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One-shot resource distillation in quantum resource theories, and W-state encoding for optical QEC

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the degree of

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Certificate of Authorship/Originality

I, Madhav Krishnan Vijayan declare that this thesis, is submitted in fulfilment of the requirements for the award of Doctor of Philosophy, in the Faculty of Engineering and Information Technology at the University of Technology Sydney.

This thesis is wholly my own work unless otherwise referenced or acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

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ABSTRACT

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Part I of this thesis studies the problem of optimally converting a single copy of an arbitrary quantum state into maximally resourceful states. Specifically, in the resource theory of coherence, the ideal rate for this conversion is found when assisted by a distant party with whom one shares a quantum state. The optimal distillation of a target pure state in a more general resource theory framework is then studied with minimal assumptions regarding the physical resource. Part II of this thesis studies the problem of error correction in linear quantum optics. An encoding using W-states is introduced which is easily implementable using current technology without feed-forward and it shown that this encoding is robust against independent dephasing errors.

Dedication

To Vijayan and Syama, my parents who always loved and supported me even when I troubled them by never going to school.

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Conference Papers

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Abbreviation

CPTP : Completely Positive Trace Preserving

LOCC: Local Operations and Classical Communication

LOCC-1: Local Operations and Classical Communication with a single round of one-way communication.

MIO : Maximally Incoherent Operations

MCS : Maximally Coherent State

QRT : Quantum Resource Theory

Nomenclature and Notation

Set of real numbers	:	$\mathbb{R}$
Hilbert space of dimension d	:	$\mathcal{H}_d$
The set of density operators acting on the Hilbert space $\mathcal{H}$	:	$\mathcal{D}(\mathcal{H})$
The set of incoherent states	:	$\mathcal{I}$
Largest eigenvalue	:	$\lambda_{max}(\cdot)$
Integer floor	:	$\lfloor \cdot \rfloor$
Projection operator onto the space of the state ρ	:	Π_ρ
ψ majorizes ϕ	:	$\psi \succ \phi$

Definitions

von-Neumann entropy	:	$S(\rho) := -\operatorname{Tr}\rho\log\rho$
Relative entropy	:	$S(\rho\|\sigma) := \operatorname{Tr}\rho\log\rho - \operatorname{Tr}\rho\log\sigma$
Relative entropy of coherence.	:	$C_r(\rho) := \min_{\delta\in\mathcal{I}} S(\rho\|\|\delta)$
Fidelity	:	$F(\rho,\sigma) := \operatorname{Tr}\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}$
Completely dephasing map	:	$\Delta(\rho) = \sum_i \|i\rangle\langle i\|\rho\|i\rangle\langle i\|$
Min-entropy	:	$S_{min}(\rho) = -\log_2(\lambda_{max}(\rho))$
Relative Rényi entropy of order 0	:	$S_0(\rho\|\sigma) = -\log_2(\operatorname{Tr}\Pi_\rho\sigma)$
ϵ-close ball	:	$b(\rho,\epsilon) = \{\sigma : \sigma \geq 0, \operatorname{Tr}[\sigma] = 1, F(\rho,\sigma) \geq 1-\epsilon\}$
Sub-normalised ϵ-close pure state ball	:	$b'_*(\rho,\epsilon) = \{\overline{\psi} : \operatorname{Tr}(\overline{\psi}) \leq 1, F(\overline{\psi},\rho) \geq 1-\epsilon\}$
ϵ-close pure state ball	:	$b_*(\rho,\epsilon) = \{\overline{\psi} : \operatorname{Tr}(\overline{\psi}) = 1, \overline{\psi} \in b'_*(\rho,\epsilon)\}$

Chapter 1

Introduction

This thesis is divided into two parts. One studying resource concentration and state transformations in quantum resource theories and the other studying optical quantum error correction. A detailed background and literature survey of each topic is given at the beginning of each of these parts.

1.1 Research Aims

Part I

- Study the problem of resource distillation from finite copies of an arbitrary state in the framework of quantum resource theories.
- Obtain the optimal rate of this resource distillation for coherence when assisted by a non-local helper.
- Show that these techniques can be generalised by abstracting the resource away in a general framework.

Part II

- Develop an optical QEC scheme that is viable for current quantum optical architectures which is passive and does not require any feed-forward.

1.2 Scope and significance

Resource theories have become an important tool in the analysis of quantum information processing. In this thesis I first study the problem of finite copy resource distillation with assistance in the resource theory of coherence. This fills a gap in our current understanding of the resource theory of coherence as well as lays the foundation for further analysis of multipartite coherence manipulation. I will then proceed to present a framework for analysing resource distillation in a general resource theory. This approach allows us to treat multiple resource theories on the same footing and make interesting statements of wide applicability. We also gain a better understanding of what are the features of known resource theories that arise from the resource theory structure as opposed to the nature of the particular resource.

Finally, I develop a quantum optical encoding scheme using W-states which is robust against independent dephasing noise. This has the following benefits — (a) W-states are highly robust against entanglement destruction due to mode loss. (b) They are easy to produce and manipulate. (b) The scheme is entirely passive, requiring no feed-forward of measurement outcomes.

1.3 Thesis Organisation

This thesis is organised as follows —

- **Chapter 1:** This introduction.
- **Chapter 2:** An introduction to resource theories is given and a literature survey of coherence and general resource theories is conducted.
- **Chapter 3:** I derive bounds for one-shot distillation of maximally coherent states from arbitrary pure states. Using this I find the optimal rate of distilling coherence

when assisted by a distant party. I then show that this finite copy result will recover the known asymptotic results in the limit of infinite copies.

- **Chapter 4:** A general resource theory framework is introduced and I consider the problem of pure state transformations in this framework. Bounds on the ideal rate are obtained in terms of a new entropic quantity G_{min} which quantifies the maximum overlap of a state with the set of free states in a resource theory.
- **Chapter 5:** The results of part I are discussed and I highlight some open problems.
- **Chapter 6:** Motivation and a literature survey of optical quantum computing are given.
- **Chapter 7:** An optical encoding using W-states is introduced. This encoding is shown to be robust against independent dephasing noise. The success probabilities and fidelities of this scheme under different operating procedures are given as a function of error parameters.
- **Chapter 8:** A summary of the findings of part II and a comparison with existing schemes is given and I discuss future directions and open problems.

Part I

Resource Theories

Chapter 2

Background and Literature Survey

2.1 Introduction

Quantum resource theories (QRTs) have become a powerful tool for analysing various topics within quantum information theory. In general they study how certain features of quantum systems behave when the physical operations and manipulations of the system are limited. For instance, quantum entanglement is a feature that emerges in multipartite quantum systems, and it is natural to consider how entanglement behaves when the spatially separated parties are restricted to local operations and classical communication (LOCC). Using resource theories, researchers are able to more precisely identify and quantify the role that certain quantum features, such as entanglement, play in the performance of different quantum computational tasks (Horodecki et al. 2009; Plenio and Virmani 2007). Beyond entanglement, the resource theoretic approach has found application in the study of quantum Shannon theory (Devetak et al. 2008), quantum thermodynamics (Gour et al. 2015; Brandao et al. 2013), shared reference frames (Gour and Spekkens 2008), and many others (Chitambar and Gour 2019). General measures such as the *relative entropy of resource* can be applied to different resource theories and carry analogous operational interpretations in each (Horodecki et al. 2002; Horodecki and Oppenheim 2013; Brandão and Gour 2015; Anshu et al. 2017).

2.2 Resource theory of Coherence

The fact that coherent superpositions of quantum states are valid physical states is an essential feature of quantum mechanics, and it is the first point to consider when identifying advantages of quantum computation over its classical counterpart (Mermin 2007). Recently, the phenomenon of coherent superpositions has received rigorous development through the lens of a quantum resource theory (Baumgratz et al. 2014; Aberg 2006; Levi and Mintert 2014; Chitambar and Gour 2016; Du et al. 2015; Winter and Yang 2016; Yadin et al. 2016). See also (Streltsov et al. 2017) for a detailed review. In the resource theory of coherence, a state is considered resourceful if it is non-diagonal in a particular fixed basis. All diagonal states are called incoherent states and they take the form $\delta = \sum_i \delta_i |i\rangle\langle i|$, where $\{|i\rangle\}_i$ is some fixed basis known as the *incoherent basis*. I will denote the set of incoherent states as $\mathcal{I}$. Note that the incoherent states essentially represent classical probability distributions encoded in some physical system, and thus the resource theory of coherence captures one of the most basic non-classical features that quantum mechanics allows.

2.2.1 Classes of incoherent operations

Like all QRTs, only certain quantum operations are permitted when characterising the operational capabilities of coherence. Several different families of allowed, or "free", operations have been proposed in the literature, and they all share the property of being non-coherence-generating; i.e. they map the set of diagonal states onto itself. I will now discuss some important classes of incoherent operations.

Maximally incoherent operations (MIO) (Aberg 2006): These are the largest class of incoherent operations and are defined as any CPTP operation that maps a diagonal state to a diagonal state. I.e., for $\Lambda \in \text{MIO}$ and $\delta \in \mathcal{I}$,

$$\Lambda(\delta) = \sum_i K_i \delta K_i^\dagger \in \mathcal{I}, \tag{2.1}$$

where $\{K_i\}$ are some Kraus operator representation of the map Λ. If we interpret this Kraus operator representation as a particular implementation of the MIO operation then we can see that the condition is weak enough to allow probabilistic generation of coherence.

This is because $\frac{1}{\operatorname{Tr} K_i \delta K_i^\dagger} K_i \delta K_i^\dagger$ need not be an incoherent state. However, the average state produced by the process has to be an incoherent state.

Incoherent operations (IO) (Baumgratz et al. 2014) : These operations are those for which there exists a Kraus operator representation $\{K_i\}$ such that even knowledge of the outcome i will not allow the generation of coherence. This is stated as, for any $\delta \in \mathcal{I}$,

$$\frac{1}{\operatorname{Tr} K_i \delta K_i^\dagger} K_i \delta K_i^\dagger \in \mathcal{I} \qquad \forall i. \tag{2.2}$$

IO Kraus operators can be shown to be of the following form (Winter and Yang 2016):

$$K_i = \sum_j c_j(i) |f_i(j)\rangle\langle j|, \tag{2.3}$$

where $f_i(j)$ is a mapping between the basis labels. In general this mapping need not be bijective.

Strictly incoherent operations (SIO) (Winter and Yang 2016; Yadin et al. 2016): These operations are those which are uninfluenced by coherence in the input state as long as we are restricted to incoherent measurements. Formally we can define this to be the property that for any state ρ and a set of incoherent Kraus operators $\{K_j\}$,

$$\langle i|K_j \rho K_j^\dagger|i\rangle = \langle i|K_j \Delta(\rho) K_j^\dagger|i\rangle \,, \tag{2.4}$$

where Δ is the completely dephasing map defined as,

$$\Delta(\rho) = \sum_i |i\rangle\langle i|\rho|i\rangle\langle i|. \tag{2.5}$$

An equivalent definition of an SIO Kraus operator is that it is an IO Kraus operator K_i with the property that $K_i^\dagger$ is also IO. This allows us to characterise an SIO Kraus operator as,

$$K_i = \sum_j c_j(i) |\pi_i(j)\rangle\langle j|, \tag{2.6}$$

where now $\pi_i(j)$ is a one-to-one permutation of the basis labels. An interesting consequence of this form is that SIO Kraus operators preserve the coherence rank of a state, which is the rank of the state after a completely dephasing channel has been applied to it.

There are several other classes of free operations discussed in the literature which I will not go into in detail but briefly describe. *Dephasing-covariant incoherent operations* (DIO) are all CPTP maps that commute with the completely dephasing operation Δ. *Physically incoherent operations* (PIO) are operations that can be implemented using an incoherent ancilla, a global incoherent unitray and incoherent measurements — in other words an incoherent Stinespring dilation. Note that in general MIO, IO and SIO do not have such 'free' dilations. *Genuinely incoherent operations* (GIO) (de Vicente and Streltsov 2016) are operations that leave incoherent states unchanged, i.e. for $\Lambda \in$ GIO and $\delta \in \mathcal{I}$, $\Lambda(\delta) = \delta$. *Fully incoherent operations* (FIO) (de Vicente and Streltsov 2016) are operations such that all Kraus operator decompositions are incoherent. Finally another class of relevant operations are the *translationally invariant operations* (TIO) (Gour and Spekkens 2008; Marvian and Spekkens 2013; Marvian et al. 2016) which were originally defined in the resource theory of asymmetry. These operations commute with time translation with respect to some Hamiltonian H such that, for $\Lambda \in$ FIO and any state ρ,

$$e^{-iHt}\Lambda(\rho)e^{iHt} = \Lambda(e^{-iHt}\rho e^{iHt}).$$

A connection to coherence is made by noticing that asymmetry with respect to a d-dimensional representation of U(1) essentially reduces to the resource theory of coherence (Piani et al. 2016).

2.2.2 Coherence quantifiers

Quantifying the amount of coherence in a quantum state is achieved through several coherence monotones and measures. A function $C : \mathcal{D}(\mathcal{H}) \to \mathbb{R}$ may be classified as a coherence monotone if it satisfies the following properties (Baumgratz et al. 2014; Streltsov et al. 2017):

(a) Non-negativity:

$$C(\rho) \geq 0 \qquad \forall \rho \in \mathcal{D}(\mathcal{H}), \tag{2.7}$$

with the equality holding only for incoherent states.

(b) Monotonicity:

$$C(\Lambda(\rho)) \leq C(\rho). \tag{2.8}$$

C is non-increasing under any incoherent operation Λ.

(c) Strong monotonicity:

$$\sum_i p_i C(\sigma_i) \leq C(\rho), \tag{2.9}$$

where $\sigma_i \propto K_i \rho K_i^\dagger$ is the post-measurement state occurring with probability $p_i = \operatorname{Tr} K_i \rho K_i^\dagger$ for some incoherent process characterised by the Kraus operators $\{K_i\}$. This implies that even if we have access to the measurement outcome i we cannot increase coherence on average.

If a function C satisfies properties (a)-(c) it is referred to as coherence monotone. For C to be a coherence measure, it must additionally satisfy the following properties*,

(d) Convexity:

$$\sum_i p_i C(\rho_i) \geq C\left(\sum_i p_i \rho_i\right) \tag{2.10}$$

(e) Uniqueness for pure states: For any pure state $|\psi\rangle\langle\psi|$,

$$C(|\psi\rangle\langle\psi|) = S(\Delta(|\psi\rangle\langle\psi|)), \tag{2.11}$$

where $S(\cdot)$ is the von-Neumann entropy.

*There is some difference in the literature regarding this. For example (Baumgratz et al. 2014) only requires properties (a)-(d) for C to be classified as a measure while (Streltsov et al. 2017) argues for all of the properties (a)-(f) to be required.

(f) Additivity :

$$C(\rho \otimes \sigma) = C(\rho) + C(\sigma) \tag{2.12}$$

I will now describe some important coherence quantifiers.

Relative entropy of coherence

The relative entropy of coherence is a distance based quantifier of coherence where the distance is the relative entropy between the state and set of incoherent states. For a state ρ it is defined as,

$$C_r(\rho) = \min_{\delta \in \mathcal{I}} S(\rho \| \delta), \tag{2.13}$$

where $S(\rho\|\delta) = \mathrm{Tr}\,\rho \log \rho - \mathrm{Tr}\,\rho \log \sigma$ is the relative entropy. It can be shown that the relative entropy of coherence can be expressed as the simple one line expression (Winter and Yang 2016; Baumgratz et al. 2014)

$$C_r(\rho) = S(\Delta(\rho)) - S(\rho). \tag{2.14}$$

The relative entropy of coherence satisfies all condition (a)-(f) to be a coherence measure.

l_1 norm of coherence

The l_1 norm of coherence is based on the l_p family of matrix norms (Baumgratz et al. 2014);

$$\|M\|_{l_p} = \left(\sum_{i,j} |M_{ij}|^p \right)^{1/p}. \tag{2.15}$$

It is defined as,

$$C_{l_1}(\rho) = \min_{\delta \in \mathcal{I}} \|\rho - \delta\|_{l_1} = \sum_{i \neq j} |\rho_{ij}|. \tag{2.16}$$

This is essentially the absolute sum of the non-diagonal or ‘coherent part’ of a density matrix. The l_1 norm of coherence satisfies properties (a)-(d) defined in section 2.2.2.

Robustness of coherence

The robustness is a geometric measure based on the idea that you can destroy the resource in a state by sufficiently mixing it with another state. The amount of this mixing using an optimal state defines the robustness. Formally we can define the robustness of coherence of a state $\rho \in \mathcal{D}(\mathcal{H})$ as (Napoli et al. 2016),

$$\mathcal{R}_c(\rho) = \min_{\sigma \in \mathcal{D}(\mathcal{H})} \left\{ s \geq 0 : \frac{\rho + s\sigma}{1+s} \in \mathcal{I} \right\}. \tag{2.17}$$

Note that unlike the definition of robustness in entanglement where the minimisation is over the free states, here the minimisation is over all states. This is because it is not possible to get a diagonal state by mixing a diagonal state with a non-diagonal state. The robustness of coherence satisfies properties (a), (b) and (d) for all incoherent operations and satisfies (c) for IO.

2.2.3 Operational tasks

I will now describe some important operational tasks in the resource theory of coherence and mention some notable results in this area. First let us definite the maximally coherent state of rank M as,

$$|\Phi_M\rangle = \frac{1}{\sqrt{M}} \sum_{i=0}^{M-1} |i\rangle \tag{2.18}$$

and the unit maximally coherent state as,

$$|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle). \tag{2.19}$$

I will refer to the corresponding density matrices as Φ_M and Φ_2 respectively.

Pure state conversion

Given two pure states ψ and ϕ in $\mathcal{H}_d$, there exists an operation $\Lambda \in$ IO such that $\Lambda(\phi) = \psi$ if and only if $\Delta(\psi) \succ \Delta(\phi)$ (Du et al. 2015). Here $a \succ b$ for d-dimensional vectors a and b is read as b majorizes a and is defined as the following conditions holding,

$$\begin{aligned} \sum_{i=1}^{d} a_i = \sum_{i=1}^{d} b_i = 1, \\ \sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow} \qquad \forall k \in \{1, 2, \ldots, d\}, \end{aligned} \tag{2.20}$$

where $a^{\downarrow}$ and $b^{\downarrow}$ are vectors with the same elements as a and b respectively but with the elements arranged in descending order. The operation that achieves this transformation is in fact an SIO operation. This implies there are certain pure states which do not allow inter-conversion via IO since there can be states ψ and ϕ where neither $\Delta(\psi) \succ \Delta(\phi)$ or $\Delta(\phi) \succ \Delta(\psi)$ are true. In the limit of arbitrarily many copies one can transform the n copies of the state ψ to m copies of the state ϕ with vanishing error using an MIO operation as long as $m/n < S(\Delta(\psi))/S(\Delta(\phi))$. This transformation is impossible for $m/n > S(\Delta(\psi))/S(\Delta(\phi))$ (Yuan et al. 2015). This implies that asymptotically any pure state ψ can be transformed reversibly to the unit maximally coherent state Φ_2 at a rate of $S(\Delta(\psi))$ since $S(\Delta(\Phi_2)) = 1$.

Distillation

The distillation of coherence is the process of generating unit maximally coherent states Φ_2 from a given arbitrary quantum state ρ using incoherent operations. This quantity which is called the distillable coherence $C_d(\rho)$ of the state is a coherence measure in the sense of satisfying properties (a)-(f) in section 2.2.2 when we consider arbitrarily many copies. The distillable coherence turns out to be equal to the relative entropy of coherence and has the simple expression

$$C_d(\rho) = S(\Delta(\rho)) - S(\rho) = C_r(\rho). \tag{2.21}$$

In the single copy regime, we are interested in the rate of this transformation allowing for an error ϵ in the traget state which we represent as $C_c(\rho, \epsilon)$. Regula et al. (2018)

showed that the pure state distillation[†] rate can be bounded using the hypothesis testing inequality as

$$C_c(\psi, \epsilon) = \min_{\sigma \in \mathcal{I}} D_H^{\epsilon}(\psi \| \sigma) - \delta, \tag{2.22}$$

where $0 \leq \delta \leq 1$ and D_H^{ϵ} is the smoothed hypothesis testing relative entropy defined as,

$$D_H^{\epsilon}(\psi \| \sigma) = -\log \min\{\mathrm{Tr}[\sigma M] : 0 \leq M \leq \mathbb{I},\ \langle \psi, M \rangle > 1 - \epsilon\}, \tag{2.23}$$

In chapter 3 I derive alternate bounds for this quantity in terms of the min-entropy.

Formation

For any d-dimensional quantum system, a maximally coherent state $|\Phi_d\rangle$ exists that can be transformed to any other d-dimensional quantum state ρ (Baumgratz et al. 2014). One can then ask the more general inverse question of distillation — given n copies of the unit maximally coherent state Φ_2 how many copies of a target state ρ can you obtain with incoherent operations and an error ϵ which is vanishing in the asymptotic limit? Winter and Yang (2016) showed that this rate is given by the coherence of formation defined as,

$$C_f(\rho) = \inf_{\{p_i, \psi_i\}} \sum_i p_i S(\Delta(\psi_i)), \tag{2.24}$$

where $\sum_i p_i \psi_i = \rho$. The one-shot pure state formation (dilution) rate for several classes of incoherent operations were given by Zhao et al. (2018) in terms of different coherence monotones.

2.3 General resource theories

While what constitutes a resource can vary widely between different QRTs, some common structure is shared among them. Broadly speaking, a QRT divides states and operations in quantum theory into ones that an experimenter has access to freely and

[†]Pure state distillation is traditionally called concentration in the literature.

ones which are costly to use; in other words, into those which are free and those which are resourceful. Where one draws this boundary usually depends on the particular experimental or physical constraints under consideration. Studying this general structure independent of specific QRTs has allowed for a better understanding of certain quantum information quantities. For example, Brandão and Gour showed that the relative entropy of resource captures the asymptotic convertibility rate between two states, when one considers resource non-generating operations in a general convex QRT (Brandão and Gour 2015). An operational interpretation for general resources was given in (Takagi et al. 2019) by showing that for any convex QRT there exists a channel discrimination task for which a resource state will strictly outperform a free state.

In this thesis I will present results in a general framework that are applicable to QRTs whose most resourceful states are pure. This includes entanglement, coherence, and magic state quantum computing theories. The meaning of "most resourceful" is ambiguous, yet it can be made more precise in both a quantitative and operational sense. Quantitatively, pure states could be regarded as being more resourceful in a QRT if they maximize some resource measure, such as the relative entropy of resource (Horodecki and Oppenheim 2013) or the robustness of resource (Brandão and Gour 2015). Alternatively, one could take an operational perspective and regard some set of pure states S as being the most resourceful in a QRT if any state ρ on a given state space can be realized by a free transformation $\varphi \to \rho$ with $\varphi := |\varphi\rangle\langle\varphi| \in S$. In entanglement theory, such sets are known as maximally entangled sets, and it is an interesting research problem to identify maximally entangled sets with minimal structure (de Vicente et al. 2013). When pure states are regarded as a precious resource, a natural task of interest is pure-state distillation. Typically, this problem is phrased as a multi-copy state conversion problem $\rho^{\otimes n} \to \varphi^{\otimes m}$, which can be interpreted as exchange n copies of ρ for m copies of φ using the free operations of the QRT. In the limit of $n \to \infty$, the smallest ratio $\frac{n}{m}$ quantifies the asymptotic distillation rate of state φ from ρ (Bennett et al. 1996). In the nonasymptotic or "one-shot" regime, the problem is to determine how many copies of φ can be obtained from an arbitrary initial state ρ up to some specified error bound (Liu et al. 2019; Regula et al. 2020). In chapter 4 I will derive bounds for this one-shot pure-state distillation problem that applies to a wide-range of QRTs. The results obtained match those obtained independently by Liu et al. (2019) but the techniques I use avoid the need for a semi-definite programming formulation and have greater mathematical simplicity.

