THE COMPLEXITY OF SOLUTION SETS TO EQUATIONS IN HYPERBOLIC GROUPS

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Abstract. We show that the full set of solutions to systems of equations and inequations in a hyperbolic group, as shortlex geodesic words (or any regular set of quasigeodesic normal forms), is an EDT0L language whose specification can be computed in $\mathsf{NSPACE}(n^2 \log n)$ for the torsion-free case and $\mathsf{NSPACE}(n^4 \log n)$ in the torsion case. Furthermore, in the presence of effective quasi-isometrically embeddable rational constraints, we show that the full set of solutions to systems of equations in a hyperbolic group remains EDT0L.

Our work combines the geometric results of Rips, Sela, Dahmani and Guirardel on the decidability of the existential theory of hyperbolic groups with the work of computer scientists including Plandowski, Jeż, Diekert and others on PSPACE algorithms to solve equations in free monoids and groups using compression, and involves an intricate language-theoretic analysis.

1. Introduction

Let G be a hyperbolic group with finite symmetric generating set S. In this paper we show that any system of equations in G has solutions which, when written as shortlex representatives over S, admit a particularly simple description as formal languages, and moreover, this description can be given in very low space complexity. Our work combines the geometric results for determining the satisfiability of equations in hyperbolic groups of Rips, Sela, Dahmani and Guirardel [12, 39], with recent tools developed in theoretical computer science which give PSPACE algorithms for solving equations in semigroups and groups [7, 14, 15, 16, 17, 28, 29].

The satisfiability of equations in torsion-free hyperbolic groups is decidable by the work of Rips and Sela [39], who reduced the problem to solving equations in free groups, and then called on Makanin's algorithm [32] for free groups. Kufleitner proved PSPACE for decidability in the torsion-free case [31], without an explicit complexity bound, by following Rips-Sela and then using Plandowski's result [35]. Dahmani and Guirardel extended Rips and Sela's work to all hyperbolic groups (with torsion), by showing that it is sufficient to solve systems in virtually free groups, which they then reduced to systems of *twisted* equations in free groups [12].

The first algorithmic description of *all* solutions to a given equation over a free group is due to Razborov [37, 36]. His description became known as a *Makanin-Razborov (MR) diagram*, and this concept was then generalised to hyperbolic groups by Reinfeld and Weidmann [38], and to relatively hyperbolic groups by Groves [25]. While in theory MR diagrams can be used to algebraically produce the solutions of an equation via composition of group homomorphisms, it is infeasible to use this approach to directly obtain solutions as freely reduced or shotlex representative

Date: October 27, 2020.

2010 *Mathematics Subject Classification.* 03D05, 20F65, 20F70, 68Q25, 68Q45 .

Key words and phrases. Hyperbolic group, Diophantine problem, existential theory, PSPACE, L-system, EDT0L, Copying Lemma.

Research supported by Australian Research Council (ARC) Project DP160100486, EPSRC grant EP/R035814/1, a Follow-On Grant from the International Centre of Mathematical Sciences (ICMS), Edinburgh, and an LMS Scheme 2 grant.

words, as cancellations in the images of these homomorphisms cannot be controlled. Also, it is extremely complicated to explicitly produce a Makanin-Razborov diagram for a given equation even for free groups, and this has been done only in very few cases ([41]). For the special case of quadratic equations, Grigorchuck and Lysionok gave efficient algorithms to obtain the solutions in [24].

Here, we describe the combinatorial structure of solution sets of systems of equations: we show that they form an *EDT0L language* over the generating set S. Roughly speaking, this means that there is a set $C \supseteq S$ and a *seed* word $c_0 \in C^*$ so that every solution can be obtained by applying to c_0 a certain set of endomorphisms of the free monoid C^*. The set of endomorphisms is described by a finite labeled directed graph. (A simple EDT0L language can be seen in Example 3.2.) Our results are that we can algorithmically construct this graph, together with the set C and c_0, and hence a finite description of the solution set, and moreover we can produce this description in PSPACE (see §3.2 for space complexity definitions). The language-theoretic characterisation of solution sets had been open for a number of years even for free groups, and while this was settled by [7], it is remarkable that the EDT0L characterisation for free groups is so robust that it holds for the much bigger class of hyperbolic groups.

More specifically, we combine Rips, Sela, Dahmani and Guirardel's approach with recent work of the authors with Diekert [7, 14] to obtain the following results. An *inequation* is simply an expression using $\neq$ instead of $=$. The acronym *qier* stands for *quasi-isometrically embeddable rational* and is defined in Subsection 4.2.

Theorem A (Torsion-free)**.** *Let G be a torsion-free hyperbolic group with finite symmetric generating set S. Let Φ be a system of equations and inequations of size n with constant size effective qier constraints (see Section 2 and Definition 4.8 for precise definitions). Then*

(1) the set of all solutions, as tuples of shortlex geodesic words over S, is EDT0L, and the algorithm which on input Φ prints a description for the EDT0L grammar runs in $\mathsf{NSPACE}(n^2 \log n)$.
(2) it can be decided in $\mathsf{NSPACE}(n^2 \log n)$ whether or not the solution set of Φ is empty, finite or infinite.

Theorem B (Torsion)**.** *Let G be a hyperbolic group with torsion, with finite symmetric generating set S. Let Φ be a system of equations and inequations of size n with constant size effective qier constraints (see Section 2 and Definition 4.8 for precise definitions). Then*

(1) the set of all solutions, as tuples of shortlex geodesic words over S, is EDT0L, and the algorithm which on input Φ prints a description for the EDT0L grammar runs in $\mathsf{NSPACE}(n^4 \log n)$.
(2) it can be decided in $\mathsf{NSPACE}(n^4 \log n)$ whether or not the solution set of Φ is empty, finite or infinite.

Theorems A and B follow from the more general Theorems 7.4 and 8.12 stated later in the paper, and have the following immediate consequence.

Corollary 1.1 (Existential theory)**.** *The existential theory with constant size effective qier constraints can be decided in $\mathsf{NSPACE}(n^2 \log n)$ for torsion-free hyperbolic groups and $\mathsf{NSPACE}(n^4 \log n)$ for hyperbolic groups with torsion.*

In this paper we produce the solution sets not only in terms of shortlex representatives, but more generally in terms of quasigeodesics: for given constants, we can obtain the full set of solutions expressed as words belonging to some regular subset of quasigeodesics surjecting to the group, such as the set of all geodesics.

Table 1 summarises the different kinds of expressing solution sets, together with the corresponding language and space complexities that we are able to prove.

In the special case of free groups, we may want to produce *all* words which represent solutions, and then such a set has a slightly higher complexity: ET0L instead of EDT0L (see §3 for definitions).

Corollary 1.2 (Full solutions in free groups)**.** *Let G be a finitely generated free group with free basis A_+, and $A = A_+ \cup \{x^{-1} \mid x \in A_+\}$ the* free basis generating set *for G. Let Φ be a system of equations and inequations of size n with constant size rational constraints.*

Then the set of all solutions, as tuples of all words over S, is ET0L, and the algorithm which on input Φ prints a description for the ET0L grammar runs in $\mathsf{NSPACE}(n \log n)$.

Note that the *word problem* for a group G is the set of solutions to the one variable equation $X = 1$, so Corollary 1.2 cannot be improved to EDT0L since it is known that the word problem for F_2 is not EDT0L (see [9, Proposition 26]; [19]). Also, it is suspected that word problems of hyperbolic groups are not ET0L unless the group is virtually free (in which case the word problem is deterministic context-free [33]), so the requirement that words are quasigeodesics in all our results apart from Corollary 1.2 most likely cannot be weakened in general.

Class of groups	gen set	solutions as	language	NSPACE	
Free	free basis	freely red words	EDT0L	$n \log n$	[7]
Free	any	unique quasigeods	EDT0L	$n \log n$	Cor. 5.7
Free	any	quasigeods	ET0L	$n \log n$	Cor. 5.7
Free	free basis	all words	ET0L	$n \log n$	Cor. 1.2
Virt free	certain	certain quasigeods	EDT0L	$n^2 \log n$	[14]
Virt free	any	unique quasigeods	EDT0L	$n^2 \log n$	Cor. 5.7
Virt free	any	quasigeods	ET0L	$n^2 \log n$	Cor. 5.7
Torsion-free hyp	any	unique quasigeods	EDT0L	$n^2 \log n$	Thm. 7.4
Torsion-free hyp	any	quasigeods	ET0L	$n^2 \log n$	Thm. 7.4
Hyp with torsion	any	unique quasigeods	EDT0L	$n^4 \log n$	Thm. 8.12
Hyp with torsion	any	quasigeods	ET0L	$n^4 \log n$	Thm. 8.12

TABLE 1. Summary of results

The above results partially extend to systems of equations and inequations subject to arbitrary quasi-isometrically embeddable rational constraints. By 'partially extend' we mean that the language-theoretic complexity is preserved, but we can no longer guarantee that the description can be computed in (non-deterministic) polynomial space. The issue here is that a Benois-type transition required to catch all solutions obeying the constraint does not preserve space complexity – the automata construction blows up (see Section 6 for further discussion). Conceivably there may be another approach to get a polynomial bound on space complexity. For now we are able to state the following:

Theorem C (Systems with quasi-isometrically embeddable rational constraints)**.** *Let G be a hyperbolic group with or without torsion, with finite symmetric generating set S. Let Φ be a system of equations and inequations with arbitrary size effective qier constraints (see Section 2 and Definition 4.8 for precise definitions). Then the set of all solutions, as tuples of shortlex geodesic words over S, is EDT0L.*

Theorem C follows from the more specific statement in Theorem 9.1 below.

EDT0L is a surprisingly low language complexity for solution sets in hyperbolic groups. EDT0L languages lie strictly in the class of indexed languages, and while containing all regular languages, they are incomparable to the context-free ones. See Figure 1. Solution sets to systems of equations are content-sensitive, since there exists a Turing machine that can simply take an input tuple of words and substitute them into the equation to check whether they form a solution, needing just linear space. However, solution sets are not context-free in general, for example, the equation $X = Y$ has solutions of the form (w, w) which, if expressed as a language $w\#w$ is a standard non-context-free example. As mentioned above, whether or not solution sets for free groups were even indexed languages was a long-standing open problem [22, 23, 27] which was resolved by [7], and the present paper radically extends this to the much larger class of hyperbolic groups.

ET0L and EDT0L languages are playing an increasingly useful role in group theory, not only in describing solution sets to equations in groups [7, 14, 16], but more generally [4, 6, 9].

Figure 1. Relationships between formal language classes (containment from left to right).

The paper is organised as follows. In Sections 2 to 4 we give the necessary background on equations, constraints, EDT0L languages, space complexity and hyperbolic groups. Section 5 describes our key trick to produce solutions as EDT0L rather than simply ET0L languages. Section 6 explains how we handle qier constraints and complexity issues for them. In Section 7 we deal with the torsion-free case. We follow Rips and Sela's approach to solving systems of equations over torsion-free hyperbolic groups by reducing them to systems over free groups via *canonical representatives*, and we apply [7] to show the language of all solutions is EDT0L. In Section 8 we use Dahmani and Guirardel's reduction of systems over torsion hyperbolic groups to systems over virtually-free groups, via canonical representatives in an appropriate (subdivision of a) Rips complex, and we apply [14] together with intricate language operations, to show the language of all solutions is EDT0L. The torsion case is more involved because it requires keeping track of generating sets, quasigeodesic words (and paths in different graphs) and translations between these, to obtain the formal language description, and to show the $\mathsf{NSPACE}(n^4 \log n)$ complexity. (See Remark 8.7 for a brief explanation.) Section 9 extends the results from the main part of the paper to systems with arbitrary qier constraints. We explain why our techniques still ensure EDT0L solutions but *a priori* not in PSPACE. In Section 10 we prove Corollary 1.2.

An extended abstract of a preliminary version of this paper was presented at the conference ICALP 2019, Patras (Greece), 8-12 July 2019 [8].

2. Preliminaries – Equations, solution sets, rational constraints

A language $L \subseteq \Sigma^*$ is *regular* if there exists some (nondeterministic) finite state automaton over Σ which accepts exactly the words in L. We abbreviate *nondeterministic finite automaton* to NFA.

Let G be a fixed group with finite symmetric generating set S, and $\pi : S^* \to G$ the natural projection map. A subset $R \subseteq G$ is *rational* if $R = \pi(L)$ for some regular language $L \subseteq S^*$.

For a word $w \in S^*$ let $|w|_S$ denote the length of w, and for $g \in G$ let $\|g\|_S = \min\{|w|_S \mid w \in S^*, \pi(w) = g\}$ (the geodesic length of g with respect to S).

Definition 2.1 (System of equations)**.** Let $\mathcal{X} = \{X_1, \dots, X_r\}$ be a set of variables, $\mathcal{A} = \{a_1, \dots, a_k\} \subseteq G$ a set of constants, and $\phi_j(\mathcal{X}, \mathcal{A}) \in (\mathcal{X}^{\pm 1}, \mathcal{A}^{\pm 1})^*$ a set of words, where $1 \leqslant j \leqslant s$, $k, r, s \geqslant 1$. For each $X \in \mathcal{X}$, let R_X be a rational subset of G. Then

$$\Phi = \{\phi_j(\mathcal{X}, \mathcal{A}) = 1\}_{j=1}^{h} \cup \{\phi_j(\mathcal{X}, \mathcal{A}) \neq 1\}_{j=h+1}^{s} \cup \{R_X \mid X \in \mathcal{X}\}$$

is a *system of equations and inequations with rational constraints* over G.

If each word $\phi_j(\mathcal{X}, \mathcal{A})$ has length l_j for all j, and each R_X is given by an NFA with $|R_X|$ states, the *size* of the system Φ, denoted $\|\Phi\|$, is defined using two parameters:

$$|\Phi|_1 := \sum_{j=1}^{s} l_j, \ |\Phi|_2 = \sum_{X \in \mathcal{X}} |R_X|, \text{ and } \|\Phi\| = (|\Phi|_1, |\Phi|_2).$$

A system is said to have *constant size rational constraints* if $|\Phi|_2$ is bounded by a constant (that might depend the fixed group and generating set). In this case we abuse notation and let the size be $\|\Phi\| = |\Phi|_1 = n$.

Definition 2.2 (Different types of solution sets)**.**

(i) A tuple $(g_1, \dots, g_r) \in G^r$ *solves* (or *is a solution of*) the system Φ if there exists a homomorphism $\sigma : F(\mathcal{X}) * G \to G$ given by $\sigma(X_i) = g_i$ which fixes G, satisfies $\sigma(X_i) \in R_{X_i}$ for each $1 \leqslant i \leqslant r$ and

$$\sigma(\phi_j(X_1, \dots, X_r, a_1, \dots, a_k)) = 1$$

$$\text{and } \sigma(\phi_l(X_1, \dots, X_r, a_1, \dots, a_k)) \neq 1$$

for all $1 \leqslant j \leqslant h$ and $h+1 \leqslant l \leqslant s$.

(ii) The *group element solutions* to Φ is the set

$$\mathrm{Sol}_G(\Phi) = \{(g_1, \dots, g_r) \in G^r \mid (g_1, \dots, g_r) \text{ solves } \Phi\}.$$

(iii) Let $\mathcal{T} \subseteq S^*$ and $\#$ a symbol not in S. The *full set of $\mathcal{T}$-solutions* is the set

$$\mathrm{Sol}_{\mathcal{T},G}(\Phi) = \{w_1\# \dots \#w_r \mid w_i \in \mathcal{T}, (\pi(w_1), \dots, \pi(w_r)) \text{ solves } \Phi\}.$$

(iv) Let $S_{\#r} = \{w_1\# \dots \#w_r \mid w_i \in S^*\}$ denote the set of all words over $S \cup \{\#\}$ which contain exactly $r-1$ $\#$ symbols. A set

$$L = \{w_1\# \dots \#w_r\} \subseteq S_{\#r}$$

is a *covering solution set* to Φ if $w_i \in S^*, 1 \leqslant i \leqslant r$, and

$$\{(\pi(w_1), \dots, \pi(w_r)) \mid (w_1\# \dots \#w_r) \in L\} = \mathrm{Sol}_G(\Phi).$$

Example 2.3. Let $G = \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1\rangle$ be the surface group of genus 2 with symmetric generating $S = \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}\}$. Let $\mathcal{X} = \{X, Y\}$ be a set of variables, $\mathcal{A} = \{a, b, c, d\}$ a set of constants, and consider the equation $\phi(\{X, Y\}, \{a, b, c, d\}) = abXcY$. Then the system consisting of a single equation

$$\Phi = \{abXcY = 1\}$$

has size 5. The pair $(a^{-1}b^{-1}, dc^{-1}d^{-1})$ is one group element solution of Φ.

Now let $R_X = R_Y = R = \{g \in G \mid \|g\|_S \text{ is even}\}$. One can prove[1] that R is a rational set in G. If

$$\Phi' = \{abXcY = 1\} \cup \{R_X, R_Y\},$$

then Φ' has no solutions: if $\sigma(X)$ and $\sigma(Y)$ are solutions of Φ' of even length then $\sigma(\phi(X, Y, \{a, b, c, d\})) = \sigma(abXcY) = ab\sigma(X)c\sigma(Y)$ has odd length and cannot represent the identity in G because G has only even length relators and only even length words can represent the identity in G.

