

ON THE GEOMETRY OF CAYLEY AUTOMATIC GROUPS

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ABSTRACT. In contrast to being automatic, being Cayley automatic *a priori* has no geometric consequences. Specifically, Cayley graphs of automatic groups enjoy a fellow traveler property. Here we study a distance function introduced by the first author and Trakuldit which aims to measure how far a Cayley automatic group is from being automatic, in terms of how badly the Cayley graph fails the fellow traveler property. The first author and Trakuldit showed that if it fails by at most a constant amount, then the group is in fact automatic. In this article we show that for a large class of non-automatic Cayley automatic groups this function is bounded below by a linear function in a precise sense defined herein. In fact, for all Cayley automatic groups which have super-quadratic Dehn function, or which are not finitely presented, we can construct a non-decreasing function which (1) depends only on the group and (2) bounds from below the distance function for any Cayley automatic structure on the group.

1. INTRODUCTION

Cayley automatic groups generalize the class of automatic groups while retaining their key *algorithmic* properties. Namely, the word problem in a Cayley automatic group is decidable in quadratic time, regular normal forms for group elements can be computed in quadratic time, and the first order theory for a (directed, labeled) Cayley graph of a Cayley automatic group is decidable. Their history traces back to S enizergues who observed that the (standard) Cayley graph for the integral Heisenberg group is FA-presentable, with a proof first appearing in [9] in 2004 (see in particular page 651), and the concept was brought to the attention of combinatorial/geometric group theorists by Kharlampovich, Khoussainov and Miasnikov in [20].

The family of Cayley automatic groups is much broader than that of automatic groups, as it includes, for example, all finitely generated nilpotent groups of nilpotency class two [20], the Baumslag-Solitar groups [5, 20], higher rank lamplighter groups [8], and restricted wreath products of the form $G \wr H$ where G is Cayley automatic and H is (virtually) infinite cyclic [4, 6].

The existence of a Cayley automatic structure for a group G appears to impose no restrictions on its geometry. This differs from the existence of an automatic structure; if a group G admits an automatic structure then the Cayley graph with respect to any finite generating set S enjoys the so-called fellow traveler property. This geometric condition requires that the normal form representatives for a pair of group elements at distance 1 in the Cayley graph $\Gamma(G, S)$ remain a uniformly bounded distance apart in this graph.

The goal of this paper is to explore the geometry of the Cayley graph of a Cayley automatic group, and in particular, to understand an analogue of the fellow traveler property for these groups. A Cayley automatic group differs from an automatic group in that the normal form for group elements is defined over a finite symbol alphabet rather than a set S of generators. However, without loss of generality we can take these symbols to be additional generators, renaming the larger generating set S , and for $g \in G$ obtain a normal form which describes a path in the Cayley graph $\Gamma(G, S)$ but (most likely) not a path to the vertex labeled g . In lieu of a fellow traveler property, we investigate the distance in $\Gamma(G, S)$ between the vertex labeled g and the endpoint of this path.

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To quantify how far a Cayley automatic structure is from being automatic, we follow the first author and Trakuldit in [7] and define the Cayley distance function f_ψ for a given Cayley automatic structure ψ , where $f_\psi(n)$ is the maximum distance between a normal form word representing $g \in G$ and the vertex labeled g , over all normal forms of word length at most n . In [7] it is shown that G is automatic if and only if this function is equivalent to a constant function, in a notion of equivalence defined below. Thus for Cayley automatic groups which are not automatic this function is always unbounded and non-decreasing. This motivates our investigation of when the Cayley distance function might be bounded below by a non-constant function, quantitatively separating G from the class of automatic groups. However, the possibility exists that a group may admit a sequence of Cayley automatic structures for which the corresponding sequence of Cayley distance functions limit to a constant function, but never contains the constant function.

In this paper we prove that such limiting behaviour is not possible in any group which is not finitely presented, or in any finitely presented group that has a super-quadratic Dehn function, as given in Definition 4.3. In each case, we construct a concrete unbounded function depending only on the group, so that the Cayley distance function for any Cayley automatic structure on the group is bounded below by this function, up to equivalence. We say that a Cayley automatic group G is f -separated if the Cayley distance function with respect to any Cayley automatic structure on G is bounded below by a function in the equivalence class of f (Definition 2.5).

Let i denote the function $i(n) = n$ on some domain $[N, \infty)$. Super-quadratic and strongly-super-polynomial functions, referred to in Theorem A below, are introduced in Definition 4.3. We prove the following.

Theorem A (Finitely presented groups). *If G is a finitely presented Cayley automatic group with super-quadratic Dehn function, then there exists an unbounded function ϕ depending only on G so that G is ϕ -separated. Furthermore, if G has strongly-super-polynomial Dehn function, then G is i -separated.*

The analogous theorem for non-finitely presented groups is as follows. A non-finitely presented group is *dense* if its irreducible relators have lengths which are “dense” in the natural numbers; see §5 for a precise definition. Wreath products are the prototypical examples of dense groups.

Theorem B (Non-finitely presented groups). *If G is a Cayley automatic group which is not finitely presented, then there is a non-decreasing step function ϕ depending only on G that is linear for infinitely many values, so that G is ϕ -separated. Furthermore, if G is dense then G is i -separated.*

We conjecture that for every Cayley automatic group that is not automatic, the distance function with respect to every Cayley automatic structure on the group is bounded below by a linear function, which is equivalent to being i -separated.

Conjecture 1. *Let G be a Cayley automatic group. Then G is either automatic or i -separated.*

Our results provide support for this conjecture by exhibiting lower bounds for Cayley distance functions for all non-automatic Cayley automatic groups whose Dehn function is super-quadratic or are not finitely presented. However, these bounds are not always equivalent to i .

While we believe the conjecture to be true, two groups for which this linear lower bound is not obvious to us are the following.

- The higher Heisenberg groups H_{2k+1} for $k \geq 2$. These are nilpotent of step 2 so they are Cayley automatic by [20, Theorem 12.4]. Since the only nilpotent automatic groups are virtually abelian [15], they are not automatic. It is proved in [1, 19, 23] that their Dehn function is quadratic.
- The higher rank lamplighter groups, or *Diestel-Leader groups*, proven to be Cayley automatic by Bérubé, Palnitkar and the third author in [8]. One can show that the Cayley automatic structure constructed in [8] has Cayley distance function equivalent to the identity function. These groups are not of type FP_∞ [2], hence not automatic. See [8] for a discussion explaining why their Dehn functions are quadratic.

The paper is organised as follows. In Section 2 we review automatic and Cayley automatic groups, define the Cayley distance function for a Cayley automatic structure, and finish with a short discussion of Dehn functions. In Section 3 we prove Proposition 3.2, which relates the Cayley distance function to the Dehn function for a finitely presented Cayley automatic group. In Section 4 we define super-quadratic, super-polynomial and strongly-super-polynomial functions and prove Theorem A. We then turn to non-finitely presented Cayley automatic groups. We introduce the notion of a dense group in Section 5, then prove Theorem B in Section 6. We include additional information about strongly-superpolynomial functions in Appendix A.

2. AUTOMATIC AND CAYLEY AUTOMATIC GROUPS

We assume that the reader is familiar with the notions of regular languages, finite automata and multi-tape synchronous automata. For more details, we refer the reader to [15]. We say a language $L \subseteq (X^*)^n$ is *regular* if it is accepted by a synchronous n -tape automaton where $n \in \mathbb{N}$ and X is a finite set, or *alphabet*.

For any group G with finite symmetric generating set $S = S^{-1}$, let $\pi: S^* \rightarrow G$ denote the canonical projection map. For $w \in S^*$ let $|w|_S$ denote the length of w as a word in the free monoid S^* .

For any $M, N \in \mathbb{N}$, let $[M, N] = \{n \in \mathbb{N} \mid M \leq n \leq N\}$ and $[M, \infty) = \{n \in \mathbb{N} \mid n \geq M\}$.

2.1. Automatic and Cayley automatic groups. We define automatic and Cayley automatic groups, and provide some standard lemmas on the invariance of the Cayley automatic structure under change of generating set.

Definition 2.1. An *automatic structure* for a group G is a pair (S, L) where

- (1) S is a finite symmetric generating set for G ;
- (2) $L \subseteq S^*$ is a regular language;
- (3) $\pi|_L: L \rightarrow G$ is a bijection;
- (4) for each $a \in S$ the binary relation

$$R_a = \{(u, v) \in L \times L \mid \pi(u)a =_G \pi(v)\} \subseteq S^* \times S^*$$

is regular, that is, recognized by a two-tape synchronous automaton.

A group is called *automatic* if it has an automatic structure with respect to some finite generating set.

It is a standard result, see, for example [15, Theorem 2.4.1], that if G is automatic then G has an automatic structure with respect to any finite generating set.

Cayley automatic groups were introduced in [20] with the motivation of allowing the language L of normal forms representing group elements to be defined over a symbol alphabet Λ rather than a generating set S for G .

Definition 2.2. A *Cayley automatic structure* for a group G is a 4-tuple (S, Λ, L, ψ) where

- (1) S is a finite symmetric generating set for G ;
- (2) Λ is an alphabet and $L \subseteq \Lambda^*$ is a regular language;
- (3) $\psi: L \rightarrow G$ is a bijection;
- (4) for each $a \in S$ the binary relation

$$R_a = \{(u, v) \in L \times L \mid \psi(u)a =_G \psi(v)\} \subseteq \Lambda^* \times \Lambda^*$$

is regular, that is, recognized by a two-tape synchronous automaton.

A group is called *Cayley automatic* if it has a Cayley automatic structure (S, Λ, L, ψ) with respect to some finite generating set S .

As for automatic groups, if G has a Cayley automatic structure (S, Λ, L, ψ) and Y is another finite generating set for G , then there exists a Cayley automatic structure $(Y, \Lambda_Y, L_Y, \psi_Y)$ for G . See [20, Theorem 6.9] for a proof of this fact; we sharpen this in Proposition 2.10 below.