Chapter 3

One-shot Assisted Coherence distillation

3.1 Introduction

The specific problem we consider here involves coherence concentration on one system under the assistance of a second party. The operational scenario is depicted in Figure 3.1. We suppose that Alice and Bob initially share some bipartite entangled state $|\psi\rangle^{AB}$, and the goal is to concentrate the largest amount of coherence on Bob's side under the constraints that (a) Alice only communicate classically with Bob, and (b) Bob can only perform incoherent operations. While this is a two-body problem as described, it generalises to a many-body one in which a large number of parties are collectively being called "Alice." Our question then finds application in classically-connected quantum networks where the goal is to concentrate coherence at one of the nodes in order to perform some quantum information processing task. For example this could be a distant lab with limited capacity to generate coherence or a quantum probe which needs to be amplified using shared entanglement. Our problem is analogous to the one of entanglement assisted generation of entanglement which has been studied in (Buscemi and Datta 2013). By studying the effect of entanglement assistance on coherence generation we aim to better understand the relationship between entanglement and coherence.

There is a strong similarity between the resource theories of coherence and entanglement, and some of these connections have been pointed out in (Chitambar and Hsieh 2016; Streltsov et al. 2015, 2016; Zhu et al. 2017) . The equivalence in structure between the coherence of assistance and the entanglement of assistance was exploited in (Chitambar et al. 2016) to find the asymptotic coherence of assistance. Inspired by previous work

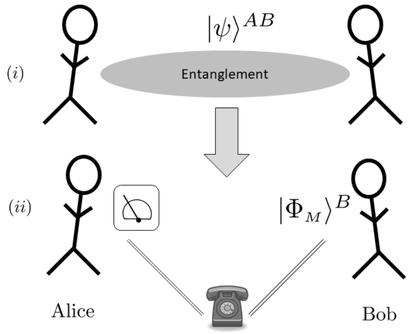

Figure 3.1 : The general task considered in this chapter involves assisted coherence concentration. In phase (i), Alice and Bob share some entangled state $|\psi\rangle^{AB}$. In phase (ii), Alice makes a measurement on her system and communicates the measurement result to Bob. Bob then performs local incoherent operations to maximize the coherence $|\Phi_M\rangle$ of his system.

on the problem of one-shot, or single-copy, assisted entanglement concentration (Buscemi and Datta 2013), we bound the *one-shot assisted coherence concentration*. In the assisted concentration scenario, Alice and Bob share a bipartite pure state $|\psi\rangle^{AB}$ and the goal is to maximize the rate of concentration of unit maximally coherent states (MCS) $|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ on Bob's side, while Bob is restricted to using incoherent operations and one-way communication is allowed from Alice to Bob. The ideal assisted concentration rate in the asymptotic setting $C_c(\psi^{AB})$, i.e., when Alice and Bob share arbitrarily many copies of the state $|\psi\rangle^{AB}$, is known to be equal to the coherence of assistance (Chitambar et al. 2016). While this rate is achievable with many copies of the state, in realistic scenarios resources are limited. Thus a more practical question is the following: if we allow for some bounded error in the process, how many copies of a maximally coherent state can we generate from just a single copy of the given pure state ψ^{AB}? While this question has been answered for concentration and dilution in the unassisted setting (Regula et al. 2018; Zhao et al. 2018, 2019), it has remained an open question for the one-shot assisted concentration paradigm, and it is one that we answer in this chapter.

An outline of our approach is as follows: We argue that Alice can prepare any pure state ensemble consistent with Bob's local density operator using just local operations and classical communication. Alice will choose to prepare the most optimal ensemble she can on Bob's system and now Bob is left with the task of concentrating this pure state ensemble using incoherent operations to create a MCS. We derive bounds on the maximum rate at which Bob can achieve this concentration in two steps. First we derive bounds for the concentration rate $C_c(\psi, \epsilon)$ for a pure state ψ using incoherent operations, where ϵ is the allowed error. While this problem has been previously solved in (Regula et al. 2018) , our approach uses different techniques. Then we generalize our pure state proof to find the bounds for the concentration rate $C_c(\mathfrak{E}, \epsilon)$ for an ensemble of pure states $\mathfrak{E} = \{p_i, \psi_i\}_i$ with error ϵ and hence find bounds upto Alice's initial optimization for the one-shot assisted concentration problem. We then show that our one-shot rate recovers the previously known rate in the appropriate limits.

This chapter is organized as follows: In section 3.2 quantities we will use for various proofs in this chapter are defined. In section 3.3 bounds on the one-shot (unassisted) concentration of MCSs from an arbitrary pure state are derived. In section 3.4 we will generalize these bounds to get the average rate of concentration from an ensemble of pure states and in section 3.5 it will be shown that in the asymptotic limit we recover the expected rate. Finally the conclusions from this section are presented in section 3.6.

3.2 Definitions

We fix a particular basis $\{|i\rangle\}_i$ in a given Hilbert space $\mathcal{H}$ as the incoherent basis and let $\mathcal{I}$ denotes the set of states which are represented by diagonal density matrices (incoherent states) in this basis. The maximally coherent state of rank M is defined with reference to this basis as,

$$|\Phi_M\rangle = \sum_{i=0}^{M-1} \frac{1}{\sqrt{M}} |i\rangle \,. \tag{3.1}$$

We use the notation ψ and $|\psi\rangle\langle\psi|$ interchangeably. We will use the fidelity measure defined as,

$$F(\rho,\sigma) := \mathrm{Tr}\left(\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}\right) = \|\sqrt{\rho}\sqrt{\sigma}\|_1. \tag{3.2}$$

The following lemmas are well-known.

Lemma 1. *For any self-adjoint operator A and B and any positive operator $0 \leq P \leq \mathbb{I}$,*

$$\mathrm{Tr}(P(A-B)) \leq \mathrm{Tr}(A-B)_+ \leq \|A-B\|_1, \tag{3.3}$$

where $(X)_+$ denotes the positive part of the operator X.

Proof: see (Bowen and Datta 2006)

Lemma 2. *For any state ρ and an operator $0 \leq \Lambda \leq \mathbb{I}$ such that $\mathrm{Tr}(\Lambda\rho) \geq 1-\epsilon$ then,*

$$\|\rho - \sqrt{\Lambda}\rho\sqrt{\Lambda}\|_1 \leq 2\sqrt{\epsilon} \tag{3.4}$$

Proof: see (Ogawa and Nagaoka 2002; Winter 1999).

We also define the following entropic quantities: for any two operators ρ and σ in a Hilbert space $\mathcal{H}$ such that $\rho,\sigma \geq 0$ and any operator P such that $0 \leq P \leq \mathbb{I}$, and $\alpha \in (0,\infty) \setminus \{1\}$,

$$S_\alpha^P(\rho\|\sigma) = \frac{1}{\alpha-1}\log_2 \mathrm{Tr}\left[\sqrt{P}\rho^\alpha\sqrt{P}\sigma^{1-\alpha}\right]. \tag{3.5}$$

Notice that for $P = \mathbb{I}$, this reduces to the relative Rényi entropy. We will be often using the quantity,

$$S_0^P(\rho\|\sigma) = \lim_{\alpha\to 0} S_\alpha^P(\rho\|\sigma) = -\log_2 \mathrm{Tr}\left[\sqrt{P}\Pi_\rho\sqrt{P}\sigma\right], \tag{3.6}$$

where Π_ρ is the projector unto the support of ρ in $\mathcal{H}$. Notice that the quantity,

$$S_0^{\mathbb{I}}(\rho\|\sigma) = S_0(\rho\|\sigma) = -\log_2(\mathrm{Tr}\,\Pi_\rho\sigma) \tag{3.7}$$

is the relative Rényi entropy of order 0. The relative entropy of coherence is defined as,

$$C_r(\rho) := \min_{\delta\in\mathcal{I}} S(\rho\|\delta) = S(\Delta(\rho)) - S(\rho), \tag{3.8}$$

where Δ is the completely dephasing operation which deletes off-diagonal terms in the reference basis, mathematically defined as $\Delta(\rho) = \sum_i |i\rangle\langle i|\rho|i\rangle\langle i|$, $S(\cdot\|\cdot) \equiv S_1^{\mathbb{I}}(\cdot\|\cdot) \equiv S_1(\cdot\|\cdot)$ is the relative entropy and $S(\cdot)$ is the von-Neumann entropy. We use $S_0(\rho\|\sigma)$ to define the min-entropy of coherence as,

$$C_{min}(\rho) = \min_{\sigma\in\mathcal{I}} S_0(\rho\|\sigma). \tag{3.9}$$

where $\mathcal{I}$ is the set of incoherent states. We also define the min-entropy as,

$$S_{min}(\rho) = -\log_2(\lambda_{max}(\rho)), \tag{3.10}$$

where $\lambda_{max}(\rho)$ is the largest eigenvalue of ρ. To define smoothed versions of these entropic quantities we define the ϵ-close ball for any state ρ and $\epsilon \geq 0$ as,

$$b(\rho,\epsilon) = \{\sigma : \sigma \geq 0, \mathrm{Tr}[\sigma] = 1, F(\rho,\sigma) \geq 1-\epsilon\}. \tag{3.11}$$

Similarly we define a ϵ-close ball of sub-normalized pure states as,

$$b'_*(\rho,\epsilon) := \{\overline{\psi} : \mathrm{Tr}(\overline{\psi}) \leq 1, F(\overline{\psi},\rho) \geq 1-\epsilon\} \tag{3.12}$$

where $\overline{\psi}$ are pure states. The normalized version of this ϵ-ball is defined as

$$b_*(\rho,\epsilon) = \{\overline{\psi} : \mathrm{Tr}(\overline{\psi}) = 1, \overline{\psi} \in b'_*(\rho,\epsilon)\}. \tag{3.13}$$

The optimal rate for concentration of coherence with assistance and asymptotically many copies of the state ψ^{AB} is known to be equal to the coherence of assistance $D_a(\rho^B)$, where $\rho^B = \mathrm{Tr}_A(\psi^{AB})$, defined as (Chitambar et al. 2016),

$$D_a(\rho^B) := \max_{\substack{\{p_i,\psi_i^B\}_i:\\ \sum_i p_i\psi_i^B=\rho^B}} \sum_i p_i C_r(\psi_i^B) = S(\Delta(\rho^B)). \tag{3.14}$$

We define the one-shot assisted coherence concentration as,

$$C_{\mathcal{O},\epsilon}^{A|B}(|\psi\rangle^{AB}) := \max_{\Lambda\in\mathcal{O}}\{\log_2 M : F^2(\Lambda^{AB\to B'}(|\psi\rangle^{AB}), \Phi_M^{B'}) \geq 1-\epsilon\}, \tag{3.15}$$

where $\mathcal{O}$ is the set of local quantum-incoherent operations with one-way classical communication (LQICC-1), $\epsilon \geq 0$, $F(\rho, \sigma)$ is the fidelity and $\Phi_M^{B'}$ is the maximally coherent state of rank M in the output Hilbert space $\mathcal{H}^{B'}$. In general LOCC can have an arbitrary number of communication rounds between the parties. But since we are restricted to communication from Alice to Bob but not the other way, one round is sufficient, see for example Buscemi and Datta (2013). The most general procedure that Alice and Bob could undertake would be for Alice to perform some POVM $\{P_i^A\}_i$ on her part of the state and communicate the result to Bob who would then apply an incoherent operation Λ_i depending on Alice's outcome. Let $\mathbb{N}$ be the set of natural numbers $\{0, 1, ..\}$, we can write down an expression or the optimal rate as:

$$C_a(\rho^B, \epsilon) := \max_{\{P_i^A\}_i} \max_{M \in \mathbb{N}} \left\{ \log_2 M : \max_{\{\Lambda_i^B\}_i} F^2 \left(\sum_i p_i \Lambda_i^B(\rho_i^B), \Phi_M^{B'} \right) \geq 1 - \epsilon \right\}, \tag{3.16}$$

which we call the one-shot coherence of assistance, where $p_i \rho_i^B = \mathrm{Tr}_A((P_i^A \otimes \mathbb{I}^B)\psi^{AB})$. Equation (3.16) can be understood as Alice performing an optimal POVM $\{P_i^A\}_i$ which prepares the state ρ_i^B with probability p_i on Bob's system. Alice then communicates the measurement outcome i to Bob who applies an optimal local incoherent operation Λ_i on his system. The one-shot coherence of assistance can be equivalently defined as,

$$C_a(\rho^B, \epsilon) := \max_{\{p_i, \psi_i^B\}_i} \max_{M \in \mathbb{N}} \left\{ \log_2 M : \max_{\{\Lambda_i^B\}_i} F^2 \left(\sum_i p_i \Lambda_i^B(\psi_i^B), \Phi_M^{B'} \right) \geq 1 - \epsilon \right\}, \tag{3.17}$$

where $\rho^B = \sum_i p_i \psi_i^B$, since without loss of generality, the maximization over POVMs $\{P_i^A\}_i$ can be restricted to rank-1 POVMs and this is equivalent to preparing any ensemble on Bob's side consistent with his reduced state ρ^B (Buscemi and Datta 2013). Thus the concentration task can be split into two parts; Alice prepares an optimal *pure* state ensemble $\{p_i, \psi_i^B\}_i$ by performing a suitable measurement and communicates the index i to Bob. Bob then performs an optimal incoherent operation on this state to distil the maximally coherent state. Then our task is reduced to finding the optimal rate of distilling the optimal pure state ensemble which will be the best achievable rate on average.

3.3 Pure state concentration

We will now derive bounds for the one-shot pure state concentration of MCSs. The one-shot coherence concentration rate for a pure state ψ, a set of incoherent operations $\mathcal{O}$ and $\epsilon \geq 0$ is defined as :

$$C_c(\psi, \epsilon) := \max_{M \in \mathbb{N}} \left\{ \log_2 M : \max_{\Lambda \in \mathcal{O}} F^2(\Lambda(\psi), \Phi_M) \geq 1 - \epsilon \right\}. \tag{3.18}$$

We will make use of the following lemma,

Lemma 3. *For any two pure states ψ and ϕ if the condition $\Delta(\psi) \succ \Delta(\phi)$ where the notation $\rho \succ \sigma$ indicates that ρ majorizes σ, then there exists an incoherent operation Λ such that*

$$\Lambda(\phi) = \psi. \tag{3.19}$$

Proof: I present a proof given in (Winter and Yang 2016) here for completeness. Let $\text{spec}(\Delta(\psi)) = \vec{p}$ and $\text{spec}(\Delta(\phi)) = \vec{q}$. The majorization condition implies that there exist permutations $\{\pi\}$ and a set of real numbers $\{\lambda_\pi : 0 \leq \lambda_\pi \leq 1, \sum_\pi \lambda_\pi = 1\}$, such that

$$\vec{q} = \sum_\pi \lambda_\pi \vec{p}_\pi \tag{3.20}$$

where $\vec{p}_\pi$ is a vector with the components of $\vec{p}$ permuted by π. An explicit construction of Λ is given in terms of its Kraus operators as,

$$\Lambda(\phi) = \sum_\pi K_\pi \phi K_\pi^\dagger, \tag{3.21}$$

where,

$$K_\pi = \sum_i \sqrt{\lambda_\pi} \sqrt{\frac{p_\pi(i)}{q(i)}} |\pi(i)\rangle\langle i| \tag{3.22}$$

where $p_\pi(i)$ and $q(i)$ are the ith components of $\vec{p}_\pi$ and $\vec{q}$ respectively. It can be verified that $\Lambda(\phi) = \psi$ by substituting the definition of the Kraus operators K_π from equation (3.22) in equation (3.21), thus proving the lemma. Since the Kraus operators K do not decrease the coherence rank of the input state, it is classified as a strictly incoherent operation (SIO) (Winter and Yang 2016).

Theorem 1. *For any pure state ψ and $\epsilon \geq 0$*

$$\max_{\overline{\psi} \in b_*(\psi,\epsilon)} S_{min}(\Delta(\overline{\psi})) - \delta \leq C_c(\psi,\epsilon) \leq \max_{\overline{\psi} \in b'_*(\psi,4\epsilon)} S_{min}(\Delta(\overline{\psi})), \tag{3.23}$$

where $0 \leq \delta \leq 1$ is a number which ensures the lower limit is the logarithm of an integer.

Proof: For any pure states ψ such that, $\Delta(\Phi_M) \succ \Delta(\psi)$, then from lemma 3 there exists an incoherent operation Λ such that $\Lambda(\psi) = \Phi_M$. Let $spec(\Delta(\Phi_M)) = (\frac{1}{M}, \frac{1}{M}, ..., \frac{1}{M})$ and $spec(\Delta(\psi)) = (\psi_1, \psi_2, , ..., \psi_d)$ Then the majorization condition implies that,

$$\sum_{i=1}^{k} \frac{1}{M} \geq \sum_{i=1}^{k} \psi_i^{\downarrow}, \qquad \forall k, d, \tag{3.24}$$

where the $\psi_i^{\downarrow}$ are the elements ψ_i in a monotonically decreasing order. Notice that in this case $\frac{1}{M} \geq \psi_{\max} \equiv \max_j \psi_j$ is sufficient to imply the majorization condition in equation (3.24) and ensuring the existence of a Λ that achieves the desired transformation. This implies that $\Lambda(\psi) = \Phi_M$ for any M such that $S_{min}(\Delta(\psi)) = -\log \lambda_{\max} \geq \log M$. In particular $M = \lfloor 2^{S_{min}(\Delta(\psi))} \rfloor$ is always achievable. Consequently, for any pure state $\overline{\psi} \in b_*(\psi, \epsilon)$ there exists an operation (in this instance an SIO) Λ such that $\Lambda(\overline{\psi}) = \Phi_{\overline{M}}$ for $\overline{M} = \lfloor 2^{S_{min}(\Delta(\overline{\psi}))} \rfloor$. Due to the monotonicity of fidelity under positive trace-preserving maps we have,

$$1 - \epsilon \leq F(\psi, \overline{\psi}) \leq F(\Lambda(\psi)\Lambda(\overline{\psi})) = F(\Lambda(\psi), \Phi_{\overline{M}}). \tag{3.25}$$

Hence, $C_c(\psi, \epsilon) \geq \log_2 \overline{M}$ for any state $\overline{\psi} \in b_*(\psi, \epsilon)$, or

$$C_c(\psi,\epsilon) \geq \max_{\overline{\psi} \in b_*(\psi,\epsilon)} \log_2 \lfloor 2^{S_{min}(\Delta(\overline{\psi}))} \rfloor. \tag{3.26}$$

For the converse, let M be the maximum of all ϵ-achievable rates for concentration of the pure state ψ, i.e., there exists an incoherent operation Λ such that $F^2(\Lambda(\psi), \Phi_M) \geq 1 - \epsilon$. Note that for any incoherent state $\gamma \in \mathcal{I}$ we have,

$$\Phi_M \Lambda(\gamma) \Phi_M = \frac{1}{M} \Phi_M, \tag{3.27}$$

since $\delta \in \mathcal{I}$ implies $\Phi_M \delta \Phi_M = \frac{1}{M}\Phi_M$. Multiplying both sides of equation (3.27) with $\Lambda(\psi)$ and taking the trace gives,

$$\mathrm{Tr}(\Lambda(\psi)\Phi_M\Lambda(\gamma)\Phi_M) = \frac{1}{M}\,\mathrm{Tr}(\Lambda(\psi)\Phi_M) \leq \frac{1}{M}, \tag{3.28}$$

where for the inequality we have used the fact that $\Lambda(\psi) \leq \mathbb{I}$. Continuing from equation (3.28)

$$\log_2 M \leq -\log_2 \mathrm{Tr}(\Phi_M\Lambda(\psi)\Phi_M\Lambda(\gamma)) = -\log_2 \mathrm{Tr}(\Lambda^*(\Phi_M\Lambda(\psi)\Phi_M)\gamma), \tag{3.29}$$

where Λ^* is the dual map of Λ such that $\mathrm{Tr}(X\Lambda(\rho)) = \mathrm{Tr}(\Lambda^*(X)\rho)$. Defining $Q := \Lambda^*(\Phi_M\Lambda(\psi)\Phi_M)$ we have,

$$\log_2 M \leq -\log_2 \mathrm{Tr}(Q\gamma) \leq -\log_2 \mathrm{Tr}(\sqrt{Q}\psi\sqrt{Q}\gamma) \leq -\log_2 \mathrm{Tr}(\widetilde{\psi}\gamma), \tag{3.30}$$

where we use the fact that $\sqrt{Q}\psi\sqrt{Q} \leq Q$ and we have introduced the sub-normalized state $|\widetilde{\psi}\rangle \equiv \sqrt{Q}\,|\psi\rangle$. Since γ is an arbitrary incoherent state, we thus have

$$\log_2 M \leq \min_{\gamma\in\mathcal{I}} \left\{-\log_2 \mathrm{Tr}(\widetilde{\psi}\gamma)\right\} = -\log_2(\lambda_{max}(\Delta(\widetilde{\psi})) = S_{min}(\Delta(\widetilde{\psi})). \tag{3.31}$$

We will now show that $\widetilde{\psi} \in b'_*(\psi, 2\epsilon)$. Note that,

$$\begin{aligned}\mathrm{Tr}(Q\psi) &= \mathrm{Tr}(\Phi_M\Lambda(\psi)\Phi_M\Lambda(\psi)) \\ &= \langle\Phi_M|\Lambda(\psi)|\Phi_M\rangle^2 = \left(F^2(\Lambda(\psi),\Phi_M)\right)^2 \geq 1-2\epsilon.\end{aligned} \tag{3.32}$$

where for the last inequality we use the fact that $F^2(\Lambda(\psi),\Phi_M) \geq 1-\epsilon$. Now we can see that,

$$F(\psi,\widetilde{\psi}) = \langle\psi|\,\sqrt{Q}\,|\psi\rangle \geq \langle\psi|\,Q\,|\psi\rangle = \mathrm{Tr}(Q\psi) \geq 1-2\epsilon, \tag{3.33}$$

where the last inequality follows from equation (3.32). This implies that $\widetilde{\psi} \in b'_*(\psi, 2\epsilon)$. From equation (3.31) we can write,

$$\log_2 M \leq S_{min}(\Delta(\widetilde{\psi})) \leq \max_{\overline{\psi}\in b'_*(\psi,2\epsilon)} S_{min}(\Delta(\overline{\psi})), \tag{3.34}$$

thus proving the theorem. Note that theorem 1 is essentially equivalent to the result given in Regula et al. (2018). Using the theory of distillation norms, the authors had shown the one-shot pure state concentration of coherence to be

$$C_c(\psi, \epsilon) = \min_{\sigma \in \mathcal{I}} D_H^{\epsilon}(\psi \| \sigma) - \delta, \tag{3.35}$$

where $0 \leq \delta \leq 1$ and $D_H^{\epsilon}(\psi\|\sigma)$ is the smoothed hypothesis testing relative entropy. That is, $D_H^{\epsilon}(\psi\|\sigma) = -\log \min\{\mathrm{Tr}[\sigma M] : 0 \leq M \leq \mathbb{I},\ F(\psi, M) > 1 - \epsilon\}$. By applying Sion's minimax theorem (Sion 1958), we see that Eq. (3.35) reduces to

$$C_c(\psi, \epsilon) = \max_{M \in B^*(\psi,\epsilon)} S_{min}(\Delta(M)), \tag{3.36}$$

where $B^*(\psi, \epsilon) = \{M : 0 \leq M \leq \mathbb{I},\ F(\psi, M) > 1 - \epsilon\}$ is the so-called operator ball around ψ. Note that $B^*(\psi, \epsilon) \supset b(\psi, \epsilon) \supset b'_*(\psi, \epsilon) \supset b_*(\psi, \epsilon)$. Our lower bound in Theorem 1 therefore implies that the maximum in Eq. (3.36) is attained by a pure state M.

