

Quantum algorithms for graph problems with cut queries

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Abstract

Let G be an n -vertex graph with m edges. When asked a subset S of vertices, a cut query on G returns the number of edges of G that have exactly one endpoint in S . We show that there is a bounded-error quantum algorithm that determines all connected components of G after making $O(\log(n)^6)$ many cut queries. In contrast, it follows from results in communication complexity that any randomized algorithm even just to decide whether the graph is connected or not must make at least $\Omega(n/\log(n))$ many cut queries. We further show that with $O(\log(n)^8)$ many cut queries a quantum algorithm can with high probability output a spanning forest for G .

En route to proving these results, we design quantum algorithms for learning a graph using cut queries. We show that a quantum algorithm can learn a graph with maximum degree d after $O(d \log(n)^2)$ many cut queries, and can learn a general graph with $O(\sqrt{m} \log(n)^{3/2})$ many cut queries. These two upper bounds are tight up to the poly-logarithmic factors, and compare to $\Omega(dn)$ and $\Omega(m/\log(n))$ lower bounds on the number of cut queries needed by a randomized algorithm for the same problems, respectively.

The key ingredients in our results are the Bernstein-Vazirani algorithm, approximate counting with “OR queries”, and learning sparse vectors from inner products as in compressed sensing.

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1 Introduction

The *cut* and *additive* functions of an n -vertex undirected graph $G = (V, E)$ are $c : 2^V \rightarrow \mathbb{N}$ defined as $c(S) = |E(S, V \setminus S)|$, and $a : 2^V \rightarrow \mathbb{N}$ defined as $a(S) = |E(S, S)|$, where $E(S, T)$ is the set of edges between sets $S, T \subseteq V$. In this paper we study quantum algorithms for several graph problems when the algorithm has oracle access to a cut or additive function for the graph. In particular, we look at the problems of learning all the edges of a graph, determining if the graph is connected, outputting a spanning tree, and determining properties of the graph such as if it is bipartite or acyclic.

Motivation to study algorithms with cut or additive query access comes from at least two different sources. Algorithms with a cut oracle have been studied for computing the minimum cut of a graph [RSW18, MN20], and the study of graph algorithms with an additive oracle began in connection with an application to genomic sequencing [GK98, ABK⁺04]. We describe the previous works on these topics and their connection to our results in turn.

Rubinfeld et al. [RSW18] give a randomized algorithm that exactly computes the size of a minimum cut in an n -vertex unweighted and undirected graph with $\tilde{O}(n)$ many cut queries¹. More recently, [MN20] generalizes this result to also give a randomized $\tilde{O}(n)$ cut query algorithm to exactly compute the size of a minimum cut in a weighted and undirected graph. These results are tight up to the polylogarithmic factors. As observed by Harvey [Har08], known lower bounds on the communication complexity of determining if a graph is connected—when divided by $\log(n)$, the number of bits needed to communicate the answer of a cut query—give lower bounds on the number of cut queries needed by an algorithm to solve connectivity, and thus also min-cut. It is known that the deterministic communication complexity of connectivity is $\Omega(n \log(n))$ [HMT88] and the randomized communication complexity is $\Omega(n)$ [BFS86], which gives the tightness of the upper bounds for the cut query complexity claimed above.

Computing the minimum cut with a cut oracle is a special case of the problem of submodular function minimization with an evaluation oracle. For an n -element set Ω , a function $f : 2^\Omega \rightarrow \mathbb{R}$ is submodular if it satisfies $f(S \cap T) + f(S \cup T) \leq f(S) + f(T)$ for all $S, T \subseteq \Omega$. As the truth table of f is exponentially large in n , submodular functions are often studied assuming access to an *evaluation oracle* for f , which for any $S \subseteq \Omega$ returns $f(S)$. The submodular function minimization problem is to compute $\min_{S \subseteq \Omega} f(S)$. The cut function is a submodular function, thus computing the minimum cut of a graph with a cut oracle is a special case of this problem². The ellipsoid method was originally used to show that submodular function minimization in general can be solved in polynomial time with an evaluation oracle [GLS81, GLS88], and the current record shows that this can be done with $O(n^3)$ many calls of the evaluation oracle [Jia20], improving on the bound of $\tilde{O}(n^3)$ by Lee, Sidford, and Wong [LSW15]. If $M = \max_{A \subseteq V} |f(A)|$ then the current best known weakly polynomial algorithm makes $O(n^2 \log(nM))$ many queries [LSW15], and the best pseudo-polynomial time algorithm makes $O(nM^3 \log(n))$ many queries [CLSW17]. The best lower bound on the number of evaluation oracle queries required to find the minimum value of a submodular

¹The $\tilde{O}()$ notation hides polylogarithmic factors in n .

²The cut function is a symmetric submodular function, $c(S) = c(V \setminus S)$. For symmetric submodular functions the minimization problem becomes to find a non-trivial minimizer $\emptyset \subset S \subset V$.

function is $\Omega(n)$ for deterministic algorithms and $\Omega(n/\log(n))$ for randomized algorithms, coming from the above mentioned communication bounds on determining if a graph is connected.

Quantum algorithms for submodular function minimization have been studied for approximating $\min_{S \subseteq \Omega} f(S)$ [HRRS19]. For the problem of exact minimization no quantum algorithm has been considered. Here we study the problem which is the source of the best known classical lower bounds on submodular function minimization, the problem of determining if a graph is connected with a cut oracle. For this special case, we show that there is a surprisingly efficient quantum algorithm: we give a quantum algorithm that determines if a graph is connected or not with high probability after $O(\log(n)^6)$ many cut queries (Theorem 44). With the same number of queries the algorithm can output the connected components of the graph, and can be extended to output a spanning forest after $O(\log(n)^8)$ many cut queries (Theorem 51). We leave deciding if minimum cut can also be computed by a quantum algorithm with a polylogarithmic number of cut queries as a tantalizing open problem.

Additive queries were originally defined because of an application to genomic sequencing. One technique for sequencing a genome is to first use shotgun sequencing to come up with random segments of DNA called *contigs*. To complete the sequencing of the genome it remains to fill the gaps between the contigs, and thus also figure out how the contigs are connected to one another. This can be done through polymerase chain reaction (PCR) techniques. To do this, one makes primers, short sequences that pair to the end of a contig. When two contigs are connected by a gap and the primers for each end of this gap are placed in solution with them, the PCR technique can connect the contigs by filling in the gap of base pairs between them. If there are n contigs, the process of putting separately all pairs of primers together with them to find the contigs connected together can take $\binom{n}{2}$ PCR experiments. This complexity led to the development of multiplex PCR, where one places many primers together with the contigs at once. The algorithmic question becomes, what is the best way to add primers to the contigs in order to fill all gaps between contigs while minimizing the number of multiplex PCR assays?

Grebinski and Kucherov nicely formalize this algorithmic problem in the language of graph theory [GK98]. They give several possible models, depending on exactly what information one assumes to learn from a multiplex PCR assay. The additive query arises if one assumes that when primers are placed together with contigs, one can read from the result the number of contigs that were paired. A weaker model only assumes that when primers are placed together with contigs, one learns whether or not there is a pairing of contigs. In this model, when one queries a subset of vertices S , one learns whether or not $|E(S, S)| > 0$. We call these *empty subgraph queries*.

In the genomic sequencing application, the primary goal is to learn all the edges in the graph. Furthermore, for this application the graph is guaranteed to have degree at most 2. Grebinski and Kucherov [GK98] show that a Hamiltonian cycle can be learned by a randomized algorithm with $O(n \log n)$ many empty subgraph queries, and [ABK⁺04] gives a nonadaptive randomized algorithm to learn a matching with $\tilde{O}(n \log n)$ many empty subgraph queries. After these works, the complexity of learning general weighted graphs with additive queries was also extensively studied [GK00, CK10, Maz10, BM10, BM11]. Notable results are that a graph of maximum degree d can be learned with $O(dn)$ many additive queries [GK00], and that an m -edge graph can be learned with $O(\frac{m \log n}{\log m})$ many additive queries [CK10, BM11]. Both of these results are tight up

to a logarithmic factor in n .

We show the additive quantum query complexity of learning a graph with maximum degree d is $\Theta(d \log(n/d))$ (see Corollary 27 and Corollary 34), and that a graph with m edges can be learned after $O(\sqrt{m \log n} + \log(n))$ many cut queries (see Corollary 27), which is tight up to logarithmic factors (see Corollary 33). It is straightforward to show that a cut query can be simulated by 3 additive queries. Somewhat surprisingly, we show that in general for weighted graphs simulating an additive query can require $\Omega(n)$ many cut queries (see Theorem 26). Nonetheless, our algorithms for learning graphs can also be made to work with cut queries, at the expense of an additional multiplicative logarithmic factor. The quantum algorithm for efficiently learning a low degree graph with cut queries is a key subroutine in our algorithm for connectivity.

1.1 Our techniques

We begin by giving an overview of our quantum algorithm for learning a graph of maximum degree d . The high level idea follows the $O(dn)$ classical algorithm given in [GK00]. The starting point is the basic principle of compressed sensing: a sparse vector can be learned from its dot product with a few random vectors. Say $y \in \{0, 1\}^n$ has at most d ones. With high probability, we can learn y from the values of $x_1^T y, \dots, x_k^T y$ for $k = 3d \log n$ where each $x_i \in \{0, 1\}^n$ is chosen randomly (see Lemma 1). Moreover, this algorithm is non-adaptive, thus given a matrix $Y \in \{0, 1\}^{n \times n}$ where each column has at most d ones, we can learn Y with high probability from the product XY where $X \in \{0, 1\}^{k \times n}$ is chosen randomly with $k = 3d \log n$. We can apply this principle to learn the adjacency matrix A_G of a graph G of maximum degree d .

Classically, a single product $x^T A_G$ can be computed with $O(n)$ cut queries. We will show how to use the Bernstein-Vazirani algorithm [BV97], to compute $x^T A_G$ with a constant number of cut queries, provided the graph G is bipartite. We can then apply this to learn a general graph G as the complete graph can be covered by $O(\log n)$ many complete bipartite graphs. This leads to an overall complexity of $\tilde{O}(d)$ many cut queries.

These results can be phrased more generally in terms of learning an m -by- n matrix A when given the ability to query $x^T A y$ for Boolean vectors $x \in \{0, 1\}^m, y \in \{0, 1\}^n$. We call such queries *matrix cut queries*, and they provide a very clean way of formulating our results³ Such queries are also used in the classical learning results [GK00, Maz10, BM11], and we find it useful to make them explicit here.

Our algorithm for connectivity is a contraction based algorithm. If we find an edge between vertices u and v , we can merge u and v into a single vertex without changing the connectivity of the graph. More generally, if we know that a set S_1 of vertices is connected, and a disjoint set S_2 of vertices is connected, and we learn that there is an edge between a vertex in S_1 and a vertex in S_2 , we can merge S_1 and S_2 without changing the connectivity of the graph.

Our algorithm for connectivity proceeds in rounds, and maintains the invariant of having a partition of the vertex set V into sets S_1, \dots, S_k , each of which is known to be connected. We call

³We term these matrix cut queries because of the relation to the cut norm of a matrix A , defined as $\max_{x \in \{0, 1\}^m, y \in \{0, 1\}^n} |x^T A y|$. The cut norm played a crucial role in the matrix decomposition results of [FK99] used for efficient approximation algorithms for maximum cut and other graph problems.

such sets S_i supervertices, and the number of S_j such that there is an edge between a vertex in S_i and a vertex in S_j the *superdegree* of S_i . A round proceeds by first in parallel approximating the superdegree of each S_i . For this we use ideas originating with Stockmeyer [Sto83] for approximately counting the number of ones in a vector $y \in \{0, 1\}^n$ with OR queries, queries that for a subset $S \subseteq \{1, \dots, n\}$ answer 1 iff there is an $i \in S$ with $y_i = 1$. We show that we can efficiently simulate the OR queries needed in the context of approximating the superdegree by cut queries. For supervertices whose superdegree is below a threshold—taken to be $\Theta(\log(n)^2)$ —we learn all of their superneighbors with polylogarithmically many cut queries using our techniques for learning a matrix with sparse columns. To take care of the supervertices with superdegree above the threshold, we randomly sample $k/2$ many of the S_j . With high probability every high superdegree supervertex will be connected to at least one S_j in the sample set, and moreover we show that we can learn edges witnessing this fact with polylogarithmically many cut queries. We then contract the supervertices according to all the edges learned in the round, and show that we either learn that the graph is disconnected or arrive at a new partition of V into at most $k/2$ many connected sets. In this way, the algorithm terminates in at most $O(\log n)$ many rounds, and as each round uses polylogarithmically many cut queries we arrive at the polylogarithmic complexity.

1.2 Related work

As far as we are aware, ours is the first work studying quantum algorithms using a cut or additive query oracle. Quantum algorithms for graph problems typically use the adjacency matrix or adjacency list input model. Early work [DHHM06, SYZ04, Zha05] gives tight bounds for many graph problems in these models. For example, it is shown in [DHHM06] that the quantum query complexity of connectivity is $\Theta(n^{3/2})$ in the adjacency matrix model and $\Theta(n)$ in the adjacency list model, the quantum query complexity of minimum spanning tree is $\Theta(n^{3/2})$ in the adjacency matrix model and $\Theta(\sqrt{nm})$ in the adjacency list model, for a graph with m edges. Focusing on the adjacency matrix model, the quantum query complexity of Bipartiteness and of Graph Matching is $\Omega(n^{3/2})$ [Zha05], and that of Scorpion graph is $\Theta(\sqrt{n})$, which is the lowest possible quantum query complexity for total graph properties [SYZ04]. Also see [MSS07, Bel12, JKM13, LMS17, Gal14] for triangle finding, [CK12] for general graph minor-closed graph properties, and [BCG⁺20] for discussions on partial graph properties.

In the time complexity model, recent work of Apers and de Wolf [AdW19] shows that a cut of size at most $(1 + \epsilon)$ -times that of the minimum cut can be found in time $\tilde{O}(\sqrt{mn}/\epsilon)$ given adjacency list access to the graph. Their work more generally shows that an ϵ -cut sparsifier of a graph with $\tilde{O}(n/\epsilon^2)$ many edges can be found in time $\tilde{O}(\sqrt{mn}/\epsilon)$. Originally defined in [BK96], an ϵ -cut sparsifier of a graph G is a reweighted subgraph H such that the value of every cut in H is a multiplicative $(1 + \epsilon)$ -approximation of the corresponding cut value in G .

In the classical setting, a recent work [RWZ20] studies a generalization of matrix cut queries where for a fixed field \mathbb{F} one can access an input matrix $A \in \mathbb{F}^{m \times n}$ through queries of the form $x^T A y$ for $x \in \mathbb{F}^m, y \in \mathbb{F}^n$. They examine the complexity of an assortment of problems from graph theory, linear algebra, and statistics in this model.

1.3 Organization

The rest of the paper is organized as follows. In Section 2, we set up the model and notation, and give some algorithmic ingredients such as degree estimation by OR queries, and a quantum primitive of vector learning that is used a number of times in later sections. In Section 3 and Section 4 we introduce several oracles for accessing a matrix or a graph, and discuss their relative powers. Some intermediate results built on these oracles will be presented, which lead to the cut oracle algorithms in the next sections: Section 5 for computing connected components by $O(\log(n)^6)$ queries, and Section 6 for computing spanning forest by $O(\log(n)^8)$ cut queries. Two applications of the latter algorithm will also be given, which test if a graph is bipartite or acyclic, also with $O(\log(n)^8)$ many cut queries. The paper concludes with several open problems.

2 Preliminaries and primitives

For a positive integer M , we denote $\{0, 1, 2, \dots, M-1\}$ by $[M]$. For two vectors $X, Y \in [M]^k$, we denote their dot product over the integers by $X \cdot Y = \sum_{i=1}^k X_i Y_i$. For a set $S \subseteq \{1, \dots, M\}$ we denote the complement of S by \bar{S} . For a set U , we let $\tilde{U} = \{\{u\} : u \in U\}$. For a string $x \in \{0, 1\}^n$ we use $|x|$ for the Hamming weight of x , i.e. the number of ones. Let $\text{OR}_n : \{0, 1\}^n \rightarrow \{0, 1\}$ denote the OR function, i.e. the function such that $\text{OR}_n(x) = 1$ iff $|x| > 0$. Let $\text{MAJ}_n : \{0, 1\}^n \rightarrow \{0, 1\}$ denote the Majority function, i.e. the function such that $\text{MAJ}_n(x) = 1$ iff $|x| \geq \lceil n/2 \rceil$. When the input length is clear from context we will drop the subscript.

In pseudocode for our algorithms we will use some Matlab-like notation. We use $\text{zeros}(k, \ell)$ and $\text{ones}(k, \ell)$ to denote the k -by- ℓ all zeros matrix and all ones matrix, respectively. For M a k -by- ℓ matrix and $H \subseteq \{1, \dots, k\}, R \subseteq \{1, \dots, \ell\}$ we use $M(H, R)$ to refer to the $|H|$ -by- $|R|$ submatrix of M given by selecting the rows of M in H and columns of M in R . We will further use the shorthand $M(i : j, k : \ell)$ for $M(H, R)$ where $H = \{i, i+1, \dots, j\}$ and $R = \{k, k+1, \dots, \ell\}$. To denote the i^{th} column of M we will use $M(:, i)$. Similarly for a vector $x \in \mathbb{R}^k$, we use $x(H) \in \mathbb{R}^{|H|}$ to denote the vector formed by selecting the coordinates in H . For column vectors $x \in \mathbb{R}^k, y \in \mathbb{R}^\ell$ we use $[x; y] \in \mathbb{R}^{k+\ell}$ for the column vector formed by vertically concatenating them. For a vector $x \in \mathbb{R}^k$ and $d \leq k$, we say that x is d -sparse if at most d coordinates in x are nonzero. We use $\ell_0(x)$ to denote the sparsity, i.e. the number of nonzero entries, of vector x .

