Interval-valued probabilistic hesitant fuzzy set-based framework for group decision-making with unknown weights information

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Abstract

This paper aims at presenting a new decision framework under an interval-valued probabilistic hesitant fuzzy set (IVPHFS) context with fully unknown weight information. At first, the weights of the attributes are determined by using the interval-valued probabilistic hesitant deviation (IVPHD) method. Later, the DMs' weights are determined by using a recently proposed evidence theory-based Bayesian approximation (ETBA) method under the IVPHFS context. The preferences are aggregated by using a newly extended generalized Maclaurin symmetric mean (GMSM) operator under the IVPHFS context. Further, the alternatives are prioritized by using an inter-valued probabilistic hesitant complex proportional assessment (IVPHCOPRAS) method. From the proposed framework, the following significances are inferred such as; *it uses a generalized preference structure that provides ease and flexibility to the decision-makers (DMs) during preference elicitation; weights are calculated systematically to mitigate inaccuracies and subjective randomness; interrelationship among attributes are effectively captured, and alternatives are prioritized from different angles by properly considering the nature of the attributes*. Finally, the applicability of the framework is validated by using green supplier selection for a leading bakery company, and from the comparison, it is observed that the framework is *useful, practical and systematic* for rational decision-making*;*

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robust and consistent from sensitivity analysis of weights and Spearman correlation of rank values, respectively.

Keywords: Bayesian approximation; COPRAS method; Deviation method; Evidence theory; Hesitant fuzzy set; Maclaurin symmetric mean.

1. Introduction

Group decision-making (GDM) is a process that involves a group of experts or decision-makers (DMs) who provide their preferences over a set of alternatives with respect to a set of competing and conflicting attributes [1]. Since the process of GDM involves human intervention, uncertainty and vagueness are implicit factors [2]. To handle such uncertainties in the process, Torra [3] presented a generalized fuzzy set called hesitant fuzzy set (HFS) that allowed DMs to enter multiple choices over a specific instance. This concept offers flexibility to DMs by generalizing the notion of classical fuzzy set [4]. Armand et al. [5] developed solutions for fractional differential equations by using a variation iteration method (VIM) under a fuzzy context with Lagrange multipliers. Later, Narayanamoorthy and Mathankumar [6] provided novel solutions to a system of linear differential equations by using VIM. Motivated by the strength of HFS, many researchers adopted the structure for multi-attribute group decision-making (MAGDM) problems [7–9]. Moreover, Rodriguez et al. [10] made an interesting analysis of HFS, its variants, and its use in MAGDM.

Though HFS is useful and handles uncertainty to some extent, the occurring probability of each element is ignored. This causes irrational decision-making, and to circumvent the issue, Xu and Zhou [11] put forward a concept called probabilistic hesitant fuzzy set (PHFS), which associates an occurring probability value for each hesitant fuzzy element (HFE). Motivated by the strength of PHFS to handle uncertainty and associate probability value for each HFE, researchers proposed new methods for MAGDM. Li and Wang [12] presented new operational laws and aggregation operators under the PHFS context for MAGDM. Yue et al. [13] extended the weighted average and ordered weighted average operators for PHFS and applied it for strategy selection. Bashir et al. [12] put forward the concept of probabilistic hesitant fuzzy preference relation (PHFPR) and presented algorithms for checking and repairing consistency. Later, Zhou and Xu [14] presented the variant of PHFPR under uncertain context and analyzed consistency measures. Jiang and Ma [15] proposed PHFS-based Frank operators for rational evaluation of transformation efficiency. Hao et al. [16] put forward a new variant of PHFS, called probabilistic dual HFS, and applied the same for risk evaluation. Finally, Gao et al. [17] developed a framework by extending a dynamic reference point under the PHFS context for emergency decision-making.

From the comprehensive analysis made above, it is clear that the PHFS concept is powerful and attractive. The main issue with PHFS is that the occurring probability cannot be provided with such precision and accuracy, as there is an implicit hesitation in the process. To circumvent the issue, Song et al. [18] introduced a new concept called interval-valued PHFS (IVPHFS), which associates interval-valued occurring probability to each HFE. As a generalization of [18], Krishankumar et al. [19] ameliorated the idea and used it for MAGDM. Readers may kindly refer [19] for clarity on the IVPHFS concept and its substantial need for MAGDM.

Based on the literature analysis, it is clear that the IVPHFS concept has just emerged, and its application to the MAGDM problem needs to be explored. The IVPHFS concept is a generalized structure that allows DMs to provide a range of values as an occurring probability (in the interval fashion) for each HFE. In this way, the hesitation in the process can be managed effectively. Following challenges can be encountered from the literature analysis:

1. The attributes' weights must be calculated systematically to avoid inaccuracies in decision-making. Zavadskas et al. [20] demonstrated the importance of systematic calculation of weights and presented an algorithm for eliminating divergent rating values from attributes.
2. DMs’ weights must be calculated for rational estimation of their reliability in preference elicitation.
3. Preferences must be aggregated by properly understanding the interrelationship among attributes, as attributes taken for evaluation are often competing and conflicting with each other.
4. Alternatives must be prioritized rationally by properly capturing the hesitation and understanding the nature of attributes.

Motivated by these challenges and to circumvent them, the following contributions are put forward:

1. To properly manage uncertainty in the process of decision-making, the IVPHFS concept is put forward. Attributes' weights are calculated in a situation where the information about attributes is unknown. The deviation method is extended under the IVPHFS context, which can effectively capture the hesitation of the DMs during preference elicitation [21].

2. Similarly, DMs' weights are calculated systematically to properly determine the reliability of each DM [22] under an unknown information context. To do so, the ETBA method is extended to the IVPHFS context.
3. Further, preferences are aggregated in a much sensible way by properly capturing the interrelationship among conflicting and competing attributes. GMSM operator is extended under the IVPHFS context for this purpose.
4. Alternatives are prioritized by extending the popular COPRAS method under IVPHFS. The COPRAS method can prioritize alternatives from different angles and promote a proper understanding of the attributes' nature during prioritization [23].
5. Finally, the applicability of the proposed framework is explored by using a green supplier selection example. The strengths and weaknesses of the framework are realized by comparison with other methods.

From the discussion made above, the following research questions are framed that are effectively answered in this paper:

1. **RQ1:** Which preference structure mitigates subjective randomness and provides flexibility to DMs by carefully handling uncertainty and vagueness in GDM problems?
2. **RQ2:** How to calculate weights of attributes and DMs systematically under unknown information context?
3. **RQ3:** How to handle/capture the interrelationship among attributes during preference aggregation from different DMs?
4. **RQ4:** How to make rational prioritization of alternatives and validate the applicability of the framework?

5. **RQ5:** What are the main advantages of the proposed framework from both theoretical and numerical perspectives?

2. Preliminaries

Some basic ideas on HFS, PHFS, and IVPHFS are discussed below:

Definition 1 [3]**:** Consider M to be a fixed set and H to be an HFS. Then, h is a function that produces values in the unit interval. It is given mathematically as,

$$H = (m, h_H(m) | m \in M) \tag{1}$$

where $h_H(m)$ is a set containing values in the unit interval.

Definition 2 [11]**:** Consider M to be a fixed set. Then, H_p is a PHFS given by,

$$H_p = \left(m, h_{H_p}(\gamma_i,\ p_i) | m \in M\right) \tag{2}$$

where $h_{H_p}(\gamma_i,\ p_i)$ is the probabilistic hesitant fuzzy element, i is the membership degree of the i^{th} instance, and p_i is the occurrence probability associated with the hesitant fuzzy element γ_i.