3.4 Coherence concentration for an ensemble of pure states

For any given pure state ensemble $\mathfrak{E} = \{p_i, \psi_i\}_i$ we define the coherence concentration for $\mathfrak{E}$ as :

$$C_c(\mathfrak{E}, \epsilon) := \max_{M \in \mathbb{N}} \left\{ \log_2 M : \max_{\{\Lambda_i\}_i} F^2 \left(\sum_i p_i \Lambda_i(\psi_i), \Phi_M \right) \geq 1 - \epsilon \right\}, \tag{3.37}$$

where Λ_i are incoherent operators. The one-shot coherence of assistance is then given by,

$$C_a(\rho, \epsilon) = \max_{\mathfrak{E}} C_c(\mathfrak{E}, \epsilon), \tag{3.38}$$

where $\mathfrak{E}$ are all possible pure state ensemble decompositions of ρ. We will now define for any ensemble $\mathfrak{E} = \{p_i, \psi_i\}_i$ the following quantity :

$$F_{min}^{\Delta}(\mathfrak{E}) := \min_i S_{min}(\Delta(\psi_i)). \tag{3.39}$$

This is an estimate of the minimum coherence that can be distilled from the ensemble $\mathfrak{E}$. Also for any ensemble $\mathfrak{E}$ and $\epsilon \geq 0$ we define the set :

$$b'(\mathfrak{E},\epsilon) := \left\{ \overline{\mathfrak{E}} = \{p_i, \overline{\psi}_i\}_i : \mathrm{Tr}(\overline{\psi}_i) \leq 1, \sum_i p_i F(\overline{\psi}_i, \psi_i) \geq 1-\epsilon \right\}. \tag{3.40}$$

and also the subset of $b(\mathfrak{E},\epsilon)$ with normalized pure states as,

$$b(\mathfrak{E},\epsilon) := \left\{ \overline{\mathfrak{E}} = \{p_i, \overline{\psi}_i\}_i \in b'(\mathfrak{E},\epsilon) : \mathrm{Tr}(\overline{\psi}_i) = 1 \right\} \tag{3.41}$$

Now we state our main result.

Theorem 2. *For any given ensemble* $\mathfrak{E} = \{p_i, \psi_i\}_i$ *of pure states, and any* $\epsilon \geq 0$,

$$\max_{\overline{\mathfrak{E}} \in b(\mathfrak{E},\epsilon)} F^{\Delta}_{min}(\overline{\mathfrak{E}}) - \delta \leq C_c(\mathfrak{E},\epsilon) \leq \max_{\overline{\mathfrak{E}} \in b'(\mathfrak{E},2\epsilon)} F^{\Delta}_{min}(\overline{\mathfrak{E}}), \tag{3.42}$$

where $0 \leq \delta \leq 1$ *is a number to ensure that the lower limit is the logarithm of an integer.*

Proof: Our proof of Theorem 2 follows in parallel to the proof of Theorem 1. For the lower bound, let $\overline{\mathfrak{E}} = \{p_i, \overline{\psi}_i\}_i$ be any ensemble such that $\overline{\mathfrak{E}} \in b(\mathfrak{E},\epsilon)$, i.e. $\sum_i p_i F(\psi_i, \overline{\psi_i}) \geq 1-\epsilon$. As in the proof of Theorem 1, we know that for each pure state ψ_i Bob can distill a maximally coherent state of length $\log_2 \left\lfloor 2^{S_{min}(\Delta(\psi_i))} \right\rfloor$ without error. Then there exists a set of incoherent operations $\{\Lambda_i\}_i$ such that $\Lambda_i(\overline{\psi}_i) = \Phi_{M(\overline{\mathfrak{E}})}$, where $M(\overline{\mathfrak{E}}) \equiv \min_i \left\lfloor 2^{S_{min}(\Delta(\overline{\psi}_i))} \right\rfloor$. This is because each $\overline{\psi}_i \in \overline{\mathfrak{E}}$ can attain a maximally coherent state of at least length $M(\overline{\mathfrak{E}})$ using incoherent operations. Then,

$$\begin{aligned} 1-\epsilon &\leq \sum_i p_i F(\psi_i, \overline{\psi}_i) \leq \sum_i p_i F(\Lambda_i(\psi_i), \Lambda_i(\overline{\psi}_i)), \\ &= \sum_i p_i F(\Lambda_i(\psi_i), \Phi_{M(\overline{\mathfrak{E}})}) = F\left(\sum_i p_i \Lambda_i\left(\psi_i\right), \Phi_{M(\overline{\mathfrak{E}})} \right), \end{aligned} \tag{3.43}$$

where the second inequality follows from the monotonicity of fidelity under CP maps. Since this holds for any $\overline{\mathfrak{E}} \in b(\mathfrak{E},\epsilon)$, we conclude that

$$\begin{aligned} C_c(\mathfrak{E},\epsilon) &\geq \max_{\overline{\mathfrak{E}} \in b(\mathfrak{E},\epsilon)} \min_i S_{min}(\Delta(\overline{\psi}_i)) - \delta, \\ &= \max_{\overline{\mathfrak{E}} \in b(\mathfrak{E},\epsilon)} F^{\Delta}_{min}(\overline{\mathfrak{E}}) - \delta, \end{aligned} \tag{3.44}$$

thus proving the direct part of the theorem.

For the converse part, suppose that $C_c(\mathfrak{E},\epsilon) = \log_2 M$. Then there exists a family of incoherent maps $\{\Lambda_i\}_i$ such that

$$1-\epsilon \le F^2\left(\sum_i p_i \Lambda_i(\psi_i), \Phi_M\right) = \sum_i p_i \left\langle \Phi_M \right| \Lambda_i(\psi_i) \left|\Phi_M\right\rangle . \tag{3.45}$$

Since each Λ_i is incoherent, for any $\gamma \in \mathcal{I}$ we have that

$$\Phi_M \Lambda_i(\gamma)\Phi_M \le \frac{1}{M}\Phi_M. \tag{3.46}$$

With $\Lambda_i(\psi_i) \le \mathbb{I}$, we can multiply both sides of the previous equation by $\Lambda_i(\psi_i)$ and take the trace to obtain

$$\begin{aligned}
\log_2 M &\le -\log \operatorname{Tr}\left[\Phi_M \Lambda_i(\psi_i)\Phi_M \Lambda_i(\gamma)\right] \\
&= -\log \operatorname{Tr}\left[\Lambda_i^* \left(\Phi_M \Lambda_i(\psi_i)\Phi_M\right)\gamma\right] \\
&= -\log \operatorname{Tr}[Q_i\gamma] \\
&\le -\log \operatorname{Tr}\left[\sqrt{Q_i}\psi_i\sqrt{Q_i}\gamma\right] \\
&\le -\log \operatorname{Tr}[\widetilde{\psi}_i\gamma],
\end{aligned} \tag{3.47}$$

where we have used the fact that $\sqrt{Q}\psi_i\sqrt{Q} \le Q$ and we have introduced the sub-normalized states $|\widetilde{\psi}_i\rangle \equiv \sqrt{Q_i}\,|\psi_i\rangle$. Define the pure state ensemble $\widetilde{\mathfrak{E}} \equiv \{p_i, \widetilde{\psi}_i\}_i$. Returning to equation (3.47), we can choose the incoherent state γ to be an eigenvector associated with the largest eigenvalue of $\Delta(\widetilde{\psi}_i)$. Using this inequality on every $|\widetilde{\psi}_i\rangle \in \widetilde{\mathfrak{E}}$, we obtain

$$\log_2 M \le \min_i S_{min}(\Delta(\widetilde{\psi}_i)) = F^{\Delta}_{min}(\widetilde{\mathfrak{E}}). \tag{3.48}$$

It remains to show that $\widetilde{\mathfrak{E}} \in b'(\mathfrak{E}, 2\epsilon)$. Using the inequality in Eq. (3.33), we have

$$\begin{aligned}
\sqrt{\sum_i p_i F(\psi_i, \widetilde{\psi}_i)} &\ge \sqrt{\sum_j p_i \operatorname{Tr}[Q_i\psi_i]} \\
&= \sqrt{\sum_i p_i \left\langle \Phi_M \right| \Lambda_i(\psi_i) \left|\Phi_M\right\rangle^2} \\
&\ge \sum_i p_i \left\langle \Phi_M \right| \Lambda_i(\psi_i) \left|\Phi_M\right\rangle \ge 1-\epsilon,
\end{aligned} \tag{3.49}$$

where the second inequality follows from the concavity of the function $f(x) = \sqrt{x}$. Hence $\sum_i p_i F(\psi_i, \widetilde{\psi}_i) \geq (1-\epsilon)^2 \geq 1 - 2\epsilon$. So we have,

$$\log_2 M \leq F^{\Delta}_{min}(\widetilde{\mathfrak{E}}) \leq \max_{\overline{\mathfrak{E}} \in b'(\mathfrak{E}, 2\epsilon)} F^{\Delta}_{min}(\overline{\mathfrak{E}}). \tag{3.50}$$

3.5 Asymptotic coherence of assistance

For a mixed state $\rho \equiv \rho^B$, its one-shot coherence of assistance is given by

$$C_a(\rho, \epsilon) = \max_{\mathfrak{E}} C_c(\mathfrak{E}, \epsilon), \tag{3.51}$$

where the maximization is over all ensemble decompositions $\mathfrak{E}$ of ρ. The coherence of assistance for ρ is defined by

$$D_a(\rho) = \max_{\mathfrak{E} = \{p_i, \psi_i\}_i} \sum_i p_i S(\Delta(\psi_i)), \tag{3.52}$$

with its regularized version being $D_a^{\infty}(\rho) = \lim_{n\to\infty} \frac{1}{n} D_a(\rho^{\otimes n})$. The asymptotic assisted coherence concentration for Alice and Bob sharing a pure state $|\psi\rangle^{AB}$ is given by (Chitambar et al. 2016),

$$D_c^{A|B}(|\psi\rangle^{AB}) = D_a^{\infty}(\rho^B) = S(\Delta(\rho^B)), \tag{3.53}$$

where $\rho^B = \mathrm{Tr}_A(|\psi\rangle^{AB})$. Let us define the asymptotic limit of the one-shot coherence of assistance as,

$$C_a^{\infty}(\rho) = \lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{1}{n} C_a(\rho^{\otimes n}, \epsilon). \tag{3.54}$$

I will now show that under this limit we recover the asymptotic expression.

Theorem 3. *For any state ρ,*

$$C_a^{\infty}(\rho) = D_a^{\infty}(\rho). \tag{3.55}$$

Lemma 4. *For any state ρ,*

$$C_a^{\infty}(\rho) \leq D_a^{\infty}(\rho). \tag{3.56}$$

Proof : Suppose ρ has support on a d-dimensional Hilbert space. From Theorem 2, we have

$$C_a(\rho^{\otimes n}, \epsilon) \leq \max_{\mathfrak{E}} \max_{\overline{\mathfrak{E}} \in b'(\mathfrak{E}, 2\epsilon)} F^{\Delta}_{min}(\overline{\mathfrak{E}}) \leq \max_{\mathfrak{E}} \max_{\overline{\mathfrak{E}} \in b'(\mathfrak{E}, 2\epsilon)} \sum_i p_i S_{min}(\Delta(\overline{\psi}_i))$$
$$\leq \max_{\mathfrak{E}} \max_{\overline{\mathfrak{E}} \in b'(\mathfrak{E}, 2\epsilon)} \sum_i p_i S(\Delta(\overline{\psi}_i)), \tag{3.57}$$

where the first maximization is taken over all ensembles $\mathfrak{E}$ generating $\rho^{\otimes n}$. To bound the last term introduce the quantum-incoherent states $\sigma^{BX} = \sum_i p_i \psi_i \otimes |i\rangle\langle i|$, $\overline{\sigma}^{BX} = \sum_i p_i \overline{\psi}_i \otimes |i\rangle\langle i|$, and note

$$\|\sigma^{BX} - \overline{\sigma}^{BX}\|_1 = \sum_i p_i \|\psi_i - \overline{\psi}_i\|_1 = 2 \sum_i p_i T(\psi_1, \overline{\psi}_i) \leq 8\epsilon + 8\sqrt{\epsilon}, \tag{3.58}$$

where $T(\psi_i, \overline{\psi}_i)$ is the trace distance and the proof of the last inequality can be found in appendix A.2. If we let Δ^B denote the dephasing map on system B then we further have $\delta := \|\Delta^B(\sigma^{BX}) - \Delta^B(\overline{\sigma}^{BX})\|_1 \leq 8\epsilon + 8\sqrt{\epsilon}$. An application of the Alicki-Fannes inequality (Alicki and Fannes 2004) to the states $\Delta^B(\sigma^{BX})$ and $\Delta^B(\overline{\sigma}^{BX})$ yields

$$\left| \sum_i p_i S(\Delta(\overline{\psi}_i)) - \sum_i p_i S(\Delta(\psi_i)) \right| \leq 4\delta n \log(d) + h(\delta), \tag{3.59}$$

where $h(\delta) := -\delta \log_2(\delta) - (1-\delta)\log_2(1-\delta)$, is the binary entropy function. Hence

$$C_a(\rho^{\otimes n}, \epsilon) \leq \max_{\mathfrak{E}} \sum_i p_i S(\Delta(\psi_i)) + 4\delta n \log(d) + h(\delta) = D_a(\rho^{\otimes n}) + 4\delta n \log(d) + h(\delta). \tag{3.60}$$

Dividing both sides by n and taking the limits $n \to \infty$, $\epsilon \to 0$ yields the desired result.

Definition 1. *We define the quantum-incoherent state corresponding to any pure state ensemble $\mathfrak{E} = \{p_i, \psi_i^B\}_i$ as,*

$$\sigma_{\mathfrak{E}}^{BZ} := \sum_i p_i \psi_i^B \otimes \pi_i^Z. \tag{3.61}$$

where π_i^Z are orthogonal rank one incoherent projectors $|i\rangle\langle i|^Z$.

I define the function $\overline{C}^{\epsilon}_{min} : \mathcal{D}(\mathcal{H}^B \otimes \mathcal{H}^Z) \to \mathbb{R}$ which is a smoothed version of $C_{min}(\cdot)$ introduced in equation (3.9) but defined for quantum-incoherent states;

$$\overline{C}^{\epsilon}_{min}(\sigma^{BZ}_{\mathfrak{E}}) := \max_{\overline{\mathfrak{E}} \in b(\mathfrak{E},\epsilon)} \min_{\nu^{BZ} \in \mathcal{I}} S_0(\overline{\sigma}^{BZ}_{\mathfrak{E}} \| \nu^{BZ}). \tag{3.62}$$

I will make use of the following lemmas,

Lemma 5. *For any state ρ^B and any $\epsilon \geq 0$,*

$$\max_{\mathfrak{E}} \overline{C}^{\frac{\epsilon}{2}}_{min}(\sigma^{BZ}_{\mathfrak{E}}) - \delta \leq C_a(\rho^B, \epsilon), \tag{3.63}$$

where the maximization is taken over all ensembles $\mathfrak{E} = \{p_i, \psi_i\}_i$ such that $\rho^B = \sum_i p_i \psi_i$, $\sigma^{BZ}_{\mathfrak{E}} := \sum_i p_i \psi_i \otimes \pi_i$ and $0 \leq \delta \leq 1$ ensures the lower limit is the logarithm of a positive integer.

Proof: Notice that,

$$\begin{aligned}
\overline{C}^{\frac{\epsilon}{2}}_{min}(\sigma^{BZ}_{\mathfrak{E}}) &:= \max_{\overline{\mathfrak{E}} \in b(\mathfrak{E},\frac{\epsilon}{2})} \min_{\nu^{BZ} \in \mathcal{I}} \left\{ -\log_2 \operatorname{Tr}\left(\Pi_{\sigma^{BZ}_{\overline{\mathfrak{E}}}} \nu^{BZ} \right) \right\}, \\
&= \max_{\{p_i, \overline{\phi}_i\}_i \in b(\mathfrak{E},\frac{\epsilon}{2})} \min_i \min_{\nu^B \in \mathcal{I}} \left\{ -\log_2 \operatorname{Tr}(\overline{\phi}_i \nu^B) \right\}, \\
&= \max_{\{p_i, \overline{\phi}_i\}_i \in b(\mathfrak{E},\frac{\epsilon}{2})} \min_i \left\{ -\log_2 \lambda_{max}(\Delta(\overline{\phi}_i)) \right\}, \\
&= \max_{\{p_i, \overline{\phi}_i\}_i \in b(\mathfrak{E},\frac{\epsilon}{2})} \min_i S_{min}(\Delta(\overline{\phi}_i)), \\
&= \max_{\overline{\mathfrak{E}} \in b(\mathfrak{E},\frac{\epsilon}{2})} F^{\Delta}_{min}(\overline{\mathfrak{E}}) \leq C_c(\mathfrak{E}, \epsilon),
\end{aligned} \tag{3.64}$$

where the inequality comes from theorem 2. Maximizing over $\mathfrak{E}$ proves the lemma.

Lemma 6. *Given a quantum-incoherent state $(\sigma^{BZ}_{\mathfrak{E}})^{\otimes n}$ and any general pure state ensemble $\mathfrak{E}_n = \{p^{(n)}_i, \psi^n_i\}_i$ such that $(\sigma^{BZ}_{\mathfrak{E}})^{\otimes n} = \sum_i p^{(n)}_i \psi^n_i$, we have*

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \max_{\mathfrak{E}_n} \overline{C}^{\epsilon}_{min}(\sigma^{B^n Z^n}_{\mathfrak{E}_n}) \geq \max_{\mathfrak{E}} C_r(\sigma^{BZ}_{\mathfrak{E}}), \tag{3.65}$$

where $C_r(\sigma)$ is relative entropy of coherence.

Proof: We need to use some results from the quantum information spectrum approach.

Definition 2. *Given a sequence of states $\hat{\rho} = \{\rho^n\}_{n=1}^{\infty}$ with $\rho^n \in \mathcal{D}(\mathcal{H}^{\otimes n})$ (set of density operators in $\mathcal{H}^{\otimes n}$) and positive operators $\hat{\sigma} = \{\sigma^n\}_{n=1}^{\infty}$ with $\sigma^n \in \mathcal{B}(\mathcal{H}^{\otimes n})$ (set of positive operators acting on $\mathcal{H}^{\otimes n}$), and defining $\Omega^n(\gamma) := \rho^n - 2^{n\gamma}\sigma^n$, the quantum spectral inf-divergence rate is defined as,*

$$\underline{D}(\hat{\rho}\|\hat{\sigma}) := \sup\left\{\gamma : \liminf_{n\to\infty} \mathrm{Tr}\left(\{\Omega^n \geq 0\}\Omega^n\right) = 1\right\}, \tag{3.66}$$

where $\{X \geq 0\}$ for a self-adjoint operator X denotes the projector unto the non-negative eigenspace of X.

Lemma 7. *Given a state ρ^n and a self-adjoint operator ω^n, for any real γ,we have,*

$$\mathrm{Tr}\left(\{\rho^n - 2^{n\gamma}\omega^n\}\omega^n\right) \leq 2^{-n\gamma}. \tag{3.67}$$

Proof : see (Datta and Renner 2009).

Lemma 8. *For any given state ρ^B, let $\mathfrak{E} = \{p_i, \psi_i\}$ denote a pure state decomposition and $\mathfrak{E}_n = \{p_{i,n}, \psi_i^n\}$ denote a pure state decomposition of the state $(\rho^B)^{\otimes n}$, then we have,*

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\max_{\mathfrak{E}_n}\overline{C}^{\epsilon}_{min}(\sigma^{B^nZ^n}_{\mathfrak{E}_n}) \geq \max_{\mathfrak{E}}\min_{\nu^{BZ}\in\mathcal{I}}\underline{D}(\hat{\sigma}^{BZ}_{\mathfrak{E}}\|\hat{\nu}^{BZ}), \tag{3.68}$$

where $\hat{\sigma}^{BZ}_{\mathfrak{E}} = \left\{(\sigma^{BZ}_{\mathfrak{E}})^{\otimes n}\right\}_{n\geq 1}$ and $\hat{\nu}^{BZ} = \{(\nu^{BZ})^{\otimes n}\}_{n\geq 1}$.