We will frequently encounter the situation where we have a set $\mathcal{S} = \{S_1, \dots, S_k\}$ and take a subset $H \subseteq \mathcal{S}$ where $H = \{S_{i_1}, \dots, S_{i_t}\}$. We use $\text{ind}(H) = \{i_1, \dots, i_t\}$ to return the indices of the elements in H .

Lemma 1. *Let r, M, d be positive integers with $M \geq 2$ and $d \leq r/2$. Let $N \leq d \binom{r}{d} M^d$ be the number of d -sparse strings in $[M]^r$. Let A be an N -by- r matrix whose rows are all the d -sparse strings in $[M]^r$. Let R be a random r -by- q Boolean matrix, with each entry chosen independently and uniformly from $\{0, 1\}$. If $q = \lceil 2d \log(eMr/d) + 2 \log(d) + \log(1/\delta) \rceil$ then $AR \bmod M$ will have distinct rows with probability at least $1 - \delta$.*

Proof. Let A_i, A_j be two different rows of A . Then for a random vector $z \in \{0, 1\}^r$, $\Pr_z[A_i z \bmod M = A_j z \bmod M] \leq 1/2$. Therefore $\Pr_R[A_i R \bmod M = A_j R \bmod M] \leq 1/2^q$. The result follows by a union bound over the $\binom{N}{2}$ pairs of distinct rows. \square

Definition 2 (OR query). For $x \in \{0, 1\}^\ell$ and $S \subseteq \{1, \dots, \ell\}$ an OR query $\text{OR}(x, S)$ returns $\bigvee_{i \in S} x_i$. For strings $x^{(1)}, \dots, x^{(k)} \in \{0, 1\}^\ell$ and a subset $S \subseteq \{1, \dots, \ell\}$, a k -OR query returns the string $a \in \{0, 1\}^k$ where $a_i = \text{OR}(x^{(i)}, S)$.

The problem of estimating the Hamming weight of a string x using OR queries was considered in the seminal work of Stockmeyer [Sto83]. We recount his basic analysis here, modifying it for our application of using k -OR queries to estimate the Hamming weight of $x^{(1)}, \dots, x^{(k)}$ in parallel.

Fact 3. For all $x \geq 1$

$$\left(1 - \frac{1}{x}\right)^x < \frac{1}{e} < \left(1 - \frac{1}{x}\right)^{x-1}$$

Definition 4. Let $x \in \{0, 1\}^\ell$. An r -out-of- ℓ sample consists of uniformly at random choosing r many elements of $\{1, \dots, \ell\}$ with replacement. An r -test consists of taking an r -out-of- ℓ sample S and querying $\text{OR}(x, S)$. When $\text{OR}(x, S) = 1$, we say that the r -test S succeeds.

Lemma 5. Let $x \in \{0, 1\}^\ell$ and suppose that $|x| = t$. The probability that an r -test S succeeds is

$$\Pr_S[\text{OR}(x, S) = 1] = 1 - \left(1 - \frac{t}{\ell}\right)^r,$$

where the probability is taken over the choice of an r -out-of- ℓ sample S . In particular,

$$1 - \exp\left(-\frac{rt}{\ell}\right) < \Pr_S[\text{OR}(x, S) = 1] < 1 - \exp\left(-\frac{rt}{\ell - t}\right)$$

Proof. As S is chosen with replacement, the probability that any element $i \in S$ satisfies $x_i = 1$ is exactly $\frac{t}{\ell}$. The first statement follows accordingly. The second statement follows by applying Fact 3 to the first statement. \square

Definition 6 (Good estimate). We say that $b \in \mathbb{R}^k$ is a *good estimate* of $c \in \mathbb{R}^k$ if $b(i)/4 \leq c(i) \leq 2b(i)$ for all $i \in \{1, \dots, k\}$.

Next we see how to use k -OR queries to estimate the Hamming weights of k given strings.

Lemma 7. Let $x^{(1)}, \dots, x^{(k)} \in \{0, 1\}^\ell$. Taking $a = 200 \left\lceil \log \left(\frac{k(\lceil \log \ell \rceil + 1)}{\delta} \right) \right\rceil$, Algorithm 1 outputs a vector b that is a good estimate of $(|x^{(1)}|, \dots, |x^{(k)}|)$ with probability at least $1 - \delta$ after making $O\left((\log(\ell) + 1) \log \left(\frac{k(\log(\ell) + 1)}{\delta} \right)\right)$ many k -OR queries.

Proof. First we argue about the complexity. There are $\lceil \log \ell \rceil + 1$ iterations of the outer for loop, and inside the loop we make a many k -OR queries. Thus the total number of k -OR queries is

$$O(a(\log(\ell) + 1)) = O\left((\log(\ell) + 1) \log \left(\frac{k(\log(\ell) + 1)}{\delta} \right)\right).$$

Algorithm 1 Approximate Count(δ)

Input: k -OR oracle for strings $x^{(1)}, \dots, x^{(k)} \in \{0, 1\}^\ell$, error bound δ
Output: A vector $b \in \mathbb{R}^k$ that is a good estimate of $(|x^{(1)}|, \dots, |x^{(k)}|)$ with probability $1 - \delta$.

- 1: $a \leftarrow 200 \left\lceil \log \left(\frac{k(\lceil \log \ell \rceil + 1)}{\delta} \right) \right\rceil$
- 2: **for** $j = 0$ to $\lceil \log \ell \rceil$ **do**
- 3: **for** $q = 1$ to a **do**
- 4: $S_q \leftarrow$ randomly choose $\min(2^j, \ell)$ many elements from $\{1, \dots, \ell\}$ with replacement
- 5: $\text{ans}_q \leftarrow (\text{OR}(x^{(1)}, S_q), \dots, \text{OR}(x^{(k)}, S_q))$
- 6: **end for**
- 7: **for** $i = 1$ to k **do**
- 8: $B(i, j) \leftarrow \text{MAJ}(\text{ans}_1(i), \dots, \text{ans}_a(i))$
- 9: **end for**
- 10: **end for**
- 11: **for** $i = 1$ to k **do**
- 12: $s \leftarrow \text{argmin}_j \{B(i, j) = 1\}$
- 13: $b(i) \leftarrow \frac{\ell}{2^s}$
- 14: **end for**

Now we argue about the error probability. Fix some $x^{(i)}$, which will henceforth be called x . Say the estimate the algorithm gives for $|x|$ is α . We will upper bound the probability that α is a bad estimate for $|x|$, and then use a union bound over the k many strings to get the final result.

Suppose that $r = 2^j$ is such that $r \geq \ell/|x|$. Then if S is a r -out-of- ℓ sample, by Lemma 5

$$\Pr_S[\text{OR}(x, S) = 1] > 1 - \exp\left(-\frac{r|x|}{\ell}\right) \geq 1 - \frac{1}{e} > 0.6 .$$

Thus by a Chernoff bound, the probability that the majority of a many r -tests is 0 is at most $\exp(-a/200)$. As the algorithm must perform an r -test for an r satisfying $\ell/|x| \leq r \leq 2\ell/|x|$, this means our estimate α will satisfy $|x| \leq 2\alpha$, except with probability at most $\exp(-a/200)$.

To bound the probability that our estimate is too large, we first need to treat the special case where $|x| \geq \ell/2$. In this case, the probability that the majority of a many 2-tests is 1 is at least $1 - \exp(-a/200)$. The algorithm is therefore correct in this case with at least this probability as it outputs a valid answer if either a majority of 1-tests or a 2-tests succeeds.

For the remainder of the proof, therefore, we assume that $|x| < \ell/2$. Suppose that $r = 2^j$ is such that $\ell/r > 4|x|$. Then if S is an r -out-of- ℓ sample, by Lemma 5

$$\Pr_S[\text{OR}(x, S) = 1] < 1 - \exp\left(-\frac{r|x|}{\ell - |x|}\right) \leq 1 - \exp\left(-\frac{2r|x|}{\ell}\right) \leq 1 - \exp(-1/2) \leq 0.4 .$$

Thus by a Chernoff bound, the probability that the majority of a many r -tests is 1 is at most $\exp(-a/200)$. By a union bound, the probability that an r -test will be 1 for any $r = 2^j$ with $r < \ell/(4|x|)$ is at most $\lceil \log \ell \rceil \exp(-a/200)$. Thus, our estimate α will satisfy $\alpha/4 \leq |x|$ except

with probability $\lceil \log \ell \rceil \exp(-a/200)$. Thus by a union bound, overall we will produce an estimate α satisfying $\alpha/4 \leq |x| \leq 2\alpha$ except with probability $(\lceil \log \ell \rceil + 1) \exp(-a/200)$.

Finally, by a union bound over the k many strings the total failure probability will be at most $k(\lceil \log \ell \rceil + 1) \exp(-a/200)$, giving the lemma. \square

Lemma 8. *Let $x^{(1)}, \dots, x^{(k)} \in \{0, 1\}^\ell$ be such that $t/8 \leq |x^{(i)}| \leq 2t$ for all $i \in \{1, \dots, k\}$. For $\delta > 0$, sample with replacement $\frac{8\ell \ln(k/\delta)}{t}$ elements of $\{1, \dots, \ell\}$, and call the resulting set R . Then*

- $\Pr_R[\exists i \in \{1, \dots, k\} : |x^{(i)}(R)| = 0] \leq \delta$,
- $\Pr_R[\exists i \in \{1, \dots, k\} : |x^{(i)}(R)| > 64 \ln(k/\delta)] \leq \delta$.

Proof. Let $x \in \{0, 1\}^\ell$ with $t/8 \leq |x| \leq 2t$. The probability that $|x(R)| = 0$ is at most δ/k by Lemma 5. Thus the first item holds by a union bound over the k many strings $x^{(1)}, \dots, x^{(k)}$.

For the second item, note $\ln(k/\delta) \leq \mathbb{E}_R[|x(R)|] \leq 16 \ln(k/\delta)$. By a Chernoff bound, the probability that $|x(R)|$ is a factor c larger than its expectation is at most $\exp(-(c-1) \ln(k/\delta)/3)$. Thus taking $c = 4$ the probability that $|x(R)| > 64 \ln(k/\delta)$ is at most δ/k . The second item then holds by a union bound over the k many strings $x^{(1)}, \dots, x^{(k)}$. \square

Next we give our quantum primitive algorithm for learning a vector from subset sums. Note that different than state learning in many previous work, here we aim to learn all entries of the vector precisely and correctly. This algorithm will be repeatedly used in various forms in later sections.

Lemma 9. *Let $x \in [M]^k$ and suppose we have an oracle that for any subset $S \subseteq [k]$ returns $\sum_{i \in S} x_i \bmod M$. Then there is a quantum algorithm which learns x with $m = \lceil \log(M) \rceil$ queries without any error.*

Proof. We can represent all the elements of $[M]$ using m bits. We first describe how we can use the oracle to compute $t \cdot x \bmod M$ for any $t \in [M]^k$. Let $t = (t_1, \dots, t_k)$ and let $t_i = \sum_{j=0}^{m-1} 2^j t_i(j)$, where $t_i(j) \in \{0, 1\}$. Then

$$t \cdot x \bmod M = \sum_{j=0}^{m-1} 2^j (t(j) \cdot x \bmod M) \bmod M$$

where $t(j) = (t_1(j), t_2(j), \dots, t_k(j)) \in \{0, 1\}^k$. Now $t(j) \cdot x \bmod M$ can be computed with one call to the oracle, thus $t \cdot x \bmod M$ can be computed with m calls to the oracle.

With a quantum algorithm we can do this in superposition over all t . Combining this with the phase kickback trick allows us to compute the inverse quantum Fourier transform of x over $(\mathbb{Z}_M)^k$. Similarly to the Bernstein-Vazirani algorithm, we can then learn x by applying the quantum Fourier transform.

More precisely, we will work with two quantum registers. In the first register we are computing over the group $(\mathbb{Z}_M)^k$, and in the second register over \mathbb{Z}_M . We will use in the second register the auxiliary state

$$|\xi_M\rangle = \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} \omega_M^j |j\rangle,$$

where ω_M is a primitive M th root of the unity. In the first register we create the uniform superposition, therefore we start with the state

$$\frac{1}{\sqrt{M^k}} \sum_{t \in [M]^k} |t\rangle |\xi_M\rangle.$$

With m queries to the oracle we compute $t \cdot x \bmod M$ that we add to the second register, creating

$$\frac{1}{\sqrt{M^k}} \sum_{t \in [M]^k} \omega_M^{-t \cdot x} |t\rangle |\xi_M\rangle.$$

Applying the quantum Fourier transform over $(\mathbb{Z}_M)^k$, we can find in the first register x . Overall the quantum algorithm uses m queries. \square

3 Learning a matrix

Let n be a positive integer, and V an ordered set of size n . For a subset S of V , we denote by $\chi_S \in \{0, 1\}^n$ is the characteristic vector of S .

Definition 10 (Matrix cut oracle). Given a matrix $A \in \mathbb{N}^{k \times \ell}$, the matrix cut oracle for A is the function $m_A : \{0, 1\}^k \times \{0, 1\}^\ell \rightarrow \mathbb{N}$ satisfying $m_A(x, y) = x^T A y$.

Lemma 11. *Let $\alpha, M \in \mathbb{N}$ and $A \in [\alpha]^{k \times \ell}$ be a matrix. There is a quantum algorithm making $\lceil \log(M) \rceil$ many matrix cut queries to A that perfectly computes $(Ay) \bmod M$ for any $y \in \{0, 1\}^\ell$.*

Proof. Let $S \subseteq \{1, \dots, k\}$. For any $y \in \{0, 1\}^\ell$ we have $\sum_{i \in S} (Ay)_i \bmod M = \chi_S^T A y \bmod M$, thus we can compute $\sum_{i \in S} (Ay)_i \bmod M$ with one matrix cut query. The result then follows from Lemma 9. \square

Corollary 12. *Let $A \in [M]^{k \times \ell}$. There is a quantum algorithm that perfectly learns A after $\min\{k, \ell\} \cdot \lceil \log(M) \rceil$ many matrix cut queries.*

Proof. If $\ell \leq k$ define $B = A$, otherwise let $B = A^T$. Let $m = \min\{k, \ell\}$ be the number of columns in B . For each standard basis vector e_1, \dots, e_m , we can learn $B e_i = B e_i \bmod M$ with $\lceil \log(M) \rceil$ matrix cut queries by Lemma 11. This tells us the i^{th} column of B . Thus we can learn B entirely with $m \lceil \log(M) \rceil$ many matrix cut queries. \square

Lemma 13. *Let $A \in [M]^{k \times \ell}$ be a matrix with d -sparse rows. There is a quantum algorithm that learns A with probability at least $1 - \delta$ after making $(4d \lceil \log(M\ell/d) \rceil + \lceil \log(1/\delta) \rceil) \lceil \log(M) \rceil$ many matrix cut queries.*

Proof. If $d \geq \ell/2$ then we use Corollary 12 to learn A perfectly with $\ell \lceil \log(M) \rceil \leq 4d \lceil \log(M\ell/d) \rceil$ many matrix cut queries.

Now assume $d < \ell/2$. By Lemma 1, taking $q = 4d \lceil \log(M\ell/d) \rceil + \lceil \log(1/\delta) \rceil$ and computing $AZ \bmod M$ for a random ℓ -by- q Boolean matrix Z allows us to learn A with probability at least $1 - \delta$. By Lemma 11 we can compute $AZ_i \bmod M$ with $\lceil \log(M) \rceil$ matrix cut queries, where Z_i the i^{th} column of Z . Thus we can learn A after $q \lceil \log(M) \rceil$ many matrix cut queries. \square

Definition 14 (degree sequence). Let $A \in [M]^{k \times \ell}$ be a matrix. The degree sequence of A is the vector $x \in \mathbb{N}^k$ such that x_i is the number of nonzero entries in the i^{th} row of A , for all $i = 1, \dots, k$.

Lemma 15 (Approximate degree sequence). Let M, k, ℓ be positive integers and $A \in [M]^{k \times \ell}$ be a matrix. There is a quantum algorithm, making $O(\log(\ell M)(\log(\ell) + 1) \log(\frac{k(\log(\ell)+1)}{\delta}))$ many matrix cut queries, that with probability at least $1 - \delta$ outputs $\vec{g} \in \mathbb{R}^k$ that is a good estimate of the degree sequence of A . If $M = 2$ then there is a quantum algorithm that outputs the degree sequence of A perfectly after $\lceil \log(\ell + 1) \rceil$ many matrix cut queries.