Definition 3 [19]**:** Consider M to be a fixed set. Then, H_{ivp} is an IVPHFS given by,

$$H_{ivp} = \left(m, h_{H_{ivp}}(\gamma_i, [p_i^l, p_i^u]) | m \in M\right) \tag{3}$$

where $h_{H_{ivp}}\left(\gamma_i, [p_i^l, p_i^u]\right)$ is the interval-valued probabilistic hesitant fuzzy element (IVPHFE), i is the membership degree of the i^{th} instance and is $[p_i^l, p_i^u]$ the occurring probability in an interval fashion which is associated with γ_i, $0 \le p_i^l \le 1, 0 \le p_i^u \le 1$ and $p_i^l \le p_i^u$.

Remark 1: For convenience, $h_{H_{ivp}}\left(\gamma_i, [p_i^l, p_i^u]\right) = h\left(\gamma_i, [p_i^l, p_i^u]\right) = h_i \forall i = 1,2, \dots, n$ is called IVPHFE. Definition 3 mentioned above is the generalization of [17].

Definition 4 [19]**:** Let h_1 and h_2 be two IVPHFEs as defined before. Then, the operations are given by,

$$h_1 \oplus h_2 = \left(\gamma_1 + \gamma_2 - \gamma_1\gamma_2, \left[p_1^l p_2^l, p_1^u p_2^u\right]\right) \quad (4)$$

$$\lambda h_2 = \left(1 - (1 - \gamma_2)^\lambda, \left[p_i^l, p_i^u\right]\right)\lambda > 0 \quad (5)$$

$$h_1 \otimes h_2 = \left(\gamma_1\gamma_2, \left[p_1^l p_2^l, p_1^u p_2^u\right]\right) \quad (6)$$

$$h_2^\lambda = \left(\gamma_2^\lambda, \left[p_i^l, p_i^u\right]\right)\lambda > 0 \quad (7)$$

3. Proposed Scientific Framework for Decision-Making

3.1 Calculation of Attributes' Weights

This section put forwards a new extension to the deviation method under the IVPHFS context. The main intention of the proposed method is to calculate the attributes' weights when the information about attributes is entirely unknown. Popular methods, like the analytical hierarchy process (AHP), entropy-based measures pose complex formulation and yield unreasonable weight values [24]. To alleviate the issue, the deviation method is extended to the IVPHFS context. The deviation method is (i) simple and straightforward, and (ii) can capture the hesitation of each DM during preference elicitation.

Motivated by these strengths, in this section, the IVPHD method is presented. The steps are as follows:

Step 1: An evaluation matrix of order $q \times s$ is constructed with IVPHFEs. Here, q represents the number of DMs and s represents the number of attributes.

Step 2: The IVPHFEs are converted to single values by using equation (8).

$$h'_{lj} = \sum_{i=1}^{\#n}\left(\frac{\gamma_i\left(p_i^l + p_i^u\right)}{2}\right) \tag{8}$$

where $\#n$ represents the number of instances.

Step 3: Deviation is calculated for each attribute by using equation (9).

$$\sigma_j = \sqrt{\frac{\sum_{t=1}^{q}\left(h'_{tj} - \overline{h'_j}\right)^2}{q-1}} \tag{9}$$

where q represents the number of DMs, σ_j' represents the mean value of the jth attribute, and j is the deviation value of the jth attribute.

Step 4: The deviation values from step 3 are normalized to calculate the weight of each attribute, given equation (10).

$$w_j = \frac{\sigma_j}{\sum_j \sigma_j} \tag{10}$$

where w_j is the weight of the j^{th} attribute, which lies between 0 and 1 and $\sum_j w_j = 1$.

3.2 Calculation of DMs' Weights

This section presents a novel method for determining the weights of the DMs when the information about each DM is entirely unknown. Koksalmis and Kabak [22] made an interesting analysis of different methods used for calculating the weights of the DMs and rightly pointed the urge for a systematic method to calculate DMs' weights. Though the methods discussed in [20] calculate DMs' weights, the issue of uncertainty is not adequately handled.

Motivated by this issue and to circumvent the same, in this paper, the ETBA method is extended to IVPHFS. Let $B = (b_1, b_2, \dots, b_m)$ be a set of m propositions with basic probability assessment P defined as $P(.): 2^{\eta} \to [0,1]$. Here P=0 and the sum of BPAs yield unity. Though the evidence theory or Dempster-Shafer theory (DST) handles uncertainty better, it must be provided with all combinations of the propositions, which is practically difficult. To nullify this limitation of DST, Voorbraak [25] put forward the idea of Bayes approximation, which is computationally attractive and properly distributes P over B. This is unlike assigning a power set of B with P.

Motivated by the efficacy of these two methods, in this paper, we extend ETBA to IVPHFS. The steps for systematic calculation of DMs' weights are given below:

Step 1: Consider $P_j^t\left(b_i^\gamma\right) = \gamma_{ij}^t$, $P_j^t\left(b_i^{p^l}\right) = \left(p_{ij}^l\right)^t$, and $P_j^t\left(b_i^{p^u}\right) = \left(p_{ij}^u\right)^t$. Here i represent the number of alternatives, j represents the number of attributes, and l represents the number of DMs.

Step 2: Weighted attributes evidence body is calculated by using equations (11,12).

$$P_j^t\left(b_i^\gamma\right) = w_j P_j^t\left(b_i^\gamma\right) \tag{11}$$

$$P_j^t\left(B^\gamma\right) = 1 - \sum_{i=1}^{m} P_j^t\left(b_i^\gamma\right) \tag{12}$$

Here, it must be noted that the same method is used for p^l and p^u.

Step 3: Bayes approximation BA is calculated for different evidence functions by using equation (13).

$$BA = \begin{cases} \frac{\sum_{D \subseteq b_i} P(D)}{\sum_{C \subseteq B} P(C)|c|} & if\ b_i\ is\ singleton \\ 0 & otherwise \end{cases} \tag{13}$$

where $|.|$ is the cardinality.

Step 4: Aggregate the evidence of alternatives for each DM by using equation (14), which is a combination rule applied attribute-wise.

$$BA^l\left(b_i^\gamma\right) = \begin{cases} \frac{\prod_{j=1}^{s} BA_j^l\left(b_i^\gamma\right)}{\sum_{\cap b_i^\gamma \neq 0} \prod_{j=1}^{s} BA_j^l\left(b_i^\gamma\right)} & if\ b_i \neq 0 \\ 0 & otherwise \end{cases} \tag{14}$$

Apply equation (14) to p^l and p^u as well. In equation (14), $BA_j^l\left(b_i^\gamma\right)$ is the Bayesian approximation value associated with the i^{th} alternative over the j^{th} attribute from the l^{th} DM's perspectives. $b_i$is the singleton value and $\sum_{\cap b_i^\gamma \neq 0} \prod_{j=1}^{s} BA_j^l\left(b_i^\gamma\right)$ is the total conflict between pieces of evidence

Step 5: The similarity matrix of order $q \times q$ is calculated using equation (15).

$$d(BA^{dm1}, BA^{dm2}) = \sqrt{\sum_{i=1}^{m}\left(BA_i^{dm1} - BA_i^{dm2}\right)^2} \ \forall dm1, dm2 \in q \tag{15}$$

$$s(BA^{dm1}, BA^{dm2}) = 1 - d(BA^{dm1}, BA^{dm2}) \ \forall dm1, dm2 \in q \tag{16}$$

where $d(BA^{dm1},\ BA^{dm2})$ is the distance between two pieces of evidence from two DMs and $s(BA^{dm1},\ BA^{dm2})$ is the similarity between two pieces of evidence from two DMs.

The similarity between DMs is high if the distance between them is low. The similarity is calculated between DMs for, p^l, and p^u separately. Determine the mean value to form a single similarity matrix of order $q \times q$.