Proof: Let $\mathfrak{E}^*$ be an ensemble such that it achieves the maximum in equation (3.68). By definition we have,

$$\begin{aligned}\max_{\mathfrak{E}_n}\overline{C}^{\epsilon}_{min}(\sigma^{B^nZ^n}_{\mathfrak{E}_n}) &= \max_{\mathfrak{E}_n}\max_{\overline{\mathfrak{E}}_n\in b(\mathfrak{E}_n,\epsilon)}\min_{\nu^{B^nZ^n}\in\mathcal{I}} S_0(\overline{\sigma}^{B^nZ^n}_{\overline{\mathfrak{E}}_n}\|\nu^{B^nZ^n}),\\ &\geq \max_{\mathfrak{E}}\max_{\overline{\mathfrak{E}}_n\in b(\mathfrak{E}^{\otimes n},\epsilon)}\min_{\nu^{B^nZ^n}\in\mathcal{I}} S_0(\overline{\sigma}^{B^nZ^n}_{\overline{\mathfrak{E}}_n}\|\nu^{B^nZ^n}),\\ &\geq \max_{\overline{\mathfrak{E}}_n\in b((\mathfrak{E}^*)^{\otimes n},\epsilon)}\min_{\nu^{B^nZ^n}\in\mathcal{I}} S_0(\overline{\sigma}^{B^nZ^n}_{\overline{\mathfrak{E}}_n}\|\nu^{B^nZ^n}),\end{aligned} \tag{3.69}$$

where $\mathfrak{E}^{\otimes n}$ is the product pure state ensemble $\{p_i, \psi_i\}^{\otimes n}$. For each $\nu^{B^nZ^n}$ and any $\gamma \in \mathbb{R}$ we define the projector,

$$P^n_\gamma \equiv P^n_\gamma(\nu^{B^nZ^n}) := \{(\sigma^{BZ}_{\mathfrak{E}^*})^{\otimes n} - 2^{n\gamma}\nu^{B^nZ^n} \geq 0\}. \tag{3.70}$$

Since $\nu^{B^nZ^n}$ are incoherent states, the projector P_γ^n also has a quantum-incoherent structure. Let $\hat{\sigma}_{\mathfrak{E}^*}^{BZ}$ be the i.i.d. (independent and identically distributed) sequence of states $\{(\sigma_{\mathfrak{E}^*}^{BZ})^{\otimes n}\}_{n=1}^{\infty}$. For a sequence $\hat{\nu}^{BZ} := \{\nu_n^{B^nZ^n}\}_{n=1}^{\infty}$ fix $\delta > 0$ and choose $\gamma \equiv \gamma(\hat{\nu}^{BZ}) := \underline{D}(\hat{\sigma}_{\mathfrak{E}^*}^{BZ} \| \hat{\nu}_n^{BZ}) - \delta$. Then from the definition of the quantum inf-divergence rate in equation (3.66), there exists an n large enough such that,

$$\operatorname{Tr}\left(P_\gamma^n (\sigma_{\mathfrak{E}^*}^{BZ})^{\otimes n}\right) \geq 1 - \epsilon, \tag{3.71}$$

for any $\epsilon \geq 0$. Here I have used the fact that the quantum inf-divergence rate can be arternatively defined as (see prop 2. in (Bowen and Datta 2006))

$$\underline{D}(\hat{\rho} \| \hat{\sigma}) := \sup\left\{\gamma : \liminf_{n\to\infty} \operatorname{Tr}\left(\{\Omega^n \geq 0\}\rho^n\right) = 1\right\}, \tag{3.72}$$

where $\Omega^n = \rho^n - 2^{n\gamma}\sigma^n$. Now I define,

$$\frac{P_\gamma^n (\sigma_{\mathfrak{E}^*}^{BZ})^{\otimes n} P_\gamma^n}{\operatorname{Tr}\left(P_\gamma^n (\sigma_{\mathfrak{E}^*}^{BZ})^{\otimes n}\right)} = \frac{\sum_i p_{i,n} \overline{\psi}_i^n \otimes \pi_i^n}{\operatorname{Tr}(P_\gamma^n (\sigma_{\mathfrak{E}^*}^{BZ})^{\otimes n})} \equiv \omega_{\mathfrak{E}'_n,\gamma}^{B^nZ^n}(\nu^{B^nZ^n}) =: \omega_{\mathfrak{E}'_n,\gamma}^{B^nZ^n}, \tag{3.73}$$

where $\pi_i^n = |i^n\rangle\langle i^n|$ and $\mathfrak{E}'_n$ is the pure state ensemble $\{p_{i,n}, \frac{\overline{\psi}_i^n}{\operatorname{Tr}(P_\gamma^n (\sigma_{\mathfrak{E}^*}^{BZ})^{\otimes n})}\}_i$ with $\overline{\psi}_i^n = \operatorname{Tr}_{Z^n}(P_\gamma^n(\psi_i^n \otimes \pi_i^n) P_\gamma^n)$. We will now show that $\mathfrak{E}'_n \in b((\mathfrak{E}^*)^{\otimes n}, \epsilon)$. Since P_γ^n has a quantum-incoherent structure, we can write it as $P_\gamma^n = \sum_i \Pi_{\gamma,i}^n \otimes \pi_i^n$. Where $\Pi_{\gamma,i}^n$ are projectors acting on the Hilbert space $(\mathcal{H}^B)^{\otimes n}$. Now we have,

$$1 - \epsilon \leq \operatorname{Tr}\left(P_\gamma^n (\sigma_{\mathfrak{E}^*}^{BZ})^{\otimes n}\right) = \operatorname{Tr}\left(\sum_i p_{i,n} \Pi_{\gamma,i}^n \psi_i^n \otimes \pi_i^n\right) = \sum_i p_{i,n} \operatorname{Tr}\left(\Pi_{\gamma,i}^n \psi_i^n\right). \tag{3.74}$$

but note that,

$$\begin{aligned} F\left(\frac{\overline{\psi}_i^n}{\operatorname{Tr}(P_\gamma^n (\sigma_{\mathfrak{E}^*}^{BZ})^{\otimes n})}, \psi_i^n\right) &= \frac{1}{\sqrt{\sum_j p_{j,n} \operatorname{Tr}(\Pi_{\gamma,j}^n \psi_i^n)}} \operatorname{Tr}\left(\sqrt{\langle\psi_i^n|\Pi_{\gamma,i}^n|\psi_i^n\rangle\langle\psi_i^n|\Pi_{\gamma,i}^n|\psi_i^n\rangle}\right), \\ &= \frac{1}{\sqrt{\sum_j p_{j,n} \operatorname{Tr}(\Pi_{\gamma,j}^n \psi_i^n)}} \operatorname{Tr}\left(\Pi_{\gamma,i}^n \psi_i^n\right). \end{aligned} \tag{3.75}$$

Hence we have,

$$\sum_i p_{i,n} F\left(\frac{\overline{\psi}_i^n}{\mathrm{Tr}(P_\gamma^n(\sigma_{\mathfrak{E}^*}^{BZ})^{\otimes n})}, \psi_i^n\right) = \sqrt{\sum_j p_{j,n}\,\mathrm{Tr}(\Pi_{\gamma,j}^n \psi_i^n)},$$
$$\geq \sum_j p_{j,n}\,\mathrm{Tr}(\Pi_{\gamma,j}^n \psi_i^n) \geq 1-\epsilon, \tag{3.76}$$

where the last inequality follows from equation (3.74). Equation (3.76) implies that $\mathfrak{E}_n' \in b((\mathfrak{E}^*)^{\otimes \mathfrak{n}}, \epsilon)$. Proceeding from equation (3.69) we have

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\left(\max_{\overline{\mathfrak{E}}_n \in b((\mathfrak{E}^*)^{\otimes n},\epsilon)}\ \min_{\nu^{B^nZ^n}\in\mathcal{I}} S_0(\sigma_{\overline{\mathfrak{E}}_n}^{B^nZ^n}\|\nu_n^{B^nZ^n})\right),\\
&\geq \lim_{n\to\infty}\frac{1}{n}\min_{\nu^{B^nZ^n}\in\mathcal{I}} S_0(\omega_{\mathfrak{E}_n',\gamma}^{B^nZ^n}\|\nu_n^{B^nZ^n}),\\
&= \lim_{n\to\infty}\frac{1}{n}\min_{\nu^{B^nZ^n}\in\mathcal{I}}\left\{-\log_2 \mathrm{Tr}\left(\Pi_{\omega_{\mathfrak{E}_n',\gamma}^{B^nZ^n}}\nu_n^{B^nZ^n}\right)\right\},\\
&\geq \lim_{n\to\infty}\frac{1}{n}\min_{\nu^{B^nZ^n}\in\mathcal{I}}\left\{-\log_2 \mathrm{Tr}\left(P_\gamma^n\nu_n^{B^nZ^n}\right)\right\},\\
&\geq \min_{\hat{\nu}^{BZ}}\gamma(\hat{\nu}^{BZ}) = \underline{D}(\hat{\sigma}_{\mathfrak{E}^*}^{BZ}\|\hat{\nu}_n^{BZ}) - \delta = \max_{\mathfrak{E}}\underline{D}(\hat{\sigma}_{\mathfrak{E}}^{BZ}\|\hat{\nu}_n^{BZ}) - \delta.
\end{aligned} \tag{3.77}$$

For the second inequality, we have used the fact that $\Pi_{\omega_{\mathfrak{E}_n',\gamma}^{B^nZ^n}} \leq P_\gamma^n$ and the third inequality follows from lemma 7. As this holds for arbitrary $\delta \geq 0$ we recover the statement of lemma 8 in the limit $\epsilon \to 0$.

Lemma 9. *For any sequence of states* $\hat{\rho} = \{\rho^{\otimes n}\}_{n\geq 1}$*,*

$$\min_{\hat{\sigma}} \underline{D}(\hat{\rho}\|\hat{\sigma}) = C_r(\rho), \tag{3.78}$$

where $\hat{\sigma} = \{\sigma^n\}_{n\geq 1}$ *with* $\sigma^n \in \mathcal{I}$ *and* $C_r(\rho) = \min_{\delta\in\mathcal{I}} S(\rho\|\delta)$ *is the relative entropy of coherence.*

Proof: Consider the family of sets $\mathcal{M} := \{\mathcal{M}_n\}_{n\geq 1}$

$$\mathcal{M}_n := \{\delta_n \in \mathcal{I}_n\}_{n\geq 1} \tag{3.79}$$

where $\mathcal{I}_n$ is the set of incoherent states in $\mathcal{H}^{\otimes n}$.

Proposition 1. *The family of sets* $\mathcal{M}$ *satisfies the conditions required to apply the generalized Stein's lemma (proposition III.1 in (Brandao and Plenio 2010))* .

Proof: see appendix A.1.

From proposition 1 we have for a given state ρ,

$$\mathcal{S}_{\mathcal{M}}^{\infty} := \frac{1}{n}\mathcal{S}_{\mathcal{M}_n}(\rho^{\otimes n}), \tag{3.80}$$

with $\mathcal{S}_{\mathcal{M}_n}(\rho^{\otimes n}) := \min_{\delta_n \in \mathcal{M}_n} S(\rho^{\otimes n}\|\delta_n)$. Let $\Omega_n(\gamma) = \rho^{\otimes n} - 2^{n\gamma}\delta_n$. Then from the generalized Stein's lemma in (Brandao and Plenio 2010) it follows that for $\gamma > \mathcal{S}_{\mathcal{M}}^{\infty}(\rho)$,

$$\lim_{n\to\infty} \min_{\delta_n \in \mathcal{M}_n} \operatorname{Tr}\left(\{\Omega_n(\gamma) \geq 0\}\Omega_n\right) = 0. \tag{3.81}$$

This implies that $\min_{\hat{\sigma}} \underline{D}(\hat{\rho}\|\hat{\sigma}) \leq S_{\mathcal{M}}^{\infty}(\rho)$. Conversely, for $\gamma < \mathcal{S}_{\mathcal{M}}^{\infty}(\rho)$,

$$\lim_{n\to\infty} \min_{\delta_n \in \mathcal{M}_n} \operatorname{Tr}\left(\{\Omega_n(\gamma) \geq 0\}\Omega_n\right) = 1, \tag{3.82}$$

which implies that $\min_{\hat{\sigma}} \underline{D}(\hat{\rho}\|\hat{\sigma}) \geq \mathcal{S}_{\mathcal{M}}^{\infty}(\rho)$. Thus we have,

$$\underline{D}(\hat{\rho}\|\hat{\sigma}) = \mathcal{S}_{\mathcal{M}}^{\infty}(\rho). \tag{3.83}$$

But by definition $\mathcal{S}_{\mathcal{M}}^{\infty}(\rho) \equiv C_r^{\infty}(\rho) := \lim_{n\to\infty} \frac{1}{n} \min_{\delta_n \in \mathcal{I}} S(\rho^{\otimes n}\|\delta_n) = C_r(\rho)$ because of the additivity of the relative entropy of coherence (Winter and Yang 2016), thus proving lemma 9. Lemma 8 and lemma 9 together prove lemma 6.

Lemma 10. *For any bipartite state* ρ^B*,*

$$C_a^{\infty}(\rho^B) \geq \lim_{n\to\infty} \frac{1}{n} D_a((\rho^B)^{\otimes n}) \equiv D_a^{\infty}(\rho). \tag{3.84}$$

Proof: Let $\mathfrak{E} = \{p_i, \psi_i\}_i$ be a pure state ensemble decomposition of ρ and $\mathfrak{E}_n = \{p_{i^n}, \psi_{i^n}^{B^n}\}_{i^n}$ be such a decomposition of $(\rho^B)^{\otimes n}$. As before I define the quantum incoherent state,

$$\sigma_{\mathfrak{E}_n}^{B^n Z^n} = \sum_i p_{i^n} \phi_{i^n}^{B^n} \otimes \pi_{i^n}^{Z^n}, \tag{3.85}$$

where $\pi_{i^n}^{Z^n} = |i^n\rangle\langle i^n|$ is the incoherent basis in $\mathcal{H}_Z^{\otimes n}$. From lemma 5 we know that,

$$C_a((\rho^B)^{\otimes n}, \epsilon) \geq \max_{\mathfrak{E}_n} \overline{C}_{min}^{\frac{\epsilon}{2}}(\sigma_{\mathfrak{E}_n}^{B^n Z^n}) - \delta_n, \tag{3.86}$$

where $0 \leq \delta_n \leq 1$. So we have,

$$\begin{aligned} C_a^{\infty}(\rho^B) &:= \lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{n}C_a((\rho^B)^{\otimes n},\epsilon) \geq \lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\max_{\mathfrak{E}_n}\overline{C}_{min}^{\frac{\epsilon}{2}}(\sigma_{\mathfrak{E}_n}^{B^nZ^n}), \\ &\geq \max_{\mathfrak{E}} C_r(\sigma_{\mathfrak{E}}^{BZ}), \end{aligned} \tag{3.87}$$

where we have used lemma 5 for tne first inequality and lemma 6 for the last inequality.

Lemma 11. *For any quantum-incoherent state $\sigma^{BZ} = \sum_i p_i\sigma_i^B \otimes \pi_i^Z$, where $\pi_i^Z = |i\rangle\langle i|^Z$ are projectors unto the incoherent basis elements and $\{\sigma_i\}_i$ are arbitrary density operators, the relative entropy of coherence of σ^{BZ} is given by,*

$$C_r(\sigma) = \sum_i p_i C_r(\sigma_i). \tag{3.88}$$

Proof:

$$\begin{aligned} C_r(\sigma^{BZ}) = C_r\left(\sum_i p_i\sigma_i^B\otimes\pi_i^Z\right) &= S\left(\Delta\left(\sum_i p_i\sigma_i^B\otimes\pi_i^Z\right)\right) - S\left(\sum_i p_i\sigma_i^B\otimes\pi_i^Z\right) \\ &= S\left(\sum_i p_i\Delta(\sigma_i^B)\otimes\pi_i^Z\right) - S\left(\sum_i p_i\sigma_i^B\otimes\pi_i^Z\right). \end{aligned} \tag{3.89}$$

We have,

$$\begin{aligned} S(\sigma^{BZ}) &= -\operatorname{Tr}\sigma^{BZ}\log\sigma^{BZ} \\ &= -\operatorname{Tr}\left(\left(\sum_i p_i\sigma_i^B\otimes\pi_i^Z\right)\log\left(\sum_j p_j\sigma_j^B\otimes\pi_j^Z\right)\right) \\ &= -\operatorname{Tr}\left(\left(\sum_{i,k} p_i\lambda_k^i|\lambda_k^i\rangle\langle\lambda_k^i|^B\otimes\pi_i^Z\right)\log\left(\sum_{j,l} p_j\lambda_l^j|\lambda_l^j\rangle\langle\lambda_l^j|^B\otimes\pi_j^Z\right)\right) \\ &= -\operatorname{Tr}\left(\sum_{i,k} p_i\lambda_k^i\log(p_i\lambda_k^i)|\lambda_k^i\rangle\langle\lambda_k^i|^B\otimes\pi_i^Z\right) \\ &= -\sum_{i,k} p_i\lambda_k^i\log(p_i\lambda_k^i) \\ &= -\sum_{i,k} p_i\lambda_k^i\log(p_i) - \sum_{i,k} p_i\lambda_k^i\log(\lambda_k^i) \\ &= -\sum_i p_i\log p_i + \sum_i p_i S(\sigma_i) \end{aligned} \tag{3.90}$$

where in the third equality I have used the spectral decomposition of $\sigma_i^B = \sum_k p_i \lambda_k^i |\lambda_k^i\rangle\langle\lambda_k^i|^B$, thus proving the lemma. Since $\sigma_{\mathfrak{C}}^{BZ}$ from equation (3.87) is a quantum-incoherent state, we use lemma 11 to get,

$$C_r\left(\sum_i p_i \phi_i^B \otimes \pi_i^Z\right) = \sum_i p_i C_r(\phi_i). \tag{3.91}$$

Hence we have,

$$C_a^\infty(\rho^B) := \lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{n}C_a((\rho^B)^{\otimes n}, \epsilon) \geq \max_{\{p_i, \phi_i^B\}} \sum_i p_i C_r(\phi_i^B) = D_a(\rho^B). \tag{3.92}$$

Lemma 4 and 10 proves theorem 3.

3.6 Conclusions

In this chapter I have derived bounds for the one-shot concentration of maximally coherent states for pure states and average rate for an ensemble of pure states. Using this I obtained bounds on the one-shot coherence of assistance and hence the assisted coherence concentration. Finally, I showed that asymptotically the one-shot quantity reduces to the correct known result. Finding the one-shot concentration rate for a more general scenario than assistance where communication is not restricted to being one-way and with multiple parties helping Bob, the so called collaboration scenario, remains an open question. These results highlight how techniques used in the resource theory of entanglement can find ready application to the resource theory of coherence. This raises an interesting question, is there a more general framework under which we can solve such problems and exploit the common resource theory structure? I will explore this in the next chapter.

Chapter 4

General One-Shot Distillation

4.1 Introduction

There is an intuitive link between the quantitative and operational resourcefulness of a state. The larger value a particular resource measure assigns to a state, the greater rate of resource distillation the state should possess. While this rule of thumb does not always hold in general, usually it is possible to bound resource distillation rates in terms of other resource measures. In this chapter, I introduce a function $G_{min}(\rho)$ that measures the overlap of the state ρ with the set of free states. One-shot distillation bounds are given in terms of this function as well as the free robustness of the state. The latter quantity measures how much mixing with another free state γ is required to erase the resourcefulness of ρ.

Robustness is an important resource monotone first used to study entanglement (Vidal and Tarrach 1999), and it has since found application in the study of general resource theories (Brandão and Gour 2015; Takagi and Regula 2019; Takagi et al. 2019). Allowing the state γ to be arbitrary and not necessarily free leads to the definition of the generalized robustness of resource. Every state will have a finite free robustness provided the set of free states has a non-empty interior. However, for affine resource theories (QRTs in which affine combinations of free states are also free) (Gour 2017) it can be shown that the free robustness will diverge for all resource states, and even for non-affine resource theories there can be states without finite free robustness (Liu et al. 2019; Regula 2017).

To make our bounds applicable to more QRTs, I consider a smoothed version of the free robustness, which I call the δ-free robustness. Roughly speaking, this quantity measures how much mixing of a free state is required to eliminate all but a $\delta \geq 0$ amount of resource from a given state. In all QRTs, including affine ones, the δ-free robustness will be finite for all states whenever $\delta > 0$. Complementing the δ-free robustness is the set of quantum operations that cannot generate more than a δ amount of resource. Bounds are given for distilling pure states using these δ-resource-generating operations. Note that studying such operations has already proven crucial to obtaining asymptotic convertibility in entanglement theory (Brandão and Plenio 2008, 2010) and more general QRTs (Brandão and Gour 2015). To my knowledge, this is the first work that investigates a trade-off in resource-distillation with respect to a relaxation on the resource-generating power of the underlying operations.

During the completion of this work I became aware of an independent work which derives bounds for the one-shot distillation rate in terms of the hypothesis testing relative entropy (Liu et al. 2019). Note that the hypothesis testing inequality is the operator smoothed version of $G_{min}(\rho)$ while I use the state smoothed version $G^{\epsilon}_{min}(\rho)$. Similarly the achievable map that Liu et al. (2019) use for mixed state transformation is a variation of the one I use for pure state distillation. The difference between these maps is that our map uses state smoothing instead of operator smoothing and is also applicable to QRTs where the free robustness need not be finite. The authors define a class of QRTs in which there exists pure reference states that have constant overlap with the set free states which is conceptually similar to the constraints on $G_{min}(\phi^m)$ I introduce through Property 1 as expressed in equation (4.19).

4.2 Definitions

Let $\mathcal{D}$ denote the collection of all quantum states for a given quantum system. A resource theory is defined by the pair $\{\mathcal{F}, \mathcal{O}\}$ where $\mathcal{F} \subset \mathcal{D}$ is called the set of free states and $\mathcal{O}$ is the set of free operations. Any state not in $\mathcal{F}$ is known as a resource state. One useful resource quantifier is the free robustness of resource, defined as

$$\mathcal{R}_f(\rho) := \inf_{\pi\in\mathcal{F}} \left\{ s \geq 0 \ : \ \frac{\rho + s\pi}{1+s} \in \mathcal{F} \right\}. \tag{4.1}$$

$\pi_\rho \in \mathcal{F}$ is referred to as an optimal state if it can be used to achieve the infimum value in the definition of $\mathcal{R}_f(\rho)$. The quantity $\mathcal{R}_f(\rho)$ has a natural operational interpretation. Suppose that an experimenter Alice has access to a resource state ρ in her laboratory. Additionally, Alice has the capability to prepare any free state $\pi \in \mathcal{F}$. With probability $\frac{1}{1+s}$ Alice forwards the state ρ to Bob, while with probability $\frac{s}{1+s}$ she prepares some free state π and sends it to Bob instead. Bob's description of the received state is thus $\frac{1}{1+s}(\rho + s\pi)$. The free robustness of ρ, quantifies the threshold value such that for any $s < \mathcal{R}_f(\rho)$, Bob's received state will assuredly still possess resource.

One drawback of the free robustness is that it is not finite in many QRTs. For example, in the resource theory of coherence, it is not possible to mix a resource state (i.e. a non-diagonal density matrix) with a free state (i.e. a diagonal density matrix) to obtain another free state. An alternative notion of robustness that does not generally suffer from this problem involves taking the infimum in equation (4.1) over all states $\mathcal{D}$ instead of over just the free states $\mathcal{F}$ (Harrow and Nielsen 2003; Steiner 2003). The resulting quantity is known as the generalized robustness $\mathcal{R}_g(\rho)$, and it has emerged as an important resource measure since its dual characterization often leads to computationally friendly resource witnesses (Brandão 2005; Piani et al. 2016; Napoli et al. 2016; Regula 2017). Here I introduce a family of robustness measures that generalises the free robustness.

For mathematical convenience, let us first recall the generalized log-robustness, which is given by

$$\mathcal{LR}_g(\rho) := \log\left[1 + \mathcal{R}_g(\rho)\right]. \tag{4.2}$$

It is not difficult to show that this quantity is sub-additive, meaning that

$$\mathcal{LR}_g(\rho \otimes \sigma) \leq \mathcal{LR}_g(\rho) + \mathcal{LR}_g(\sigma), \tag{4.3}$$

as long as $\mathcal{F}$ is closed under tensor products. Next let us define the set of δ-free states,

$$\mathcal{F}^{\delta} := \{\rho \ : \ \mathcal{LR}_g(\rho) \leq \delta\}, \tag{4.4}$$

which from sub-additivity satisfies

$$\rho, \sigma \in \mathcal{F}^{\delta} \quad \Rightarrow \quad \rho \otimes \sigma \in \mathcal{F}^{2\delta} \tag{4.5}$$

provided $\mathcal{F}$ is closed under tensor products. Then for $\delta \in [0, +\infty]$, I define the δ-free robustness as

$$\mathcal{R}^{\delta}(\rho) := \inf_{\pi \in \mathcal{F}^{\delta}} \left\{ s \geq 0 \ : \ \frac{\rho + s\pi}{1+s} \in \mathcal{F}^{\delta} \right\}, \tag{4.6}$$

from which we recover $\mathcal{R}_f(\rho) = \mathcal{R}^0(\rho)$. We can likewise consider the δ-free log-robustness,

$$\mathcal{LR}^{\delta}(\rho) := \log[1 + \mathcal{R}_f^{\delta}(\rho)]. \tag{4.7}$$

It can be shown easily that the δ-free robustness for any state $\rho \in \mathcal{S}$ is finite if $\delta > 0$ and the generalized robustness is finite. Indeed, using convexity of the generalized robustness, we can see that for any state ρ and free state γ,

$$\begin{aligned} R_g\left(\frac{\rho + s\gamma}{1+s}\right) &\leq \frac{1}{1+s} R_g(\rho) + \frac{s}{1+s} R_g(\gamma), \\ &= \frac{1}{1+s} R_g(\rho). \end{aligned} \tag{4.8}$$

Equation (4.8) implies that the resource in any state ρ as quantified by the generalized robustness, can be made arbitrarily small by mixing sufficiently with a free state provided the generalized robustness of ρ is finite. In other words for any $\delta > 0$, there exists some finite positive number s^* such that $\frac{1}{1+s^*}(\rho + s^*\pi) \in \mathcal{F}^{\delta}$.

