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GLOBALLY EXPONENTIALLY STABLE PERIODIC SOLUTION IN A GENERAL DELAYED PREDATOR-PREY MODEL UNDER DISCONTINUOUS PREY CONTROL STRATEGY

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Abstract. This paper studies the solution behaviour of a general delayed predator-prey model with discontinuous prey control strategy. The positiveness and boundeness of the solution of the system is firstly investigated using the comparison theorem. Then the sufficient conditions are derived for the existence of positive periodic solutions using the differential inclusion theory and the topological degree theory. Furthermore, the positive periodic solution is proved to be globally exponentially stable by employing the generalized Lyapunov approach. The global finite-time convergence is also discussed for the system state. Finally, the numerical simulations of four examples are given to validate the correctness of the theoretical results.

1. **Introduction.**

Predator-prey systems are of significant importance in many fields such as biology, physics, chemical technology, population models and economy. The dynamic behavior analysis of predator-prey systems plays an important role in the design and application of the predator-prey model (see[1, 21]). The periodic solution and the stability of the positive equilibrium are the main concern for the predator-prey systems.

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From the control method perspectives, there exist several types of control strategies for the predator-prey models, including the economic threshold control (see[6, 22]), the sliding mode control (see[5, 36]), and the feedback control (see[11, 38]). For the economic threshold control strategy, harvesting is carried out when the population density of prey is above a certain harvesting economic level, while harvesting is prohibited when the population density of prey is below the harvesting economic level. As pointed out by [22], owing to the presence of time delays, it is difficult for managers to carry out the economic threshold control strategy efficiently. Discontinuous prey harvesting control strategy is an effective control method of the economic threshold control methodologies and has attracted significant research interest over the past decade. The main purpose of the harvesting control is to ensure the existence of globally asymptotically periodic solution of the predator-prey systems to maintain enough natural resources and at the same time to avoid the extinction of species.

When a discontinuous control strategy is implemented to the predator-prey system, the resulting dynamic model becomes discontinuous predator-prey models. Generally speaking, there are two types of discontinuous predator-prey models, one is a delayed predator-prey model with control strategy, and the other is a non-autonomous system with control strategy ([4, 32, 9, 42, 27, 14, 17, 20, 36]). Cai et al.[2, 3] studied the periodic dynamical behavior of a class of delayed Filippov system under the framework of differential inclusion in set-valued analysis. Duan et al.[9, 7] considered a delay Lasota-Wazewska model with discontinuous harvesting policy. Luo et al. [32] gave the almost periodicity of the delayed predator-prey model with mutual interference and discontinuous harvesting policy. Wang et al.[37] investigated a non-autonomous Hassell-Varley type delayed prey-predator system with non-selective harvesting control strategy. Martin et al. [33] studied the predator-prey models with delay and prey harvesting. Fan et al.[10] examined a non-autonomous delayed ratio-dependent predator-prey system by using the continuation theorem of coincidence degree theory. Liu et al. [28] presented the dynamics of a stochastic regime-switching distributed delays predator-prey model with harvesting. Luo et al. [31] analyzed the global boundedness of the solutions in a Beddington DeAngelis type predator-prey model with nonlinear prey taxis and random diffusion. Liu et al. [29] studied the nonlinear dynamic behavior in a predator-prey model with harvesting. Chakraborty et al. [4] investigated a prey-predator type fishery model incorporating partial closure for the populations. Song et al. [35] analysed a non-autonomous ratio-dependent three species predator-prey system with additional food for predator. Lu et al. [30] considered a non-constant eco-epidemiological model with SIS-type infectious disease in prey. Zhang et al. [41] studied a stochastic non-autonomous Lotka-Volterra predator-prey model with impulsive effects. Jiang et al. [19] investigated a stochastic non-autonomous competitive Lotka-Volterra model in a polluted environment. Zuo et al. [44] examined a stochastic non-autonomous Holling-Tanner predator-prey system with impulsive effect. Li et al. [26] gave the dynamics of a non-autonomous Beddington-DeAngelis type density-dependent predator-prey system.

Although the periodic solution of a predator-prey model ([2, 3, 33, 32, 37, 13, 15, 18, 40, 23, 24, 16]) was extensively studied, to the best of our knowledge, the general delayed predator-prey model with discontinuous prey harvesting control has not yet been considered. Due to the characteristic of discontinuous harvesting control strategy, the existing results obtained for the optimal continuous harvesting policy, threshold policy and weighted escapement policy cannot be directly applied to the general delayed predator-prey model with discontinuous harvesting control strategy. New analysis needs to be performed for the periodic solution of a general delayed predator-prey model with discontinuous harvesting control strategy.

Motivated by the above brief discussions and inspired work [43, 38, 33], the main focus of this paper is on the periodic solution analysis of a general delayed predator-prey model with discontinuous harvesting control strategy. The dynamical behavior of the general delayed system with discontinuous prey harvesting control is more complex than that of a predator-prey model without delay considered in [43, 38, 33], because of the complexity of the structure of the delayed system with discontinuous prey harvesting control. The novelty of this paper lies in three aspects. Firstly, the regularity and visibility analysis of the general delayed predator-prey model is conducted by using the principle of differential inclusion. Secondly, it is found that there exists a periodic solution for the non-autonomous delayed predator-prey model by using the principle of topological degree and set value mapping. Furthermore, it is shown that the solution of the general delayed predator-prey system is globally exponentially stable by utilizing the Mawhin-like coincidence theorem. Numerical simulations of four examples are presented to demonstrate the correctness of the theoretical results. The obtained results show that the general predator-prey model with delay and discontinuous prey harvesting introduced in this paper can have more dynamical behaviors than the conventional models without delay.

The rest of this paper is organized as follows. Section 2 presents a general delayed predator-prey model with discontinuous prey harvesting control strategy and some preliminaries for the analysis. Section 3 performs the periodic solution analysis of the general delayed predator-prey model with discontinuous prey harvesting control strategy. In Section 4, four illustrative examples and their simulations are provided to demonstrate the effectiveness of the theoretical results obtained. Concluding remarks are given in Section 5.

2. **Preliminaries.**

This section briefly introduces the general delayed predator-prey model with discontinuous prey control strategy and provides some definitions and lemmas for the analysis. Throughout this paper, we set a positive continuous function of ω-period on a compact interval of R (see[2]) as:

$$\bar{\eta} = \frac{1}{\omega}\int_0^\omega \eta(t)\mathrm{d}t, \eta^M = \max_{t\in[0,\omega]}\eta(t), \eta^L = \min_{t\in[0,\omega]}\eta(t). \tag{2.1}$$

2.1. **Model description.**

This paper considers the discontinuous prey control strategy introduced in the first equation and a time delay τ in the interplay term $y(t)Q(x(t))$ of the second equation of the predator-prey system as:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = r_1(t)x(t)g(x(t),K(t)) - k_1(t)P(t,x(t))y(t) - \varepsilon_1(t)h_1(x(t))x(t), \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = y(t-\tau)k_1(t)Q(t,x(t-\tau)) - y(t)\delta(t), \end{cases} \tag{2.2}$$

where the change rate of the predators depends on the number of prey and predator at a certain previous time (see [43, 33, 2] and [39]); $x(t)$ and $y(t)$ represent the population densities of the prey and predator, respectively; $r_1(t)$ denotes the specific growth rate of the prey, and $K(t)$ stands for the carrying capacity of the prey; $\delta(t)$ is the death rate of the predator and $k_1(t)$ is a positive constant describing the effects of the capture rate; $r_1(t), k_1(t), \delta(t), \varepsilon_1(t), K(t), b_1(t), c(t)$ are positive continuous ω-periodic functions.

The other parameters and functions involved in the equation when $\tau = 0$ are satisfied the following assumptions (see [43, 34, 33] for more details):

(H1): $g(x(t, K(t))$ represents the net growth rate of the prey and satisfies $g(K(t), K(t)) = 0$, $g(0, K(t)) > 0$ $g'_x(x(t), K(t)) < 0$, $\lim_{K\to\infty} g'_x(x(t), K(t)) = 0$.(for $x(t) > 0, K(t) > 0$). In addition, there exists a positive constant G and a continuous function ξ such that $g(x(t), K(t)) - g(x_1(t), K(t)) = g'_x(\xi, K(t))(x(t) - x_1(t))$, $|g'_x(\xi, K(t))| < G$.

(H2): $P(t, x(t))$ is a continuously differentiable predator response function and satisfies $P(0, x(0)) = 0, P(t, x(t)) = c(t)x(t)p(t, x(t)) > 0$ for $x(t) > 0$, $\lim_{\substack{x\to\infty\\ t\to\infty}} p(t, x(t)) < \infty$. Moreover, there exists a positive constantP_0 and a continuous function ν such that $p(t, x(t)) - p(t, x_1(t)) = p'_x(t, \nu)(x(t) - x_1(t))$, $|p'_x(t, \nu)| < P_0$.

(H3): $Q(t, x(t)) = b(t)P(t, x(t))$ denotes continuously differentiable predator response function and satisfies $Q(0, x(0)) = 0, Q(t, x(t)) > 0$ for $x(t) > 0$. There also exists a positive constant P_0 and a continuous function ν such that $p(t, x(t)) - p(t, x_1(t)) = p'_x(t, \nu)(x(t) - x_1(t))$, $|p'_x(t, \nu)| < P_0$.

(H4): h_1 is continuous except on a countable set of isolate points $\{\rho_k\}$, where there exist finite right and left limits, $h_1^+(\rho_k)$ and $h_1^-(\rho_k)$, with $h_1^+(\rho_k) > h_1^-(\rho_k)$. In addition, h_1 has a finite number of discontinuous points on any compact interval R, $\forall x(t) \in [0, \infty)$, $0 \leqslant h_1(x(t)) \leqslant 1$ and $h_1(0) = h_1(0^+) = 0$.

2.2. **Preliminaries.** Let $([0, \omega], \zeta)$ denote the Lebesgue measurable space, $R^n (n \geqslant 1)$ as an Euclidean space, and $\mathbb{K}v(R^n)$ as all nonempty compact subsets of Euclidean space, then the metric ζ is defined by

$$\zeta(c, d) = \max\{\iota(c, d), \iota(d, c), c, d \in \mathbb{K}v(R^n)\}, \tag{2.3}$$

where $\iota(c, d) = \sup\{\mathrm{dist}(x, c) : x \in c\}, \iota(d, c) = \sup\{\mathrm{dist}(y, d) : y \in d\}$, $\mathbb{K}v(R^n)$ represents a complete metric space with the Hausdorff metric ζ.

Definition 2.1. [8] Given a set-valued function $F : X \to P(Y)$, a function $f : X \to Y$ is said to be a selector for F if $f(x) \in F(x)$ for all $x \in X$.

Definition 2.2. [8] Let X and Y be topological Hausdorff spaces and $P(Y)$ be all nonempty subsets of Y. We say that $F : X \to P(Y)$ is upper semicontinuous at $x \in X$ if for every neighborhood U of $F(x)$, there exists a neighborhood V of x such that $F(\hat{x}) \subset U$ for every $\hat{x} \in V$. A set-valued function $F : X \to P(Y)$ is upper semi-continuous on X if it is upper semi-continuous at every $x \in X$.