Step 6: By using the single similarity matrix from step 5, calculate support and credit, which is given by equations (17,18).

$$support_l = \sum_{\substack{l=1 \\ l \neq l}}^{q} s\left(BA^{dml}, BA^{dm(l+1)}\right) \tag{17}$$

$$credit_l = \frac{support_l}{\sum_l support_l} \tag{18}$$

From equation (18), the creditability of each DM is determined, and this provides the relative importance/reliability of the DM. The credit value is in the unit interval and$\sum_l credit_l = 1$.

3.3 Aggregation of IVPHFEs

This section puts forward a new extension to GMSM under IVPHFS. The GMSM operator is a generalized version of an MSM operator that can effectively represent other operators as special cases. The operator can efficiently capture the interrelationship among other attributes, which is lacking in many averages and geometric type operators. The operator also considers the relative importance (weight) of each DM and risk appetite value associated with each DM.

Motivated by these advantages, in this paper, the GMSM operator is extended to the IVPHFS context. Previously, operators viz., geometric [19], and Muirhead mean [26] are extended to IVPHFS, which are special cases of GMSM operator. So, the proposed operator is highly generalized and powerful in capturing the interrelationship among attributes.

Definition 5: IVPHFEs are aggregated using newly proposed interval-valued probabilistic hesitant GMSM (IVPHGMSM) operator, which is a mapping from $X^n \rightarrow X$ and is given by,

$$IVPHGMSM^{(\lambda_1,\lambda_2,\ldots,\lambda_r)}(h_1,h_2,\ldots,h_q) = \begin{pmatrix} \left(1-\left(\prod_{l=1}^{q}\left(1-\prod_{g=1}^{r}\gamma_{kj}^{\lambda_g}\right)^{\omega^l}\right)\right)^{\frac{1}{\sum_g \lambda_g}}, \\ \left[\begin{array}{c} \left(1-\left(\prod_{l=1}^{q}\left(1-\prod_{g=1}^{r}\left(p_{kj}^{l}\right)^{\lambda_g}\right)^{\omega^l}\right)\right)^{\frac{1}{\sum_g \lambda\lambda_g}}, \\ \left(1-\left(\prod_{l=1}^{q}\left(1-\prod_{g=1}^{r}\left(p_{kj}^{u}\right)^{\lambda_g}\right)^{\omega^l}\right)\right)^{\frac{1}{\sum_g \lambda\lambda_g}} \end{array}\right] \end{pmatrix} \quad (19)$$

where λ_1, λ_2,…,λ_r is the risk appetite values which can have values from the set $\{1,2,\dots,q\}$, ω^l is the weight of the l^{th} DM and $r = \left\lceil \frac{q}{2} \right\rceil$ with $\lceil . \rceil$ being the ceil operator.

The operator in equation (19) aggregates the HFEs, followed by the lower and upper occurring probability values. The risk appetite is defined as the amount of risk pursued, taken, or retained by an organization, which is given by ISO 31000 standards.

Property 1: Idempotent

If $h_i \; \forall i = 1,2,\dots,q = h$, then $IVPHGMSM^{(\lambda_1,\lambda_2,\dots,\lambda_q)}(h_1, h_2, \dots, h_q) = h$.

Proof:

$$IVPHGMSM^{(\lambda_1,\lambda_2,\dots,\lambda_r)}(h_1, h_2, \dots, h_q) = \left(\begin{array}{c} \left(1 - \left(\prod_{l=1}^{q}\left(1 - \prod_{g=1}^{r} \gamma_{kj}^{\lambda_g}\right)^{\omega^l}\right)\right)^{\frac{1}{\sum_g \lambda_g}}, \\ \left[\begin{array}{c} \left(1 - \left(\prod_{l=1}^{q}\left(1 - \prod_{g=1}^{r} \left(p_{kj}^{l}\right)^{\lambda_g}\right)^{\omega^l}\right)\right)^{\frac{1}{\sum_g \lambda_g}}, \\ \left(1 - \left(\prod_{l=1}^{q}\left(1 - \prod_{g=1}^{r} \left(p_{kj}^{u}\right)^{\lambda_g}\right)^{\omega^l}\right)\right)^{\frac{1}{\sum_g \lambda_g}} \end{array} \right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left(1-\left(\prod_{l=1}^{q}\left(1-\prod_{g=1}^{r}\gamma_{kj}^{\lambda_1+\lambda_2+\cdots+\lambda_g}\right)^{\omega^1+\omega^2+\cdots+\omega^q}\right)\right)^{\frac{1}{\sum_g \lambda_g}}, \\ \left[\begin{array}{c} \left(1-\left(\prod_{l=1}^{q}\left(1-\prod_{g=1}^{r}\left(p_{kj}^{l}\right)^{\lambda_1+\lambda_2+\cdots+\lambda_g}\right)^{\omega^1+\omega^2+\cdots+\omega^q}\right)\right)^{\frac{1}{\sum_g \lambda_g}}, \\ \left(1-\left(\prod_{l=1}^{q}\left(1-\prod_{g=1}^{r}\left(p_{kj}^{u}\right)^{\lambda_1+\lambda_2+\cdots+\lambda_g}\right)^{\omega^1+\omega^2+\cdots+\omega^q}\right)\right)^{\frac{1}{\sum_g \lambda_g}} \end{array}\right] \end{array} \right)$$

By expanding the terms, we get, $\{\gamma, [\, p^l, p^u]\} = h$ ■

Property 2: Commutative

Consider $h_i' \,\forall i = 1,2,\dots,q$ which is any permutation of $h_i \forall i = 1,2,\dots,q$, then $IVPHGMSM^{(\lambda_1,\lambda_2,\dots,\lambda_r)}\left(h_1, h_2, \dots, h_q\right) = IVPHGMSM^{(\lambda_1,\lambda_2,\dots,\lambda_q)}\left(h_1', h_2', \dots, h_q'\right)$

Proof:

$$IVPHGMSM^{(\lambda_1,\lambda_2,\dots,\lambda_r)}(h'_1,h'_2,\dots,h'_q)=\left(\begin{array}{c}\left(1-\left(\prod_{l=1}^{q}\left(1-\prod_{g=1}^{r}\gamma_{kj}^{\prime\lambda_g}\right)^{\omega^l}\right)\right)^{\frac{1}{\sum_g\lambda_g}},\\ \left[\left(1-\left(\prod_{l=1}^{q}\left(1-\prod_{g=1}^{r}(p_{kj}^{\prime l})^{\lambda_g}\right)^{\omega^l}\right)\right)^{\frac{1}{\sum_g\lambda_g}},\right.\\ \left.\left(1-\left(\prod_{l=1}^{q}\left(1-\prod_{g=1}^{r}(p_{kj}^{\prime u})^{\lambda_g}\right)^{\omega^l}\right)\right)^{\frac{1}{\sum_g\lambda_g}}\right]\end{array}\right)$$

Property 3: Monotonicity

Let $h=h_t$ and $h^*=h_t^*$ be two IVPHFSs $\forall t=1,2,\dots,q$ with a constraint that $h_t^*\geq h_t$. Then, $IVPHGMSM^{(\lambda_1,\lambda_2,\dots,\lambda_r)}(h_1,h_2,\dots,h_q)\leq IVPHGMSM^{(\lambda_1,\lambda_2,\dots,\lambda_q)}(h_1^*,h_2^*,\dots,h_q^*)$.