Note that a Cayley automatic structure $(S, S, L, \pi|_L)$ for G , that is, one in which the symbol alphabet is in fact a generating set, and the natural projection gives a bijection from L to G , is simply an automatic structure for G .

A priori the symbol alphabet Λ has no relation to a generating set for G . However it is straightforward to show that if (S, Λ, L, ψ) is a Cayley automatic structure for G , then there exists another Cayley automatic structure (S', S', L, ψ) for G where $S' = \Lambda \cup \Lambda^{-1} \cup S$, and so we can always associate a Cayley graph with respect to a generating set which includes symbol letters from the Cayley automatic structure. This is proven in [7] and in Proposition 2.10 below. In this case, a word $w \in L$ labels a path from 1_G to $\pi(w)$ in the Cayley graph $\Gamma(G, S)$. It is crucial to note that in general $\pi(w) \neq \psi(w)$.

Definition 2.3. Let (G, S) be a group with Cayley automatic structure (S, S, L, ψ) . The *Cayley distance function* corresponding to ψ is defined to be

$$h_{S,\psi}(n) = \max\{d_S(\pi(w), \psi(w)) \mid w \in L^{\leq n}\}$$

where d_S is the word metric on G with respect to S and

$$L^{\leq n} = \{w \in L \mid |w| \leq n\}.$$

Let \mathcal{F} be the following set of non-decreasing functions:

$$\mathcal{F} = \{f: [N, \infty) \rightarrow \mathbb{R}^+ \mid N \in \mathbb{N} \wedge \forall n (n \in \text{dom} f \Rightarrow f(n) \leq f(n+1))\}.$$

Note that if G is a group with Cayley automatic structure (S, S, L, ψ) and Cayley distance function $h_{S,\psi}$, then $h_{S,\psi} \in \mathcal{F}$.

We introduce the following partial order on \mathcal{F} .

Definition 2.4. Let $f, g \in \mathcal{F}$. We say that $g \preceq_1 f$ if there exist positive integers K, M and N such that $[N, \infty) \subseteq \text{dom} g \cap \text{dom} f$ and $g(n) \leq Kf(Mn)$ for every integer $n \geq N$. We say that $g \approx_1 f$ if $g \preceq_1 f$ and $f \preceq_1 g$.

The subscript in Definition 2.4 serves to distinguish this equivalence from the equivalence on Dehn functions discussed in § 2.3. It is clear from the definition that both \preceq_1 and \approx_1 are transitive.

Definition 2.5. Let $f, g \in \mathcal{F}$, and let G be a Cayley automatic group. We say that g is *f-separated* if $f \preceq_1 g$, and that G is *f-separated* if for every Cayley automatic structure (S, S, L, ψ) on G with distance function $h_{S,\psi}$, $h_{S,\psi}$ is *f-separated*.

Note that since \preceq_1, \approx_1 are both transitive, if G is *f-separated* and $h \approx_1 f$ then G is *h-separated* also.

Let \mathbf{z} denote the zero function $\mathbf{z}(n) = 0$ on some domain $[N, \infty)$, and \mathbf{i} the function $\mathbf{i}(n) = n$ on some domain $[N, \infty)$. We note that if $f \in \mathcal{F}$ and $f(n) = 0$ for infinitely many values of $n \in \text{dom} f$ then $f = \mathbf{z}$ on its domain, because $f \in \mathcal{F}$ is non-decreasing.

The next lemma will be used repeatedly in the proofs below, and is a fact about certain types of functions which are easily seen to be related or equivalent under the definition above.

Lemma 2.6. Let $A, B, C, D \in \mathbb{R}$ with $A, D \geq 1$ and $B, C \geq 0$. Let $f, g \in \mathcal{F}$ with $f(n) \leq Dg(An+B) + C$. If $g \neq \mathbf{z}$, then $f(n) \preceq_1 g(n)$. Moreover, if $h(n) = Df(An+B) + C$ and $f \neq \mathbf{z}$, then $h \in \mathcal{F}$ and $h \approx_1 f$.

Proof. If g is bounded, so $g(n) \leq E$ for all $n \in \text{dom} g$ for some fixed constant E , then f is bounded as well. Since $g(n) = 0$ for at most finitely many values of $n \in \text{dom} g$, we have $f(n) \preceq_1 g(n)$ (possibly increasing N_0). If f is bounded and $f(n) = 0$ for at most finitely many values of $n \in \text{dom} f$, it follows immediately that $h \approx_1 f$.

For the remainder of the proof, we assume that g is not bounded. There is a constant N_0 so that for $n \geq N_0$ we have $An \geq B$. As $g \in \mathcal{F}$, it follows that for $n \geq N_0$ we have $Dg(An+B) + C \leq Dg(2An) + C$. Since g is not bounded, there is a constant N_1 so that for $n \geq N_1$ we have $g(2An) \geq C$. Then for $n \geq \max(N_0, N_1)$ we have $f(n) \leq Dg(An+B) + C \leq Dg(2An) + C \leq (D+1)g(2An)$ and thus $f \preceq_1 g$.

Letting $g = f \in \mathcal{F}$ the above reasoning shows that $h \preceq_1 f$. As it is clear that $h(n) = Df(An + B) + C \in \mathcal{F}$ and $f \preceq_1 h$, it follows that $f \approx_1 h$, as desired. \square

Note that \approx_1 defines an equivalence relation on the set \mathcal{F} and \preceq_1 then gives a partial ordering on the resulting set of equivalence classes. The poset of equivalence classes of elements of \mathcal{F} has a minimal element $[\mathbf{z}]$. It follows from the previous lemma that all bounded functions $f \in \mathcal{F}$ for which $f(n) = 0$ for at most finitely many values of $n \in \text{dom} f$ are in the same equivalence class. Furthermore, every $f \in \mathcal{F}$ can be compared to a constant function. In contrast, we show that the partial ordering is not a linear ordering, that is, there are functions in \mathcal{F} which cannot be compared to the identity function \mathbf{i} under \preceq_1 .

Lemma 2.7. *Let $c \in \mathbb{N}$ and $g : \mathbb{N} \rightarrow \mathbb{N}$ the constant function $g(n) = c$. Let $f \in \mathcal{F}$ be any function. Then either $f \preceq_1 g$ or $g \preceq_1 f$.*

Proof. If there is some $D \in \mathbb{R}^+$ so that $f(n) \leq D = \left(\frac{D}{c}\right)c = \left(\frac{D}{c}\right)g(n)$ for all $n \in \text{dom} f$ then $f \preceq_1 g$. If not, then for all $D \in \mathbb{R}^+$ there is an integer $N_D \in \text{dom} f$ so that $f(n) > D$ for $n > N_D$. In particular, there is an integer $N_c \in \text{dom} f$ so that $f(n) > c = g(n)$ for all $n \in \text{dom} f \cap [N_c, \infty)$. Thus $g \preceq_1 f$. \square

Lemma 2.8 demonstrates that not every function $f \in \mathcal{F}$ is \mathbf{i} -separated.

Lemma 2.8. *There exists a function $f \in \mathcal{F}$ so that $\mathbf{i} \not\preceq_1 f$ and $f \not\preceq_1 \mathbf{i}$.*

Proof. Let $n_0 = 2$ and define the infinite sequence of integers $n_{i+1} = n_i^2 = 2^{2^i}$. Consider the step function $f: \mathbb{N} \rightarrow \mathbb{R}^+$ defined by

$$f(x) = \begin{cases} n_{2i} & n_{2i} \leq x < n_{2i+1}, \\ n_{2i+2} & n_{2i+1} \leq x < n_{2i+2}. \end{cases}$$

Suppose $f \preceq_1 \mathbf{i}$. Then $\exists N_0, K, M$ so that $f(x) \leq K\mathbf{i}(Mx) = KMx$ for all $x \geq N_0$. However,

$$f(n_{2i+1}) = n_{2i+2} = n_{2i+1}^2 \leq KMn_{2i+1}$$

which implies that $n_{2i+1} \leq KM$ for sufficiently large i , a contradiction. Thus $f \not\preceq_1 \mathbf{i}$.

Conversely suppose $\mathbf{i} \preceq_1 f$. Then $\exists N_0, K, M$ so that $x \leq Kf(Mx)$ for all $x \geq N_0$. This means $\frac{s}{M} \leq Kf(s)$ for all $s = Mx \geq MN_0$, which implies that $M \lfloor \frac{s}{M} \rfloor \leq KMf(s)$ for all $s \geq MN_0$. However,

$$M \left\lfloor \frac{n_{2i}^2 - 1}{M} \right\rfloor = M \left\lfloor \frac{n_{2i+1} - 1}{M} \right\rfloor \leq KMf(n_{2i+1} - 1) = KMn_{2i}$$

and thus $\left\lfloor \frac{n_{2i}^2 - 1}{M} \right\rfloor \leq Kn_{2i}$. Therefore, $\frac{n_{2i}^2 - 1}{M} \leq Kn_{2i} + 1$, so $n_{2i} \leq KM + \frac{M+1}{n_{2i}}$ which is a contradiction for sufficiently large i . Thus $\mathbf{i} \not\preceq_1 f$. \square

2.2. Invariance under change of generating set and change of structure. Here we describe how robust both the Cayley automatic structure and the function $h_{S,\psi}$ are to, respectively, change in generating set and change in structure. First we recall the following standard fact.

Lemma 2.9. *Let G be a Cayley automatic group and S a finite symmetric generating set for G . Let (S, Λ, L, ψ) be a Cayley automatic structure for G . Then for any $w \in S^*$,*

$$L_w = \{(u, v) \in L^2 \mid \psi(v) =_G \psi(u)w\}$$

is regular.