We adopt a reasonable definition from Filippov (see [8, 2, 9]) and consider the following delayed differential equation with discontinuous right-hand side

$$z'(t) = f(t, z(t - \tau), z(t)). \tag{2.4}$$

A set-valued map $F : R^n \times R^n \to R^n$ associated with system (2.2) is given by

$$F(t, y, z(t)) = \bigcap_{\delta>0} \bigcap_{\eta(N)=0} \mathbb{K}[f(t, y, [z(t) - \delta, z(t) + \delta] \setminus N], \tag{2.5}$$

where $\mathbb{K}[I]$ denotes the convex hull of I for set $I \subset R$, and $\eta(N)$ represents Lebesgue measure of set N. A solution in Filippov's sense is an absolutely continuous function

$z(t)$, $t \in (0,T)$, which satisfies differential inclusion and for almost all $t \in I$, $z(t)$ satisfies the differential inclusion,

$$z'(t) \in F(t, z(t-\tau), z(t)) \text{ for almost all (a.a.) } t \in (0,T). \tag{2.6}$$

Definition 2.3. A Filippov solution of system (2.2) $u(t) = (x(t), y(t))^T$ is a density solution on any compact interval of $(-\infty, T) \to R$ if

1). $u(t)$ is continuous on $(-\infty, T)$, and absolutely continuous on $[0,T)$;

2). For almost all $t \in [0,T), u(t) = (x(t), y(t))^T$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} \in r_1(t)x(t)g(x(t),K(t)) - c(t)k_1(t)p(t,x(t))x(t)y(t) - \varepsilon_1(t)\mathbb{K}[h_1(x(t))]x(t), \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = k_1(t)c(t)b(t)p(t,x(t-\tau))x(t-\tau)y(t-\tau) - \delta(t)y(t), \end{cases} \tag{2.7}$$

where $r_1(t)x(t)g(x(t),K(t)) - c(t)k_1(t)p(t,x(t))y(t) - \varepsilon_1(t)\mathbb{K}[h_1(x(t))]x(t) \triangleq f_1(t,u(t))$, $k_1(t)c(t)b(t)p(t,x(t-\tau))x(t-\tau)y(t-\tau) - \delta(t)y(t) \triangleq f_2(t,u(t))$. It is clear that the map $(t,u) \hookrightarrow (f_1(t,u(t)), f_2(t,u(t))^T$ is upper semi-continuous. Then there exists a function $\gamma_1(t) \in \mathbb{K}[h_1(x(t))]$ such that

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = r_1(t)x(t)g(x(t),K(t)) - c(t)k_1(t)p(t,x(t))x(t)y(t) - \varepsilon_1(t)\gamma_1(t)x(t), \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = k_1(t)c(t)b(t)p(t,x(t-\tau))x(t-\tau)y(t-\tau) - \delta(t)y(t), \end{cases} \tag{2.8}$$

for almost everywhere $t \in [0,T)$ any bounded measurable function $\gamma_1(t)$ satisfying (2.8) is defined by the population density solution $u(t)$. With Definition 2.3, it is easy to notice that the population density $u(t)$ is a solution to the discontinuous system (2.2). Clearly, $\gamma_1(t)$ is a harvesting policy function, and $u(t)$ denotes the population densities of the predator and prey.

Definition 2.4. (Initial value problem (IVP)) For any continuous and bounded function $g = (g_1, g_2)^T : (-\infty, 0] \to R^2$ and any measurable selection $\psi_1 : (-\infty, 0] \to R$, such that $\psi_1 \in \mathbb{K}[h_1(g_1(\alpha))]$ for a.e. $\alpha \in (-\infty, 0]$ by an initial value problem of system (2.2) with condition $[g, \psi_1]$, we consider the following problem, look for a couple of functions $[u(t), \gamma_1(t)]$; $(-\infty, T] \to R \times R$, such that $u(t)$ is a solution of model (2.2) on $(-\infty, T)$ for some $T > 0$ (T might be $+\infty$), $\gamma_1(t)$ is a harvesting solution of $u(t)$, and

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = r_1(t)x(t)g(x(t),K(t)) - c(t)k_1(t)p(t,x(t))x(t)y(t) - \varepsilon_1(t)\gamma_1(t)x(t) \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = k_1(t)c(t)b(t)p(t,x(t-\tau))x(t-\tau)y(t-\tau) - \delta(t)y(t), \text{for a.e.} \alpha \in [0,T), \\ \gamma_1(\alpha) \in \mathbb{K}[h_1(g_1(\alpha))], \text{for a.e.} \alpha \in [0,T), \\ u(\alpha) = g(\alpha) > 0, \ \forall \alpha \in (-\infty, 0]. \\ \gamma_1(\alpha) = \psi_1(\alpha) > 0, \text{for a.e.} \alpha \in (-\infty, 0]. \end{cases} \tag{2.9}$$

Definition 2.5. [9] Let $u^*(t) = (x^*(t), y^*(t))^T$ be a solution to the given IVP of system (2.2), if for any solution $u(t)$ there exist constants $M > 0$, $\lambda > 0$ and $t_1 > 0$ such that

$$||u(t) - u^*(t)|| < Me^{-\lambda(t-t_1)} \text{ for } t > t_1 > 0, \tag{2.10}$$

then solution $u^*(t)$ is said to be globally exponentially stable.

Next we adopt some lemmas and definitions for the periodic dynamical behavior of the inclusion (2.7) ([2, 3, 4]).

Definition 2.6. A solution $u(t)$ of the given IVP of system (2.2) on $[0,+\infty)$ is a periodic solution with period ω if $u(t+\omega)=u(t)$, for all $t \geqslant 0$.

Lemma 2.7. [2, 25, 3]*Suppose that $f: R\times R^n \to \mathbb{K}v(R^n)$ is upper semi-continuous and ω-periodic in t. If*

1). there exists a bounded, continuous open set $\Omega \subset C_\omega$, ω-periodic map: $R\to R^n$, such that for any positive number $0<\lambda\leqslant 1$ each ω-periodic solution $u(t)$ of the following inclusion

$$\frac{\mathrm{d}u}{\mathrm{d}t} \in \lambda f(t,u) \tag{2.11}$$

satisfies $u \notin \partial\Omega$ if u exists;

2). each solution $u\in R^n$ of the following inclusion

$$0 \in \frac{1}{\omega}\int_0^\omega f(t,u)\mathrm{d}t = g_0(u) \tag{2.12}$$

satisfies $u\notin \partial\Omega\cap R^n$; and

3). $\deg(g_0,\Omega\cap R^n,0)\neq 0$,

then inclusion (2.7) has at least one ω-periodic solution with $u\in\bar{\Omega}$.

A local Lipchitz function $V(u)$: $R^n\to R$ is said to be regular, if for each $u\in R^n$ and $v\in R^n$, there exists the usual (right-sided) directional derivative of $V(u,v)$ at u in the direction v

$$V'(u,v)^+ = \lim_{t\to 0^+}\frac{V(u+tv)-V(u)}{t},$$

and $V(u)$ admits a strict derivative at u, provided that for each v, the following equation holds:

$$\widehat{V'}(u,v) = \lim_{t\to 0^+}\sup_{y\to u}\frac{V(y+tv)-V(y)}{t},$$

then $V'(u,v)^+ = \widehat{V'}(u,v)$.

Now, we briefly introduce a chain rule for computing the time derivative of the composed function $V(u(t)):[0,+\infty)\to R$.

Lemma 2.8. *(Chain Rule)*[2, 3] *Assume that $V(u)$: $R^n\to R$ is C-regular and $u(t)$:$[0,+\infty)\to R$ is absolutely continuous on any compact interval of $[0,+\infty)$. Then, $u(t)$ and $V(u(t))$ are differentiable for almost all $t\in[0,+\infty)$, and*

$$\frac{\mathrm{d}V(u(t))}{\mathrm{d}t} = \left\langle \varrho(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right\rangle, \forall \varrho(t)\in\partial V(u(t)), \tag{2.13}$$

where $\partial V(u(t))$ is the Clark generalized gradient of V at $u(t)$.

Lemma 2.9. [8] *Let $u(t)$ be a solution of system (2.2), which is defined on $[0,T)$, $T\in(0,+\infty]$. Then, the function $|u(t)|$, is absolutely continuous and*

$$\frac{\mathrm{d}}{\mathrm{d}t}|u(t)| = v^T(t)u'(t) = \sum_{i=1}^n v_i(t)u_i'(t), \text{ for a.a.} t\in[0,T),$$

with

$$v_i(t) = \begin{cases} \mathrm{sign}(u_i(t)), & \text{if } u_i(t)\neq 0,\\ \text{arbitrary chosen in } [-1,1], & \text{if } u_i(t)=0.\end{cases}$$

Lemma 2.10. *(Positivity) Under the assumptions* (H1)-(H4)*, suppose that every positive initial value* $u(\alpha) = (x_0, y_0) > 0$ *is continuous on* $t_1 - \tau \leqslant \alpha \leqslant +\infty$*, then the solution of system (2.2) satisfies* $u(t) > 0$ *for* $t \in [t_1, +\infty)$.

Proof. By applying the same method as Proposition 3.1 in [2], we can obtain the conclusion of Lemma 2.10. The detailed proof for Lemma 2.10 is provided in Appendix A. □

Lemma 2.11. *All the solutions of system (2.2) with the positive initial condition* $u_{t_1} = (x_0, y_0)$ *are ultimately bounded.*

Proof. From the first equation of system (2.9) and Eq.(2.1), under assumption (H1), if $x(t) \geqslant K^M$, then we have $r_1^M x_1(t) g(x(t), K(t)) < 0$, and

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \leqslant -c^L k_1^L p(t, x(t)) x(t) y(t). \tag{2.14}$$

Integrating both sides of Eq.(2.14) with respect to t gives rise to

$$x(t) \leqslant K^M \exp\left\{ -\int_{t_1}^{t} c^L k_1^L p(s, x(s)) y(s) \mathrm{d}s \right\},$$

where $c^L k_1^L p(s, x(s)) y(s) > 0$, and t_1 is a positive number. By using assumption (H2) and Lemma 2.10, we know that $x(t) < K^M$ for $t > t_1$, then $x(t)$ is bounded for all $t > 0$.

Next, we prove that $y(t)$ is ultimately bounded for $t \in [t_1, +\infty)$. This discussion will be divided into two cases below:

Case 1. If $x_0 \leqslant K^M$, then we have $x(t) < K^M$ on $t_1 - \tau \leqslant t \leqslant +\infty$. By using the assumption (H1), it easy to know that $g(t, x(t)) > 0$. By comparing the coefficients of the non-autonomous system (2.9), then we have the following comparison system

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} \leqslant r_1^M x(t) g(x(t), K(t)) - c^L k_1^L p(t, x(t)) x(t) y(t), \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} \leqslant k_1^M c^M b^M p(t, x(t-\tau)) x(t-\tau) y(t-\tau) - \delta^L y(t). \end{cases} \tag{2.15}$$

By taking the right-hand side term of inequality (2.15) as the new two-dimensional system, we can obtain a new delayed system

$$\begin{cases} \dfrac{\mathrm{d}z_1(t)}{\mathrm{d}t} = r_1^M z_1(t) g(z_1(t), K(t)) - c^L k_1^L p(t, z_1(t)) z_1(t) z_2(t), \\ \dfrac{\mathrm{d}z_2(t)}{\mathrm{d}t} = k_1^M c^M b^M p(t, z_1(t-\tau)) z_1(t-\tau) z_2(t-\tau) - \delta^L z_2(t). \end{cases} \tag{2.16}$$

Due to $z_1(t) > 0, z_2(t) > 0$ for all $t \in [t_1, +\infty)$, by letting $w(t) = \ell_2 z_1(t) + \ell_1 z_2(t+\tau), c^L k_1^L = \ell_1, k_1^M c^M b^M = \ell_2$, based on assumptions (H1)-(H4), there exist positive constants K^M and g^M such that $|z_1(t)| < K^M, |g(z_1(t), K(t))| < g^M$. Taking the derivative on both sides of the equality $w(t)$ yields

$$\begin{aligned} w'(t) &= \ell_2 z_1'(t) + \ell_1 z_2'(t+\tau) \\ &\leqslant -\delta^L \ell_1 z_2(t+\tau) + r_1^M \ell_2 z_1(t) g(z_1(t), K(t)) \\ &\leqslant -\delta^L(\ell_1 z_2(t+\tau) + \ell_2 z_1(t)) + \delta^L \ell_2 z_1(t) + \ell_2 r_1^M z_1(t) g(z_1(t), K(t)) \\ &\leqslant -\delta w(t+\tau) + \ell_2 \delta^L z_1(t) + \ell_2 r_1^M z_1(t) g(z_1(t), K(t)) \\ &\leqslant -\delta^L w(t) + \ell_2 \delta^L K^M + \ell_2 r_1^M K^M g^M, \end{aligned} \tag{2.17}$$

we can easily know that $w'(t) \leqslant \ell_2\delta^L K^M + \ell_2 r_1^M K^M g^M - \delta^L w$ for all $t \geqslant T$.