Example 2.4. Let G be a hyperbolic group with finite symmetric generating set S. If Φ consists of the single equation $X = 1$ then $\mathrm{Sol}_G(\Phi) = \{1\}$ and $\mathrm{Sol}_{S^*,G}(\Phi)$ is the word problem of G (believed to not be ET0L if G is not virtually free).

3. Preliminaries – languages and space complexity

An alphabet is a finite set. We use the notation $\mathscr{P}(A)$ for the power set (set of all subsets) of a set A, and $|A|$ for the size of (number of elements in) the set A.

3.1. **ET0L and EDT0L languages.** Let C be an alphabet. A *table* for C is a finite subset of $C \times C^*$ which includes at least one element (c, v) for each $c \in C$. A table t is *deterministic* if for each $c \in C$ there is exactly one $v \in C^*$ with $(c, v) \in t$. If (c, v) is in some table t, we say that (c, v) is a *rule* for c. Applying a rule (c, v) to a letter c means replacing c by v.

If t is a table and $u \in C^*$ then we write $u \longrightarrow^t v$ to mean that v is obtained by applying rules from t to each letter of u. That is, $u = a_1 \dots a_n$, $a_i \in C$, $v = v_1 \dots v_n$, $v_i \in C^*$, and $(a_i, v_i) \in t$ for $1 \leqslant i \leqslant n$. Note that when t is deterministic then the word v obtained from u by applying t is unique. In this case we can write $v = t(u)$ instead of $u \longrightarrow^t v$.

If H is a set of tables and $r \in H^*$ then we write $u \longrightarrow^r v$ to mean that there is a sequence of words $u = v_0, v_1, \dots, v_n = v \in C^*$ such that $v_{i-1} \longrightarrow^{t_i} v_i$ for $1 \leqslant i \leqslant n$ where $r = t_1 \dots t_n$. If $A \subseteq H^*$ we write $u \longrightarrow^A v$ if $u \longrightarrow^r v$ for some $r \in A$.

Definition 3.1 ([2])**.** Let Σ be an alphabet. We say that $L \subseteq \Sigma^*$ is an *ET0L* language if there is an alphabet C with $\Sigma \subseteq C$, a finite set $H \subset \mathscr{P}(C \times C^*)$ of tables, a regular language $A \subseteq H^*$ and a fixed word $c_0 \in C^*$ such that

$$L = \{w \in \Sigma^* \mid c_0 \longrightarrow^A w\}.$$

In the case when every table $t \in H$ is deterministic, i.e. each $h \in A$ is in fact a homomorphism, we write $L = \{h(c_0) \in \Sigma^* \mid h \in A\}$ and say that L is *EDT0L*. The set A is called the *rational control*, the word c_0 the *seed* and C the *extended alphabet*[2].

Example 3.2. [3] The language $L = \{a^{n^2} \mid n \in \mathbb{N}\}$ over the alphabet $\Sigma = \{a\}$ is EDT0L but not context-free. The extended alphabet is $C = \{s, t, u, a\}$, seed word is $c_0 = tsa$,

$$\begin{aligned} \phi_1 &= \{(s, su)\} \\ \phi_2 &= \{(t, at), (u, ua^2)\} \\ \phi_3 &= \{(s, \varepsilon), (t, \varepsilon), (u, \varepsilon)\} \end{aligned}$$

where we use here the convention that ϕ_i fixes the elements in C not explicitly specified, $H = \{\phi_1, \phi_2, \phi_3 : C \to C^*\}$, and M the automaton in Figure 2. We have $A = L(M) = (\phi_1\phi_2)^*\phi_3$. One can check that $(\phi_1\phi_2)^i(tsa) = a^i tsua^2ua^4u \dots ua^{2i}a$

[1] By Proposition 4.6 below, the language L_1 of all geodesics is regular, which we can intersect with the regular language L_2 of all words of even length. Then $R = \pi(L_1 \cap L_2)$.

[2] The letters E,D,T,L stand for *extended, deterministic, table, Lindenmayer* respectively, and 0 is the number zero standing for 0-*interaction*.

[3] We thank Alex Levine for providing this example.

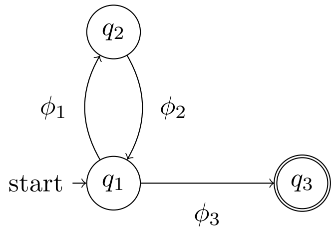


FIGURE 2. NFA for the rational control for the EDT0L grammar in Example 3.2.

which is sent to $a^{(i+1)^2}$ by applying ϕ_3. It follows that the language of the EDT0L system is $\{a^{n^2} \mid n \in \mathbb{N}_+\}$.

3.2. **Space complexity for E(D)T0L.** Let $s\colon \mathbb{N} \to \mathbb{N}$ be a function. Recall an algorithm is said to run in $\mathsf{NSPACE}(s(n))$ if it can be performed by a non-deterministic Turing machine with a read-only input tape, a write-only output tape, and a read-write work tape, with the work tape restricted to using $\mathcal{O}(s(n))$ squares on input of size n. We use the notation $L(\mathcal{M})$ to denote the language accepted by the automaton $\mathcal{M}$. The following definition formalises the idea of producing a description of some E(D)T0L language (such as the solution set of some system of equations) in $\mathsf{NSPACE}(s(n))$, where the language is the output of a computation with input (such as a system of equations) of size n. We say an algorithm runs in PSPACE if it runs in $\mathsf{NSPACE}(s(n))$ for some polynomial function s.

Remark 3.3. Every $\mathsf{NSPACE}(s(n))$ algorithm (with $s(n) \in \Omega(\log n)$) can be simulated by a deterministic algorithm using at most working space $s(n)^2$ (Savitch's Theorem), and also by a deterministic Turing machine which uses a time bound in $2^{\mathcal{O}(s(n))}$, see [34] for more details. Thus, every PSPACE algorithm can be implemented such that it runs in deterministic singly exponential time $2^{\mathrm{poly}(n)}$.

Definition 3.4. Let Σ be a (fixed) alphabet and $s\colon \mathbb{N} \to \mathbb{N}$ a function. If there is an $\mathsf{NSPACE}(s(n))$ algorithm that on input Ω of size n outputs (prints out) a specification of an ET0L language $L_\Omega \subseteq \Sigma^*$, then we say that L_Ω is *ET0L in* $\mathsf{NSPACE}(s(n))$.

Here a specification of L_Ω consists of

(1) an extended alphabet $C \supseteq \Sigma$,
(2) a seed word $c_0 \in C^*$,
(3) a finite list of nodes of an NFA $\mathcal{M}$, labeled by some data, some possibly marked as initial and/or final,
(4) a finite list $\{(u, v, h)\}$ of edges of $\mathcal{M}$ where u, v are nodes and $h \in \mathscr{P}(C \times C^*)$ is a table

such that $L_\Omega = \{w \in \Sigma^* \mid c_0 \to^{L(\mathcal{M})} w\}$.

A language L_Ω is *EDT0L in* $\mathsf{NSPACE}(s(n))$ if, in addition, every table h labelling an edge of $\mathcal{M}$ is deterministic.

Note that the entire print-out is not required to be in $O(s(n))$ space. If $|C| \in \mathcal{O}\left(\frac{s(n)}{\log n}\right)$ then we can write out and store the entire extended alphabet as binary strings within our space bound, but in general this is just a convenience and not essential.

Previous results of the authors with Diekert can now be restated as follows.

Theorem 3.5 ([7, Theorem 2.1])**.** *The set of all solutions in a free group to a system of equations of size n, with constant size rational constraints, as reduced words over the free generating set, is EDT0L in* $\mathsf{NSPACE}(n \log n)$.

Theorem 3.6 ([14, Theorem 45])**.** *The set of all solutions in a virtually free group to a system of equations of size n, with constant size rational constraints, as words in* standard normal forms[4] *over a certain finite generating set, is EDT0L in* $\mathsf{NSPACE}(n^2 \log n)$.

The results in these papers are more general, in that they handle rational constraints of certain non-constant sizes within the same space bound. We will return to the issue of non-constant sized constraints in Section 9.

Remark 3.7. In our applications below we have Ω representing some system of equations and inequations with (constant size) rational constraints, with $|\Omega| = n$, and we construct algorithms where the extended alphabet C has size $|C| \in \mathcal{O}(n)$ in the torsion-free case and $|C| \in \mathcal{O}(n^2)$ in the torsion case. This means we can write down the entire alphabet C as binary strings within our space bounds. Moreover, each element (c, v) of any table we construct has v of (fixed) bounded length, so we can write down entire tables within our space bounds.

3.3. **Closure properties.** It is well known (see for example [40, Theorem V.1.7]) that the class of ET0L languages is closed under homomorphism, inverse homomorphism, finite union, and intersection with regular languages (a *full AFL*), and that EDT0L is closed under all except inverse homomorphism. Here we show that the space complexity of an E(D)T0L language is preserved by these operations.

Let us denote an E(D)T0L system by the 4-tuple $(C, \Sigma, c_0, \mathcal{M})$ consisting of extended alphabet C, alphabet $\Sigma \subseteq C$, seed word $c_0 \in C^*$, and NFA $\mathcal{M}$ with edges labeled by tables in $\mathscr{P}(C \times C^*)$, so that $L = \{w \in \Sigma^* \mid c_0 \to^{L(\mathcal{M})} w\}$ is the corresponding E(D)T0L language.

Proposition 3.8. *Let Σ, Γ be finite alphabets of fixed size, $\mathcal{N}$ an NFA of fixed size with $L(\mathcal{N}) \subseteq \Sigma^*$, and $\psi \colon \Sigma^* \to \Gamma^*$, $\varphi \colon \Gamma^* \to \Sigma^*$ homomorphisms. Suppose there is a function $s \colon \mathbb{N} \to \mathbb{N}$ and* $\mathsf{NSPACE}(s(n))$ *algorithms that on inputs Ω_1, Ω_2, each of size in $\mathcal{O}(n)$, prints out specifications for E(D)T0L languages $L_{\Omega_1}, L_{\Omega_2} \subseteq \Sigma^*$. (That is, $L_{\Omega_1}, L_{\Omega_2}$ are E(D)T0L in* $\mathsf{NSPACE}(s(n))$*.) Then*

(1) (union) $L_{\Omega_1} \cup L_{\Omega_2}$
(2) (homomorphism) $\psi(L_{\Omega_1})$
(3) (intersection with regular) $L_{\Omega_1} \cap L(\mathcal{N})$

are E(D)T0L in $\mathsf{NSPACE}(s(n))$ *and*

(4) (inverse homomorphism) $\varphi^{-1}(L_{\Omega_1})$

is ET0L in $\mathsf{NSPACE}(s(n))$.

The proof is straightforward keeping track of complexity in the standard proofs, for example [40, Theorem V.1.7] and [3, 10]. We include a proof here for completeness.

Proof. To prove item (1), assume the E(D)T0L language L_{Ω_i} is given by the 4-tuple $(C_i, \Sigma, c_i, \mathcal{M}_i)$ which can be constructed in $\mathsf{NSPACE}(s(n))$, for $i = 1, 2$. Let p_i be the start state for $\mathcal{M}_i$. Construct a new E(D)T0L system $(C_1 \cup C_2, \Sigma, c_*, \mathcal{M}')$ where c_* is a new seed word consisting of a single letter, t_i are two deterministic tables defined by the rule (c_*, c_i), and p_* the start state for $\mathcal{M}'$. The automaton $\mathcal{M}'$ is obtained by printing two edges (p_*, p_i) labeled t_i then printing all edges for $\mathcal{M}_1, \mathcal{M}_2$. Printing these can be done in space $s(n)$ given printing the data

[4] a particular quasigeodesic normal form, see Proposition 5.6.

for $\mathcal{M}_1, \mathcal{M}_2$ can. Moreover since the two additional tables are deterministic, the resulting language is EDT0L if both L_{Ω_i} are.

For the remaining items, assume the E(D)T0L language L_{Ω_1} is given by the 4-tuple $(C, \Sigma, c_0, \mathcal{M})$ which can be constructed in $\mathsf{NSPACE}(s(n))$, with p_0 the start state for $\mathcal{M}$.

For item (2), print $\mathcal{M}$ with the following modifications. Print a new unique accept state p_{accept} and for each previous accept state of $\mathcal{M}$, print an edge from it to p_{accept} labeled by the homomorphism (deterministic table) $\psi|_\Sigma$. The resulting language is $\psi(L_{\Omega_1})$ and the additional space required is in $\mathcal{O}(s(n))$ given that printing the data for $\mathcal{M}$ requires this much space. If L_{Ω_1} is EDT0L, since the table $\psi|_\Sigma$ added is deterministic, then so is $\psi(L_{\Omega_1})$.

To prove item (3), assume $\mathcal{N}$ has states $Q = \{q_0, \ldots, q_r\}$, with start state q_0. Without restriction (since $\mathcal{N}$ has constant size) we may assume $\mathcal{N}$ is deterministic and has a unique accept state q_{accept}. Construct a new E(D)T0L system $(C_*, \Sigma_*, [q_0, c_*, q_{\text{accept}}], \mathcal{M}')$ to accept $L_{\Omega_1} \cap L(\mathcal{N})$ in $\mathsf{NSPACE}(s(n))$ as follows.

The states for $\mathcal{M}'$ are the states of $\mathcal{M}$ plus a new start state p_*. The new extended alphabet is

$$C_* = \{[q_i, c, q_j] \mid c \in C, q_i, q_j \in Q\} \cup \{[q_0, c_*, q_{\text{accept}}]\}$$

with seed word the single letter $[q_0, c_*, q_{\text{accept}}]$ where $c_* \notin C$ is a new symbol. Define

$$\Sigma_* = \{[q_i, a, q_j] \mid a \in \Sigma, q_i, q_j \in Q, (q_i, q_j, a) \text{ is an edge of } \mathcal{N}\}.$$

For each $(c, v) \in (C \cup \{c_*\}) \times C^*$, define $\mathfrak{r}(c, v)$ to be the following set: if $v = a_1 \ldots a_n$ with $a_i \in C$, then

$$\mathfrak{r}(c, v) = \left\{ \left([q_{i_0}, c, q_{i_n}], [q_{i_0}, a_1, q_{i_1}][q_{i_1}, a_2, q_{i_2}] \ldots [q_{i_{n-1}}, a_n, q_{i_n}]\right) \;\middle|\; q_{i_j} \in Q \right\}.$$

For each $(c_*, x) \in \mathfrak{r}(c_*, c_0)$ print an edge from p_* to p_0 labeled by the deterministic table which sends c_* to x (and leaves remaining letters fixed). For each edge (p_s, p_t, h) printed by the algorithm producing $\mathcal{M}$, print a constant number of new edges from p_s to p_t labeled by tables obtained from h by replacing each element $(c, v) \in h$ by some choice from $\mathfrak{r}(c, v)$. Since $\mathcal{N}$ is constant size, each edge can be printed in $\mathcal{O}(s(n))$ space, that is, the same space needed to print edges for $\mathcal{M}$. Each table printed is deterministic if the original h was.

The language K of this system is the set of all strings over Σ_* of the form

$$[q_0, a_1, q_{i_1}][q_{i_1}, a_2, q_{i_2}] \ldots [q_{i_{n-1}}, a_n, q_{\text{accept}}]$$

where $a_1 \ldots a_n \in L_\Omega$, q_0, q_{accept} are the unique start and accept states of $\mathcal{N}$, and q_{i_j} are any possible choice of states of $\mathcal{N}$. Finally, define the homomorphism $\tau : \Sigma_* \to \Sigma$ by $\tau([q_i, a, q_j]) = a$, then by construction $\tau(K) = L_{\Omega_1} \cap L(\mathcal{N})$, and by item (1) $\tau(K)$ is E(D)T0L in $\mathsf{NSPACE}(s(n))$.

To prove item (4), we have $\varphi \colon \Gamma^* \to \Sigma^*$ and L_{Ω_1} given by the 4-tuple $(C, \Sigma, c_0, \mathcal{M})$. Assume without loss of generality that $\Gamma \cap C = \emptyset$. Let $K \subseteq (\Gamma \cup \Sigma)^*$ defined by

$$K = \{y \in (\Gamma \cup \Sigma)^* \mid y = z_0 x_1 z_1 \ldots x_k z_k, x_1 \ldots x_k \in L_{\Omega_1}, x_i \in \Sigma, z_i \in \Gamma^*\}$$

be a "padded" copy of L_{Ω_1}. Define a new extended alphabet $C' = C \cup \Gamma$, and define a non-deterministic table $h_0 = \{(a, xay) \mid a \in \Sigma, x, y \in \Gamma \cup \{\epsilon\}\}$ (and fixes any $a \in C' \setminus \Sigma$). Each table labelling an edge in $\mathcal{M}$ can be viewed as a table in $\mathscr{P}(C' \times (C')^*)$ (again by convention tables are the identity on letters not in Σ). Modify $\mathcal{M}$ by adding loops labelled by h_0 to each accept state to obtain $\mathcal{M}'$. The resulting system $(C', \Gamma \cup \Sigma, c_0, \mathcal{M}')$ is ET0L in the same space complexity as the initial system since the modifications depend only on Γ (which is fixed size) and C, and the language of this system in K.