Proof. Let $w = s_1 \dots s_n$ where $s_i \in S$ for $1 \leq i \leq n$. As (S, Λ, L, ψ) is a Cayley automatic structure for G , for each $s \in S$ there is a synchronous 2-tape automaton \mathbf{M}_s which accepts the language

$$L(\mathbf{M}_s) = \{(u, v) \in L^2 \mid \psi(v) =_G \psi(u)s\}.$$

Let \mathbf{M}'_i be a synchronous $(n+1)$ -tape automaton accepting

$$(z_0, \dots, z_{i-1}, u, v, z_{i+2}, \dots, z_n)$$

where $z_j \in \Lambda^*$, $j \in [0, i-1] \cup [i+1, n]$, $u, v \in L$ and $\psi(v) = \psi(u)s_i$. We construct \mathbf{M}'_i from \mathbf{M}_{s_i} by replacing each edge labeled $(a, b) \in \Lambda^2$ by the finite number of edges labeled

$(x_0, \dots, x_{i-1}, a, b, x_{i+2}, \dots, x_n) \in \Lambda^{n+1}$ for all possible choices of $x_j \in \Lambda$ where $0 \leq j \leq i-1$ and $i+2 \leq j \leq n$.

Then

$$L_w = \bigcap_{i=1}^n L(M'_i)$$

which is regular since this is a finite intersection, then apply a homomorphism to project onto the first and last factors. \square

Proposition 2.10. *Let G be a Cayley automatic group and S a finite symmetric generating set for G .*

- (1) *If (S, Λ, L, ψ) is a Cayley automatic structure for G , then so is (S', S', L, ψ) where $S' = \Lambda \cup \Lambda^{-1} \cup S$*
- (2) *If (S, S, L, ψ) is a Cayley automatic structure for G with Cayley distance function $h_{S,\psi}$, and Y is a finite symmetric generating set for G , then there exists a language $L' \subseteq Y^*$ and a bijection $\psi': L' \rightarrow G$ so that (Y, Y, L', ψ') is a Cayley automatic structure for G with Cayley distance function $h_{Y,\psi'} \approx_1 h_{S,\psi}$.*

Proof. (1) Suppose $\langle S \mid R \rangle$ is a presentation for G . For each $a \in \Lambda$ choose an element $g_a \in G$, and choose a word $u_a \in S^*$ with $\pi(u_a) = g_a$. Note that this choice is arbitrary; the element g_a corresponding to the symbol letter a could be any group element. Let Λ^{-1} be the disjoint set $\{a^{-1} \mid a \in \Lambda\}$; we will not use these letters, but include them to ensure our new generating set is symmetric. Since Λ is finite, there is a bound on the length of all u_a words. We have $S' = \Lambda \cup \Lambda^{-1} \cup S$ and G is presented by $\langle S' \mid R \cup \{a = u_a \mid a \in \Lambda\} \rangle$.

With this new generating set, we have the same language L which is regular, and the map $\psi: L \rightarrow G$. For each $s \in S$ there is an automaton M_s recognizing multiplication by s , and it follows from Lemma 2.9 that there is an analogous 2-tape automaton M_a for each $a \in \Lambda^{\pm 1}$.

(2) We have $L \subseteq S^*$ is a regular language in bijective correspondence with G . For each $s \in S$, choose a word $u_s \in Y^*$ with $s =_G u_s$ and for each $y \in Y$, choose a word $v_y \in S^*$ with $y =_G v_y$. Let $M_1 = \max\{|v_y|_S \mid y \in Y\}$ and $M_2 = \max\{|u_s|_Y \mid s \in S\}$.

Let the monoid homomorphism $\rho: S^* \rightarrow Y^*$ be defined by $\rho(s) = u_s$. It follows that $L' = \rho(L)$ is a regular language in bijection with G , where $\psi': L' \rightarrow G$ defined by $\psi' = \psi \circ (\rho|_L)^{-1}$ is a bijection. Note that for all $w \in L$ we have $\pi(w) = \pi(\rho(w))$ and $\psi(w) = \psi'(\rho(w))$. For each $w \in L^{\leq n}$ we claim that

$$(1) \quad d_S(\pi(w), \psi(w)) \leq M_1 h_{Y,\psi'}(M_2 n)$$

To see this, we argue as follows.

- Under ρ , the path labeled w from 1_G to $\pi(w)$ in $\Gamma(G, S)$ is mapped to a path labeled $\rho(w)$ from 1_G to $\pi(\rho(w)) = \pi(w)$ in $\Gamma(G, Y)$, and this path has length at most $M_2 n$, replacing each letter s of the path by u_s . See Figure 1.
- By definition, the distance from $\pi(\rho(w))$ to $\psi'(\rho(w))$ in $\Gamma(G, Y)$ is at most $h_{Y,\psi'}(M_2 n)$ since this is the maximum such distance over all possible words in $(L')^{\leq M_2 n}$.
- Then in $\Gamma(G, Y)$ we have a path from $\pi(w)$ to $\psi(w)$ of length at most $h_{Y,\psi'}(M_2 n)$ in the letters from Y ; call it γ . Replacing each of these letters y by u_y we obtain a path $\rho^{-1}(\gamma)$ in $\Gamma(G, S)$ from $\pi(w)$ to $\psi(w)$ of length at most $M_1 h_{Y,\psi'}(M_2 n)$.

Since Equation (1) is true for all $w \in L^{\leq n}$, it follows that $h_{S,\psi} \preceq_1 h_{Y,\psi'}$.

Similarly for each $w' = \rho(w) \in (L')^{\leq n}$ the same argument shows that

$$d_Y(\pi(w'), \psi'(w')) \leq M_2 h_{S,\psi}(M_1 n)$$

as $\pi(w') = \pi(w)$ and $\psi(w) = \psi'(w')$ are the same vertices.

Thus $h_{Y,\psi'} \preceq_1 h_{S,\psi}$ and it follows that $h_{Y,\psi'} \approx_1 h_{S,\psi}$ \square

Remark 2.11. Note that a given group may admit many different Cayley automatic structures whose Cayley distance functions are not equivalent under \preceq_1 . Part (2) of Proposition 2.10 proves that given one Cayley automatic structure for a group G with respect to a generating set S , we

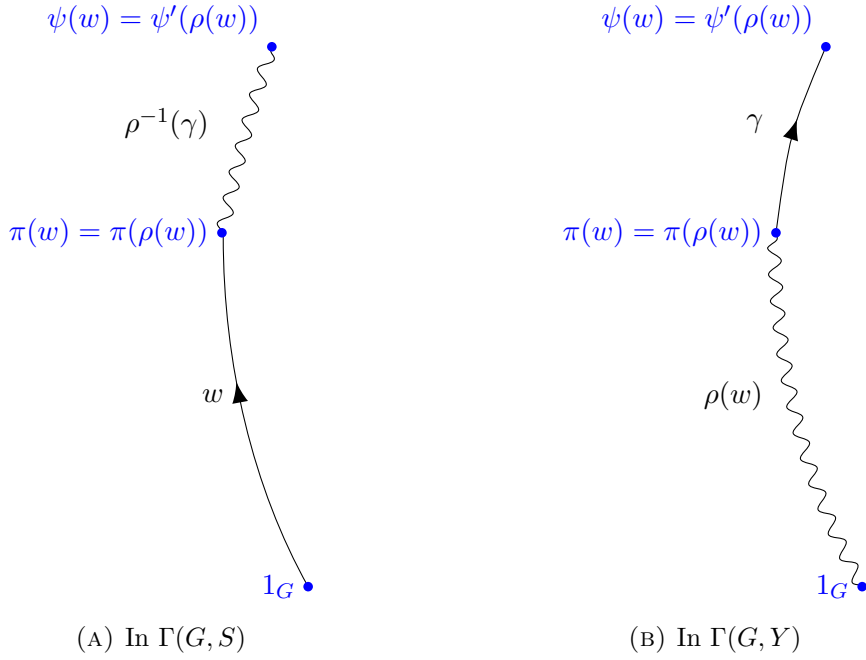


FIGURE 1. Drawing $w \in L$ and $\rho(w) \in L'$ in each Cayley graph.

can create a new Cayley automatic structure for G over a generating set Y so that both Cayley distance functions are equivalent under \preceq_1 .

2.3. Dehn functions. Let $\mathcal{P} = \langle X \mid R \rangle$ be a finite presentation of a group G and F_X the free group on X . If $w \in F_X$ is equal to 1_G in G , then there exist $N \in \mathbb{N}$, $r_i \in R$ and $u_i \in F_X$ such that

$$w_{F_X} = \prod_{i=1}^N u_i^{-1} r_i^{\epsilon_i} u_i$$

We define the *area* of w , denoted $A_{\mathcal{P}}(w)$, to be the minimal $N \in \mathbb{N}$ so that w has such an expression.

Definition 2.12. The *Dehn function* of a presentation is the function $\delta_{\mathcal{P}}: \mathbb{N} \rightarrow \mathbb{N}$ given by

$$\delta_{\mathcal{P}}(n) = \max\{A_{\mathcal{P}}(w) \mid w \in F_X, w =_G 1_G, |w| \leq n\}$$

Note that if f is a Dehn function then $f \in \mathcal{F}$. It is standard to define the following partial order on Dehn functions.

Definition 2.13. For $f, g \in \mathcal{F}$ we define $f \preceq_2 g$ if there exists a constant $C > 0$ so that $f(n) \leq Cg(Cn) + Cn$ for all $n \in \mathbb{N}$. We write $f \approx_2 g$ if $f \preceq_2 g$ and $g \preceq_2 f$.

Recall that each presentation of a group G can give rise to a different Dehn function. It is a standard fact that all Dehn functions on a group G are equivalent under the relation \approx_2 . Thus we can consider the equivalence class of these functions as a quasi-isometry invariant of the group. In particular, we can refer to a group as possessing a linear, quadratic or exponential Dehn function, for example.

Recall that there are no groups with Dehn function equivalent to n^α for $\alpha \in (1, 2)$ [10, 18, 21].

3. FINITE PRESENTABILITY AND DEHN FUNCTIONS

We start with the following observation.