Furthermore,

$$\lim_{t\longrightarrow\infty} w(t) \leqslant \frac{\ell_2\delta^L K^M + \ell_2 r_1^M K^M g^M}{\delta^L} \Rightarrow \ell_2 z_1(t) + \ell_1 z_2(t+\tau) \leqslant \frac{\ell_2\delta^L K^M + \ell_2 r_1^M K^M g^M}{\delta^L}. \tag{2.18}$$

Then by letting $\imath = t+\tau \geqslant T$, we know that

$$z_2(\imath) \leqslant \frac{\ell_2 r_1^M K^M g^M}{\ell_1 \delta^L} \tag{2.19}$$

and consequently $z_1(t), z_2(t)$ are ultimately bounded.

Case 2. If $x_0 > K^M$, where x_0 is a positive initial value, we prove that the $y(t)$ is bounded if $x_0 > x(t) > K^M$. By using the assumption (H1), we know that $g(x(t), K(t)) < 0$. A similar procedure to that of Case 1 can be used to prove Case 2, and we have

$$z_2(t) \leqslant \frac{\ell_2 r_1^L K^M g^L}{\ell_1 \delta^L}. \tag{2.20}$$

Then, from Eqs.(2.15), (2.19) and (2.20), we know that $x(t), y(t)$ are also ultimately bounded for $t \in [t_1, +\infty)$, where $t_1 = \max\{t_0, T\}$. Hence, the $y(t)$ is ultimately bounded, i.e., for any positive solution $u(t)$ of model (2.2), there are positive constants $K^M, \frac{\ell_2 r_1^M K^M g^M}{\ell_1\delta^L}$ and t_1 such that $0 < x(t) \leqslant \max\{x_0, K^M\}, 0 < y(t) \leqslant \max\{y_0, \frac{\ell_2 r_1^M K^M g^M}{\ell_1\delta^L}\}$ for $t \in [t_1, +\infty)$, which completes the proof of Lemma 2.11. □

3. Main results.

3.1. Existence of the periodic solution.

Theorem 3.1. *Suppose that assumptions* (H1)-(H4) *hold and that* $\overline{r}g^L - \overline{\varepsilon_1} > 0$, *then system (2.2) admits an ω-period periodic solution.*

Proof. Theorem 3.1 can be proved in three steps. Firstly, we know that the solution $u(t) = (x(t), y(t))$ of system (2.2) is non-negative for all $t \geqslant 0$, then we introduce the change of variables

$$u_1(t) = \ln[x(t)], \ u_2(t) = \ln[y(t)] \tag{3.1}$$

into the discontinuous system (2.2), and we have

$$\begin{cases} \dfrac{\mathrm{d}u_1(t)}{\mathrm{d}t} \in r_1(t) g(e^{u_1(t)}, K(t)) - c(t)k_1(t)p(t, e^{u_1(t)})e^{u_2(t)} - \varepsilon_1(t)\mathbb{K}[h_1(e^{u_1(t)})], \\ \dfrac{\mathrm{d}u_2(t)}{\mathrm{d}t} = k_1(t)c(t)b(t)p(t, e^{u_1(t-\tau)})e^{u_2(t-\tau)-u_2(t)}e^{u_1(t-\tau)} - \delta(t), \end{cases} \tag{3.2}$$

where all functions satisfying assumptions (H1)-(H4). We know that $u_1(t) = \ln x(t) : (0,\infty) \to R$, and $u_2(t) = \ln y(t) : (0,\infty) \to R$ are absolutely continuous, then we can say that $x(t) = e^{u_1(t)}$ and $y(t) = e^{u_2(t)}$ are both absolutely continuous. In order to prove the existence of ω-periodic solution $x(t), y(t)$ of system (2.7), the existence of ω-periodic solution $u(t) = (u_1(t), u_2(t))^T$ of system (3.2) needs to be proved. Thus, in order to prove Theorem 3.1, we have to show that Eq.(2.7) has one periodic solution of positive period ω.

Now, let us define

$$C_\omega = \{u(t) \in C(R, R^2) : u(t+\omega) = u(t)\}, \parallel u(t) \parallel_{c_\omega} = \sum_{i=1}^{2} \max_{t\in[0,\omega]} \mid u_i(t) \mid, \tag{3.3}$$

then we know that C_ω is a Banach space. Let $f(t, u(t)) = (f_1(t, u(t)), f_2(t, u(t)))^T$ for $u(t) = (x(t), y(t)) \in C_\omega$, where

$$\begin{aligned} f_1(t, u(t)) &= r_1(t)g(e^{u_1(t)}, K(t)) - c(t)k_1(t)p(t, e^{u_1(t)})e^{u_2(t)} - \varepsilon_1(t)\mathbb{K}[h_1(e^{u_1(t)})], \\ f_2(t, u(t)) &= k_1(t)c(t)b(t)p(t, e^{u_1(t-\tau)})e^{u_2(t-\tau)-u_2(t)}e^{u_1(t-\tau)} - \delta(t). \end{aligned} \tag{3.4}$$

Obviously, $f(t, u(t)) : R \times R^2 \to \mathrm{K}v(R^2)$ is a upper semi-continuous under assumptions (H1)-(H4). From Lemma 2.7, we need to find a bounded and open set Ω corresponding to the inclusion $u'(t) \in \lambda f(t, u(t)), \lambda \in (0, 1]$, that is

$$\begin{cases} \dfrac{\mathrm{d}u_1(t)}{\mathrm{d}t} \in \lambda[r_1(t)g(e^{u_1(t)}, K(t)) - c(t)k_1(t)p(t, e^{u_1(t)})e^{u_2(t)} - \varepsilon_1(t)\mathbb{K}[h_1(e^{u_1(t)})]], \\ \dfrac{\mathrm{d}u_2(t)}{\mathrm{d}t} = \lambda[k_1(t)c(t)b(t)p(t, e^{u_1(t-\tau)})e^{u_2(t-\tau)-u_2(t)}e^{u_1(t-\tau)} - \delta(t)]. \end{cases} \tag{3.5}$$

Suppose that $(u_1(t), u_2(t))^T$ is a periodic solution of the inclusion (3.5) with an arbitrary ω period and there exits a certain $h \in [0, 1]$. Due to the measurable selection theorem (see[2]), we can find a function $(u_1(t), u_2(t))^T : [0, +\infty) \to R^2$ such that $\gamma_1(t) \in \mathbb{K}[h_1(e^{u_1(t)})]$, for almost everywhere $t \in [0, T)$ and

$$\begin{cases} \dfrac{\mathrm{d}u_1(t)}{\mathrm{d}t} = h[r_1(t)g(e^{u_1(t)}, K(t)) - c(t)k_1(t)p(t, e^{u_1(t)})e^{u_2(t)} - \varepsilon_1(t)\gamma_1(t)], \\ \dfrac{\mathrm{d}u_2(t)}{\mathrm{d}t} = h[k_1(t)c(t)b(t)p(t, e^{u_1(t-\tau)})e^{u_2(t-\tau)-u_2(t)}e^{u_1(t-\tau)} - \delta(t)]. \end{cases} \tag{3.6}$$

Integrating (3.6) over the interval $[0, \omega)$, $0 \leqslant \gamma_1(t) \leqslant \sup\limits_{\gamma_1 \in \mathbb{K}(u(t))} \gamma_1(t) \leqslant 1$ results in

$$\int_0^\omega c(t)k_1(t)p(t, e^{u_1(t)})e^{u_2(t)}\mathrm{d}t + \int_0^\omega \varepsilon_1(t)\gamma_1(t)\mathrm{d}t = \int_0^\omega g(t, e^{u_1(t)})r(t)\mathrm{d}t, \tag{3.7}$$

$$\int_0^\omega \delta(t)\mathrm{d}t = \int_0^\omega k_1(t)c(t)b(t)p(t, e^{u_1(t-\tau)})e^{u_2(t-\tau)-u_2(t)}e^{u_1(t-\tau)}\mathrm{d}t. \tag{3.8}$$

From Eqs.(2.7)-(2.8) and assumption (H4), we can obtain

$$\begin{aligned} \int_0^\omega \mid u_1'(t) \mid \mathrm{d}t &\leqslant \int_0^\omega c(t)k_1(t)p(t, e^{u_1(t)})e^{u_2(t)}\mathrm{d}t + \int_0^\omega \varepsilon_1(t)\gamma_1(t)\mathrm{d}t + \int_0^\omega r(t)g^M\mathrm{d}t \\ &= 2\overline{r}g^M\omega, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \int_0^\omega \mid u_2'(t) \mid \mathrm{d}t &\leqslant \int_0^\omega \delta(t)\mathrm{d}t + \int_0^\omega k_1(t)c(t)b(t)p(t, e^{u_1(t-\tau)})e^{u_2(t-\tau)-u_2(t)}e^{u_1(t-\tau)}\mathrm{d}t \\ &= 2\int_0^\omega \delta(t)\mathrm{d}t = 2\overline{\delta}\omega. \end{aligned} \tag{3.10}$$

By noting that $u(t) = (u_1(t), u_2(t))^T$, there exist $\pi_i, \zeta_i \in [0, \omega)$ such that

$$u_i(\pi_i) = \max_{t\in[0,\omega]} u_i(t),\ u_i(\zeta_i) = \min_{t\in[0,\omega]} u_i(t), i = 1, 2. \tag{3.11}$$

From Eq.(3.7), we have that $\int_0^\omega c(t)k_1(t)p(t,e^{u_1(t)})e^{u_2(t)}\mathrm{d}t < \overline{r}g^M\omega$, which implies

$$u_2(\pi_2) < \ln\frac{\overline{r}g^M}{c^Lp^Lk_1^L}. \tag{3.12}$$

From (3.10) and (3.12), we can derive that

$$u_2(t) < u_2(\pi_2) + \int_0^\omega \mid u_2'(t) \mid \mathrm{d}t < 2\overline{\delta}\omega + \ln\frac{\overline{r}g^M}{c^Lp^Lk_1^L}. \tag{3.13}$$

It follows from (3.9) that

$$\int_0^\omega \mid u_1'(t) \mid \mathrm{d}t < 2\overline{r}g^M\omega. \tag{3.14}$$

Using assumptions (H1)-(H4) and differential mean value theorem, there exist constants $\hbar$ and $\wp$ such that $u(t-\tau)-u(t) = -u'(\hbar)\tau$ and $e^{-u'(\hbar)\tau} \leqslant \wp$. Then from (3.10), (3.7) and (3.6), we can obtain

$$\begin{aligned}&\int_0^\omega k_1(t)c(t)b(t)p(t,e^{u_1(t-\tau)})e^{u_2(t-\tau)-u_2(t)}e^{u_1(t-\tau)}\mathrm{d}t\\ =&\int_0^\omega k_1(t)c(t)b(t)p(t,e^{u_1(t-\tau)})e^{-u'(\hbar)\tau}e^{u_1(t-\tau)}\mathrm{d}t < \int_0^\omega \delta(t)\mathrm{d}t,\end{aligned} \tag{3.15}$$

$$\int_0^\omega e^{u_1(t-\tau)}\mathrm{d}t < \frac{\overline{\delta}\omega}{\wp p^Lk_1^Lc^Lb^L}, \tag{3.16}$$

there exists $t-\tau = \pi_1 \in [0,\omega]$ such that

$$u_1(\pi_1) < \ln\frac{\overline{\delta}}{\wp p^Lk_1^Lc^Lb^L}. \tag{3.17}$$

From (3.16) and (3.17), we have

$$u_1(t) < u_1(\pi_1) + \int_0^\omega \mid u_1'(t) \mid \mathrm{d}t < 2\overline{r}g^M\omega + \ln\frac{\overline{\delta}}{\wp p^Lk_1^Lc^Lb^L}. \tag{3.18}$$