For a given QRT $\{\mathcal{F}, \mathcal{O}\}$, equation (4.4) provides a relaxation on the set of free states. A corresponding relaxation can be made on the free operations. Following the lead of Brandão and Plenio (2008, 2010); Brandão and Gour (2015), let $\mathcal{O}^{\delta}$ denote the set of

δ-resource-generating (δ-RG) operations as the full collection of operations that act invariantly on $\mathcal{F}^\delta$; i.e.

$$\Lambda \in \mathcal{O}^\delta \quad \Leftrightarrow \quad \Lambda(\gamma) \in \mathcal{F}^\delta \qquad \forall \gamma \in \mathcal{F}^\delta. \tag{4.9}$$

We are interested in the problem of converting a given state ρ to multiple copies of some pure state φ using the δ-resource-generating operations of the theory. More precisely, for an initial state ρ and a target state $\varphi = |\varphi\rangle\langle\varphi|$, the one-shot distillation rate of conversion for parameters $\epsilon, \delta \geq 0$ is defined as

$$\mathcal{D}^{\delta,\epsilon}(\rho, \varphi) := \max_{m \in \mathbb{N}} \left\{ m : \sup_{\Lambda \in \mathcal{O}^\delta} F^2(\Lambda(\rho), \varphi^{\otimes m}) \geq 1 - \epsilon \right\}. \tag{4.10}$$

Here, the fidelity between two states is given by

$$F(\rho, \sigma) := \mathrm{Tr}\left(\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}\right) = \|\sqrt{\rho}\sqrt{\sigma}\|_1, \tag{4.11}$$

which for a pure state $\sigma = |\varphi\rangle\langle\varphi|$ has the form $F(\rho, \varphi) = \sqrt{\langle\varphi|\,\rho\,|\varphi\rangle}$. I will use the notation $\varphi^{\otimes m}$ and φ^m interchangeably.

To obtain bounds on $\mathcal{D}^{\delta,\epsilon}(\rho, \varphi)$, I first define a quantity $G_{min}(\rho)$ as a measure of the maximum overlap between a positive operator ρ and the set of free states $\mathcal{F}$,

$$G_{\min}(\rho) = \inf_{\gamma \in \mathcal{F}} \left\{ -\log \mathrm{Tr}(\rho\gamma) \right\}. \tag{4.12}$$

We will want a smoothing of $G_{\min}(\rho)$ similar to what we had in chapter 3 for S_{min}. Let us denote the sub-normalised ϵ-ball around a state ρ by

$$b'(\rho, \epsilon) = \left\{ \mathbb{I} \geq \overline{\rho} \geq 0 : F(\overline{\rho}, \rho) \geq 1 - \epsilon \right\}. \tag{4.13}$$

The pure state ball around a state ρ is as before given by

$$b_*(\rho, \epsilon) = \left\{ \overline{\psi} \in b'(\rho, \epsilon) \text{ s.t. } \overline{\psi} \text{ is pure} \right\}. \tag{4.14}$$

Then the state-smoothed version of $G_{min}(\rho)$ is defined as

$$G^\epsilon_{min}(\rho) = \max_{\overline{\rho} \in b'(\rho, \epsilon)} G_{min}(\overline{\rho}). \tag{4.15}$$

The pure state smoothed version $G^{\epsilon}_{min,*}(\rho)$ has a similar meaning except with the maximization taken over $b_*(\rho,\epsilon)$ instead of $b'(\rho,\epsilon)$. If $b_*(\rho,\epsilon)$ is an empty set we define $G^{\epsilon}_{min,*}(\rho) = 0$.

4.3 General Distillation Bounds

As described above, the essential ingredients to a resource theory are the sets of free states $\mathcal{F}$ and free operations $\mathcal{O}$. Most QRTs will have additional structure on these objects, such as convexity or closure of $\mathcal{F}$ under partial trace. We wish to bound $\mathcal{D}^{\delta,\epsilon}(\rho,\varphi)$ with as few assumptions on the QRT as possible. For our upper bound, we only require that $G_{\min}$ is an extensive resource measure for pure states. More precisely, we make the following singular assumption:

Property 1. *For every pure state φ, there exists a constant $c(\varphi)$ such that*

$$G_{\min}(\varphi^{\otimes m}) = \inf_{\gamma\in\mathcal{F}} -\log \operatorname{Tr}\left(\varphi^{\otimes m}\gamma\right) \geq m \cdot c(\varphi) \tag{4.16}$$

for all $m \in \mathbb{N}$.

In thermodynamics, an extensive property is additive under the addition of more systems, for example the total heat contained in a system is the sum of the heat contained in each subsystem. Equation (4.16) expresses this condition in a general QRT for the quantity $G_{\min}$ and multiple copies of a pure state. This extensive property holds for the QRTs for entanglement, coherence and purity. I now present the first result.

Theorem 4. *Let $\epsilon, \delta \geq 0$ be arbitrary. For any resource theory satisfying property 1,*

$$\frac{G^{2\sqrt{2\epsilon}}_{\min}(\rho) + \log(1+\delta)}{c(\varphi)} \geq \mathcal{D}^{\delta,\epsilon}(\rho,\varphi). \tag{4.17}$$

Moreover, if ρ is a pure state, this bound can be tightened to read

$$\frac{G^{2\epsilon}_{\min,*}(\rho) + \log(1+\delta)}{c(\varphi)} \geq \mathcal{D}^{\delta,\epsilon}(\rho,\varphi). \tag{4.18}$$

Proof. Let m be the highest rate achievable with error ϵ. This implies that there exists a δ-resource generating operation $\Lambda \in \mathcal{O}^\delta$ such that $F^2(\Lambda(\rho), \varphi^m) \geq 1 - \epsilon$. Property 1 can be equivalently stated as,

$$\varphi^m \gamma \varphi^m \leq \frac{1}{2^{mc(\varphi)}} \varphi^m \qquad \forall \gamma \in \mathcal{F}. \tag{4.19}$$

To see this, note that using the definition of G_{min} and the statement of property 1 we have,

$$\min_{\gamma \in \mathcal{F}} -\log(\mathrm{Tr}(\varphi^m \gamma)) \geq mc(\varphi), \tag{4.20}$$

$$\implies -\log(\mathrm{Tr}(\varphi^m \gamma)) \geq mc(\varphi) \quad \forall \gamma \in \mathcal{F}, \tag{4.21}$$

$$\implies \mathrm{Tr}(\varphi^m \gamma) \leq 2^{-mc(\varphi)}, \tag{4.22}$$

$$\implies \langle \varphi^m | \gamma | \varphi^m \rangle \leq 2^{-mc(\varphi)}, \tag{4.23}$$

$$\implies \langle \varphi^m | \gamma | \varphi^m \rangle \, \varphi^m \leq 2^{-mc(\varphi)} \varphi^m, \tag{4.24}$$

$$\implies \langle \varphi^m | \gamma | \varphi^m \rangle \, |\varphi^m\rangle\langle\varphi^m| \leq 2^{-mc(\varphi)} \varphi^m, \tag{4.25}$$

$$\implies |\varphi^m\rangle \, \langle \varphi^m | \gamma | \varphi^m \rangle \, \langle\varphi^m| \leq \frac{1}{2^{mc(\varphi)}} \varphi^m. \tag{4.26}$$

$$\implies \varphi^m \gamma \varphi^m \leq \frac{1}{2^{mc(\varphi)}} \varphi^m, \tag{4.27}$$

where I have used the notation $\varphi^m = |\varphi^m\rangle\langle\varphi^m|$. Starting from the final expression, all the steps can be reversed to obtain the initial expression hence proving the equivalence.

Since $\Lambda \in \mathcal{O}^\delta$, for every $\gamma \in \mathcal{F}$, there exists some $\pi \in \mathcal{F}$ and $\sigma \in \mathcal{D}$ such that $\Lambda(\gamma) = (1 + \delta)\pi - \delta\sigma$. Then from equation (4.19), it follows that

$$\varphi^m \Lambda(\gamma) \varphi^m \leq \frac{1 + \delta}{2^{mc(\varphi)}} \varphi^m. \tag{4.28}$$

Multiplying both sides of this by $\Lambda(\rho)$ and taking the trace yields

$$\mathrm{Tr}(\Lambda(\rho)\varphi^m \Lambda(\gamma)\varphi^m) \leq \frac{1 + \delta}{2^{mc(\varphi)}} \mathrm{Tr}(\Lambda(\rho)\varphi^m) \leq \frac{1 + \delta}{2^{mc(\varphi)}}. \tag{4.29}$$

Using the cyclic property of trace and denoting the dual map of Λ as Λ^* gives

$$mc(\varphi) - \log(1+\delta) \leq -\log \mathrm{Tr}(\varphi^m \Lambda(\rho) \varphi^m \Lambda(\gamma)), \tag{4.30}$$

$$= -\log \mathrm{Tr}(\Lambda^*(\varphi^m \Lambda(\rho) \varphi^m)\gamma), \tag{4.31}$$

$$= -\log \mathrm{Tr}(Q\gamma) \leq -\log \mathrm{Tr}(\sqrt{Q}\rho\sqrt{Q}\gamma), \tag{4.32}$$

where in the last inequality I use the fact that $\overline{\rho} := \sqrt{Q}\rho\sqrt{Q} \leq Q$. Since γ is an arbitrary free state, we can say that

$$mc(\varphi) \leq \min_{\gamma \in \mathcal{F}} \{-\log \mathrm{Tr}(\overline{\rho}\gamma)\} = G_{min}(\overline{\rho}) + \log(1+\delta). \tag{4.33}$$

I will now show that $\overline{\rho} \in b'(\rho, 2\sqrt{2\epsilon})$. Note that,

$$\mathrm{Tr}(Q\rho) = \mathrm{Tr}(\varphi^m \Lambda(\rho) \varphi^m \Lambda(\rho))) = \langle \varphi^m | \Lambda(\rho) | \varphi^m \rangle^2, \tag{4.34}$$

$$= \left(F^2(\Lambda(\rho), \varphi^m)\right)^2 \geq 1 - 2\epsilon. \tag{4.35}$$

where for the last inequality I use the fact that $F^2(\Lambda(\rho), \varphi^m) \geq 1 - \epsilon$. From the gentle measurement lemma (Winter 1999) we know that, $\|\rho - \overline{\rho}\|_1 \leq 2\sqrt{2\epsilon}$. This implies that

$$F^2(\rho, \overline{\rho}) \geq 1 - 2\sqrt{2\epsilon} \tag{4.36}$$

and $\overline{\rho} \in b'(\rho, 2\sqrt{2\epsilon})$. In equation (4.34), replacing the mixed state ρ with the pure state ψ we have,

$$\mathrm{Tr}(Q\psi) \geq 1 - 2\epsilon. \tag{4.37}$$

Note that $\overline{\psi} = \sqrt{Q}\psi\sqrt{Q}$, hence

$$F(\psi, \overline{\psi}) = \langle \psi | \sqrt{Q} | \psi \rangle \geq \langle \psi | Q | \psi \rangle = \mathrm{Tr}(Q\psi) \geq 1 - 2\epsilon. \tag{4.38}$$

Hence $\overline{\psi} \in b_*(\psi, 2\epsilon)$ and we see that,

$$mc(\varphi) \leq \max_{\overline{\psi} \in b_*(\psi, 2\epsilon)} G_{min}(\overline{\psi}) + \log(1+\delta). \tag{4.39}$$

□

We next consider the achievability of pure-state distillation. While Theorem 4 holds for a wide class of QRTs, including ones that are non-convex, our lower bound on $\mathcal{D}^{\delta,\epsilon}(\rho,\varphi)$ applies only for convex QRTs whose free states are closed under tensor products. Before stating this, we observe a property of the δ-free log-robustness which holds in such QRTs. Unlike the generalized log-robustness, $\mathcal{LR}^{\delta}$ does not appear to be sub-additive in general. However, we can at least provide the following bound.

Proposition 2.

$$\begin{aligned}\mathcal{LR}^{m\delta}(\rho^{\otimes m}) &\leq \log[1+(1+2\mathcal{R}^{\delta}(\rho))^m]-1 \\ &\leq m\log[1+2\mathcal{R}^{\delta}(\rho)] \end{aligned} \tag{4.40}$$

for every integer m and $\delta > 0$.

Proof. Let $\rho = (1+s)\pi - s\sigma$, where $s = \mathcal{R}^{\delta}(\rho)$ and $\pi, \sigma \in \mathcal{F}^{\delta}$. We can then write $\rho^{\otimes m}$ as a linear combination of operators belonging to $\mathcal{F}^{m\delta}$, each of which is an m-part tensor product of the π and σ. From the definition, $\mathcal{LR}^{m\delta}(\rho^{\otimes m})$ is no greater than the logarithm of the positive weight in this linear combination. The positive weight can be written as

$$\frac{1}{2}\left[((1+s)+s)^m + ((1+s)-s)^m\right] = \frac{1}{2}(1+(1+2s)^m).$$

Taking a logarithm establishes the first inequality in (4.40), and the second follows by observing $1 \leq (1+2\mathcal{R}^{\delta}(\rho))^m$. □

We use this inequality to establish a lower bound $\mathcal{D}^{\sigma,\epsilon}(\rho,\varphi)$.

Proposition 3. *Consider any QRT in which the set of free states $\mathcal{F}$ is convex. For any $\delta, \epsilon \geq 0$,*

$$\mathcal{D}^{\delta,\epsilon}(\rho,\varphi) \geq m \tag{4.41}$$

for any positive integer m satisfying

$$G^{2\epsilon}_{\min,*}(\rho) \geq m\log[1+2\mathcal{R}^{\delta/m}(\varphi)].$$

Remark 1. *For the special case that* $\delta = 0$, *one can take* $m = \left\lfloor \frac{G^{2\epsilon}_{\min,*}(\rho)}{\log[1+2\mathcal{R}^0(\varphi)]} \right\rfloor$ *so that*

$$\mathcal{D}^{0,\epsilon}(\rho, \varphi) \geq \left\lfloor \frac{G^{2\epsilon}_{\min,*}(\rho)}{\log[1 + 2\mathcal{R}^0(\varphi)]} \right\rfloor . \tag{4.42}$$

Proof. Let $m > 0$ satisfy $G^{2\epsilon}_{\min,*}(\rho) \geq m \log[1 + 2\mathcal{R}^{\delta/m}(\varphi)]$. We will follow a standard approach of introducing a simple measure-and-prepare map that does the job (see, for example, (Rains 2001)). Consider the CPTP map

$$\Lambda(\omega) = \mathrm{Tr}[(I - \overline{\psi})\omega]\pi_{\varphi^m} + \mathrm{Tr}[\overline{\psi}\omega]\varphi^m, \tag{4.43}$$

where π_{φ^m} is an optimal state chosen in the definition of $\mathcal{R}^{\delta/m}(\varphi^m)$ and $\overline{\psi}$ is an optimal state chosen in the definition of $G^{2\epsilon}_{\min,*}(\rho)$. We first verify that

$$F(\varphi^m, \Lambda(\rho)) \geq F^2(\varphi^m, \Lambda(\rho)) = \mathrm{Tr}(\varphi^m \Lambda(\rho)), \tag{4.44}$$

$$\geq \mathrm{Tr}[\overline{\psi}\rho] \geq 1 - 2\epsilon, \tag{4.45}$$

where we use the fact that $\overline{\psi} \in b_*(\rho, 2\epsilon)$. Next, we use Eq. (4.40)

$$G^{2\epsilon}_{\min,*}(\rho) \geq m \log[1 + 2\mathcal{R}^{\delta/m}(\varphi)] \geq \mathcal{LR}^{\delta}(\varphi^m), \tag{4.46}$$

along with the definition of $G^{2\epsilon}_{\min,*}(\rho)$ to conclude that

$$-\log \mathrm{Tr}[\overline{\psi}\gamma] \geq \mathcal{LR}^{\delta}_f(\varphi^m) \implies \mathrm{Tr}[\overline{\psi}\gamma] \leq [1 + \mathcal{R}^{\delta}(\varphi^m)]^{-1}$$

for any $\gamma \in \mathcal{F}$. Hence

$$\Lambda(\gamma) = \mathrm{Tr}[(I - \overline{\psi})\gamma]\pi_{\varphi^m} + \mathrm{Tr}[\overline{\psi}\gamma]\varphi^m \in \mathcal{F}^{\delta}. \tag{4.47}$$

Convexity of $\mathcal{F}$ has been used here to ensure that $\mathcal{R}^{\delta}(\varphi^m)[1 + \mathcal{R}^{\delta}(\varphi^m)]^{-1}\pi_{\varphi^m} + [1 + \mathcal{R}^{\delta}(\varphi^m)]^{-1}\varphi^m$ remains free under any mixing with π_{φ^m}. □

Notice that the lower bound in proposition 3 would be tighter if we could replace the δ-free robustness in equation (4.41) with the generalized robustness. However doing so would no longer ensure that the measure-and-prepare map of equation (4.43) always generates a sufficiently small amount of resource. This problem can be overcome in the many-copy setting where one can invoke the Generalized Quantum Stein's Lemma (Brandao and Plenio 2010), and this is essentially the high-level approach taken in Refs. (Brandão and Plenio 2008, 2010; Brandão and Gour 2015) to obtain asymptotic reversibility of resource transformations.

4.4 Examples

In many resource theories there exists a maximally resourceful unit pure state φ, such as the Bell state $|\varphi_e\rangle := \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$ for entanglement or the uniform superposition state $|\varphi_c\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ ($|\Phi_2\rangle$ from chapter 3) for coherence. The one-shot distillation rate of the resource is the optimal rate at which one can convert a single copy of a given state into several copies of the maximally resourceful unit state under some error threshold. Equation (4.39) immediately recovers known results for the one-shot concentration rate in entanglement (Buscemi and Datta 2013) and the coherence rate from chapter 3.

Let us first recall that min-entropy of a state ρ is defined as.

$$S_{\min}(\rho) = -\log(\lambda_{\max}(\rho)), \tag{4.48}$$

where $\lambda_{max}(\rho)$ is the largest eigenvalue of ρ. In the QRT of entanglement, $G_{\min}(\varphi^{AB}) = S_{\min}(\mathrm{Tr}_A\, \varphi^{AB})$ while in the QRT of coherence, $G_{\min}(\varphi^A) = S_{\min}(\Delta(\varphi^A))$, where Δ is the completely dephasing map. To see this notice that for any bipartite pure state,

$$|\varphi\rangle^{AB} = \sum_i \sqrt{\lambda_i} U\, |\lambda_i\rangle^A\, |\lambda_i\rangle^B \tag{4.49}$$

the minimisation in equation (4.12) is achieved by the product state $\gamma = U|\lambda_{max}\rangle\langle\lambda_{\max}|U^\dagger \otimes |\lambda_{max}\rangle\langle\lambda_{\max}|$, where $|\lambda_{max}\rangle$ is the eigenvector corresponding to the largest eigenvalue of $\mathrm{Tr}_A\, \varphi^{AB}$ (see Appendix B.1). Similarly for coherence the minimisation is achieved by the largest eigenvector of $\Delta(\varphi)$. Let Γ represent the partial trace operation or the completely

Table 4.1 : Value of $G_{min}(\psi)$ in different theories

R.Theory	Entanglemnet	Coherence
$G_{min}(\psi)$	$S_{min}(\rho_\psi)$	$S_{min}(\Delta(\psi))$

dephasing map. It can be easily verified that $S_{min}(\Gamma(\varphi^m)) = mS_{min}(\Gamma(\varphi))$. Comparing with equation (4.16), the quantity $c(\varphi) = S_{min}(\Gamma(\varphi)) = 1$ for these resource theories.

We define the ideal rate of one-shot distillation of entanglement for an arbitrary pure state ρ to many copies of the maximally entangled state φ_e using δ-entanglement-generating operations as $E^{\delta,\epsilon}(\rho, \varphi_e)$. Similarly the ideal rate of one-shot coherence distillation is defined to be $C^{\delta,\epsilon}(\rho, \varphi_c)$, where φ_c is the maximally coherent state.

Corollary 1. *The one-shot pure state concentration rate for entanglement $E^{\delta,\epsilon}(\psi^{AB}, \varphi_e)$ using δ-entanglement-generating operations and the one-shot concentration rate of coherence $C^{\delta,\epsilon}(\psi, \varphi_c)$ using δ-coherence-generating operations are given by,*

$$E^{\delta,\epsilon}(\psi^{AB}, \varphi_e) \leq \max_{\overline{\psi}^{AB} \in b_*(\psi^{AB}, 2\epsilon)} S_{min}(\rho_{\overline{\psi}^{AB}}) + \log(1+\delta), \tag{4.50}$$

$$C^{\delta,\epsilon}(\psi, \varphi_c) \leq \max_{\overline{\psi} \in b_*(\psi, 2\epsilon)} S_{min}(\Delta(\psi)) + \log(1+\delta), \tag{4.51}$$

respectively, where $\rho_{\overline{\psi}^{AB}} = \mathrm{Tr}_A(\psi^{AB})$ is the reduced density matrix of $\overline{\psi}^{AB}$ and $\Delta(\psi) = \sum_i |i\rangle\langle i|\psi|i\rangle\langle i|$ is the completely dephased version of ψ in the incoherent basis.

In the limit of $\delta = 0$ we recover previously known results regarding the one-shot concentration of coherence and entanglement in (Buscemi and Datta 2013; Vijayan et al. 2018; Regula et al. 2018). Proposition 3 implies that for the resource theory of entanglement, the upper-bound given in Theorem 4 is tight for prefect transformations recovering the known result in (Buscemi and Datta 2013) as shown below.

Corollary 2. *For the resource theory of entanglement the perfect transformation $\psi \to \varphi_e^m$,* *where φ_e is the unit maximally entangled state is achievable with a free operation $\Lambda \in \mathcal{O}$ with a rate*

$$E^{0,0}(\psi^{AB}, \varphi_e) = G_{min}(\psi^{AB}) = S_{min}(\rho_{\psi^{AB}}), \tag{4.52}$$

where $\rho_{\psi^{AB}} = \mathrm{Tr}_B(\psi^{AB})$.

Proof. From proposition 3 we know that there exists a free-operation Λ in the limit $\delta, \epsilon \to 0$ which performs the transformation $\psi \to \varphi_e^m$ if,

$$\begin{aligned} G_{min}(\psi) &\geq m \log(1 + 2\mathcal{R}_f(\varphi_e)) \geq \mathcal{LR}(\varphi_e^m) \\ &= \mathcal{LR}_g(\varphi_e^m) = m, \end{aligned} \tag{4.53}$$

where we have used equation (4.46) and the fact that the free log-robustness of entanglement is equal to the generalized log-robustness of entanglement $\mathcal{LR}_g(\rho)$ for pure states and for the maximally entangled state of rank 2^m the generalized robustness is equal to m (Harrow and Nielsen 2003; Steiner 2003). Combining equations (4.53) and (4.39) in the limit $\delta, \epsilon \to 0$ gives the desired result. □

Remark 2. *For any dimension $d \geq 2$, the δ-free robustness of coherence $\mathcal{R}_f^\delta(\varphi^m)$ is achieved by the maximally mixed state $\mathbb{I}_d = \frac{1}{d}\sum_i |i\rangle\langle i|$, where $m = \log d$.*

Proof. Let the optimal incoherent state achieving $\mathcal{R}_f^\delta(\varphi^m)$ be π_{φ^m}. The twirling operation T for a state ρ is defined as an empirical average over all possible incoherent basis permutations of ρ. Notice that for the maximally coherent state $T(\varphi^m) = \varphi^m$. From the definition of δ-resource robustness we have,

$$\rho = \frac{\varphi^m + \mathcal{R}_f^\delta(\varphi^m)\pi_{\varphi^m}}{1 + \mathcal{R}_f^\delta(\varphi^m)} \in \mathcal{I}^\delta, \tag{4.54}$$

where $\mathcal{I}^\delta$ is the set of δ-incoherent states. Applying the twirling operation on both sides of equation (4.54) we have,

$$T(\rho) = \frac{\varphi^m + \mathcal{R}_f^\delta(\varphi^m)T(\pi_{\varphi^m})}{1 + \mathcal{R}_f^\delta(\varphi^m)} \in \mathcal{I}^\delta. \tag{4.55}$$

The last inclusion follows from the fact that coherence is invariant under the twirling operation. Equation (4.55) implies that if mixing $\mathcal{R}_f^\delta(\varphi^m)$ amount of π_{φ^m} with φ^m gives you a state in $\mathcal{I}^\delta$ then mixing $\mathcal{R}_f^\delta(\varphi^m)$ amount of $T(\pi_{\varphi^m})$ will also give you a state in $\mathcal{I}^\delta$. For any incoherent state γ, $T(\gamma)$ will be the completely mixed state $\frac{\mathbb{I}_d}{d}$. We can see this by noticing that the state $T(\gamma)$ is permutation invariant by virtue of the twirling operation and the only permutation invariant incoherent state is the maximally mixed state. $\square$

4.5 Conclusions

As mentioned in the introduction, the general one-shot distillation problem has been studied in Ref. (Liu et al. 2019) in a more exhaustive manner. However, the techniques used in this chapter are different, and a salient point of this work is the relative mathematical simplicity of the techniques. Currently these results are confined to pure state distillation and a future direction would be to see if these techniques can find the most general mixed state transformation bounds.

