

A sublinear time quantum algorithm for s - t minimum cut on dense simple graphs

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Abstract

An s - t minimum cut in a graph corresponds to a minimum weight subset of edges whose removal disconnects vertices s and t . Finding such a cut is a classic problem that is dual to that of finding a maximum flow from s to t . In this work we describe a quantum algorithm for the minimum s - t cut problem on undirected graphs. For an undirected graph with n vertices, m edges, and integral edge weights bounded by W , the algorithm computes with high probability the weight of a minimum s - t cut in time $\tilde{O}(\sqrt{mn}^{5/6}W^{1/3} + n^{5/3}W^{2/3})$, given adjacency list access to G . For simple graphs this bound is always $\tilde{O}(n^{11/6})$, even in the dense case when $m = \Omega(n^2)$. In contrast, a randomized algorithm must make $\Omega(m)$ queries to the adjacency list of a simple graph G even to decide whether s and t are connected.

1 Introduction

Let G be a graph with n vertices and m edges, and let s, t be two vertices of G . The problem of determining the maximum amount of flow $\lambda_{st}(G)$ that can be routed from s to t while respecting the capacity constraints of G is one of the most fundamental problems in theoretical computer science, whose study goes back at least to the 1950s and the pioneering work of Ford and Fulkerson [FF56]. By the max-flow min-cut theorem, $\lambda_{st}(G)$ is equal to the minimum weight of a cut separating s and t in G . From a maximum s - t flow one can compute a minimum s - t cut in linear time, but no such reduction is known the other way around. The Goldberg-Rao [GR98] algorithm, with running time $O(\min\{\sqrt{m}, n^{2/3}\}m \log(n) \log(W))$, where W is the largest weight of an edge, stood as the best bound for both problems for many years. In the past decade, however, beginning with work of Christiano et al. [CKM⁺11] there has been tremendous progress in max-flow algorithms by incorporating techniques from continuous optimization [She13, Mađ13, KLOS14, Pen16, LS14, LS20, GLP21]. With a recent $\tilde{O}((m + n^{3/2}) \log W)$ time max-flow algorithm by Brand et al. [vdBLL⁺21], there are now near-linear time randomized max-flow algorithms for dense graphs with polynomial weights.

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Given its fundamental nature, there has been a surprising lack of work on quantum algorithms for the exact maximum flow or minimum s - t cut problem. As far as we are aware, the only work on the quantum complexity of these problems is by Ambainis and Špalek [AŠ06], who gave a quantum algorithm for max flow in a directed graph with integral capacities bounded by $W \leq n^{1/4}$ with running time $\tilde{O}(\min\{n^{7/6}\sqrt{m}W^{1/3}, m\sqrt{n}W\})$, given adjacency list access to G . This bound is completely subsumed by current classical randomized algorithms.

More recent work of Apers and de Wolf [AdW20] gives a $\tilde{O}(\sqrt{mn}/\varepsilon)$ time quantum algorithm for finding a $(1 + \varepsilon)$ -approximation of the minimum s - t cut.¹ This algorithm works by first constructing an ε -cut sparsifier H of the input graph G with a quantum algorithm in time $\tilde{O}(\sqrt{mn}/\varepsilon)$. An ε -cut sparsifier is a re-weighted subgraph of G that has only $\tilde{O}(n/\varepsilon^2)$ edges but for which the weight of every cut agrees with that of G up to a factor of $1 \pm \varepsilon$. One can then run a classical randomized s - t minimum cut algorithm on the sparse graph H to find an approximate s - t minimum cut of G . The quantum sparsification algorithm also plays a key role in our work.

In this paper, we give a quantum algorithm that computes $\lambda_{st}(G)$ on dense simple graphs faster than is possible classically.² More generally, for an undirected and integral weighted graph G with maximum weight W , we give a quantum algorithm with adjacency list access to G that with high probability outputs $\lambda_{st}(G)$ in time $\tilde{O}(\sqrt{mn}^{5/6}W^{1/3} + n^{5/3}W^{2/3})$ (see Theorem 15). Thus for unweighted graphs the running time is $\tilde{O}(n^{11/6})$ even for dense instances with $m \in \Omega(n^2)$ edges. On the other hand, it is easy to show that in the worst case a randomized algorithm requires $\Omega(m)$ queries to the adjacency list of a graph with m edges even to decide whether s and t are connected (see Theorem 20). Our algorithm also works with adjacency matrix access to G , in which case its running time becomes $\tilde{O}(n^{11/6}W^{1/3})$ (see Corollary 16). In addition to $\lambda_{st}(G)$, our algorithm can output the edges of a minimum s - t cut and the corresponding bipartition of the vertex set. It does not however compute a maximum s - t flow, and finding a sublinear quantum algorithm for this problem remains a tantalizing open question.

Our quantum algorithm follows an algorithm of Rubinfeld, Schramm and Weinberg (RSW) [RSW18], which computes a minimum s - t cut in the *cut query* model. In the cut query model one can query a subset of vertices S and receives as an answer the total weight of edges with exactly one endpoint in S . While designed for cut query complexity, we show the algorithm also has a surprisingly time-efficient implementation that is well-suited for quantum speedups. The crux of the algorithm is a procedure to transform the input graph G into a sparser graph G' while preserving minimum s - t cuts. We use two quantum tools to speed up this procedure. The first is to use the quantum algorithm from [AdW20] to find an ε -cut sparsifier of G faster than a classical sparsification algorithm. The second is to use Grover's algorithm to learn all the edges in the sparser graph G' . Once we explicitly know the graph G' we can use a classical randomized algorithm to compute a minimum s - t cut in G' . This blueprint is similar to the global min cut algorithm from [AL21], which is also based on a cut query algorithm by [RSW18]. There as well

¹The bound quoted in [AdW20, Claim 9] for finding a $(1 + \varepsilon)$ -approximate minimum s - t cut is $\tilde{O}(\sqrt{mn}/\varepsilon + n/\varepsilon^5)$. However, plugging into the proof the improved randomized approximate s - t minimum cut algorithm of [ST18] with running time $\tilde{O}(m + \sqrt{mn}/\varepsilon)$ improves this to $\tilde{O}(\sqrt{mn}/\varepsilon)$.