By recalling the definition of $\mathbb{K}[h_1(e^{u_1(t)})]$ and the property of $h_1((e^{u_1(t)}))$ in assumption (H4), we can easily notice that $0 \leqslant \gamma_1(t) \leqslant \sup_{\gamma_1\in\mathbb{K}[h_1(e^{u_1(t)})]}\gamma_1(t) \leqslant 1$. According to assumptions (H1)-(H4), we can derive from (3.5) that,

$$\begin{aligned}&\overline{r}g^L\omega < \int_0^\omega c(t)k_1(t)p(t,e^{u_1(t)})e^{u_2(t)}\mathrm{d}t + \int_0^\omega \varepsilon_1(t)\gamma_1(t)\mathrm{d}t,\\ &\overline{r}g^L\omega < \int_0^\omega c(t)k_1(t)p(t,e^{u_1(t)})e^{u_2(t)}\mathrm{d}t + \int_0^\omega \varepsilon_1(t)\mathrm{d}t\\ &\Longrightarrow \overline{r}g^L\omega - \overline{\varepsilon_1}\omega < \int_0^\omega c(t)k_1(t)p(t,e^{u_1(t)})e^{u_2(t)}\mathrm{d}t,\\ &\overline{r}g^L\omega - \overline{\varepsilon_1}\omega < \int_0^\omega c^Mk_1^Mp^Me^{u_2(t)}\mathrm{d}t \Longrightarrow \overline{r}g^L\omega - \overline{\varepsilon_1}\omega < c^Mk_1^Mp^Me^{u_2(t)}\omega,\end{aligned} \tag{3.19}$$

which yields

$$u_2(\zeta_2) \geqslant \ln\frac{\overline{r}g^L - \overline{\varepsilon_1}}{c^Mk_1^Mp^M}. \tag{3.20}$$

From (3.20) and (3.10), we have

$$u_2(t) \geqslant u_2(\zeta_2) - \int_0^\omega \mid u_2'(t) \mid \mathrm{d}t \geqslant \ln\frac{\overline{r}g^L - \overline{\varepsilon_1}}{c^Mk_1^Mp^M} - 2\overline{\delta}\omega. \tag{3.21}$$

Combining (3.13) and (3.21) leads to

$$B_2 \triangleq \max_{t\in[0,\omega]} \mid u_2(t) \mid < \max\left\{\left|\ln\frac{\overline{r}g^L-\overline{\varepsilon_1}}{c^M k_1^M p^M}\right| + 2\overline{\delta}\omega, \left|\ln\frac{\overline{r}g^M}{\overline{c}p^L}\right| + 2\overline{\delta}\omega\right\}. \tag{3.22}$$

Similarly, using assumptions (H1)-(H4), we can obtain from (3.6) that

$$\begin{aligned}\int_0^\omega \delta(t)\mathrm{d}t &= \int_0^\omega k_1(t)c(t)b(t)p(t,e^{u_1(t-\tau)})e^{u_2(t-\tau)-u_2(t)}e^{u_1(t-\tau)}\mathrm{d}t \\ &= \int_0^\omega k_1(t)c(t)b(t)p(t,e^{u_1(t-\tau)})e^{-u'(\hbar)\tau}e^{u_1(t-\tau)}\mathrm{d}t,\end{aligned} \tag{3.23}$$

$$\overline{\delta}\omega \leqslant \int_0^\omega k_1^M c^M b^M p^M e^{-u'(\hbar)\tau}e^{u_1(t-\tau)}\mathrm{d}t \Rightarrow \overline{\delta}\omega \leqslant \int_0^\omega \wp k_1^M c^M b^M p^M e^{u_1(t-\tau)}\mathrm{d}t. \tag{3.24}$$

Therefore, there exists a positive number $\zeta_1 = t-\tau$, and from (3.24) we can derive that

$$u_1(\zeta_1) \geqslant \ln\frac{\overline{\delta}}{\wp k_1^M c^M b^M p^M}. \tag{3.25}$$

From (3.10) and (3.25), we have

$$u_1(t) \geqslant u_1(\zeta_1) - \int_0^\omega \mid u_1'(t) \mid \mathrm{d}t \geqslant \ln\frac{\overline{\delta}}{\wp k_1^M c^M b^M p^M} - 2\overline{r}g^M\omega. \tag{3.26}$$

Combining (3.26) and (3.18) implies

$$B_1 \triangleq \max_{t\in[0,\omega]} \mid u_1(t) \mid < \max\left\{\left|\ln\frac{\overline{\delta}}{\wp k_1^M c^M b^M p^M}\right| + 2\overline{r}g^M\omega, \left|\ln\frac{\overline{\delta}}{\wp p^L k_1^L c^L b^L}\right| + 2\overline{r}g^M\omega\right\}. \tag{3.27}$$

Consider the following model of inclusion

$$\begin{cases} 0 \in \overline{r}g(e^{u_1}) - \overline{ck_1}p(e^{u_1})e^{u_2} - \overline{\varepsilon_1}\mathbb{K}[h_1(e^{u_1})], \\ 0 = \overline{ck_1b}p(e^{u_1(t-\tau)})e^{-u'(\hbar)\tau}e^{u_1(t-\tau)} - \overline{\delta}, \end{cases} \tag{3.28}$$

where $\overline{r}g(e^{u_1}) = \frac{1}{\omega}\int_0^\omega rg(e^{u_1(t)})\mathrm{d}t$, $\overline{ck_1}p(e^{u_1})e^{u_2} = \frac{1}{\omega}\int_0^\omega c(t)p(t,e^{u_1(t)})e^{u_2}\mathrm{d}t$, $\overline{ck_1b}p(e^{u_1(t-\tau)}) \times e^{-u'(\hbar)\tau}e^{u_1(t-\tau)} = \frac{1}{\omega}\int_0^\omega k_1(t)c(t)b(t)p(e^{u_1(t-\tau)})e^{-u'(\hbar)\tau}e^{u_1(t-\tau)}\mathrm{d}t$. It is easy to notice that all solutions of Eq.(3.28) are bounded. By denoting $B = B_0 + B_1 + B_2$, each solution $u = (u_1,u_2)^T \in R^2$ of Eq.(3.28) satisfies $\mid u_1 \mid + \mid u_2 \mid < B$ and the inequality

$$\begin{aligned}&\max\left\{\left|\ln\frac{\overline{\delta}}{\wp k_1^M c^M b^M p^M}\right| + 2\overline{r}g^M\omega, \left|\ln\frac{\overline{\delta}}{\wp k_1^L c^L b^L p^L}\right| + 2\overline{r}g^M\omega\right\} \\ &\quad + \max\left\{\left|\ln\frac{\overline{r}g^L-\overline{\varepsilon_1}}{c^M k_1^M p^M}\right| + 2\overline{\delta}\omega, \left|\ln\frac{\overline{r}g^M}{\overline{c}p^L}\right| + 2\overline{\delta}\omega\right\} < B.\end{aligned} \tag{3.29}$$

Here B_0 is sufficiently large. Let $\Omega = (u_1(t),u_2(t))^T \in C_\omega :\| (u_1(t),u_2(t))^T \|_{C_\omega} < B$. Then, it is easy to know that Ω is an open bounded compact set of C_ω and $u \notin \partial\Omega$ for any $\lambda \in (0,1]$. The proof of Condition 1) of Lemma 2.7 is now complete.

In the second step, we need to prove that Condition 2) of Lemma 2.7 is satisfied. Suppose that there exists a solution $u(t)$ of the inclusion (3.28), then $u(t)$ is a constant vector with $|u_1| + |u_2| = B$. That is,

$$0 \notin \frac{1}{\omega}\int_0^\omega f(t,u)\mathrm{d}t = g_0(u) = \begin{pmatrix} \overline{r}g(e^{\overline{u_1}}) - \overline{ck_1}p(e^{\overline{u_1}})e^{\overline{u_2}} - \frac{1}{\omega}\int_0^\omega \overline{\varepsilon_1}\mathbb{K}[h_1(e^{u_1})\mathrm{d}t] \\ \frac{1}{\omega}\int_0^\omega \overline{ck_1b}p(e^{u_1})e^{-u'(\hbar)\tau}e^{u_1(t-\tau)}\mathrm{d}t - \overline{\delta} \end{pmatrix}. \tag{3.30}$$

This leads to a contradiction to Eq.(3.28). This completes proof of Condition 2) of Lemma 2.7.

In the third step, we need to prove that Condition 3) in Lemma 2.7 is satisfied. Define the continuous homotopy map $\varphi : \Omega \cap R^2 \times [0,1] \to C_\omega$:

$$\begin{aligned} \varphi(u_1,u_2,h) = &\begin{pmatrix} \overline{r}g(e^{\overline{u_1}}) \\ -\overline{\delta} \end{pmatrix} \\ &+ h\begin{pmatrix} \overline{r}g(e^{\overline{u_1}}) - \frac{1}{\omega}\int_0^\omega \overline{ck_1}p(e^{u_1})e^{u_2} + \overline{\varepsilon_1}\mathbb{K}[h_1(e^{u_1})\mathrm{d}t] \\ -\overline{\delta} + \frac{1}{\omega}\int_0^\omega \overline{k_1cb}p(e^{u_1})\mathrm{d}t \end{pmatrix}. \end{aligned} \tag{3.31}$$

where $h \in [0,1]$ is a parameter. If $u(t) = (u_1(t), u_2(t))^T \in \partial\Omega \cap R^2$, then we know that $u = (u_1, u_2)$ is a constant vector in R^2 with $\mid u_1 \mid + \mid u_2 \mid = B$. We will show that when $u \in \partial\Omega \cap R^2, 0 \notin (u_1,u_2)^T \notin \varphi(u_1,u_2,h)$. If not , then there is a vector $u = (u_1,u_2)^T \notin \varphi(u_1,u_2,h)$ with $\mid u_1 \mid + \mid u_2 \mid = B$, such that $0 \in (u_1,u_2,h)$, that is

$$\begin{cases} 0 \in \overline{r}g(e^{\overline{u_1}}) + h[\overline{r}g(e^{\overline{u_1}}) + \dfrac{1}{\omega}\displaystyle\int_0^\omega \overline{ck_1}p(e^{u_1})e^{u_2} + \overline{\varepsilon_1}\mathbb{K}[h_1(e^{u_1})\mathrm{d}t], \\ 0 = -\overline{\delta} + h[-\overline{\delta} + \dfrac{1}{\omega}\displaystyle\int_0^\omega \overline{ck_1b}p(e^{u_1})e^{-u'(\hbar)\tau}e^{u_1(t-\tau)}\mathrm{d}t]. \end{cases} \tag{3.32}$$

Based on Eq.(3.29), we can obtain that

$$\begin{aligned} &\max\left\{\left|\ln\frac{\overline{\delta}}{\wp k_1^M c^M b^M p^M}\right| + 2\overline{r}g^M\omega, \left|\ln\frac{\overline{\delta}}{\wp k_1^L c^L b^L p^L}\right| + 2\overline{r}g^M\omega\right\} \\ &+ \max\left\{\left|\ln\frac{\overline{r}g^L - \overline{\varepsilon_1}}{c^M k_1^M p^M}\right| + 2\overline{\delta}\omega, \left|\ln\frac{\overline{r}g^M}{\overline{c}p^L}\right| + 2\overline{\delta}\omega\right\} < B. \end{aligned}$$