Now consider the regular language $S = \{\varphi(y_1)y_1 \ldots \varphi(y_n)y_n \mid n \geqslant 1, y_i \in \Gamma\}$. Then Γ is of fixed size (not part of the input); also, the automaton accepting S

consists of state q_0 which is both the start and unique accept state, and paths starting and ending at q_0 labeled by $\varphi(y_i)y_i$, so is of fixed constant size. Thus by item (3) $S \cap K$ is ET0L in $\mathsf{NSPACE}(s(n))$. Finally, define a homomorphism $\tau : (\Gamma \cup \Sigma)^* \to \Sigma$ by $\tau(a) = a$ if $a \in \Gamma$ and $\tau(a) = \epsilon$ if $a \in \Sigma$. Then by item (2) $\tau(K \cap S)$ is ET0L in $\mathsf{NSPACE}(s(n))$, and by construction $\tau(K \cap S) = \phi^{-1}(L_\Omega)$. □

Notation 3.9. Let Σ be an alphabet with $\# \notin \Sigma$. Let $\mathcal{T} \subseteq \Sigma^*$ and $r \in \mathbb{N}$. Define $\mathcal{T}_{\#r} = \{u_1\# \dots \#u_r \mid u_i \in \mathcal{T}, 1 \leqslant i \leqslant r\}$. (Note that this agrees with the usage in Definition 2.2 item (iv).)

Proposition 3.10 (Projection onto a factor)**.** *Let Σ be an alphabet with $\# \notin \Sigma$, and $1 \leqslant k \leqslant \ell$. Suppose for each input Ω there is a language $L_\Omega \subseteq (\Sigma \cup \{\#\})^*$ and an integer $r_\Omega \geqslant 0$ so that $L_\Omega \subseteq (\Sigma^*)_{\#r_\Omega}$. In that case we can define*

$$L_{\Omega,k,\ell} = \{u_k\# \dots \#u_\ell \mid u_1\# \dots \#u_k\# \dots \#u_\ell\# \dots \#u_{r_\Omega} \in L_\Omega\}$$

if $\ell \leqslant r_\Omega$ and $L_{\Omega,k,\ell} = \emptyset$ otherwise. Then if L_Ω is E(D)T0L in $\mathsf{NSPACE}(s(n))$, then $L_{\Omega,k,\ell}$ is E(D)T0L in $\mathsf{NSPACE}(s(n))$.

Proof. Assume the E(D)T0L language L_Ω is given by the 4-tuple $(C, \Sigma\cup\{\#\}, c_0, \mathcal{M})$ which can be constructed in $\mathsf{NSPACE}(s(n))$. Construct a new E(D)T0L system $(C_*, \Sigma \cup \{\#\}, [1, c_*, r_\Omega], \mathcal{M}')$ as follows. The new alphabet is

$$C_* = \{[i, c, j], [1, c_*, r_\Omega] \mid c \in C, 1 \leqslant i \leqslant j \leqslant r_\Omega\} \cup \Sigma \cup \{\#\},$$

where $[1, c_*, r_\Omega]$ is a new seed letter, $c_* \notin C$. The notation $[i, c, j]$ is intended to indicate that the word in $(\Sigma \cup \{\#\})^*$ eventually produced in L_Ω by following some tables starting with c will be $v_i\#u_{i+1}\# \dots u_{j-1}\#v_j$, where v_i is a suffix of u_i and v_j is a prefix of u_j.

For each $(c, v) \in (C \cup \{c_*\}) \times C^*$, define $\mathfrak{r}(c, v)$ to be the following set: if $v = a_1 \dots a_n$ with $a_i \in C$, then $\mathfrak{r}(c, v) =$

$$\{([i, c, j], [i, a_1, s_1][i_1, a_2, i_2] \dots [s_{n-1}, a_n, j] \mid 1 \leqslant i \leqslant s_1 \leqslant \dots \leqslant s_{n-1} \leqslant j \leqslant r_\Omega\}.$$

Let p_0 be the start node of $\mathcal{M}$. Make a new start node p_*, and for each element $([1, c_*, r_\Omega], x) \in \mathfrak{r}(c_*, c_0)$ print an edge from p_* to p_0 labeled by the table which sends $[1, c_*, r_\Omega]$ to x. For each edge in $\mathcal{M}$ labeled by table h, print all possible edges labeled by tables obtained from h by replacing each rule (c, v) in h by some choice of element of $\mathfrak{r}(c, v)$. Note that the space required to print each edge is $\mathcal{O}(s(n))$. For each final state of $\mathcal{M}$, print an edge to a new final state q_* labeled by the homomorphism which for all $k \leqslant i \leqslant \ell, k \leqslant j < \ell, c \in \Sigma$ sends $[i, c, i]$ to c and $[j, \#, j + 1]$ to $\#$ (and is constant on all other letters).

The resulting language is exactly the set of factors of L_Ω as required, the space to print C_* and $\mathcal{M}'$ is $\mathcal{O}(s(n))$, and all tables labelling edges in $\mathcal{M}'$ are deterministic whenever all tables were in $\mathcal{M}$. □

4. Preliminaries – Hyperbolic groups

Recall that the *Cayley graph* for a group G with respect to a finite symmetric generating set S is the directed graph $\Gamma(G, S)$ with vertices labeled by $g \in G$ and a directed edge (g, h) labeled by $s \in S$ whenever $h =_G gs$. Let $\ell_{\Gamma(G,S)}(p)$, $i(p)$ and $f(p)$ be the length, initial and terminal vertices of a path p in the Cayley graph, respectively. A path p is *geodesic* if $\ell_{\Gamma(G,S)}(p)$ is minimal among the lengths of all paths q with the same endpoints. If x, y are two points in $\Gamma(G, S)$, we define $d_S(x, y)$ to be the length of a shortest path from x to y in $\Gamma(G, S)$. We use $d(x, y)$ and $\ell(p)$ if the group G and set S are clear from the context.

Definition 4.1 (δ-hyperbolic group (Gromov))**.** Let G be a group with finite symmetric generating set S, and let $\delta \geqslant 0$ be a fixed real number. If p, q, r are geodesic paths in $\Gamma(G, S)$ with $f(p) = i(q), f(q) = i(r), f(r) = i(p)$, we call $[p, q, r]$ a *geodesic triangle*. A geodesic triangle is *δ-slim* if p is contained in a δ-neighbourhood of $q \cup r$, that is, every point on one side of the triangle is within δ of some point on one of the other sides. We say (G, S) is *δ-hyperbolic* if every geodesic triangle in $\Gamma(G, S)$ is δ-slim. We say (G, S) is *hyperbolic* if it is δ-hyperbolic for some $\delta \geqslant 0$.

Lemma 4.2 (Dehn presentation, [1, Theorems 2.12 and 2.16])**.** *G is hyperbolic if and only if there is a finite list of pairs of words $(u_i, v_i) \in S^* \times S^*$ with $|u_i| > |v_i|$ and $u_i =_G v_i$ such that the following holds: if $w \in S^*$ is equal to the identity of G then it contains some u_i as a subword.*

This gives an algorithm to decide whether or not a word $w \in S^*$ is equal to the identity: while $|w|_S > 0$, look for some u_i subword. If there is none, then $w \neq_G 1$. Else replace u_i by v_i (which is shorter). This procedure is called *Dehn's algorithm*. The following is immediate.

Lemma 4.3. *Dehn's algorithm runs in linear space.*

Definition 4.4 (Quasigeodesic)**.** Let (X, d) be a metric space. For $\lambda \geqslant 1, \mu \geqslant 0$ real numbers, a path p in (X, d) is an (X, λ, μ)-*quasigeodesic* if for any subpath q of p we have $\ell(q) \leqslant \lambda d(i(q), f(q)) + \mu$. Let $Q_{X,\lambda,\mu}$ be the set of all (X, λ, μ)-quasigeodesics.

When (X, d) is the Cayley graph of a group G with respect to a generating set S, we use the notation (G, S, λ, μ)-*quasigeodesic*, and denote by

$$Q_{G,S,\lambda,\mu} = Q_{\Gamma(G,S),\lambda,\mu} \cap S^*$$

the set of all (λ, μ)-quasigeodesic paths between vertices of the Cayley graph, or equivalently, paths corresponding to words over S.

Lemma 4.5 (Change of generating set is a quasi-isometry; folklore)**.** *Let S_1, S_2 be two finite symmetric generating sets for a group G, $\pi_i : S_i^* \mapsto G$ the natural projection maps, and $\psi : S_1^* \to S_2^*$ the monoid morphism which satisfies $\pi_1(s) = \pi_2(\psi(s))$ for all $s \in S_1$. Then for each $\lambda \geqslant 1, \mu \geqslant 0$ there exist λ', μ' so that if p is a (G, S_1, λ, μ)-quasigeodesic, then $\psi(p)$ is a (G, S_2, λ', μ')-quasigeodesic.*

Throughout this article, we assume G is a fixed hyperbolic group with finite generating set S which we treat as a constant for complexity purposes. We also assume we are given the constant δ, the finite list of pairs (u_i, v_i) for Dehn's algorithm, and any other constants depending only on the group, for example the constants λ_G, μ_G in Proposition 7.1 below.

4.1. **Languages in hyperbolic groups.** Let Λ be an alphabet and \$ a symbol not in Λ. An *asynchronous 2-tape automaton* is a finite state automaton $\mathcal{M}$ with alphabet

$$\mathcal{X} = \left\{ \begin{pmatrix} s \\ t \end{pmatrix}, \begin{pmatrix} s \\ \$ \end{pmatrix}, \begin{pmatrix} \$ \\ s \end{pmatrix} \;\middle|\; s, t \in \Lambda \right\}.$$

The 2-tape automaton is *synchronous* if it only accepts words

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdots \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

with either $x_1 \dots x_n \in \Lambda^*$ and $y_1 \dots y_n \in \Lambda^*\{\$\}^*$; or $x_1 \dots x_n \in \Lambda^*\{\$\}^*$ and $y_1 \dots y_n \in \Lambda^*$.

Define homomorphisms $\xi_1, \xi_2 \colon \mathcal{X} \to (\Lambda \cup \Lambda)^*$ by

$$\xi_1 \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{cases} x & x \in \Lambda \\ 1 & x = \$ \end{cases} \quad \text{and} \quad \xi_2 \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{cases} y & y \in \Lambda \\ 1 & y = \$ \end{cases}.$$

We say the pair $(u,v) \in \Lambda^2$ is accepted by the (synchronous or asynchronous) 2-tape automaton if there is a word

$$\binom{x_1}{y_1}\dots\binom{x_n}{y_n}$$

accepted by $\mathcal{M}$ such that $u = \xi_1(x_1 \dots x_n)$ and $v = \xi(y_1 \dots y_n)$. See [21, Definition 1.4.5 and §7.1] and [26, §2] for equivalent formulations.

It is standard practice for finitely generated groups to fix an order on the generators, order which can be extended to (a lexicographic order for) all words over that generating set, and then for each group element to choose a *shortlex representative*; that is, choose the smallest word, in the lexicographic order, among all the words representing the group element.

Proposition 4.6 (See [21, 26])**.** *Let G be a fixed hyperbolic group with finite generating set S, and $\lambda \geqslant 1, \mu \geqslant 0$ constants with $\lambda \in \mathbb{Q}$. Then the following sets are regular languages.*

(1) The set of all geodesics over S.
(2) The set of all shortlex geodesics over S.
(3) The set of quasigeodesics $Q_{G,S,\lambda,\mu} \subseteq S^*$.

Furthermore, the set of all pairs of words $(u,v) \in Q^2_{G,S,\lambda,\mu}$ such that $u =_G v$ is accepted by an asynchronous 2-tape automaton.

Note that Holt and Rees [26] show more: if $\lambda \notin \mathbb{Q}$ then the set $Q_{G,S,\lambda,\mu}$ is never regular, and in the special case that (λ,μ) are "exact" then the 2-tape automaton is synchronous. We do not need these sharper results here.

4.2. **Quasi-isometrically embedded rational constraints.** Let G be a group generated by a symmetric set S, and let $\pi : S^* \to G$ be the natural projection as before. Recall that for an element $g \in G$ we denote the geodesic length of g with respect to the word metric for S by $\|g\|_S$.

Definition 4.7 (see [12])**.** (1) A regular language $\widetilde{R} \subseteq S^*$ is (λ,μ)-*quasi-isometrically embedded* in G if for any $w \in \widetilde{R}$

$$\|\pi(w)\|_S \geqslant \frac{1}{\lambda}\,|w|_S - \mu.$$

(2) A rational set $R \subseteq G$ is *quasi-isometrically embeddable* in G if there exist real numbers $\lambda \geqslant 1$ and $\mu \geqslant 0$ and a (λ,μ)-quasi-isometrically embedded regular language $\widetilde{R} \subseteq S^*$ such that $\pi(\widetilde{R}) = R$. In this case we say that R is *quasi-isometrically embeddable rational*, abbreviated as *qier*.

For the purposes of our complexity and computability results, we also define the following.

Definition 4.8 (Effective and explicit)**.** A rational set R is said to be *effective quasi-isometrically embeddable rational* (abbreviated as *effective qier*) if we are given an NFA A such that there exist constants $\lambda \geqslant 1, \mu \geqslant 0$ with $L(A) \subseteq Q_{G,S,\lambda,\mu}$ and $\pi(L(A)) = R$. The set R is said to be *explicit qier* if, in addition, the constants $\lambda \geqslant 1, \mu \geqslant 0$ are also given.

In [12], decidability statements use the notion of effective qier as defined here. Moreover, Dahmani and Guirardel show that given an effective qier set, there is an algorithm to compute constants λ,μ such that the set is explicit qier with respect to these constants, see Section 6 below.

4.3. **The Rips complex.** For any metric space $X = (X, d)$ and constant r, the *Rips complex* with parameter r is the simplicial complex whose vertices are the points of X and whose simplices are the finite subsets of X whose diameter is at most r. More specifically, we will need the Rips complex which is based on the Cayley graph of a hyperbolic group, as follows.

Definition 4.9 (Rips Complex)**.** Let G be a hyperbolic group with finite generating set S. For a fixed constant r, the Rips complex $\mathcal{P}_r(G)$ is a simplicial complex defined as the collection of sets with diameter $\leqslant r$ (with respect to the d_S-distance in $\Gamma(G, S)$):

$$\mathcal{P}_r(G) = \{Y \subset G \mid Y \neq \emptyset, \;\; \mathrm{diam}_S(Y) \leqslant r\},$$

where each set $Y \in \mathcal{P}_r(G)$ of cardinality $k+1$ is identified with a k-simplex whose vertex set is Y.

The Rips complex of a hyperbolic group has several important properties, relevant to this paper. The group G acts properly discontinuously on the Rips complex $\mathcal{P}_r(G)$, the quotient $\mathcal{P}_r(G)/G$ is compact, and $\mathcal{P}_r(G)$ is contractible. In Section 8 we will work with the barycentric subdivision of $\mathcal{P}_r(G)$, which we recall next.

Definition 4.10 (Barycentric subdivision)**.** Let σ be a simplicial complex. For a simplex $\tau = \{v_0, v_1, \dots, v_q\} \in \sigma$ denote by b_τ its *barycentre*, and for two simplices α, β in σ write $\alpha < \beta$ to denote that α is a face of β.

The *barycentric subdivision* B_σ of a simplicial complex σ is the collection of all simplices whose vertices are $b_{\sigma_0}, \dots, b_{\sigma_r}$ for some sequence $\sigma_0 < \cdots < \sigma_r$ in σ.

The set of vertices of B_σ is the set of all barycentres of simplices of σ, B_σ has the same dimension as σ, and any vertex in $B_\sigma \setminus \sigma$ is connected to a vertex in σ.

5. Doubling and copying

In computing the full solution set to equations as shortlex geodesic words, we will need to take inverse homomorphism. Even though in general the image under an inverse homomorphism of an EDT0L language is just ET0L, because of the special structure of solution sets we can apply the *Copying Lemma* of Ehrenfeucht and Rozenberg [18] to show the statement in Proposition 5.5. This is indeed a *trick* – without it we would only be able to state our main structural results (solutions as shortlex geodesics) at ET0L languages, whereas EDT0L is a much smaller class and hence we have stronger statements.

Lemma 5.1 (Copying Lemma, [18, Theorem 1])**.** *Let Σ_1, Σ_2 be two finite disjoint alphabets, $K_1 \subseteq \Sigma_1^*$ and $K_2 \subseteq \Sigma_2^*$. Let f be a bijective function from K_1 onto K_2. Let $K = \{wf(w) \mid w \in K_1\}$. If K is ET0L, then K, K_1, K_2 are each EDT0L.*

Lemma 5.2 (Copying in NSPACE)**.** *Let $s\colon \mathbb{N} \to \mathbb{N}$ be a function. Let Σ_1, Σ_2 be two finite disjoint alphabets, and $f\colon \Sigma_1 \to \Sigma_2$ a bijection which extends to a monoid homomorphism $f\colon \Sigma_1^* \to \Sigma_2^*$ of the same name. Suppose that on input Ω languages $(K_\Omega)_1 \subseteq \Sigma_1^*$ and*

$$K_\Omega = \{wf(w) \mid w \in (K_\Omega)_1\}$$

are produced. If K_Ω is ET0L in $\mathsf{NSPACE}(s(n))$, then $K_\Omega, (K_\Omega)_1$ (and $f((K_\Omega)_1)$) are each EDT0L in $\mathsf{NSPACE}(s(n))$.