Proof:

Let $h_t=(\gamma_i,[p_i^l,p_i^u])$ and $h_t^*=(\gamma_i^*,[p_i^{*l},p_i^{*u}])\forall i=1,2,\dots,\#n$ as defined before. Now, $\gamma^*=\left(1-\left(\prod_{l=1}^{q}\left(1-\prod_{g=1}^{r}\gamma_{kj}^{*\lambda_g}\right)^{\omega^l}\right)\right)^{\frac{1}{\sum_g\lambda_g}}$, $[p^{*l},p^{*u}]=\left[\left(1-\left(\prod_{l=1}^{q}\left(1-\prod_{g=1}^{r}(p_{kj}^{\prime l})^{\lambda_g}\right)^{\omega^l}\right)\right)^{\frac{1}{\sum_g\lambda_g}},\left(1-\left(\prod_{l=1}^{q}\left(1-\prod_{g=1}^{r}(p_{kj}^{\prime u})^{\lambda_g}\right)^{\omega^l}\right)\right)^{\frac{1}{\sum_g\lambda_g}}\right]$. Similarly h_t is defined as before. Since $h_t^*\geq h_t$, $s(h_t)\leq s(h_t^*)\ \forall t=1,2,\dots,q$ or $\sigma(h_t)\geq\sigma(h_t^*)\ \forall t=1,2,\dots,q$ and from the formulation presented above, $h^*\geq h$. Thus, $IVPHGMSM^{(\lambda_1,\lambda_2,\dots,\lambda_r)}(h_1,h_2,\dots,h_q)\leq IVPHGMSM^{(\lambda_1,\lambda_2,\dots,\lambda_q)}(h_1^*,h_2^*,\dots,h_q^*)$ ■

Property 4: Bounded

Let h_t be IVPHFEs $\forall t = 1,2, \dots, q$ as defined before. Then, $h^- \leq IVPHGMSM^{(\lambda_1, \lambda_2, \dots, \lambda_r)}(h_1, h_2, \dots, h_q) \leq h^+$ with $h^- = min_i(h_i)$ and $h^+ = max_t(h_t)\ \forall t = 1,2, \dots q$.

Proof:

From the idempotent property and monotonicity property of IVPHFGMSM operator, it can be deduced that $IVPHGMSM^{(\lambda_1, \lambda_2, \dots, \lambda_r)}(h_1, h_2, \dots, h_q) \geq IVPHGMSM^{(\lambda_1, \lambda_2, \dots, \lambda_r)}(h^-, h^-, \dots, h^-)$ and $IVPHGMSM^{(\lambda_1, \lambda_2, \dots, \lambda_r)}(h_1, h_2, \dots, h_l) \leq IVPHGMSM^{(\lambda_1, \lambda_2, \dots, \lambda_r)}(h^+, h^+, \dots, h^+)$ yields $h^- \leq IVPHGMSM^{(\lambda_1, \lambda_2, \dots, \lambda_r)}(h_1, h_2, \dots, h_q) \leq h^+$ ■

Theorem 1: The aggregation of IVPHFEs by using an IVPHFGMSM operator yields a result that is also an IVPHFE.

Proof:

To prove that the aggregated result is an IVPHFE, we need to show that the result obeys Definition 3. From property 4, it is inferred that $h^- \leq IVPHGMSM^{(\lambda_1, \lambda_2, \dots, \lambda_r)}(h_1, h_2, \dots, h_q) \leq h^+$ with $h^- = min_t(h_t)$ and $h^+ = max_t(h_t)\ \forall t = 1,2, \dots q$. By generalizing, we get $h^- = 0 \leq IVPHGMSM^{(\lambda_1, \lambda_2, \dots, \lambda_q)}(h_1, h_2, \dots, h_q) \leq 1 = h^+$. Hence, aggregation of IVPHFEs by using IVPHFGMSM operator provides an IVPHFE ■

3.4 Proposed IVPHCOPRAS Method

This section puts forward a new extension to COPRAS under IVPHFS. As mentioned earlier, COPRAS is a powerful method, which prioritizes alternatives by properly capturing the nature of attributes. Also, it provides ranking from different angles. Zavadskas et al. [27] framed the genesis for the COPRAS method and demonstrated its applicability to select suitable dwellers for house wall [28]. Vahdani et al. [29], Mondal et al. [30], and Gorabe et al. [31] extended the COPRAS method for robot selection in industries. Valipour et al. [32] put forward an integrated SWARA-COPRAS method for a real case study in Iran. Zavadskas et al. [33,34] proposed a grey-COPRAS method for strategic decision-making. Hajiagha et al. [35] and Wang et al. [36] extended COPRAS to interval-valued intuitionistic fuzzy context for investor selection risk mitigation, respectively. Bielinski et al. [37] used the COPRAS method for territory conversion that is abandoned. Further, Bausys et al. [38] used the COPRAS method to select an apt location for setting natural gas terminal under the neutrosophic fuzzy context. Krishankumar et al. [39] proposed a new decision model with double hierarchy hesitant fuzzy linguistic information for green supplier selection. Ayrim et al. [40] put forward a new extension to COPRAS under a stochastic decision environment and applied it to the cargo transport company selection. Roy et al. [41] provided a new extension to COPRAS under a rough set context for web-based hotel selection. Stefano et al. [42], Zavadskas et al. [43], and Mardani et al. [44] made an interesting review on the COPRAS method and discussed its strengths and applicability on different MAGDM problems.

From the brief literature investigation, it is inferred that (i) COPRAS method is popular for solving many MAGDM problems; (ii) IVPHFS-based COPRAS method is not yet explored; (iii) nature of attributes are considered during prioritization and (iv) alternatives are prioritized from the context of benefit type, cost type, and a linear combination of benefit and cost types.

Motivated by these inferences, in this paper, we present the systematic procedure for IVPHFS-based COPRAS ranking method:

Step 1: Obtain an aggregated matrix of order $m \times s$, where m represents the number of objects/alternatives and s represents the number of attributes from section 3.3. Also, obtain the attributes' weight vector of order $1 \times s$ from section 3.1 as input for prioritization.

Step 2: Apply equations (20,21) to calculate the parameters of COPRAS under the IVPHFS context.

$$T_k = \sum_{i=1}^{\#n} \left(\sum_{j=1}^{\#b} w_j \gamma_{kj}^i + \sum_{j=1}^{\#b} 1 - \left(1 - \left(p_{kj}^i\right)^i\right)^{w_j} + \sum_{j=1}^{\#b} 1 - \left(1 - \left(p_{kj}^u\right)^i\right)^{w_j} \right) \tag{20}$$

$$R_k = \sum_{i=1}^{\#n} \left(\sum_{j=1}^{\#c} w_j \gamma_{kj}^i + \sum_{j=1}^{\#c} 1 - \left(1 - \left(p_{kj}^i\right)^i\right)^{w_j} + \sum_{j=1}^{\#c} 1 - \left(1 - \left(p_{kj}^u\right)^i\right)^{w_j} \right) \tag{21}$$

where $\#b$ represents the number of attributes belonging to the benefit type, $\#c$ represents the number of attributes belonging to the cost type and w_j represents the weight of the j^{th} attribute.

Step 3: Derive the final prioritization order by using equation (22). This equation is a linear combination of equations (20,21).

$$Q_k = \eta T_k + (1 - \eta) \frac{\sum_k R_k}{R_k \left(\frac{1}{\sum_k R_k} \right)} \tag{22}$$

where Q_k is the final prioritization value for the k^{th} alternative, and η denotes the strategy value adopted by the DM, which is in range 0 to 1.

Before demonstrating the practical use of the proposed framework, it is worth understanding the framework's working model. Fig.1 depicts the working model of the proposed framework. Initially, DMs provide two sets of matrices. In the first set, q matrices of order $m \times s$ are provided. Then, in the second set, one matrix of order $q \times s$ is provided. From this matrix, attributes' weights are determined, and the details are given in section 3.1. Later, the first set (q matrices of order $m \times n$) is utilized to determine the weights of the DMs, and its details are given in section 3.2. By using the DMs' weight vector of order $1 \times q$, q matrices of order $m \times s$ are aggregated into a single matrix of order $m \times s$ (refer section 3.3 for details). By utilizing the aggregated matrix of order $m \times s$ and the attributes' weight vector of order $1\times s$, alternatives are prioritized to obtain a vector of order $1 \times m$ (refer section 3.4 for details). Finally, the strengths and weaknesses of the proposed framework are investigated by comparison with other methods (kindly refer to section 5 for more information).