Lemma 3.1 ([20], Lemma 8.2; [14], Lemma 8). *Let (S, Λ, L, ψ) be a Cayley automatic structure for G . Then there are constants $m, e \in \mathbb{N}$, depending on the Cayley automatic structure, with $m \geq 1$ so that for each $u \in L$,*

$$|u| \leq md_S(1_G, \psi(u)) + e$$

where d_S denotes the word metric in G with respect to the generating set S .

Proof. For each $x \in S$ let M_x be a synchronous 2-tape automaton accepting

$$\{(u, v) \in L^2 \mid \psi(v) =_G \psi(u)x\},$$

and let $|M_x|$ denote the number of states in M_x , and $m = \max\{|M_x| \mid x \in S\}$. Let $u_0 \in L$ be such that $\psi(u_0) = 1_G$, and $e = |u_0|$.

For $u \in L$ let $x_1 \dots x_k$ be a geodesic for $\psi(u)$ where $x_i \in S$, so $k = d_S(1_G, \psi(u))$. Define $u_i \in L$ by $\psi(u_i) =_G x_1 \dots x_i$. Then for $i = 1, \dots, k$ we have

$$\||u_i| - |u_{i-1}|\| \leq m.$$

If this difference in length was greater than m , the path accepted by the two-tape automaton would end with a sequence of $\$$ symbols in one coordinate of length greater than m . One could then apply the pumping lemma to this path, and contradict the fact that ψ is a bijection.

It then follows from the triangle inequality that

$$\begin{aligned} |u| &= \||u_k| - |u_{k-1}| + |u_{k-1}| - \dots - |u_1| + |u_1| - |u_0| + |u_0|\| \\ &\leq \|(|u_k| - |u_{k-1}|)| + \dots + \|(|u_1| - |u_0|)| + |u_0|\| \\ &\leq mk + e \end{aligned}$$

which establishes the bound. \square

The following proposition relates the Cayley distance function to fillings of loops in the Cayley graph of a Cayley automatic group.

Proposition 3.2. *Let (S, S, L, ψ) be a Cayley automatic structure for G with Cayley distance function $h_{S, \psi}$. There exist constants $c, d, \varsigma, n_0, D \in \mathbb{N}$, depending on the Cayley automatic structure, so that the following holds.*

- (1) *For every $w \in S^*$ with $w =_G 1_G$ and $|w| \geq n_0$, there exist $w_i, \rho_i \in S^*$ with $w_i =_G 1_G$ for $1 \leq i \leq k$ so that*

$$w =_{F_S} \prod_{i=1}^k \rho_i w_i \rho_i^{-1} \quad \text{and} \quad |w_i| \leq 4h_{S, \psi}(c|w| + d) + \varsigma.$$

- (2) *If G is finitely presented, and δ is the Dehn function with respect to a fixed presentation $\langle S \mid R \rangle$, then*

$$\delta(n) \leq Dn^2\delta(f(n))$$

for all $n \geq n_0$, where $f \approx_1 h_{S, \psi}$.

Note that the constants D, c , and ς in the statement of the proposition depends only on the Cayley automatic structure and not on the Dehn function δ .

Proof. Let $m = \max_{s \in S}\{|M_s|\}$ be the maximum number of states in any two-tape synchronous automaton accepting R_s as in Definition 2.2 in the Cayley automatic structure for G and $u_0 \in L$ the word representing the identity element of G of length e as in Lemma 3.1. Without loss of generality we assume that m is even, so that all arguments of the function $h_{S, \psi}$ below are integers.

Choose a loop in $\Gamma(G, S)$ based at 1_G labeled by the path $w = s_1 \dots s_n$ where $s_i \in S$ and $\pi(w) = 1_G$. For each $g_i = \pi(s_1 \dots s_i)$ let $u_i \in L$ be such that $\psi(u_i) = g_i$. For $1 \leq i \leq n$, as $d(1_G, g_i) \leq n/2$, it follows from Lemma 3.1 that

$$(2) \quad |u_i|_S \leq mn/2 + e$$

so the distance from $\pi(u_i)$ to g_i is at most $h_{S, \psi}(mn/2 + e)$. Let γ_i be a path from $\pi(u_i)$ to g_i of length at most this bound. We will describe how to fill ‘‘corridors’’ having perimeter $u_i \gamma_i s_i \gamma_{i+1}^{-1} u_{i+1}^{-1}$ with relators of bounded perimeter. See Figure 2 for an example of such a corridor.

Let $u_i = a_{i,1} a_{i,2} \dots a_{i,|u_i|_S}$ and for $0 \leq j \leq |u_i|_S$ define $p_{ij} = \pi(a_{i,1} a_{i,2} \dots a_{i,j})$ to be the point in $\Gamma(G, S)$ corresponding to the prefix of u_i of length j . If $j > |u_i|_S$ then let $p_{ij} = \pi(u_i)$.

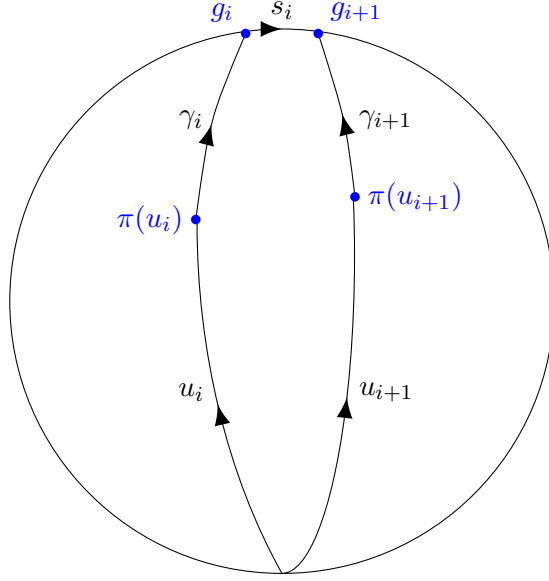


FIGURE 2. The exterior of the figure is labeled by a loop $w = s_1 s_2 \cdots s_n$ with $w =_G 1$. The figure depicts a corridor whose sides are labeled by the closed path $u_i \gamma_i s_i \gamma_{i+1}^{-1} u_{i+1}^{-1}$.

We know that the pair (u_i, u_{i+1}) is accepted by M_{s_i} . Consider the state of M_{s_i} which is reached upon reading the input $\{(a_{i,l}, a_{i+1,l})\}_{l=1}^j$ where $a_{i,l}, a_{i+1,l} \in S \cup \{\$\}$. There must be a path of length at most m in M_{s_i} from this state to some accept state of M_{s_i} . Denote the labels along this path by $\{(b_{i,j,r}, b_{i+1,j,r})\}_{r=1}^m$ where $b_{i,j,r}, b_{i+1,j,r} \in S \cup \{\$\}$ and we insert the padding symbol $\$$ in both coordinates if the path has length less than m . Then if $x_{i,j}$ denotes the concatenation $\{a_{i,j}\}_{j=1}^l \{b_{i,j,r}\}_{r=1}^m$, and $x_{i+1,j}$ denotes $\{a_{i+1,l}\}_{l=1}^j \{b_{i+1,j,r}\}_{r=1}^m$, then $\psi(x_{i,j})s_i = \psi(x_{i+1,j})$, and both these points, as well as $\pi(x_{i,j})$ and $\pi(x_{i+1,j})$ are depicted in Figure 3.

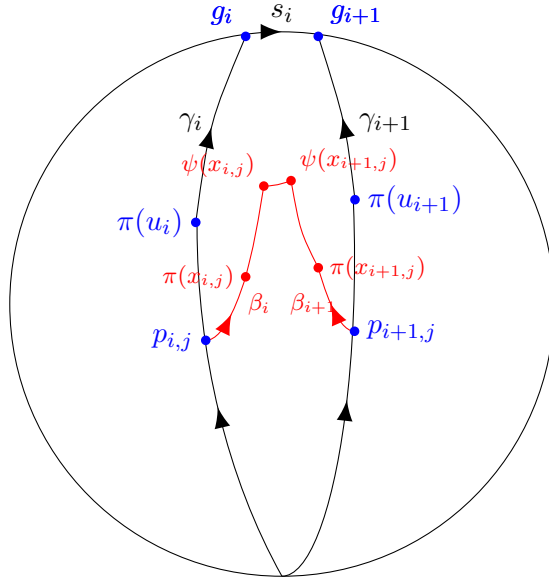


FIGURE 3. Depiction of a path in $\Gamma(G, S)$ between $p_{i,j} = \pi(a_{i,1}a_{i,2} \cdots a_{i,j})$ and $p_{i+1,j} = \pi(a_{i+1,1}a_{i+1,2} \cdots a_{i+1,j})$, where $u_i = a_{i,1}a_{i,2} \cdots a_{i,|u_i|_S}$ is such that $\psi(u_i) = g_i$.

Thus there is a path in $\Gamma(G, S)$ from $p_{i,j}$ to $p_{i+1,j}$ of length at most $2m + 2h_{S,\psi}(mn/2 + e + m) + 1$ consisting of the following segments, as shown in Figure 3:

- $\beta_i = \{b_{i,j,r}\}_{r=1}^r$ from $p_{i,j}$ to $\pi(x_{i,j})$ of length at most m ,
- a path from $\pi(x_{i,j})$ to $\psi(x_{i,j})$ of length at most $h_{S,\psi}(mn/2 + e + m)$,
- an edge labeled s_i from $\psi(x_{i,j})$ to $\psi(x_{i+1,j})$,
- a path from $\psi(x_{i,j})s_i = \psi(x_{i+1,j})$ to $\pi(x_{i+1,j})$ of length at most $h_{S,\psi}(mn/2 + e + m)$,
- $\beta_{i+1} = \{b_{i+1,j,r}\}_{r=1}^r$ from $\pi(x_{i+1,j})$ to $p_{i+1,j}$ of length at most m .