It follows from (3.32) that $\mid u_1 \mid + \mid u_2 \mid < B$ which is a contradiction. Clearly, the algebraic equation $\psi(u_1,u_2,0) = 0$ has a unique solution $u^* = (x_0,y_0)^2 = (u_1^*, u_2^*)^T$ which satisfies $\overline{r}g(e^{\overline{u_1}}) = \frac{1}{\omega}\int_0^\omega rg(t, e^{u_1(t)})dt, \overline{c}\psi(e^{u_1}) = \frac{1}{\omega}\int_0^\omega k_1(t)c(t)p(e^{u_1(t)})dt$, $\overline{k}\phi(e^{u_1}) = \frac{1}{\omega}\int_0^\omega c(t)b(t)k_1(t)p(t, e^{u_1(t-\tau)})e^{u_2(t-\tau)-u_2(t)]}e^{u_1(t-\tau)}dt$. Therefore, by applying the homotopy invariance and using the property of the topological degree, we have

$$\begin{aligned} \deg\{g_0, \Omega \cap R^n, 0\} &= \deg\{\varphi(u_1,u_2,1), \Omega \cap R^n, 0\} \\ &= \deg\{\varphi(u_1,u_2,0), \Omega \cap R^n, 0\} = \mathrm{sign}\begin{vmatrix} \overline{r}g(e^{\overline{u_1}}) & 0 \\ * & \overline{\delta} \end{vmatrix} = 1 \neq 0. \end{aligned} \tag{3.33}$$

where $\deg.(\cdot,\cdot,\cdot,\cdot)$ is the topological degree, which is an upper semi-continuous set-valued map (see [2]). Then, Condition 3) in Lemma 2.7 holds. This indicates that all the conditions in Lemma 2.7 are satisfied. Hence, under assumptions (H1)-(H4), if $\overline{r}g^L - \overline{\varepsilon_1} > 0$ holds, then system (2.2) has a periodic solution which is ω period. The proof of Theorem 3.1 is complete. □

Now, we are ready to state that system (2.2) has a unique ω periodic solution $x(t), y(t)$, which is globally exponentially stable.

3.2. **Uniqueness and global exponential stability.**

Theorem 3.2. *Under the assumptions of Theorem 3.1, suppose that* $\max\limits_{t\in[0,,\omega]}[r(t)\rho_1 + c(t)k_1(t)\rho_2 - \varepsilon_1(t) + c(t)b(t)k_1(t)\rho_3 - \delta(t)] < 0$, *then system (2.2) admits a unique* ω*-periodic solution* $u(t) = (x(t), y(t))$, *which is globally exponentially stable, where* $\rho_1 = (GM_0 + g^M), \rho_2 = (M_0P_0 - L - M_0L), \rho_3 = (M_0L + M_0P_0 + L)$, $M_0 = \max\{K^M, \frac{\ell_2 r_1^M K^M g^M}{\ell_1\delta^L}, x_0, y_0\}$, $g^M = \max\limits_{t\in[0,,\omega]}|g(t, x(t))|$ *and* $L = \max\limits_{t\in[0,,\omega]}|p(t, x(t))|$, *where* (x_0, y_0) *is a positive initial value.*

Proof. From the condition of Theorem 3.2, there exists a positive constant $\varepsilon > 0$ such that

$$\varepsilon + \frac{r(t)\rho_1 + c(t)k_1(t)\rho_2 - \varepsilon_1(t) + c(t)b(t)k_1(t)\rho_3 - \delta(t)}{2} < 0. \tag{3.34}$$

From Lemma 2.11, we know that there exist a constant $M_0 = \max\left\{K^M, \frac{\ell_2 r_1^M K^M g^M}{\ell_1\delta^L}, x_0, y_0\right\}$ with a positive initial condition (x_0, y_0) and a t such that for all $t \geqslant t_2$, $0 < x^*(t), y^*(t), x(t), y(t) \leqslant M_0$. From (2.9) and assumptions (H1)-(H4), we know that $|g(t, x(t))| < g^M$ and $|p(t, x(t))| < L$, where $\gamma_1(t) \in \overline{co}[h_1(t, x(t))], \gamma_1^*(t) \in \overline{co}[h_1(t, x^*(t))]$, then we can obtain

$$\begin{aligned}
&\dot{x}(t) - \dot{x}^*(t)\\
&= r(t)[g(x_(t), K(t))x(t) - g(x^*(t), K(t))x^*(t)] - c(t)k_1(t)[p(t, x(t))y(t)x(t)\\
&- p(t, x^*(t))y^*(t)x^*(t)] - \varepsilon_1(t)[\gamma_1(t)x(t) - \gamma_1^*(t)x^*(t)],\\
&= r(t)[g(x_(t), K(t))x(t) - g(x^*(t), K(t))x(t) + g(x^*(t), K(t))x(t) - g(x^*(t), K(t))x^*(t)]\\
&- c(t)k_1(t)y(t)[p(t, x(t))x(t) - p(t, x^*(t))x^*(t)] - c(t)k_1(t)p(t, x^*(t))x^*(t)[(y(t) - y^*(t))]\\
&- \varepsilon_1(t)[\gamma_1(t)x(t) - \gamma_1^*(t)x^*(t)]\\
&= r(t)[g(x_(t), K(t))x(t) - g(x^*(t), K(t))x(t) + g(x^*(t), K(t))x(t)\\
&- g(x^*(t), K(t))x^*(t)] - c(t)k_1(t)y(t)[p(t, x(t))x(t) - p^*(t, x(t))x(t)]\\
&- c(t)k_1(t)y(t)[p(t, x^*(t))x(t) - p(t, x^*(t)x^*(t)]\\
&- c(t)k_1(t)p(t, x^*(t))x^*(t)[y(t) - y^*(t)] - \varepsilon_1(t)[\gamma_1(t)x(t) - \gamma_1^*(t)x^*(t)]\\
&= r(t)[g(x_(t), K(t))x(t) - g(x^*(t), K(t))x(t) + g(x^*(t), K(t))x(t) - g(x^*(t), K(t))x^*(t)]\\
&- c(t)k_1(t)y(t)x(t)[p(t, x(t)) - p(t, x^*(t))] - c(t)k_1(t)y(t)p(t, x^*(t))[x(t) - x^*(t)]\\
&- c(t)k_1(t)p(t, x^*(t))x^*(t)[y(t) - y^*(t)] - \varepsilon_1(t)[\gamma_1(t)x(t) - \gamma_1^*(t)x^*(t)]\\
&= r(t)x(t)[g(x_(t), K(t)) - g(x^*(t), K(t))] + r(t)g(x^*(t), K(t)))[x(t) - x^*(t)]\\
&- c(t)k_1(t)y(t)x(t)[p(t, x(t)) - p(t, x^*(t))] - c(t)k_1(t)y(t)p(t, x^*(t))[x(t) - x^*(t)]\\
&- c(t)k_1(t)p(t, x^*(t))x^*(t)[y(t) - y^*(t)] - \varepsilon_1(t)[\gamma_1(t)x(t) - \gamma_1^*(t)x^*(t)].
\end{aligned} \tag{3.35}$$

There also exist continuous functions τ_2, τ_3 such that $g(x(t), K(t)) - g(x^*(t), K(t)) = g'(\tau_2, K(t))(x(t) - x^*(t))$ and $p(t, x(t)) - p(t, x^*(t)) = p'(t, \tau_3)(x(t) - x^*(t))$ by using assumptions (H1)-(H3). Then we have $|g'(\tau_2, K(t))| \leqslant G$ and $|p'(t, \tau_3| \leqslant P_0$. Accordingly, Eq.(3.35) can be rewritten as

$$\begin{aligned}
\dot{x}(t) - \dot{x}^*(t) = {} & [r(t)(g'(\tau_2, K(t))x(t) + g(x(t), K(t)) - c(t)k_1(t)y(t)(p'(t, \tau_3)x(t)\\
& + p(t, x^*(t)))][x(t) - x^*(t)] - c(t)k_1(t)p(t, x^*(t))x^*(t)[(y(t) - y^*(t)]\\
& - \varepsilon_1(t)[\gamma_1(t)x(t) - \gamma_1^*(t)x^*(t)].
\end{aligned}$$

By using Lemmas 2.10 and 2.11 and assumption(H1)-(H3), we have

$$\begin{aligned}\dot{x}(t)-\dot{x}^*(t) &< [(r(t)(GM_0+g^M)+c(t)k_1(t)(M_0P_0-L)](x(t)-x^*(t))\\ &\quad -c(t)k_1(t)LM_0(y(t)-y^*(t))-\varepsilon_1(t)[\gamma_1(t)x(t)-\gamma_1^*(t)x(t)\\ &\quad +\gamma_1^*(t)x(t)-\gamma_1^*(t)x^*(t)]\\ &< [(r(t)(GM_0+g^M)+c(t)k_1(t)(M_0P_0-L)-\varepsilon_1(t)](x(t)-x^*(t))\\ &\quad -c(t)k_1(t)M_0L(y(t)-y^*(t))-\varepsilon_1(t)x(t)[\gamma_1(t)-\gamma_1^*(t)]\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}\dot{y}(t)-\dot{y}^*(t) &= c(t)b(t)k_1(t)[p(t-\tau,x)y(t-\tau)x(t-\tau)\\ &\quad -p(t-\tau,x^*)y^*(t-\tau)x^*(t-\tau)]-\delta(t)[y(t)-y^*(t)]\\ &= c(t)b(t)k_1(t)[p(t-\tau,x)y(t-\tau)x(t-\tau)-p(t-\tau,x^*)y(t-\tau)x^*(t-\tau)\\ &\quad +p(t-\tau,x^*)y(t-\tau)x^*(t-\tau)-p(t-\tau,x^*)y^*(t-\tau)x^*(t-\tau)]\\ &\quad -\delta(t)[y(t)-y^*(t)]\\ &= c(t)b(t)k_1(t)y(t-\tau)[p(t-\tau,x)x(t-\tau)-p(t-\tau,x^*)x^*(t-\tau)]\\ &\quad +c(t)b(t)k_1(t)p(t-\tau,x^*)x^*(t-\tau)[y(t-\tau)-y^*(t-\tau)]\\ &\quad -\delta(t)[y(t)-y^*(t)].\end{aligned} \tag{3.37}$$

Furthermore, there exists a continuous function τ_4 such that $p(t,x(t-\tau))-p(t,x^*(t-\tau))=p'(t,\tau_4)(x(t-\tau)-x^*(t-\tau))$ by using assumptions (H1)-(H3). Then we have $|p'(t,\tau_4)|<P_0$. Subsequently, Eq.(3.37) can be rewritten as

$$\begin{aligned}\dot{y}(t)-\dot{y}^*(t) &\leqslant c(t)b(t)k_1(t)(M_0P_0+L)[x(t-\tau)-x^*(t-\tau)]\\ &\quad +c(t)b(t)k_1(t)M_0L[y(t-\tau)-y^*(t-\tau)]-\delta^L[y(t)-y^*(t)].\end{aligned} \tag{3.38}$$

There exists a $t\in[t_1-\tau,+\infty]$ (see[9]) such that

$$\begin{aligned}\dot{y}(t)-\dot{y}^*(t) &\leqslant c(t)b(t)k_1(t)(M_0P_0+L)[x(t)-x^*(t)]+c(t)b(t)k_1(t)M_0L[y(t))-y^*(t)]\\ &\quad -\delta^L(y(t)-y^*(t)).\end{aligned} \tag{3.39}$$

Moreover, by letting $x(t)=u_1(t), y(t)=u_2(t)$, we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_i(t)-u_i^*(t)|=\partial|u_i(t)-u_i^*(t)|\,(\dot{u_i}(t)-\dot{u_i}^*(t))=v_i(t)\,(\dot{u_i}(t)-\dot{u_i}^*(t))\,, \tag{3.40}$$

where

$$v_i(t)=\begin{cases}0 & \text{if } u_1(t)-u_1^*(t)=\gamma_1(t)-\gamma_1^*(t)=0,\\ \operatorname{sign}(\gamma_1(t)-\gamma_1^*(t)) & \text{if } u_1(t)=u_1^*(t) \text{ and } \gamma_1(t)\neq\gamma_1^*(t),\\ \operatorname{sign}(u_i(t)-u_i^*(t)) & \text{if } u_i(t)\neq u_i^*(t).\end{cases} \tag{3.41}$$

Obviously, it follows from (3.41) that

$$v_i(t)(u_i(t)-u_i^*(t))=|u_i(t)-u_i^*(t)|, i=1,2, v_1(t)(\gamma_1(t)-\gamma_1^*(t))=|\gamma_1(t)-\gamma_1^*(t)|. \tag{3.42}$$

Consider the Lyapunov function $V(t)$:

$$V(t)=|x(t)-x^*(t)|e^{\varepsilon t}+|y(t)-y^*(t)|e^{\varepsilon t}. \tag{3.43}$$

Clearly, $V(t)$ is absolutely continuous. By virtue of Lemma 2.8, from (3.42)-(3.46) we can determine the derivative of $V(t)$ along the trajectory of (2.2) with the initial condition $u(0) = (x(0), y(0)) > 0$.