²A simple graph is an undirected and unweighted graph with no self loops and at most one edge between any pair of vertices.

a cut sparsifier of G is used to identify edges that can be contracted while preserving (non-trivial) global minimum cuts, and then Grover search is applied to learn the edges in the sparser contracted graph.

The contracted graph in the RSW procedure is obtained by contracting edges that cannot participate in any minimum s - t cut. To get an intuition for the idea, suppose that G is a simple graph and first imagine that we compute a maximum s - t flow F in G . If we let G_F be the graph whose edge weights are those of G minus the flow F , then s and t become disconnected in G_F —otherwise we could route more flow from s to t . Contracting all connected components of G_F then results in a sparser graph while preserving any edge that is part of a minimum s - t cut. This routine is also how one can compute a minimum s - t cut from a maximum s - t flow: the connected component of s in G_F gives one side of a vertex bipartition corresponding to a minimum s - t cut.

Of course we did not save anything in this example as we had to compute a maximum s - t flow in G . What RSW actually do is to first obtain an ε -cut sparsifier H of G . This is where we use the $\tilde{O}(\sqrt{mn}/\varepsilon)$ time quantum sparsification algorithm from [AdW20]. As H has only $\tilde{O}(n/\varepsilon^2)$ edges it is much cheaper to carry out the above plan on H instead: we compute a maximum flow F in H and consider the graph H_F whose weights are those of H minus the flow F . As cuts of H only approximate those in G we cannot fully contract the connected components in H_F and hope to preserve all minimum s - t cut of G . However, a key insight of RSW is that we can still safely contract induced subgraphs of H_F that are highly connected. Doing so results in the desired contraction G' of G that is relatively sparse—one can argue it has only $O(\varepsilon n^2)$ edges—yet preserves all minimum s - t cuts. We can learn the $O(\varepsilon n^2)$ edges of G' among the m edges of G using Grover search in time $\tilde{O}(n\sqrt{m\varepsilon})$. We then again run a randomized max flow algorithm on the now explicitly known G' to compute $\lambda_{st}(G)$. Balancing the terms \sqrt{mn}/ε from the sparsifier computation and $n\sqrt{m\varepsilon}$ for learning the edges of G' leads to the choice $\varepsilon = n^{-1/3}$, and a $\tilde{O}(\sqrt{mn}^{5/6})$ running time for the quantum steps of the algorithm. With this choice of ε the classical parts of the algorithm can be performed in near-linear time in the sizes of H and G' , thus in time $\tilde{O}(n/\varepsilon^2 + \varepsilon n^2) = \tilde{O}(n^{5/3})$. This gives the overall running time of $\tilde{O}(\sqrt{mn}^{5/6} + n^{5/3}) \in \tilde{O}(n^{11/6})$ in the unweighted case.

2 Preliminaries

2.1 Graph notation and background

Let V be a finite set and $V^{(2)}$ the set of all subsets of V of cardinality 2. We represent a weighted undirected graph as a pair $G = (V, w)$ where $w : V^{(2)} \rightarrow \mathbb{R}$ is a non-negative function. We let $V(G)$ be the vertex set of G and $E(G) = \{e \in V^{(2)} : w(e) > 0\}$ be the set of edges of G . We extend the weight function to sets $S \subseteq V^{(2)}$ by $w(S) = \sum_{e \in S} w(e)$. We say that G is *simple* if $w : V^{(2)} \rightarrow \{0, 1\}$ and in this case also denote G as $G = (V, E)$, where E is the set of edges. For a subset $X \subseteq V$ we write $G[X]$ for the induced subgraph on X .

For a subset $X \subseteq V$ we use the shorthand $\bar{X} = V \setminus X$, and we say X is *non-trivial* if $\emptyset \neq X \subsetneq V$. For disjoint sets $X, Y \subseteq V$ we use $E(X, Y)$ for the set of edges with one endpoint in X and one endpoint in Y . For a non-trivial set X , let $\Delta_G(X) = \{\{i, j\} \in E(G) : i \in X, j \in \bar{X}\}$

be the set of edges of G with one endpoint in X and one endpoint in \overline{X} . A *cut* of G is a set of the form $\Delta_G(X)$ for some non-trivial set X . We call X and \overline{X} the *shores* of the cut $\Delta_G(X)$. For two distinguished vertices $s, t \in V$ we call $\Delta_G(X)$ an s - t cut if $s \in X, t \notin X$. When we speak of the shore of an s - t cut we always refer to the shore containing s . We let $\lambda_{st}(G)$ be the minimum weight of an s - t cut of G .

By the famous max-flow min-cut theorem [FF56], $\lambda_{st}(G)$ is equal to the maximum amount of flow that can be routed from s to t in G . While all the graphs G in this paper will be undirected, for flows it is most natural to think of an undirected graph G as a directed graph with each edge directed in both directions. Without loss of generality an s - t flow F in G can be defined to use an edge in at most one direction. For $e \in E(G)$ it will be convenient for us to let $F(e)$ be the flow along edge e in the direction it is used by the flow F .

We will need the following result about the total weight of a flow from Rubinstein, Schramm, and Weinberg [RSW18].

