{l_{T_{act}}} - C_{p_{l_i}}(t) \right), \tag{10}$$

where $k_{p_{l_i}} = k_{1_{l_i}} k_{2_{l_i}}$, and $C_{p_{l_i}}(t_{l_i}) = 0$. Now Equation 10 is trivially solved to obtain

$$C_{p_{l_i}}(t) = \frac{URR_l}{w_{l_{T_{act}}}} \left[1 - e^{-k_{p_{l_i}}(t - t_{l_i})} \right],$$

and differentiating obtains the production curve

$$P_{l_i}(t) = k_{p_{l_i}} \frac{URR_l}{w_{l_{T_{act}}}} e^{-k_{p_{l_i}}(t - t_{l_i})}.$$

Let the initial production of the i-th well, in the l-th reservoir be $P_{0_{l_i}}$, $(P_{l_i}(t_{l_i}) = P_{0_{l_i}} \ \forall i)$ then the production curve for the i-th well is (Arps, 1945)

$$P_{l_i}(t) = P_{0_{l_i}} e^{-P_{0_{l_i}} w_{l_{T_{act}}} (t - t_{l_i})/URR_l}.$$

Hence the cumulative production for the l-th reservoir, $C_{p_l}(t)$, is determined iteratively by Equation 11

$$C_{p_l}(t+1) = C_{p_l}(t) + \sum_{i=1}^{\lceil w_l(t) \rceil} \left(\frac{P_{l_i}(t) + P_{l_i}(t+1)}{2} \right), \tag{11}$$

with the initial condition $C_{p_l}(t_l) = 0$. The world's cumulative production, $C_p(t)$, is trivially the sum of the cumulative production of the reservoirs,

$$C_p(t) = \sum_l C_{p_l}(t)$$

For ease of use we will assume that all wells in all reservoirs have the same initial production, P_0, that is $P_0 = P_{0_{l_i}}$, it is also assumed that $k_w = k_{w_l}$.

4 Results and Discussion

Bauquis (2003) indicates that URR estimates for conventional oil have remained constant at between 2-3 trillion barrels (318-477 TL) for the time period of 1973-2000. A Pessimistic case will assume that the URR is 318 TL (2 trillion barrels); the Optimistic case will assume the URR to be 477 TL (3 trillion barrels). An ideal case is also made where the URR is determined from the actual backdated discoveries data from Wells (2005b). We have several constants, which need to be defined. For the discovery model we have URR, t_t and b_t, for the number of wells model its k_w, and w_{l_T} and for the production of a well we need P_0. The variables for the discovery model were calculated by fitting the model to the actual data from Wells (2005b) using the coefficient of determination, R^2, for more details see Appendix B. The cumulative discoveries as a function of time is shown in Figure 1.

Figure 1 Hereabouts

In order to determine valid estimates for k_w, w_{l_T}, and P_0, it was necessary to find some literature data. The best literature found to date is from EIA (2007), which has incomplete well and production data for all U.S. states. By analyzing the EIA (2007) data, we assumed $P_0 = 18.3$ ML/year, $k_w = 10.7$ and $w_{l_T} = 0.072 URR_l/P_0$ respectively, for more details see Appendix C. With the constants determined the world model is shown in Figure 2;

and compared to actual production data from BP (2006); DeGolyer and MacNaughton Inc. (2006); CAPP (2006); Williams (2003); Moritis (2005).

Figure 2 Hereabouts

The resulting model of production matches the production data with a reasonable precision up to the 1979 oil crisis (year 119 in Figure 2) with an R^2 value in all three cases of greater than 0.98. The theoretical models when fitted to the asymmetric exponential model, have a slightly positive rate difference of $\Delta r \approx 0.02$ year^{-1}, which agrees with the statistical analysis of Brandt (2007), who indicated a median rate difference of $\Delta r = 0.05$ year^{-1} see Appendix A. for more details. The theoretical models are approximately symmetrical and have R^2 values of great that 0.95 when compared to Hubbert curves with the same URR fitted to production data prior to 1979, with the Ideal case compared to the Hubbert curve having an R^2 value of 0.995.