Proof. Following Steps 1–4 in the proof of [18, Theorem 1], we see that each nondeterministic table in the grammar for K is replaced by a finite number (independent of Ω) of deterministic tables, with symbols superscripted by $(1), (2), (m)$, $(m:1)$, and $(m:2)$, where the letter m stands for "middle" (not an integer). It follows that all tables for the EDT0L grammars for $K_\Omega, (K_\Omega)_1$ and $f((K_\Omega)_1)$ can be printed in $\mathcal{O}(s(n))$ space. □

We will also make use of the following fact.

Lemma 5.3 (Doubling in NSPACE). *Let $s\colon \mathbb{N} \to \mathbb{N}$ be a function. Let Σ_1, Σ_2 be two finite disjoint alphabets, and $f\colon \Sigma_1 \to \Sigma_2$ a bijection which extends to a monoid homomorphism $f\colon \Sigma_1^* \to \Sigma_2^*$ of the same name. If $L_\Omega \subseteq \Sigma_1^*$ is EDT0L in $\mathsf{NSPACE}(s(n))$ then $(L_\Omega)_D = \{wf(w) \mid w \in L_\Omega\}$ is EDT0L in $\mathsf{NSPACE}(s(n))$.*

Proof. Modify the grammar for L_Ω as follows. Replace the seed word c_0 by $c_0f(c_0)$, and for each rule (a, u) in a table, add $(f(a), f(u))$ to the table. These modifications are clearly in the same space bound and tables remain deterministic since alphabets are disjoint. □

Note that if the language L_Ω in Lemma 5.3 were ET0L and not EDT0L, then the proof no longer works, since tables in the ET0L grammar could make different substitutions to letters in the prefix than in the suffix. For this reason[5] we cannot simply double any ET0L language and then apply Lemma 5.1 to prove that it is EDT0L. We remark that [20, Theorem 3.3] is a slightly stronger statement of copying, but we don't use this here.

Here is our key technical result for languages of words over hyperbolic groups. It will show how we can build, from a given covering solution set of quasigeodesic words which is ET0L, an ET0L language consisting of all words in some regular language $\mathcal{T}$ which correspond to one of the covering solutions, without increasing the amount of space required. In the case that $\mathcal{T}$ is a set of normal forms for the group (unique representative for each element), the Copying Lemma (Lemma 5.2) ensures that the resulting language is in fact EDT0L.

First we fix some notation to be used throughout the paper.

Notation 5.4. Let G be a hyperbolic group with finite symmetric generating set S and natural projection map $\pi : S^* \to G$. Let $S_\dagger = \{x_\dagger \mid x \in S\}$ be a disjoint copy of the alphabet S where every letter is marked with a subscript †, and define a bijective function $f\colon S \to S_\dagger$ by $f(x) = x_\dagger$, which we can extend to the free monoid homomorphism $f\colon S^* \to S_\dagger^*$. Then $\pi_\dagger = \pi \circ f^{-1}$ is a map from $S_\dagger^*$ to G, and we can formally consider $S_\dagger$ as a generating set for G with projection map $\pi_\dagger$.

Proposition 5.5 (Covering to full sets). *Let G be a hyperbolic group with finite symmetric generating set S. For $\#$ a symbol not in S, extend the bijection $f\colon S \to S_\dagger$ defined in Notation 5.4 to include $f(\#) = \#$.*

Let $\lambda, \lambda', \mu, \mu' \in \mathbb{R}$ be given fixed constants with $\lambda, \lambda' \geqslant 1$, $\mu, \mu' \geqslant 0$ and $\lambda' \in \mathbb{Q}$. Let $\mathcal{T} \subseteq Q_{G,S,\lambda',\mu'}$ be a fixed regular language of (G, S, λ', μ')-quasigeodesics, and r be a fixed positive integer.

Suppose some language

$$L_{\mathrm{cover}} \subseteq \{u_1\#\dots\#u_r f(v_1\#\dots\#v_r) \mid u_i, v_i \in Q_{G,S,\lambda,\mu}, u_i =_G v_i, 1 \leqslant i \leqslant r\}$$

is ET0L in $\mathsf{NSPACE}(s(n))$. Then

(1) if the projection $\pi\colon \mathcal{T} \to G$ is a surjection, then

$$L_{\mathcal{T}} = \{w_1\#\dots\#w_r \mid \exists u_1\#\dots\#u_r f(v_1\#\dots\#v_r) \in L_{\mathrm{cover}}, w_i =_G u_i, w_i \in \mathcal{T}\}$$

is ET0L in $\mathsf{NSPACE}(s(n))$.

(2) if the projection $\pi\colon \mathcal{T} \to G$ is a bijection, then

$$L_{\mathcal{T}} = \{w_1\#\dots\#w_r \mid \exists u_1\#\dots\#u_r f(v_1\#\dots\#v_r) \in L_{\mathrm{cover}}, w_i =_G u_i, w_i \in \mathcal{T}\}$$

is EDT0L in $\mathsf{NSPACE}(s(n))$.

Moreover, in both cases $L_{\mathcal{T}}$ is finite (resp. empty) if and only if L_{cover} is finite (resp. empty).

[5] and more obviously, because we know EDT0L is a proper subclass of ET0L [20]

Note the requirement that $\lambda' \in \mathbb{Q}$, this is because we make use of the fact that the set of all quasigeodesics in a hyperbolic group gives an asynchronous automatic structure which only holds when the constant λ' is rational by [26]. The value of λ' is independent of λ (can be larger or smaller).

Proof. Let $\Sigma_1 = S \cup \{\#\}$, and let $\Sigma_2 = S_\dagger \cup \{\#_\dagger\}$ be a disjoint copy of Σ_1. Choose $\lambda_*, \mu_* \geqslant 0$ so that $\lambda_* \in \mathbb{Q}$, $\lambda_* \geqslant \max\{\lambda, \lambda'\}$ and $\mu_* \geqslant \max\{\mu, \mu'\}$. Recall that $\mathcal{T}_{\#r} = \{u_1\# \dots \#u_r \mid u_i \in \mathcal{T}\}$. Observe that if $\mathcal{T}$ is regular and of fixed constant size, then so is $\mathcal{T}_{\#r}$: take r disjoint copies of the automaton accepting $\mathcal{T}$ and join by $\#$ transitions each accept state of the ith copy to the start state of the $(i+1)$st copy.

Let $\$$ denote a 'padding symbol' that is distinct from $\Sigma_1 \cup \Sigma_2$, and let

$$\mathcal{P} = \left\{ \binom{s}{t}, \binom{s}{\$}, \binom{\$}{s}, \binom{\#}{\#}, \binom{s_\dagger}{t_\dagger}, \binom{s_\dagger}{\$}, \binom{\$}{s_\dagger}, \binom{\#_\dagger}{\#_\dagger} \;\middle|\; s, t \in S, s_\dagger, t_\dagger \in \Sigma_2 \right\}.$$

Define a homomorphism $\psi \colon \mathcal{P}^* \to (\Sigma_1 \cup \Sigma_2)^*$ by

$$\psi\left(\binom{x}{y}\right) = \begin{cases} 1 & x = \$ \\ x & x \in \Sigma_1 \cup \Sigma_2 \end{cases}$$

Then $\psi^{-1}(L_{\text{cover}})$ is ET0L in $\mathsf{NSPACE}(s(n))$ by Proposition 3.8(4), and consists of strings of letters which we can view as two parallel strings: the top string is a word from L_{cover} with $\$$ symbols inserted, and the bottom string can be any word in $\Sigma_1 \cup \Sigma_2 \cup \{\$\}$ with $\#, \#'$ occurring in exactly the same positions as the top string, and no two $\$$ symbols in the same position top and bottom.

Let $\mathcal{M}$ be the asynchronous 2-tape automaton which accepts all pairs $(u, v) \in Q_{G,S,\lambda_*,\mu_*}$ with $u =_G v$ as guaranteed by Proposition 4.6. Construct a new automaton $\mathcal{M}_1$ by adding an edge labeled $\binom{\#}{\#}$ from each accept state of $\mathcal{M}$ to the start state. The new automaton $\mathcal{M}_1$ accepts padded pairs of (G, S, λ_*, μ_*)-quasigeodesic words (u_i, v_i) with $u_i =_G v_i$, and each pair is separated by $\binom{\#}{\#}$. Make a copy $\mathcal{M}_2$ of $\mathcal{M}_1$ where each letter from Σ_1 is replaced by its corresponding marked letter from Σ_2. The automaton $\mathcal{M}_2$ accepts padded pairs of $(\Gamma(G, S_\dagger), \lambda_*, \mu_*)$-quasigeodesic words (u_i, v_i) with $u_i =_G v_i$, and each pair is separated by $\binom{\#_\dagger}{\#_\dagger}$.

Make a new automaton $\mathcal{M}_3$ by attaching edges labeled ε from the accept states of $\mathcal{M}_1$ to the start state of $\mathcal{M}_2$. Note that the size of $\mathcal{M}_3$ is constant.

We now take $L_1 = L(\mathcal{M}_3) \cap \psi^{-1}(L_{\text{cover}})$. This is again ET0L in $\mathsf{NSPACE}(s(n))$ by Proposition 3.8(3). The language L_1 can be seen as pairs of strings, the top string a padded version of a string in L_{cover} which has the form

$$w_1\# \dots \#w_r f(z_1)\#_\dagger \dots \#_\dagger f(z_r)$$

and the bottom of the form

$$u_1\# \dots \#u_r f(v_1)\#_\dagger \dots \#_\dagger f(v_r)$$

with $w_i =_G u_i =_G v_i$ and u_i, v_i (G, S, λ_*, μ_*)-quasigeodesics for all $1 \leqslant i \leqslant r$.

Define a homomorphism $\xi \colon L_1 \to (\Sigma_1 \cup \Sigma_2)^*$ by

$$\xi\left(\binom{x}{y}\right) = \begin{cases} 1 & y = \$ \\ y & y \in \Sigma_1 \cup \Sigma_2 \end{cases}$$

Then by Proposition 3.8(2), $\xi(L_1)$ is ET0L in $\mathsf{NSPACE}(s(n))$, and consists of all possible strings of the form

$$u_1\# \dots \#u_r f(v_1)\#_\dagger \dots \#_\dagger f(v_r) = u_1\# \dots u_r f(v_1\# \dots \#v_r)$$

with u_i, v_i $G, S, \lambda, \mu)$-quasigeodesics, $u_i =_G v_i$, and such that there exists some $w_1\# \dots \#w_r f(z_1\# \dots \#z_r) \in L$ with $w_i =_G u_i$.

Finally, $L_2 = \xi(L_1) \cap \mathcal{T}_{\#r}$ is ET0L in $\mathsf{NSPACE}(s(n))$ since $\mathcal{T}_{\#r}$ is regular and constant size, and since $\mathcal{T}$ a subset of $Q_{G,S,\lambda',\mu'} \subseteq Q_{G,S,\lambda_*,\mu_*}$ since $\lambda' \leqslant \lambda_*$ and $\mu' \leqslant \mu_*$, L_2 consists of all words of the form

$$u_1\# \dots u_r f(v_1\# \dots \#v_r)$$

for all possible $u_i, v_i \in \mathcal{T}$ where $u_i =_G v_i$ such that there exists some

$$w_1\# \dots \#w_r f(z_1\# \dots \#z_r) \in L$$

with $w_i =_G u_i$. Apply the homomorphism $\chi\colon (\Sigma_1 \cup \Sigma_2)^* \to \Sigma_1^*$ defined by

$$\chi(x) = \begin{cases} 1 & x \in \Sigma_2 \\ x & x \in \Sigma_1 \end{cases}$$

to obtain $\chi(L_2) = L_{\mathcal{T}}$ is ET0L in $\mathsf{NSPACE}(s(n))$. This establishes the first item.

If $\mathcal{T}$ is in bijection with G, we have u_i and v_i are identical words, thus $L_{\mathcal{T}}$ consists of words of the form $u_1\# \dots \#u_r f(u_1\# \dots \#u_r)$. Thus by Lemma 5.2, $L_{\mathcal{T}}$ is EDT0L in $\mathsf{NSPACE}(s(n))$. This establishes the second item.

For the final result, let

$$L_G = \{(g_1, \dots, g_r) \mid \exists w_1\# \dots \#w_r f(z_1\# \dots \#z_r) \in L[w_i =_G g_i]\}.$$

Then since for each group element there are a finite number of (G, S, λ, μ)-quasigeodesic words representing it, L_{cover} is finite (resp. empty) if and only if L_G is finite (resp. empty), and since for each group element there are a finite number of (G, S, λ', μ')-quasigeodesic words representing it, L_G is finite (resp. empty) if and only if $L_{\mathcal{T}}$ is finite (resp. empty). □

5.1. **Alternative generating sets for free and virtually free groups.** Previous work [7, 14] expresses the full set of (group element) solutions to systems in terms of specific normal forms: freely reduced words in free groups or the standard normal forms of [14, Definition 14.1] for virtually free groups. However, we can express the solutions more generally, in terms of arbitrary languages of quasigeodesics, as E(D)T0L in the same space complexity as in [7, 14], by using Proposition 5.5.

We first observe that the standard normal forms used in [14] are quasigeodesics.

Proposition 5.6 (Standard normal forms are quasigeodesics)**.** *Let V be a virtually free group, F a free normal subgroup of V, and H a finite quotient that satisfy the exact short sequence*

$$1 \to F \to V \to H \to 1.$$

Consider a symmetric generating set $Y = A \cup (H \setminus \{1_H\})$ for V, where A is a free basis generating set for F.

The set $\{uh \mid u \in A^*$ freely reduced over $A, h \in H \setminus 1_H\}$ is a set of unique (V, Y, λ_Y, μ_Y)-quasigeodesic representatives for V for some $\lambda_Y \geqslant 1, \mu_Y \geqslant 0$.

Proof. Since F is normal in V, for each $a \in A, h \in H$ we can write $hah^{-1} = u_{a,h}$ where $u_{a,h} \in A^*$ is some fixed choice of word. Let $m = \max\{|u_{a,h}| \mid a \in A, h \in H\}$ and $h'_i =_V h_0 h_1 \dots h_i$.

Suppose $g \in V$ has standard normal form uh, and suppose $v = h_0 a_1 h_1 \cdots a_k h_k$ is a geodesic for g, where $a_i \in A$ and $h_i \in H \cup \{1_H\}$. We have $|v|_Y \geqslant k$. Now push the h_i to the right:

$$h_0 a_1 h_1 a_2 \dots h_k =_V u_{a_1,h_0} h_0 h_1 a_2 \dots h_k =_V u_{a_1,h_0} h'_1 a_2 \dots h_k =$$
$$u_{a_1,h_0} u_{a_2,h'_1} h'_2 \dots h_k = \cdots =_V u'h,$$

where $|u'h|_Y \leqslant mk + 1 \leqslant m|v|_Y + 1 = m\,\|g\|_Y + 1$. Since $|u|_Y \leqslant |u'|_Y$ we get that $\|g\|_Y \leqslant |uh|_Y \leqslant m\,\|g\|_Y + 1$, so uh is quasigeodesic in V for some constants $\lambda_Y \geqslant 1, \mu_Y \geqslant 0$. □

Then Proposition 5.5 implies the following, which extends previous results for free and virtually free groups by both allowing arbitrary generating sets S instead of very specific ones, and by expressing the solutions in terms of more general quasigeodesics instead of particular normal forms.

Corollary 5.7 (Free and virtually free with arbitrary generating sets)**.** *Let G be a free [respectively virtually free] group with finite symmetric generating set S. Let $\mathcal{T} \subseteq Q_{G,S,\lambda,\mu}$ be a regular language of (G, S, λ, μ)-quasigeodesic words over S surjecting to G for some fixed arbitrary values of $\lambda \geqslant 1, \mu \geqslant 0, \lambda \in \mathbb{Q}$.*

Let Φ be a system of equations and inequations of size n with constant size rational constraints, as in Definition 2.1. Then

(1) *the full set of $\mathcal{T}$-solutions is ET0L, and the algorithm which on input Φ prints a description for the ET0L grammar runs in $\mathsf{NSPACE}(n \log n)$ [resp. $\mathsf{NSPACE}(n^2 \log n)$];*
(2) *if $\mathcal{T}$ is in bijection with G, then the full set of $\mathcal{T}$-solutions is EDT0L, and the algorithm which on input Φ prints a description for the EDT0L grammar runs in $\mathsf{NSPACE}(n \log n)$ [resp. $\mathsf{NSPACE}(n^2 \log n)$].*

Proof. Let S_1 be a free basis generating set for a free group F and R_1 the set of all freely reduced words over $S_1 \cup S_1^{-1}$. We have R_1 are $(G, S_1 \cup S_1^{-1}, 1, 0)$-quasigeodesics. By Theorem 3.5 the set of all solutions to Φ as words in R_1 is EDT0L in $\mathsf{NSPACE}(n \log n)$. Applying the homomorphism which sends each generator to a word in S^*, we obtain (by Proposition 3.8(2) and Lemma 4.5) a covering solution of words over S^* of (G, S, λ', μ')-quasigeodesics for some λ', μ'. Next, apply Lemma 5.3 to obtain a doubled copy which is EDT0L also in $\mathsf{NSPACE}(n \log n)$. We can now input this into Proposition 5.5 to obtain the result.