Lemma 1 ([RSW18, Lemma 5.4]). *Let $G = (V, w)$ be a graph with integral weights from $[0, W]$ and let $s, t \in V$. Let F be a non-circular s - t flow of value f in G . Then the total weight of flow $\sum_{e \in E(G)} F(e) \leq 10 \cdot n \sqrt{fW}$.*

Finally, we will also need a very recent and very fast randomized algorithm for computing max flows due to van den Brand, Lee, Liu, Saranurak, Sidford, Song and Wang [vdBLL⁺21].

Theorem 2 ([vdBLL⁺21, Corollary 7.9]). *Let G be a directed graph with n vertices, m edges, and integral edge weights bounded by W . There is a randomized algorithm that with high probability computes a maximum s - t flow in G in time $\tilde{O}((m + n^{3/2}) \log W)$.*

2.2 Quantum models and background

We will work with the two most standard quantum models for graph algorithms, the adjacency list and adjacency matrix models. We refer the reader to [AL21, Section 2.3] for definitions of these models.

Our quantum algorithm can be viewed as a classical algorithm with two quantum primitives. The first is a quantum algorithm to compute a sparsifier [AdW20] described in the next section, and the second is the following application of Grover's algorithm.

Theorem 3 (cf. [AL21, Theorem 13]). *Given $t, N \in \mathbb{N}$ with $1 \leq t \leq N$ and oracle access to $x \in \{0, 1\}^N$, there is a quantum algorithm such that*

- *if $|x| \leq t$ then the algorithm outputs x with certainty, and*
- *if $|x| > t$ then the algorithm reports so with probability at least $9/10$.*

The algorithm makes $O(\sqrt{tN})$ queries to x and has time complexity $O(\sqrt{tN} \log(N))$.

2.3 Sparsifiers and strong components

A key component of our algorithm will be a cut sparsifier of a graph.

Definition 4 (cut sparsifier). Let $G = (V, w_G)$ be a weighted graph. An ε -cut sparsifier $H = (V, w_H)$ of G is a reweighted subgraph of G , i.e. $w_H(e) > 0$ only if $w_G(e) > 0$, satisfying

$$(1 - \varepsilon)w_G(\Delta_G(X)) \leq w_H(\Delta_H(X)) \leq (1 + \varepsilon)w_G(\Delta_G(X))$$

for every $X \subseteq V$.

We will use a quantum algorithm to construct a cut sparsifier of a graph by Apers and de Wolf [AdW20].

Theorem 5 ([AdW20, Theorem 1]). *Let G be a weighted and undirected graph with n vertices and m edges. There exists a quantum algorithm that outputs with high probability the explicit description of an ε -cut sparsifier H of G with $\tilde{O}(n/\varepsilon^2)$ edges in time $\tilde{O}(\sqrt{mn}/\varepsilon)$ in the adjacency list model and time $\tilde{O}(n^{3/2}/\varepsilon)$ in the adjacency matrix model. Moreover, if G has integral weights then so does H , and if the largest weight of an edge of G is τ then the largest weight of an edge of H is $\tilde{O}(\tau\varepsilon^2n)$.*

Proof. The first part of the statement is directly from Theorem 1 of [AdW20]. The “moreover” part follows from an examination of Algorithm 1 of [AdW20]. There it can be seen that when an edge is added to the sparsifier it is always done so with a power of 4 times its original weight. The power of 4 is at most the number of times Algorithm 1 is run, which depends on the error parameter ε . To achieve an ε -cut sparsifier Algorithm 1 is run $k = \log(n\varepsilon^2)/2 + O(\log \log n)$ times, thus an edge of G is placed in the sparsifier with weight at most its original weight times $4^k = \tilde{O}(\varepsilon^2n)$. \square

Apers and de Wolf actually show how to construct a *spectral sparsifier*. However, we will not need the stronger properties of a spectral sparsifier.

We will also need some definitions and results relating to the minimum weight of cuts in induced subgraphs of G .

Definition 6 (k -strong component). A graph G is k -connected if the weight of each cut in G is at least k . A k -strong component of G is a maximal k -connected vertex induced subgraph of G . Individual vertices are defined to be ∞ -strong components.

Definition 7 (edge strength). The strength of an edge e , denoted k_e is the maximum value of k such that a k -strong component contains both endpoints of e . We say that e is k -strong if its strength is k or more, and k -weak otherwise.

In our s - t mincut algorithm we would like to efficiently partition the vertices of G into k -strong components. One way to do this would be to compute the strength of every edge, and then find the connected components of the graph amongst edges whose strength is at least k . However, Benczúr and Karger remark that even quickly approximating the edge strength of all edges is an open question [BK15]. For the application of giving a randomized algorithm to construct a cut sparsifier they come up with a way of estimating edge strengths which will still suffice for our purpose.

Theorem 8 (Theorem 6.5 [BK15]). *Let $G = (V, w)$ be a weighted graph with n vertices and m edges. There is a deterministic algorithm that in time $O(m \log^3(n))$ can produce edge strength estimates \tilde{k}_e such that*

1. $\tilde{k}_e \leq k_e$ for all $e \in E(G)$ and
2. $\sum_{e \in E(G)} \frac{w(e)}{\tilde{k}_e} = O(n)$.

From this theorem we can obtain the following corollary.

Definition 9 (k -strong partition). For a graph $G = (V, w)$, a k -strong partition of G is a partition $\mathcal{P} = \{S_1, \dots, S_t\}$ of V such that

1. The subgraph $G[S_i]$ is k -connected for each $i = 1, \dots, t$.
2. $\sum_{\{i,j\} \in [t]^{(2)}} w(S_i, S_j) = O(kn)$.