The theoretical model was ammended by use of a technique in Mohr and Evans (2007), to account for the 1979 oil crisis. The method in Mohr and Evans (2007) has four key components: first the original theoretical curve is used to model oil production prior to the anomaly (1979 oil crisis). Second, simple linear or low order polynomials are fitted to the production data from the anomaly to the present day. Three, a polynomial is used to extend the recent production trend, and smoothly rejoin the original theoretical model, in the future. Four, the model returns to the original theoretical model, shifted a certain distance into the future to ensure the area under the graph (URR) is the same. Modifying the theoretical production curve using the literature method in (Mohr and Evans, 2007), allowed for the 1979 oil crisis to be factored, for more details see Appendix D. The amended model is shown in Figure 3 and indicates that the ideal case will peak in 2013, at 13.3 GL/d (83.5 mb/d). The optimistic case peaks in 2025 at 14.1 GL/d (88.8 mb/d), and the pessimistic case peaks in 2010, at 13 GL/d (81.8 mb/d).

Figure 3 Hereabouts

Whilst the theoretical model matches the data with reasonable accuracy $R^2 > 0.98$, it is important to note several gross simplifications. The assumption that P_0 and k_w are constants for all wells and reservoirs is too simplistic. Also instead of modeling four US states and using these values to estimate P_0 and k_w it would be better to use a large data set of reservoir data, to determine the average P_0 and k_w for each reservoir, unfortunately such data was not found.

5 Conclusion

A model has been developed to model oil production using simple theoretical logic. The model accurately replicates the actual discovery and production trends, whilst remaining theoretical. The model produces a bell curve, which is slightly asymmetric with a slightly larger rate of increase compared to the rate of decrease ($\Delta r = 0.002$ year^{-1}). The model validates Hubberts empirical model which indicates that oil production follows a symmetric bell curve. The theoretical model indicates that conventional oil production will peak somewhere between 2010 and 2025, with the ideal case peaking in 2013, at 13.3 GL/d (83.5 mb/d).

Nomenclature

Functions

$C_d(t)$	The Cumulative discoveries for the world as a function of time (TL)
$C_p(t)$	The Cumulative production for the world as a function of time (TL)
$C_{p_l}(t)$	The Cumulative production for the reservoir l as a function of time (TL)
$C_{p_{l_i}}(t)$	The Cumulative production for the i-th well in reservoir l (TL)
$k(t)$	The technology function (-)
$p(t)$	The expected discovery percentage function (-)
$P'(t)$	The Production function as used in Brandt (2007) (b/year)
$P_{l_i}(t)$	The production in the i-th well of reservoir l (TL/year)
$Pr_{l_i}(t)$	The pressure in the i-th well of reservoir l as a function of time (Pa)
R^2	The Coefficent of determination (-)
$w_l(t)$	The number of wells in operation for the reservoir l as a function of time (-)

$y_d(t)$	The yearly discoveries function (TL/year)

Variables

b_t	The slope constant for the technology function (year^{-1})
$k_{1_{l_i}}$	The proportionality constant relating production to pressure, in the i-th well (TL/Pa.year)
$k_{2_{l_i}}$	The proportionality constant relating pressure to remaining reserves (Pa/TL)
$k_{p_{l_i}}$	The proportionality constant relating the production to the remaining reserves (year^{-1})
k_w	The proportionality constant in the wells model (-)
k_{w_l}	The proportionality constant for reservoir l in the wells model (-)
P_0	The initial production of the wells in all reservoirs (TL/year)
P_{0_l}	The initial production of the wells in reservoir l (TL/year)
$P_{0_{l_i}}$	The initial production from the i-th well in reservoir l (TL/year)
r_{dec}	The rate of decrease, as used by Brandt (2007) (year^{-1})
r_{inc}	The rate of increase, as used by Brandt (2007) (year^{-1})
Δr	The difference between the rate of increase and rate of decrease, as used by Brandt (2007) (year^{-1})
t	Time (year)
t_l	The year the l-th reservoir is found (year)
t_{l_i}	The year the i-th well comes on-line in reservoir l (year)
T_{peak}	The Peak year for the production curve as used in Brandt (2007) (year)
T_{start}	The start year for the production curve as used in Brandt (2007) (year)
t_t	The year the technology function reaches 0.5 (year)
URR	The Ultimate Recoverable Resources (TL)
URR_l	The Ultimately Recoverable Resources for the reservoir l, (TL)
w_{l_T}	The total number of wells for reservoir l, if cumulative production was infinite (-)
$w_{l_{T_{act}}}$	The total number of wells for reservoir l given the cumulative production is finite (-)