In Figure 4 the paths between $p_{i,j}$ and $p_{i+1,j}$ for all $1 \leq j < \max\{|u_i|_S, |u_{i+1}|_S\}$ are depicted for one corridor. Between $p_{|u_i|}$ and $p_{|u_{i+1}|}$ we use that existing path $\gamma_i s_i \gamma_{i+1}^{-1}$.

These corridors create two types of cells. The first type are created from two of these paths and their connecting edges, for some j and $j+1 < \max\{|u_i|_S, |u_{i+1}|_S\}$. This creates a cell with perimeter at most

$$2(2m + 2h_{S,\psi}(mn/2 + e + m) + 1) + 2 = 4m + 4h_{S,\psi}(mn/2 + e + m) + 4,$$

where the additional $+2$ accounts for the single edges $a_{i,j+1}$ between $p_{i,j}$ and $p_{i,j+1}$, and $a_{i+1,j+1}$ between $p_{i+1,j}$ and $p_{i+1,j+1}$, which lie on the paths u_i and u_{i+1} , respectively, and are not part of the paths previously constructed.

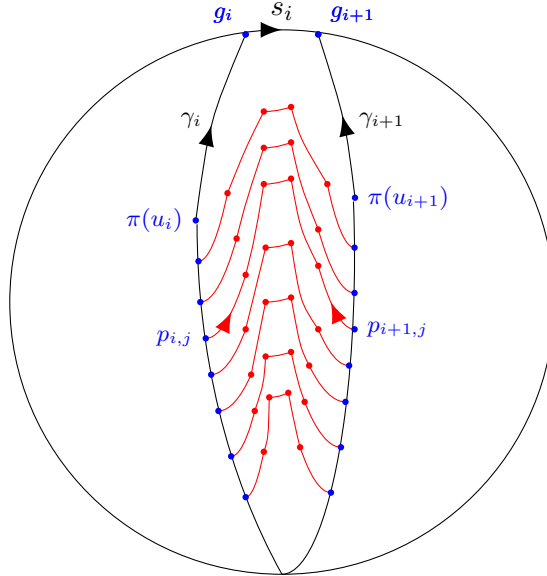


FIGURE 4. Filling the corridors created by the paths $u_i \gamma_i$ with cells of bounded perimeter.

The second type is the “top” cell created by the path from $p_{i,|u_i|-1}$ to $p_{i+1,|u_{i+1}|-1}$ together with the path $a_{i,|u_i|-1} \gamma_i s_i \gamma_{i+1}^{-1} a_{i+1,|u_{i+1}|-1}$. This cell has perimeter at most

$$\begin{aligned} & (2m + 2h_{S,\psi}(mn/2 + e + m) + 1) + 2 + (2h_{S,\psi}(mn/2 + e) + 1) \\ &= 2m + 2h_{S,\psi}(mn/2 + e + m) + 2h_{S,\psi}(mn/2 + e) + 4 \\ &\leq 4m + 4h_{S,\psi}(mn/2 + e + m) + 4 \end{aligned}$$

where the terms in the first line come, respectively, from

- the path from $p_{i,|u_i|-1}$ to $p_{i+1,|u_{i+1}|-1}$,
- the two edges labeled $a_{i,|u_i|-1}$ and $a_{i+1,|u_{i+1}|-1}$, and
- the path $\gamma_i s_i \gamma_{i+1}^{-1}$.

To obtain the inequality, note that $h_{S,\psi} \in \mathcal{F}$ and $m \geq 1$, so $2m + 4 \leq 4m + 4$ and

$$2h_{S,\psi}(mn/2 + e + m) + 2h_{S,\psi}(mn/2 + e) \leq 4h_{S,\psi}(mn/2 + e + m).$$

Setting $c = m/2$, $d = e + m$ and $\varsigma = 4m + 4$ proves the first claim in the proposition.

To prove the second claim in the proposition, we count the total number of cells required to subdivide the initial loop into cells of bounded perimeter. This will yield the inequality involving the Dehn function δ .

It follows from Lemma 3.1 that $|u_i| \leq mn/2 + e$ for all $1 \leq i \leq n$. Each corridor is filled by at most $(mn/2 + e)$ cells, each of perimeter at most $4h_{S,\psi}(cn + d) + \varsigma$, where c, d and ς depend on m and e . For a fixed finite presentation $\langle S \mid R \rangle$ for G with Dehn function δ , each cell constructed above can be filled by at most $\delta(4h_{S,\psi}(cn + d) + \varsigma)$ cells with perimeter labeled by a relator from the set R .

With n corridors, there are $n \cdot (mn/2 + e) = n(cn + e)$ such cells to fill. Thus an upper bound on the number of relators required to fill w is

$$\begin{aligned} n(cn + e) \cdot \delta(4h_{S,\psi}(cn + d) + \varsigma) &= n^2 \left(c + \frac{e}{n}\right) \cdot \delta(4h_{S,\psi}(cn + d) + \varsigma) \\ &\leq n^2(c + e) \cdot \delta(4h_{S,\psi}(cn + d) + \varsigma). \end{aligned}$$

Setting $D = c + e$ and noting that it follows from Lemma 2.6 that $4h_{S,\psi}(cn + d) + \varsigma \approx_1 h_{S,\psi}(n)$ proves the second claim of the proposition. \square

4. SEPARATING FINITELY PRESENTED CAYLEY AUTOMATIC GROUPS FROM AUTOMATIC GROUPS

In this section we prove Theorem A. First we introduce the following notion.

Definition 4.1. Let $f, g \in \mathcal{F}$. We say that $f \ll g$ if there exists an unbounded function $t \in \mathcal{F}$ such that $ft \preceq_1 g$.

Example 4.2. If $g(n) = n^c$ with $c > 2$ and $f(n) = n^2$ then $f \ll g$. Take $t(n) = n^{c-2}$. Then $t \in \mathcal{F}$ is an unbounded function and

$$f(n)t(n) = n^c \preceq_1 g(n).$$

Next we define the following.

Definition 4.3. A function $f \in \mathcal{F}$ is *super-quadratic* if for all constants $M > 0$ we have $f(n) \leq Mn^2$ for at most finitely many $n \in \mathbb{N}$. A non-zero function $f \in \mathcal{F}$ is *strongly-super-polynomial* if $n^2 f \ll f$.

Example 4.4. The functions $n^2 \ln n$ and n^c for $c > 2$ are super-quadratic; the functions e^n and $n^{\ln n}$ are strongly-super-polynomial.

Ol'shanskii introduces the notion of a function being *almost quadratic* in [22]; our definition of a super-quadratic function is the same as being not almost quadratic. However, our notion of a strongly-super-polynomial is stronger than the more standard definition of a super-polynomial function given, for example, in [17]:

Definition 4.5. A function $f: \mathbb{N} \rightarrow \mathbb{R}$ is *super-polynomial* if

$$\lim_{n \rightarrow \infty} \frac{\ln f(n)}{\ln n} = \infty.$$

In Lemma A.2 we give an example of a function in \mathcal{F} which satisfies the above limit but is not strongly-super-polynomial. However Proposition A.3 justifies our use of “strongly” since it shows that every strongly-super-polynomial function is super-polynomial.

Proposition A.1 shows that f is strongly-super-polynomial if and only if $n^c f \ll f$ for any $c > 0$, that is, there is nothing special about the choice of the exponent 2 in Definition 4.3.

Lemma 4.6. A function $f \in \mathcal{F}$ is super-quadratic if and only if $n^2 \ll f$.

Proof. Assume first that $n^2 \ll f$. Since $n^2 \ll f$, there exist an unbounded function $t \in \mathcal{F}$ and integer constants $K, N > 0$ and $M \geq 1$ such that $n^2 t(n) \leq Kf(Mn)$ for all $n \geq N$. Assume

that for some $M' \geq 1$ there exist infinitely many $n_i \in \mathbb{N}$, $i \geq 1$ with $1 \leq n_i < n_{i+1}$ for which $f(n_i) \leq M'n_i^2$. Let $n_i = k_i M + r_i$, where k_i is an integer and $0 \leq r_i < M$. Then we have:

$$\begin{aligned} \frac{1}{K} k_i^2 t(k_i) &\leq f(Mk_i) \leq f(n_i) \leq M'n_i^2 = M'M^2 k_i^2 + 2M'Mr_i k_i + M'r_i^2 \leq \\ &M'M^2 k_i^2 + 2M'M^2 k_i + M'M^2 \leq M'M^2 (k_i + 1)^2 \end{aligned}$$

for all $k_i \geq N$. Therefore, $t(k_i) \leq KM'M^2 \frac{(k_i+1)^2}{k_i^2} \leq 2KM'M^2$ for all $k_i \geq \max\{N, 3\}$, where the 3 follows from the simple observation that if $k \geq 3$ then $\frac{(k+1)^2}{k^2} \leq 2$. This contradicts the fact that t is an unbounded function.

Now assume that f is super-quadratic. Then for each integer $i \geq 1$ the set

$$\{m \mid \forall n [n \geq m \implies f(n) \geq in^2]\}$$

is non-empty. Let $m_i = \min\{m \mid \forall n [n \geq m \implies f(n) \geq in^2]\}$. We define a function $t(n)$ as follows: for $0 \leq n < m_1$, let $t(n) = 0$, and for $m_i \leq n < m_{i+1}$ with $i \geq 1$, let $t(n) = i$. By construction, $t(n)$ is a nondecreasing and unbounded function. As $n^2 t(n) \leq f(n)$, it follows that $n^2 \ll f$. \square

In [25] Ol'shanskii gives an example of a finitely presented group which has Dehn function bounded above by $c_1 n^2$ for infinitely many values of n , bounded below by $c_2 n^2 \log' n \log' \log' n$ for infinitely many values of n , where $\log'(n) = \max\{\log_2 n, 1\}$, and bounded between $c_3 n^2$ and $c_4 n^2 \log' n \log' \log' n$ for all $n \in \mathbb{N}$. Since this Dehn function is not super-quadratic, it follows that Theorem A below does not apply to this example.

We now prove Theorem A.