Proof. Define $x^{(1)}, \dots, x^{(k)} \in \{0, 1\}^\ell$ by $x^{(i)}(j) = 1$ if $A(i, j) > 0$ and $x^{(i)}(j) = 0$ otherwise. The degree sequence of A is then $(|x^{(1)}|, \dots, |x^{(k)}|)$. We apply Lemma 7 to approximate $(|x^{(1)}|, \dots, |x^{(k)}|)$ and therefore the degree sequence of A . To do this, for a subset $S \subseteq \{1, \dots, \ell\}$, we need to serve the k -OR query $(\text{OR}(x^{(1)}, S), \dots, \text{OR}(x^{(k)}, S))$. Let χ_S be the characteristic vector of S . As the entries of A are at most $M - 1$ in magnitude, the entries of $A\chi_S$ are at most $\ell(M - 1)$. Therefore $A\chi_S = A\chi_S \bmod (\ell(M - 1) + 1)$. We can thus compute $A\chi_S$ with $\lceil \log((\ell(M - 1) + 1)) \rceil$ many queries by Lemma 11. Computing $A\chi_S$ suffices to serve the k -OR query since $(\text{OR}(x^{(1)}, S), \dots, \text{OR}(x^{(k)}, S)) = (\min\{(A\chi_S)_1, 1\}, \dots, \min\{(A\chi_S)_k, 1\})$. Thus by Lemma 7 we can output a good estimate of the degree sequence of A with probability at least $1 - \delta$ after $O\left(\log(\ell M)(\log(\ell) + 1) \log\left(\frac{k(\log(\ell)+1)}{\delta}\right)\right)$ many matrix cut queries.

For the case $M = 2$, note that the degree sequence of A is the vector $A\vec{1}$, where $\vec{1}$ is the vector of all ones. As the entries of $A\vec{1}$ are at most ℓ by Lemma 11 we can compute the degree sequence with $\lceil \log(\ell + 1) \rceil$ many matrix cut queries. \square

Theorem 16. Let $A \in [M]^{k \times \ell}$ be a matrix with m many nonzero entries. There is a quantum algorithm to learn A with probability at least $1 - \delta$ after making

$$O\left(\sqrt{m \log(M\ell)} \log(M) + \log(\ell M)(\log(\ell) + 1) \log\left(\frac{k(\log(\ell) + 1)}{\delta}\right)\right)$$

many matrix cut queries. When $M = 2$, a better bound of $O(\sqrt{m \log(\ell)} + \log(\ell + 1) + \log(1/\delta))$ many matrix cut queries holds.

Proof. First we use Lemma 15 to compute $g \in \mathbb{R}^k$ which is a good approximation of the degree sequence of A except with probability $\delta/2$. This takes

$$O\left(\log(\ell M)(\log(\ell) + 1) \log\left(\frac{k(\log(\ell) + 1)}{\delta}\right)\right)$$

many queries. If $M = 2$ then we can exactly compute the degree sequence of A with $\lceil \log(\ell + 1) \rceil$ many matrix cut queries. We now assume that g is a good approximation and add $\delta/2$ to our total error bound.

Let $d \in \mathbb{N}$ be a degree threshold that will be chosen later, and define $L = \{i : g(i) \leq d\}$ and $H = \{i : g(i) > d\}$. As g is a good approximation of the degree sequence, rows whose indices are in L have at most $2d$ nonzero entries. We use Lemma 13 to learn the submatrix $A(L, 1 : \ell)$ with probability at least $1 - \delta/2$ with $O((d \log(M\ell/d) + \log(1/\delta)) \log(M))$ many matrix cut queries.

For each $i \in H$, the i^{th} row of A must have at least $d/4$ many nonzero entries, as g is a good approximation of the degree sequence of A . Thus $|H| \leq 4m/d$. By Corollary 12 we can learn the submatrix $A(H, 1 : \ell)$ with $4m \lceil \log(M) \rceil / d$ many matrix cut queries.

Setting $d = \sqrt{\frac{m}{\log(M\ell)}}$ the total number of queries becomes

$$O\left(\sqrt{m \log(M\ell)} \log(M) + \log(\ell M)(\log(\ell) + 1) \log\left(\frac{k(\log(\ell) + 1)}{\delta}\right)\right).$$

In the case $M = 2$, the number of queries becomes $O(\sqrt{m \log(\ell)} + \log(\ell + 1) + \log(1/\delta))$. \square

4 Learning graphs

A *weighted graph* is a couple $G = (V, w)$, where V is the set of vertices, $V^{(2)}$ is the set of subsets of V with cardinality 2, and $w : V^{(2)} \rightarrow \mathbb{N}$ is the weight function. We assume that we have a total ordering $v_1 < v_2 < \dots < v_{|V|}$ on the elements of V . The set of edges is defined as $E = \{e \in V^{(2)} : w(e) > 0\}$, therefore weighted graphs are undirected and without self-loops. We can also think of them as multi-graphs, where the number of edges between vertices u and v is $w(\{u, v\})$. When the range of w is $\{0, 1\}$ we will call a weighted graph a *simple graph*, or just a *graph* and denote it by $G = (V, E)$. If G is a weighted graph on n vertices, the *adjacency matrix* of G is an n -by- n matrix A_G with zeros on the diagonal and $A_G(i, j) = w(\{v_i, v_j\})$, for $i \neq j$. Observe that A_G is a symmetric matrix.

For an edge $e = \{u, v\}$, we say that u and v are the endpoints of e . The degree $\deg(v)$ of a vertex v is the number of edges for which v is an endpoint. For $S, T \subseteq V$ sets of vertices, we denote by $E(S, T)$ the set of edges with one endpoint in S and the other endpoint in T . (More precisely, $E(S, T) = \{e \in E : |e \cap (S \cup T)| = 2, |e \cap S| \geq 1, |e \cap T| \geq 1\}$.) We extend the weight function w to sets of vertices $S, T \subseteq V$ by $w(S, T) = \sum_{e \in E(S, T)} w(e)$.

A *bipartite weighted graph* is a triple $G = (V_1, V_2, w)$, where $V_1 = \{v_1, \dots, v_k\}$ and $V_2 = \{v_{k+1}, \dots, v_{k+\ell}\}$ are the disjoint sets of respectively left and right vertices, and $w : V_1 \times V_2 \rightarrow \mathbb{N}$ is the bipartite weight function. A bipartite graph $G = (V_1, V_2, w)$ can of course also be viewed as a graph $G' = (V', w')$ with vertex set $V = V_1 \cup V_2$, where V_1 and V_2 are independent sets, and weight function w' over $V^{(2)}$, where

$$w'(\{v_i, v_j\}) = \begin{cases} 0, & \text{if } 1 \leq i, j \leq k \text{ or } k+1 \leq i, j \leq k+\ell, \\ w(v_{\min\{i, j\}}, v_{\max\{i, j\}}), & \text{otherwise.} \end{cases}$$

Consistently with the general case, for $V'_1 \subseteq V_1$ and $V'_2 \subseteq V_2$, we extend w as $w(V'_1, V'_2) = \sum_{u \in V'_1, v \in V'_2} w(u, v)$. The set of edges is defined as $E = \{e \in V_1 \times V_2 : w(e) > 0\}$.

The biadjacency matrix of G is a $k \times \ell$ matrix B_G where $B_G(i, j - k) = w(\{v_i, v_j\})$ for $1 \leq i \leq k, k+1 \leq j \leq k+\ell$. Similarly to the bipartite weight function, the biadjacency matrix B_G is a condensed description of the $(k+\ell) \times (k+\ell)$ adjacency matrix A_G of G , where $A_G(i, j) = w'(\{v_i, v_j\})$, for $i \neq j$.

We will look at four different oracles for accessing a graph, the matrix cut oracle, the disjoint matrix cut oracle, the additive oracle, and the cut oracle. The comparison of their definitions and relative power is illustrated in Fig. 1.

Definition 17 (Matrix cut oracle and disjoint matrix cut oracle for a graph). Let $G = (V, w)$ be a weighted graph. The matrix cut oracle for G is the matrix cut oracle for the adjacency matrix A_G of G . The disjoint matrix cut oracle for G is the matrix cut oracle for G with the restriction that it can only be queried on strings $x, y \in \{0, 1\}^{|V|}$, where $x_i y_i = 0$, for all $1 \leq i \leq n$.

If we consider the strings $x, y \in \{0, 1\}^{|V|}$ as characteristic vectors of the sets $X, Y \subseteq V$, the restriction on the domain of the disjoint matrix cut oracle is that it is only defined if $X \cap Y = \emptyset$, which explains its name.

Beside the matrix oracles we will look at two more oracle models for accessing a graph, the additive oracle and the cut oracle.

Definition 18 (Additive oracle). Let $G = (V, w)$ be a weighted graph. The additive oracle $a : [2]^V \rightarrow \mathbb{N}$ returns $a(S) = \sum_{\{u,v\} \in S^{(2)}} w(\{u, v\})$ for any subset $S \subseteq V$.

Definition 19 (Cut oracle). Let $G = (V, w)$ be a weighted graph. The cut oracle $c : [2]^V \rightarrow \mathbb{N}$ returns $c(S) = w(S, V \setminus S)$ for any subset $S \subseteq V$.

Learning graphs with an additive oracle have been extensively studied in the classical case [GK98, GK00, ABK⁺04, Maz10, BM10, BM11]. For their proofs many of these papers actually work with matrix cut queries, and we find it useful to explicitly define these here. The cut oracle has also been studied in the classical case in the context of computing the minimum cut of a graph [RSW18, MN20]. We see in the next section that the cut oracle is the weakest of all these models, and our main algorithmic results will be for the cut oracle model.

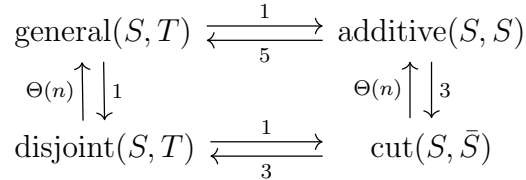


Figure 1: Illustration of oracles for comparison. The arrows indicate reductions, and $A \xrightarrow{s} B$ means that a query to oracle B can be simulated by s queries to oracle A .

4.1 Relationships between oracles

In this section we examine the power of the four oracles we have defined for graphs. In essence we show that if we consider the relative power of these oracles up to a constant overhead, then the disjoint matrix cut oracle for graphs and the cut oracle have the same power, the matrix cut oracle for graphs and the additive oracle have the same power, and the power of the latter group is greater than the power of the former.

Definition 20 (Constant-reduction between oracles). Let O_1 and O_2 be graph oracles. We say that O_1 is constant-reducible to O_2 if there exist a positive integer k such that for every weighted graph $G = (V, w)$, every query of O_1 to G can be computed with k queries of O_2 to G . If O_1 and O_2 are mutually constant-reducible to each other, then they are called constant-equivalent.

The first lemma shows the constant-equivalence of the disjoint matrix cut oracle for graphs and the cut oracle.

Lemma 21. *The disjoint matrix cut oracle for graphs and the cut oracle are constant-equivalent. In particular, the cut oracle can simulate with 3 queries a query of the disjoint matrix cut oracle for graphs.*

Proof. The cut oracle is obviously 1-reducible to the disjoint matrix cut oracle. For the reverse direction, let $G = (V, w)$ be a weighted graph with $|V| = n$ and let A_G be its adjacency matrix. Let $x, y \in \{0, 1\}^n$ be the characteristic vectors of $X, Y \subseteq V$ where X and Y are disjoint. Then

$$\begin{aligned} x^T A_G y &= w(X, Y) \\ &= \frac{1}{2} (c(X) + c(Y) - c(X \cup Y)) . \end{aligned}$$

□

Lemma 22. *Let G be a bipartite graph. A matrix cut query to the biadjacency matrix of G can be simulated by 3 cut queries.*

Proof. Let B_G be the biadjacency matrix of G with k left and ℓ right vertices, and let $x \in \{0, 1\}^k, y \in \{0, 1\}^\ell$. We define the vectors $\bar{x}, \bar{y} \in \{0, 1\}^{k+\ell}$ as $\bar{x} = [x; 0^\ell]$ and $\bar{y} = [0^k; y]$. Then $x^T B_G y = \bar{x}^T A_G \bar{y}$, where A_G is the adjacency matrix of G . Since x and y are the characteristic vectors of disjoint sets in $V_1 \cup V_2$, by Lemma 21 we can compute $x^T B_G y$ with 3 queries to the cut oracle for G . □

The constant-equivalence of the matrix cut oracle for graphs and the additive oracle was essentially shown by Grebinski and Kucherov in Theorem 4 of [GK00]. For completeness we reproduce here the proof.

Lemma 23. *The matrix cut oracle for graphs and the additive oracle are constant-equivalent. In particular, the latter oracle can simulate with 5 queries a query of the former.*

Proof. Let $G = (V, w)$ be a weighted graph on n -vertices, and let A_G be its adjacency matrix. The additive oracle is 1-reducible to the matrix cut oracle for A_G because for every set $X \subseteq V$ with characteristic vector $x \in \{0, 1\}^n$, we have $a(X) = \frac{1}{2} x^T A_G x$.

For the reverse direction we show that the matrix cut oracle to A_G can be simulated with 5 queries to the additive oracle for G . For this let $X, Y \subseteq V$ be arbitrary sets. We consider the characteristic vectors x_-, y_- and z of respectively $X \setminus Y, Y \setminus X$ and $X \cap Y$. Then

$$\begin{aligned} x^T A_G y &= (x_- + z)^T A_G (y_- + z) \\ &= x_-^T A_G y_- + x_-^T A_G z + z^T A_G y_- + z^T A_G z \\ &= \frac{1}{2} ((x_- + y_-)^T A_G (x_- + y_-) + x^T A_G x + y^T A_G y) - x_-^T A_G x_- - y_-^T A_G y_- . \end{aligned}$$

To go from the second line to the last line we used that A_G is symmetric, that the sets $X \setminus Y, Y \setminus X$ and $X \cap Y$ are pairwise disjoint, and finally the fact that if u_1 and u_2 are the characteristic vectors of the disjoint sets $U_1, U_2 \subseteq V$ then $u_1^T A_G u_2 = a(U_1 \cup U_2) - a(U_1) - a(U_2)$. \square

Since the matrix cut oracle for graphs is by definition at least as strong as the disjoint matrix cut oracle for graphs, Lemma 21 and Lemma 23 imply that the cut oracle is constant reducible to the additive oracle. The following lemma shows that the simulation in fact can be done by 3 queries.

Lemma 24. *Let $G = (V, w)$ be a weighted graph. The cut oracle to G is 3-reducible to the additive oracle for G .*

Proof. For any $S \subseteq V$

$$c(S) = a(V) - a(S) - a(V \setminus S) .$$

\square

We now turn to the question of simulating an additive oracle with a cut oracle. In an n -vertex weighted graph $G = (V, w)$, we can simulate an additive oracle with at most $3n$ applications of a cut oracle. This is because for any $S \subseteq V$,

$$a(S) = \frac{1}{2} \sum_{v \in S} w(v, S \setminus \{v\})$$

and each $w(v, S \setminus \{v\})$ can be computed with 3 cut queries by Lemma 21. Note that this algorithm works no matter how large the weights are. We next show that for weighted graphs with sufficiently large weights this trivial algorithm is nearly tight, and in the worst case $\Omega(n)$ cut queries are needed to simulate an additive query.

To do this, we first need a specific form of the Fredholm alternative.

Lemma 25 (Fredholm Alternative). *Let $A \in \{0, 1\}^{N \times k}$ have independent columns, and let $b \in \{0, 1\}^N$. Suppose that $Ax = b$ has no solution $x \in \mathbb{R}^k$. Then the integer vector $\hat{y} = \det(A^T A)(I - A(A^T A)^{-1} A^T)b \in \mathbb{Z}^N$ satisfies*

1. $\hat{y}^T A = \vec{0}$,
2. $\hat{y}^T b \neq 0$, and
3. $\|\hat{y}\|_\infty \leq N^{k+1/2} k^{k/2}$.

Proof. As A has independent columns, $A^T A$ is invertible. Let $b_c = A(A^T A)^{-1} A^T b$ be the orthogonal projection of b onto the column space of A and $y = (I - A(A^T A)^{-1} A^T)b$ be the orthogonal projection onto the left nullspace. Then $b = b_c + y$ and, as $Ax = b$ has no solution, $y \neq \vec{0}$. As y is in the left nullspace, we have $y^T A = \vec{0}$. Also $y^T b = y^T (b_c + y) = \|y\|^2 \neq 0$ as $y^T b_c = 0$.

Now $\|y\| \leq \|b\| \leq \sqrt{N}$, thus also $\|y\|_\infty \leq \sqrt{N}$. For an invertible matrix B , by Cramer's rule $B^{-1} = \text{adj}(B) / \det(B)$, where $\text{adj}(B)$ is the adjugate matrix of B . Thus $\det(A^T A) \cdot (A^T A)^{-1} = \text{adj}(A^T A)$ is an integer matrix and $\hat{y} = \det(A^T A)y$ is an integer vector. As \hat{y} is just a nonzero

scalar factor of y , we have $\hat{y}^T A = 0$ and $\hat{y}^T b \neq 0$. Finally, $\|\hat{y}\|_\infty \leq \sqrt{N} \cdot \det(A^T A) \leq N^{k+1/2} k^{k/2}$, as the entries of $A^T A$ are at most N and Hadamard's inequality gives that $\det(B) \leq N^k k^{k/2}$ for a k -by- k matrix B whose entries are bounded by N . \square

Theorem 26. *Any deterministic cut query algorithm on weighted graphs with vertex set V must make at least $|V|/2$ queries to compute $a(V)$.*

Proof. Let G be a weighted graph with vertex set V where $|V| = n$. We will show that at least $n/2$ cut queries are needed to compute the total edge weight $a(V)$ in G .