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Appendix A. The rate difference

The rate difference, Δr, as defined by (Brandt, 2007) is

$$\Delta r = r_{inc} - r_{dec},$$

where the rate of increase r_{inc} and rate of decrease r_{dec} are determined by fitting Equation A.1 to the production data (Brandt, 2007).

$$P'(t) = \begin{cases} e^{r_{inc}(t-T_{start})} & \text{if } t \leq T_{peak} \\ P'(T_{peak})e^{-r_{dec}(t-T_{peak})} & \text{if } t > T_{peak} \end{cases} \tag{A.1}$$

where $P'(t)$ is production in (barrels/year) and T_{peak} is the year the production peaks (year) and r_{inc} and r_{dec} are the rate constants (year^{-1}) and T_{start} is the year production was 1 barrel a year Brandt (2007). In calculating the rate difference of the theoretical model equation A.1 was altered to

$$P'(t) = \begin{cases} P(40)e^{r_{inc}(t-40)} & \text{if } t \leq T_{peak} \\ P'(T_{peak})e^{-r_{dec}(t-T_{peak})} & \text{if } t > T_{peak} \end{cases} \tag{A.2}$$

where $P(40)$ is the production of oil estimated by the theoretical model in the year 1900. Using equation A.2 the rate difference for the Pessimistic case was 0.0184 (years^{-1}), Optimistic case was 0.0179 (years^{-1}) and the Ideal case was 0.0217 (years $^{-1}$).

Appendix B. Coefficient of Determination

The coefficient of determination, R^2, was used to measure the accuracy of the discovery model to the data.

$$R^2 = \sum_n \frac{[y_f(n) - \bar{y_a}]^2 - [y_f(n) - y_a(n)]^2}{[y_f(n) - \bar{y_a}]^2}$$

For the Pessimistic case $URR = 2$ trillion barrels and for the Optimistic case $URR = 3$ trillion barrels. For the ideal case, the URR was a variable. The best fit was found by varying b_t, t_t (and URR for ideal case) to obtain the highest R^2 value. The constants are shown in Table B.1. The actual data for the Pessimistic and Optimistic cases was truncated to the year 1966, as the Optimists claim that oil reserves found in the past will grow.

Table B.1 hereabouts

Appendix C. determining the constants

The method to determine valid estimates for the constants k_w, w_{l_T}, and P_0 in the reservoir production model is given in this section. The best data found is unfortunately state based rather than reservoir based data from EIA (2007). The production model was used with several cycles to model the production as a function of time, and the number of wells as a function of cumulative production for various states as shown in Figures C.1 $-$ C.4.

Figures C.1 $-$ C.4 hereabouts

Note our model assumes that no wells are shut down, and instead exponentially decay and although are still on-line, are in reality producing no significant quantity of oil. This is the reason for the poor fit of the well model for Nevada and South Dakota. Now observe that there is only one sensible option for the w_{l_T} constants, since these values need to match the actual total wells. The k_w values and P_0 values determine the rate of increase in the wells model and are determined by trial and error so that the wells model and production model fit the data as accurately as possible. Whilst the values used produce reasonably accurate results, we need to check that the initial production values P_0 correspond to the actual initial production. Unfortunately the initial production for all the wells is not known, however the number of wells as a function of size and time is known EIA (2007) and the model's predictions were compared the actual data for the four states, as shown in Figures C.5 $-$ C.8 (Note that the size of the wells from EIA (2007) is explained in Table C.1).