Theorem A (Finitely presented groups). *If G is a finitely presented Cayley automatic group with super-quadratic Dehn function, then there exists an unbounded function ϕ depending only on G so that G is ϕ -separated. Furthermore, if G has strongly-super-polynomial Dehn function, then G is i -separated.*

Proof. Fix a presentation for G and let δ be the Dehn function arising from this presentation. Fix a Cayley automatic structure (S, S, L, ψ) for G . If $n^2 \ll \delta$, there is an unbounded function $t(n) \in \mathcal{F}$ and positive constants $K, M, N_0 \in \mathbb{N}$ so that $n^2 t(n) \leq K\delta(Mn)$ for all $n \geq N_0$.

From Proposition 3.2 we know that there are constants $N_1, D \geq 1$ and a function $f \in \mathcal{F}$ so that for $n \geq N_1$, we have $\delta(n) \leq Dn^2 \delta(f(n))$ where $f \approx_1 h_{S, \psi}(n)$.

Combining these equations we have that for all $n \geq \max\{N_0, N_1\}$

$$n^2 t(n) \leq K\delta(Mn) \leq KDM^2 n^2 \delta(f(Mn))$$

and dividing both sides by $KDM^2 n^2$ we obtain

$$(3) \quad \frac{t(n)}{KDM^2} \leq \delta(f(Mn)).$$

Define

$$\phi(n) = \min \left\{ m \mid \frac{t(n)}{KDM^2} \leq \delta(m) \right\}.$$

It is immediate from Equation (3) that in the definition of $\phi(n)$ we have $\phi(n) \leq f(Mn)$ for all $n \geq \max\{N_0, N_1\}$, and hence $\phi \preceq_1 f$. Since $t \in \mathcal{F}$ is unbounded, it follows that $\phi \in \mathcal{F}$ and ϕ is unbounded.

Now assume that the inequality $n^2 \delta \ll \delta$ is satisfied. Therefore, there exist integer constants $K, M, N_0 > 0$ and an unbounded function $t \in \mathcal{F}$ such that

$$(4) \quad n^2 \delta(n) t(n) \leq K\delta(Mn)$$

for all $n \geq N_0$. It follows from statement (2) of Proposition 3.2. that there exists a function $f \approx_1 h_{S, \psi}$ and integer constants $N_1 \geq 0$ and $D > 0$ for which the inequality

$$\delta(n) \leq Dn^2 \delta(f(n))$$

holds for all $n \geq N_1$. This implies that $\delta(Mn) \leq DM^2n^2\delta(f(Mn))$ for all $n \geq N_1$. Combining this with the inequality in (4) we obtain that

$$n^2\delta(n)t(n) \leq K\delta(Mn) \leq DKM^2n^2\delta(f(Mn))$$

for all $n \geq \max\{N_0, N_1\}$. Therefore,

$$\delta(n)\tau(n) \leq \delta(f(Mn))$$

for all $n \geq \max\{N_0, N_1\}$, where $\tau(n) = \frac{t(n)}{DKM^2}$. Let

$$m_0 = \min\{n \in \text{dom}(\tau) \subseteq \mathbb{N} \mid \tau(n) \geq 2\};$$

such m_0 exists because $\tau(n)$ is unbounded. Therefore,

$$2\delta(n) \leq \delta(f(Mn))$$

for all $n \geq \max\{N_0, N_1, m_0\}$. Let $d_0 = \min\{n \mid \delta(n) \geq 1\}$.

If $f(Mn) < n$ for some $n \geq \max\{N_0, N_1, m_0, d_0\}$ then $2\delta(n) \leq \delta(f(Mn)) \leq \delta(n)$, which is a contradiction. Thus for all $n \geq \max\{N_0, N_1, m_0, d_0\}$ we must have that

$$n \leq f(Mn).$$

From this we obtain that $\mathbf{i} \preceq_1 f$. As $f \preceq_1 h_{S,\psi}$ it follows that $\mathbf{i} \preceq_1 h_{S,\psi}$, and we conclude that G is \mathbf{i} -separated. \square

5. DENSE GROUPS

We introduce a property of some infinitely presented groups which will allow us to obtain sharper lower bounds on the Cayley distance function of such a Cayley automatic group. This property will be shown to be independent of generating set and the prototypical examples of groups with this property are restricted wreath products.

Recall that F_X denotes the free group generated by a set X .

Definition 5.1 (Densely generated). Let G be a group with finite generating set X . We say that G is *densely generated* by X if there exist constants $E, F, N_0 \in \mathbb{N}$, $1 \leq E < F$ such that for all $n \geq N_0$ there is a word $w_n \in (X \cup X^{-1})^*$ which has the following properties:

- $w_n =_G 1_G$,
- $En \leq |w_n| \leq Fn$, and
- for any collection of words $u_i, \rho_i \in (X \cup X^{-1})^*$, $1 \leq j \leq k$ with $u_i =_G 1_G$ and

$$w_n =_{F_X} \prod_{i=1}^k \rho_i u_i \rho_i^{-1},$$

we have $|u_j| > n$ for some $1 \leq j \leq k$.

In other words, for every interval $[En, Fn]$ there is a loop w_n whose length lies in that interval which cannot be filled by loops all having length at most n . It follows that if G is densely generated by X then every presentation for G over X is infinite.

The following lemma shows that being densely generated is independent of the choice of finite generating set.

Lemma 5.2. *If G is densely generated by X and Y is another finite generating set for G then G is densely generated by Y .*

Proof. Let $|\cdot|_X$ denote the length of a word in $(X \cup X^{-1})^*$ and $|\cdot|_Y$ denote the length of a word in $(Y \cup Y^{-1})^*$.

For each $x \in X$ choose a nonempty word $v_x \in (Y \cup Y^{-1})^*$ with $x =_G v_x$. Let $M_1 = \max_{x \in X} \{|v_x|_Y\}$, and $\tau: (X \cup X^{-1})^* \rightarrow (Y \cup Y^{-1})^*$ be the monoid homomorphism defined by $\tau(x) = v_x$. For each $y \in Y$ choose a nonempty word $q_y \in (X \cup X^{-1})^*$ with $y =_G q_y$. Let $M_2 = \max_{y \in Y} \{|q_y|_X\}$, and $\kappa: (Y \cup Y^{-1})^* \rightarrow (X \cup X^{-1})^*$ the monoid homomorphism defined by $\kappa(y) = q_y$.

As G is densely generated by X , there exist fixed constants $E, F, N_0 \in \mathbb{N}$ as in Definition 5.1. Suppose G is not densely generated by Y . Then for all constants $E', F', N'_0 \in \mathbb{N}$ there exist some $s \geq N'_0$ so that all words equal to 1_G of length between $E's$ and $F's$ can be filled by cells of perimeter at most s .

Choose $E' = EM_2$ and $F' = M_1M_2F$, and $N'_0 = \max\{N_0, M_1M_2 + 1\}$. Let $s_0 \geq N'_0$ be chosen so that with respect to these constants, all words equal to 1_G of length between $E's_0$ and $F's_0$ can be filled by cells of perimeter at most s_0 .

As G is densely generated by X , choose $n = M_2s_0$. There must be a word $w_n \in (X \cup X^{-1})^*$ so that $w_n =_G 1_G$ and whose length satisfies

$$E(M_2s_0) \leq |w_n| \leq F(M_2s_0)$$

which cannot be filled by cells all of perimeter at most n . Then $\tau(w_n)$ labels a path in $\Gamma(G, Y)$ so that

$$EM_2s_0 \leq |\tau(w_n)| \leq M_1FM_2s_0.$$

Note that the map τ does not decrease length, as $\tau(w_n)$ is obtained by substitution, with no free reduction.

Since $E's_0 \leq |\tau(w_n)| \leq F's_0$, by our choice of s_0 we can fill this word by cells of perimeter at most s_0 . Now consider this van Kampen diagram as a subgraph of $\Gamma(G, Y)$. Map the entire subgraph, edge-by-edge, into $\Gamma(G, X)$ by applying the map κ ; the boundary of the new subgraph consists of paths of the form $\kappa(\tau(x_i))$, where $w = x_1x_2 \cdots x_n$. These paths form the boundary of the subgraph in $\Gamma(G, X)$ connecting the original vertices on the path labeled by w_n , and have length at most M_1M_2 . We have thus created cells of the form $\kappa(\tau(x_i))x_i^{-1}$ of perimeter at most $1 + M_1M_2$. These boundary cells, together with the copy of the van Kampen diagram, provide a filling of w_n .

In summary, the filling we have created has cells of two types:

- the boundary cells, of perimeter $1 + M_1M_2$, and
- images of the cells in the van Kampen diagram in $\Gamma(G, Y)$, which had perimeter at most s_0 ; after applying the homomorphism κ , the image of such a cell has perimeter at most M_2s_0 .

Note that we chose $N'_0 = \max\{M_2M_1 + 1, N_0\}$ so all of these cells have perimeter at most $M_2s_0 = n$. This contradicts the existence of w_n . Thus G is also densely generated by Y . \square

A group is called *dense* if it is densely generated by some, hence any, finite generating set. This definition is inspired by Baumslag's paper [3] about wreath products $G \wr H$. We prove in Proposition 5.3 that if H is infinite then $G \wr H$ is dense.