Let Sym_n^0 be the vector space of symmetric n -by- n matrices with zeros along the diagonal. Let the map $\text{symvec} : \text{Sym}_n^0 \rightarrow \mathbb{R}^{\binom{n}{2}}$ be defined by $\text{symvec}(C) = [C(2:n, 1); C(3:n, 2); \dots; C(n, n-1)]$, which is an isomorphism between the two vector spaces.

Let $B = J - I$ where J is the n -by- n all ones matrix and I is the n -by- n identity matrix. Let $b = \text{symvec}(B)$. Then $a(V) = \text{symvec}(A_G)^T b$ where A_G is the adjacency matrix of G . For a subset $X \subseteq V$ of the vertices, let $x \in \{0, 1\}^n$ be the characteristic vector of X , and let $\bar{x} \in \{0, 1\}^n$ be the characteristic vector of $V \setminus X$. Then for the cut value of X we have $c(X) = \text{symvec}(A_G)^T \text{symvec}(x\bar{x}^T + \bar{x}x^T)$. Observe that the rank of the matrix $x\bar{x}^T + \bar{x}x^T$ is 2.

Consider a deterministic cut query algorithm making $k < n/2$ many queries. Let $m = n^{2k+1} k^{k/2}$, and let G_1 be the graph whose adjacency matrix is $A_{G_1} = m \cdot B$. We set $a_1 = \text{symvec}(A_{G_1})$. Suppose that the algorithm makes queries $X_1, \dots, X_k \subseteq V$ in G_1 , with respective characteristic sequences $x_1, \dots, x_k \in \{0, 1\}^n$. Let $d_i = \text{symvec}(x_i \bar{x}_i^T + \bar{x}_i x_i^T)$ for $i = 1, \dots, k$, and let D be the $\binom{n}{2}$ -by- k matrix whose i th column is d_i .

As the rank of B is n and $k < n/2$, there is no solution x to the equation $Dx = b$, since otherwise we could express B as a linear combination of k matrices of rank 2. Let $\hat{y} \in \mathbb{Z}^{\binom{n}{2}}$ be the vector given by Lemma 25 satisfying $\hat{y}^T D = \vec{0}$ and $\hat{y}^T b \neq 0$. Then by the choice of m and by Lemma 25, $a_1 + \hat{y}$ will be a non-negative integer vector. Let G_2 be the weighted graph whose adjacency matrix satisfies $\text{symvec}(A_{G_2}) = a_1 + \hat{y}$.

Now $a_1^T D = (a_1 + \hat{y})^T D$ means that the graphs G_1 and G_2 give the same answers on all queries. Yet $a_1^T b \neq (a_1 + \hat{y})^T b$, meaning that the total sum of weights in G_1 and in G_2 are different. \square

4.2 Learning graphs with an additive oracle

Lemma 23 immediately lets us apply Lemma 13 and Theorem 16 to learning graphs with additive queries.

Corollary 27. *Let $G = (V, w)$ be a weighted graph on $n > 1$ vertices.*

1. *If every vertex has degree at most d then there is a quantum algorithm that learns A with probability at least $1 - \delta$ after making $O((d \log(Mn/d) + \log(1/\delta)) \log(M))$ many additive oracle queries to G .*
2. *If G has m edges then there is a quantum algorithm that learns G with probability at least $1 - \delta$ after making*

$$O\left(\sqrt{m \log(Mn)} \log(M) + \log(Mn) \log(n) \log\left(\frac{n \log(n)}{\delta}\right)\right)$$

many additive oracle queries to G . When $M = 2$, a better bound of $O(\sqrt{m \log(n)} + \log(n) + \log(1/\delta))$ many additive queries holds.

Note that this result implies that one can learn a matching or Hamiltonian cycle using $O(\log n)$ quantum additive queries, in contrast to the $\Theta(n)$ additive queries needed deterministically [GK00].

4.3 Learning graphs with a cut oracle

Since by Theorem 26 a cut oracle cannot efficiently simulate an additive oracle (or a matrix cut oracle), we must do more work to adapt the results from Section 3 to the case of a cut oracle. We first show how to learn a general weighted graph with weights bounded by M with $O(n \log(M))$ many cut queries.

Theorem 28. *Let $G = (V, w)$ be a weighted graph with n vertices and edge weights in $[M]$. There is a quantum algorithm that learns G perfectly after making $3n \lceil \log(M) \rceil$ cut queries. In particular, a simple graph can be learned perfectly by a quantum algorithm making $3n$ cut queries.*

Proof. Let A_G be the adjacency matrix of G , and let v_1, \dots, v_n be the order of the vertices labelling the rows and columns. Let $B_i = A_G(i, \{1, \dots, i-1\} \cup \{i+1, \dots, n\})$ be the i^{th} row of A_G with the i^{th} entry (which must be 0) removed. Then $x^T B_i y$ can be computed with 3 cut queries by Lemma 22 for any $x \in \{0, 1\}, y \in \{0, 1\}^{n-1}$. Thus by Corollary 12 we can learn B_i with $3 \lceil \log M \rceil$ many cut queries. Doing this in turn for each v_i gives the result.

The “in particular” statement follows by taking $M = 2$. □

Next we show how to learn low-degree and sparse graphs more efficiently. We start out in the bipartite case.

Lemma 29. *Let $G = (V_1, V_2, w)$ be a weighted bipartite graph with $|V_1| = k, |V_2| = \ell$, and edge weights in $[M]$.*

1. *If $\deg(u) \leq d$ for every $u \in L$ then there is a quantum algorithm that learns G with probability at least $1 - \delta$ after making $O((d \log(M\ell/d) + \log(1/\delta)) \log M)$ many cut queries.*
2. *If G has m edges then there is a quantum algorithm that learns G with probability at least $1 - \delta$ after making*

$$O\left(\sqrt{m \log(M\ell)} \log(M) + \log(\ell M)(\log(\ell) + 1) \log\left(\frac{k(\log(\ell) + 1)}{\delta}\right)\right)$$

many cut queries. When $M = 2$, a better bound of $O(\sqrt{m \log(\ell)} + \log(\ell + 1) + \log(1/\delta))$ many cut queries holds.

Proof. By Lemma 22 any matrix cut query to the biadjacency matrix of G can be computed by 3 queries to the cut oracle for G . Item (1) therefore follows from Lemma 13 and item (2) follows from Theorem 16. □

Now we extend the bipartite algorithms to the general case.

Theorem 30. *Let $G = (V, w)$ be a weighted n -vertex graph, $n > 1$, with edge weights in $[M]$.*

1. *If $\deg(v) \leq d$ for every $v \in V$ then there is a quantum algorithm that learns G with probability at least $1 - \delta$ after making $O((d \log(Mn/d) + \log(\log(n)/\delta)) \log(M) \log(n))$ many cut queries.*
2. *If G has m edges then there is a quantum algorithm that learns G with probability at least $1 - \delta$ after making*

$$O\left(\left(\sqrt{m \log(Mn)} \log(M) + \log(Mn)(\log(n)) \log\left(\frac{n \log(n)}{\delta}\right)\right) \log(n)\right)$$

many cut oracle queries to G . When $M = 2$, a better bound of $O((\sqrt{m \log(n)} + \log(n) + \log(\log(n)/\delta)) \log(n))$ many cut queries holds.

Proof. Let $r = \lceil \log(n) \rceil$. We suppose that the vertices are labeled with r -bit binary strings. We define r many weighted bipartite graphs $G_i = (L_i, R_i, w_i)$, for $i = 1, \dots, r$, derived from G as follows. L_i consists of the vertices of V whose i^{th} bit is 0, whereas R_i consists of the vertices whose i^{th} bit is 1. The weight function w_i in G_i is defined as

$$w_i(u, v) = \begin{cases} w(u, v) & \text{if } (u, v) \in L_i \times R_i \\ 0 & \text{otherwise .} \end{cases}$$

Let B_{G_i} be the biadjacency matrix of G_i . Let $x \in \{0, 1\}^{|L_i|}$, $y \in \{0, 1\}^{|R_i|}$ and $X = \{v \in V : x_i(v) = 1\}$, $Y = \{v \in V : y_i(v) = 1\}$, and let χ_X and χ_Y be the characteristic sequence of respectively X and Y in V . Then $x^T B_{G_i} y = \chi_X^T A_{G_i} \chi_Y$, and since X and Y are disjoint sets in V we can compute it with 3 cut queries to G by Lemma 21.

We first prove item (1). As G has degree at most d , each B_i will also have degree at most d . Therefore by applying the first item of Lemma 29 with error bound δ/r we can learn B_i with $O((d \log(Mn/d) + \log(\log(n)/\delta)) \log(M))$ many cut queries. Doing this for each B_i in turn gives item (1).

Similarly for item (2), as G has at most m edges so will each B_i . Learning each B_i in turn by the second item of Lemma 29 with error bound δ/r gives the bound in item (2). \square

4.4 Lower bounds

In the classical case, an $\Omega(n)$ lower bound is known on the number of cut queries needed by a deterministic algorithm to determine if a graph is connected. As observed by Harvey (Theorem 5.9 in [Har08]), an $\Omega(n)$ cut query lower bound for connectivity follows from the deterministic communication complexity lower bound of $\Omega(n \log n)$ for connectivity by [HMT88], and the fact that the answers to cut queries can be communicated with $O(\log n)$ bits. Clearly this lower bound applies to all 4 of the graph oracles we have discussed, as all of them have answers that can be communicated with $O(\log(n))$ bits. For randomized communication protocols a $\Omega(n)$ lower bound is known for connectivity [BFS86], giving an $\Omega(n/\log(n))$ lower bound on the number of cut queries needed to solve connectivity.

The quantum case behaves differently. There is an analogous communication complexity result: [IKL⁺12] show that the bounded-error quantum communication complexity of connectivity is $\Omega(n)$. However, the standard way to turn a query algorithm into a communication protocol, developed by Buhrman, Cleve, and Wigderson (BCW) [BCW98], involves Alice and Bob sending back and forth the query and answer state. In the case of cut queries, this is a state of the form $\sum_{S \subseteq \{0,1\}^n} \alpha_S |S\rangle |b_S\rangle$ that is $n + \log(n)$ -qubits. Thus a k -cut query quantum algorithm for connectivity gives a kn qubit communication protocol for connectivity via the BCW simulation. Our polylogarithmic cut query algorithm for connectivity shows an example of a (partial) function where this blow-up from the BCW simulation is nearly optimal. We note that Chakraborty et al. [CCMP19] recently gave the first example of a *total* function showing that the BCW blow-up can be necessary.

We now show that our quantum algorithms for learning a general graph with $O(n)$ cut queries Theorem 28, a graph of maximum degree d with $\tilde{O}(d)$ cut queries, and an m -edge graph with $\tilde{O}(\sqrt{m})$ cut queries Theorem 30 are all tight, even if the algorithms are equipped with the stronger additive oracle.

Recall that the inner product function $IP_N : \{0, 1\}^N \times \{0, 1\}^N \rightarrow \{0, 1\}$ on strings of length N is defined as $IP_N(x, y) = \text{PARITY}(x \wedge y)$. The bounded-error quantum communication complexity of IP_N is $\Omega(N)$ [Kre95].

Lemma 31. *Suppose there is a bounded-error quantum query algorithm that learns a graph on n vertices with k many additive queries. Then there is a bounded-error quantum communication protocol for $IP_{\binom{n}{2}}$ with $O(kn)$ qubits of communication.*

Proof. We consider $[n]^{\binom{2}{2}}$, the set of possible edges in a graph with vertex set $[n]$. Alice and Bob first agree on a bijective mapping $f : [\binom{n}{2}] \rightarrow [n]^{\binom{2}{2}}$. Now say that Alice receives $x \in \{0, 1\}^{\binom{n}{2}}$ and Bob receives $y \in \{0, 1\}^{\binom{n}{2}}$, and they wish to compute $IP_{\binom{n}{2}}(x, y)$. Bob first creates a graph G_y with vertex set $[n]$ and edge set $E = \{f(e) : y_e = 1\}$. Alice now runs the quantum query algorithm to learn G_y by sending the query states to Bob, who serves each query and sends the state back to Alice. As the query state is $n + \log(n)$ qubits, this will take communication $O(kn)$ if there are k queries. \square

Theorem 32. *Any quantum algorithm to learn an n -vertex graph with bounded-error must make $\Omega(n)$ many additive queries.*

Proof. The theorem is immediate from Lemma 31 and Kremer's $\Omega(n^2)$ lower bound on the quantum communication complexity of the inner product on $\binom{n}{2}$ bits. \square

Corollary 33. *For any n and $m \leq \binom{n}{2}$ there is a family of n -vertex graphs $\mathcal{G}_{n,m}$ with at most m edges such that any quantum algorithm requires $\Omega(\sqrt{m})$ many additive queries to learn a graph from $\mathcal{G}_{n,m}$ with bounded-error.*

Proof. Let $\mathcal{G}_{n,m}$ be the family of graphs that are arbitrary among the first k vertices, where k is the smallest integer such that $\binom{k}{2} \geq m$, and all remaining $n - k$ vertices are isolated. By Theorem 32 $\Omega(k) = \Omega(\sqrt{m})$ many additive queries are needed to learn a graph from this family. \square

Corollary 34. *For any n and $d \leq n$ there is a family of n -vertex graphs $\mathcal{G}_{n,d}$ with each vertex having degree at most d such that any quantum algorithm requires $\Omega(d \log(n/d))$ many additive queries to learn a graph from $\mathcal{G}_{n,d}$ with bounded-error.*

Proof. Let $\mathcal{G}_{n,d}$ be the set of all d regular graphs on n -vertices. It is known that $|\mathcal{G}_{n,d}| \geq \left(\frac{n}{2ed}\right)^{nd/2}$ (see the argument above Theorem 4 in [GK00], or the very general results in [LW19]). Letting $\ell = \lfloor \frac{nd}{2} \log\left(\frac{n}{2ed}\right) \rfloor$, Alice and Bob can therefore agree on a bijection between $\{0, 1\}^\ell$ and a subset of $\mathcal{G}_{n,d}$. As in the proof of Lemma 31, they then can use an algorithm to learn graphs in $\mathcal{G}_{n,d}$ with k additive queries to solve IP_ℓ with quantum communication complexity $O(kn)$. Thus $k = \Omega(\ell/n) = \Omega(d \log(n/d))$. \square

5 A quantum algorithm for computing connected components with cut queries

In this section, we give a quantum algorithm that outputs the connected components of an n -vertex graph after making $O(\log(n)^6)$ cut queries. By Lemma 24, this implies the same result with respect to additive queries.

We first give a high level overview of the algorithm. Let G be an n -vertex graph with vertex set V and A_G its adjacency matrix. For this high level overview, we will assume that we have matrix cut query access to A_G . This case contains all of the conceptual ideas needed and eliminates several technical issues that arise in the cut query case.

The algorithm proceeds in rounds. In every round, we maintain a partition of V . In a general round, we represent the partition as $\{S_1, \dots, S_k, C_1, \dots, C_t\}$ and let $\mathcal{S} = \{S_1, \dots, S_k\}$ and $\mathcal{C} = \{C_1, \dots, C_t\}$. We refer to sets of vertices as supervertices. The algorithm maintains the invariants that each C_j is a connected component, i.e. the induced subgraph on C_j is connected and $|E(C_j, V \setminus C_j)| = 0$, and that for each S_i the induced subgraph on S_i is connected. As the sets C_j are connected components, they require no further processing and thus in a round we focus only on the sets in \mathcal{S} . We initially set $\mathcal{S} = \tilde{V} = \{\{v\} : v \in V\}$ and $\mathcal{C} = \emptyset$.

An important concept for the algorithm will be the adjacency matrix of the weighted graph among the sets of \mathcal{S} . For $\mathcal{S} = \{S_1, \dots, S_k\}$ define a k -by- k matrix $A_{\mathcal{S}}$ where $A_{\mathcal{S}}(i, j) = |E(S_i, S_j)|$. A key fact is that matrix cut query access to A_G allows us to implement matrix cut queries on $A_{\mathcal{S}}$. For $x, y \in \{0, 1\}^k$ define $u, v \in \{0, 1\}^n$ to be the characteristic vectors of the sets $\cup_{i:x_i=1} S_i$ and $\cup_{i:y_i=1} S_i$ respectively. Then we have $x^T A_{\mathcal{S}} y = u^T A_G v$.

A general round proceeds as follows. We take a degree threshold $d = \Theta(\log(n)^2)$. The superdegree of a supervertex S_i is the number of $j \neq i$ such that $|E(S_i, S_j)| > 0$. Supervertices with superdegree at most d we call low superdegree, and supervertices with superdegree greater than d we call high superdegree.

1. Estimate the superdegree of all supervertices by Lemma 15 ($O(\log(n)^3)$ queries).
2. For low superdegree supervertices learn all their neighbors S_j by Lemma 13 ($O(\log(n)^4)$ queries taking $d = \Theta(\log(n)^2)$).