Table C.1 hereabouts

Figures C.5 $-$ C.8 hereabouts

The Figures C.5 $-$ C.8 indicate a reasonable fit and hence the initial well productions P_0 can be assumed to be reasonable estimates. The constants k_w, P_0 and w_{l_T} used in the state models are shown in Table C.2

Table C.2 hereabouts

Now if we ignore the outlier of 32 for South Dakota, the average for k_w is 10.7 and this value is assumed to be constant in the world model; including the outlier the average becomes 12. By plotting w_T versus URR_l/P_0 we obtained the linear relation $w_{l_T} = 0.072 URR_l/P_0$ which is shown in Figure C.9. The linear relationship is expected, as increasing the size of the reservoir would increase the total number of wells needed. Equally if we have two reservoirs of the same size we could either have a small number of wells with a large initial production P_0 or a large number of wells with a small initial production P_0. Hence the linear relationship between w_{l_T} and URR_l/P_0 was expected.

The value for P_0 appears to have a great deal of variability. However by analyzing the other US state wells sizes from EIA (2007), we observe that Alaska along with Federal Pacific and Federal Gulf, have abnormally large wells compared to the rest of the US, we hence considered the Alaskan well production data as an outlier and ignored the data. Taking the initial production from Nevada, South Dakota and Alabama, we obtain an average of 18.3 ML/year, which places it in category 16 in the EIA sizes. Hence we assumed that the values of the constants were $P_0 = 18.3$ ML/year, $k_w = 10.7$ and $w_{l_T} = 0.072 URR_l / P_0$.

Figure C.9 hereabouts

Appendix D. Amended model

The amended model $C_{p_{mod}}(t)$ is determined from the method explained in Mohr and Evans (2007), and formally is:

$$C_{p_{mod}}(t) = \begin{cases} C_p(t) & \text{if } t \le 118 \\ f_1(t) & \text{if } 118 < t \le 123 \\ f_2(t) & \text{if } 123 < t \le 129 \\ f_3(t) & \text{if } 129 < t \le 145 \\ f_4(t) & \text{if } 145 < t \le t_2 \\ C_p(t + (t_1 - t_2)) & \text{if } t_2 < t \end{cases} .$$

Now $f_1(t)$, $f_2(t)$ and $f_3(t)$ are small polynomials fitted to the production data using least squares method, and formally are:

$$f_1(t) = -0.82t + 121.7$$

$$f_2(t) = 0.43t - 32.3$$

$$f_3(t) = 0.007t^2 - 1.6t + 109.5$$

the $f_4(t)$ is a 3rd degree polynomial. The polynomial was determined by the literature method explained generally in Mohr and Evans (2007). Specifically $f_4(t)$ is the 3rd degree polynomial such that the following equations are solved:

$$f_4(145) = p(145)$$

$$f_4'(145) = p'(145)$$

$$f_4''(145) = p''(145)$$

$$f_4(t_2) = C_p(t_1)$$

$$f_4'(t_2) = C_p'(t_1)$$

$$\int_{t_0}^{t_1} C_p(t)dt = \int_{145}^{t_2} p(t)dt.$$

Where $t_0 \approx 138$ and $p(t)$ is a polynomial to replicate the long term historic trend and is $p(t) = -0.004t^2 + 1.5t - 104.6$. With the list of equations solved, we obtain $t_1 = 153.5$, $t_2 = 162$ and $f_4(t) = -0.0012t^3 + 0.51t^2 - 73.4t + 3515.6$ for the ideal case. For the Pessimistic case it was $t_1 = 147.2$, $t_2 = 153.6$, and $f_4(t) = -0.0041t^3 + 1.77t^2 - 256.2t + 12353.8$. $t_1 = 177.8$, $t_2 = 196.4$, and $f_4(t) = -0.00009t^3 + 0.036t^2 - 4.3t + 178.8$ for the optimistic case.