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= v_1\left(\dot{x}(t) - \dot{x}^*(t)\right) + v_2\left(\dot{y}(t) - \dot{y}^*(t)\right) + \varepsilon|x(t) - x^*(t)|e^{\varepsilon t} + \varepsilon|y(t) - y^*(t)|e^{\varepsilon t}\\ &\leqslant \varepsilon|x(t) - x^*(t)|e^{\varepsilon t} + \varepsilon|y(t) - y^*(t)|e^{\varepsilon t} + [r(t)(GM_0 + g^M)\\ &\quad + c(t)k_1(t)(M_0P_0 - L) - \varepsilon_1(t) + c(t)b(t)k_1(t)(M_0P_0 + L)]|x(t) - x^*(t)|e^{\varepsilon t}\\ &\quad + [-c(t)k_1(t)M_0L + c(t)b(t)k_1(t)M_0L - \delta(t)]|y(t) - y^*(t)|e^{\varepsilon t}\\ &\quad - \varepsilon_1(t)x(t)|(\gamma_1(t) - \gamma_1^*(t)|e^{\varepsilon t}\\ &\leqslant (\varepsilon + r(t)(GM_0 + g^M) + c(t)k_1(t)(M_0P_0 - L)\\ &\quad - \varepsilon_1(t) + c(t)b(t)k_1(t)(M_0P_0 + L))|x(t) - x^*(t)|e^{\varepsilon t}\\ &\quad + (\varepsilon - c(t)k_1(t)M_0L + c(t)b(t)k_1(t)M_0L - \delta(t))|y(t) - y^*(t)|e^{\varepsilon t}.\end{aligned} \tag{3.44}$$

Then it follows from (3.44) that

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leqslant &[(2\varepsilon + (r(t)(GM_0 + g^M) + c(t)k_1(t)(M_0P_0 - L) - \varepsilon_1(t) - c(t)k_1(t)M_0L\\ &\quad + c(t)b(t)k_1(t)(M_0P_0 + L) + c(t)b(t)k_1(t)M_0L - \delta(t)]M_0e^{\varepsilon t}.\end{aligned} \tag{3.45}$$

Accordingly, when $t > t_2$, there exists a positive constant $\varepsilon > 0$ such that

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leqslant \left[\varepsilon + \frac{[r(t)\rho_1 + c(t)k_1(t)\rho_2 - \varepsilon_1(t) + c(t)b(t)k_1(t)\rho_3 - \delta(t)]}{2}\right] M_0e^{\varepsilon t} < 0$$

holds, where $M_0 = max\{K^M, \frac{\ell_2 r_1^M K^M g^M}{\ell_1 \delta^L}, x_0, y_0\}$, $\rho_1 = (GM_0 + g^M)$, $\rho_2 = (M_0P_0 - L - M_0L)$, $\rho_3 = (M_0L + M_0P_0 + L)$.

Hence, system (2.2) admits a unique ω-periodic solution $u(t) = (x(t), y(t))$, which is globally exponentially stable. This completes the proof Theorem 3.2. □

Remark 1. In system (2.2), the function $g(x(t), K(t))$ is the net growth rate of the prey, for instance, the logistic growth with $g(x(t), K(t)) = r(t)(1 - \frac{x(t)}{K(t)})$ which satisfies all the conditions. $p(t, x(t))$ is the so-called predator functional response, and Condition 2 includes the commonly used functional response ([43, 34, 2]), namely Holling I type with $p(t, x(t)) = m(t)x(t)$, Holling II type with $p(t, x(t)) = \frac{m(t)x(t)}{a(t)+x(t)}$, Holling III type with $p(t, x(t)) = \frac{m(t)x(t)^2}{a(t)+x(t)^2}$, Ivlev type with $p(t, x(t)) = \alpha(t)x(t)(1 - e^{-\beta(t)x(t)})$, Monod-Haldane type with $p(t, x(t)) = \frac{m(t)x(t)}{a(t)+b(t)x(t)+x(t)^2}$, Holling IV type with $p(t, x(t)) = \frac{m(t)x(t)}{a(t)+x(t)^2}$, and some other equivalent forms (see[34]). From the point of view of biology, we only restrict our attention to system (2.2) in R_{2+}. For system (2.2), a global bifurcation and the existence uniqueness and the non-existence of limit cycles in certain ranges of parameters for the general predator-prey system were studied in the absence of the impulsive harvesting [43].

3.3. **Global convergence.**

Theorem 3.3. *Under the assumptions of Theorem 3.2, any harvesting solution $(u(t), \gamma_1(t))$ is globally convergent to the harvesting equilibrium point $(u^*(t), \gamma_1^*(t))$ in measure.*

Proof. Firstly, it is assumed that the positive harvesting equilibrium point of system(2.2) is denoted by $(u^*(t), \gamma_1^*(t)) \in R^2$. By differentiating the Lyapunov function, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}V(t)\\
&= v_1\left(\dot{x}(t) - \dot{x^*}(t)\right) + v_2\left(\dot{y}(t) - \dot{y^*}(t)\right) + \varepsilon|x(t) - x^*(t)|e^{\varepsilon t} + \varepsilon|y(t) - y^*(t)|e^{\varepsilon t}\\
&\leqslant \varepsilon|x(t) - x^*(t)|e^{\varepsilon t} + \varepsilon|y(t) - y^*(t)|e^{\varepsilon t} + [\varepsilon + r(t)(GM_0 + g^M)\\
&+ c(t)k_1(t)(M_0P_0 - L) - \varepsilon_1(t) + c(t)b(t)k_1(t)(M_0P_0 + L)]|x(t) - x^*(t)|e^{\varepsilon t}\\
&+ [\varepsilon - c(t)k_1(t)M_0L + c(t)b(t)k_1(t)M_0L - \delta(t)]|y(t)) - y^*(t)|e^{\varepsilon t}\\
&- \varepsilon_1(t)x(t)|\gamma_1(t) - \gamma_1^*(t)|e^{\varepsilon t} \leqslant -\varepsilon_1^L M_0|\gamma_1(t) - \gamma_1^*(t)|e^{\varepsilon t},
\end{aligned}
\tag{3.46}
$$

where $\zeta = \varepsilon_1^L M_0 e^{\varepsilon t} > 0$.

Integrating both sides of inequality(3.46) with respect to t yields

$$
V(t) - V(T) \leqslant -\zeta \int_T^t |\gamma_1(s) - \gamma_1^*(s)|\mathrm{d}s. \tag{3.47}
$$

Clearly, $V(t)$ is monotonically non-increasing. Then there exists a limit number N_0 such that $\lim\limits_{t\to+\infty} V(t) = N_0$. Subsequently we have $V(\mu(t)) - N_0 \geqslant 0$ and

$$
\frac{1}{\zeta}(V(t) - V(T)) \geqslant \int_T^{+\infty} |\gamma_1(s) - \gamma_1^*(s)|\mathrm{d}s. \tag{3.48}
$$

For any $\varepsilon > 0$, by letting $E_\varepsilon = \{t \in [T, +\infty)|\gamma_1(t) - \gamma_1^*(t)| > \varepsilon\} < \varepsilon$, (see[2]), we can obtain

$$
\frac{1}{\zeta}(V(t) - V(T)) \geqslant \int_T^{+\infty} |\gamma_1(s) - \gamma_1^*(s)|\mathrm{d}s \geqslant \int_T^{+\infty} |\gamma_1(t) - \gamma_1^*(t)|\mathrm{d}s \geqslant \varepsilon\mu(E_\varepsilon). \tag{3.49}
$$

From $\mu(E_\varepsilon) < +\infty$, we know that $\forall\varepsilon > 0, \mu\{t \in [T, +\infty)\}|\gamma_1(t) - \gamma_1^*(t)| < \varepsilon\} = +\infty$, i.e., for $t \to +\infty$, $\gamma_1^*(t) \in R$ is an almost classification point of $\gamma_1(t)$. By using Proposition 2 in [23], it is easy to see solution $(u(t), \gamma_1(t))$ of system (2.2) converges to equilibrium point $(u^*(t), \gamma_1^*(t))$ in measure, as $t \to +\infty$, that is, $\mu \lim\limits_{t\to+\infty} \gamma_1 = \gamma_1^*$. This completes the proof of Theorem 3.3. $\square$

Theorem 3.4. *Under the conditions of Theorem 3.3, any solution $(x(t), y(t))$ of the predator-prey system (2.2) converges to equilibrium point $(x^*(t), y^*(t))$ in finite time.*

Proof. If $h(x^{*-}(t)) - \gamma_1^* < 0 < h(x^{*+}(t)) - \gamma_1^*$. Let $h^-(x^*(t)) = \gamma_1^*(t) - h(x^{*-}(t))$, $h^+(x^*(t)) = \gamma_1^*(t) - h(x^{*+}(t))$, $\Delta = \min\{h^+(x^*(t)), h^-(x^*(t))\}$, and

$$
H(x(t)) = h_1(\mu(t) + x^*) - \gamma_1^*(t),\ \mu(t) = x(t) - x^*(t). \tag{3.50}
$$

Obviously, $H(x_1(t))$ is compact continuous interval in $[0, \infty)$ except on a countable set of isolate points $\{\rho_k\}$, and $\overline{\gamma}(t) = \gamma_1(t) - \gamma_1^*(t) \in \mathbb{K}[h_1(x(t))]$. It is clear that $\Delta > 0$ by assumption (H4). Because $\lim\limits_{\rho\to+0^-} H(\rho) = H(0^-) \leqslant -\triangle$, and $\lim\limits_{\rho\to+0^+} H(\rho) = H(0^+) \geqslant \triangle$, there exists a sufficiently small positive constant ε such that $|H(x_1(t))| \geqslant \Delta$ and $\forall 0 < |\mu(x)| \leqslant \varepsilon$. Since the positive equilibrium point

$(x(t), y(t))$ of system(2.2) globally asymptotically converges to $(x^*(t), y^*(t))$, there exists a positive number $T_\varepsilon > 0$ such that

$$|\mu(t)| = |x(t) - x^*(t)| \leqslant \varepsilon, \tag{3.51}$$

from (3.50) yields $\| \overline{\gamma}(t) \| = |\gamma_1(t) - \gamma_1^*(t)| \geqslant \Delta$. For Eq.(3.47), there exists a positive number T^{**}, such that

$$\frac{V(\mu(T^{**}))}{dt} \leqslant -\zeta|\gamma_1(t) - \gamma_1^*(t)| \leqslant -\zeta\Delta. \tag{3.52}$$

Integrating both sides of inequality (3.52) with respect to t leads to

$$V(\mu(t), t) \leqslant V(\mu(T^{**}), T^{**}) - \zeta\Delta(t - T^{**}). \tag{3.53}$$

Combing (3.43) and (3.53) yields

$$M|\mu(t)| \leqslant V(\mu(T^{**}), T^{**}) - \zeta\Delta(t - T^{**}). \tag{3.54}$$

There exists a positive number T^* such that $t \geqslant T^* = T^{**} + \frac{V(\mu(T^{**}),T^{**})}{\zeta\Delta}$. Then we have

$$M|x(t) - x^*(t)| = M|\mu(t)| = |V(\mu(t))| \leqslant 0. \tag{3.55}$$

from the conditions of Eq.(3.43), we have $x(t) = x^*(t)$ and $y(t) = y^*(t)$ for $t \geqslant T^*$. This completes the proof of Theorem 3.4. □

4. **Numerical simulations.** This section gives numerical simulation results to demonstrate the theoretical results obtained in Section 3. Four specific biological models will be compared to show the correctness of the theoretical results developed in Section 4.