3. Randomly sample $\mathcal{R} \subseteq \mathcal{S}$ satisfying $|\mathcal{R}| \leq |\mathcal{S}|/2$. With high probability for every high degree supervertex S_i there is an $R \in \mathcal{R}$ with $|E(S_i, R)| > 0$, and we again use Lemma 13 to find such an R for every high degree S_i with $O(\log(n)^4)$ queries.
4. Merge all supervertices S_i, S_j for which we know $|E(S_i, S_j)| > 0$ and update \mathcal{S} and \mathcal{C} accordingly.

We show that with high probability after this merging step the number of supervertices in \mathcal{S} is at most half of what it was at the beginning of the round. This follows because with high probability every high degree supervertex will be merged with an element of \mathcal{R} . For low degree supervertices, we learn all of their neighbors. Therefore if the connected component of a low degree supervertex S_i only contains low degree supervertices, this step will learn its entire connected component and we can add this component to \mathcal{C} . Otherwise there is some low degree supervertex S_j in the connected component of S_i that is connected to a high degree supervertex S_ℓ . We will learn this connection in Step (2), and learn a neighbor of S_ℓ in \mathcal{R} in step (3). This means that in Step (4) S_i will be merged into a supervertex that also contains some element of \mathcal{R} . Thus every supervertex for which we have not already learned its connected component will be merged with a supervertex in \mathcal{R} with high probability and at the end of the round the total number of supervertices in \mathcal{S} is reduced by at least half. The algorithm thus terminates after $O(\log n)$ rounds and the total complexity becomes $O(\log(n)^5)$ many matrix cut queries.

We have seen that cut queries cannot efficiently simulate matrix cut queries in general. However, by Lemma 22, cut queries can simulate matrix cut queries on *bipartite* weighted graphs. The trick we use to adapt the above algorithm to the cut query case is to always work with bipartite graphs. For a graph G we associate $\lceil \log n \rceil$ many bipartite graphs (L_i, R_i, E_i) for $i \in \lceil \log n \rceil$, where $L_i = \{v : v_i = 0\}$, $R_i = \{v : v_i = 1\}$ and $E_i = \{(u, v) \in V \times V : \{u, v\} \in E, u \in L_i, v \in R_i\}$. Note that every edge of G appears as an edge in some (L_i, R_i, E_i) . We essentially run steps (1)–(3) above on each (L_i, R_i, E_i) separately, then incorporate all the information learned in the merge step in (4). Having to iterate over these $O(\log(n))$ many bipartite graphs results in an extra multiplicative logarithmic factor in the complexity, resulting in the claimed bound of $O(\log(n)^6)$ cut queries.

5.1 Auxiliary subroutines

In this subsection we go over some auxiliary subroutines that will be used in the cut query algorithm for connectivity. We will make use of the following definition.

Definition 35 (Supervertex, Superdegree). A *supervertex* is a subset of $S \subseteq V$ of vertices. We say that a supervertex S is connected if the subgraph induced on S is connected.

We say that a set of supervertices $\mathcal{S} = \{S_1, \dots, S_k\}$ is *valid* if $S_i \cap S_j = \emptyset$ for all $i \neq j$. We say that two valid sets of supervertices $\mathcal{S} = \{S_1, \dots, S_k\}$ and $\mathcal{T} = \{T_1, \dots, T_\ell\}$ are *disjoint* if $S_i \cap T_j = \emptyset$ for all $i \in \{1, \dots, k\}, j \in \{1, \dots, \ell\}$.

We say that there is a *superedge* between supervertices S_1, S_2 if $|E(S_1, S_2)| > 0$. Given two disjoint valid sets of supervertices $\mathcal{S} = \{S_1, \dots, S_k\}$ and $\mathcal{T} = \{T_1, \dots, T_\ell\}$, the *superdegree* of S_i into \mathcal{T} , denoted $\text{deg}_{\mathcal{T}}(S_i)$, is the number of $j \in [\ell]$ such that $|E(S_i, T_j)| > 0$.

Definition 36. Let $\mathcal{S} = \{S_1, \dots, S_k\}$ and $\mathcal{T} = \{T_1, \dots, T_\ell\}$ be disjoint valid sets of supervertices of an n -vertex graph G . Define the weighted biadjacency matrix between \mathcal{S} and \mathcal{T} as the matrix $B \in [n^2]^{k \times \ell}$ where $B(i, j) = |E(S_i, T_j)|$.

Lemma 37. Let $\mathcal{S} = \{S_1, \dots, S_k\}$ and $\mathcal{T} = \{T_1, \dots, T_\ell\}$ be disjoint valid sets of supervertices of an n -vertex graph G . Let B be the weighted biadjacency matrix between \mathcal{S} and \mathcal{T} . A matrix cut query to B can be answered with 3 cut queries to G .

Proof. For $x \in \{0, 1\}^k, y \in \{0, 1\}^\ell$ we have $x^T B y = |E(\cup_{i:x(i)=1} S_i, \cup_{j:y(j)=1} T_j)|$ and thus can be computed with 3 cut queries by Lemma 21. \square

Corollary 38 (Approximate Degree Sequence). *There is a quantum algorithm Approximate Degree Sequence($\mathcal{S}, \mathcal{T}, \delta$) that takes as input disjoint valid sets of supervertices $\mathcal{S} = \{S_1, \dots, S_k\}$ and $\mathcal{T} = \{T_1, \dots, T_\ell\}$ of an n -vertex graph G and an error parameter δ , and with probability at least $1 - \delta$ outputs $\vec{g} \in \mathbb{R}^k$ that is a good estimate of $(\deg_{\mathcal{T}}(S_1), \dots, \deg_{\mathcal{T}}(S_k))$. The number of cut queries made is $O(\log(\ell n)(\log(\ell) + 1) \log(k(\log(\ell) + 1)/\delta))$.*

Proof. Let B be the weighted biadjacency matrix between \mathcal{S} and \mathcal{T} . By Lemma 37 a matrix cut query to B can be answered by 3 cut queries to G . The result then follows from Lemma 15. \square

Algorithm 2 Approximate Degree Sequence($\mathcal{S}, \mathcal{T}, \delta$)

Input: Disjoint valid sets of supervertices $\mathcal{S} = \{S_1, \dots, S_k\}, \mathcal{T} = \{T_1, \dots, T_\ell\}$ and an error parameter δ

Output: Vector $\vec{g} \in \mathbb{R}^k$ that is a good estimate of $(\deg_{\mathcal{T}}(S_1), \dots, \deg_{\mathcal{T}}(S_k))$.

Algorithm 3 Learn Low($\mathcal{S}, \mathcal{T}, h, \delta$)

Input: Disjoint valid sets of supervertices $\mathcal{S} = \{S_1, \dots, S_k\}, \mathcal{T} = \{T_1, \dots, T_\ell\}$, a degree parameter h such that $\deg_{\mathcal{T}}(S_i) \leq h$ for all $S_i \in \mathcal{S}$, and an error parameter δ .

Output: A k -by- ℓ matrix B where $B(i, j) = |E(S_i, T_j)|$.

Corollary 39 (Learn Low). *There is a quantum algorithm Learn Low($\mathcal{S}, \mathcal{T}, h, \delta$) that takes as input disjoint valid sets of supervertices $\mathcal{S} = \{S_1, \dots, S_k\}$ and $\mathcal{T} = \{T_1, \dots, T_\ell\}$ of an n -vertex graph G , a degree parameter h such that $\deg_{\mathcal{T}}(S_i) \leq h$ for all $S_i \in \mathcal{S}$, and an error parameter δ . With probability at least $1 - \delta$ Learn Low($\mathcal{S}, \mathcal{T}, h, \delta$) outputs the weighted biadjacency matrix between \mathcal{S} and \mathcal{T} . The number of cut queries made is $O((h \log(n/h) + \log(1/\delta)) \log(n))$.*

Proof. Let B be the weighted biadjacency matrix between \mathcal{S} and \mathcal{T} . The rows of B have at most h nonzero entries by assumption and all entries are $O(n^2)$. By Lemma 37 we can answer a matrix cut query to B by 3 cut queries to G . The result then follows from Lemma 13. \square

We accomplish Step (3) in the high level description of the algorithm by a routine called Reduce High. We first need a sampling lemma.

Lemma 40. Let $\mathcal{S} = \{S_1, \dots, S_k\}$ and $\mathcal{T} = \{T_1, \dots, T_\ell\}$ be disjoint valid sets of supervertices of an n -vertex graph G , and suppose $t/8 \leq \deg_{\mathcal{T}}(S_i) \leq 2t$. Randomly sample with replacement $\frac{16\ell \ln(kn)}{t}$ many supervertices in \mathcal{T} , and call the resulting set \mathcal{R} . Then, except with probability $O(n^{-2})$, the superdegree of each S_i into \mathcal{R} will be at least 1 and at most $192 \ln(n)$.

Proof. Define $x^{(i)} \in \{0, 1\}^\ell$ for $i \in \{1, \dots, k\}$ by $x^{(i)}(j) = 1$ iff $|E(S_i, T_j)| > 0$. Then the lemma follows by applying Lemma 8 to $x^{(1)}, \dots, x^{(k)}$ with $\delta = 1/n^2$ and using the fact that $k \leq n$. \square

Lemma 41 (Reduce High). The quantum algorithm *Reduce High*($\mathcal{S}, \mathcal{T}, d, \vec{g}$) (Algorithm 4) takes as input disjoint valid sets of supervertices $\mathcal{S} = \{S_1, \dots, S_k\}$ and $\mathcal{T} = \{T_1, \dots, T_\ell\}$ of an n -vertex graph G , a degree parameter d such that $\deg_{\mathcal{T}}(S_i) \geq d$ for all $S_i \in \mathcal{S}$, and a vector $\vec{g} \in \mathbb{R}^k$ that is a good estimate of $(\deg_{\mathcal{T}}(S_1), \dots, \deg_{\mathcal{T}}(S_k))$. *Reduce High* ($\mathcal{S}, \mathcal{T}, d, \vec{g}$) makes $O(\log(n)^4)$ many cut queries and with probability at least $1 - O(\log(n)/n^2)$ outputs a k -by- ℓ Boolean matrix B such that

1. Every row of B has at least one nonzero entry.
2. If $B(i, j) \neq 0$ then $B(i, j) = |E(S_i, T_j)|$.
3. B has at most $\frac{256\ell \ln(n)}{d}$ many nonzero columns.

Proof. The algorithm is given in Algorithm 4. First we bound the number of queries made. We do $O(\log n)$ iterations of the for loop, and within each iteration queries only occur in Line 5, which is a call *Learn Low* with an $O(\log n)$ degree parameter. Each such call to *Learn Low* requires $O(\log(n)^3)$ many cut queries by Corollary 39 Thus overall the number of queries is $O(\log(n)^4)$.

Now we show the correctness. As $\vec{g} \in \mathbb{R}^k$ provides good estimates and $\deg_{\mathcal{T}}(S_i) \geq d$ for all $S_i \in \mathcal{S}$, we have $g(i) \geq d/2$ for all $i \in \{1, \dots, k\}$. Also $g(i) \leq 4\ell$, thus each S_i will be put into \mathcal{H}_q for some value of q in the loop.

Let us now consider a particular iteration of the for loop when $q = j$. Let $\mathcal{H}_j = \{S_i \in \mathcal{S} : 2^{j-1} < g(i) \leq 2^j\}$. As \vec{g} contains good estimates, this means $2^{j-3} < \deg_{\mathcal{T}}(S_i) \leq 2^{j+1}$ for all $S_i \in \mathcal{H}_j$. We are thus in the setting of Lemma 8 with $t = 2^j$. As \mathcal{R}_j is a $\frac{16\ell \ln(|\mathcal{H}_j|n)}{2^j}$ -sample from \mathcal{T} the conclusion of Lemma 8 gives that the superdegree of every vertex in \mathcal{H}_j into \mathcal{R}_j is between 1 and $192 \ln(n)$, except with probability $O(n^{-2})$. We now assume we are in this good case. Then the upper bound on the degree passed to *Learn Low* is valid, and *Learn Low* will return the weighted biadjacency matrix between \mathcal{H}_j and \mathcal{R}_j with probability $1 - O(n^{-2})$ by Corollary 39

As every S_i is in \mathcal{H}_q for one call of the for loop, each row of B will have at least one nonzero entry. Further, as each call to *Learn Low* returns a correct weighted biadjacency matrix between \mathcal{H}_q and \mathcal{R}_q except with probability $1/n^2$, B will satisfy item (2) except with probability at most $O(\log(n)/n^2)$.

Finally, the only columns of B that can be nonzero are those indexed by sets that appeared in

\mathcal{R}_q at some point of the algorithm. As

$$\begin{aligned} |\cup_q \mathcal{R}_q| &\leq 32\ell \ln(n) \cdot \left(\sum_{q=\lfloor \log d \rfloor - 1}^{\lceil \log \ell \rceil + 2} \frac{1}{2^q} \right) \\ &\leq 32\ell \ln(n) \cdot \frac{4}{d} \sum_{j=0}^{\infty} \frac{1}{2^j} \\ &\leq \frac{256\ell \ln(n)}{d}, \end{aligned}$$

the total number of nonzero columns of B is at most $\frac{256\ell \ln(n)}{d}$. \square

Algorithm 4 Reduce High($\mathcal{S}, \mathcal{T}, d, \vec{g}$)

Input: Disjoint valid sets of supervertices $\mathcal{S} = \{S_1, \dots, S_k\}$, $\mathcal{T} = \{T_1, \dots, T_\ell\}$ of an n -vertex graph G , a degree parameter d such that $\deg_{\mathcal{T}}(S_i) \geq d$ for all $S_i \in \mathcal{S}$, and a vector $\vec{g} \in \mathbb{R}^k$ such that $\vec{g}(i)/4 \leq \deg_{\mathcal{T}}(S_i) \leq 2\vec{g}(i)$ for all $i \in \{1, \dots, k\}$.

Output: A k -by- ℓ matrix B satisfying the conditions of Lemma 41.

- 1: $B \leftarrow \text{zeros}(k, \ell)$
 - 2: **for** $q = \lfloor \log d \rfloor - 1$ to $\lceil \log \ell \rceil + 2$ **do**
 - 3: $\mathcal{H}_q = \{S_i \in \mathcal{S} : 2^{q-1} < \vec{g}(i) \leq 2^q\}$
 - 4: $\mathcal{R}_q \leftarrow$ Randomly choose $\frac{16\ell \ln(|\mathcal{H}_q|n)}{2^q}$ supervertices in \mathcal{T} , with replacement
 - 5: $B(\text{ind}(\mathcal{H}_q), \text{ind}(\mathcal{R}_q)) \leftarrow \text{Learn Low}(\mathcal{H}_q, \mathcal{R}_q, 192 \ln(n), 1/n^2)$
 - 6: **end for**
 - 7: **return** B
-

Lemma 42. Let $\mathcal{S} = \{S_1, \dots, S_k\}$ be a valid set of supervertices, \mathcal{A} a list of k -by- k weighted adjacency matrices, and $\text{low} \in \{0, 1\}^k$ a Boolean vector with the following properties:

1. Every supervertex in \mathcal{S} is connected.
2. For every $A \in \mathcal{A}$ if $A(i, j) > 0$ there is a superedge between S_i and S_j .
3. For every (i, j) such that $\text{low}(i) = 1$ and there is a superedge between S_i and S_j , there is an $A \in \mathcal{A}$ with $A(i, j) > 0$.

Then the algorithm Contract($\mathcal{S}, \mathcal{A}, \text{low}$) given in Algorithm 5 outputs sets of supervertices \mathcal{S}' , \mathcal{C} such that every $U \in \mathcal{S}'$ is connected, every $W \in \mathcal{C}$ is a connected component, and $\mathcal{S}' \cup \mathcal{C}$ is a partition of $\cup_{S \in \mathcal{S}} S$. Moreover, for every $U \in \mathcal{S}'$ there is an $S_i \subseteq U$ with $\text{low}(i) = 0$.

Proof. We first show that all supervertices in \mathcal{S}' , \mathcal{C} are connected. This follows because each $S_i \in \mathcal{S}$ is connected and we only merge two supervertices U and W when there is an $S_i \subseteq U, S_j \subseteq W$ and an $A \in \mathcal{A}$ with $A(i, j) > 0$. As by hypothesis $A(i, j) > 0$ implies there is a superedge between S_i and S_j this means that U and W are in fact connected.

Algorithm 5 Contract($\mathcal{S}, \mathcal{A}, \text{low}$)

Input: Valid set of connected supervertices $\mathcal{S} = \{S_1, \dots, S_k\}$, a list of k -by- k weighted adjacency matrices $\mathcal{A} = (A_1, \dots, A_m)$ with rows and columns labeled by elements of \mathcal{S} , a vector $\text{low} \in \{0, 1\}^k$ indicating if each set S_i is low.