For systems over virtually free groups, let Y be the generating set used in [14] and R_1 the set of standard normal forms over Y. By Proposition 5.6 we have that standard normal forms over Y are quasigeodesics. By Theorem 3.6 the set of all solutions to Φ as words in R_1 is EDT0L in $\mathsf{NSPACE}(n^2 \log n)$. Applying the homomorphism which sends each letter of Y to a word in S^*, we obtain (by Proposition 3.8(2) and Lemma 4.5) a covering solution of words over S^* of (G, S, λ', μ')-quasigeodesics for some λ', μ'. Next, apply Lemma 5.3 to obtain a doubled copy which is EDT0L also in $\mathsf{NSPACE}(n^2 \log n)$. We can now input this into Proposition 5.5 to obtain the result. □

6. A key fact about quasi-isometrically embeddable rational constraints and complexity

In this paper we will make use of the following result of Dahmani and Guirardel, which we restate here together with a note about complexity.

Proposition 6.1 (see [12, Proposition 9.4])**.** *Let G be a hyperbolic group with finite symmetric generating set S, and let $\mathcal{R} \subseteq G$ be an effective qier set (recall this means we are given an NFA A such that there exist constants $\lambda_0 \geqslant 1, \mu_0 \geqslant 0$ with $L(A) \subseteq Q_{G,S,\lambda_0,\mu_0}$ and $\pi(L(A)) = \mathcal{R}$). Then for $\lambda \geqslant 1, \mu \geqslant 0$ fixed given constants, there is an algorithm which computes an automaton A' that accepts the language*

$$\widetilde{\mathcal{R}} = \pi^{-1}(\mathcal{R}) \cap Q_{G,S,\lambda,\mu}.$$

If the set $\mathcal{R}$ is just effective qier and not explicit, then in general there is no computable bound on the number of steps that the subroutine must make before it terminates.

Proof. The main statement is proved in full detail in [12, Proposition 9.4]. We start by assuming we have an NFA A such that there exist constants $\lambda_0 \geqslant 1, \mu_0 \geqslant 0$ with

$L(A) \subseteq Q_{G,S,\lambda_0,\mu_0}$ and $\pi(L(A)) = \mathcal{R}$. We don't assume we are given the λ_0, μ_0 *a priori*, just that they exist. A key step in their algorithm is to calculate some constants which can play the roles of λ_0, μ_0, which is done by enumerating larger and larger values of constants λ_1, μ_1 until a certain condition is satisfied. The proof of [12, Proposition 9.4] gives no bound on how many steps or how much space this would take; all it needs is that the enumeration is guaranteed to terminate since we are promised that $L(A) \subseteq Q_{G,S,\lambda_0,\mu_0}$ for some λ_0, μ_0. The automaton A' is then constructed using constants λ', μ' which themselves depend on λ_1, μ_1. The size of A' is a function of the constants λ', μ' and $|A|$. If $|A|$ is considered a considered in the context of a larger algorithm, then even though this function may be large (or even not bounded, as we discuss in the next paragraph), the complexity of this subroutine to compute A' is considered to be constant. In the case of an explicit qier set, where the λ_0, μ_0 are also specified, and also considered constants in the context of a larger algorithm, then it is clear that the function taking inputs $|A|, \lambda_0, \mu_0$ is also constant. In the case that λ_0, μ_0 are not given, it may seem dishonest to claim the subroutine makes no contribution, but since this is a standard convention in complexity theory and it leads to stronger statements, we follow this convention.

Suppose hypothetically there was a computable function $s : \mathbb{N} \to \mathbb{R}$ such that on input (G, λ, μ, A), some algorithm was able to compute an automaton A' accepting $\widetilde{\mathcal{R}} = \pi^{-1}(R) \cap Q_{G,S,\lambda,\mu}$. This leads to a contradiction: let H be a finitely generated subgroup of G with generators $u_1, \ldots, u_n \in S^*$ and construct its Stallings graph, which is an automaton $\mathcal{M}$ with $\pi(L(\mathcal{M})) = H$ (by folding the bouquet of loops labeled by the u_i). Now *a priori* we do not know whether or not H is quasi-isometrically embedded. Consider the system Φ consisting of a single equation $X = X$ and constraint $X \in L(\mathcal{M})$. Run the hypothetical algorithm to solve Φ via constructing the automata accepting all (G, S, λ, μ)-quasigeodesic words corresponding to the constraint, and if it does not produce an answer within space $\mathcal{O}(s(n))$ (equivalently in time $2^{\mathcal{O}(s(n))}$ steps) then we know H was not quasi-isometrically embedded, and if it does then H is quasi-isometrically embedded and we have even found the appropriate constants. However, by [5] the problem of deciding whether or not H is quasi-isometrically embedded for an arbitrary hyperbolic group is undecidable, so there is no computable bound on the number of steps needed to determine whether H is not quasi-isometrically embedded. □

In this paper we consider systems of equations and inequations with qier constraints given effectively by an NFA for each variable, as in Definition 2.1. If we consider the sizes of the NFAs to be constant, with the input size depending only on the size of the equations and inequations, then when running the algorithm in the previous proposition as a subroutine inside a larger algorithm, it is standard practice in complexity theory to consider the subroutine as requiring only constant space since its complexity depends solely on the sizes of the input NFAs. That is, even though *a priori* we cannot bound the number of steps that may be needed to compute λ_0, μ_0, the subroutine only depends on items considered to be constant. For this reason, the statements of our main theorems refer to constant size effective qier constraints, rather than just explicit.

7. Equations and their solutions in torsion-free hyperbolic groups

In this section we prove Theorem 7.4, which characterises the solutions of equations in torsion-free hyperbolic groups as EDT0L in PSPACE. The first and main part of the algorithm for getting the solutions is Proposition 7.3, where by Theorem 7.2 we obtain a covering solution set of quasigeodesic words, but not yet as shortlex or other representatives. Proposition 5.5 then allows us get the solutions as shortlex representatives from the set produced in Proposition 7.3. The algorithm relies on the existence and properties of canonical representatives, explained below, which guarantee that the solutions of any system in a torsion-free hyperbolic group with generating set S can be found by solving an associated system in the free group on S.

7.1. **Canonical representatives.** Canonical representatives of elements in torsion-free hyperbolic groups were defined by Rips and Sela in [39], and we refer the reader to [39] for their construction, and to [8, Appendix C] for a basic exposition.

Let G be a torsion-free hyperbolic group with generating set S, and let $g \in G$. For a fixed integer $T \geqslant 1$, called the *criterion*, the canonical representative $\theta_T(g)$ of g with respect to T is a (G, S, λ, μ)-quasigeodesic word over S which satisfies $\theta_T(g) =_G g$, $\theta_T(g)^{-1} = \theta_T(g^{-1})$. If T is well-chosen, a number of combinatorial stability properties make canonical representatives with respect to T particularly suitable for solving triangular equations in hyperbolic groups, as in Theorem 7.2.

It is essential for the language characterisations in our main results that canonical representatives with respect to T are (G, S, λ, μ)-quasigeodesics, with λ and μ depending only on δ, and not on T.

Proposition 7.1 (see [11, Proposition 3.4])**.** *There are constants $\lambda_G \geqslant 1$ and $\mu_G \geqslant 0$ depending only on G such that for any criterion T and any element $g \in G$ the canonical representative $\theta_T(g)$ of g is a (G, S, λ_G, μ_G)-quasigeodesic.*

The following result provides the main reduction of a system of equations in a torsion-free hyperbolic group to a system over the free group with the same basis.

Theorem 7.2 ([39, Theorem 4.2, Corollary 4.4])**.** *Let G be a torsion-free δ-hyperbolic group generated by set S, and let $\Phi = \{\phi_j(\mathcal{X}, \mathcal{A}) = 1\}_{j=1}^{q}$ be a system of triangular equations, that is, $\phi_j(\mathcal{X}, \mathcal{A}) = X_{(j,1)}X_{(j,2)}X_{(j,3)} = 1$, where $1 \leqslant j \leqslant q$ and $(j, a) \in \{1, \dots l\}$.*

There exists an effectively computable criterion T, a constant $b = b(\delta, q)$ depending on δ and linearly on q such that: if $(g_1, \dots, g_l) \in Sol_G(\Phi)$, then there exist $y_{(j,a)}, c_{(j,a)} \in F(S)$, $1 \leqslant j \leqslant q, 1 \leqslant a \leqslant 3$, such that

$$|c_{(j,a)}|_S \leqslant b \text{ and } c_{(j,1)}c_{(j,2)}c_{(j,3)} =_G 1,$$

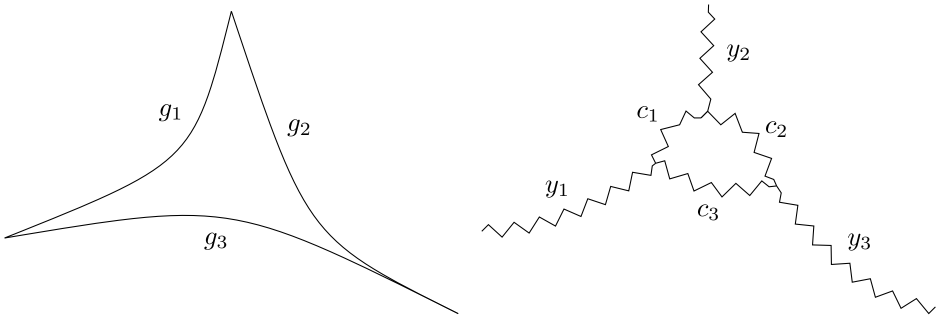
for which the canonical representatives satisfy:

$$\begin{aligned} \theta_T(g_{(j,1)}) &= y_{(j,1)}c_{(j,1)}(y_{(j,2)})^{-1}, \\ \theta_T(g_{(j,2)}) &= y_{(j,2)}c_{(j,2)}(y_{(j,3)})^{-1}, \\ \theta_T(g_{(j,3)}) &= y_{(j,3)}c_{(j,3)}(y_{(j,1)})^{-1}. \end{aligned}$$

The following explains how Theorem 7.2 is employed.

7.2. **Reduction from torsion-free hyperbolic to free groups: an overview.** The first part of the algorithm that finds solutions to a system of equations and inequations is to reduce it to a triangular one, that is, where each equation has length 3. In the free group $F(S)$, a triangular equation such as $XY = Z$ has a solution in reduced words over S if and only if there exist words P, Q, R with $X = PQ, Y = Q^{-1}R, Z = PR$ where no cancellation occurs between P and Q, Q^{-1} and R, and P and R. (This allows the translation of equations in free groups into equations in free monoids with involution, as in [7, Lemma 4.1]).

In a hyperbolic group, the direct reduction of a triangular equation to a system of cancellation-free equations is no longer possible. Instead of looking for geodesic solutions (g_1, g_2, g_3) in G to the equation $XY = Z$, represented by a geodesic triangle in the Cayley graph of G as in Figure 3a, one looks for solutions as in Theorem 7.2, illustrated in Figure 3b. That is, one finds the solution g_i via some canonical representative $\theta_T(g_i)$, where T is computable and guaranteed to exist; we henceforth write $\theta(-)$ for $\theta_T(-)$. By Theorem 7.2 there exist $y_i, c_i \in F(S)$ such that the equations $\theta(g_1) = y_1c_1y_2$, $\theta(g_2) = y_2^{-1}c_2y_3$, $\theta(g_3) = y_3^{-1}c_3y_1^{-1}$ are cancellation-free (no cancellation occurs between y_1 and c_1, c_1 and y_2 etc.) equations over $F(S)$. Moreover, Theorem 7.2 implies that the y_i's should be viewed as 'long' prefixes and suffixes that coincide, and the $c_1c_2c_3$ as a 'small' inner circle with circumference in $\mathcal{O}(n)$ (see Figure 3b).



(A) Using geodesics (B) Using canonical representatives

Figure 3. Solutions to $XY = Z$ in a hyperbolic group.

We now prove Proposition 7.3, which together with Proposition 5.5 gives Theorem 7.4. This proposition provides a covering set of solutions (that is, every solution as tuple of group elements is found, but not necessarily written as shortlex representatives etc) that can then be processed via Proposition 5.5 to give the EDT0L formal language description in the space complexity claimed in Theorem 7.4.

Proposition 7.3. *Let G be a torsion-free hyperbolic group with finite symmetric generating set S. Let $\#$ be a symbol not in S, and let $\Sigma = S \cup \{\#\}$. Let Φ be a system of equations and inequations of size n (without rational constraints) over variables $X_1, \ldots, X_r$, as in Definition 2.1.*

(1) There exists a language $L = \{w_1\# \ldots \#w_r\}$ over Σ such that
  (a) L is a covering solution set,
  (b) $w_i \in Q_{G,S,\lambda,\mu}, 1 \leqslant i \leqslant r$ where $\lambda = \lambda_G, \mu = \mu_G$ are the (fixed) constants from Proposition 7.1,
  (c) L is EDT0L in $\mathsf{NSPACE}(n^2 \log n)$.

(2) The system Φ has infinitely many solutions if and only if L is infinite, and no solution if and only if L is empty.

Proof. We produce a language L of quasigeodesic words over S as above such that any tuple $(\pi(w_1), \ldots, \pi(w_r))$ arising from some $w_1\# \ldots \#w_r$ in L is in the solution set $\mathrm{Sol}_G(\Phi)$ (soundness). We then prove (using Theorem 7.2) that any solution in $\mathrm{Sol}_G(\Phi)$ is the projection, via π, of some element in L (completeness).

Steps 1 and 2: Preprocessing

Step 1 [Remove inequations] We first transform Φ into a system consisting entirely of equations by adding a variable X_N to $\mathcal{X}$ and replacing any inequation $\varphi_j(\mathcal{X}, \mathcal{A}) \neq 1$ by $\varphi_j(\mathcal{X}, \mathcal{A}) = X_N$, with the constraint $X_N \neq_G 1$.

Step 2 [Triangulate] We transform each equation into several equations of length 3, by introducing new variables. This can always be done (see the discussion in [7, §4]), and it produces approximately $\sum_{i=1}^{s} l_i \in \mathcal{O}(n)$ triangular equations with set of variables $\mathcal{Z}$ where $|\mathcal{Z}| \in \mathcal{O}(n)$ and $\mathcal{X} \subset \mathcal{Z}$. From now on assume that the system Φ consists of $q \in \mathcal{O}(n)$ equations of the form $X_jY_j = Z_j$ where $1 \leqslant j \leqslant q$.

Steps 3 and 4: Lifting Φ to the free group on S

Step 3 Let $b \in \mathcal{O}(q) = \mathcal{O}(n)$ be the constant from Theorem 7.2 that depends on δ and linearly on q. We run in lexicographic order through all possible tuples of words $\mathbf{c} = (c_{11}, c_{12}, c_{13}, \ldots, c_{q1}, c_{q2}, c_{q3})$ with $c_{ji} \in S^*$ and $|c_{ji}|_S \leqslant b$.

Step 4 For each tuple $\mathbf{c}$ we use Dehn's algorithm to check whether $c_{j1}c_{j2}c_{j3} =_G 1$; if this holds for all j we construct a system $\Phi_{\mathbf{c}}$ of equations of the form

$$X_j = P_j c_{j1} Q_j,\ Y_j = Q_j^{-1} c_{j2} R_j,\ Z_j = P_j c_{j3} R_j, \quad 1 \leqslant j \leqslant q, \tag{1}$$

where P_i, Q_i, Z_i are new variables. This new system has size $\mathcal{O}(n^2)$. In order to avoid an exponential size complexity we write down each system $\Phi_{\mathbf{c}}$ one at a time, so the space required for this step is $\mathcal{O}(n^2)$.

Let $\mathcal{Y} \supset \mathcal{Z} \supset \mathcal{X}$, $|\mathcal{Y}| = m$, be the new set of variables, including all the P_i, Q_i, Z_i.

Soundness Any solution to $\Phi_{\mathbf{c}}$ in the free group $F(S)$ is clearly a solution to Φ in the original hyperbolic group G when restricting to the original variables $\mathcal{X}$. That is, if $(w_1, \ldots, w_m) \subseteq F^m(S)$ is a solution to $\Phi_{\mathbf{c}}$, then $(\pi(w_1), \ldots, \pi(w_r))$ is a solution to Φ in G. This shows soundness.