Another open question is whether these bounds reproduce the asymptotic results in (Brandão and Gour 2015) under the usual regularisation procedure. This requires further investigation of the asymptotic properties of $G_{min}(\rho)$ and $\mathcal{LR}^\delta(\rho)$. One technical challenge in this direction is that we are constrained to use the free robustness instead of the generalized robustness to ensure that the direct map is a free map. To directly apply the Generalized Quantum Stein's Lemma of Ref. (Brandao and Plenio 2010) for obtaining asymptotic results (Brandão and Gour 2015), a connection needs to be made between the δ-free robustness and the generalized robustness.

It is also of interest to explore what the nature of trade off between error δ in the used operation and error ϵ in the final state is and if they have some operational interpretation. Clearly these quantities must be inversely related since increasing δ allows you to use a larger set of operations which can get you closer to the target state and thus reducing ϵ. Operationally one can interpret a non-zero δ to represent the resource consumed to perform the given task. Seeing whether this allows us to define a new resource measure and more quantitative statements regarding specific QRTs are left to future work.

For the lower bounds we have assumed that the QRT must be convex, an improvement to these bounds would be to find a way to relax this constraint to include non-convex QRTs as well like we do in our upper bounds.

Chapter 5

Discussion

5.1 Main Results

In part I of this thesis I started by studying the resource theory of coherence and answered the question of what is the ideal rate of one-shot assisted coherence distillation. In the process I obtained bounds for one-shot distillation of maximally coherent states from both pure states and pure state ensembles. The obtained rates were shown to reproduce known results in the asymptotic regime of no error and arbitrarily many copies. I then presented a general framework for analysing resource theories and showed that with a minimal set of assumptions we can obtain one-shot pure state distillation bounds. I also introduce the δ-free robustness and show how to incorporate imperfection in the free operations into the analysis. These results provide a powerful approach towards handling multiple resource theories under one framework and can serve as a guide for future studies in unifying resource theories.

5.2 Open problems

A natural extension of our results in assisted distillation of coherence in the bipartite setting is considering assistance in a multipartite setting. In this case Bob is tasked with distilling coherence in his system while being assisted by multiple correlated parties. The rate of assistance here must be less than the bipartite case since if we consider all the helpers as one system this reduces to the results we have obtained. The exact rates in this case remains an open question.

While we have obtained one-shot distillation bounds in a general resource framework, the next step of showing this reproduces the asymptotic results of Brandão and Gour (2015) is left to future work. The challenge here is to find a way to apply the Quantum Stein's Lemma (Brandao and Plenio 2010) to our map which uses the δ-free robustness. Almost certainly there must be a trade-off between error in the final state of the transformation ϵ and the error in the free operation used δ, an exact expression for this trade-off is of interest to determine. An improvement on the results of chapter 4 would be to relax the constraint of convexity of the free states for the lower bounds as it was done for the upper bounds.

5.3 Future Directions

The general resource theory framework is a powerful tool to understand resource theories. However most of the results in this area are of an information theoretic nature and not readily applicable to real world systems. However the idea of different operations being restricted and classes of states being more costly is a natural assumption for any real world quantum technology platform. A worthwhile project would be to explore how to apply these ideas to optimisation problems in quantum hardware. For example in the circuit model of quantum computation it is possible to decompose a given quantum circuit using different universal gate sets which are in principle equivalent. However depending on the nature of the physical system used for this implementation some gates are naturally going to be more desirable than others. Since what is costly and not costly is highly platform dependent, a framework for abstracting away this detail and performing important optimisation tasks at an abstract level could become a powerful tool for the development of quantum technology.

Part II

Optical Quantum Error Correction

Chapter 6

Introduction

In this part of the thesis, we now turn our attention to the problem of optical quantum error correction in passive linear optical systems. A major difference from part I is that here we are primarily interested in infinite dimensional Hilbert spaces rather than finite dimensional ones. We present an encoding scheme for such systems using W-states which we go on to show is robust against uncorrelated dephasing noise.

6.1 Overview

Before presenting the main results I will introduce common tools for analysing infinite dimensional systems in section 6.2 and provide background and motivation to the research problem in section 6.3. I will then discuss different classes of multipartite entangled states and elaborate on why W-states are a good candidate for encoding and define the W-basis in section 7.1. In section 7.2 I introduce the W-state based encoding using only linear optics and single-photon inputs and describe how to post-select to filter random phase errors. This section will also describe the linear optics error model that we will consider. Then in section 7.3 the success probability of the protocol is computed along with the fidelity of the output logical state with the input logical state. I will then go on to show how the performance improves as the level of encoding increases. Section 7.4 discusses how to implement qubit gates on the logical qubit while in the W-state encoding. Finally I discuss some of the implications of this work and make some concluding remarks in section 7.5.

6.2 Preliminaries

Photons can be understood to be mode excitation in a quantised electromagnetic field. Any optical pure state can be represented as,

$$|\psi\rangle = \sum_{k=0}^{\infty} c_k |k\rangle , \tag{6.1}$$

where c_k are complex amplitudes and the state $|k\rangle$ is a state with exactly k photons. The basis formed by all states $|k\rangle$ for $k \in \{0, 1, \dots\}$ is called the Fock basis or photon number basis. The state $|0\rangle$ is the vacuum state with no photons. This is the state that is responsible for the so called zero point energy of empty space. We denote this state with the special symbol $|\Omega\rangle$. We will use the formalism of creation and annihilation operators which is ubiquitous in the study of infinite dimensional systems. The creation operation $a^\dagger$ is defined through its action on the Fock basis as

$$\hat{a}^\dagger |k\rangle = \sqrt{k+1}\, |k+1\rangle . \tag{6.2}$$

The annihilation operator $\hat{a}$ is similarly defined as

$$\hat{a} |k\rangle = \sqrt{k}\, |k-1\rangle . \tag{6.3}$$

They also obey the canonical commutation relation $\hat{a}\hat{a}^\dagger - a^\dagger \hat{a} = 1$. Using these operators we can express the state in equation (6.1) in the Heisenberg representation as

$$|\psi\rangle = \sum_{k=0}^{\infty} \frac{c_k}{\sqrt{k!}} (\hat{a}^\dagger)^k |\Omega\rangle . \tag{6.4}$$

We will make of use this form and represent the unitary evolution of the state as the evolution of the creation an annihilation operators using the relation,

$$\hat{U} |\psi\rangle = \sum_{k=0}^{\infty} \frac{c_k}{\sqrt{k!}} (\hat{U}\hat{a}^\dagger \hat{U}^\dagger)^k |\Omega\rangle , \tag{6.5}$$

where $\hat{U}$ is any unitary operator.

6.3 Background and Motivation

Within quantum information processing systems, the ability to detect and correct errors is a prerequisite for fault-tolerant quantum computing. In the standard approach to solving this problem, one first constructs error-detection circuits, upon which we build error-correction capabilities. Finally, keeping the physical errors below the so called error threshold the protocol is applied in a recursive fashion to suppress errors completely in the asymptotic limit (Shor 1996; Preskill 1998; Nielsen and Chuang 2000). In the absence of the initial error detection stage, such a construction for mitigating errors cannot function.

The standard framework when considering optical quantum error correction is in the context of universal quantum computation (Knill et al. 2001). Given that this model is universal, multiple layers of error correcting codes can be implemented. In general this requires large, but sub-exponential, resource overheads each with sub-threshold error rates. Although such constructions are essential for realising the full potential of quantum computing, it remains a distant target. Hence there is currently a pursuit to find utility for achievable near-term devices with post-classical capabilities, even if not universal (Harrow and Montanaro 2017; Lund et al. 2017). This has lead to the alternative target where universality is discarded as a requirement and the sole purpose is demonstrating some form of quantum computational advantage with pragmatically reasonable resources.

An important example of such a paradigm whose quantum power has been proven to be more than classical but not fully universal is boson-sampling. Boson-sampling is the set of problems that can be constructed from the preparation and measurement of individual bosons subject to evolution via a network of passive random linear interferometers (Aaronson and Arkhipov 2011, 2014). Some quantum resources come cheaper than others within these models, in particular, additional modes prepared with vacuum states within the boson-sampling paradigm are considered to have much lower cost than additional modes prepared with single photons. However, given that this model is *passive*, one may suspect that it is not possible to perform any kind of error correction without leaving the constraints of the model, and hence dealing with errors defaults back to the requirements associated with universal models.

Marshman *et al.* (Marshman et al. 2018) have shown that, for boson-sampling, it is possible to detect the presence of random phase errors without leaving the paradigm and that the conditional state on detecting the error has a lower error than would otherwise be the case. This was done using a redundant encoding of the passive linear interferometer with a particular network chosen for encoding and decoding of input single photons. The presence of the photon within a particular mode was used as the error detection mechanism. This is distinct from the considerations of (Arkhipov 2015) for errors within unitary networks as there it was assumed that there was no redundancy utilizing additional resources.

I extend this result by considering single photon encoding that involve W-state path entanglement encoding of photonic qubits encoded in dual-rail form. These states can be generated from single photons through passive linear interferometers, and resemble a generalisation of an optical fan-out operation. It has desirable properties for error correction such as the maintenance of path entanglement when single systems are lost. The expansion in mode number can be conceptually related to conventional error-correction schemes based on redundancy, such as Shor's original 3-qubit code (Shor 1995), however the code space that is used in this work is much more robust against mode loss than most current QEC codes which use stabilizer states such as the GHZ state. I will show that this encoding yields an improvement on local dephasing errors (Rohde and Ralph 2006) much like that of the previous work but also show that photon loss is the constraining factor in the heralded fidelity for this localised noise model. This performance is shown to be independent of the type of distribution underlying the random phase errors provided that the errors acting on different modes are independent (i.e., no correlated errors), identical (all modes are treated the same) and the characteristic function for the distribution is well defined. Under these conditions any level of encoding will improve fidelity when conditioned on detecting no error and with a large enough encoding we can fully mitigate the dephasing error.

The idea of error filtration in a passive linear optic network has been explored in (Gisin et al. 2005; Li et al. 2007; Jiang et al. 2017). Broadly these schemes transmit a photon through a linear optical network such that some measurement outcomes will indicate an uncorrupted state in some output. I formalise this intuition by giving an explicit code space and show how it is robust against mode loss and i.i.d. dephasing noise.

Other schemes have explored the use of probabilistic gates to protect against transmission loss such as (Ewert and van Loock 2017) where optical Bell measurements are used along with a parity encoding. However the encoding states used for these schemes are highly entangled GHZ-like states. These states cannot be deterministically prepared using passive linear optics without introducing active feed-forward and additional photons to accommodate the higher level of encoding — making such schemes highly impractical using present-day technology.

Chapter 7

W-state encoding for linear optics

7.1 Introduction

An inherent feature of any kind of multi-qubit entangled state is that, by virtue of its entanglement, loss or decoherence of a single constituent qubit diminishes its degree of entanglement, similarly reducing its purity (or conversely, increasing its collective entropy). Some entangled states are more robust than others in this respect and, as discussed below, the W-states are a quintessential example of entangled states with this robustness property. Note that the resultant state following a partial trace operation upon a qubit (equivalent to loss when using single photon encoding) is independent of anything done only to traced out qubits prior to the partial tracing operation. Therefore considering loss via partial trace is completely sufficient to understand the worst-case degradation of an entangled state under any kind of local noise process.

7.1.1 GHZ states

The worst-case scenario is the GHZ state, a maximally-entangled n-qubit state of the form,

$$|\mathrm{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}), \tag{7.1}$$

whereby all qubits are collectively perfectly correlated. That is, measurement of any one (in the computational Z-basis), reveals the equivalent measurement outcome of all others. However, this directly implies that losing access to this information similarly implies loss of knowledge of the others. Loss or dephasing directly correspond to such loss of information. For this reason, dephasing a single qubit, or losing it outright, implies complete decoherence of the entire n-qubit state. Specifically, partial tracing out a single qubit from a GHZ state leaves behind the hopelessly mixed stated,

$$\begin{aligned}\mathrm{Tr}_i(|\mathrm{GHZ}_n\rangle\langle\mathrm{GHZ}_n|) = &\frac{1}{2}|0\rangle^{\otimes n-1}\langle 0|^{\otimes n-1}\\ &+\frac{1}{2}|1\rangle^{\otimes n-1}\langle 1|^{\otimes n-1},\end{aligned} \tag{7.2}$$

where the partial trace is performed upon any qubit i.

7.1.2 Cluster states

Cluster (or graph) states (Raussendorf and Briegel 2001; Raussendorf et al. 2003) are a highly useful class of states, enabling universal quantum computation using the measurement based model for quantum computing (MBQC). Despite being more computationally useful than GHZ states, they are far less entangled, and hence far more robust against localised noise processes. For example, by measuring out the immediate neighbours of a lost qubit from within a graph state, a reduced, yet perfect graph state is recovered, given by the sub-graph of the original graph, with the neighbourhood of the lost qubit removed.

7.1.3 W-states

An especially robust (and so far not particularly useful) class of entangled states are the W-states (Zeilinger et al. 1997; Dür et al. 2000), given by the equal superposition of a single excitation across n-sites. In qubit form this can be expressed,

$$\begin{aligned}|W_n\rangle = &\frac{1}{\sqrt{n}}(|1,0,0,\ldots\rangle + |0,1,0,0,\ldots\rangle\\ &+|0,0,1,0,\ldots\rangle + |0,0,0,1,\ldots\rangle + \ldots).\end{aligned} \tag{7.3}$$

Alternately, this can be expressed in terms of creation or excitation operators, $\hat{a}_i^\dagger$ for the ith site,

$$|W_n\rangle = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\hat{a}_i^\dagger|\Omega\rangle, \tag{7.4}$$

where $|\Omega\rangle$ is the collective ground or optical vacuum state. The latter representation is the one we will focus on here, given its direct applicability to photonic encoding.

These states exhibit complete permutational symmetry under qubit interchange. That is, the state is invariant under any permutation $\hat{\pi} \in S_n$ in the symmetric group,

$$\hat{\pi} |W_n\rangle = |W_n\rangle . \tag{7.5}$$

Tracing out a single qubit from a W-state yields,

$$\begin{aligned} \mathrm{Tr}_i(|W_n\rangle \langle W_n|) = &\frac{n-1}{n} |W_{n-1}\rangle \langle W_{n-1}| \\ &+ \frac{1}{n} |0\rangle^{\otimes n-1} \langle 0|^{\otimes n-1} . \end{aligned} \tag{7.6}$$

That is, upon loss of a single qubit, with probability $p = (n-1)/n$ it simply undergoes a reduction in its level of encoding to a $|W_{n-1}\rangle$ state, preserving its W-type structure entirely, otherwise collapsing to the $|0\rangle^{\otimes n-1}$ state. This implies that for large n, W-states are extremely robust (indeed almost invariant) against single-qubit loss. As discussed earlier, this directly implies similar single-qubit robustness against other noise channels.

Note that atomic ensemble qubits (Duan et al. 2001) are a direct alternate physical manifestation of W-type encoding, whereby an ensemble (or cloud) of collectively-addressed atomic qubits undergo *collective excitation*, mathematically of the form given in Eq. (7.4). This approach to realising physical qubits has attracted much attention, especially as good candidates for quantum memories, given their notably high coherence lifetimes, often at room temperature, which can be intuitively associated with the described robustness of their underlying W-type entanglement structure – if a few atoms go missing from the cloud, little is lost.

The n-qubit W-state can be easily generalised to an entire orthonormal W-basis, by appropriately manipulating the phase relationships within the n terms in the superposition. One way in which to choose these phases is by taking the elements from the Quantum Fourier Transform (QFT) matrix, or generalised Hadamard matrices, both of which have equal $1/\sqrt{n}$ amplitudes across all matrix entries, with phase relationships ensuring orthonormality.

These different phase relationships do not change the earlier observations about the states' robustness against local noise. This immediately leads to the intuition, that by choosing such a W-basis for encoding logical qubits, the encoded logical qubit must inherit via linearity these same robustness characteristics. This makes them a direct candidate for optical encoding, given that photonic implementation of QFT mappings may be implemented via passive linear optics, in the absence of any active control, and is realisable with today's technology integrated wave-guide architectures across a large number of modes.

7.2 Protocol

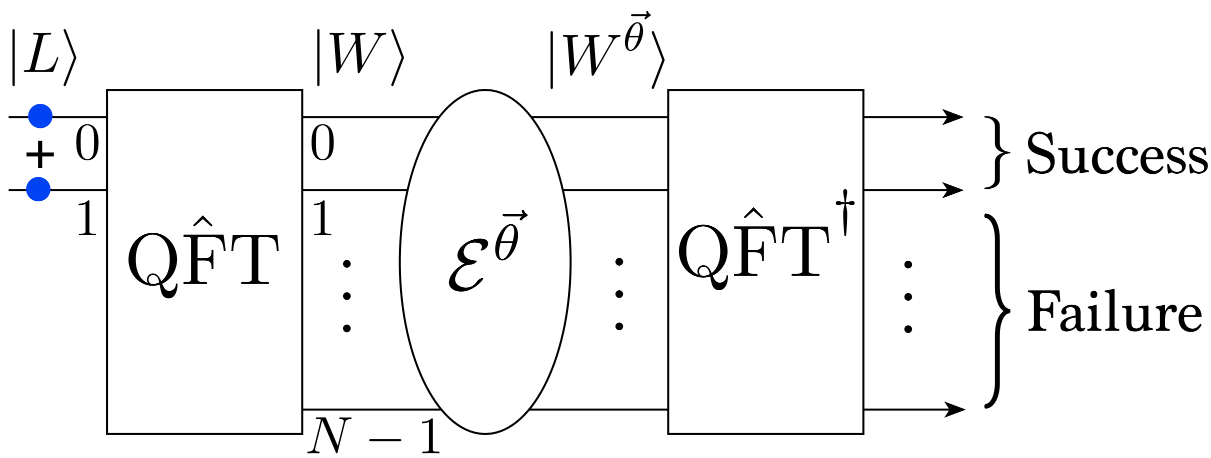

Figure 7.1 : Photonic W-state error-correction and -detection protocol. Encoding of a single dual-rail photonic qubit proceeds via a Quantum Fourier Transform ($Q\hat{F}T$), which maps the 2-mode encoding across a larger number of redundant modes. The independent dephasing noise channel is denoted by $\mathcal{E}$. Decoding proceeds via the inverse Quantum Fourier Transform ($Q\hat{F}T^{\dagger}$). Post-selection upon detecting the single photon within the desired 2 output modes defining the single qubit, projects the logical state into one with reduced noise action.

The error-detection and correction protocol is shown in Fig. 7.1. Consider N optical modes, the first two of which contain a single photon state, defining a dual-rail-encoded photonic qubit. This qubit can defines a *logical* qubit,

$$\begin{aligned} |L\rangle &= \alpha\,|0\rangle_L + \beta\,|1\rangle_L \\ &= (\alpha\hat{a}_0^\dagger + \beta\hat{a}_1^\dagger)\,|\Omega\rangle\,, \end{aligned} \tag{7.7}$$

where $|\Omega\rangle$ is the N mode vacuum state. To W-encode the logical qubit we pass the N modes through a linear optical network implementing the N-mode quantum Fourier transform,

$$\hat{a}_i^\dagger \to \hat{W}_i^\dagger = \sum_{j=0}^{N-1} \hat{\mathrm{Q}}_{ij}\hat{a}_j^\dagger, \tag{7.8}$$

where,

$$\hat{\mathrm{Q}}_{jk} = \frac{\omega_N^{jk}}{\sqrt{N}}; \qquad j,k \in \{0,1,\ldots,N-1\}, \tag{7.9}$$

are matrix elements of the N-dimensional quantum Fourier transformation operator $\hat{\mathrm{Q}}$ with $\omega_N = e^{\frac{2\pi i}{N}}$. This transforms the *logical* qubit to the *encoded* qubit,

$$|W\rangle = (\alpha\hat{W}_0^\dagger + \beta\hat{W}_1^\dagger)\,|\Omega\rangle\,, \tag{7.10}$$

which represents the same state of quantum information, but in expanded form. Next the W-encoded state passes through a noisy channel that independently adds random phases to each optical mode,

$$\hat{a}_j^\dagger \to e^{i\theta_j}\hat{a}_j^\dagger, \tag{7.11}$$

where the θ_j represent random variables, whose distribution is considered arbitrary at this point, that form a vector $\vec{\theta}$ describing the phases applied to each mode. The state $|W^{\vec{\theta}}\rangle$ denotes the W-encoded state following application of the phase noise channel. We now apply the decoding operation (the inverse QFT operation), and the first two output modes represent the decoded logical state, $\hat{\rho}_L$. Because of the noise in the channel we are not guaranteed to observe the photon strictly within the first two modes. Thus we post-select and treat cases where photons are found in the other modes as heralding a failure. The intuition is then that for the heralded success cases the phase noise errors would have been filtered out.