Output: Sets of supervertices $\mathcal{S}', \mathcal{C}$, where each supervertex in \mathcal{S}' , \mathcal{C} is connected, and moreover the supervertices in \mathcal{C} are connected components.

```
1:  $L \leftarrow [(S_i, \text{low}(i)) : S_i \in \mathcal{S}]$ 
2: for  $A \in \mathcal{A}$  do
3:   for  $(i, j) \in [k]^{(2)}$  do
4:     if  $A(i, j) > 0$  then
5:       Pop  $(U, \text{flag1}) \in L$  such that  $S_i \subseteq U$ 
6:       Pop  $(W, \text{flag2}) \in L$  such that  $S_j \subseteq W$ 
7:        $U \leftarrow U \cup W$ 
8:        $\text{lowFlag} = \text{flag1} \wedge \text{flag2}$ 
9:       Append  $(U, \text{lowFlag})$  to  $L$ 
10:    end if
11:  end for
12: end for
13:  $\mathcal{S}' \leftarrow \emptyset$ 
14:  $\mathcal{C} \leftarrow \emptyset$ 
15: for  $(U, \text{lowFlag}) \in L$  do
16:   if  $\text{lowFlag} = 0$  then
17:      $\mathcal{S}' \leftarrow \mathcal{S}' \cup \{U\}$ 
18:   else
19:      $\mathcal{C} \leftarrow \mathcal{C} \cup \{U\}$ 
20:   end if
21: end for
22: return  $\mathcal{S}', \mathcal{C}$ 
```

Algorithm 6 $\text{Shrink}(\mathcal{S}, d)$

Input: A valid set of connected supervertices $\mathcal{S} = \{S_1, \dots, S_k\}$, and a degree parameter d .

Output: A set of connected supervertices \mathcal{S}' , and a set \mathcal{C} of connected components.

```
1: low  $\leftarrow$  ones( $k, 1$ )
2:  $\mathcal{A} \leftarrow []$ 
3: for  $j = 1$  to  $\lceil \log(k) \rceil$  do
4:   for  $b \in \{0, 1\}$  do
5:      $\mathcal{L}_{j,b} = \{S_t \in \mathcal{S} : t_j = b\}$ 
6:      $\mathcal{R}_{j,b} = \{S_t \in \mathcal{S} : t_j = 1 - b\}$ 
7:      $\vec{g} \leftarrow \text{Approximate Degree Sequence}(\mathcal{L}_{j,b}, \mathcal{R}_{j,b}, 1/n)$ 
8:      $\mathcal{H} \leftarrow \{S_t \in \mathcal{L}_{j,b} : \vec{g}(t) \geq d\}$ 
9:     low(ind( $\mathcal{H}$ )) = 0
10:     $B_{j,b} \leftarrow \text{zeros}(k, k)$ 
11:     $B_{j,b}(\text{ind}(\mathcal{H}), \text{ind}(\mathcal{R}_{j,b})) \leftarrow \text{Reduce High}(\mathcal{H}, \mathcal{R}_{j,b}, d/4, \vec{g}(\text{ind}(\mathcal{H})))$ 
12:     $\mathcal{L} \leftarrow \{S_t \in \mathcal{L}_{j,b} : \vec{g}(t) < d\}$ 
13:     $C_{j,b} \leftarrow \text{zeros}(k, k)$ 
14:     $C_{j,b}(\text{ind}(\mathcal{L}), \text{ind}(\mathcal{R}_{j,b})) \leftarrow \text{Learn Low}(\mathcal{L}, \mathcal{R}_{j,b}, 2d, 1/n)$ 
15:    Append  $B_{j,b}, C_{j,b}$  to  $\mathcal{A}$ 
16:   end for
17: end for
18:  $(\mathcal{S}', \mathcal{C}) \leftarrow \text{Contract}(\mathcal{S}, \mathcal{A}, \text{low})$ 
19: return  $(\mathcal{S}', \mathcal{C})$ 
```

Algorithm 7 Connectivity algorithm with cut queries

Input: Cut oracle for a graph G on n vertices

Output: Connected components of G

```
1:  $\mathcal{S} \leftarrow \tilde{V}$ 
2: ConComp  $\leftarrow \emptyset$ 
3: repeat
4:    $(\mathcal{S}, \mathcal{C}) \leftarrow \text{Shrink}(\mathcal{S}, 1024 \lceil \log n \rceil^2)$ 
5:   ConComp  $\leftarrow \text{ConComp} \cup \mathcal{C}$ 
6: until  $\mathcal{S} = \emptyset$ 
7: return ConComp
```

Next we show that every $W \in \mathcal{C}$ is a connected component. Suppose for a contradiction that this is not the case and therefore there is a $w \in S_i \subseteq W$ and $u \in S_j \subseteq (\cup_{t=1}^k S_t) \setminus W$ such that $\{u, w\}$ is an edge of G . It must be the case that S_i is low, as otherwise the lowFlag for W would have been set to 0 and W would have been placed in \mathcal{S}' . Thus S_i must be low and therefore by hypothesis for some $A \in \mathcal{A}$ it is the case that $A(i, j) > 0$. This means that at some point in Contract a set containing S_i would have been merged with a set containing S_j , a contradiction to the fact that $S_j \subseteq (\cup_{t=1}^k S_t) \setminus W$.

The fact that $\mathcal{S} \cup \mathcal{C}$ is a partition of $\cup_{S \in \mathcal{S}} S$ follows because at all times $\cup_{(U, \text{flag}) \in L} U$ is equal to $\cup_{S \in \mathcal{S}} S$. This is true when L is first defined, and is preserved when sets are popped from L , merged, and put back into L .

Finally, the ‘‘moreover’’ statement holds as if $\text{low}(i) = 1$ for all $S_i \in U$ then the lowFlag variable for U will be set to 1 and therefore U will be placed into \mathcal{C} on Line 21. \square

5.2 The shrink subroutine

We now package Approximate Degree Sequence, Learn Low, Reduce High, and Contract together into our algorithm for finding the connected components of a graph.

Lemma 43. *Let \mathcal{S} be a valid set of connected supervertices and $d \in \mathbb{N}$ be a degree parameter given as input to Algorithm 6. Then except with probability $O(\log(n)/n)$ the following two statements hold.*

1. *Algorithm 6 outputs sets of supervertices \mathcal{S}' , \mathcal{C} such that*

- (a) $|\mathcal{S}'| \leq 512 \lceil \log(|\mathcal{S}|) \rceil \ln(n) |\mathcal{S}| / d$.
- (b) *All supervertices in \mathcal{S}' are connected.*
- (c) *All supervertices in \mathcal{C} are connected components.*
- (d) $\mathcal{S}' \cup \mathcal{C}$ *is a partition of $\cup_{S \in \mathcal{S}} S$.*

2. *The total number of cut queries made is $O(\log(n)^5 + d \log(n)^3)$.*

Proof. All of Approximate Degree Sequence, Reduce High, and Learn Low have error probability at most $O(1/n)$. As they are called at most $O(\log n)$ times, with probability at least $1 - O(\log n/n)$ they will all return as promised. We now argue correctness assuming this is the case.

Let us establish that the hypotheses of Lemma 42 hold when Contract is called on Line 18. By assumption all supervertices in \mathcal{S} are connected, thus Item (1) holds. Also, as we are in the case where Reduce High and Learn Low perform correctly, Item (2) holds. If $\text{low}(i) = 1$ in the call to Contract then for all j, b for which $S_i \in \mathcal{L}_{j,b}$ it holds that $\vec{g}(i) < d$, and thus the corresponding call to Learn Low learns all neighbors of S_i in $\mathcal{R}_{j,b}$. As this is true for all j, b , we learn all neighbors of S_i , meaning that Item (3) also holds. As the hypotheses to Lemma 42 hold, this means all supervertices in \mathcal{S}' are connected, \mathcal{C} contains connected components, and $\mathcal{S}' \cup \mathcal{C}$ is a partition of $\cup_{S \in \mathcal{S}} S$, establishing Items 1(b),(c),(d).

We now turn to establish Item 1(a) of the lemma. Let

$$\mathcal{R} = \{S_i \in \mathcal{S} : \exists j \in \{1, \dots, \lceil \log |\mathcal{S}| \rceil\}, b \in \{0, 1\} \text{ such that } B_{j,b}(:, i) \neq \vec{0}\} .$$

In words, \mathcal{R} is the set of all S_i for which the i^{th} column of some $B_{j,b}$ matrix is nonzero. By Lemma 41, $|\mathcal{R}| \leq 512 \lceil \log(|\mathcal{S}|) \rceil \ln(n) |\mathcal{S}| / d$. We show that for every $U \in \mathcal{S}'$ there is an $R \in \mathcal{R}$ with $R \subseteq U$. This will establish Item 1(a) since \mathcal{S}' is a valid set of supervertices.

By the “moreover” statement of Lemma 42, for every $U \in \mathcal{S}'$ there is an $S_i \subseteq U$ with $\text{low}(i) = 0$. Thus for some j, b it holds that $S_i \in \mathcal{L}_{j,b}$ and the degree of S_i into $\mathcal{R}_{j,b}$ is at least $d/4$. By Lemma 41 the corresponding call to Reduce High on Line 11 will find a neighbor $R \in \mathcal{R}$ of S_i . Therefore, in the call to Contract a set containing S_i will be merged with a set containing R and therefore $R \subseteq U$.

Finally, the total number of iterations from the two for loops is $O(\log n)$. Let us now look at the complexity of each iteration of the inner for loop. Approximate Degree Sequence with error probability at most $1/n$ takes $O(\log(n)^3)$ many queries by Corollary 38. Each call to Reduce High takes $O(\log(n)^4)$ many queries by Lemma 41. Each call to Learn Low with error probability $1/n$ takes $O(d \log(n)^2)$ many queries by Corollary 39. This gives the complexity $O(\log(n)^5 + d \log(n)^3)$ as claimed in item (2). \square

Theorem 44. *Let G be a graph with vertex set V where $|V| = n$. There is a quantum algorithm that outputs the connected components of G with error probability at most $O(\log(n)^2/n)$ after making $O(\log(n)^6)$ many cut queries. In particular, the algorithm determines if G is connected or not with the same number of cut queries.*

Proof. The algorithm is given by Algorithm 7. Shrink is called with a degree parameter d of $1024 \lceil \log(n) \rceil^2$, thus by item 1(a) of Lemma 43 the size of the set \mathcal{S} will decrease by a factor of at least 2 in each iteration. Therefore the number of iterations of the repeat loop will be $O(\log n)$. This, together with item (3) of Lemma 43 gives a bound on the total number of queries of $O(\log(n)^6)$.

Now we argue correctness. By Lemma 43, at each stage of the algorithm we maintain the invariant that \mathcal{S} contains connected supervertices, ConComp contains connected components, and $\mathcal{S} \cup \text{ConComp}$ is a partition of V . The repeat-until loop will terminate as $|\mathcal{S}|$ is halved with every iteration. When it does terminate $\mathcal{S} = \emptyset$, thus at this stage ConComp is a partition of V by sets that are connected components. \square

6 Spanning forest

In this section we show that Algorithm 7 to find connected components can be extended to give an algorithm that finds a spanning forest and still only makes polylogarithmically many cut queries. The key idea for this is to find witnesses for superedges found in Algorithm 7.

Definition 45 (Witness). Let $G = (V, E)$ be a graph and S_i, S_j two supervertices of G connected by a superedge. We say that $\{u, v\}$ is a witness for this superedge if $u \in S_i, v \in S_j$ and $\{u, v\} \in E$.

Let T_1, T_2 be spanning trees for connected supervertices S_1, S_2 , and let $\{u, v\}$ be a witness for a superedge between S_1 and S_2 . Then $T_1 \cup T_2 \cup \{\{u, v\}\}$ is a spanning tree for $S_1 \cup S_2$. The spanning forest algorithm proceeds in the same framework as Algorithm 7, but now we maintain a spanning tree for each supervertex in $\mathcal{S} = \{S_1, \dots, S_k\}$ and only merge supervertices when we have a witness for a superedge between them. In this way, we are able to maintain spanning trees for every supervertex in \mathcal{S} as the algorithm proceeds. The main new difficulty is to find witnesses for the superedges discovered in Algorithm 7, as there we only discovered the existence of an edge between supervertices. However, we show that one can still manage to find enough witnesses to guarantee the size of \mathcal{S} shrinks by a factor of $1/2$ in each round.

First we show how to find a witness for each superedge in a bipartite graph if both sides of supervertices have low superdegree.

Algorithm 8 Witness Low-Low($\mathcal{S}, \mathcal{T}, h, \delta$)

Input: Disjoint valid sets of supervertices $\mathcal{S} = \{S_1, \dots, S_k\}, \mathcal{T} = \{T_1, \dots, T_\ell\}$, a degree parameter h such that $\deg_{\mathcal{T}}(S_i) \leq h$ for all $S_i \in \mathcal{S}$ and $\deg_{\mathcal{S}}(T_j) \leq h$ for all $T_j \in \mathcal{T}$, and an error parameter δ .

Output: A $|\cup_{i=1}^k S_i|$ -by- $|\cup_{i=1}^\ell T_i|$ Boolean matrix C such that $C(u, v) = 1$ implies $\{u, v\} \in E$ and for every superedge (S_i, T_j) there is a witness $\{u, v\}$ with $C(u, v) = 1$.

- 1: $U \leftarrow \cup_{i=1}^k S_i, U = \{u_1, \dots, u_{|U|}\}$
 - 2: $Y \leftarrow \cup_{i=1}^\ell T_i$
 - 3: $C \leftarrow \text{zeros}(|U|, |Y|)$
 - 4: $B \leftarrow \text{Learn Low}(\tilde{U}, \mathcal{T}, h, \delta/2)$ ▷ Recall $\tilde{U} = \{\{u\} : u \in U\}$
 - 5: $X \leftarrow \{u_a \in U : (\exists j (B(a, j) > 0) \text{ AND } (u_a \in S_i, u_b \in S_i, b < a) \Rightarrow B(b, j) = 0))\}$
 - 6: $D \leftarrow \text{Learn Low}(\tilde{Y}, \tilde{X}, h^2, \delta/2)$
 - 7: $C(\text{ind}(X), :) = D^T$
 - 8: **return** C
-

Lemma 46 (Witness Low-Low). *Let $\mathcal{S} = \{S_1, \dots, S_k\}, \mathcal{T} = \{T_1, \dots, T_\ell\}$ be disjoint valid sets of supervertices in an n -vertex graph $G = (V, E)$. Suppose that $\deg_{\mathcal{T}}(S_i) \leq h$ for all $S_i \in \mathcal{S}$ and $\deg_{\mathcal{S}}(T_j) \leq h$ for all $T_j \in \mathcal{T}$. Algorithm Witness Low-Low($\mathcal{S}, \mathcal{T}, h, \delta$) (Algorithm 8) makes $O((h^2 \log(n/h) + \log(1/\delta)) \log(n))$ many cut queries and finds a witness for every superedge between \mathcal{S} and \mathcal{T} , except with probability δ .*

Proof. Let $U = \cup_{i=1}^k S_i$, and $Y = \cup_{i=1}^\ell T_i$. Let $u_1 < \dots < u_{|U|}$ be an ordering of the elements of U . Let B be a $|U|$ -by- ℓ matrix where $B(a, j) = |E(u_a, T_j)|$ for $a \in \{1, \dots, |U|\}, j \in \{1, \dots, \ell\}$. Note that every row of B has at most h nonzero entries. By Corollary 39, $\text{Learn Low}(\tilde{U}, \mathcal{T}, h, \delta/2)$ will return B except with probability at most $\delta/2$, and makes $O((h \log(n/h) + \log(1/\delta)) \log(n))$ many cut queries. Now let $X = \{u_a \in U : \exists j (B(a, j) > 0 \text{ AND } (u_a \in S_i, u_b \in S_i, b < a) \Rightarrow B(b, j) = 0)\}$. In other words, for every i, j for which there exists $u_a \in S_i$ with $B(a, j) > 0$ we choose the least such u_a to put in the set X . Note that $|X \cap S_i| \leq h$ for every $S_i \in \mathcal{S}$.

Let D be the $|Y|$ -by- $|X|$ biadjacency matrix of the graph between Y and X . Every row will have at most h^2 ones, since $\deg_{\mathcal{S}}(T_j) \leq h$ and $|X \cap S_i| \leq h$. $\text{Learn Low}(\tilde{Y}, \tilde{X}, h^2, \delta/2)$ learns D

with $O((h^2 \log(n/h) + \log(1/\delta)) \log(n))$ many cut queries with error probability $\delta/2$. By learning D , for every superedge (S_i, T_j) we find a $u \in S_i, v \in T_j$ with $\{u, v\} \in E$. The total error probability is at most δ . \square

Next we show that we can find witnesses for supervertices on the left hand side of a bipartite graph between supervertices where all supervertices on the left hand side have low superdegree.

Algorithm 9 Witness Low-High($\mathcal{S}, \mathcal{T}, h$)

Input: Disjoint valid sets of supervertices $\mathcal{S} = \{S_1, \dots, S_k\}, \mathcal{T} = \{T_1, \dots, T_\ell\}$, a degree parameter h such that $\deg_{\mathcal{T}}(S_i) \leq h$ for all $S_i \in \mathcal{S}$.