Now let $\lambda_G \geqslant 1, \mu_G \geqslant 0$ be the constants provided by Proposition 7.1. Note that if a word $w \in S^*$ is a (G, S, λ_G, μ_G)-quasigeodesic and satisfies $w =_G 1$ then $|w|_S \leqslant \mu_G$. We can construct an NFA $\mathcal{N}$ which accepts all words in S^* equal to 1 in the hyperbolic group G of length at most μ_G in constant space (using for example Dehn's algorithm), and in our next step we will use this rational constraint to handle the variable X_N added in Step 1 (to remove inequalities).

Steps 5 and 6: Solving equations with constraints in $F(S)$

Step 5 We now run the algorithm from [7] (which we will refer to as the CDE algorithm) which takes input $\Phi_{\mathbf{c}}$ of size $\mathcal{O}(n^2)$, plus the rational constraint $X_N \notin L(\mathcal{N})$, plus for each $Y \in \mathcal{Y}$ the rational constraint that the solution for Y is a word in Q_{G,S,λ_G,μ_G}. Since these constraints have constant size (depending only on the group G, not the system Φ), they do not contribute to the $\mathcal{O}(n^2)$ size of the input to the CDE algorithm.

We modify the CDE algorithm to ensure every node printed by the algorithm includes the additional label $\mathbf{c}$. (This ensures the NFA we print for each system $\Phi_{\mathbf{c}}$ is distinct.) This does not affect the complexity since $\mathbf{c}$ has size in $\mathcal{O}(n^2)$. Let $\mathcal{C}$ be the set of all tuples $\mathbf{c}$.

Step 6 We run the modified CDE algorithm described above (adding label $\mathbf{c}$ for each node printed) to print an NFA (possibly empty) for each $\Phi_{\mathbf{c}}$, which is the rational control for an EDT0L grammar that produces all solutions as freely reduced words in $F(S)$, which correspond to solutions as (G, S, λ_G, μ_G)-quasigeodesics to the same system $\Phi_{\mathbf{c}}$ in the hyperbolic group.

Completeness If $(g_1, \ldots, g_r) \in G^r$ is a solution to Φ in the original hyperbolic group, Theorem 7.2 guarantees that there exist canonical representatives $w_i \in Q_{G,S,\lambda_G,\mu_G}$ with $w_i =_G g_i$, which have freely reduced forms $u_i =_G w_i$ for $1 \leqslant i \leqslant m$, and our construction is guaranteed to capture any such collection of words. More specifically, if $(w_1, \ldots, w_r)$ is a solution in canonical representatives to Φ then $(u_1, \ldots, u_r, \ldots u_{|\mathcal{Y}|})$ will be included in the solution to $\Phi_{\mathbf{c}}$ produced by the CDE algorithm, with u_i the reduced forms of w_i for $1 \leqslant i \leqslant m$. This shows completeness once we union the grammars from all systems $\Phi_{\mathbf{c}}$ together.

Step 7 [Producing the solutions in G] Add a new start node with edges to each of the start nodes of the NFA's with label $\mathbf{c}$, for all $\mathbf{c} \in \mathcal{C}$. We obtain an automaton which is the rational control for the language L in the statement of the proposition. This rational control makes L an EDT0L language of

(G, S, λ, μ)-quasigeodesics as required, where $\lambda := \lambda_G, \mu := \mu_G$. The space needed is exactly that of the CDE algorithm on input $\mathcal{O}(n^2)$, which is $\mathsf{NSPACE}(n^2 \log n)$.

To see why (2) in the statement of the proposition holds, note that when we call on the CDE algorithm above, this will be able to report if the system over $F(S)$ has infinitely many, finitely many, or no solutions. If CDE reports finitely many or no solutions then we are done, as L will be finite or empty, respectively. If CDE returns infinitely many solutions, note that there are only finitely many words over S per group element in G in the solution set because (in Step 5) we run CDE with the rational constraint that only (G, S, λ_G, μ_G)-quasigeodesics are part of output, so we end up with infinitely many solutions in $\mathrm{Sol}_G(\Phi)$. □

We can now prove our main result about torsion-free hyperbolic groups.

Theorem 7.4 (Torsion-free)**.** *Let G be a torsion-free hyperbolic group with finite symmetric generating set S. Let $\#$ be a symbol not in S, and let $\Sigma = S \cup \{\#\}$. Let Φ be a system of equations and inequations of size n with constant size effective qier constraints over variables $X_1, \dots, X_r$, as in Section 2 and Definition 4.8.*

Then we have the following.

(1) For any $\lambda' \geqslant 1, \mu' \geqslant 0, \lambda' \in \mathbb{Q}$ and for any regular language $\mathcal{T} \subseteq Q_{G,S,\lambda',\mu'}$ such that the projection $\pi : \mathcal{T} \to G$ is a surjection, the full set of $\mathcal{T}$-solutions

$$Sol_{\mathcal{T},G}(\Phi) = \{(w_1\# \dots \#w_r) \mid w_i \in \mathcal{T}, (\pi(w_1), \dots, \pi(w_r)) \text{ solves } \Phi\}$$

is ET0L in $\mathsf{NSPACE}(n^2 \log n)$.

(2) For any $\lambda' \geqslant 1, \mu' \geqslant 0, \lambda' \in \mathbb{Q}$ and for any regular language $\mathcal{T} \subseteq Q_{G,S,\lambda',\mu'}$ such that the projection $\pi : \mathcal{T} \to G$ is a bijection, the full set of $\mathcal{T}$-solutions

$$Sol_{\mathcal{T},G}(\Phi) = \{(w_1\# \dots \#w_r) \mid w_i \in \mathcal{T}, (\pi(w_1), \dots, \pi(w_r)) \text{ solves } \Phi\}$$

is EDT0L in $\mathsf{NSPACE}(n^2 \log n)$.

(3) It can be decided in $\mathsf{NSPACE}(n^2 \log n)$ whether or not $Sol_G(\Phi)$ is empty, finite or infinite.

In particular, if $\mathcal{T}$ is the set of all shortlex geodesics over S, then Theorem A follows immediately.

Proof. Suppose R_{X_i}, $X_i \in \mathcal{X}$, are the constant size effective qier constraints that are part of the system Φ. By Proposition 6.1 ([12, Proposition 9.4]) the languages $\widetilde{R_{X_i}} = \pi^{-1}(R_{X_i}) \cap Q_{G,S,\lambda_G,\mu_G}$ are regular and one can compute automata accepting them in constant space. (Recall, the complexity for this step only depends on the constant size R_X, so from the point of view of our result the complexity is constant.)

Let Φ' be the system consisting only of the equations and inequations from Φ. Apply Proposition 7.3 to Φ' to obtain an EDT0L language L of covering solutions to Φ'. Then intersect the language L of covering solutions with $\widetilde{R_{X_1}}\# \dots \#\widetilde{R_{X_r}}$ to ensure that the solutions belong to the input constraints, and call this covering solution set L_c (to point out that constraints have been taken into account). The language L_c is EDT0L, as the intersection of an EDT0L language with a regular language, and stays in the same space complexity of $\mathsf{NSPACE}(n^2 \log n)$ by Proposition 3.8.

We then apply Lemma 5.3 to the covering set L_c to double it and get a set which satisfies the conditions and plays the role of the set L_{cover} in Proposition 5.5. Then Proposition 5.5 applied to this set shows (1) that the set of solutions written as words in a regular language (of quasigeodesics) in bijection with the group is EDT0L, and (2) written as words in a regular language (of quasigeodesics) surjecting onto the group is ET0L. Part (3) also follows immediately from Proposition 5.5. □

8. Equations and their solutions in hyperbolic groups with torsion

In the case of a hyperbolic group G with torsion, the general approach of Rips and Sela can still be applied, but the existence of canonical representatives is not always guaranteed (see Delzant [13, Rem.III.1]). To get around this, Dahmani and Guirardel 'fatten' the Cayley graph $\Gamma(G,S)$ of G to a larger graph $\mathcal{K}$ which contains $\Gamma(G,S)$ (in fact $\Gamma(G,S)$ with midpoints of edges included), and solve equations in G by considering equalities of paths in $\mathcal{K}$. More precisely, $\mathcal{K}$ is the 1-skeleton of the barycentric subdivision of a Rips complex of G, as explained below.

Let $P_{50\delta}(G)$ be the Rips complex whose set of vertices is G, and whose simplices are subsets of G of diameter at most 50δ (see Definition 4.9). Then let $\mathcal{B}$ be the barycentric subdivision of $P_{50\delta}(G)$ and let $\mathcal{K} = \mathcal{B}^1$ the 1-skeleton of $\mathcal{B}$. By construction, the vertices of $\mathcal{K}$ (and $\mathcal{B}$) are in 1-to-1 correspondence with the simplices of $P_{50\delta}(G)$, so we can identify G, viewed as the set of vertices of $\Gamma(G,S)$, with a subset of vertices of $\mathcal{K}$.

Remark 8.1. The graphs $\mathcal{K}$ and $\Gamma(G,S)$ are quasi-isometric, and any vertex in $\mathcal{K}$ that is not in G is at distance 1 (in $\mathcal{K}$) from a vertex in G.

Definition 8.2. Let γ, γ' be paths in $\mathcal{K}$.
- (i) We denote by $i(\gamma)$ the initial vertex of γ, by $f(\gamma)$ the final vertex of γ, and by $\overline{\gamma}$ the reverse of γ starting at $f(\gamma)$ and ending at $i(\gamma)$.
- (ii) We say that γ is *reduced* if it contains no backtracking, that is, no subpath of length 2 of the form $e\overline{e}$, where e is an edge.
- (iii) We write $\gamma\gamma'$ for the concatenation of γ, γ' if $i(\gamma') = f(\gamma)$.
- (iv) Two paths in $\mathcal{K}$ are *homotopic* if one can obtain a path from the other by adding or deleting backtracking subpaths. Each *homotopy class* $[\gamma]$ has a unique reduced representative.

Let V be the set of all homotopy classes $[\gamma]$ of paths γ in $\mathcal{K}$ starting at 1_G and ending at a point in $\Gamma(G,S)$ which corresponds to some element of G, that is,

$$V := \{[\gamma] \mid \gamma \in \mathcal{K}, i(\gamma) = 1_G, f(\gamma) \in G\}. \tag{2}$$

For $[\gamma]$, $[\gamma'] \in V$ define their product $[\gamma][\gamma'] := [\gamma_v\gamma']$, where $\gamma_v\gamma'$ denotes the concatenation of γ and the translate ${}_v\gamma'$ of γ' by $v = f(\gamma)$; we will abuse notation and write $\gamma\gamma'$ for the concatenation of γ and the translate of γ' by $f(\gamma)$, without additional mention of the translation. Let $[\gamma]^{-1}$ be the homotopy class of ${}_{v^{-1}}\overline{\gamma}$.

The set V is then a group that projects onto G via the final vertex map f, that is, $f : V \twoheadrightarrow G$ is a surjective homomorphism. Since G has an action on $\mathcal{K}$ induced by the natural action on its Rips complex, V will act on $\mathcal{K}$ as well. This gives rise to an action of V onto the universal cover T (which is a tree) of $\mathcal{K}$, and [12, Lemma 9.9] shows that the quotient T/V is a finite graph (isomorphic to $\mathcal{K}/G$) of finite groups, and so V is virtually free.

We assume that the algorithmic construction (see [12, Lemma 9.9]) of a presentation for V is part of the preprocessing of the algorithm, will be treated as a constant, and will not be included in the complexity discussion.

The first step in solving a system Φ of (triangular) equations in G is to translate Φ into identities between (λ_1, μ_1)-quasigeodesic paths in $\mathcal{K}$ (λ_1 and μ_1 are defined after Proposition 8.4) that have start and end points in G. These paths can be seen as the analogues of the canonical representatives from the torsion-free case. This can be done by Proposition 9.8 [12] (see Proposition 8.4). The second step in solving Φ is to express the equalities of quasigeodesic paths in $\mathcal{K}$ in terms of equations in the virtually free group V based on $\mathcal{K}$. Finally, Proposition 9.10 [12] (see Proposition 8.5) shows it is sufficient to solve the systems of equations in V in order to obtain the solutions of the system Φ in G.

Notation 8.3. Let $\lambda_0 = 400\delta m_0$, where m_0 is the size of a ball of radius 50δ in $\Gamma(G, S)$, $\mu_0 = 8$ and $b = b(\delta, |\Phi|)$ is a linear function of the size of the system Φ (b is a multiple of the constant '*bp*' defined in [39, Theorem 4.2]).

Proposition 8.4 ([12, Proposition 9.8]). *Let G be a δ-hyperbolic group generated by S, $\mathcal{K}$ the graph defined above, and $\Phi = \{\phi_j(\mathcal{X}, \mathcal{A}) = 1\}_{j=1}^q$ a system of triangular equations on variables $\{X_1, \ldots, X_l\}$. That is, $\phi_j(\mathcal{X}, \mathcal{A}) = X_{(j,1)}X_{(j,2)}X_{(j,3)} = 1$, where $1 \leqslant j \leqslant q$. Let λ_0, $\mu_0 = 8$ and b be as in Notation 8.3.*

If $(g_1, \ldots, g_l) \in Sol_G(\Phi)$, then there exist paths $y_{(j,a)}, c_{(j,a)} \in \mathcal{K}$, $1 \leqslant j \leqslant q, 1 \leqslant a \leqslant 3$ and $(j, a) \in \{1, \ldots l\}$, such that

$$\ell_{\mathcal{K}}(c_{(j,a)}) \leqslant b \text{ and } f(c_{(j,1)}c_{(j,2)}c_{(j,3)}) = 1 \in G,$$

and there exist $(\mathcal{K}, \lambda_0, \mu_0)$-quasigeodesic paths $\gamma(g_{(j,a)})$ joining 1_G to $g_{(j,a)}$ in $\mathcal{K}$ such that $\gamma(g_{(j,a)}^{-1}) = {}_{g_{(j,a)}^{-1}}\overline{\gamma(g_{(j,a)})}$ and

$$\gamma(g_{(j,1)}) = y_{(j,1)}c_{(j,1)}(y_{(j,2)})^{-1},$$
$$\gamma(g_{(j,2)}) = y_{(j,2)}c_{(j,2)}(y_{(j,3)})^{-1},$$
$$\gamma(g_{(j,3)}) = y_{(j,3)}c_{(j,3)}(y_{(j,1)})^{-1}.$$

For λ_0, μ_0 as in Proposition 8.4, let λ_1, μ_1 be such that any path $\alpha\gamma\alpha'$ in $\mathcal{K}$, where $\ell_{\mathcal{K}}(\alpha) = \ell_{\mathcal{K}}(\alpha') = 1$ and γ is a $(\mathcal{K}, \lambda_0, \mu_0)$-guasigeodesic, is a $(\mathcal{K}, \lambda_1, \mu_1)$-quasigeodesic. Define

(3) $Q_{\mathcal{K},V,\lambda_1,\mu_1}$[6] $= \{\gamma \in \mathcal{K} \mid \gamma \text{ is in } V \text{ and is a reduced } (\mathcal{K}, \lambda_1, \mu_1)\text{−quasigeodesic}\}$,

that is, $Q_{\mathcal{K},V,\lambda_1,\mu_1}$ is the set of those paths in $Q_{\mathcal{K},\lambda_1,\mu_1}$ with start and endpoints in G. Then for any $L > 0$ let

$$V_{\leqslant L} = \{[\gamma] \in V \mid \gamma \text{ reduced and } \ell_{\mathcal{K}}(\gamma) \leqslant L\}.$$

The next proposition is a refinement of Proposition 8.4 in the sense that it makes the same statements about equalities of paths, but this time all the paths are in V; this setting allows us to work with equations in the virtually free group V.

Proposition 8.5 ([12, Proposition 9.10]). *Let G be a δ-hyperbolic group, $\mathcal{K}$, λ_1, μ_1, $b = b(\delta, n)$ and $\Phi = \{\phi_j(\mathcal{X}, \mathcal{A}) = 1\}_{j=1}^q$ a system on l variables as above. That is, Φ consists of equations $\phi_j(\mathcal{X}, \mathcal{A}) = X_{(j,1)}X_{(j,2)}X_{(j,3)} = 1$, where $1 \leqslant j \leqslant q$ and $(j, a) \in \{1, \ldots l\}$.*

- *(i) If $(g_1, \ldots, g_l) \in Sol_G(\Phi)$, then there exist $v_{(j,a)}, y_{(j,a)} \in Q_{\mathcal{K},V,\lambda_1,\mu_1}$ and $c_{(j,a)} \in V_{\leqslant b}$ such that $f(c_{(j,1)}c_{(j,2)}c_{(j,3)}) = 1$ in G, $f(v_{(j,a)}) = g_{(j,a)}$ and $v_{(j,a)} = y_{(j,a)}c_{(j,a)}(y_{(j,a+1)})^{-1}$, where $a + 1$ is computed modulo 3.*
- *(ii) Conversely, suppose a set of elements $\{v_1, \ldots, v_l\} \subset V$ is given and for every $1 \leqslant j \leqslant q, 1 \leqslant a \leqslant 3$ there are $y_{(j,a)} \in V, c_{(j,a)} \in V_{\leqslant\kappa}$, where $(j, a) \in \{1, \ldots l\}$, satisfying $v_{(j,a)} = y_{(j,a)}c_{(j,a)}(y_{(j,a+1)})^{-1}$ and $f(c_{(j,1)}c_{(j,2)}c_{(j,3)}) = 1$, then the $g_{(j,a)} = f(v_{(j,a)})$ give a solution of Φ in G.*

In the virtually free group V we will use the results of Diekert and the second author from [14]. Let Y be the generating set of V, and let $\mathcal{T} \subseteq Y^*$ be the set of "standard normal forms" for V over Y as in Proposition 5.6. By Proposition 5.6 the set $\mathcal{T}$ consists of (λ_Y, μ_Y)-quasigeodesics over Y, where λ_Y, μ_Y are constants depending on Y. Let

$$\mathrm{Sol}_{\mathcal{T},V}(\Psi) = \{w_1\# \ldots \#w_r \mid w_i \in \mathcal{T}, (\pi(w_1), \ldots, \pi(w_r)) \text{ solves } \Psi \text{ in } V\}$$

[6]Notation similar to that in Definition 4.4, but the type of paths is different: these are quasi-geodesics in the graph $\mathcal{K}$ that represent elements in V, not paths in a Cayley graph of V.

be the set of $\mathcal{T}$-solutions in V of a system Ψ of size $|\Psi| = \mathcal{O}(k)$; by [14], $\mathrm{Sol}_{\mathcal{T},V}(\Psi)$ consists of (λ_Y, μ_Y)-quasigeodesics over Y and is EDT0L in $\mathsf{NSPACE}(k^2 \log k)$.