Example 4.1. Holling I type. Let $g(t, x(t)) = (1 - 0.2\sin(\frac{t}{6})) - (1 - 0.5\cos(\frac{t}{6}))x(t)$, $r(t) = 1$, $p(t, x(t)) = x(t)(1 - 0.1\sin(\frac{t}{6}))$, $k_1(t)c(t)b(t) = 2(1 - 0.1\cos(\frac{t}{6}))$, $\delta(t) = 1 - 0.1\sin(\frac{t}{6})$. Consider the non-autonomous predator-prey Holling I type model with discontinuous harvesting policy:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t)(1 - 0.2\sin(\dfrac{t}{6})) - (1 - 0.5\cos(\dfrac{t}{6}))x(t)x(t) \\ \qquad\qquad - x(t)(1 - 0.1\sin(\dfrac{t}{6}))y(t) - 0.3h_1(x(t))x(t), \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = 2(1 - 0.1\cos(\dfrac{t}{6}))y(t-1)x(t-1) - (1 - 0.1\sin(\dfrac{t}{6}))y(t) \end{cases} \tag{4.1}$$

where $h_1(x(t)) = \begin{cases} 0 & \text{if } 0 \leqslant x(t) \leqslant 0.83, \\ 1 & \text{if } x(t) \geqslant 0.83, \end{cases}$

Fig.1 and Fig.2 show the numerical simulation results for the non-autonomous and the corresponding Holling I type system (4.1). Fig.1 demonstrates the periodic solution of the non-autonomous Holling I type system with discontinuous prey control strategy, while Fig.2 displays that the solution converging to an equilibrium for the corresponding autonomous Holling I type system whose coefficients are constant.

Example 4.2. Holling II type. Let $g(t, x(t)) = 1.3 - 0.2\sin(\frac{t}{5}) - (1.3 - 0.2\sin(\frac{t}{5}))$ $x(t)$, $r = 1$, $p(t, x(t)) = (0.2 - 0.1\cos(\frac{t}{5}))x(t)/(1 + 0.1\cos(\frac{t}{5}))x(t)$, $k_1(t)c(t)b(t) =$

$2-0.1\cos(\frac{t}{5})$, $\delta(t)=1-0.1\sin(\frac{t}{5})$, the Holling II type system (2.2) is given as:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t)(1.3-0.2\sin(\dfrac{t}{5}))-(1.3-0.2\sin(\dfrac{t}{5}))x(t)x(t) \\ \qquad -\dfrac{(x(t)(0.2-0.1\cos(\frac{t}{5}))y(t))}{(1+0.1\cos(\frac{t}{5})x(t))}-0.2h_1(x(t))x(t), \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = \dfrac{((2-0.1\cos(\frac{t}{5}))y(t-1)x(t-1))}{(1+0.1\cos(\frac{t}{5})(x(t-1))}-(1-0.1\sin(\dfrac{t}{5}))y(t), \end{cases} \tag{4.2}$$

where $h_1(x(t))=\begin{cases} 0 & \text{if } 0\leqslant x(t)\leqslant 0.61, \\ 1 & \text{if } x(t)\geqslant 0.61, \end{cases}$

Fig.3 shows the periodic solution of the non-autonomous Holling II type system (4.2) under discontinuous prey control strategy. While for the corresponding autonomous Holling II type system (4.2), Fig.4 displays the trajectory converging to an equilibrium for the system under the condition that the coefficients of the Holling II type system (4.2) are constant.

Example 4.3. Holling III type. Let $g(t,x(t))=(1-0.2\sin(\frac{t}{6}))-(1.3-0.1\sin(\frac{t}{6}))$ $x(t)$, $r(t)=1$, $p(t,x(t))=((0.2-0.1\cos(\frac{t}{6}))x(t))/(1+0.2\cos(\frac{t}{6})x(t)x(t))$, $k_1(t)c(t)b(t)=3-0.2\cos(\frac{t}{6})$, $\delta(t)=1-0.1\sin(\frac{t}{6})$, the Holling III type equation (2.2) is given by

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t)(1-0.2\sin(\dfrac{t}{6}))-(1.3-0.1\sin(\dfrac{t}{6}))x(t)x(t) \\ \qquad -\dfrac{(x(t)(0.2-0.1\cos(\frac{t}{6}))y(t))}{(1+0.2\cos(\frac{t}{6})x(t)x(t))}-0.3h_1(x(t))x(t), \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = \dfrac{((3-0.2\cos(\frac{t}{6}))y(t-1)x(t-1))}{(1+0.2\cos(\frac{t}{6})(x(t-1)x(t-1))}-(1-0.1\sin(\dfrac{t}{6}))y(t), \end{cases} \tag{4.3}$$

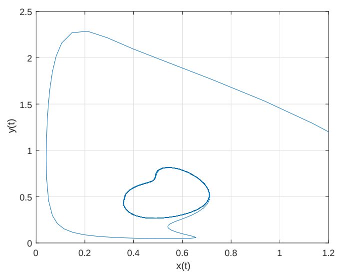
where $h_1(x(t))=\begin{cases} 0 & \text{if } 0\leqslant x(t)\leqslant 0.27, \\ 1 & \text{if } x(t)\geqslant 0.27. \end{cases}$

Similarly, Fig.5 and Fig.6 show the periodic solution of the non-autonomous Holling III type system (4.3) and the trajectory converging to the equilibrium of the corresponding autonomous Holling III type system with constant coefficients, respectively.

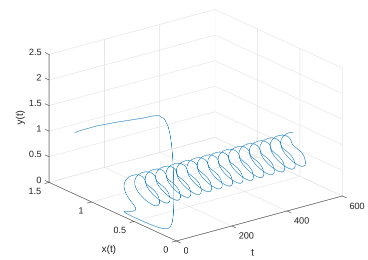
Example 4.4. Holling Ivlev type. Let $g(t,x(t))=(1-0.1\sin(\frac{t}{8}))-(0.6-0.1\sin(\frac{t}{8}))x(t)$, $r(t)=1$, $p(t,x(t))=x(t)(0.6-0.1\cos(\frac{t}{8}))(1-\exp(-0.8x(t))$, $\delta(t)=1-0.1\sin(\frac{t}{8})$,$k_1(t)c(t)b(t)=2-0.1\cos(\frac{t}{8})$,the Holling Ivlev type Eq.(2.2) is expressed as

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = (1-0.1\sin(\dfrac{t}{8}))x(t)-(0.6-0.1\sin(\dfrac{t}{8}))x(t)x(t) \\ \qquad -x(t)(0.6-0.1\cos(\dfrac{t}{8}))y(t)(1-\exp(-0.8x(t)))-0.3h_1(x(t))x(t), \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = x(t-1)(2-0.1\cos(\dfrac{t}{8}))y(t-1)(1-\exp(-0.8x(t-1))) \\ \qquad -(1-0.1\sin(\dfrac{t}{8}))y(t), \end{cases} \tag{4.4}$$

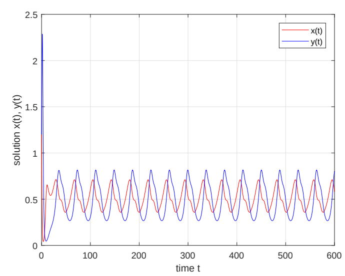
where $h_1(x(t))=\begin{cases} 0 & \text{if } 0\leqslant x(t)\leqslant 0.73, \\ 1 & \text{if } x(t)\geqslant 0.73. \end{cases}$



(a)

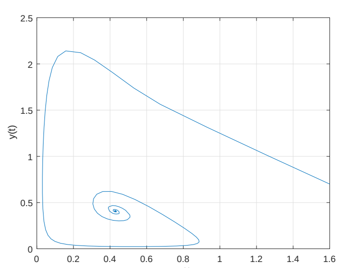


(b)



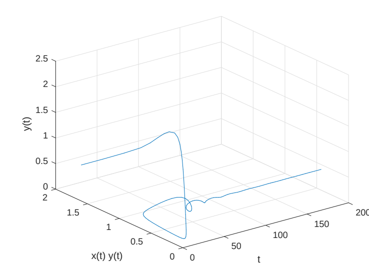
(c)

FIGURE 1. Periodic solution of Holling I type non-autonomous system; (a) phase portraits of the state variables $x(t)$ and $y(t)$, (b) trajectory in three-dimensional space, and (c) trajectories of the state variables $x(t)$ and $y(t)$ with time.

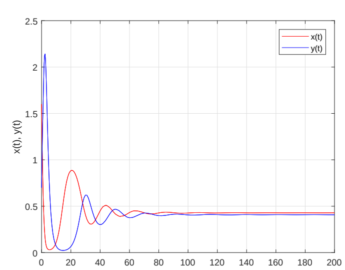


4

(a)

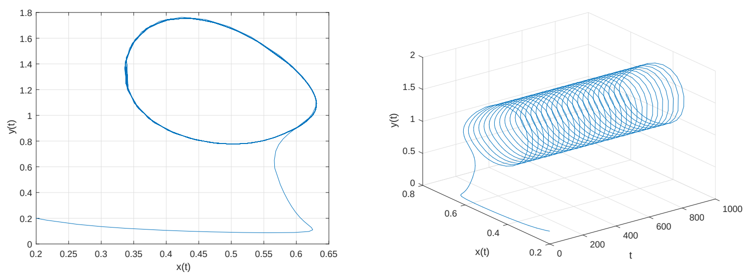


(b)

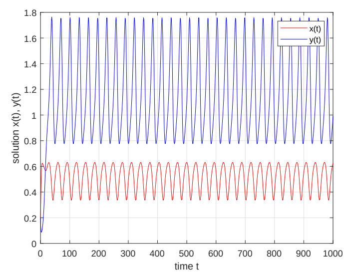


(c)

FIGURE 2. The trajectory converging to an equilibrium of the corresponding Holling I type autonomous system; (a) phase portrait of the state variables $x(t)$ and $y(t)$, (b) trajectory of the state in three-dimensional space, and (c) trajectories of the variables $x(t)$ and $y(t)$ with time.