4. Illustrative example: Green Supplier Selection

This section puts forward an illustrative example of green supplier selection, for a leading bakery company, to demonstrate the applicability of the proposed framework. Consider the company PXA (name anonymous) in Chennai, which produces top-class cakes, cookies, and snacks. PXA gets its raw materials from the suppliers, which are then prepared as a final product by using a skilled workforce and high-tech equipment. The company decides on expanding its market in the country by renovating some of its work lines. The primary raw material for the company is 'milk,' which is dominantly used for preparing snacks and cakes. The company sets an annual meeting and identifies an urge for transformation from classical supply chains to green supply chains by adhering to ISO 14000 and 14001. This idea eventually brings eco-friendly preparation of snacks and cakes. PXA feels that it is substantial to go green for a healthy and clean ecosystem, and so, a systematic selection process is adopted. The company initially constitutes a panel of three experts/DMs who participate in the systematic decision-making process. Let $D = (d_1, d_2, \dots, d_q)$ be a set of DMs, who rate a set of green suppliers $G = (g_1, g_2, \dots, g_m)$ over a set of competing/conflicting attributes $A = (a_1, a_2, \dots, a_s)$. The company authorities carefully choose three experts/DMs for effective selection. The DMs analyze different green suppliers and based on the Delphi method, and four green suppliers are shortlisted. All these green suppliers follow green standards specified by ISO. Also, from the literature analysis, many attributes are initially chosen for evaluation, which is finally revised to five attributes. They are the quality of material a_1, customer/client service a_2, customer relationship a_3, the delivery time of material a_4, and total cost a_5. Based on the voting method, the attributes are finalized by the DMs. Attributes a_1, a_2, and a_3 belong to the benefit type and attributes a_4 and a_5 belong to the cost type. DMs plan to use the IVPHFS environment for rating green suppliers over a set of evaluation attributes.

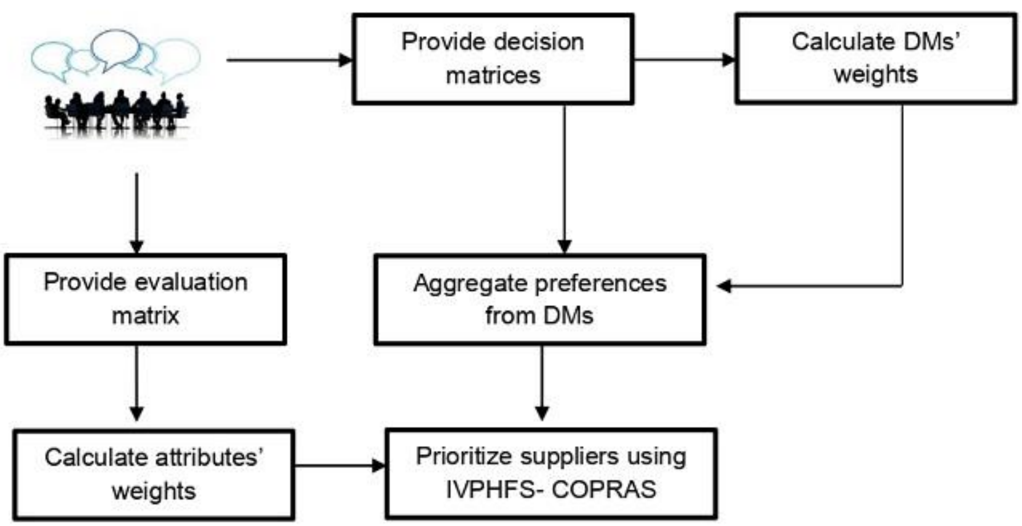

Fig.1 Proposed scientific decision framework.

The systematic procedure for effective selection of green suppliers is given below:

Step 1: Each DM provides his/her preference information over the green suppliers with respect to the evaluation attributes. So, three matrices of order 4x5 are obtained with IVPHFS information.