Proposition 5.3. *Let G and H be finitely generated groups. If G is nontrivial and H is infinite, then $G \wr H$ is dense.*

Proof. For $x, y \in G$ let $x^y = y^{-1}xy$, and $[x, y] = x^{-1}y^{-1}xy$. Let $G = \langle X \mid P \rangle$ and $H = \langle Y \mid Q \rangle$ be presentations of the groups G and H , where $X \subseteq G$ and $Y \subseteq H$ are finite generating sets. For each $h \in H$ choose a geodesic word $u_h \in (Y \cup Y^{-1})^*$ with $\pi(u_h) =_G h$, and let $U = \{u_h \mid h \in H\}$. Then the wreath product $G \wr H$ has presentation

$$G \wr H = \langle X \cup Y \mid P \cup Q \cup \{[a_1^u, a_2^v] \mid a_1, a_2 \in X, u, v \in U, u \neq v\} \rangle.$$

Let $B_{H,Y}(n)$ denote the ball of radius n in the group H with respect to the generating set Y . For a given positive integer m , define the set of relators

$$R_m = P \cup Q \cup \{[a_1^u, a_2^v] \mid u, v \in U, u \neq v, \pi(u), \pi(v) \in B_{H,Y}(m)\}.$$

For any set $S \subseteq H$, define the relation $T_S = \{(s_1h, s_2h) \mid s_1, s_2 \in S, h \in H\}$. Now we mimic Baumslag's argument for proving the non-finite presentability of wreath products, presented in Lemma 3 of [3], see also [12]. Baumslag constructs a group $\mathfrak{G}_{G,H,S}$ generated by G and H with the following properties:

- $G^{h_1} \cap G^{h_2} = \{1_{\mathfrak{G}_{G,H,S}}\}$ for all $h_1, h_2 \in H$ with $h_1 \neq h_2$, and
- $[G^{h_1}, G^{h_2}] = \{1_{\mathfrak{G}_{G,H,S}}\}$ if and only if $(h_1, h_2) \in T_S$.

Note that instead of requiring all conjugacy classes to commute in $\mathfrak{G}_{G,H,S}$, we only require this when the conjugating elements form a pair in the relation T_S .

Choosing $S = B_{H,Y}(n)$ for any fixed n , it follows that in $\mathfrak{G}_{G,H,S}$ we have $[G, G^h] \neq 1_{\mathfrak{G}_{G,H,S}}$ for any h for which $(1_H, h) \notin T_S$. In particular, this holds for any $h \in B_{H,Y}(2n+1) \setminus B_{H,Y}(2n)$. Therefore there is a relation $[a_1, a_2^h]$ in $R_{2n+1} \setminus R_{2n}$ which cannot be obtained as a product of conjugates of the relations from R_n .

Now observe that every loop $w \in (X \cup X^{-1} \cup Y \cup Y^{-1})^*$ of length $|w| \leq n$ in the wreath product $G \wr H$ can be represented as a product of conjugates of relations from R_n . Therefore, the loop of length $8n+8$, given by the relation $[a_1, a_2^h]$, cannot be decomposed into smaller loops of length less or equal than n . Thus, $G \wr H$ is dense. \square

6. SEPARATING NON-FINITELY PRESENTED CAYLEY AUTOMATIC GROUPS FROM AUTOMATIC GROUPS

The proof of Theorem B relies on the following proposition.

Proposition 6.1. *Let G be a non-finitely presented group with finite generating set S . Then there exists a non-decreasing step function $\phi_{G,S} \in \mathcal{F}$, depending on G and S , and an infinite sequence of integers $\{n_i\}$ such that $\phi_{G,S}(n_i) = n_i$ and for any Cayley automatic structure (S, S, L, ψ) on G ,*

- (1) $\phi_{G,S} \preceq_1 h_{S,\psi}$, and
- (2) if G is dense then $\mathfrak{i} \preceq_1 h_{S,\psi}$.

Proof. Since G is not finitely presented, there exists an infinite sequence of words $w_i \in S^*$ so that

- $w_i =_G 1_G$,
- if $w_i = \prod_{j=1}^k \rho_j u_j \rho_j^{-1}$ for some $k \in \mathbb{N}$ and $u_j, \rho_j \in S^*$ for $1 \leq j \leq k$, then $|u_j| \geq |w_i|$ for at least one value of j , and
- $|w_i| = l_i$, and $l_i < l_{i+1}$.

Define $\phi_{G,S} \in \mathcal{F}$ by $\phi_{G,S}(n) = l_i$ for $l_i \leq n < l_{i+1}$.

Let c, d, ς , and n_0 be the constants from Proposition 3.2. Then for any $i \in \mathbb{N}$ with $l_i \geq n_0$ we can decompose w_i into loops $u_{i,j}$ using the algorithm described in Proposition 3.2, and illustrated in Figures 2, 3 and 4, so that $|u_{i,j}| \leq 4h_{S,\psi}(cl_i + d) + \varsigma$. Our choice of w_i ensures that for some j we have $\phi_{G,S}(l_i) = l_i = |w_i| \leq |u_{i,j}|$.

Suppose $l_i \leq n < l_{i+1}$. It follows that for this choice of j ,

$$\phi_{G,S}(n) = l_i \leq |u_{i,j}| \leq 4h_{S,\psi}(cl_i + d) + \varsigma \leq 4h_{S,\psi}(cn + d) + \varsigma.$$

It then follows from Lemma 2.6 that $\phi_{G,S}(n) \preceq_1 h_{S,\psi}(n)$.

Now suppose that G is densely generated by X , so there exist constants E, F and N_0 so that for all $n \geq N_0$ there exists a loop $w_n =_G 1_G$ so that

- $En \leq |w_n| \leq Fn$, and
- w_n cannot be subdivided into loops all of whose lengths are bounded above by n . That is, if we write $w_n = \prod_{i=1}^k \rho_i u_i \rho_i^{-1}$ where each $u_i =_G 1_G$ then for some i we have $|u_i| \geq n$.

Again it follows from Proposition 3.2 that there are constants c, d, ς and n_0 so that for $n \geq \max(n_0, N_0)$, each u_j in the above decomposition of w_n we have $|u_j| \leq 4h_{S,\psi}(cFn + d) + \varsigma$. Since G is dense, it follows that for some j we have

$$n \leq |u_j| \leq 4h_{S,\psi}(cFn + d) + \varsigma.$$

As this is true for every $n \in \mathbb{N}$ with $n \geq \max(n_0, N_0)$, it follows from Lemma 2.6 that $\mathfrak{i} \preceq_1 h_{S,\psi}$. \square

We now prove Theorem B.

Theorem B (Non-finitely presented groups). *If G is a Cayley automatic group which is not finitely presented, then there is a non-decreasing step function ϕ depending only on G that is*

linear for infinitely many values, so that G is ϕ -separated. Furthermore, if G is dense then G is i -separated.

Proof. Suppose (Y, Y, L, ψ) is a Cayley automatic structure for G with respect to some arbitrary finite generating set Y .

If G is dense, it follows from part (2) of Proposition 6.1 that $i \preceq_1 h_{Y, \psi}$.

Next, choose a finite generating set S for G . It follows from part (1) of Proposition 6.1 that the function $\phi_{G, S}$ is such that $\phi_{G, S} \preceq_1 h_{S, \psi'} \approx_1 h_{Y, \psi}$. Setting $\phi = \phi_{G, S}$ gives a step function which suffices to prove the theorem. \square

7. CONCLUSION

Theorems A and B together imply that the only possible candidates for non-automatic Cayley automatic groups where the geometry comes close to resembling that of an automatic group (in the coarse sense considered here) are groups with quadratic or almost quadratic Dehn function. The class of groups with quadratic Dehn function is a wild and interesting collection (Gersten referred to the class as a “zoo” in [16]), including the following groups.

- (1) The higher Heisenberg groups H_{2k+1} for $k > 2$ [1, 19, 23].
- (2) The higher rank lamplighter groups, or Diestel-Leader groups [8].
- (3) Stallings’ group and its generalizations [11, 13]. These examples are not of type FP_∞ , hence not automatic; it is not known whether they admit Cayley automatic structures.
- (4) Thompson’s group F , which is not known to be automatic or Cayley automatic though is 1-counter-graph automatic [14].
- (5) An example of Ol’shanskii and Sapir [24] which has quadratic Dehn function and unsolvable conjugacy problem.

Ol’shanskii [25, 22] also gives an example of a group whose Dehn function is almost quadratic, and so Theorem A does not apply to this group. It is not known whether this example or the unsolvable conjugacy example of Ol’shanskii and Sapir are Cayley automatic. Note that our definition of a super-quadratic function is equivalent to saying the function is not almost quadratic, as shown in Lemma 4.6.

Progress towards proving Conjecture 1 takes two forms. First, one must improve the exhibited bounding functions given for groups with super-quadratic Dehn function and for non-dense non-finitely presented groups to show that these groups are i -separated. Second, one must prove that non-automatic Cayley automatic groups with quadratic and almost quadratic Dehn function are i -separated. We have some optimism for progress on the first part, and find the second more difficult.

APPENDIX A. FURTHER REMARKS ON STRONGLY-SUPER-POLYNOMIAL FUNCTIONS

In this appendix we give more details on strongly-super-polynomial and super-polynomial functions.

Proposition A.1. *Let $c, d \in \mathbb{R}$ such that $0 < c < d$. Then for a function $f \in \mathcal{F}$, we have $n^c f \ll f$ if and only if $n^d f \ll f$.*

Proof. Assume first that $n^d f \ll f$. Then there exists an unbounded function $t \in \mathcal{F}$ and integer constants $K, M > 0$ and $N \geq 0$ such that $n^c f(n)t(n) \leq Kf(Mn)$ for all $n \geq N$. Let $\tau(n) = n^{d-c}t(n)$. Then $\tau \in \mathcal{F}$ and as t is unbounded, so is τ . Writing $n^c f(n)\tau(n) \leq Kf(Mn)$, it follows that $n^c f \ll f$.