Corollary 10. *Let $G = (V, w)$ be a weighted graph with n vertices and m edges. Given k there is a deterministic algorithm that in time $O(m \log^3(n))$ can produce a k -strong partition of G .*

Proof. Let \tilde{k}_e be the estimates of edge strengths obtained from Theorem 8. Let G' be the graph obtained from G by deleting all edges e with $\tilde{k}_e < k$. Using depth first search we can find all connected components S_1, \dots, S_t of G' in time $O(m+n)$. We claim that each $G[S_i]$ is k -connected. If not then let $(A, S_i \setminus A)$ be the shores of a cut in $G[S_i]$ of weight less than k . Any edge e in this cut has $\tilde{k}_e \leq k_e < k$ and the cut disconnects $G[S_i]$. This contradicts the fact that vertices in A and $S_i \setminus A$ are connected in G' .

Now we show item (2). Any edge e crossing sets of the partition has $\tilde{k}_e < k$ thus it suffices to bound the weight of such edges. We have

$$\sum_{e: \tilde{k}_e < k} w(e) \leq k \sum_{e: \tilde{k}_e < k} \frac{w(e)}{\tilde{k}_e} = O(kn) ,$$

by item (2) of Theorem 8. □

3 Main algorithm

Our algorithm is based on the following algorithm by Rubinfeld, Schramm, and Weinberg [RSW18, Algorithm 5.1], who used it to give a randomized cut query algorithm for the minimum s - t cut problem.

Algorithm 1 Algorithm for s - t -mincut on a weighted graph G

Input: Oracle access to a weighted graph $G = (V, w_G)$, vertices $s, t \in V$, and a parameter $0 < \varepsilon < 1/3$.

Output: The shore of a minimum s - t cut in G and the value $\lambda_{st}(G)$.

- 1: Compute an ε -cut sparsifier H of G .
 - 2: Compute a maximum s - t flow F in H and subtract F from H . Denote the result as H' .
 - 3: Compute a $3\varepsilon n$ -strong partition $\mathcal{P} = \{A_1, \dots, A_t\}$ of H' .
 - 4: Let G' be the graph formed from G by contracting the vertices in each A_i , and let $a, b \in \mathcal{P}$ be the sets containing s, t respectively. Compute a minimum a - b cut $\Delta_{G'}(X')$ in G' and return $\lambda_{ab}(G')$ and the shore $X = \cup_{A \in X'} A$.
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For completeness we show correctness of the template, largely following the discussion of [RSW18]. In the next section we analyze the running time for a quantum algorithm that implements this algorithm with adjacency list access to G .

Theorem 11. *If every step of Algorithm 1 is performed correctly then the algorithm returns a minimum s - t cut of G .*

We first prove two claims from which the proof of Theorem 11 will easily follow.

Claim 12. *Let $G = (V, w_G)$ be a weighted graph and $s, t \in V$. For $0 < \varepsilon < 1/3$, let $H = (V, w_H)$ be an ε -cut sparsifier of G and let F be a maximum s - t flow in H . Form the graph $H' = (V, w_{H'})$ where $w_{H'}(e) = w_H(e) - F(e)$ for all $e \in V^{(2)}$. Let \mathcal{P} be a $3\varepsilon n$ -strong partition of H' . Then for any minimum s - t cut $\Delta_G(X)$ of G and all $A \in \mathcal{P}$ it holds that either $A \subseteq X$ or $A \subseteq \overline{X}$.*

Proof. Let $f_G = \lambda_{st}(G)$ and $f_H = \lambda_{st}(H)$. As H is an ε -cut sparsifier of G we have $f_G \leq f_H/(1 - \varepsilon)$. Let $\Delta_G(X)$ be a minimum s - t cut of G , i.e. such that $s \in X, t \notin X$ and $w_G(\Delta_G(X)) = f_G$. Then $w_H(X) \leq (1 + \varepsilon)f_G \leq \frac{1+\varepsilon}{1-\varepsilon}f_H$. Further, we have $w_H(X) = w_{H'}(X) + w_F(X)$ by definition of H' and F . We must have $w_F(X) \geq f_H$ since there is a flow of value f_H from s to t in H . Thus $w_H(X) \geq f_H + w_{H'}(X)$. Comparing inequalities we have $w_{H'}(X) \leq \frac{2\varepsilon}{1-\varepsilon}f_H < 3\varepsilon n$, as $\varepsilon < 1/3$.

Now let $A \in \mathcal{P}$ and let $B = A \cap X$. If B is nontrivial, i.e. $\emptyset \neq B \subsetneq A$, then there is a cut of $H'[A]$ of weight less than $3\varepsilon n$, a contradiction to the assumption that \mathcal{P} is a $3\varepsilon n$ -strong partition of H' . Thus B must be trivial and either $A \subseteq X$ or $A \subseteq \overline{X}$. \square

Claim 13. *Let G, H, H' and ε be as in Claim 12. Let $\mathcal{P} = \{A_1, \dots, A_t\}$ be a $3\varepsilon n$ -strong partition of H' and G' be the graph formed from G by contracting vertices in the same set of \mathcal{P} . Let $a, b \in \mathcal{P}$ be the sets containing s and t respectively. Then $\lambda_{st}(G) = \lambda_{ab}(G')$. Moreover, if X' is the shore of a minimum a - b cut of G' then $X = \cup_{A \in X'} A$ is the shore of a minimum s - t cut of G .*

Proof. Let $G = (V, w_G)$ and note that by the definition of G' we have $G' = (\mathcal{P}, w)$ where $w(A_i, A_j) = w_G(E(A_i, A_j))$ for $i \neq j$.