Table 1 IVPHFS information from each DM

Green suppliers	Evaluation attributes				
	a_1	a_2	a_3	a_4	a_5
			d_1		
g_1	$\begin{Bmatrix}0.24,[0.25,0.52]\\0.64,[0.16,0.62]\end{Bmatrix}$	$\begin{Bmatrix}0.35,[0.26,0.62]\\0.26,[0.62,0.74]\end{Bmatrix}$	$\begin{Bmatrix}0.74,[0.85,0.85]\\0.35,[0.23,0.36]\end{Bmatrix}$	$\begin{Bmatrix}0.16,[0.26,0.73]\\0.8,[0.36,0.46]\end{Bmatrix}$	$\begin{Bmatrix}0.52,[0.2,0.26]\\0.36,[0.63,0.63]\end{Bmatrix}$
g_2	$\begin{Bmatrix}0.26,[0.46,0.57]\\0.52,[0.26,0.73]\end{Bmatrix}$	$\begin{Bmatrix}0.25,[0.47,0.47]\\0.83,[0.27,0.8]\end{Bmatrix}$	$\begin{Bmatrix}0.77,[0.36,0.72]\\0.26,[0.17,0.53]\end{Bmatrix}$	$\begin{Bmatrix}0.42,[0.36,0.62]\\0.16,[0.36,0.57]\end{Bmatrix}$	$\begin{Bmatrix}0.68,[0.58,0.69]\\0.38,[0.76,0.8]\end{Bmatrix}$
g_3	$\begin{Bmatrix}0.26,[0.40,0.66]\\0.27,[0.27,0.45]\end{Bmatrix}$	$\begin{Bmatrix}0.74,[0.42,0.76]\\0.25,[0.16,0.17]\end{Bmatrix}$	$\begin{Bmatrix}0.26,[0.47,0.47]\\0.57,[0.68,0.69]\end{Bmatrix}$	$\begin{Bmatrix}0.74,[0.42,0.8]\\0.35,[0.52,0.64]\end{Bmatrix}$	$\begin{Bmatrix}0.25,[0.26,0.63]\\0.35,[0.26,0.26]\end{Bmatrix}$
g_4	$\begin{Bmatrix}0.64,[0.74,0.75]\\0.8,[0.25,0.3]\end{Bmatrix}$	$\begin{Bmatrix}0.55,[0.2,0.62]\\0.16,[0.36,0.62]\end{Bmatrix}$	$\begin{Bmatrix}0.27,[0.32,0.72]\\0.63,[0.74,0.75]\end{Bmatrix}$	$\begin{Bmatrix}0.63,[0.26,0.62]\\0.27,[0.42,0.53]\end{Bmatrix}$	$\begin{Bmatrix}0.15,[0.61,0.61]\\0.36,[0.47,0.67]\end{Bmatrix}$
			d_2		
g_1	$\begin{Bmatrix}0.72,[0.27,0.77]\\0.38,[0.19,0.72]\end{Bmatrix}$	$\begin{Bmatrix}0.74,[0.58,0.8]\\0.72,[0.28,0.72]\end{Bmatrix}$	$\begin{Bmatrix}0.8,[0.22,0.55]\\0.17,[0.72,0.73]\end{Bmatrix}$	$\begin{Bmatrix}0.26,[0.36,0.63]\\0.73,[0.48,0.55]\end{Bmatrix}$	$\begin{Bmatrix}0.49,[0.27,0.72]\\0.42,[0.19,0.31]\end{Bmatrix}$
g_2	$\begin{Bmatrix}0.26,[0.42,0s.61]\\0.47,[0.27,0.28]\end{Bmatrix}$	$\begin{Bmatrix}0.8,[0.26,0.47]\\0.83,[0.27,0.36]\end{Bmatrix}$	$\begin{Bmatrix}0.73,[0.15,0.42]\\0.18,[0.42,0.73]\end{Bmatrix}$	$\begin{Bmatrix}0.17,[0.23,0.53]\\0.52,[0.33,0.83]\end{Bmatrix}$	$\begin{Bmatrix}0.82,[0.26,0.72]\\0.26,[0.26,0.52]\end{Bmatrix}$
g_3	$\begin{Bmatrix}0.62,[0.17,0.62]\\0.46,[0.26,0.62]\end{Bmatrix}$	$\begin{Bmatrix}0.26,[0.47,0.74]\\0.83,[0.16,0.27]\end{Bmatrix}$	$\begin{Bmatrix}0.33,[0.6,0.63]\\0.27,[0.58,0.73]\end{Bmatrix}$	$\begin{Bmatrix}0.19,[0.14,0.37]\\0.46,[0.39,0.72]\end{Bmatrix}$	$\begin{Bmatrix}0.25,[0.63,0.72]\\0.13,[0.47,0.74]\end{Bmatrix}$
g_4	$\begin{Bmatrix}0.28,[0.2,0.36]\\0.61,[0.44,0.7]\end{Bmatrix}$	$\begin{Bmatrix}0.27,[0.32,0.72]\\0.46,[0.57,0.82]\end{Bmatrix}$	$\begin{Bmatrix}0.27,[0.57,0.77]\\0.27,[0.23,0.45]\end{Bmatrix}$	$\begin{Bmatrix}0.64,[0.67,0.69]\\0.82,[0.63,0.75]\end{Bmatrix}$	$\begin{Bmatrix}0.36,[0.27,0.27]\\0.72,[0.31,0.74]\end{Bmatrix}$
			d_3		
g_1	$\begin{Bmatrix}0.24,[0.36,0.53]\\0.56,[0.54,0.79]\end{Bmatrix}$	$\begin{Bmatrix}0.8,[0.11,0.16]\\0.63,[0.57,0.72]\end{Bmatrix}$	$\begin{Bmatrix}0.27,[0.47,0.72]\\0.74,[0.37,0.53]\end{Bmatrix}$	$\begin{Bmatrix}0.86,[0.63,0.52]\\0.27,[0.74,0.83]\end{Bmatrix}$	$\begin{Bmatrix}0.26,[0.27,0.35]\\0.57,[0.26,0.69]\end{Bmatrix}$
g_2	$\begin{Bmatrix}0.57,[0.72,0.74]\\0.48,[0.27,0.57]\end{Bmatrix}$	$\begin{Bmatrix}0.8,[0.16,0.36]\\0.46,[0.36,0.62]\end{Bmatrix}$	$\begin{Bmatrix}0.63,[0.27,0.43]\\0.73,[0.27,0.47]\end{Bmatrix}$	$\begin{Bmatrix}0.74,[0.1,0.62]\\0.36,[0.73,0.8]\end{Bmatrix}$	$\begin{Bmatrix}0.63,[0.1,0.26]\\0.4,[0.45,0.66]\end{Bmatrix}$
g_3	$\begin{Bmatrix}0.8,[0.36,0.63]\\0.75,[0.16,0.68]\end{Bmatrix}$	$\begin{Bmatrix}0.73,[0.37,0.57]\\0.79,[0.3,0.37]\end{Bmatrix}$	$\begin{Bmatrix}0.64,[0.72,0.74]\\0.55,[0.37,0.75]\end{Bmatrix}$	$\begin{Bmatrix}0.16,[0.36,0.73]\\0.56,[0.67,0.73]\end{Bmatrix}$	$\begin{Bmatrix}0.74,[0.47,0.7]\\0.62,[0.19,0.24]\end{Bmatrix}$
g_4	$\begin{Bmatrix}0.26,[0.63,0.63]\\0.26,[0.4,0.66]\end{Bmatrix}$	$\begin{Bmatrix}0.26,[0.19,0.28]\\0.64,[0.26,0.53]\end{Bmatrix}$	$\begin{Bmatrix}0.46,[0.25,0.75]\\0.68,[0.17,0.74]\end{Bmatrix}$	$\begin{Bmatrix}0.37,[0.56,0.73]\\0.7,[0.3,0.55]\end{Bmatrix}$	$\begin{Bmatrix}0.17,[0.11,0.37]\\0.73,[0.21,0.26]\end{Bmatrix}$

Table 1 presents the decision matrices from each DM with IVPHFS information. There are four green suppliers rated over five attributes. These matrices are taken as input for calculating DMs' weights. It must be noted that these matrices adopt IVPHFS information, which associates probability values in the interval fashion to each element. For the ease of understanding, let us consider an entry from Table 1, DM d_1 rates green supplier g_1 based on attribute a_1 as 0.24 and 0.64 with their respective occurrence probability in the interval [0.25,0.52] and [0.16,0.62]. The values 0.24 and 0.62 are membership values that represent the degree of preferences.

Step 2: An attribute weight calculation matrix is obtained, which is of order 3x5 with IVPHFS information. Apply section 3.1 to calculate the weights of the attributes. It is a vector of order $1 \times s$.

Table 2 provides the preferences of each DM on each attribute, which are used for calculating the weights of attributes. The weight values are given by $(0.06, 0.07, 0.08, 0.51, 0.28)$ when the IVPHFD method is applied.

Table 2 IVPHFEs for attributes' weights calculation

DMs	Evaluation attributes				
	a_1	a_2	a_3	a_4	a_5
d_1	$\left\{\begin{matrix} 0.8, [0.1,0.33] \\ 0.55, [0.41,0.91] \end{matrix}\right\}$	$\left\{\begin{matrix} 0.35, [0.05,0.34] \\ 0.72, [0.25,0.27] \end{matrix}\right\}$	$\left\{\begin{matrix} 0.53, [0.56,0.97] \\ 0.84, [0.13,0.51] \end{matrix}\right\}$	$\left\{\begin{matrix} 0.75, [0.22,0.48] \\ 0.11, [0.51,0.91] \end{matrix}\right\}$	$\left\{\begin{matrix} 0.49, [0.43,0.61] \\ 0.97, [0.56,0.92] \end{matrix}\right\}$
d_2	$\left\{\begin{matrix} 0.58, [0.27,0.5] \\ 0.39, [0.53,0.77] \end{matrix}\right\}$	$\left\{\begin{matrix} 0.37, [0.13,0.42] \\ 0.61, [0.59,0.78] \end{matrix}\right\}$	$\left\{\begin{matrix} 0.46, [0.58,0.94] \\ 0.04, [0.51,0.93] \end{matrix}\right\}$	$\left\{\begin{matrix} 0.84, [0.58,1] \\ 0.5, [0.33,0.64] \end{matrix}\right\}$	$\left\{\begin{matrix} 0.59, [0.48,0.8] \\ 0.29, [0.65,0.82] \end{matrix}\right\}$
d_3	$\left\{\begin{matrix} 0.13, [0.88,1] \\ 0.6, [0.1,0.42] \end{matrix}\right\}$	$\left\{\begin{matrix} 0.74, [0.14,0.72] \\ 0.31, [0.27,0.89] \end{matrix}\right\}$	$\left\{\begin{matrix} 0.67, [0.26,0.62] \\ 0.71, [0.05,0.45] \end{matrix}\right\}$	$\left\{\begin{matrix} 0.49, [0.11,0.47] \\ 0.02, [0.22,0.24] \end{matrix}\right\}$	$\left\{\begin{matrix} 0.07, [0.06,0.45] \\ 0.4, [0.36,0.4] \end{matrix}\right\}$

Table 2 provides preference information related to each attribute by each DM. These are used to calculate the weights of attributes. Equations (8,9) are applied to determine the deviation of each attribute, which is further normalized using equation (10) to calculate the weights of the attributes, and it is given by 0.06, 0.07, 0.08, 0.51, and 0.28, respectively.