Now assume that $n^c f \ll f$. Then there exists an unbounded function $t \in \mathcal{F}$ and integer constants $K, M > 0$ and $N \geq 0$ such that $n^c f(n)t(n) \leq Kf(Mn)$ for all $n \geq N$. Therefore, the inequality

$$n^c t(n) \leq K \frac{f(Mn)}{f(n)}$$

holds for all $n \geq N$. This implies that the inequality

$$(5) \quad (M^k n)^c t(M^k n) \leq K \frac{f(M^{k+1}n)}{f(M^k n)}$$

holds for all integers $k \geq 0$ and $n \geq N$. Let $k_0 = \lfloor \frac{d}{c} \rfloor$; then we have $d \leq (k_0 + 1)c$. Allowing k to take all values between 1 and k_0 in (5) and multiplying the resulting inequalities together yields

$$\begin{aligned} & n^c t(n) (Mn)^c t(Mn) \dots (M^{k_0} n)^c t(M^{k_0} n) \\ & \leq K^{k_0+1} \frac{f(Mn)}{f(n)} \frac{f(M^2 n)}{f(Mn)} \dots \frac{f(M^{k_0+1} n)}{f(M^{k_0} n)} = K^{k_0+1} \frac{f(M^{k_0+1} n)}{f(n)}. \end{aligned}$$

It follows that

$$(6) \quad M' n^{c(k_0+1)} \tau(n) \leq K^{k_0+1} \frac{f(M^{k_0+1} n)}{f(n)}$$

holds for all $n \geq N$, where $\tau(n) = t(n)t(Mn) \dots t(M^{k_0} n)$ and $M' = M^c M^{2c} \dots M^{k_0 c}$. Since $M' \geq 1$, the inequality in (6) implies that

$$(7) \quad n^{c(k_0+1)} f(n) \tau(n) \leq K^{k_0+1} f(M^{k_0+1} n)$$

for all $n \geq N$. By construction, $\tau(n)$ is both an element of \mathcal{F} and an unbounded function. Therefore, $n^{c(k_0+1)} f \ll f$. As $d \leq (k_0 + 1)c$ it follows from the initial argument in the proof that $n^d f \ll f$. \square

The following lemma presents an example of a function which is super-polynomial but not strongly-super-polynomial.

Lemma A.2. *For given real number $\alpha > 1$, let $f_\alpha: \mathbb{N} \rightarrow \mathbb{R}$ be the function defined by $f_\alpha(n) = \alpha^{(\ln n)^{1.5}}$. Then*

- (1) $f \in \mathcal{F}$,
- (2) f_α is super-polynomial, and
- (3) f_α is not strongly-super-polynomial.

Proof. It is clear that $f_\alpha \in \mathcal{F}$ since $\alpha > 1$.

We have

$$\lim_{n \rightarrow \infty} \frac{\ln(\alpha^{(\ln n)^{1.5}})}{\ln n} = \lim_{n \rightarrow \infty} \frac{(\ln n)^{1.5} \ln(\alpha)}{\ln n} = \ln(\alpha) \lim_{n \rightarrow \infty} (\ln n)^{0.5} = \infty$$

so f_α is super-polynomial. Note that $\alpha > 1$ so $\ln \alpha > 0$.

Suppose (for contradiction) that f_α is strongly-super-polynomial, so there is an unbounded function $t \in \mathcal{F}$ and positive integer constants K, M and N such that $n^2 f_\alpha(n) t(n) \leq K f_\alpha(Mn)$ for all $n \geq N$. This means

$$t(n) \leq \frac{K \alpha^{(\ln(Mn))^{1.5} - (\ln n)^{1.5}}}{n^2}$$

Taking the logarithm of this inequality we obtain:

$$\ln(t(n)) \leq \ln K + [(\ln(Mn))^{1.5} - (\ln n)^{1.5}] \ln \alpha - 2 \ln n$$

which we can write as

$$\ln(t(n)) \leq \ln K + [(\ln(M) + \ln(n))(\ln(Mn))^{0.5} - (\ln(n))(\ln(n))^{0.5}] \ln \alpha - 2 \ln n$$

which becomes

$$\ln(t(n)) \leq \ln K + (\ln(M))(\ln(Mn))^{0.5} \ln \alpha + (\ln(n)) [(\ln(Mn))^{0.5} - (\ln n)^{0.5}] \ln(\alpha) - 2$$

which becomes

$$\ln(t(n)) \leq \ln K + (\ln(M))(\ln(Mn))^{0.5} \ln \alpha + \ln(\alpha)(\ln(n)) \left[(\ln(Mn))^{0.5} - (\ln n)^{0.5} - \frac{2}{\ln(\alpha)} \right]$$

Now if we let $B = \ln(Mn)^{0.5} - (\ln n)^{0.5}$ then $B(\ln(Mn)^{0.5} + (\ln n)^{0.5}) = \ln(Mn) - (\ln n) = \ln M$ and so

$$B = \frac{\ln M}{(\ln(Mn))^{0.5} + (\ln n)^{0.5}}.$$

Thus

$$\ln(t(n)) \leq \ln K + (\ln(M))(\ln(Mn))^{0.5} \ln \alpha + \ln(\alpha)(\ln(n)) \left[\left(\frac{\ln M}{(\ln(Mn))^{0.5} + (\ln n)^{0.5}} \right) - \frac{2}{\ln(\alpha)} \right]$$

which becomes

$$\ln(t(n)) \leq \ln K + \ln \alpha \ln(M) \left[(\ln(Mn))^{0.5} + (\ln(n)) \left[\left(\frac{1}{(\ln(Mn))^{0.5} + (\ln n)^{0.5}} \right) - \frac{2}{\ln(\alpha) \ln M} \right] \right]$$

which becomes

$$\ln(t(n)) \leq \ln K + \ln \alpha \ln(M) \left[\frac{(\ln(Mn))}{(\ln(Mn))^{0.5}} + (\ln(n)) \left[\left(\frac{1}{(\ln(Mn))^{0.5} + (\ln n)^{0.5}} \right) - \frac{2}{\ln(\alpha) \ln M} \right] \right]$$

which becomes

$$\ln(t(n)) \leq \ln K + \ln \alpha \ln(M) \left[\frac{(\ln(M) + \ln(n))}{(\ln(Mn))^{0.5}} + (\ln(n)) \left[\left(\frac{1}{(\ln(Mn))^{0.5} + (\ln n)^{0.5}} \right) - \frac{2}{\ln(\alpha) \ln M} \right] \right]$$

which becomes

$$\begin{aligned} \ln(t(n)) &\leq \ln K \\ &+ \ln \alpha \ln(M) \ln(n) \left[\left[\frac{\ln(M)}{(\ln(n) \ln(Mn))^{0.5}} + \frac{1}{(\ln(Mn))^{0.5}} + \frac{1}{(\ln(Mn))^{0.5} + (\ln n)^{0.5}} \right] \right. \\ &\left. - \frac{2}{\ln(\alpha) \ln M} \right]. \end{aligned}$$

The expression in the inside square brackets is going to 0 as $n \rightarrow \infty$, so eventually it will be less than $\frac{2}{\ln(\alpha) \ln M}$, contradicting the fact that $t \in \mathcal{F}$. \square

The next proposition proves that any function which is strongly-super-polynomial is also super-polynomial.

Proposition A.3. *Let $f \in \mathcal{F}$ be a non-zero function. If f is strongly-super-polynomial, then f is super-polynomial.*

Proof. Since f is strongly-super-polynomial, by Proposition A.1 we have $n^c f \ll f$ for any arbitrary $c > 0$ we wish to choose. So for any $c > 0$ there are positive constants K_c, M_c and N_c and an unbounded function $t_c \in \mathcal{F}$ so that the inequality

$$n^c f(n) t_c(n) \leq K_c f(M_c n)$$

holds for all $n \geq N_c$. Therefore, for all $n \geq N_c M_c$ we have

$$\left\lfloor \frac{n}{M_c} \right\rfloor^c f \left(\left\lfloor \frac{n}{M_c} \right\rfloor \right) t_c \left(\left\lfloor \frac{n}{M_c} \right\rfloor \right) \leq K_c f \left(M_c \left\lfloor \frac{n}{M_c} \right\rfloor \right) \leq K_c f(n).$$

Taking the logarithm of this inequality we obtain

$$c \ln \left\lfloor \frac{n}{M_c} \right\rfloor + \ln f \left(\left\lfloor \frac{n}{M_c} \right\rfloor \right) + \ln t_c \left(\left\lfloor \frac{n}{M_c} \right\rfloor \right) \leq \ln K_c + \ln f(n).$$

which becomes

$$c \ln \left\lfloor \frac{n}{M_c} \right\rfloor + \ln f \left(\left\lfloor \frac{n}{M_c} \right\rfloor \right) + \ln t_c \left(\left\lfloor \frac{n}{M_c} \right\rfloor \right) - \ln K_c \leq \ln f(n).$$

Now since f and t_c are both unbounded functions, there exists some $N'_c \geq N_c M_c$ so that

$$\ln f \left(\left\lfloor \frac{n}{M_c} \right\rfloor \right) + \ln t_c \left(\left\lfloor \frac{n}{M_c} \right\rfloor \right) - \ln K_c \geq 0$$

for all $n \geq N'_c$. Thus we have

$$c \ln \left\lfloor \frac{n}{M_c} \right\rfloor \leq \ln f(n)$$

for all $n \geq N'_c$.

Dividing sides by $\ln n$, we obtain that

$$(8) \quad c \frac{\ln \left\lfloor \frac{n}{M_c} \right\rfloor}{\ln n} \leq \frac{\ln f(n)}{\ln n}$$

for all $n \geq N'_c$.

Now choose $I_c \in \mathbb{N}$ so that $I_c \geq N'_c$ and $\frac{\ln \lfloor \frac{n}{M_c} \rfloor}{\ln n} > 0.9$ for all $n \geq I_c$. Observe that the limit $\lim_{n \rightarrow \infty} \frac{\ln \lfloor \frac{n}{M_c} \rfloor}{\ln n} = 1$ and the term $\frac{\ln \lfloor \frac{i}{M_c} \rfloor}{\ln i}$ is increasing, so such a value I_c exists.

Then from this observation and equation (8) we get

$$0.9c \leq c \frac{\ln \lfloor \frac{n}{M_c} \rfloor}{\ln n} \leq \frac{\ln f(n)}{\ln n}$$

for all $n \geq I_c$.

Since $c > 0$ can be arbitrary, this shows that the limit of $\frac{\ln f(n)}{\ln n}$ must go to ∞ . \square

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