Let us show that $\lambda_{st}(G') \leq \lambda_{st}(G)$. Let $\Delta_G(X)$ be a minimum s - t cut in G . For every $A_i \in \mathcal{P}$ we either have $A_i \subseteq X$ or $A_i \subseteq \overline{X}$ by Claim 12. Note that in particular this means $a \subseteq X$ and $b \cap X = \emptyset$. From X we define a set X' where for every $A_i \in \mathcal{P}$ we put A_i in X' iff $A_i \subseteq X$. Note that $\Delta_{G'}(X')$ is an a - b cut of G' and $w_G(\Delta_G(X)) = w(\Delta_{G'}(X'))$, thus $\lambda_{ab}(G') \leq \lambda_{st}(G)$.

We next show the general fact that contraction cannot decrease the minimum s - t cut value. For any set $X' \subseteq \mathcal{P}$ we can define a set $X \subseteq V$ by $X = \cup_{A \in X'} A$. Further $w(\Delta_{G'}(X')) = w_G(\Delta_G(X))$ by the definition of w . Thus $\lambda_{ab}(G') \geq \lambda_{st}(G)$.

This establishes $\lambda_{st}(G) = \lambda_{ab}(G')$. To see the “moreover” part of the claim, let X' be a minimum a - b cut of G' . We have just shown that $w(\Delta_{G'}(X')) = w_G(\Delta_G(X))$ for $X = \cup_{A \in X'} A$, thus X will be a minimum s - t cut of G . \square

Proof of Theorem 11. If the steps of the algorithm are implemented correctly then H, H' satisfy the hypotheses of Claim 12. We can therefore invoke Claim 12 and Claim 13 to conclude that $\lambda_{st}(G) = \lambda_{ab}(G')$ and that X will be the shore of a minimum s - t cut in G . \square

3.1 Implementation by a quantum algorithm

We now discuss the implementation of Theorem 15 by a quantum algorithm with adjacency list access to G . For the running time it is important to have an upper bound on the number of edges in the contracted graph G' formed in step 4 of Algorithm 1. We do this by means of the following claim.

Claim 14. *Let G, H, H' and ε be as in Claim 12, with the following additional conditions:*

1. *The largest edge weight of G is $W \geq 1$.*
2. *H has integral weights and the largest edge weight is $W_H \in O(\varepsilon^2 n W)$.*

Let $\mathcal{P} = \{A_1, \dots, A_t\}$ be a $3\varepsilon n$ -strong partition of H' . Then $\sum_{i=1}^t w_G(\Delta_G(A_i)) = O(\varepsilon n^2 W)$.

Proof. As H is an ε -cut sparsifier of G we have

$$\begin{aligned} w_G(\Delta_G(A_i)) &\leq \frac{w_H(\Delta_H(A_i))}{1 - \varepsilon} \\ &= \frac{w_{H'}(\Delta_H(A_i)) + w_F(\Delta_H(A_i))}{1 - \varepsilon} . \end{aligned}$$

By the definition of a $3\varepsilon n$ -strong partition (Definition 9), we have $\sum_{i=1}^t w_{H'}(\Delta_{H'}(A_i)) = O(\varepsilon n^2)$. Note also that $w_{H'}(\Delta_{H'}(A_i)) = w_{H'}(\Delta_H(A_i))$ because $e \in E(H')$ implies $e \in E(H)$, and so $\sum_{i=1}^t w_{H'}(\Delta_H(A_i)) = O(\varepsilon n^2)$. By Lemma 1 we also have that

$$\begin{aligned} \sum_{i=1}^t w_F(\Delta_H(A_i)) &\leq 20n \sqrt{\lambda_{st}(H) W_H} \\ &= O(\varepsilon n^2 W) , \end{aligned}$$

using that $\lambda_{st}(H) \leq (1 + \varepsilon)nW$ and $W_H \in O(\varepsilon^2 n W)$. As $W \geq 1$ this term dominates and $\sum_{i=1}^t w_G(\Delta_G(A_i)) = O(\varepsilon n^2 W)$. \square

Theorem 15. *Let G be a graph with n vertices and m edges where all edge weights are integral and the largest weight of an edge is W . Given adjacency list access to G , there is a quantum algorithm that with high probability computes $\lambda_{st}(G)$ and the shore of a minimum s - t cut of G with running time $\tilde{O}(\sqrt{mn}^{5/6}W^{1/3} + n^{5/3}W^{2/3})$ and number of queries $\tilde{O}(\sqrt{mn}^{5/6}W^{1/3})$.*

Proof. We follow the algorithm given in Algorithm 1 with the choice $\varepsilon = (nW)^{-1/3}$. As the choice of ε depends on W , we first need to know what the maximum weight of an edge of G is. We can do this by the quantum minimum finding routine of Dürr and Høyer [DH96] with high probability in time $\tilde{O}(\sqrt{m})$. With that settled, we now go over each step of the algorithm to explain how it is implemented and its running time.

Line 1 We run quantum algorithm of Apers and de Wolf [AdW20] given in Theorem 5 to find an ε -cut sparsifier H of G with $\tilde{O}(n/\varepsilon^2)$ edges. As all edge weights of G are integral and the largest weight of an edge is W , by the “moreover” part of this theorem the edge weights of H are integral and the largest weight of an edge of H is $O(\varepsilon^2 nW)$. This step succeeds with high probability and takes time $\tilde{O}(\sqrt{mn}/\varepsilon)$.