Output: A $|\cup_i S_i|$ -by- $|\cup_j T_j|$ matrix C such that $C(a, b) = 1$ implies $(u_a, v_b) \in E$ and for every S_i with $\deg_{\mathcal{T}}(S_i) > 0$ there is a $u_a \in S_i, v_b \in \cup_j T_j$ such that $C(a, b) = 1$.

```

1:  $U \leftarrow \cup_{i=1}^k S_i$ 
2:  $Y \leftarrow \cup_{i=1}^{\ell} T_i$ 
3:  $C \leftarrow \text{zeros}(|U|, |Y|)$ 
4:  $B \leftarrow \text{Learn Low}(\tilde{U}, \mathcal{T}, h, 1/n)$   $\triangleright \tilde{U} = \{\{u\} : u \in U\}$ 
5:  $X \leftarrow \{u_a \in U : (\exists j B(a, j) > 0) \text{ AND } (u_a \in S_i, u_b \in S_i, b < a) \Rightarrow \forall c B(b, c) = 0\}$ 
6: for  $u \in X$  do
7:    $j^*(u_a) = \arg \max_{j \in \{1, \dots, \ell\}} B(a, j)$ 
8: end for
9: for  $q = 0, \dots, \lceil \log |Y| \rceil$  do
10:    $X_q \leftarrow \{u \in X : 2^{q-1} < B(u, j^*(u)) \leq 2^q\}$ 
11:    $R_q \leftarrow \text{Randomly sample } \lceil 16|Y| \ln(|X_q|n)/2^q \rceil$  many elements from  $Y$  with replacement
12:    $C(\text{ind}(X_q), \text{ind}(R_q)) \leftarrow \text{Learn Low}(X_q, R_q, 192h \ln(n), 1/n)$ 
13: end for
14: return  $C$ 

```

Lemma 47 (Witness Low-High). *Let $\mathcal{S} = \{S_1, \dots, S_k\}$ and $\mathcal{T} = \{T_1, \dots, T_\ell\}$ be disjoint valid sets of supervertices in an n -vertex graph $G = (V, E)$, and suppose that $\deg_{\mathcal{T}}(S_i) \leq h$ for all $S_i \in \mathcal{S}$. Algorithm Witness Low-High($\mathcal{S}, \mathcal{T}, h$) (Algorithm 9) makes $O(h \log(n)^4)$ many cut queries and except with probability $O(\log(n)/n)$ outputs a $|\cup_i S_i|$ -by- $|\cup_j T_j|$ matrix C such that*

1. $C(a, b) = 1$ implies $\{u_a, v_b\} \in E$
2. For every S_i with $\deg_{\mathcal{T}}(S_i) > 0$ there is a $u_a \in S_i, v_b \in \cup_j T_j$ such that $C(a, b) = 1$.

Proof. Let $U = \cup_{i=1}^k S_i$ and let $u_1 < \dots < u_{|U|}$ be an ordering of the elements of U . The algorithm first performs $B \leftarrow \text{Learn Low}(\tilde{U}, \mathcal{T}, h, 1/n)$. By the correctness of Learn Low from Corollary 39, except with probability $1/n$, it will hold that $B(a, j) = |E(u_a, T_j)|$ for all $a \in \{1, \dots, |U|\}, j \in \{1, \dots, \ell\}$. This step takes $O(h \log(n/h) \log(n))$ cut queries. We henceforth assume that this step was performed correctly. Define $X = \{u_a \in U : (\exists j B(a, j) > 0) \text{ AND } ((u_a \in S_i, u_b \in S_i, b < a) \Rightarrow \forall c B(b, c) = 0)\}$. In other words, for every S_i we take the first $u_a \in S_i$ for which there is a $j \in \{1, \dots, \ell\}$ such that $B(a, j) > 0$, if such a u_a exists. Let $j^*(u_a) = \arg \max_{j \in \{1, \dots, \ell\}} B(a, j)$ for each $u_a \in X$. For each $u \in X$, we are going to learn a $v \in T_{j^*(u)}$ such that $\{u, v\} \in E$.

Fix a value of q in the for loop starting on line 9. For every $u_a \in X_q$ and $j \in \{1, \dots, \ell\}$ we have $B(a, j) \leq B(a, j^*(u_a)) \leq 2^q$. For $\delta = 1/n^2$, R_q is formed by randomly sampling with replacement $\lceil 8|Y| \ln(|X_q|/\delta)/2^q \rceil$ elements of Y . Thus by Lemma 8 and a union bound over $j \in \{1, \dots, \ell\}$, except with probability $1/n$, we have

$$|E(u, R_q \cap T_j)| \leq 192 \ln(n) \text{ for all } j \in \{1, \dots, \ell\} \text{ and } u \in X_q . \quad (1)$$

Similarly, except with probability $1/n$, we also have

$$|E(u, R_q \cap T_j)| > 0 \text{ for all } j \in \{1, \dots, \ell\} \text{ and } u \in X_q \text{ with } 2^{q-1} < B(u, j) .$$

In particular, except with probability $1/n$

$$|E(u, R_q \cap T_j^*(u))| > 0 \text{ for all } u \in X_q . \quad (2)$$

We now add $O(1/n)$ to the error probability and assume for the rest of the proof that Eq. (1) and Eq. (2) hold. Next the algorithm performs

$$C(\text{ind}(X_q), \text{ind}(R_q)) \leftarrow \text{Learn Low}(X_q, R_q, 192h \ln(n), 1/n) .$$

By Eq. (1), the upper bound on the degree passed to Learn Low is valid, thus by Corollary 39 Learn Low returns the biadjacency matrix between X_q and R_q , except with probability $1/n$. In particular, except with probability $1/n$, if $C(a, b) = 1$ for $a \in \text{ind}(X_q), b \in \text{ind}(R_q)$ then $\{u_a, v_b\} \in E$ and by Eq. (2) for every $u_a \in X_q$ there is $v_b \in R_q$ such that $C(a, b) = 1$. This step requires $O(h \log(n)^2 \log(n/h))$ many cut queries.

Finally, by a union bound over $q = 0, \dots, \lceil \log |Y| \rceil$, the probability of an error in any iteration of the for loop is at most $O(\log(n)/n)$. As every $u \in X$ will be in X_q for some value of q , for every S_i with $\deg_{\mathcal{T}}(S_i) > 0$ we will find a $u \in S_i$ and $v \in Y$ with $\{u, v\} \in E$. The total error probability is $O(\log(n)/n)$, and total number of queries is $O(h \log(n)^4)$. \square

Next we show how to find witnesses for supervertices of high superdegree.

Lemma 48 (Witness Reduce High). *Let $\mathcal{S} = \{S_1, \dots, S_k\}, \mathcal{T} = \{T_1, \dots, T_\ell\}$ be disjoint valid sets of supervertices of an n -vertex graph G . Let $d \in \mathbb{N}$ be a degree parameter and $\vec{g} \in \mathbb{R}^k$ be a good estimate of $(\deg_{\mathcal{T}}(S_1), \dots, \deg_{\mathcal{T}}(S_k))$. Further suppose that $\deg_{\mathcal{T}}(S_i) \geq d$ for all $S_i \in \mathcal{S}$. On input $\mathcal{S}, \mathcal{T}, d, \vec{g}$, Witness Reduce High (Algorithm 10) makes $O(\log(n)^5)$ many cut queries and with probability at least $1 - O(\log(n)^2/n)$ outputs a $|\cup_i S_i|$ -by- $|\cup_j T_j|$ Boolean matrix C such that*

1. For every $S_i \in \mathcal{S}$, there exists $u \in S_i$ and $v \in \cup_j T_j$ such that $C(u, v) = 1$.
2. If $C(u, v) = 1$ then $\{u, v\} \in E$.
3. $|\{T_j : \exists u \in \cup_i S_i, \exists v \in T_j : C(u, v) = 1\}| \leq \frac{256\ell \log(n)}{d}$.

Algorithm 10 Witness Reduce High($\mathcal{S}, \mathcal{T}, d, \vec{g}$)

Input: Disjoint valid sets of supervertices $\mathcal{S} = \{S_1, \dots, S_k\}$, $\mathcal{T} = \{T_1, \dots, T_\ell\}$ of an n -vertex graph G , a degree parameter d such that $\deg_{\mathcal{T}}(S_i) \geq d$ for all $S_i \in \mathcal{S}$, and a vector $\vec{g} \in \mathbb{Z}^k$ such that $\vec{g}(i)i/4 \leq \deg_{\mathcal{T}}(S_i) \leq 2\vec{g}(i)$ for all $i \in \{1, \dots, k\}$.

Output: A $|\cup_{i=1}^k S_i|$ -by- $|\cup_{i=1}^\ell T_i|$ Boolean matrix C such that $C(u, v) = 1$ implies $\{u, v\} \in E$ and for every S_i there is a $u \in S_i, v \in \cup_j T_j$ such that $C(u, v) = 1$.

- 1: $C \leftarrow \text{zeros}(|\cup_{i=1}^k S_i|, |\cup_{i=1}^\ell T_i|)$
 - 2: **for** $q = \lfloor \log d \rfloor - 1$ **to** $\lceil \log \ell \rceil + 2$ **do**
 - 3: $\mathcal{H}_q = \{S_i \in \mathcal{S} : 2^{q-1} < \vec{g}(i) \leq 2^q\}$
 - 4: $\mathcal{R}_q \leftarrow$ Randomly choose $\frac{16\ell \ln(|\mathcal{H}_q|n)}{2^q}$ supervertices in \mathcal{T} , with replacement
 - 5: $C(\text{ind}(\mathcal{H}_q), \text{ind}(\mathcal{R}_q)) \leftarrow$ Witness Low-High($\mathcal{H}_q, \mathcal{R}_q, 192 \ln(n)$)
 - 6: **end for**
 - 7: **return** B
-

Proof. We first show item (2). By Lemma 47, except with probability $O(\log(n)/n)$, the call to Witness Low-High on Line 5 will only return valid edges in G . As there are $O(\log n)$ iterations of the for loop, item (2) will therefore hold except with probability $O(\log(n)^2/n)$.

We now show items (1) and (3), assuming that all calls to Witness Low-High return correctly. The only difference between Reduce High and Witness Reduce High is that in Line 5 of Witness Reduce-High we call Witness Low-High instead of Learn Low. As Witness Low-High returns correctly, by Lemma 47 this means we find a witness for every superedge found by Learn Low, and therefore a witness for every superedge found by Reduce High. By Item (1) of Lemma 41 for every $S_i \in \mathcal{S}$ Reduce High finds a $T_j \in \mathcal{T}$ such that (S_i, T_j) is a superedge. Finding a witness for each of these superedges gives Item (1) here.

Furthermore, we do not find witnesses for any superedges not found in Reduce High. Thus Item (3) of Lemma 41 implies Item (3) here.

Finally, we bound the number of queries made. There are $O(\log n)$ iterations of the for loop, and in each iteration queries are only made in the call to Witness Low-High. Each of these calls take $O(\log(n)^4)$ cut queries, thus the total number of cut queries is $O(\log(n)^5)$. \square

Next we show how to contract supervertices by using the edges found so far.

Lemma 49. Let $\mathcal{S} = \{S_1, \dots, S_k\}$ be a valid set of supervertices with $N = |\cup_{i=1}^k S_i|$, $\mathcal{P} = \{P_1, \dots, P_k\}$ a set of spanning trees, \mathcal{A} a list of N -by- N weighted adjacency matrices, and $\text{low} \in \{0, 1\}^k$ a Boolean vector with the following properties:

1. Every supervertex in \mathcal{S} is connected, and a spanning tree for $S_i \in \mathcal{S}$ is given by $P_i \in \mathcal{P}$.
2. For every $A \in \mathcal{A}$ if $A(u, v) = 1$ then $\{u, v\} \in E$.
3. For every $i \in \{1, \dots, k\}$ with $\text{low}(i) = 1$, it holds that for every $j \in \{1, \dots, k\}$ such that there is a superedge between S_i and S_j , there is a $u \in S_i, v \in S_j$ and $A \in \mathcal{A}$ with $A(u, v) = 1$.

The algorithm *Witness Contract*($\mathcal{S}, \mathcal{P}, \mathcal{A}, \text{low}$) given in Algorithm 11 makes no queries and outputs sets of supervertices $\mathcal{S}' = \{S'_1, \dots, S'_\ell\}$, $\mathcal{C} = \{C_1, \dots, C_t\}$ and sets of spanning trees $\mathcal{P}_{\mathcal{S}'} = \{P'_1, \dots, P'_\ell\}$, $\mathcal{P}_{\mathcal{C}} = \{Q_1, \dots, Q_t\}$ such that

- Each $S'_i \in \mathcal{S}'$ is connected and a spanning tree for it is given by $P'_i \in \mathcal{P}_{\mathcal{S}'}$.
- Each $C_i \in \mathcal{C}$ is a connected component and a spanning tree for it is given by $Q_i \in \mathcal{P}_{\mathcal{C}}$.
- $\mathcal{S}' \cup \mathcal{C}$ is a partition of $\cup_{S \in \mathcal{S}} S$.
- For every $U \in \mathcal{S}'$ there is an $S_i \subseteq U$ with $\text{low}(i) = 0$.

Proof. We first show that each $S'_i \in \mathcal{S}'$ is connected and a spanning tree for it is given by $P'_i \in \mathcal{P}_{\mathcal{S}'}$. By assumption each $S_i \in \mathcal{S}$ is connected with a spanning tree given by $P_i \in \mathcal{P}$. We maintain this invariant because we only merge two supervertices U, W when there is a $u \in U, w \in W$ and a $A \in \mathcal{A}$ with $A(u, w) = 1$. By assumption if $A(u, w) = 1$ then $\{u, w\} \in E$ and thus $U \cup W$ is connected when U, W are. Furthermore, if T_1 is a spanning tree for U and T_2 is a spanning tree for W then $T_1 \cup T_2 \cup \{\{u, w\}\}$ is a spanning tree for $U \cup W$.

The same argument shows that each $C_i \in \mathcal{C}$ is connected with a spanning tree given by $P_i \in \mathcal{P}_{\mathcal{C}}$.

Next we show that every $W \in \mathcal{C}$ is a connected component. Suppose for a contradiction that this is not the case and therefore there is a $w \in S_i \subseteq W$ and $u \in S_j \subseteq (\cup_{t=1}^k S_t) \setminus W$ such that $\{u, w\}$ is an edge of G . It must be the case that S_i is low, as otherwise the lowFlag for W would have been set to 0 and W would have been placed in \mathcal{S}' . Thus S_i must be low and therefore by hypothesis for some $w' \in S_i, u' \in S_j$ and $A \in \mathcal{A}$ it is the case that $A(w', u') = 1$. This means that at some point in *Witness Contract* a set containing S_i would have been merged with a set containing S_j , a contradiction to the fact that $S_j \subseteq (\cup_{t=1}^k S_t) \setminus W$.

The fact that $\mathcal{S} \cup \mathcal{C}$ is a partition of $\cup_{S \in \mathcal{S}} S$ follows because at all times $\cup_{(U, T, \text{flag}) \in L} U$ is equal to $\cup_{S \in \mathcal{S}} S$. This is true when L is first defined, and is preserved when sets are popped from L , merged, and put back into L .

Finally, the “moreover” statement holds as if $\text{low}(i) = 1$ for all $S_i \in U$ then the lowFlag variable for U will be set to 1 and therefore U will be placed into \mathcal{C} on Line 21. \square

The last lemma guarantees the shrinkage of the number of supervertices.

Lemma 50. *Let $\mathcal{S} = \{S_1, \dots, S_k\}$ be a valid set of connected supervertices, $\mathcal{P} = \{P_1, \dots, P_k\}$ be a set of spanning trees where P_i is a spanning tree for S_i , and $d \in \mathbb{N}$ be a degree parameter. *Witness Shrink*($\mathcal{S}, \mathcal{P}, d$) given by Algorithm 12 has the following properties:*

1. *Except with probability $O(\log(n)^3/n)$, it outputs sets of supervertices $\mathcal{S}', \mathcal{C}$ and sets of spanning trees $\mathcal{P}_{\mathcal{S}'}, \mathcal{P}_{\mathcal{C}}$ such that*
 - (a) $|\mathcal{S}'| \leq 512 \lceil \log(|\mathcal{S}|) \rceil \ln(n) |\mathcal{S}| / d$.
 - (b) *For every supervertex in \mathcal{S}' there is a spanning tree for it in $\mathcal{P}_{\mathcal{S}'}$.*
 - (c) *Every supervertex in \mathcal{C} is a connected component and has a spanning tree for it in $\mathcal{P}_{\mathcal{C}}$.*

Algorithm 11 Witness Contract($\mathcal{S}, \mathcal{P}, \mathcal{A}, low$)

Input: Valid set of connected supervertices $\mathcal{S} = \{S_1, \dots, S_k\}$, a set of spanning trees $\mathcal{P} = \{P_1, \dots, P_k\}$, a list of N -by- N weighted adjacency matrices $\mathcal{A} = (A_1, \dots, A_m)$ with rows and columns labeled by elements of $[N]$, a vector $low \in \{0, 1\}^k$ indicating if each set S_i is low.