The fidelity of the decoded state compared to the input logical qubit $|L\rangle$ is

$$F_N = \langle L| \hat{\rho}_L |L\rangle \,, \tag{7.12}$$

where $|L\rangle$ is implicitly a function of the superposition parameters α and β. Note that the overlap between two states is invariant under common unitary operations. As the encoding and decoding operations are unitary it suffices to consider the fidelity of the W-encoded state,

$$F_N = \langle W|W^{\vec{\theta}}\rangle \langle W^{\vec{\theta}}|W\rangle \,, \tag{7.13}$$

where F_N is used here to show that the fidelity will depend on the number of modes used for the encoding N. Eq. (7.13) assumes knowledge of the phase errors in each mode as represented by $\vec{\theta}$ but these are of course unknown. However we can model them as independent random variables acting on each mode separately according to some arbitrary distribution p,

$$p(\vec{\theta}) = \prod_{j=0}^{N-1} p_i(\theta_j), \tag{7.14}$$

where p_j is the distribution for mode j. The encoded state after application of the error channel on average is given by,

$$\hat{\rho}_W = \int p(\vec{\theta})\, |W^{\vec{\theta}}\rangle \langle W^{\vec{\theta}}| \; d\vec{\theta}. \tag{7.15}$$

The fidelity between the output and input of the error channel is given by,

$$F_N = \langle W| \left(\int p(\vec{\theta}) |W^{\vec{\theta}}\rangle\langle W^{\vec{\theta}}| d\vec{\theta} \right) |W\rangle \,. \tag{7.16}$$

As with all quantum operations, the noise channel is a linear map on the state space. Let the channel map be denoted by $\mathcal{L}_{\vec{\theta}}$, then we have for the encoded qubit state,

$$\begin{aligned} |W^{\vec{\theta}}\rangle &= \mathcal{L}_{\vec{\theta}}(|W\rangle) = \alpha\mathcal{L}_{\vec{\theta}}(|W_0\rangle) + \beta\mathcal{L}_{\vec{\theta}}(|W_1\rangle) \\ &= \alpha\, |W_0^{\vec{\theta}}\rangle + \beta\, |W_1^{\vec{\theta}}\rangle , \end{aligned} \tag{7.17}$$

where $|W_k\rangle = \hat{W}_k^\dagger\,|\Omega\rangle$, following the definition in Eq. (7.8),

$$|W_k^{\vec{\theta}}\rangle = \sum_j^{N-1} \hat{\mathrm{Q}}_{kj} e^{i\theta_j} a_j^\dagger\,|\Omega\rangle\,. \tag{7.18}$$

These equations can now be used to compute $\hat{\rho}_W$,

$$\begin{aligned} \hat{\rho}_W &= \int p(\vec{\theta}) \left(|W^{\vec{\theta}}\rangle\langle W^{\vec{\theta}}| \right) d\vec{\theta} \\ &= \int p(\vec{\theta}) \left(\sum_{i,j=0}^{1} c_i c_j^* |W_i^{\vec{\theta}}\rangle\langle W_j^{\vec{\theta}}| \right) d\vec{\theta}, \end{aligned} \tag{7.19}$$

where $c_0 = \alpha$ and $c_1 = \beta$. Substituting the definition of $|W_i^{\vec{\theta}}\rangle$ from Eq. (7.18) and defining,

$$k_{m,n} = (\alpha + \beta\omega_N^m)(\alpha + \beta\omega_N^n)^*, \tag{7.20}$$

we obtain,

$$\hat{\rho}_W = \frac{1}{N} \int p(\vec{\theta}) \begin{pmatrix} k_{0,0} & k_{01}e^{i(\theta_0-\theta_1)} & k_{0,2}e^{i(\theta_0-\theta_2)} & \dots & k_{0,(N-1)}e^{i(\theta_0-\theta_{N-1})} \\ k_{1,0}e^{i(\theta_1-\theta_0)} & k_{1,1} & k_{12}e^{i(\theta_1-\theta_2)} & \dots & \\ k_{2,0}e^{i(\theta_2-\theta_0)} & k_{2,1}e^{i(\theta_2-\theta_1)} & k_{2,2} & \dots & \\ \vdots & & & \ddots & \\ k_{(N-1),0}e^{i(\theta_{N-1}-\theta_0)} & & & & k_{(N-1),(N-1)} \end{pmatrix} d\vec{\theta}, \tag{7.21}$$

where the matrix within the integral represents the state after the noise channel in the photon number basis. The characteristic function of a probability distribution $p(x)$ is defined as,

$$\phi_{p(x)}(z) = \int_{-\infty}^{\infty} p(x) e^{ixz} dx. \tag{7.22}$$

If we assume all θ_j are identically and independently distributed as $p(\theta)$ then we have,

$$\lambda = \int p(\vec{\theta}) e^{i(\theta_j - \theta_k)} d\vec{\theta} = |\phi_{p(\theta)}(1)|^2, \tag{7.23}$$

whenever the indices j and k are different. Thus,

$$\begin{aligned} \hat{\rho}_W &= \frac{\lambda}{N} \sum_{i,j=0}^{N-1} (1-\delta_{ij}) k_{i,j} |i\rangle\langle j| + \frac{1}{N} \sum_{i}^{N-1} k_{i,i} |i\rangle\langle i| \\ &= \lambda \left(|W\rangle\langle W| - \Delta(|W\rangle\langle W|) \right) + \Delta(|W\rangle\langle W|) \\ &= \lambda |W\rangle\langle W| + (1-\lambda) \Delta(|W\rangle\langle W|), \end{aligned} \tag{7.24}$$

where we have used the fact that,

$$\sum_{i,j=0}^{N-1} \frac{k_{i,j}}{N} |i\rangle\langle j| = |W\rangle\langle W|, \tag{7.25}$$

and Δ is the completely dephasing map in the photon number basis defined as,

$$\Delta(\hat{\rho}) = \sum_i \langle i|\hat{\rho}|i\rangle \, |i\rangle\langle i|. \tag{7.26}$$

We can see from Eq. (7.24) that the our error channel is essentially a dephasing channel with dephasing parameter λ. To analyze our protocol further we will choose a particular error model by assuming that the phase error in each mode is distributed as a Gaussian with mean μ and variance δ^{2*},

$$p(\theta) = \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{(\theta-\mu)^2}{2\delta^2}}. \tag{7.27}$$

This is a natural choice when we do not have any knowledge about the nature of the processes that generates the errors beyond that many underlying random distributions

*This assumption allows for values of θ that are larger than single multiples of 2π, but the theory used here does not need to be changed to incorporate this. The operations used in defining the phase shift channel are periodic and hence having larger value of phase does not invalidate this description. However, it does mean that there is no unique probability density function for any given distribution on the range $[0, 2\pi)$.

average to give a final contribution to the error from the central limit theorem. The characteristic function of a normal distribution is given by,

$$\phi_{p(\theta)}(z) = e^{-\frac{\delta^2 z^2}{2} + i\mu z}. \tag{7.28}$$

This leads to $\lambda = e^{-\delta^2}$ and,

$$\hat{\rho}_W = e^{-\delta^2}|W\rangle\langle W| + (1 - e^{-\delta^2})\Delta\left(|W\rangle\langle W|\right). \tag{7.29}$$

We can interpret the error channel as performing the identity operation with probability $e^{-\delta^2}$ and the Fock basis dephasing operation with probability $(1 - e^{-\delta^2})$. This channel is thus a dephasing channel with probability of no error occurring $p = e^{-\delta^2}$. In practical terms, the variance δ^2 of the phase error will depend on the physical implementation of the quantum channel. For fibre-optic cables we would generally expect the variance to increase with the length of the cable L or equivalently the propagation time of the photon in cable $t_p = L/v$, where v is the propagation velocity of the photon in the fibre. If we model the variance as increasing linearly with propagation time, i.e,

$$\delta^2 = \frac{t_p}{T_2}, \tag{7.30}$$

where T_2 is a constant defining a characteristic time for the dephasing channel, we can write down our error channel in the standard dephasing channel notation as,

$$\begin{aligned}\hat{\rho}_W &= \mathcal{E}_{t_p}^{\text{dephasing}}(|W\rangle\langle W|) \\ &= e^{-t_p/T_2}|W\rangle\langle W| + (1 - e^{-t_p/T_2})\Delta(|W\rangle\langle W|).\end{aligned} \tag{7.31}$$

7.3 Error heralding

In implementing the protocol as described in Fig. 7.1, we can perform the post-selection in two different ways:

- **Presence heralding**: Success is assumed based upon the detection of exactly one photon between the output modes 0 and 1, which define the logical qubit space.

Note that in the absence of quantum non-demolition measurements, this is necessarily destructive, limiting its applicability. The heralding operator is effectively the projector

$$\hat{\Pi}_{\text{presence}} = \hat{a}_0^\dagger |\Omega\rangle \langle\Omega| \hat{a}_0 + \hat{a}_1^\dagger |\Omega\rangle \langle\Omega| \hat{a}_1. \tag{7.32}$$

- **Absence heralding**: Success is inferred via the detection of no photons in any of the remaining modes outside the logical qubit space. This is non-destructive on the logical qubit, broadening its utility. However, photon loss contributes to the occurrence of this signature, implying higher error rates on the remaining logical qubit. The heralding operator is equivalently a projection given by

$$\hat{\Pi}_{\text{absence}} = \hat{\mathbb{I}} - \sum_{i=2}^{N-1} \hat{a}_i^\dagger |\Omega\rangle \langle\Omega| \hat{a}_i. \tag{7.33}$$

7.3.1 Heralding probability

Absence heralding

We define the absence heralded probability P_{H_a} as the probability that no photons are detected in modes 2 to $(N-1)$ and $\hat{\rho}_{\text{out}}$ to be the N-mode state of the system at the end of the protocol before the final measurement. Assuming a uniform loss model parameterized by η, for our choice of input states the probability of detecting the photon in mode m is

$$(1-\eta) \cdot \langle\Omega|\hat{a}_m \hat{\rho}_{\text{out}} \hat{a}_m^\dagger|\Omega\rangle \,, \tag{7.34}$$

where $|\Omega\rangle$ is the global vacuum state. Using this expression for the probability of detection under loss we can see that,

$$\begin{aligned}
P_{H_a} &= \text{Pr(Photon is in modes 0 or 1)} \\
&+ \text{Pr(Photon is in modes 2-}(N-1)) \\
&\times \text{Pr(Loss in modes 2-}(N-1)), \\
&= \sum_{i=0}^{1} \langle\Omega|\hat{a}_i\hat{\rho}_{\text{out}}\hat{a}_i^\dagger|\Omega\rangle + \eta\left(1-\sum_{i=0}^{1}\langle\Omega|\hat{a}_i\hat{\rho}_{\text{out}}\hat{a}_i^\dagger|\Omega\rangle\right), \\
&= \eta + (1-\eta)\left(\sum_{i=0}^{1}\langle\Omega|\hat{a}_i\hat{\rho}_{\text{out}}\hat{a}_i^\dagger|\Omega\rangle\right), \\
&= \eta + (1-\eta)\left(\sum_{i=0}^{1}\langle\Omega|\hat{a}_i\hat{\mathrm{Q}}^\dagger\hat{\rho}_W\hat{\mathrm{Q}}\hat{a}_i^\dagger|\Omega\rangle\right), \\
&= \eta + (1-\eta)\left(\sum_{i=0}^{1}\langle W_i|\hat{\rho}_W|W_i\rangle\right). \qquad (7.35)
\end{aligned}$$

Using Eq. (7.24) we can write,

$$\begin{aligned}
\langle W_k|\hat{\rho}_W|W_k\rangle &= \lambda\,|\langle W_k|W\rangle|^2 \\
&+ (1-\lambda)\,\langle W_k|\Delta(|W\rangle\langle W|)|W_k\rangle\,. \qquad (7.36)
\end{aligned}$$

The first term on the R.H.S. of Eq. (7.36) is simply,

$$\begin{aligned}
|\langle W_0|W\rangle|^2 &= |\alpha|^2, \\
|\langle W_1|W\rangle|^2 &= |\beta|^2, \qquad (7.37)
\end{aligned}$$

which are the values of k that preserve the encoded qubit. The second term can be calculated as,

$$\begin{aligned}
&\langle W_k|\Delta(|W\rangle\langle W|)|W_k\rangle = \\
&\frac{1}{N^2}\sum_{m,n=0}^{1}\sum_{j,q,p=0}^{N-1}\omega_N^{(m-n)j+(q-p)k}c_m c_n^*\delta_{pj}\delta_{jq}, \\
&= \frac{1}{N^2}\sum_{m,n=0}^{1}\sum_{j=0}^{N-1}\omega_N^{(m-n)j}c_m c_n^* \\
&= \frac{1}{N}\sum_{m,n=0}^{1}\delta_{mn}c_m c_n^* = \frac{1}{N}, \qquad (7.38)
\end{aligned}$$

where in the first equality we have used the fact that,

$$|W_k\rangle = \sum_{q=0}^{N-1} \hat{\mathrm{Q}}_{kq} \hat{a}_q^\dagger |\Omega\rangle \,, \tag{7.39}$$

where $\hat{\mathrm{Q}}_{kp}$ are the coefficients of the quantum Fourier transform operator defined in equation (7.9) and

$$\Delta(|W\rangle\langle W|) = \sum_{m,n=0}^{1} c_m c_n^* \sum_{j=0}^{N-1} \omega_N^{(m-n)j} \hat{a}_j^\dagger |\Omega\rangle\langle\Omega| \hat{a}_j. \tag{7.40}$$

This implies that the photon in the error state is equally spread over all the modes after decoding. If the state contains an error, the heralding will detect it with a probability of $\frac{N-2}{N}$ and miss it with probability $\frac{2}{N}$. So there will be a linear advantage in error detection with the number of modes. Substituting these results in Eq. (7.35) we get,

$$\begin{aligned} P_{H_a} &= \eta + (1-\eta)\left[\lambda|\alpha|^2 + \frac{1}{N}(1-\lambda)\right] \\ &\quad + (1-\eta)\left[\lambda|\beta|^2 + \frac{1}{N}(1-\lambda)\right] \\ &= \eta + (1-\eta)\left[\lambda + \frac{2}{N}(1-\lambda)\right]. \end{aligned} \tag{7.41}$$

If we assume the Gaussian error model in Eq. (7.27) this will reduce to,

$$P_{H_a} = \eta + (1-\eta)\left[e^{-\delta^2} + \frac{2}{N}(1-e^{-\delta^2})\right]. \tag{7.42}$$

As the number of modes N increases the heralded probability will decrease, this is because the probability of the error state being in the modes 1 and 2 is inversely proportional to N. As we connected the phase error variance to a T_2 time via Eq. (7.30), we can also reparameterize the loss probability as $\eta = 1 - e^{-tp/T_1}$. In terms of the T_1 and T_2 parameters and propagation time t_p, the absenence heralded probability can be written as,

$$\begin{aligned} P_{H_a} &= (1 - e^{-t_p/T_1}) \\ &\quad + e^{-t_p/T_1}\left[e^{-t_p/T_2} + \frac{2}{N}(1-e^{-t_p/T_2})\right]. \end{aligned} \tag{7.43}$$

Presence heralding

The presence heralded case is the case where we post-select on there being no photon loss. The presence heralded fidelity is the probability of getting a photon in modes 1 and 2 and this can be easily seen to be,

$$P_{H_p} = P_{H_a} - \eta. \tag{7.44}$$

7.3.2 Heralded fidelity

Absence heralding

The absence heralded state is the state in the output modes 0 and 1 when no photons are detected in the modes modes 2 – $(N-1)$. This can happen in two mutually exclusive ways; either the photon is lost and there is no photon in any mode or there is no loss and our negative measurement of modes 2 – $(N-1)$ projects the quantum state into the subspace spanned by $a_0^\dagger\,|\Omega\rangle$ and $a_1^\dagger\,|\Omega\rangle$. So, the absence heralded state is given by,

$$\hat{\rho}_{H_a} = (1-\eta)\frac{\hat{\Pi}_{0,1}\hat{\rho}_{\text{out}}\hat{\Pi}_{0,1}}{\text{Tr}(\hat{\Pi}_{0,1}\hat{\rho}_{\text{out}})} + \eta|\Omega\rangle\langle\Omega|, \tag{7.45}$$

where,

$$\hat{\Pi}_{0,1} = a_0^\dagger|\Omega\rangle\langle\Omega|a_0 + a_1^\dagger|\Omega\rangle\langle\Omega|a_1, \tag{7.46}$$

is the projector on to the subspace of modes 1 and 2. But we observe that,

$$\begin{aligned}\text{Tr}(\hat{\Pi}_{0,1}\hat{\rho}_{\text{out}}) &= \langle\Omega|\hat{a}_0\hat{\rho}_{\text{out}}\hat{a}_0^\dagger|\Omega\rangle + \langle\Omega|\hat{a}_1\hat{\rho}_{\text{out}}\hat{a}_1^\dagger|\Omega\rangle \\ &= \langle\Omega|\hat{a}_0\hat{\text{Q}}^\dagger\hat{\text{Q}}\hat{\rho}_{\text{out}}\hat{\text{Q}}^\dagger\hat{\text{Q}}\hat{a}_0^\dagger|\Omega\rangle \\ &+ \langle\Omega|\hat{a}_1\hat{\text{Q}}^\dagger\hat{\text{Q}}\hat{\rho}_{\text{out}}\hat{\text{Q}}^\dagger\hat{\text{Q}}\hat{a}_1^\dagger|\Omega\rangle \\ &= \langle W_0|\hat{\rho}_W|W_0\rangle + \langle W_1|\hat{\rho}_W|W_1\rangle \\ &= \lambda + \frac{2}{N}(1-\lambda). \end{aligned} \tag{7.47}$$

The fidelity of the heralded state with our logical input state is given by,

$$\begin{aligned} F_{H_a} &= \langle L|\hat{\rho}_{H_a}|L\rangle \\ &= \langle L|\hat{\mathrm{Q}}^\dagger\hat{\mathrm{Q}}\hat{\rho}_{H_a}\hat{\mathrm{Q}}^\dagger\hat{\mathrm{Q}}|L\rangle \\ &= (1-\eta)\frac{\langle W|\hat{\rho}_W|W\rangle}{\mathrm{Tr}(\hat{\Pi}_{W_0,W_1}\rho_W)} \\ &= \frac{1-\eta}{\lambda+\frac{2}{N}(1-\lambda)}\cdot F(|W\rangle\,,\hat{\rho}_W), \end{aligned} \tag{7.48}$$

where,

$$\hat{\Pi}_{W_0,W_1} = |W_0\rangle\langle W_0| + |W_1\rangle\langle W_1|, \tag{7.49}$$

and in the third equality we have used the fact that $\hat{\Pi}_{0,1}|\Omega\rangle\langle\Omega|\hat{\Pi}_{0,1} = 0$. We can interpret Eq. (7.48) as saying that heralding improves the output fidelity of our protocol by a factor of,

$$\frac{1-\eta}{\lambda+\frac{2}{N}(1-\lambda)}. \tag{7.50}$$

We have,

$$F(|W\rangle\,,\hat{\rho}_W) = \lambda + (1-\lambda)\,\langle W|\Delta(|W\rangle\langle W|)|W\rangle\,. \tag{7.51}$$

Notice that,

$$\begin{aligned} \langle W|\Delta(|W\rangle\langle W|)|W\rangle &= \mathrm{Tr}(|W\rangle\langle W|\Delta(|W\rangle\langle W|)) \\ &= \sum_{i=0}^{N-1}([|W\rangle\langle W|]_{ii})^2, \end{aligned} \tag{7.52}$$

where, $[|W\rangle\langle W|]_{ii}$ are diagonal elements of the the state $|W\rangle\langle W|$ in the computational basis. We know that,

$$
\begin{aligned}
&[|W\rangle\langle W|]_{ii} = \\
&\langle\Omega|\hat{a}_i\left(\sum_{m,n=0}^{1} c_m c_n^* \sum_{k,j=0}^{N-1} \hat{\mathrm{Q}}_{mk}\hat{a}_k^\dagger|\Omega\rangle\langle\Omega|\hat{a}_j\hat{\mathrm{Q}}_{nj}^*\right)\hat{a}_i^\dagger|\Omega\rangle \\
&= \frac{1}{N}\sum_{m,n=0}^{1}\sum_{k,j=0}^{N-1} c_m c_n^* \omega_N^{mk-nj}\delta_{ik}\delta_{ji} \\
&= \frac{1}{N}\sum_{m,n=0}^{1} c_m c_n^* \omega_N^{(m-n)i}.
\end{aligned}
\tag{7.53}
$$

Using the above expression we obtain,

$$
\begin{aligned}
&\langle W|\Delta(|W\rangle\langle W|)|W\rangle \\
&= \sum_{i=0}^{N-1}\left(\frac{1}{N}\sum_{m,n=0}^{1} c_m c_n^* \omega_N^{(m-n)i}\right)\left(\frac{1}{N}\sum_{p,q=0}^{1} c_p c_q^* \omega_N^{(p-q)i}\right) \\
&= \frac{1}{N^2}\sum_{i=0}^{1}\sum_{\substack{m,n,\\p,q}=0}^{1} c_m c_p c_n^* c_q^* \omega_N^{(m-n+p-q)i} \\
&= \frac{1}{N}\sum_{\substack{m,n,\\p,q}=0}^{1} c_m c_p c_n^* c_q^* \delta_{m-n+p,q} \\
&= \frac{1}{N}\sum_{\substack{m,n,p=0\\m-n+p\geq 0}}^{1} c_m c_p c_n^* c_{m-n+p}^* \\
&= \frac{|\alpha|^4+|\beta|^4+4|\alpha|^2|\beta|^2}{N} \\
&= \frac{1+2|\alpha|^2|\beta|^2}{N}.
\end{aligned}
\tag{7.54}
$$

Therefore,

$$
F_{H_a} = (1-\eta)\cdot\frac{\lambda+(1-\lambda)\frac{1+2|\alpha|^2|\beta|^2}{N}}{\lambda+\frac{2}{N}(1-\lambda)}.
\tag{7.55}
$$

In terms of Bloch variables θ and ϕ where $\alpha=\cos\frac{\theta}{2}$, $\beta=e^{i\phi}\sin\frac{\theta}{2}$ as, it can be seen that,

$$
F_{H_a} = (1-\eta)\cdot\frac{e^{-\delta^2}+(1-e^{-\delta^2})(\frac{2+\sin^2\theta}{2N})}{e^{-\delta^2}+\frac{2}{N}(1-e^{-\delta^2})}.
\tag{7.56}
$$

In the limit of large N the heralded fidelity will approach $(1-\eta)$. This implies that the only error will be from photon loss. In terms of the dephasing and amplitude damping channel parameters T_2 and T_1 and a propagation time t_p this can be written as,

$$F_{H_a} = e^{-t_p/T_1} \cdot \frac{e^{-t_p/T_2} + (1 - e^{-t_p/T_2})(\frac{2+\sin^2\theta}{2N})}{e^{-t_p/T_2} + \frac{2}{N}(1 - e^{-t_p/T_2})}. \tag{7.57}$$

Presence heralding

In the presence heralded case, we are post selecting the case where there is no photon loss in the system, so the post measurement state in this scenario will be,

$$\hat{\rho}_{H_p} = \frac{\hat{\Pi}_{0,1}\hat{\rho}_{\text{out}}\hat{\Pi}_{0,1}}{\text{Tr}(\hat{\Pi}_{0,1}\hat{\rho}_{\text{out}})} \tag{7.58}$$

this just improves the fidelity by a factor of $(1-\eta)$ giving, the fidelity as,

$$F_{H_p} = \frac{F_{H_a}}{1-\eta}. \tag{7.59}$$

The heralded probability is plotted in Fig. 7.2 as a function of δ and T_2 with fixed values of η and T_1 respectively. From these plots it is evident that the heralded fidelity improves with the number of modes N. The choice of the parameters values T_1 and η do not influence the ordering of these plots.

Heralding type	**Probability**	**Fidelity**
Presence	$P_{H_p} = (1-\eta)[\lambda + \frac{2}{N}(1-\lambda)]$	$F_{H_p} = \frac{N\lambda + (1-\lambda)(1+2\lvert\alpha\beta\rvert^2)}{(N-2)\lambda+2}$
Presence $(N \to \infty)$	$P_{H_p} = (1-\eta)\lambda$	$F_{H_p} = 1$
Absence	$\eta + P_{H_p}$	$(1-\eta)F_{H_p}$

Table 7.1 : Heralding probabilities and post-selected logical qubit fidelities of a single photon qubit under the W-state encoding protocol, according to the two different modes of post-selection operation. Note that here $\lambda = |\phi_{p(\theta)}(1)|^2$ as defined in equation (7.23).

7.3.3 Probability and fidelity plots

The heralding probability and associated post-selected state fidelities are shown as a function of the channel parameters in Fig. 7.2, and the respective analytic expressions in Tab. 7.1. Note that while we are specifically plotting for a Gaussian noise model, the qualitative features can be expected to be the same for any i.i.d. error model. This is because depolarizing parameter λ is related to the characteristic function $\phi_{p(\theta)}(z)$ through equation (7.23). A function and it's Fourier transform will have their variances inversely related like quadrature variances so even if the exact expressions for the fidelity and heralding probabilities might vary we can expect the qualitative behaviour to remain the same and the average map to be a depolarizing channel.