In the following we will need to translate between elements in V over the generating set Y, and elements in G over S, passing through the graph $\mathcal{K}$, in a way that preserves the EDT0L characterisation of languages. We introduce Notation 8.6 to facilitate this translation.

Notation 8.6. Let Z be some generating set of V and let $\pi\colon Z^* \to V$ be the standard projection map from words to group elements in V.

(i) [From (V, Z) to $\mathcal{K}$] For each generator $z_i \in Z$ denote by p_i the unique reduced path in $\mathcal{K}$ corresponding to z_i, that is, with $i(p_i) = 1_G$ and $f(p_i) \in G$ such that $z_i =_V [p_i]$. For any word $w = z_{i_1} \dots z_{i_k}$ over Z, denote by p_w the path obtained by concatenating the paths p_{i_j}, that is,

$$p_w = p_{i_1} \dots p_{i_k}, \tag{4}$$

where $i(p_w) = 1_G$, $f(p_w) \in G$, and $w =_V [p_w]$.

(ii) For $w \in Z^*$ and $g \in V$ such that $\pi(w) = g$, denoted by $p_g = p_{\pi(w)}$ the unique reduced path in $\mathcal{K}$ homotopic to p_w (which might be unreduced).

(iii) [From (V, Z) to (G, S)] For each $z_i \in Z$, choose a geodesic path/word γ_i in $\Gamma(G, S)$ such that $i(\gamma_i) = 1_G$ and $f(\gamma_i) = f(p_i) \in G$, where p_i is the reduced path representing z_i in $\mathcal{K}$ (see (i)), and let $\sigma : Z \mapsto S^*$ be the map given by $\sigma(z_i) = \gamma_i$. This can be extended to the substitution $\sigma : Z^* \mapsto S^*$ that associates to each word $w = z_{i_1} \dots z_{i_k}$ over Z a path/word γ_w in $\Gamma(G, S)$ obtained by concatenation, that is,

$$\sigma(w) = \gamma_w = \gamma_{i_1} \dots \gamma_{i_k}, \tag{5}$$

where $i(\gamma_w) = 1_G$ and $f(\gamma_w) = f(p_w) \in G$.

Remark 8.7. Finding all the group element solutions of a system Φ over G, written as some set of words over S that are not necessarily quasigeodesic or in a normal form, is possible by (a) the work of Dahmani and Guirardel [12], who reduce this problem to solving equations in the virtually free group V defined above, and by (b) the algorithm of Diekert and the second author for V [14]. However, obtaining the solutions as an EDT0L language of (quasigeodesic) normal forms in G requires a lot more effort, and is considerably more intricate than in the torsion-free case.

In the torsion-free case it is sufficient to solve equations in the free group $F(S)$ on the same generating set as G and it is guaranteed (by [39]) that the solutions from the free group contain a quasigeodesic for each group element solution. In the torsion case it is necessary to solve equations in V, whose generating set Y is completely different from that of G, and furthermore, V contains G with infinite index. Simply taking the solutions over Y given by the DE algorithm does not guarantee to produce the quasigeodesics in the graph $\mathcal{K}$ specified in Proposition 8.5 (1), and therefore does not guarantee quasigeodesic solutions in G over S. We therefore need to produce a larger set of quasigeodesics in V over Y representing the solution set of [14], in order to be certain that there is at least a quasigeodesic with parameters from Proposition 8.5 (1) per solution. This blow-up of the solution set is ET0L, and we can ultimately get EDT0L for the solution set in G only by employing the doubling and copying tricks of Proposition 5.5, which makes Proposition 8.8 more technical than its torsion-free counterpart.

Proposition 8.8. *Let G be a hyperbolic group with finite symmetric generating set S. For $\#$ a symbol not in S, extend the bijection $f\colon S \to S_\dagger$ defined in Notation 5.4 to include $f(\#) = \#$. Let Φ be a system of equations and inequations (without* *rational constraints) of size n over variables $X_1, \ldots, X_r$ (see Definition 2.1). Then there exist $\lambda_G \geqslant 1, \mu_G \geqslant 0$ which depend only on G, and a language*

$$L \subseteq \{w_1\# \ldots \#w_r f(z_1\# \ldots \#z_r) \mid w_i, z_i \in Q_{G,S,\lambda,\mu}, \pi(w_i) = \pi(z_i), 1 \leqslant i \leqslant r\}$$

such that

(1) $\{w_1\# \ldots \#w_r \mid w_1\# \ldots \#w_r f(z_1\# \ldots \#z_r) \in L\}$ *is a covering solution set for* Φ,
(2) L *is ET0L in* $\mathsf{NSPACE}(n^4 \log n)$.

Moreover, the system Φ has infinitely many solutions if and only if L is infinite, and no solutions if and only if L is empty.

Proof. The algorithm to produce the solutions for Φ is similar to the torsion-free case, except that the virtually free group V and the graph $\mathcal{K}$ will play the role of the free group on S and the Cayley graph of G from the torsion-free case, instead.

Steps 1 and 2: Preprocessing

We triangulate Φ (Step 1) and introduce a new variable with rational constraint to deal with the inequations (Step 2), exactly as in proof of Proposition 7.3. From now on assume that the system Φ consists of $q \in \mathcal{O}(n)$ equations of the form $X_jY_j = Z_j$ where $1 \leqslant j \leqslant q$.

Steps 3 and 4: Lifting Φ to the virtually free group V

Step 3 Let $b \in \mathcal{O}(q) = \mathcal{O}(n)$ be the constant defined in Proposition 8.5 that depends on δ and linearly on q. Recall that

$$V_{\leqslant b} = \{[\gamma] \in V \mid \gamma \text{ reduced and } \ell_{\mathcal{K}}(\gamma) \leqslant b\},$$

and let Y be the generating set of V as in Proposition 5.6.

One lifts the system Φ in G to a finite set of systems $\Phi_{\mathbf{c}}$ in the virtually free group V, one system for each q-tuple $\mathbf{c} = (c_{11}, c_{12}, c_{13}, \ldots, c_{q1}, c_{q2}, c_{q3})$ of triples (c_{j1}, c_{j2}, c_{j3}) with $c_{ij} \in V_{\leqslant b}$ and such that $f(c_{j1}c_{j2}c_{j3}) = 1_G$, as in Proposition 8.5. We enumerate these tuples by enumerating triples of words (v_{j1}, v_{j2}, v_{j3}) over the generating set Y of V with $\ell_{(V,Y)}(v_{ij}) \leqslant b_Y$, where $b_Y \in \mathcal{O}(q)$ depends on b, as in Lemma 8.9(ii). By Lemma 8.9(ii) the set of corresponding path triples $(p_{v_{j1}}, p_{v_{j2}}, p_{v_{j3}})$ in $\mathcal{K}$ (see Notation 8.6 – Equation (4)) contains all triples (c_{j1}, c_{j2}, c_{j3}) with $c_{ji} \in V_{\leqslant b}$, up to homotopy. Then for each triple (v_{j1}, v_{j2}, v_{j3}) we check whether $f(v_{j1}v_{j2}v_{j3}) = 1_G$, which is done by checking whether in the Cayley graph $\Gamma(G, S)$ the identity $\gamma_{v_{j1}}\gamma_{v_{j2}}\gamma_{v_{j3}} =_G 1$ holds, using the Dehn algorithm in G.

Step 4 For each tuple $\mathbf{c}$ we construct a system $\Phi_{\mathbf{c}}$ of equations in V of the form

$$X_j = P_jc_{j1}Q_j,\ Y_j = Q_j^{-1}c_{j2}R_j,\ Z_j = P_jc_{j3}R_j, \quad 1 \leqslant j \leqslant q, \tag{6}$$

where P_i, Q_i, Z_i are new variables. This new system has size $\mathcal{O}(n^2)$ since it has $\mathcal{O}(q)(= \mathcal{O}(n))$ equations, each of length in $\mathcal{O}(q)$, and the c_{ij}'s inserted have length in $\mathcal{O}(q)$. In order to avoid an exponential size complexity we write down each system $\Phi_{\mathbf{c}}$ one at a time, so the space required for this step is $\mathcal{O}(n^2)$.

Steps 5 and 6: Solving equations in V and then in G

Steps 5 and 6 produce an ET0L covering solution set of quasigeodesics for Φ in G. However, the technical Steps 5A and 6A are additionally needed in order to generate an EDT0L language of solutions in Theorem 8.12.

Step 5 [Solve in V] For each system $\Phi_{\mathbf{c}}$ over V we run the modified DE algorithm (inserting the additional label $\mathbf{c}$ for each node printed) to print an NFA for each $\Phi_{\mathbf{c}}$. We obtain the set of solutions $\mathrm{Sol}_{\mathcal{T},V}(\Phi_{\mathbf{c}})$ as an EDT0L language in $\mathsf{NSPACE}((q^2)^2 \log(q^2)) = \mathsf{NSPACE}(n^4 \log n)$ of (λ_Y, μ_Y)-quasigeodesics over Y. (These solutions are written in the "standard normal forms" of [14, Definition 14.1], which are quasigeodesics by Proposition 5.6.)

Step 5A [Duplicate solutions from V] This step is needed to get the solutions in G as EDT0L at the very end; if omitted, the solutions will be ET0L only.

Apply the (Doubling) Lemma 5.3 to obtain an EDT0L language of the form $\{p_1\#\dots\#p_r f(p_1\#\dots\#p_r)\}$ where $p_1\#\dots\#p_r \in \mathrm{Sol}_{\mathcal{T},V}(\Phi_{\mathbf{c}})$ is a tuple of words in standard normal form and f is the bijection $f : Y \to Y_\dagger = \{y_\dagger \mid y \in Y \cup \{\#\}\}$ as in the lemma. We can assume this language is

$$L' \subseteq \left\{p_1\#\dots\#p_r q_1\#_\dagger\dots\#_\dagger q_r \mid p_i \in S^*, q_i \in S_\dagger^*, \pi(p_i) = \pi_\dagger(q_i)\right\},$$

and it consists of actual solutions $p_1\#\dots\#p_r$, together with their *shadows* $q_1\#_\dagger\dots\#_\dagger q_r$.

Step 6 [Produce an alternative covering solution set of $\Phi_{\mathbf{c}}$ in V, then apply substitution map to words in this set to get quasigeodesic solutions in G]

Let $\lambda_1' \geqslant 1, \mu_1' \geqslant 0$ be the the constants in Lemma 8.9 (computable from (λ_1, μ_1), which were defined in (3)), and let $\mathcal{T}' = Q_{V,Y,\lambda_1',\mu_1'}$ be the set of (λ_1', μ_1')-quasigeodesics in V over Y. Then the set $\mathrm{Sol}_{\mathcal{T}',V}(\Phi_{\mathbf{c}})$ of all $(V, Y, \lambda_1', \mu_1')$-quasigeodesics which represent solutions of $\Phi_{\mathbf{c}}$ is ET0L by Proposition 5.5 (1). Furthermore, by Corollary 8.10 this set contains at least one word over Y for each solution in $Q_{\mathcal{K},V,\lambda_1,\mu_1}$.

Then for σ as in (5), $\sigma(\mathrm{Sol}_{\mathcal{T}',V}(\Phi_{\mathbf{c}}))$ is ET0L since ET0L is preserved by substitutions, and by Proposition 8.5 the set $\bigcup_{\mathbf{c}} \sigma(\mathrm{Sol}_{\mathcal{T}',V}(\Phi_{\mathbf{c}}))$ projects onto $\mathrm{Sol}_G(\Phi)$, so it is a covering solution set of Φ in G. By Lemma 8.11 there exist constants λ, μ such that all words in $\bigcup_{\mathbf{c}} \sigma(\mathrm{Sol}_{\mathcal{T}',V}(\Phi_{\mathbf{c}}))$ are (G, S, λ, μ)-quasigeodesic so we obtained an ET0L covering solution set for Φ contained in $Q_{G,S,\lambda,\mu}$.

Step 6A The operations in Step 6 are being performed not only on the actual solutions $\mathrm{Sol}_{\mathcal{T},V}(\Phi_{\mathbf{c}})$, but also on their shadows (see Step 5A), that is, on the entire set L', to obtain a set L as required in the proposition. The solutions and their shadows are not in bijection as words, but they represent the same tuples of elements in V and then G, respectively. This will be sufficient for us to later apply Proposition 5.5 to get Theorem 8.12.

To see why the final statement of the proposition holds, note that when we call on the DE algorithm above, this will be able to report if the system over V has infinitely many, finitely many, or no solutions. If DE reports finitely many or no solutions then we are done, as L will be finite or empty, respectively. If DE returns infinitely many solutions, note that there are only finitely many words over S per group element in G in the solution set because only (G, S, λ, μ)-quasigeodesics are part of output, so we end up with infinitely many solutions in $\mathrm{Sol}_G(\Phi)$. □

Lemma 8.9. *Let $\alpha \geqslant 1, \beta, L \geqslant 0$ be constants.*

(i) There exist $\alpha' \geqslant 1, \beta' \geqslant 0$ (computable from α, β and Y) such that for any $c \in Q_{\mathcal{K},V,\alpha,\beta}$, there is a word w on Y representing c such that $w \in Q_{V,Y,\alpha',\beta'}$.

(ii) There exists $L' \geqslant 0$ (computable from L and Y) such that for any $c \in V$ with $\ell_{\mathcal{K}}(p_c) \leqslant L$ (recall that p_c is the reduced path representing c in $\mathcal{K}$), there is a word $w \in Y^*$, $w =_V c$ with $\ell_{(V,Y)}(w) \leqslant L'$.

Proof. Consider the generating set $Z = Y \cup V_{\leqslant 3}$ for V.

(i) Let γ_v be the reduced path in $\mathcal{K}$ corresponding to some $v \in V$. We will construct a path γ'_v in $\mathcal{K}$, homotopic to γ_v, such that $\ell_{\mathcal{K}}(\gamma_v) \leqslant \ell_{\mathcal{K}}(\gamma'_v) \leqslant 2\ell_{\mathcal{K}}(\gamma_v)$ and $\frac{1}{M}\ell_{\mathcal{K}}(\gamma_v) \leqslant |\gamma'_v|_Z \leqslant M\ell_{\mathcal{K}}(\gamma_v)$, where $M = \max\{\ell_{\mathcal{K}}(p_y) \mid y \in Y\}$ (that is, M is the maximal length of a generator in Y with respect to the associated reduced path length in $\mathcal{K}$).

The vertices in $\mathcal{K}$ belong either to G, call those G-vertices, or to $\mathcal{K} \setminus G$, call them $\neg G$-vertices. Modify the path γ_v as follows. If γ_v contains subpaths that are in $\{p_y \mid y \in Y\} \cup V_{\leqslant 3}$ then leave them as they are. All other subpaths will have some maximal sequences of $\neg G$-vertices $A_1, A_2, \dots A_t$, $t \geqslant 3$. By Remark 8.1, for any $\neg G$-vertex there is a G-vertex at distance 1 in $\mathcal{K}$, so to every A_i whose γ_v-neighbours are both in $\neg G$ choose some G-vertex B_i at distance 1 from A_i in $\mathcal{K} \setminus \gamma_v$, and attach the backtrack $[A_iB_i, B_iA_i]$ to the path γ_v to obtained the unreduced path γ'_v. Then γ'_v is a concatenation of paths in $V_{\leqslant 3}$ and $\{p_y \mid y \in Y\}$, each of which is a generator of V belonging to Z. This shows that one can obtain a path γ'_v in $\mathcal{K}$ for v which can be written as a word w over Z, and $\frac{1}{M}\ell_{\mathcal{K}}(\gamma_v) \leqslant |w|_Z \leqslant \ell_{\mathcal{K}}(\gamma_v) < M\ell_{\mathcal{K}}(\gamma_v)$.