Line 2 For this step we run the classical maximum s - t flow algorithm of [vdBLL⁺21] quoted in Theorem 2 to compute a max flow F in H . This algorithm requires a graph with integral weights, which is satisfied by H . This step again succeeds with high probability, and runs in time $\tilde{O}((n^{3/2} + n/\varepsilon^2) \log(\varepsilon^2 nW)) = \tilde{O}(n^{5/3}W^{2/3})$ by the choice of ε . As $\lambda_{st}(H) \leq (1 + \varepsilon)nW$, the total weight of edges in F is $O(\varepsilon n^2 W)$ by Lemma 1. We can compute $H' = H - F$ in time $\tilde{O}(n/\varepsilon^2)$ as this is the number of edges in H .

Line 3 To find a $3\varepsilon n$ -strong partition of H' we use the deterministic algorithm from Corollary 10. This step runs in near-linear time in the number of edges of H' , thus in time $\tilde{O}(n/\varepsilon^2)$.

Line 4 Let $\mathcal{P} = \{A_1, \dots, A_t\}$ be the $3\varepsilon n$ -strong partition of H' computed in the previous step. The graph G' is formed by contracting vertices in the same set of this partition. By Claim 14 the graph G' has $O(\varepsilon n^2 W)$ edges, assuming all previous steps have succeeded. Consider the concatenation of all the lists in the adjacency list representation of G . This gives a vector of dimension $2m$ and defines an ordering of edges of G where every edge appears twice. Define a string $x \in \{0, 1\}^{2m}$ using the same ordering where $x(e) = 1$ if the endpoints of e are in distinct sets of the partition \mathcal{P} and $x(e) = 0$ otherwise. A query to a bit of x can be answered with a single query to the adjacency list of G . By Theorem 3 via Grover search in time $\tilde{O}(n\sqrt{m\varepsilon W})$ with high probability we can determine if x has at most $O(\varepsilon n^2 W)$ ones, and if so learn the positions of all these ones. If the number of ones in x is larger than $O(\varepsilon n^2 W)$ then we abort. With high probability we do not abort and once we learn the positions of all the ones in x we can then classically learn the weight of the corresponding edges in G with $O(\varepsilon n^2 W)$ more queries. Then we have explicitly learned the graph G' .

Once we have an explicit description of G' we can run the max flow algorithm of [vdBLL⁺21] to compute a maximum a - b flow F' in G' where $a, b \in \mathcal{P}$ are the sets of the partition containing s, t

respectively. This takes time $\tilde{O}(n^{3/2} + \varepsilon n^2 W) = \tilde{O}(n^{5/3} W^{2/3})$ and succeeds with high probability. The flow of F' gives $\lambda_{ab}(G')$. To also compute the shore of a minimum a - b cut of G' we consider the residual graph of the flow F' and compute the connected component containing a . This can be done deterministically by breadth-first search in time $O(\varepsilon n^2 W)$.

Correctness and total running time Each step individually succeeds with high probability and so with high probability all steps will be correct. We can thus invoke Theorem 11 to conclude that the algorithm is correct with high probability.

To analyze the running time note that the quantum parts of the algorithm give terms with \sqrt{mn}/ε and $n\sqrt{m\varepsilon W}$, and the classical steps have complexities $n^{3/2}$, n/ε^2 and $\varepsilon n^2 W$. The choice $\varepsilon = (nW)^{-1/3}$ makes the quantum terms equal to $n^{5/6} W^{1/3}$, makes the $n^{3/2}$ term low order, and makes the remaining classical terms equal to $n^{5/3} W^{2/3}$. Thus overall the running time is $\tilde{O}(\sqrt{mn} n^{5/6} W^{1/3} + n^{5/3} W^{2/3})$. Queries to G are only made in the quantum steps as all classical steps work on graphs explicitly constructed by the algorithm. Thus the total number of queries is $\tilde{O}(\sqrt{mn} n^{5/6} W^{1/3})$. \square

We have the following result for the adjacency matrix model.

Corollary 16. *Let G be a graph with n vertices and m edges where all edge weights are integral and the largest weight of an edge is W . Given oracle access to the adjacency matrix of G , there is a quantum algorithm that computes $\lambda_{st}(G)$ and the shore of a minimum s - t cut of G with high probability in time $\tilde{O}(n^{11/6} W^{1/3})$.*

Proof. As in the proof of Theorem 15 we first compute the maximum weight of an edge of G . This can be done by the quantum minimum finding routine of Dürr and Høyer [DH96] with high probability in time $\tilde{O}(n)$. If $W \geq \sqrt{n}$ then we simply run the randomized max-flow algorithm of [vdBLL⁺21] from Theorem 2 with running time $\tilde{O}(n^2 \log(W))$.

If $W < \sqrt{n}$ then we follow from the proof of Theorem 15. Only minor modifications are needed in the lines 1 and 4 where we require query access to the graph. In particular, we again use Algorithm 1 with the same value $\varepsilon = (nW)^{-1/3}$.

In Line 1 we run the adjacency matrix version of the sparsifier algorithm from [AdW20] (quoted in Theorem 5), which takes time $\tilde{O}(n^{3/2}/\varepsilon) = \tilde{O}(n^{11/6} W^{1/3})$. In Line 4 we find the $O(\varepsilon n^2 W)$ edges of G' by applying Theorem 3 over the $O(n^2)$ entries of the adjacency matrix, rather than the $O(m)$ entries of the adjacency list. This requires time $\tilde{O}(n^2 \sqrt{\varepsilon W}) = \tilde{O}(n^{11/6} W^{1/3})$.

The remaining steps are performed classically on graphs explicitly constructed by the algorithm and work in the same time $\tilde{O}(n^{5/3} W^{2/3})$ as before. As $W < \sqrt{n}$ in this case this term is dominated by the $\tilde{O}(n^{11/6} W^{1/3})$ term.