Step 3: Aggregate the matrices from step 1 by using the operator proposed in section 3.3. DMs' weights are obtained from the method proposed in section 3.2. A single matrix of order 4x5 is obtained (refer to Table 3).

By applying equations (11-14), aggregated pieces of evidence are calculated and they are given by $BA^1(g_1) = \{0.34, [0.33,0.33]\}$, $BA^1(g_2) = \{0.34, [0.33,0.33]\}$, $BA^1(g_3) = \{0.32, [0.34,0.33]\}$, $BA^1(g_4) = \{0.34, [0.35,0.35]\}$; $BA^2(g_1) = \{0.33, [0.32,0.33]\}$, $BA^2(g_2) = \{0.33, [0.33,0.32]\}$, $BA^2(g_3) = \{0.35, [0.33,0.34]\}$, $BA^2(g_4) = \{0.32, [0.33,0.34]\}$; $BA^3(g_1) = \{0.33, [0.33,0.32]\}$, $BA^3(g_2) = \{0.32, [0.33,0.34]\}$, $BA^3(g_3) = \{0.37, [0.36,0.34]\}$, and $BA^3(g_4) = \{0.31, [0.31,0.31]\}$.

By applying equations (15,16), the similarity measure is determined, which is shown below. Using the similarity values between DMs, support, and credit are determined by using equations (17,18).

$$s(BA^{dm1}, BA^{dm2}) = \begin{pmatrix} - & \{0.95, [0.96,1]\} & \{0.99, [0.98,0.97]\} \\ \{0.95, [0.96,1]\} & - & \{0.94, [0.94,0.97]\} \\ \{0.99, [0.98,0.97]\} & \{0.94, [0.94,0.97]\} & - \end{pmatrix}$$

The credit value for each DM is given by $(0.34,0.33,0.33)$, which provides the weight value of each DM. The IVPHFGMSM operator uses these values for aggregation of preferences with $\lambda_1 = 1$ and $\lambda_2 = 2$, which is shown in Table 3.

Table 3 Aggregated IVPHFE for decision-making

Green suppliers	Evaluation attributes				
	a_1	a_2	a_3	a_4	a_5
g_1	$\begin{Bmatrix} 0.53, [0.3,0.64] \\ 0.55, [0.39,0.72] \end{Bmatrix}$	$\begin{Bmatrix} 0.7, [0.42,0.65] \\ 0.61, [0.54,0.73] \end{Bmatrix}$	$\begin{Bmatrix} 0.7, [0.67,0.75] \\ 0.56, [0.54,0.59] \end{Bmatrix}$	$\begin{Bmatrix} 0.66, [0.38,0.67] \\ 0.7, [0.59,0.68] \end{Bmatrix}$	$\begin{Bmatrix} 0.45, [0.25,0.54] \\ 0.47, [0.46,0.6] \end{Bmatrix}$
g_2	$\begin{Bmatrix} 0.43, [0.58,0.65] \\ 0.49, [0.27,0.6] \end{Bmatrix}$	$\begin{Bmatrix} 0.72, [0.35,0.44] \\ 0.77, [0.31,0.67] \end{Bmatrix}$	$\begin{Bmatrix} 0.72, [0.28,0.58] \\ 0.54, [0.32,0.61] \end{Bmatrix}$	$\begin{Bmatrix} 0.57, [0.27,0.59] \\ 0.4, [0.56,0.76] \end{Bmatrix}$	$\begin{Bmatrix} 0.73, [0.42,0.63] \\ 0.36, [0.6,0.69] \end{Bmatrix}$
g_3	$\begin{Bmatrix} 0.66, [0.34,0.64] \\ 0.58, [0.24,0.6] \end{Bmatrix}$	$\begin{Bmatrix} 0.66, [0.42,0.71] \\ 0.74, [0.23,0.29] \end{Bmatrix}$	$\begin{Bmatrix} 0.48, [0.62,0.64] \\ 0.5, [0.58,0.72] \end{Bmatrix}$	$\begin{Bmatrix} 0.55, [0.35,0.7] \\ 0.47, [0.56,0.7] \end{Bmatrix}$	$\begin{Bmatrix} 0.55, [0.5,0.69] \\ 0.47, [0.35,0.55] \end{Bmatrix}$

g_4	$\begin{Bmatrix} 0.48,[0.62,0.64] \\ 0.66,[0.38,0.61] \end{Bmatrix}$	$\begin{Bmatrix} 0.41,[0.25,0.61] \\ 0.5,[0.44,0.7] \end{Bmatrix}$	$\begin{Bmatrix} 0.36,[0.43,0.75] \\ 0.59,[0.55,0.69] \end{Bmatrix}$	$\begin{Bmatrix} 0.58,[0.56,0.68] \\ 0.7,[0.5,0.63] \end{Bmatrix}$	$\begin{Bmatrix} 0.26,[0.45,0.47] \\ 0.66,[0.37,0.64] \end{Bmatrix}$

Step 4: Use the aggregated matrix from step 3 and attributes' weight vector from step 2 to prioritize green suppliers by using the method provided in section 3.4.

Table 4 Prioritization values from COPRAS method

Green suppliers	COPRAS prioritization parameters		
	T_k	R_k	Q_k
g_1	(0.52,0.5) (3.56,3.57)	(1.38,1.67) (1.18,0.93)	(2.04,4.62)
g_2	(0.41,0.4) (3.3,3.26)	(1.4,1.67) (0.98,1.31)	(1.94,4.42)
g_3	(0.48,0.4) (3.53,3.09)	(1.55,1.48) (1.14,1.06)	(1.96,3.91)
g_4	(0.43,0.48) (3.48,3.51)	(1.47,1.6) (1.05,1.1)	(1.99,4.57)

Table 4 depicts the parameter values of the COPRAS method under the IVPHFS context that are used for prioritization. It can be noted that the aggregated matrix has two instances of IVPHFEs for rating green suppliers based on a specific attribute. Values are determined under both biased and unbiased weights of attributes. The first part of the Q_k values is determined under biased weights, and the second part is determined under unbiased weights. The first row of T_k has two instances under a biased context, while the second row of T_k has two instances under unbiased context. A similar idea is followed by R_k as well. The procedure is followed for other green suppliers also. The instances are added respectively to form Q_k under both biased and unbiased weight contexts. Table 4 shows the Q_k values at $\eta = 0.5$. Prioritization order under biased weights is given by $g_1 \succ g_4 \succ g_3 \succ g_2$, and under unbiased weights is given by $g_1 \succ g_4 \succ g_2 \succ g_3$. Later, Step 5 provides details on the sensitivity analysis.

Step 5: Perform sensitivity analysis over attributes' weights and strategy values to understand their effect on prioritization order. From the analysis of attributes' weights, it is clear that prioritization order changes with the change of weight values. However, the top-ranked supplier remains unchanged. Similarly, strategy values are varied from 0.1 to 0.9 in equation (22), and their effect is realized from Fig. 2.