Now suppose γ_c is an (α, β)-quasigeodesic in $\mathcal{K}$. Then the modified path γ'_c is homotopic to γ_c and corresponds to a word w over Z as above; moreover, all subwords of w have length proportional to the corresponding subpaths of γ'_c, so w is a $(\alpha_M, \beta_M)-$quasigeodesic over Z, where (α_M, β_M) depend on α, β and M, and M depends on Y. Since changing generating sets (from Z to Y) is a quasi-isometry, w is an $(V, Y, \alpha', \beta')-$quasigeodesic , and (α', β') depend on Y and the initial (α, β).

(ii) This is an immediate application of (i). ☐

The next corollary ensures that by considering a sufficiently large set of quasigeodesics in V over Y we capture all the quasigeodesic solutions in $\mathcal{K}$ guaranteed to exist by Proposition 8.5 (1).

Corollary 8.10. *There exist λ'_1, μ'_1 (depending on λ_1, μ_1 as above, and Y) such that for any path $v \in Q_{\mathcal{K},V,\lambda_1,\mu_1}$ there is a $(V, Y, \lambda'_1, \mu'_1)$-quasigeodesic word over Y representing v.*

The following lemma shows that the substitution $\sigma : Y^* \mapsto S^*$ (5) will produce quasigeodesics in G from an appropriate set of quasigeodesics in V, so the quasigeodesic solutions in V given by [14] can be transformed into quasigeodesic solutions in the hyperbolic group G in the proof of Proposition 8.8.

Lemma 8.11. *Let $w \in Q_{V,Y,\lambda'_1,\mu'_1}$ be such that the reduced $\mathcal{K}$-path $p_{\pi(w)}$ is in $Q_{\mathcal{K},V,\lambda_1,\mu_1}$. Then the unreduced path p_w is quasigeodesic in $\mathcal{K}$ (for appropriate constants). In particular, there exist $\lambda \geqslant 1, \mu \geqslant 0$ (computable from λ'_1, μ'_1 and Y in Corollary 8.10) such that $\sigma(w) \in Q_{G,S,\lambda,\mu}$.*

Proof. Consider the generating set $Z = Y \cup V_{\leqslant 3}$ for V and let λ_Z, μ_Z be such that any $(V, Y, \lambda'_1, \mu'_1)$-quasigeodesic is (V, Z, λ_Z, μ_Z)-quasigeodesic . We will show that for $(a_{\mathcal{K}}, b_{\mathcal{K}}) = (\lambda_1, \mu_1 + M\mu_Z)$ the (unreduced) path p_w is $(a_{\mathcal{K}}, b_{\mathcal{K}})$-quasigeodesic in $\mathcal{K}$, where $M = \max\{\ell_{\mathcal{K}}(p_y) \mid y \in Y\}$, that is, M is the maximal length of a generator in Y with respect to the associated reduced path length in $\mathcal{K}$.

We say that a subpath s_w of p_w is a maximal backtrack if $p_w = ps_wp'$, s_w is homotopic to an empty path (via the elimination of backtrackings), and s_w is not contained in a longer subpath of p_w with the same property. This implies there is a point A on p_w such that s_w starts and ends at A, and such a maximal backtrack traces a tree in $\mathcal{K}$. We can then write $p_w = a_1s_1a_2 \dots s_{n-1}a_n$, where a_i are (possibly empty) subpaths of p_w and s_i are maximal backtracks; thus $p_{\pi(w)} = a_1a_2 \dots a_n$. If $\ell_{\mathcal{K}}(s_i) \leqslant M\mu_Z$ for all i, then the result follows immediately. Otherwise there exists an s_i with $\ell_{\mathcal{K}}(s_i) > M\mu_Z$, and we claim that we can write s_i in terms of a word over Z that is not a $(\mathcal{K}, \lambda_Z, \mu_Z)$-quasigeodesic, which contradicts the assumption that w is quasigeodesic.

To prove the claim, suppose $i(s_i) = f(s_i) = A$. We have two cases: in the first case $A \in G$ then $\pi(s_i) =_V 1$ and s_i corresponds to a subword v of w for which $\ell_{\mathcal{K}}(p_v) \geqslant M\mu_Z$. But v represents a word over Z, so $\ell_{\mathcal{K}}(p_v) \leqslant \ell_Z(v)M$, and altogether $M\mu_Z \leqslant \ell_{\mathcal{K}}(p_v) \leqslant \ell_Z(v)M$. Since $|v|_Z = 0$ and $v \in Q_{V,Z,\lambda_Z,\mu_Z}$, $\ell_Z(v) \leqslant \mu_Z$, which contradicts $\ell_Z(v) \geqslant \mu_Z$ from above.

In the second case $A \notin G$, so take a point $B \in G$ at distance 1 from A in $\mathcal{K}$ (this can always be done), and modify the word w to get w' over Z so that $p_{w'}$ in $\mathcal{K}$ includes the backtrack $[AB, BA]$ off the path p_w. Also modify s_i to obtain a new backtrack s_i'. Clearly $\pi(p_w) = \pi(p_{w'})$ and $\pi(s_i) = \pi(s_i')$, and s_i' becomes a maximal backtrack of $p_{w'}$ which can be written as a word over the generators Z that represents the trivial element in V. We can the apply then argument from the first case.

Finally, recall from (5) that the path $\sigma(w)$ replaces each subpath $p_i \in \mathcal{K}$ (corresponding to a generator z_i) of p_w by another path γ_i with the same endpoints. and if we add midpoints to the edges in $\sigma(w)$ to get a path $\sigma'(w)$, we can view $\sigma'(w)$ as a path in $\mathcal{K}$, not just $\Gamma(G, S)$. Since by construction $\sigma'(w)$ and p_w fellow travel, and p_w is a quasigeodesic in $\mathcal{K}$, we get that $\sigma'(w)$ is quasigeodesic in $\mathcal{K}$, with constants depending on the fixed choice of γ_i's from (5). Thus there exist $\lambda \geqslant 1, \mu \geqslant 0$ such that $\sigma(w)$ is a (λ, μ)-quasigeodesic over S in the hyperbolic group G since $\mathcal{K}$ and $\Gamma(G, S)$ are quasi-isometric, and the path $\sigma(w)$ is obtained from the quasigeodesic $\sigma'(w)$ by removing the midpoints of edges. $\square$

We can now prove our main result about hyperbolic groups with torsion.

Theorem 8.12 (Torsion). *Let G be a hyperbolic group with torsion, with finite symmetric generating set S. Let $\#$ be a symbol not in S, and let $\Sigma = S \cup \{\#\}$. Let Φ be a system of equations and inequations of size n with constant size effective qier constraints, over variables $X_1, \ldots, X_r$, as in Section 2 and Definition 4.8.*

Then we have the following.

(1) *For any $\lambda' \geqslant 1, \mu' \geqslant 0, \lambda' \in \mathbb{Q}$ and for any regular language $\mathcal{T} \subseteq Q_{G,S,\lambda',\mu'}$ such that the projection $\pi : \mathcal{T} \to G$ is a surjection, the full set of $\mathcal{T}$-solutions*
$$Sol_{\mathcal{T},G}(\Phi) = \{(w_1\#\ldots\#w_r) \mid w_i \in \mathcal{T}, (\pi(w_1), \ldots, \pi(w_r)) \text{ solves } \Phi\}$$
is ET0L in $\mathsf{NSPACE}(n^4 \log n)$.

(2) *For any $\lambda' \geqslant 1, \mu' \geqslant 0, \lambda' \in \mathbb{Q}$ and for any regular language $\mathcal{T} \subseteq Q_{G,S,\lambda',\mu'}$ such that the projection $\pi : \mathcal{T} \to G$ is a bijection, the full set of $\mathcal{T}$-solutions*
$$Sol_{\mathcal{T},G}(\Phi) = \{(w_1\#\ldots\#w_r) \mid w_i \in \mathcal{T}, (\pi(w_1), \ldots, \pi(w_r)) \text{ solves } \Phi\}$$
is EDT0L in $\mathsf{NSPACE}(n^4 \log n)$.

(3) *It can be decided in $\mathsf{NSPACE}(n^2 \log n)$ whether or not the solution set of Φ is empty, finite or infinite.*

In particular, if $\mathcal{T}$ is the set of all shortlex geodesics over S, then Theorem B follows immediately.

Proof. The proof follows that of the torsion-free case (Theorem 7.4) with some necessary additional details. Suppose R_{X_i}, $X_i \in \mathcal{X}$, are the constant size qier constraints that are part of the system Φ. By Proposition 6.1 ([12, Proposition 9.4]) the languages $\widetilde{R_{X_i}} = \pi^{-1}(R_{X_i}) \cap Q_{G,S,\lambda_G,\mu_G}$ are regular and one can compute automata accepting them, and as before, this is done in constant space.

Let Φ' be the system consisting only of the equations and inequations from Φ. Apply Proposition 8.8 to Φ' to obtain an ET0L language
$$L \subseteq \{w_1\#\ldots\#w_r f(z_1\#\ldots\#z_r) \mid w_i, z_i \in Q_{G,S,\lambda,\mu}, \pi(w_i) = \pi(z_i), 1 \leqslant i \leqslant r\}$$

where $\{w_1\#\dots\#w_r \mid w_1\#\dots\#w_r f(z_1\#\dots\#z_r) \in L\}$ is a covering solution to Φ'. Note that a key difference at this point to the torsion-free case is that we have doubled the language earlier, and our resulting language is not necessarily deterministic ET0L yet.

Intersect the language L with the regular language $\widetilde{R_{X_1}}\#\dots\#\widetilde{R_{X_r}}S_\dagger^*\#\dots\#S_\dagger^*$ where $S_\dagger$ is as in Notation 5.4 and call the resulting language L_c. The language L_c is ET0L, as the intersection of an ET0L language with a regular language, and stays in the same space complexity of $\mathsf{NSPACE}(n^2\log n)$ by Proposition 3.8(3). By construction we have

$$L_c \subseteq \{w_1\#\dots\#w_r f(z_1\#\dots\#z_r) \mid w_i, z_i \in Q_{G,S,\lambda,\mu}, \pi(w_i) = \pi(z_i) \in R_{X_i}, 1 \leqslant i \leqslant r\}$$

and $\{w_1\#\dots\#w_r \mid w_1\#\dots\#w_r(z_1\#\dots\#z_r) \in L_c\}$ is a covering solution to Φ, that is, solves Φ' and obeys the constraints.

Proposition 5.5 applied to L_c shows that (1) the set of solutions written as words in a regular language $\mathcal{T} \subseteq Q_{G,S,\lambda',\mu'}$ surjecting onto the group is ET0L, and (2) the set of solutions written as words in a regular language $\mathcal{T} \subseteq Q_{G,S,\lambda',\mu'}$ in bijection with the group is EDT0L. Part (3) also follows immediately from Proposition 5.5. □

9. Non-explicit constraints

In this section we consider systems of equations with effective quasi-isometrically embeddable rational constraints that are not necessarily constant size with respect to the size of equations. In the proofs of Theorems 7.4 (torsion-free) and 8.12 (the torsion case), when we had to deal with constant size qier constraints, we applied Proposition 6.1 ([12, Proposition 9.4]) to replace the automaton for each rational constraint R_{X_i} by another automaton accepting all (G, S, λ, μ)-quasigeodesic words representing group elements which belong to the rational subset R_{X_i}; then we could intersect them with our covering solution set to obtain solutions which obey the constraints. Since the constraints were assumed to be constant size, this operation was considered to have no contribution to the complexity. Now if we consider constraints whose size is considered as contributing to the input size, Proposition 6.1 shows that we have no hope in general to state any computable upper bound on the size required to obtain the automata accepting $\widetilde{\mathcal{R}}_i = \pi^{-1}(R_{X_i}) \cap Q_{G,S,\lambda,\mu}$.

We therefore state Theorem 9.1 in terms of existence and decidability only, without any space complexity bound.

Theorem 9.1 (Systems with arbitrary effective qier constraints)**.** *Let G be a hyperbolic group with or without torsion, with finite symmetric generating set S. Let $\#$ be a symbol not in S. Let Φ be a finite system of equations and inequations over variables $X_1, \dots, X_r$, with effective qier constraints as in Section 2 and Definition 4.8.*

Then we have the following.

(1) *For any $\lambda' \geqslant 1, \mu' \geqslant 0, \lambda' \in \mathbb{Q}$ and for any regular language $\mathcal{T} \subseteq Q_{G,S,\lambda',\mu'}$ such that the projection $\pi : \mathcal{T} \to G$ is a surjection, the full set of $\mathcal{T}$-solutions*

$$Sol_{\mathcal{T},G}(\Phi) = \{(w_1\#\dots\#w_r) \mid w_i \in \mathcal{T}, (\pi(w_1), \dots, \pi(w_r)) \text{ solves } \Phi\}$$

is ET0L.

(2) *For any $\lambda' \geqslant 1, \mu' \geqslant 0, \lambda' \in \mathbb{Q}$ and for any regular language $\mathcal{T} \subseteq Q_{G,S,\lambda',\mu'}$ such that the projection $\pi : \mathcal{T} \to G$ is a bijection, the full set of $\mathcal{T}$-solutions*

$$Sol_{\mathcal{T},G}(\Phi) = \{(w_1\#\dots\#w_r) \mid w_i \in \mathcal{T}, (\pi(w_1), \dots, \pi(w_r)) \text{ solves } \Phi\}$$

is EDT0L.

(3) It is decidable whether or not the solution set of Φ *is empty, finite or infinite.*

Proof. In the proofs of Theorems 7.4 (torsion-free) and 8.12 (the torsion case) the space required by the algorithm which constructs the automata accepting the languages $\widetilde{R_{X_i}}$ is no longer constant, so do the same steps but ignore the space calculation. Propositions 7.3 and 8.8 produce an EDT0L (in the torsion-free case), respectively ET0L (in the torsion case) covering solution set to a system without any constraints (except say for the inequations). One then employs [12, Proposition 9.4] to obtain finite state automata which encode the constraints for each variable as words over $Q_{G,S,\lambda',\mu'}$ for effectively computed values of λ', μ'. Intersect the EDT0L language with these languages to obtain a covering solution set to the system Φ including constraints. Then apply Proposition 5.5 to obtain the full set of $\mathcal{T}$-solutions as E(D)T0L as appropriate.

By [30] it is decidable whether the language produced by some given ET0L system is empty, finite, or infinite. So it is decidable to conclude whether or not the solution set is empty, finite, or infinite since we have an effective construction of the NFA for the EDT0L grammar, and then we can decide finiteness/emptiness using [30]. Equivalently, it follows from our construction and our previous work [7, 14] that the solutions in the hyperbolic group are empty, finite or infinite if and only if the corresponding solution sets in the associated (virtually) free group are. □

10. Additional result for free groups

In this section we prove Corollary 1.2. This is a direct corollary of [7] and does not rely on any other results in the present paper.

Corollary 1.2 (Full solutions in free groups)**.** *Let* G *be a finitely generated free group with free basis* A_+*, and* $A = A_+ \cup \{x^{-1} \mid x \in A_+\}$ *the* free basis generating set *for* G*. Let* Φ *be a system of equations and inequations of size* n *with constant size rational constraints.*

Then the set of all solutions, as tuples of all words over S*, is ET0L, and the algorithm which on input* Φ *prints a description for the ET0L grammar runs in* $\mathsf{NSPACE}(n \log n)$*.*

Proof. Let $(C, A \cup \{\#\}, c_0, \mathcal{M})$ be the EDT0L system from [7]. Assume $c_0 = a_1 \dots a_s \in C^*$. We construct a new system $(C', A\cup\{\#\}, c_0', \mathcal{M}')$ where $C' = C\cup\{\diamond\}$, $c_0' = \diamond a_1 \diamond \cdots \diamond a_s \diamond$ and $\mathcal{M}'$ is obtained from $\mathcal{M}$ as follows. Replace each rule $(a, u_1 \dots u_k)$ in each table labeling an edge of $\mathcal{M}$ by $(a, u_1 \diamond \cdots \diamond u_k)$. Let t be the non-deterministic table $t = \{(\diamond, \diamond), (\diamond, \diamond a \diamond a^{-1} \diamond) \mid a \in A\}$ (and is constant on all other letters of C') which inserts cancelling pairs aa^{-1} at certain places between A-letters in a word. For each accept state q of $\mathcal{M}$, print a new edge $(q, q_{\text{unreduce}}, \varepsilon)$ to a new state q_{unreduce}, and print a loop $(q_{\text{unreduce}}, q_{\text{unreduce}}, t)$. Finally print an edge $(q_{\text{unreduce}}, q_{\text{accept}}, \phi)$ to a new unique accept state q_{accept} where ϕ is the table (homomorphism) which sends $\diamond$ to the empty string. These modifications can be printed in the same space bound, and produce a grammar which accepts all possible tuples words obtained from tuples of shortlex words by inserting aa^{-1} pairs (iterating the loop t at q_{unreduce} reverses the process of free reduction). □

11. Acknowledgements

We sincerely thank Yago Antolin, Alex Bishop, François Dahmani, Volker Diekert, Derek Holt, Jim Howie and Alex Levine for invaluable help with this work. We also thank the anonymous reviewer for their careful reading and comments.

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