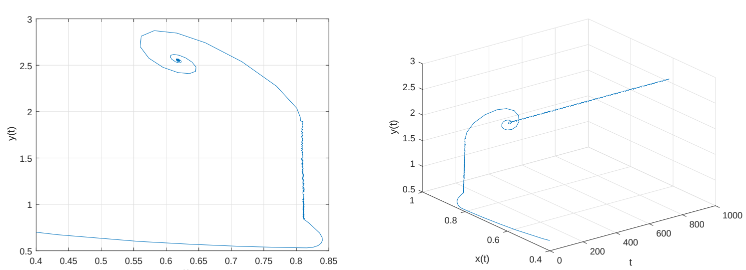


(a) (b)



(c)

FIGURE 3. Periodic solution of Holling II type non-autonomous system; (a) phase portraits of the state variables $x(t)$ and $y(t)$, (b) trajectory in three-dimensional space, and (c) trajectories of the state variables $x(t)$ and $y(t)$ with time.



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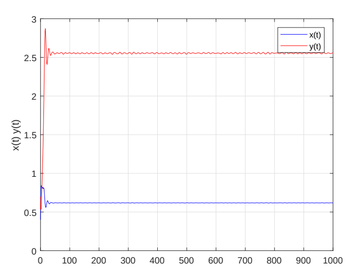
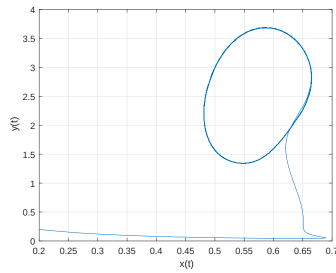
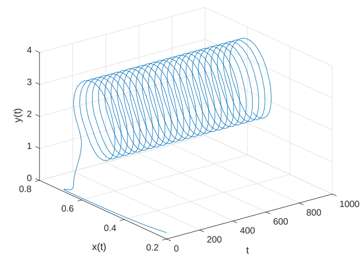


(c)

FIGURE 4. The trajectory converging to an equilibrium of the corresponding Holling II type autonomous system; (a) phase portrait of the state variables $x(t)$ and $y(t)$, (b) trajectory of the state in three-dimensional space, (c) trajectories of the variables $x(t)$ and $y(t)$ with time.



(a)



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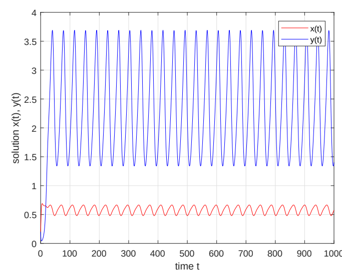
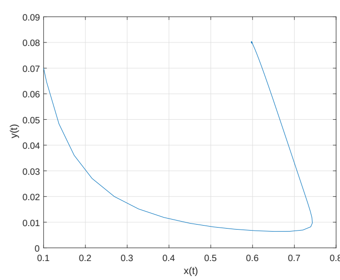
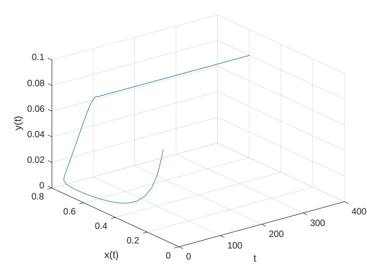


(c)

FIGURE 5. Periodic solution of Holling III type non-autonomous system; (a) phase portraits of the state variables $x(t)$ and $y(t)$, (b) trajectory of the state in three-dimensional space, and (c) trajectories of the state variables $x(t)$ and $y(t)$ with time.



(a)



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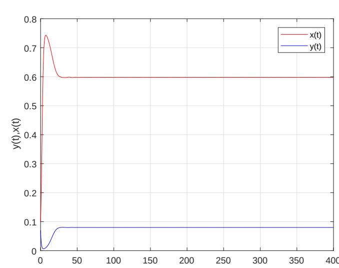
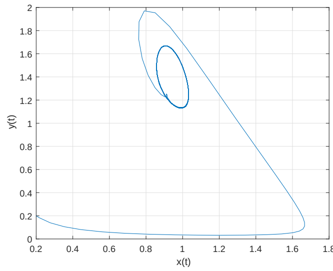
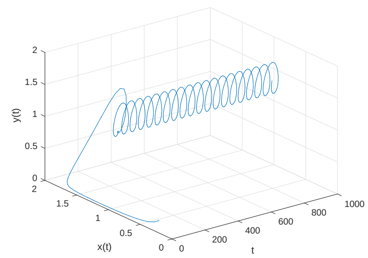


(c)

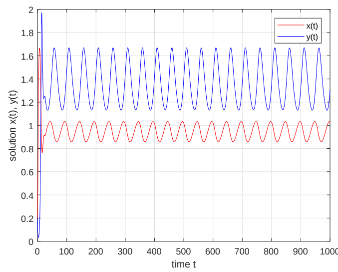
FIGURE 6. The trajectory converging to an equilibrium of the corresponding Holling III type autonomous system; (a) phase portrait of the state variables $x(t)$ and $y(t)$. (b) trajectory of the state in three-dimensional space, and (c) trajectories of the variables $x(t)$ and $y(t)$ with time.



(a)

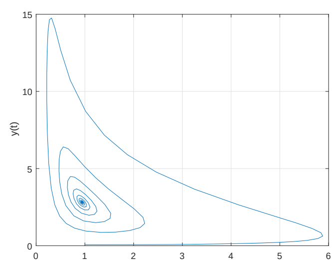


(b)

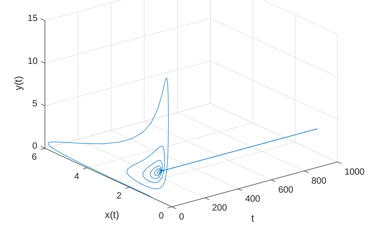


(c)

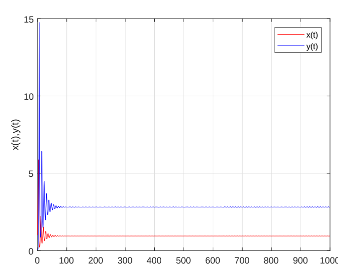
Figure 7. Periodic solution of Ivlev type non-autonomous system; (a) phase portraits of the state variables $x(t)$ and $y(t)$, (b)trajectory in three-dimensional space, and (c) trajectories of the state variables $x(t)$ and $y(t)$ with time.



(a)



(b)



(c)

Figure 8. The trajectory converging to an equilibrium of the corresponding Ivlev type autonomous system; (a) phase portrait of the state variables $x(t)$ and $y(t)$, (b) trajectory of the state in three-dimensional space, and (c) trajectories of the variables $x(t)$ and $y(t)$ with time.

Fig.7 and Fig.8 demonstrate the numerical simulation results for the periodic solution of the non-autonomous Holling Ivlev type system under discontinuous control strategy and for the equilibrium of the corresponding autonomous Holling Ivlev type systems with constant coefficients, respectively.

By analyzing Examples 4.1-4.4, we can easily know that the conditions of Theorems 3.1-3.4 are all satisfied. Thus, the non-autonomous predator-prey systems have a unique globally asymptotically stable ω-periodic solution. The numerical simulation results shown in Fig.1 for Holling I type system (4.1), Fig.3 for Holling II type system (4.2), Fig.5 for Holling III type system (4.3), and Fig.7 for Holling Ivlev type system (4.4) demonstrate that there is a unique globally exponentially stable ω-periodic solution of the non-autonomous delayed models with discontinuous prey control strategy. The numerical results are in excellent agreement with the theoretical results of Theorems 3.1-3.4. While for the corresponding autonomous predator-prey models, periodic solutions do not exist in the systems but an equilibrium under the selected parameter regions. The existence of the globally exponentially stable periodic solution is preferred to maintain the sustainable development of the ecosystem.

5. **Summary and discussion.** Under the condition of the predator-prey system with ecological sustainability, when the number of prey is above a certain level, we should remove some preys. In real world, given the fact that an excess of preys cannot be discovered in time, then we should consider the delay effects occurring in taking actions to remove a certain number of preys. Based on the previous literature on discontinuous harvesting control strategy, we proved the positiveness and boundedness of the solutions of the general delayed model with discontinuous prey harvesting control strategy. By using the degree theory, set-valued mapping and differential inclusion theory, we analyzed the periodic solutions of the general non-autonomous system. More interestingly, both theoretical and numerical results demonstrated that Example 4.1 (Holling I type), Example 4.2 (Holling II type), Example 4.3 (Holling III type), and Example 4.4 (Holling Ivlev type) all exist a periodic solution under discontinuous control strategy. The periodic solutions of the models were found to be exponentially asymptotically stable. The periodic solution can well maintain the sustainable development of the ecosystem.

Compared with the existing studies in the literature, system (2.2) is more general and considers the delay effects in harvesting preys. The discontinuous harvesting control strategy is simple and easy to be implemented in real world applications. It is worth pointing out that our research results on the corresponding autonomous systems have been obtained before, but few studies were focused on the non-autonomous delayed predator-prey model. In this paper, the classical autonomous ordinary differential theory could not be applied to perform the qualitative analysis of the general delayed predator-prey model (such as equilibrium point analysis and sliding bifurcation analysis). This will be our future research topics.

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Appendix A. *Proof* of Lemma 2.10. From the definition of the solution for system(2.2), $u(t)$ is a solution of differential inclusion (2.7). Clearly, under Condition (H4), we know that $\bar{co}[h_1(0)] = 0$, and $h_1(x(t))$ is continuous at $u(t) = 0$. Due to the continuity of h_1 at $u(t) = 0$, there are positive constants D_1 and D_2 such that when $\mid x(t) \mid< D_1$, $\mid y(t) \mid< D_2$, $x(t)$ and $y(t)$ are continuous, and the inclusion (2.7) becomes the dynamic system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(t)[r(t)g(t,x(t)) - c(t)p(t,x(t))y(t) - \varepsilon_1(t)h_1(t,x(t))], \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(t)[k_1(t)c(t)b(t)p(t,t-\tau)x(t-\tau)y(t-\tau)y(t)^{-1} - \delta(t)]. \end{cases} \tag{5.1}$$

Hence, $u(t) > 0$ for all $t \in [0,+\infty)$. If not, we let $t_1^* = \inf\{t \mid x(t) = 0\}$, $t_2^* = \inf\{t \mid y(t) = 0\}$, then $t_i^* > 0 (i = 1,2)$ and $x(t_1^*) = 0$, $y(t_2^*) = 0$. From the continuity of $x(t), y(t)$ on $[0,+\infty)$, there is a positive constant δ_i such that $t_i^* - \delta_i > 0$ and $0 < x(t) < D_1$, $0 < y(t) < D_1$ for $t \in [t_i^* - \delta_i, t_i^*) i = 1,2$. Next, by multiplying both sides of Eq.(5.1) by $\frac{1}{x(t)}, \frac{1}{y(t)}$ for all $t \in [t_i^* - \delta_i, t_i^*)$ and integrating with the interval $t \in [t_i^* - \delta_i, t_i^*)$, we have

$$\begin{cases} 0 = x(t_1^*) = x(t_1^* - \delta_1)\mathrm{e}^{\int_{t_1^*-\delta_1}^{t_1^*}[r(s)g(s,x(s)) - c(s)p(s,x(s))y(s) - \varepsilon_1(s)h_1(s,x(s))]\mathrm{d}s} > 0, \\ 0 = y(t_2^*) = y(t_2^* - \delta_2)\mathrm{e}^{\int_{t_2^*-\delta_2}^{t_2^*}[k_1(s)c(s)b(s)p(s,s-\tau)x(s-\tau)y(s-\tau)y(s)^{-1} - \delta(s)]\mathrm{d}s} > 0, \end{cases} \tag{5.2}$$

which is a contradiction to the previous assumption. Hence, $u(t) = (x(t), y(t)) > 0$ for all $t \in [0,T)$ and $T \to \infty$, which completes the proof of Lemma 2.10.

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