Output: Sets of supervertices \mathcal{S}' , \mathcal{C} and sets of spanning trees for them $\mathcal{P}_{\mathcal{S}'}$, $\mathcal{P}_{\mathcal{C}}$, where each supervertex in \mathcal{S}' , \mathcal{C} is connected, and moreover the supervertices in \mathcal{C} are connected components.

```
1:  $L \leftarrow [(S_i, P_i, low(i)) : i \in \{1, \dots, k\}]$ 
2: for  $A \in \mathcal{A}$  do
3:   for  $\{u, v\} \in [N]^{(2)}$  do
4:     if  $A(u, v) = 1$  then
5:        $Pop(U, T_1, flag1) \in L$  such that  $u \in S_i \subseteq U$ 
6:        $Pop(W, T_2, flag2) \in L$  such that  $v \in S_j \subseteq W$ 
7:        $U \leftarrow U \cup W$ 
8:        $T = T_1 \cup T_2 \cup \{\{u, v\}\}$ 
9:        $lowFlag = flag1 \wedge flag2$ 
10:      Append  $(U, T, lowFlag)$  to  $L$ 
11:     end if
12:   end for
13: end for
14:  $\mathcal{S}' \leftarrow \emptyset, \mathcal{P}_{\mathcal{S}'} \leftarrow \emptyset$ 
15:  $\mathcal{C} \leftarrow \emptyset, \mathcal{P}_{\mathcal{C}} \leftarrow \emptyset$ 
16: for  $(U, T, lowFlag) \in L$  do
17:   if  $lowFlag = 0$  then
18:      $\mathcal{S}' \leftarrow \mathcal{S}' \cup \{U\}$ 
19:      $\mathcal{P}_{\mathcal{S}'} \leftarrow \mathcal{P}_{\mathcal{S}'} \cup \{T\}$ 
20:   else
21:      $\mathcal{C} \leftarrow \mathcal{C} \cup \{U\}$ 
22:      $\mathcal{P}_{\mathcal{C}} \leftarrow \mathcal{P}_{\mathcal{C}} \cup \{T\}$ 
23:   end if
24: end for
25: return  $\mathcal{S}', \mathcal{P}_{\mathcal{S}'}, \mathcal{C}, \mathcal{P}_{\mathcal{C}}$ 
```

Algorithm 12 Witness Shrink($\mathcal{S}, \mathcal{P}, d$)

Input: A valid set of connected supervertices $\mathcal{S} = \{S_1, \dots, S_k\}$, a set of spanning trees $\mathcal{P} = \{P_1, \dots, P_k\}$ where P_i is a spanning tree for S_i , and a degree parameter d .

Output: Sets of supervertices \mathcal{S}, \mathcal{C} and corresponding sets of spanning trees $\mathcal{P}_{\mathcal{S}}, \mathcal{P}_{\mathcal{C}}$.

```
1:  $low \leftarrow \text{ones}(k, 1)$ 
2:  $\mathcal{A} \leftarrow []$ 
3:  $N \leftarrow |\cup_{i=1}^k S_i|$ 
4: for  $j = 1$  to  $\lceil \log(k) \rceil$  do
5:   for  $b \in \{0, 1\}$  do
6:      $\mathcal{L}_{j,b} = \{S_t \in \mathcal{S} : t_j = b\}$ 
7:      $\mathcal{R}_{j,b} = \{S_t \in \mathcal{S} : t_j = 1 - b\}$ 
8:      $\vec{g} \leftarrow \text{Approximate Degree Sequence}(\mathcal{L}_{j,b}, \mathcal{R}_{j,b}, 1/n^2)$ 
9:      $\mathcal{H} \leftarrow \{S_t \in \mathcal{L}_{j,b} : \vec{g}(t) \geq d\}$ 
10:     $low(\text{ind}(\mathcal{H})) = 0$ 
11:     $B_{j,b} \leftarrow \text{zeros}(N, N)$ 
12:     $B_{j,b}(\cup_{S_t \in \mathcal{H}} S_t, \cup_{S_t \in \mathcal{R}_{j,b}} S_t) \leftarrow \text{Witness Reduce High}(\mathcal{H}, \mathcal{R}_{j,b}, d/4, \vec{g}(\text{ind}(\mathcal{H})))$ 
13:     $\mathcal{L} \leftarrow \{S_t \in \mathcal{L}_{j,b} : \vec{g}(t) < d\}$ 
14:     $\vec{f} \leftarrow \text{Approximate Degree Sequence}(\mathcal{R}_{j,b}, \mathcal{L}, 1/n^2)$ 
15:     $\mathcal{R}_{j,b}^+ \leftarrow \{S_t \in \mathcal{R}_{j,b} : \vec{f}(t) \geq 16d\}$ 
16:     $\mathcal{R}_{j,b}^- \leftarrow \{S_t \in \mathcal{R}_{j,b} : \vec{f}(t) < 16d\}$ 
17:     $C_{j,b}^+(\cup_{S_t \in \mathcal{L}} S_t, \cup_{S_t \in \mathcal{R}_{j,b}^+} S_t) \leftarrow \text{Witness Low-High}(\mathcal{L}, \mathcal{R}_{j,b}^+, 2d)$ 
18:     $HasHighNeighbor \leftarrow \{i : \exists u \in S_i \in \mathcal{L}, \exists v : C_{j,b}^+(u, v) = 1\}$ 
19:     $low(HasHighNeighbor) = 0$ 
20:     $C_{j,b}^-(\cup_{S_t \in \mathcal{L}} S_t, \cup_{S_t \in \mathcal{R}_{j,b}^-} S_t) \leftarrow \text{Witness Low-Low}(\mathcal{L}, \mathcal{R}_{j,b}^-, 32d, 1/n)$ 
21:    Append  $B_{j,b}, C_{j,b}^+$ , and  $C_{j,b}^-$  to  $\mathcal{A}$ .
22:   end for
23: end for
24:  $\mathcal{S}, \mathcal{P}, \mathcal{C}, \mathcal{P}_{\mathcal{C}} \leftarrow \text{Witness Contract}(\mathcal{S}, \mathcal{P}, \mathcal{A})$ 
25: return  $\mathcal{S}, \mathcal{P}, \mathcal{C}, \mathcal{P}_{\mathcal{C}}$ 
```

(d) $\mathcal{S}' \cup \mathcal{C}$ is a partition of $\cup_{S \in \mathcal{S}} S$.

2. The total number of cut queries made is $O(d \log(n)^5 + d^2 \log(n)^3)$.

Proof. We first prove item (2). Queries are only made in the calls to Approximate Degree Sequence, Witness Reduce High, Witness Low-High, and Witness Low-Low. Each of these routines is called $2 \lceil \log |\mathcal{S}| \rceil = O(\log n)$ many times. The query cost is dominated by Witness Low-High, which takes $O(d \log(n)^4)$ many queries, and Witness Low-Low which requires $O(d^2 \log(n)^2)$ queries, resulting in a total of $O(d \log(n)^5 + d^2 \log(n)^3)$ cut queries.

The error probability of the call to Approximate Degree Sequence is $1/n^2$, Witness Reduce High is $O(\log(n)^2/n)$, Witness Low-High is $O(\log(n)/n)$, and Witness Low-Low is $1/n$. Thus the probability that an error occurs in any of these routines over the course of Witness Shrink is $O(\log(n)^3/n)$. We now argue the points in item (1) hold assuming all of these routines always return correctly.

We first establish that the hypotheses of Lemma 49 hold when Witness Contract is called on Line 24. By assumption the supervertices in \mathcal{S} are connected and spanning trees for them are given in \mathcal{P} . The matrices in \mathcal{A} are produced in calls to Witness Reduce High, Witness Low-High, and Witness Low-Low. As we are in the case that all of these algorithms return correctly it follows by Lemma 46, Lemma 47, and Lemma 48 that for every $A \in \mathcal{A}$ if $A(u, v) = 1$ then $\{u, v\} \in E$. Finally, we need to establish that if $low(i) = 1$ then \mathcal{A} contains a witness for every superedge of S_i . Suppose that $low(i) = 1$ and S_i has a superedge with S_t . For some value of j, b in the for loop we will have $S_i \in \mathcal{L}_{j,b}, S_t \in \mathcal{R}_{j,b}$. As $low(i) = 1$ and Witness Low-High returns correctly, S_i has no neighbors in $\mathcal{R}_{j,b}^+$. Thus it must be the case that $S_t \in \mathcal{R}_{j,b}^-$. As \vec{f}, \vec{g} are good estimates because Approximate Degree Sequence returns correctly, the degree bound in the call to Witness Low-Low is valid and therefore by Lemma 46 a witness for the (S_i, S_t) superedge will be found in the call to Witness Low-Low.

We have now established the hypotheses to Lemma 49 and thus can invoke the conclusion of Lemma 49 which implies Items 1(b),(c),(d) of the current lemma.

It remains to show Item 1(a). Let

$$\mathcal{R} = \{S_t \in \mathcal{S} : \exists j \in \{1, \dots, \lceil \log k \rceil\}, b \in \{0, 1\}, v \in S_t \text{ such that } B_{j,b}(\cdot, v) \neq \vec{0}\}.$$

By Lemma 48, and the fact that the number of iterations of the for loop is $2 \lceil \log |\mathcal{S}| \rceil$, we have $|\mathcal{R}| \leq 512 \lceil \log |\mathcal{S}| \rceil \log(n) |\mathcal{S}| / d$. In Witness Contract, a supervertex W will be put into \mathcal{S}' iff for some $S_i \in W$ we have $low(i) = 0$. In the next paragraph we show that in that case Witness Shrink finds witnesses to certify that an element of \mathcal{R} is in the connected component of S_i . This means that Witness Contract will merge S_i into a set W containing an element of \mathcal{R} and therefore the number of supervertices in \mathcal{S}' can be upper bounded by $|\mathcal{R}|$ and will give Item 1(a).

Take an S_i with $low(i) = 0$ and consider the iteration j, b of the for loop where $low(i)$ is set to zero. There are two ways this can happen. The first is if $\vec{g}(S_i) \geq d$. In this case S_i will be placed into \mathcal{H} and a witness for a neighbor in \mathcal{R} will be found in the call to Witness Reduce High by Lemma 48. The second case is that a witness is found for a superedge between S_i and an element $S_\ell \in \mathcal{R}_{j,b}^+$. In the $j, 1-b$ iteration of the for loop, $S_\ell \in \mathcal{L}_{j,1-b}$ and moreover $\vec{g}(\ell) \geq d$, as $\vec{f}(\ell) \geq 16d$ for S_ℓ to be placed in $\mathcal{R}_{j,b}^+$ and both are good estimates. Therefore by the previous

argument a witness for a neighbor of S_ℓ with an element of \mathcal{R} will be found in the call to Witness Reduce High. Thus we have witnesses that S_i is connected to S_ℓ and that S_ℓ is connected to an element of \mathcal{R} . \square

Finally we can give the algorithm of finding a spanning forest with polylogarithmic many cut queries.

Algorithm 13 Spanning Forest with cut queries

Input: Cut oracle for a graph G on n vertices.

Output: A set $\mathcal{P} = \{P_1, \dots, P_t\}$ containing a spanning tree for every connected component of G .

- 1: $\mathcal{S} \leftarrow \tilde{V}, \mathcal{P}_\mathcal{S} \leftarrow \tilde{V}$
 - 2: $\mathcal{P} \leftarrow \emptyset, \text{ConComp} \leftarrow \emptyset$
 - 3: **repeat**
 - 4: $(\mathcal{S}, \mathcal{P}_\mathcal{S}, \mathcal{C}, \mathcal{P}_\mathcal{C}) \leftarrow \text{WitnessShrink}(\mathcal{S}, \mathcal{P}_\mathcal{S}, 1024 \lceil \log n \rceil^2)$
 - 5: $\text{ConComp} \leftarrow \text{ConComp} \cup \mathcal{C}$
 - 6: $\mathcal{P} \leftarrow \mathcal{P} \cup \mathcal{P}_\mathcal{C}$
 - 7: **until** $\mathcal{S} = \emptyset$
 - 8: **return** \mathcal{P}
-

Theorem 51. *Given cut query access to an n -vertex graph G , there is quantum algorithm (Algorithm 13) making $O(\log(n)^8)$ queries that outputs a spanning forest for G with probability $1 - O(\log(n)^4/n)$.*

Proof. We first argue by induction that the i^{th} time Witness Shrink is called the hypothesis to Lemma 50 is satisfied with probability at least $1 - (i - 1) \log(n)^3/n$.

The first time Witness Shrink is called, $\mathcal{S} = \tilde{V}$ and $\mathcal{P}_\mathcal{S} = \tilde{V}$. Thus $\mathcal{P}_\mathcal{S}$ provides valid spanning trees for each supervertex in \mathcal{S} and the hypothesis to Lemma 50 is satisfied. Now suppose the inductive assumption holds the i^{th} time Witness Shrink is called. Then after the call to Witness Shrink we know that except with probability $\log(n)^3/n$ the output \mathcal{S}, \mathcal{P} satisfy that each supervertex in \mathcal{S} is connected and has a valid spanning tree for it given in \mathcal{P} . Thus in the $(i + 1)^{\text{th}}$ call to Witness Shrink the hypothesis to Lemma 50 holds with probability at least $1 - i \log(n)^3/n$.

If the hypothesis to Lemma 50 holds in the call to Witness Shrink, then by the choice of the degree parameter $d = 1024 \lceil \log n \rceil^2$ the size of \mathcal{S} will reduce by a factor of $1/2$ with every iteration of the repeat until loop. Thus with probability at least $1 - \log(n)^4/n$ every call to Witness Shrink will return correctly and the number of iterations will be at most $O(\log(n))$. In this case the total number of queries made will be $O(\log(n)^8)$.

If every call to Witness Shrink returns correctly, then the algorithm maintains the invariant that $\mathcal{S} \cup \text{ConComp}$ is a partition of V , every supervertex in ConComp is a connected component and has a valid spanning tree in $\mathcal{P}_\mathcal{C}$. At the end of the algorithm $\mathcal{S} = \emptyset$ thus ConComp contains all connected components of G and $\mathcal{P}_\mathcal{C}$ is a spanning forest. \square

6.1 Applications

With the ability to compute a spanning forest we can also easily solve some other graph problems in the cut query model.

Theorem 52. *There is a quantum algorithm to determine if an n -vertex graph G is bipartite that makes $O(\log(n)^8)$ many cut queries and succeeds with probability at least $1 - O(\log(n)^4/n)$.*

Proof. We first invoke Theorem 51 to find a spanning forest for G with $O(\log(n)^8)$ many cut queries and success probability $1 - O(\log(n)^4/n)$. We then color each root of a spanning tree red, and proceed to color all the remaining vertices blue and red such that no two vertices connected in a spanning tree have the same color. The graph is then bipartite if and only if there is no edge of G between two vertices of the same color.

We can check if there is an edge between two red vertices with $O(\log(n))$ cut queries. Let S be the set of red vertices. We consider $O(\log n)$ bipartite graphs (L_i, R_i, E_i) where $L_i = \{v \in S : v_i = 0\}$, $R_i = \{v \in S : v_i = 1\}$ and $E_i = \{(u, v) : u \in L_i, v \in R_i, \{u, v\} \in E\}$. Then with 3 cut queries we can check if $|E(L_i, R_i)| > 0$. An edge between two red vertices will be present in at least one of these bipartite graphs, thus this process will determine if there is an edge between two red vertices. We then do the same procedure for the blue vertices. \square

Similarly we can check if a graph is acyclic.

Theorem 53. *There is a quantum algorithm to determine if an n -vertex graph G is acyclic that makes $O(\log(n)^8)$ many cut queries and succeeds with probability at least $1 - O(\log(n)^4/n)$.*

Proof. We first check that the graph is bipartite, i.e. that it has no odd cycles, using Theorem 52. If it is bipartite, then it remains to check that it also has no even cycles.

To check for even cycles we do the same procedure as in the proof of Theorem 52: we find a spanning forest and color the vertices of the spanning trees red and blue. Then the graph will have no even cycles if and only if there are no additional edges between red and blue vertices than those present in the spanning trees. Let the set of red vertices be R and the set of blue vertices be B . With three cut queries we determine $|E(B, R)|$. We then compare this to the number of edges between blue and red vertices that in the spanning forest. There is no even cycle if and only if these numbers are the same. \square

Remark In both applications, after finding a spanning forest, the problem essentially becomes testing graph emptiness: In bipartite testing we need to check the vertices of the same color form an empty graph, and in acyclic graph testing we need to check that there is no edge other than those in the found spanning forest. If a small constant error is tolerated (as opposed to the $\tilde{O}(1/n)$ one obtained in the above two proofs), then testing graph emptiness can be done in a constant number of queries. Indeed, a query $E(S, V \setminus S)$ for a random subset $S \subseteq V$ returns a positive integer as long as one edge exists, and repeating this $\lceil \log(1/\epsilon) \rceil$ times gives an error probability of at most ϵ .

7 Concluding remarks

In this paper we investigate the power of additive and cut queries on graphs, and demonstrate that quantum algorithms using these oracles that can solve certain graph problems with surprisingly low query cost. Some open questions are left for future investigation, and we list a few of them here.

1. The most pressing problem left open by this work is the quantum complexity of minimum cut with a cut oracle. Can this be solved with a polylogarithmic number of queries?
2. Classically, the best known lower bounds on the query complexity of minimizing a submodular function with an evaluation oracle can be shown via the connectivity problem. We have ruled connectivity out as a candidate for a good quantum lower bound, and in fact we do not know of any nontrivial lower bound on the quantum query complexity of minimizing a submodular function. As a modest challenge, can one give an example of a submodular function whose minimization problem requires $\Omega(n)$ many evaluation queries by a quantum algorithm?
3. Another interesting problem is the maximization of a submodular function. This problem is NP-hard in general and classically exponentially large query lower bounds are known. For example, [FMV11] show that $\exp(\epsilon^2 n/8)$ many evaluation oracle queries can be needed by a randomized algorithm even to find a $(\frac{1}{2} + \epsilon)$ -approximation to the maximum value of a submodular function. What is the quantum query complexity of submodular function maximization?

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