Finally, for $W \geq \sqrt{n}$ we have $n^2 \log(W) = \tilde{O}(n^{11/6} W^{1/3})$ thus we can say that in all cases the running time of the algorithm is $\tilde{O}(n^{11/6} W^{1/3})$. \square

Remark 17. *After finding the shore of a minimum s - t cut via Theorem 15 or Corollary 16, one can also learn the edges in the cut by a quantum algorithm in the same asymptotic running time. If the maximum weight of an edge is W then in a graph with integral weights the number of edges*

in a minimum s - t cut is at most nW . We can learn these $O(nW)$ edges by Grover's algorithm (Theorem 3) in time $\tilde{O}(\sqrt{mnW})$ in the adjacency list model or time $\tilde{O}(n^{3/2}\sqrt{W})$ in the adjacency matrix model. In both cases this is low order to the $\tilde{O}(n^{5/3}W^{2/3})$ term.

4 Randomized lower bound

In this section we show that a randomized algorithm that computes $\lambda_{st}(G)$ with success probability at least $9/10$ on simple graphs G with m edges must make $\Omega(m)$ queries to the adjacency list of G in the worst case. This holds true even for just determining if s and t are connected in G , which we call the USTCON problem.

Definition 18 (USTCON). In the USTCON problem one is given adjacency list access to an undirected graph $G = (V, E)$ and two distinguished vertices $s, t \in V$. The problem is to determine if s and t are connected in G .

To show a lower bound on USTCON we will use the classical adversary method for randomized query complexity [Aar06, LM08, AKP⁺21], a randomized analog of the quantum adversary method [Amb02]. We will just need a simple unweighted version of the method, adapted from [Aar06, Theorem 4.3].

Lemma 19 (cf. [Aar06, Theorem 4.3]). *For a finite set Σ and $S \subseteq \Sigma^n$, let $f : S \rightarrow \{0, 1\}$. Let $X \subseteq f^{-1}(0)$ and $Y \subseteq f^{-1}(1)$. Let $R \subseteq X \times Y$ be such that for every $x \in X$ there are at least t different $y \in Y$ such that $(x, y) \in R$ and for every $x \in X$ and $k \in [n]$ there are at most ℓ different $y \in Y$ such that $(x, y) \in R$ and $x_k \neq y_k$. Then any randomized algorithm that computes f with success probability $9/10$ must make $\Omega(t/\ell)$ queries.*

Theorem 20. *A randomized algorithm that solves USTCON with success probability at least $9/10$ on graphs with m edges requires $\Omega(m)$ queries.*

Proof. First we show the lower bound in the dense case where $m = \Omega(n^2)$ using Lemma 19. We construct sets of negative instances X and positive instances Y . Suppose that n is even. Excluding s and t we partition the remaining $n - 2$ vertices into two sets A and B each of size $(n - 2)/2$. In all instances, s will be connected to all vertices in A and t will be connected to all vertices in B . X will consist of a single negative instance G where we additionally put a clique on the vertices in A and a clique on the vertices in B and no further edges. Thus s and t will not be connected in G . Note that apart from s and t , every vertex has degree $(n - 2)/2$ in G .

For Y we create a family of $2\binom{n-2}{2}$ positive instances. A positive instance will be labeled by $a = \{a_1, a_2\} \in A^{(2)}$, $b = \{b_1, b_2\} \in B^{(2)}$ and a bit $c \in \{0, 1\}$. Associated to a, b, c we construct a graph $G_{a,b,c}$ as follows. Take the graph G and remove the edges a and b . If $c = 0$ then add edges $\{\min\{a_1, a_2\}, \min\{b_1, b_2\}\}$ and $\{\max\{a_1, a_2\}, \max\{b_1, b_2\}\}$ to form $G_{a,b,c}$. Otherwise if $c = 1$ then add edges $\{\min\{a_1, a_2\}, \max\{b_1, b_2\}\}$ and $\{\max\{a_1, a_2\}, \min\{b_1, b_2\}\}$ to form $G_{a,b,c}$. In both cases all vertices in $A \cup B$ have degree $(n - 2)/2$ in $G_{a,b,c}$ and there will be two edges connecting A and B so $\lambda_{st}(G_{a,b,c}) = 2$. This completes the description of the instances.

The degree sequences of all instances are the same, thus we may assume this is known to the algorithm and the algorithm does not need to make any degree queries. We thus focus on queries to the name of the i^{th} neighbor of a vertex v . For this it is important to specify the ordering of vertices given in the adjacency lists. For all vertices we will use the same ordering in their list. Let $k = (n - 2)/2$ and label the vertices of A as a_1, a_2, \dots, a_k and the vertices of B as b_1, b_2, \dots, b_k . We use the ordering $s < t < a_1 < b_1 < \dots < a_k < b_k$.

We are now ready to show the lower bound using Lemma 19. We put G in relation with all $2^{\binom{n-2}{2}}$ of the $G_{a,b,c}$. Now consider how many of the $G_{a,b,c}$ differ from G in a specific location of the adjacency list, specifically consider the i^{th} neighbor of a vertex $a_j \in A$. In G the i^{th} neighbor of a_j is

1. s if $i = 1$.
2. a_{i-1} if $i \leq j$.
3. a_i if $i > j$.

The i^{th} neighbor of a_j in $G_{a,b,c}$ will be the same unless

1. $i \leq j$ and $a = \{a_{i-1}, a_j\}$.
2. $i > j$ and $a = \{a_j, a_i\}$.