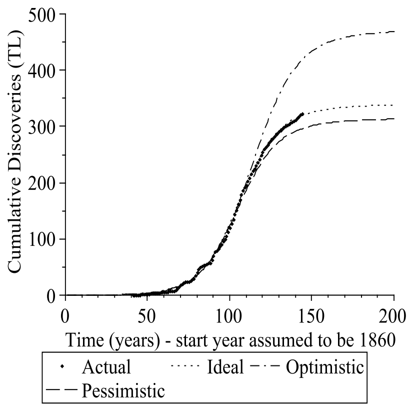

Fig. 1 The modeled cumulative discoveries as a function of time (the year 1860 is assumed as the start year $t = 0$.)

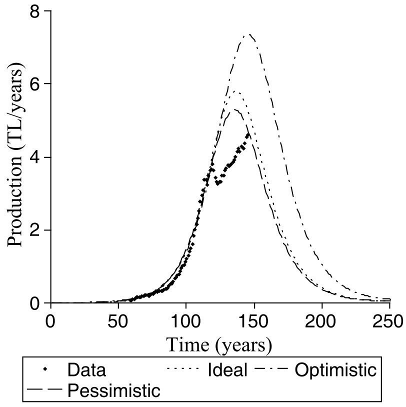

Fig. 2 The production model compared to the actual data

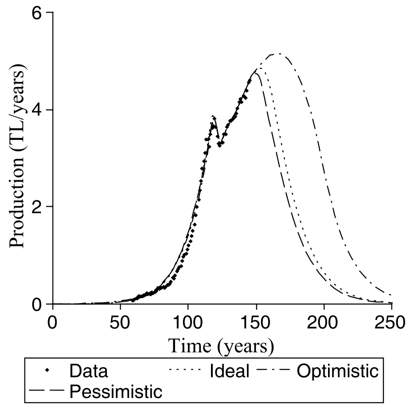

Fig. 3 Results of the modified model compared to the actual production data

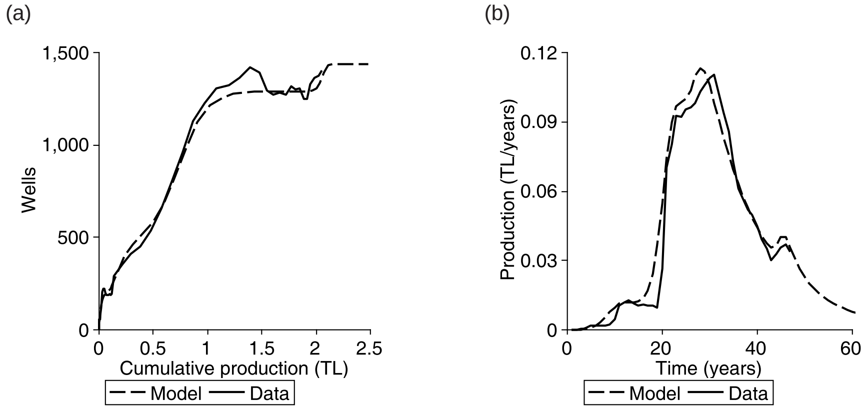

Fig. C.1 a) The number of wells as a function of cumulative production for Alaska and b) Production as a function of time for Alaska

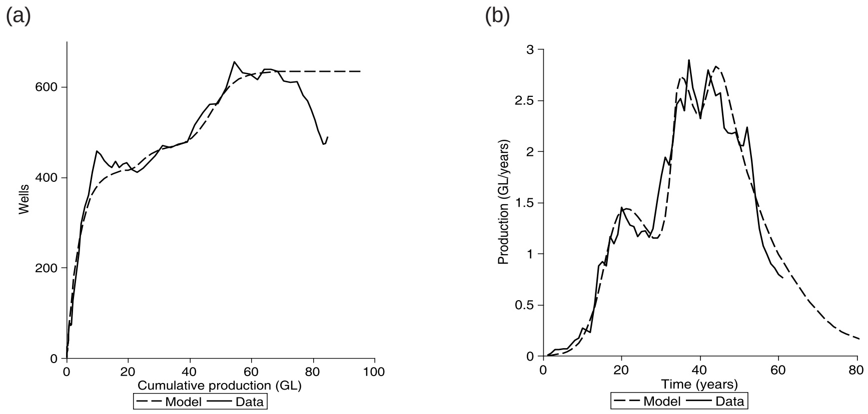

Fig. C.2 a) The number of wells as a function of cumulative production for Alabama and b) Production as a function of time for Alabama