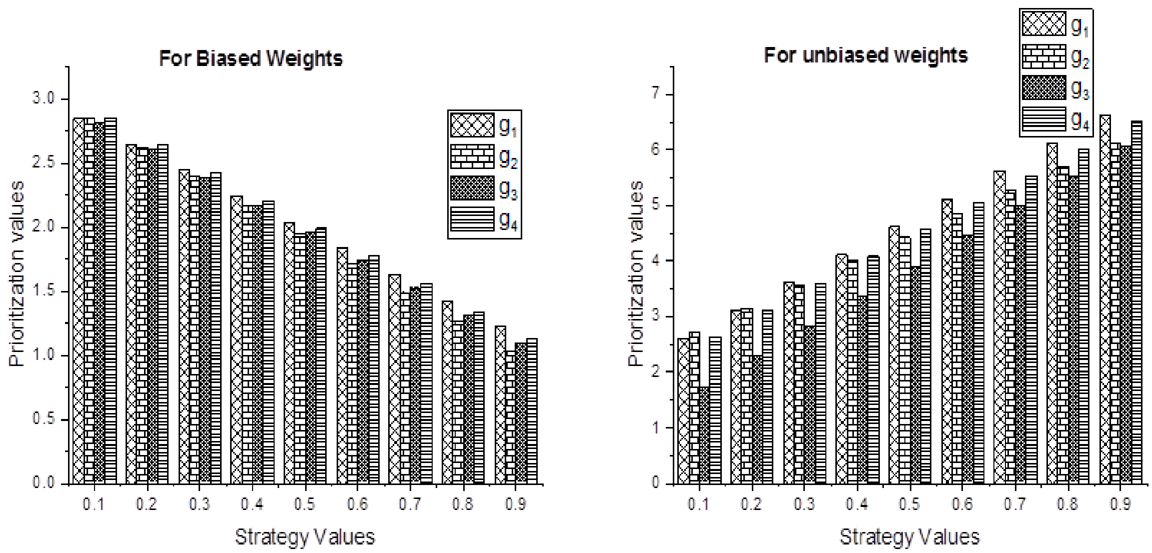

Fig.2 Sensitivity analysis of strategy values – (a) Biased weights and (b) Unbiased weights

From Fig.2, it is clear that there is a competition between suppliers g_1, g_2, and g_4 with unbiased attributes' weights and competition between suppliers g_2 and g_3 with biased attributes' weights, and the final prioritization order is given by $g_1 \succ g_4 \succ g_3 \succcurlyeq g_2$ for biased weights and $g_1 \succcurlyeq g_4 \succcurlyeq g_2 \succ g_3$ for unbiased weights. Finally, from the cumulative result of the sensitivity analysis of strategy values for both biased and unbiased attributes' weights, supplier g_1 is considered a suitable alternative from the set of suppliers. The proposed framework is robust even after adequate changes are made to the attributes' weights and strategy values (first and second ranking positions do not change).

5. Comparative Investigation of Frameworks

This section presents the comparative investigation of different frameworks under the IVPHFS and PHFS context to retain the homogeneity in the process. Frameworks proposed by Krishankumar et al. [19], Li and Wang [12], and Xu and Zhou [11] are considered for investigation.

Table 5 Prioritization values from proposed and state-of-the-art methods

Green suppliers	Methods			
	Proposed	[11]	[12]	[19]
g_1	1	2	1	1
g_2	3	3	4	2
g_3	2	4	3	3
g_4	4	1	2	4

Table 5 presents the prioritization order from different methods. Spearman correlation [45] is applied to determine the consistency of the proposed framework. From Fig.3, it is inferred that the proposed framework is highly consistent with its close counterpart method [19] and moderately consistent with the other two methods. The consistency is affected due to the information loss in methods [11] and [12], which is caused during the conversion from IVPHFS to PHFS.

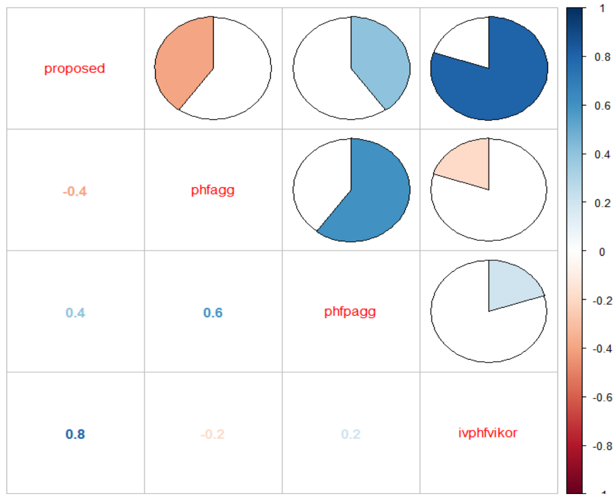

Fig.3 Corrplot for consistency analysis - Spearman Correlation

Table 6 Analysis of different factors - Proposed vs. Others

Factors	**Methods**			
	Proposed	[19]	[11]	[12]
Input	IVPHFS		PHFS	
Attributes' weight	Calculated by the programming model	Calculated by SV method	Not calculated. Directly provided	
DMs' weights	Calculated by ET-BA method	Not calculated. Directly provided		
Aggregation	GMSM operator	Simple weighted geometry operator	Weighted arithmetic/geometric operator	Prioritized weighted arithmetic/geometric operator
Interrelationship among attributes	Captured effectively	Not captured		
Capturing uncertainty	Done effectively with the help of IVPHFS information		Information loss occurs during the conversion process	
Prioritization method	COPRAS	VIKOR	Aggregation-based method	
Generalizability	Gained from interval probabilities		The specific case of IVPHFS	

Table 6 provides the investigation of different prioritization methods. From the analysis, we infer the following:

1. IVPHFS information is a generalization of PHFS, which associates interval-valued probability for each HFE. This preference style provides flexibility to the DMs during preference elicitation.

2. Unlike state-of-the methods, the proposed framework systematically calculates attributes' weights by effectively utilizing the partial information on each attribute.
3. Also, DMs' weights are systematically calculated by properly handling uncertainty.
4. Preferences are aggregated by effectively capturing the interrelationship among attributes. Systematically calculated DMs' weights are utilized for aggregation of preferences.
5. The proposed framework is consistent with other state-of-the-art methods. Moreover, suppliers are prioritized by properly considering the nature of attributes.

Some weaknesses of the framework are:

1. Manual elicitation of probability value is difficult, so a systematic calculation of probability values would enhance the flexibility of the preferred style.
2. Moreover, experts/DMs must be trained for effective elicitation of preferences.

Some managerial implications are provided below:

1. The proposed framework is ready-to-use with a flexible preference style.
2. Organizations can easily make decisions with effective mathematical support and can also revert to the decisions in the latter part of the process.
3. Human intervention is mitigated by systematical calculation of weights.
4. From the organizations' point of view, it is a supportive tool that could effectively contribute to decision-making pertaining to risk management, inventory management,

etc. On the other hand, from the customers' point of view, it can act as an aiding tool by providing suggestions related to purchasing.

6. Conclusion

In this paper, we aimed at proposing a new decision framework under the IVPHFS context with fully unknown weight information. The framework integrates novel methods for weight calculation of attributes and DMs, aggregation of preferences, and prioritization of alternatives. From the comparative study, it is evident that the proposed framework is *robust* even after adequate changes are made to the weights and strategy values and *consistent* with state-of-the-art methods. Sensitivity analysis (refer to Fig.2) and Spearman correlation (refer to Fig.3) is adopted to realize the robustness and consistency of the proposed framework. The close competition among alternatives g_1, g_2, and g_4 are brought out for efficient planning and backup management. Finally, it is inferred that IVPHFEs (proposed and Ref.[18]) are flexible and generalized values, which upon conversion to PHFEs (Refs. [9,10]) causes information loss, and it is reflected in Fig.3 by producing lower consistency values.

As a part of future work, the weaknesses mentioned above will be addressed. Also, the IVPHFS preference style will be used for developing new decision frameworks, which will address missing data during preference elicitation and provide sensible prioritization of alternatives. Finally, plans are made to adopt advanced variational theories [46–49] to decision-making with generalized preference structures and combine machine learning and recommender concepts for solving complex real-time problems.

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