In either case, the number of a, b, c for which $G_{a,b,c}$ differs from G on the name of the i^{th} neighbor of a_j is $2^{\binom{n-2}{2}}$, as one only has free choice of the edge b and the bit c . A similar argument holds when considering the i^{th} neighbor of a vertex $b_j \in B$. Thus for any position of the adjacency list the number of $G_{a,b,c}$ that differ from G is at most $2^{\binom{n-2}{2}}$ and by Lemma 19 we obtain a lower bound of $\binom{n-2}{2} = \Omega(m)$, proving the theorem in the dense case.

Finally, let us treat the general case of graphs with at most m edges. We choose the largest integer p such that $2(p + \binom{p}{2}) \leq m$ and then take disjoint sets of vertices A and B both of size p . We then repeat the construction from the dense case on s, t, A, B . The lower bound will be $\binom{p}{2} = \Omega(m)$ as desired. \square

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References

- [Aar06] Scott Aaronson. Lower bounds for local search by quantum arguments. *SIAM Journal of Computing*, 35(4):804–824, 2006.
- [AdW20] Simon Apers and Ronald de Wolf. Quantum speedup for graph sparsification, cut approximation and Laplacian solving. In *Proceedings of the 61st IEEE Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE, 2020.

- [AKP⁺21] Andris Ambainis, Martins Kokainis, Krisjanis Prusis, Jevgenijs Vihrovs, and Aleksajs Zajakins. All classical adversary methods are equivalent for total functions. *ACM Transactions on Computational Theory*, 13(1):7:1–7:20, 2021.
- [AL21] Simon Apers and Troy Lee. Quantum complexity of minimum cut. In *Proceedings of the 36th Computational Complexity Conference (CCC)*. LIPICS, 2021.
- [Amb02] Andris Ambainis. Quantum lower bounds by quantum arguments. *Journal of Computer and System Sciences*, 64(4):750–767, 2002.
- [AŠ06] Andris Ambainis and Robert Špalek. Quantum algorithms for matching and network flows. In *Proceedings of the 23rd Symposium on Theoretical Aspects of Computer Science (STACS)*, pages 172–183. Springer, 2006.
- [BK15] András A. Benczúr and David R. Karger. Randomized approximation schemes for cuts and flows in capacitated graphs. *SIAM Journal on Computing*, 44(2):290–319, 2015.
- [CKM⁺11] Paul Christiano, Jonathan A. Kelner, Aleksander Mądry, Daniel A. Spielman, and Shang-Hua Teng. Electrical flows, laplacian systems, and faster approximation of maximum flow in undirected graphs. In *Proceedings of the 43rd ACM Symposium on Theory of Computing (STOC)*, pages 273–282. ACM, 2011.
- [DH96] Christoph Dürr and Peter Høyer. A quantum algorithm for finding the minimum. *CoRR*, quant-ph/9607014, 1996.
- [FF56] Lester R. Ford and Delbert R. Fulkerson. Maximal flow through a network. *Canadian Journal of Mathematics*, 8:399–404, 1956.
- [GLP21] Yu Gao, Yang P. Liu, and Richard Peng. Fully dynamic electrical flows: Sparse maxflow faster than Goldberg-Rao. In *IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE, 2021.
- [GR98] Andrew V. Goldberg and Satish Rao. Beyond the flow decomposition barrier. *Journal of the ACM*, 45(5):783–797, 1998.
- [KLOS14] Jonathan A. Kelner, Yin Tat Lee, Lorenzo Orecchia, and Aaron Sidford. An almost-linear-time algorithm for approximate max flow in undirected graphs, and its multi-commodity generalizations. In *Proceedings of the 25th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 217–226. SIAM, 2014.
- [LM08] Sophie Laplante and Frédéric Magniez. Lower bounds for randomized and quantum query complexity using kolmogorov arguments. *SIAM Journal on Computing*, 38(1):46–62, 2008.

- [LS14] Yin Tat Lee and Aaron Sidford. Path finding methods for linear programming: Solving linear programs in $\tilde{O}(\sqrt{\text{rank}})$ iterations and faster algorithms for maximum flow. In *Proceedings of the 55th IEEE Annual Symposium on Foundations of Computer Science*, pages 424–433. IEEE, 2014.
- [LS20] Yang P. Liu and Aaron Sidford. Faster energy maximization for faster maximum flow. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing, (STOC)*, pages 803–814. ACM, 2020.
- [Mađ13] Aleksander Mađry. Navigating central path with electrical flows: From flows to matchings, and back. In *Proceedings of the 54th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 253–262. IEEE, 2013.
- [Pen16] Richard Peng. Approximate undirected maximum flows in $O(m \text{ polylog}(n))$ Time. In Robert Krauthgamer, editor, *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2016)*, pages 1862–1867. SIAM, 2016.
- [RSW18] Aviad Rubinfeld, Tselil Schramm, and S. Matthew Weinberg. Computing exact minimum cuts without knowing the graph. In *Proceedings of the 9th Innovations in Theoretical Computer Science Conference (ITCS)*, pages 39:1–39:16. LIPICS, 2018.
- [She13] Jonah Sherman. Nearly maximum flows in nearly linear time. In *54th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 263–269. IEEE, 2013.
- [ST18] Aaron Sidford and Kevin Tian. Coordinate methods for accelerating ℓ_∞ regression and faster approximate maximum flow. In *59th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 922–933. IEEE, 2018.
- [vdBLL⁺21] Jan van den Brand, Yin Tat Lee, Yang P. Liu, Thatchaphol Saranurak, Aaron Sidford, Zhao Song, and Di Wang. Minimum cost flows, MDPs, and ℓ_1 -regression in nearly linear time for dense instances. In *53rd Annual ACM SIGACT Symposium on Theory of Computing (STOC)*, pages 859–869. ACM, 2021.