7.4 Single-qubit unitary operations

Once in the encoded basis we can directly perform single-qubit unitary operations, without decoding and encoding again. To see this, consider the single qubit operation,

$$|\psi_{\text{out}}\rangle_L = \hat{U}\,|\psi_{\text{in}}\rangle_L\,, \tag{7.60}$$

in the logical qubit basis. In the encoded photonic basis, this can be expressed,

$$|\psi_{\text{out}}\rangle_L = \hat{Q}^\dagger\hat{Q}[\hat{U}\oplus\hat{\mathbb{I}}_{N-2}]\hat{Q}^\dagger\hat{Q}\,|\psi_{\text{in}}\rangle_L\,, \tag{7.61}$$

where we have inserted the identity operation $\hat{\mathbb{I}}_{N-2}$ on the ancillary input photonic modes, and $\hat{\mathbb{I}} = \hat{Q}^\dagger\hat{Q}$. This yields the equivalent redundantly-encoded photonic unitary operation (i.e between encoding and decoding),

$$|\psi_{\text{out}}\rangle_{\text{enc}} = \tilde{U}\,|\psi_{\text{in}}\rangle_{\text{enc}}\,, \tag{7.62}$$

where,

$$\tilde{U} = \hat{Q}[\hat{U}\oplus\hat{\mathbb{I}}_{N-2}]\hat{Q}^\dagger, \tag{7.63}$$

is the redundantly-encoded equivalent of the logical 2-qubit operation, obtained by conjugating with $\hat{Q}$.

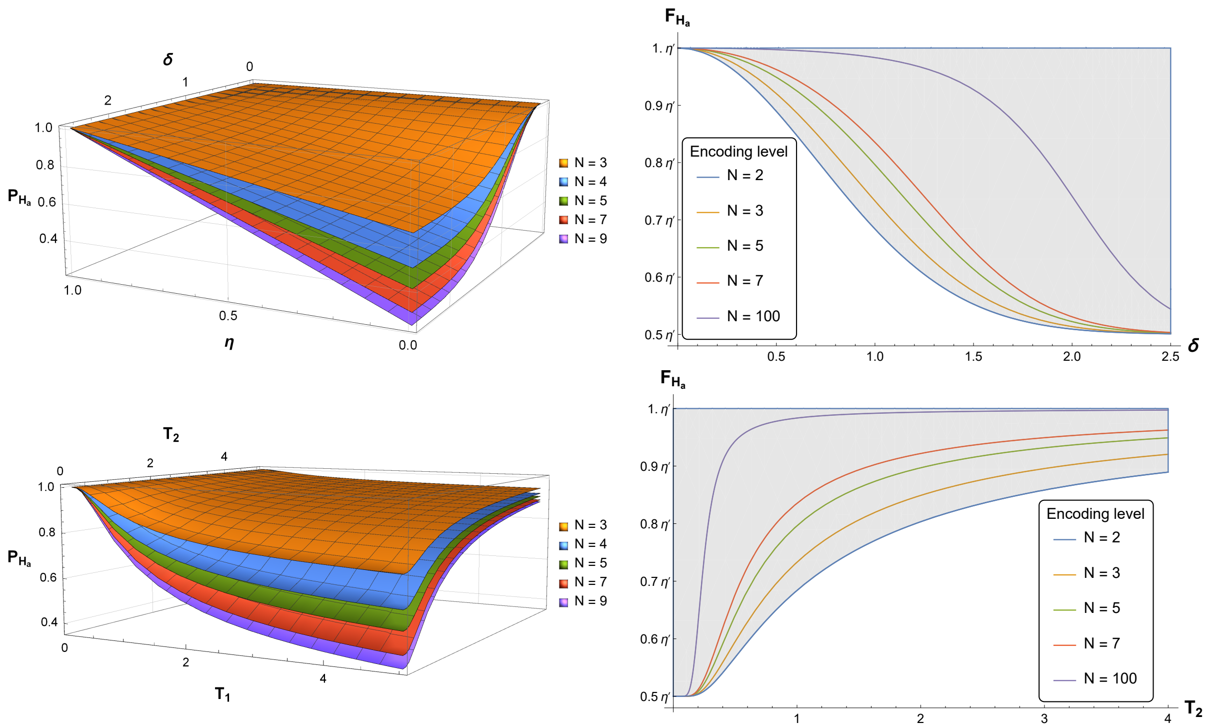

Figure 7.2 : Analytic heralded error-correction results for the *absence heralding* technique. Results for *presence heralding* are given by simple transformations of these results (shift by η for P_H, and scale by $(1-\eta)$ for F_H). (left) Heralding probability and (right) post-selected fidelity, parameterized in terms of loss-rate, and dephasing in terms of (top) phase-variance δ, (bottom) T_2-time for $t_p = 1$.

7.5 Discussion

7.5.1 Pros and cons

The W-state encoding scheme has the following advantages:

1. It can be implemented in a number of quantum memory architectures such as atomic ensembles, optical cavities and delay lines.

2. Any independent uncorrelated dephasing noise can be corrected for with sufficient levels of encoding. The physical source of these phase errors will depend on the particular architecture. For example, in a delay line, temporal mismatch between photon arrival times will manifest as a phase error. If this is influenced by thermal fluctuations, it will be manifested as a dephasing error reflecting our theoretical calculations of the post error state. In an optical cavity array, the source of the phase error could be decay rate mismatch between cavities. All these cases are consistent with our model.

3. Normally to correct phase mismatch one could either thermally or mechanically isolate the system or use a high intensity source to periodically measure phase errors and actively correct it using feedback. Our scheme mitigates this.

4. Because we only need passive linear optics without feed-forward, this is quite scalable with present-day technology, notably integrated photonic waveguide chips.

5. Robustness against mode loss. Standard QECs require the use of entangling gates such as CNOTs and the code state themselves can be highly entangled such as the GHZ states. These states however are not robust against loss in the sense of a partial trace operation while the W-state encoding will robust against such loss and increasingly so with higher levels of encoding.

The disadvantages of this scheme are:

1. Inability to correct correlated phase fluctuations (e.g a uniform phase shift across all redundant memory cells).

2. A multiplier in production cost and resource overhead, determined by the degree of redundancy.

7.5.2 Robustness against different noise models

While I have used a Gaussian noise model for detailed analysis in Section 7.2, this is by no means an absolute requirement on many of the results we present. As mentioned in Section 7.3, the property that determines the output state is the characteristic function for the random variables in the noise model for the unitary errors evaluated at $z = 1$. Due to the nature of the characteristic function, this value will be well defined in virtually all possible distributions, even ones that do not have a well defined moment generating function. The exact details of specific properties of the scheme will change under different distributions (e.g. the T_1 and T_2 decoherence factors identified here won't be well defined in general), but the analysis from the point of view of the encoded state will be essentially the same as what we have presented here.

7.6 Conclusion

I have proposed a passive linear optics encoding, using W-states which have the property of being strongly robust against entanglement degradation from qubit loss. This encoding was shown to be robust against any dephasing error modelled as an uncorrelated independent and identically distributed dephasing process on each subsystem. It was shown that the effective error probability is inversely related to the level of encoding N, vanishing in the large N limit. The loss rate upper-bounds the fidelity and success probability, but its effect does not scale with N, given that uniform losses can be commuted through passive linear optics systems.

The protocol is naturally suited to optical quantum memories (e.g via atomic ensembles, cavities, or delay lines), where the dominant error processes are independent dephasing and loss. Single-qubit operations are readily implementable within the encoded basis using conjugated passive linear optics operations.

Chapter 8

Discussion

8.1 Main Results

In part II of the thesis I presented a quantum optical encoding scheme using W-states. This encoding was shown to be robust against independent dephasing noise and success rates and fidelities of the protocol are given for a Gaussian noise model. The scheme has several advantages such as

(a) Code words are robust against mode loss.

(b) States can be easily encoded using simple fan out style beam splitter operations.

(c) Is completely passive and requires no measurement feed-forward.

(d) Easily implementable using current technology (an experimental implementation is under-way in collaboration with the quantum photonics group at the Indian Institute of Science, Bangalore)

8.2 Comparison with other schemes

It is important to clarify the distinction between this protocol, which can be regarded as a form of error filtration, and the more general concept of fault-tolerance where gate errors are accommodated for. Here we have assumed that our encoding and decoding operations are ideal, and all the dephasing errors are associated with the channel between them. Furthermore, we are not considering full quantum computations, but rather the storage or communication of just a single photonic qubit.

While future work might consider the effects of errors in the encoding and decoding errors in this protocol, the presented analysis is nonetheless reasonably well justified in most practical circumstances. Current linear optics technology, both using discrete elements or in integrated wave-guides, has become extremely mature and precise, enabling passive linear networks to be implemented with very high degrees of fidelity. On the other hand, photonic qubits communicated over long-distance links, via any medium, or which are held in quantum memories by coupling them to non-optical physical systems, are far more likely to contribute to these noise processes.

A further distinction between this scheme and conventional error correction schemes, is that we don't rely on any notion of code concatenation to asymptotically improve error thresholds. Instead, we directly expand our level of encoding by increasing the number of optical modes in the fan-out operation implemented by the QFT encoding operation. Unlike most well known codes whereby error syndrome measurements are used to apply feed-forward corrections to encoded qubits, this protocol does not rely on any form of active correction via syndrome extraction. Rather, dephasing noise is effectively mapped to non-determinism, such that upon success the effective dephasing rate has been reduced.

The final important distinction between this scheme and conventional QEC schemes, is that we do not create our encoded state via the introduction of additional qubits (i.e photons), but via the the introduction of additional optical modes, where the number of photons is preserved. Owing to these conceptual differences compared to more familiar QEC and fault-tolerance techniques, this scheme as presented is especially suited to the context of photonic quantum communication or storage via coupling into quantum memories.

8.3 Open problems

The W-state encoding we introduce in chapter 7 allows for the implementation of arbitrary unitary gates as discussed in section 7.4. However to achieve universal quantum computing in the sense of KLM (Knill et al. 2001) we need to able to implement probabilistic gates; how to do this in the W-state encoding is a promising line of research.

8.4 Future Directions

Extending the W-state encoding scheme to qudits has received some attention in (Ramakrishnan et al. 2020). However, applying the encoding for multiple qubits is an open problem. There are two natural ways to do this, either using a global W-state encoding on the full system or applying the encoding to each qubit individually. Comparing these schemes to see if one has an advantage is an avenue that is being explored.

Appendix A

Resource theory of coherence

A.1 $\mathcal{M}$ satisfies generalized Stein's lemma

For the generalized Stein's lemma to hold for a family of sets $\mathcal{M}$ the following conditions need to be met (Brandao and Plenio 2010)

1. Each $\mathcal{M}_n$ must be closed and convex.
2. Each $\mathcal{M}_n$ contains $\sigma^{\otimes n}$ for a full rank state $\sigma \in \mathcal{D}(\mathcal{H})$.
3. If $\rho \in \mathcal{M}_{n+1}$, then $\text{Tr}_k(\rho) \in \mathcal{M}_n$, for every $k \in \{1, ..., n+1\}$.
4. If $\rho \in \mathcal{M}_n$ and $\nu \in \mathcal{M}_m$, then $\rho \otimes \nu \in \mathcal{M}_{n+m}$.
5. If $\rho \in \mathcal{M}_n$ then $P_\pi \rho P_\pi \in \mathcal{M}_n$ for every $\pi \in S_n$, where P_π is the representation of a permutation π in $\mathcal{H}^{\otimes n}$ and S_n is symmetric group of order n.

The set of incoherent states in $\mathcal{H}^{\otimes n}$ will be convex and closed satisfying the first condition. $\delta^{\otimes n} \in \mathcal{I}_n$ satisfying condition 2. $\text{Tr}_k(\delta_{n+1}) \in \mathcal{I}_n$ where $\delta_{n+1} \in \mathcal{I}_{n+1}$ for any $k \in \{1, ..., n+1\}$ satisfying condition 3. $\delta_n \otimes \nu_m \in \mathcal{I}_{m+n}$ when $\delta_n \in \mathcal{I}_n$ and $\nu_m \in \mathcal{I}_m$ hence condition 4. is satisfied. Finally the permutation operation is just a relabelling of the incoherent basis hence the set of incoherent states will be closed under such a permutation and condition 5. is satisfied.

A.2 Upper bound for trace distance

Given that $\sum_i p_i F(\psi_i, \overline{\psi}_i) \geq 1 - 2\epsilon$, and $\mathrm{Tr}(\psi_i) = 1, \mathrm{Tr}(\overline{\psi}_i) \leq 1$ we want to find an upper bound in terms of ϵ for the expression,

$$\sum_i p_i T(\psi_i, \overline{\psi}_i), \tag{A.1}$$

where T is the trace distance. Let $|\overline{\psi}_i\rangle = a\,|\psi_i\rangle + b\,|\psi_i^{\perp}\rangle$, where $\langle\psi_i|\psi_i^{\perp}\rangle = 0$ and $|a_i|^2 + |b_i|^2 = \delta_i \leq 1$. We have,

$$T(\psi_i, \overline{\psi}_i) := \frac{1}{2}\|\psi_i - \overline{\psi}_i\| = \sum_j |\lambda_j^i|, \tag{A.2}$$

where $\{\lambda_j^i\}_j$ are the eigenvalues of the operator $\psi_i - \overline{\psi}_i$. Expressing this operator in the basis $\{|\psi_i\rangle\,, |\psi_i^{\perp}\rangle\}$ we get

$$\psi_i - \overline{\psi}_i = (1 - |a_i|^2)|\psi_i\rangle\langle\psi_i| - |b|^2|\psi_i^{\perp}\rangle\langle\psi_i^{\perp}| - ab^*|\psi_i\rangle\langle\psi_i^{\perp}| - a^*b|\psi_i^{\perp}\rangle\langle\psi_i|, \tag{A.3}$$

or in matrix form,

$$\begin{pmatrix} 1 - |a_i|^2 & -a_i b_i^* \\ -a_i^* b_i & |b_i|^2 \end{pmatrix}. \tag{A.4}$$

If we calculate the eigenvalues of the above matrix we get,

$$\lambda_{\pm}^i = \frac{1 - \delta_i}{2} \pm \frac{1}{2}\sqrt{(1 + \delta_i)^2 - 4|a_i|^2}. \tag{A.5}$$

Using equation (A.2) we have,

$$\begin{aligned} T(\psi_i, \overline{\psi}_i) = &\left|\frac{1 - \delta_i}{2} + \frac{1}{2}\sqrt{(1 + \delta_i)^2 - 4|a_i|^2}\right| \\ &+ \left|\frac{1 - \delta_i}{2} - \frac{1}{2}\sqrt{(1 + \delta_i)^2 - 4|a_i|^2}\right|, \quad \text{(A.6)} \\ \leq\ & |1 - \delta_i| + \left|\sqrt{(1 + \delta_i)^2 - 4|a_i|^2}\right|. \quad \text{(A.7)} \end{aligned}$$

Now we have,

$$\sum_i p_i T(\psi_i, \overline{\psi}_i) \leq \sum p_i |1 - \delta_i| + \sum_i p_i \left|\sqrt{(1 + \delta_i)^2 - 4|a_i|^2}\right|. \tag{A.8}$$

We know that $\delta_i \leq 1$ so we can find an upper bound by substituting $\delta_i = 1$ in the second term in the above equation, i.e.,

$$\sum_i p_i T(\psi_i, \overline{\psi}_i) \leq \sum_i p_i |1 - \delta_i| + \sum_i p_i \left| \sqrt{4 - 4|a_i|^2} \right|, \tag{A.9}$$

$$\leq \sum_i p_i (1 - \delta_i) + 2 \sum_i p_i \sqrt{1 - |a_i|^2}, \tag{A.10}$$

$$\leq \sum_i p_i (1 - \delta_i) + 2 \sqrt{1 - \sum_i p_i |a_i|^2}. \tag{A.11}$$

Notice that

$$F^2(\psi_i, \overline{\psi}_i) = | \langle \psi_i | \overline{\psi}_i \rangle |^2 = |a_i|^2 \leq \delta_i. \tag{A.12}$$

So,

$$\sum_i p_i T(\psi_i, \overline{\psi}_i) \leq \sum_i p_i (1 - \delta_i) + 2 \sqrt{1 - \sum_i p_i F^2(\psi_i, \overline{\psi}_i)}. \tag{A.13}$$

Consider the following inequality,

$$1 - 2\epsilon \leq \sum_i p_i F(\psi_i, \overline{\psi}_i) \leq \sqrt{\sum_i p_i F^2(\psi_i, \overline{\psi}_i)}. \tag{A.14}$$

So we have,

$$\sum_i p_i F^2(\psi_i, \overline{\psi}_i) \geq 1 - 4\epsilon. \tag{A.15}$$

Since $\delta_i \geq F^2(\psi_i, \overline{\psi}_i)$, we also have,

$$\sum_i p_i \delta_i \geq 1 - 4\epsilon. \tag{A.16}$$

An upper bound for equation (A.13) can be found by substituting the values $\sum_i p_i \delta_i = 1 - 4\epsilon = \sum_i p_i F^2(\psi_i, \overline{\psi}_i)$, so we get,

$$\sum_i p_i T(\psi_i, \overline{\psi}_i) \leq (1 - (1 - 4\epsilon)) + 2\sqrt{1 - (1 - 4\epsilon)}, \tag{A.17}$$

$$= 4\epsilon + 4\sqrt{\epsilon}. \tag{A.18}$$

Appendix B

General resource theory

B.1 Converse for of one-shot concentration of entanglement and coherence

B.1.1 Entanglement

Notice that for any separable state can be expressed as $\gamma = \sum_i q_i \rho_i \otimes \sigma_i$. We will show that the maximally entangled state

$$|\Phi_e\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right), \tag{B.1}$$

satisfies property 1 with $c(\Phi_e) = 1$. I.e.,

$$\Phi_e^m \gamma \Phi_e^m \leq \frac{1}{2^m} \Phi_e^m, \tag{B.2}$$

Using the definitions directly we have,

$$\begin{aligned} \Phi_e^m \gamma \Phi_e^m &= \langle \Phi_e^m \rangle \gamma \Phi_e^m \Phi_e^m, \\ &\leq \alpha \Phi_e^m, \end{aligned} \tag{B.3}$$

where,

$$\alpha = \max_{\gamma \in \text{SEP}} \text{Tr}(\Phi_e^m \gamma). \tag{B.4}$$

Now using the definition of γ we can write,

$$\begin{aligned}
&\max_{\gamma\in\text{SEP}} \text{Tr}(\Phi_e^m\gamma) \\
&= \max_{\{\rho_k,\sigma_k\}} \frac{1}{2^m}\sum_k p_k \sum_{i,j} \langle j\rangle\, \rho_k i \,\langle j\rangle\, \sigma_k i, && \text{(B.5)}\\
&\leq \frac{1}{2^m}\max_{\rho,\sigma}\sum_{i,j} \langle j\rangle\, \rho i \,\langle j\rangle\, \sigma i, && \text{(B.6)}\\
&= \frac{1}{2^m}\max_{\rho,\sigma}\sum_{i,j} \langle j\rangle\, \rho i \,\langle i\rangle\, \sigma^T j, && \text{(B.7)}\\
&= \frac{1}{2^m}\max_{\rho,\sigma}\text{Tr}(\rho\sigma^T), && \text{(B.8)}\\
&\leq \frac{1}{2^m}, && \text{(B.9)}
\end{aligned}$$

where in the last inequality I have used the fact that $\rho, \sigma^T \leq \mathbb{I}$. Thus property 1 is satisfied.

From equation (4.18) the ideal rate of one-shot concentration for a pure state ρ using a δ-entanglement-generating operation is bounded as,

$$E^{\delta,\epsilon}(\rho, \Phi_e^m) \leq \max_{\overline{\psi}\in b_*(\psi,2\epsilon)} G_{min}(\overline{\psi}) + \log(1+\delta). \quad \text{(B.10)}$$

Let $\overline{\psi}$ be the state that achieves this maximisation. Then,

$$E^{\delta,\epsilon}(\psi^{AB}, \Phi_e^m) \leq \min_{\gamma\in\text{SEP}} \left\{-\log \text{Tr}(\overline{\psi}\gamma)\right\} + \log(1+\delta), \quad \text{(B.11)}$$

where SEP is the set of separable states. Since $\overline{\psi}$ is really a bipartite pure state $\overline{\psi}^{AB}$, we can write it as a purification of it's reduced density matrix $\overline{\rho}^B = \text{Tr}_A(\overline{\psi}^{AB})$. Let $\overline{\rho}^B = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$, then there is some unitary U such that,

$$|\overline{\psi}^{AB}\rangle = \sum_i \sqrt{\lambda_i} U \,|\lambda_i\rangle^A \,|\lambda_i\rangle^B . \quad \text{(B.12)}$$

Now the trace in equation (B.11) is maximized by a product state $\gamma = \sigma\otimes\delta$, since any convex combination can only decrease the trace. So,

$$\begin{aligned}
&\max_{\gamma} \text{Tr}(\overline{\psi}^{AB}\gamma) \\
&= \max_{\sigma,\delta}\sum_{i,j} \sqrt{\lambda_i\lambda_j}\, \langle\lambda_j\rangle\, U^\dagger\sigma U \lambda_j \,\langle\lambda_i\rangle\, \delta\lambda_i. && \text{(B.13)}
\end{aligned}$$

The above sum is maximized by choosing $\sigma = U|\lambda_{max}\rangle\langle\lambda_{\max}|U^\dagger$ and $\delta = |\lambda_{max}\rangle\langle\lambda_{\max}|$ where $|\lambda_{max}\rangle$ is understood to be the eigenvector with the largest eigenvalue. So,

$$\max_{\gamma\in\text{SEP}} \text{Tr}(\overline{\psi}^{AB}\gamma) = \lambda_{max}(\overline{\rho}^B). \tag{B.14}$$

Using this,

$$\begin{aligned} E^{\delta,\epsilon}(\psi^{AB},\Phi_e) &\leq -\log\lambda_{max}(\overline{\rho}^B) + \log(1+\delta) && \text{(B.15)} \\ &= S_{min}(\overline{\rho}^B) + \log(1+\delta). && \text{(B.16)} \end{aligned}$$

B.1.2 Coherence

For the unit maximally coherent state,

$$|\Phi_c\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right), \tag{B.17}$$

notice that,

$$\begin{aligned} \Phi_c^m\gamma\Phi_c^m &= \text{Tr}(\Phi_c^m\gamma)\Phi_c^m, && \text{(B.18)} \\ &= \frac{1}{2^m}\Phi_c^m, && \text{(B.19)} \end{aligned}$$

where in the second line we have used the fact that for any incoherent state γ and pure state ψ, $\text{Tr}(\psi\gamma) = \lambda_{max}(\Delta(\psi))$, where λ_{max} gives the largest eigenvalue and Δ is the completely dephasing map in the incoherent basis. Thus the maximally coherent state also satisfies property 1 with $c(\Phi_c) = 1$. Following the same line for reasoning as for entanglement, the ideal concentration rate with error ϵ for an optimal state $\overline{\psi}$ using δ-incoherence-generating operations is bounded as,

$$\begin{aligned} C^{\delta,\epsilon}(\rho,\Phi_c) &\leq \min_{\gamma\in\mathcal{I}}\left\{-\log\text{Tr}(\overline{\psi}\gamma)\right\} + \log(1+\delta), \\ &= -\log(\lambda_{max}(\Delta(\overline{\psi})) + \log(1+\delta), \\ &= S_{min}(\Delta(\overline{\psi})) + \log(1+\delta). \end{aligned} \tag{B.20}$$

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