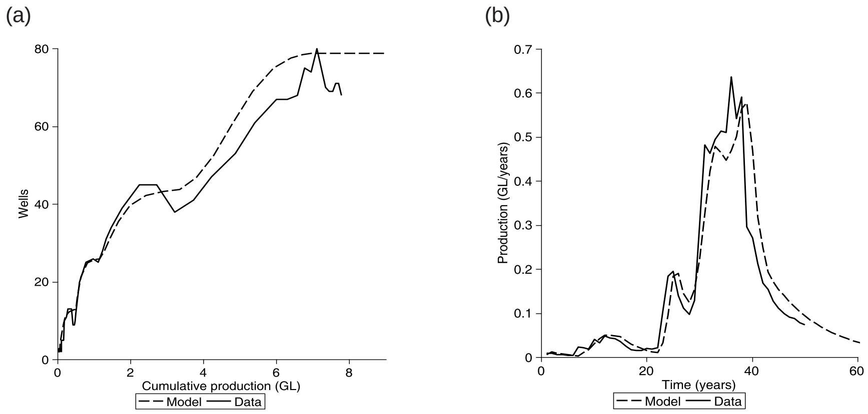

Fig. C.3 a) The number of wells as a function of cumulative production for Nevada and b) Production as a function of time for Nevada

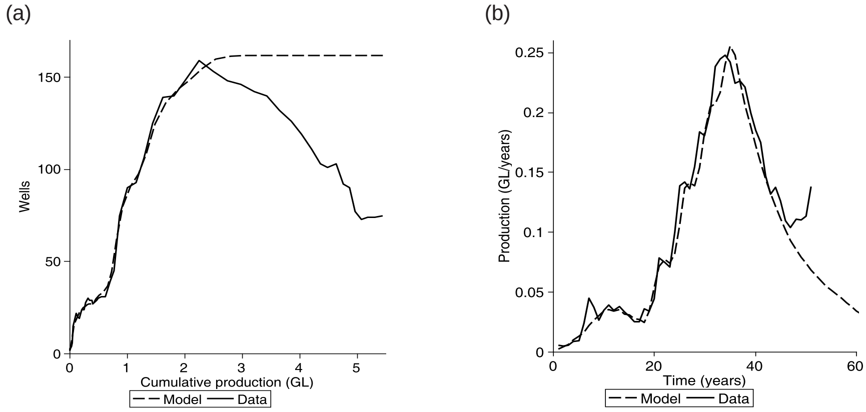

Fig. C.4 a) The number of wells as a function of cumulative production for South Dakota and b) Production as a function of time for South Dakota

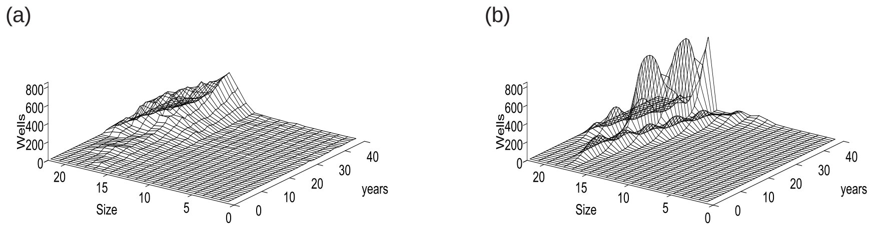

Fig. C.5 The number of wells as a function of size and time for Alaska. a) Actual and b) Model

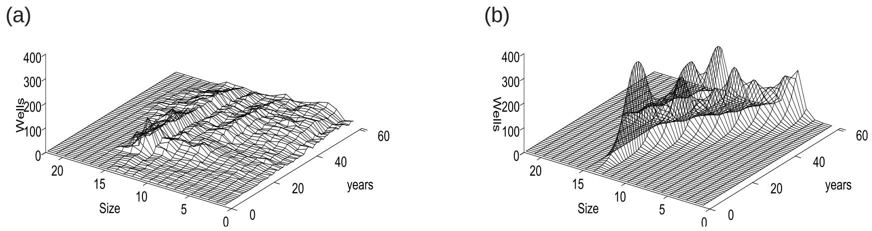

Fig. C.6 The number of wells as a function of size and time for Alabama. a) Actual and b) Model

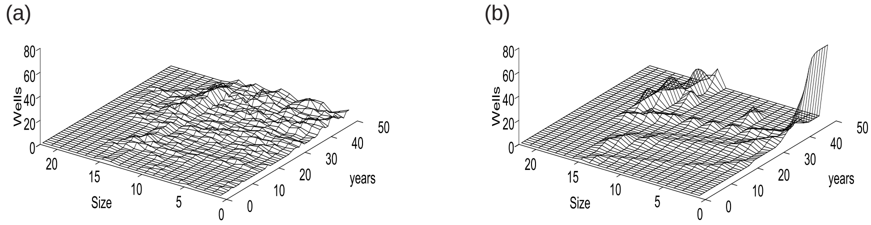

Fig. C.7 The number of wells as a function of size and time for Nevada. a) Actual and b) Model

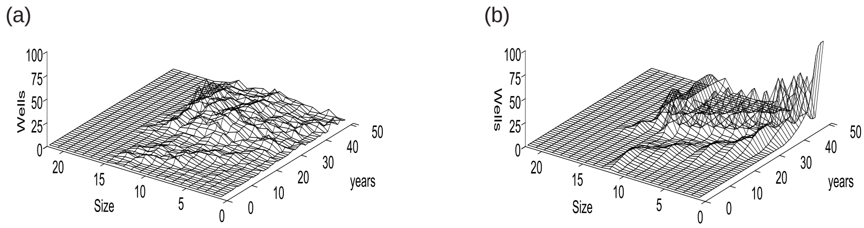

Fig. C.8 The number of wells as a function of size and time for South Dakota. a) Actual and b) Model

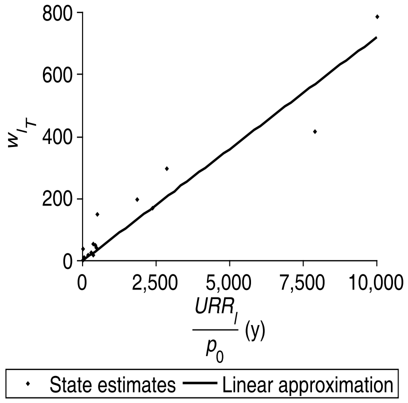

Fig. C.9 w_{l_T} versus URR_l/P_0

Table B.1 The URR, b_t and t_t for the 3 cases

Case	URR TL (trillion barrels)	b_t year^{-1}	t_t year
Pessimistic	318 (2)	0.0413	135.3
Optimistic	477 (3)	0.0372	148.9
Ideal	343 (2.16)	0.0421	135.4

Table C.1 The size of wells from EIA (2007)

Category	Size barrels/day		
1	0	–	1
2	1	–	2
3	2	–	4
4	4	–	6
5	6	–	8
6	8	–	10
7	10	–	12
8	12	–	15
9	15	–	20
10	20	–	25
11	25	–	30
12	30	–	40
13	40	–	50
14	50	–	100
15	100	–	200
16	200	–	400
17	400	–	800
18	800	–	1600
19	1600	–	3200
20	3200	–	6400
21	6400	–	12800
22		$>$	12800

Table C.2 The constants for various states

State	URR_l (GL)	k_w	P_0 $\times 10^{-3}$(GL/year)	w_{l_T}
Alaska	174.9	9	95	200
	1271.9	11	445	300
	953.9	15	95	790
	79.5	11	159	150
Alabama	41.3	10	5	420
	27.0	15	59	45
	30.2	13	13	170
Nevada	0.1	-[a]	8	2
	0.5	5	10	11
	0.8	7	30	13
	6.4	14	35	18
	0.8	2	51	39
South Dakota	0.6	6	2	28
	0.6	12	13	6
	0.8	15	2	56
	1.4	16	3	52
	2.7	32	8	20

[a] $w_l(t) = 2$