

DO HETEROGENEOUS BELIEFS DIVERSIFY MARKET RISK?

CARL CHIARELLA*, ROBERTO DIECI** AND XUE-ZHONG HE*

*School of Finance and Economics
University of Technology, Sydney
PO Box 123 Broadway
NSW 2007, Australia

**Department of Mathematics for Economics and Social Sciences
University of Bologna
Viale Q. Filopanti 5, I-40126
Bologna, Italy

ABSTRACT. It is believed that diversity is good for our society, but is it good for financial markets? In particular, does the diversity with respect to beliefs among investors reduce the market risk of risky assets? The current paper aims to answer this question. Within the standard mean-variance framework, we introduce heterogeneous beliefs not only in risk preferences and expected payoffs but also in variances/covariances. By aggregating heterogeneous beliefs into a market consensus belief, we obtain CAPM-like equilibrium price and return relationships under heterogeneous beliefs. We show that the market aggregate behaviour is in principle a weighted average of heterogeneous individual behaviours. The impact of heterogeneity on the market equilibrium price and risk premium is examined in general. In particular, we give a positive answer to the question in the title by considering some special structure in heterogeneous beliefs. In addition, we provide an explanation of Miller's long standing hypothesis on the relation between a stock's risk and the divergence of opinions.

Key words: Heterogeneous beliefs, CAPM, mean-variance analysis, diversification, Miller's hypothesis.

JEL Classification: G12, D84.

We are grateful to seminar participants at Peking University, at the University of Technology, Sydney, and to participants at the First International Conference on Asia-Pacific Financial Markets (CAFM) 2006 (Seoul) and the Inaugural All China Economics (ACE) International Conference 2006 (Hong Kong) for helpful comments and suggestions. Financial support from the Australian Research Council (ARC) under Discovery Grant (DP0773776) for Chiarella and He is gratefully acknowledged. Dieci acknowledges support from MIUR under the project PRIN-2004137559.

Corresponding author: Xuezhong (Tony) He, School of Finance and Economics, University of Technology, Sydney, PO Box 123 Broadway, NSW 2007, Australia. Email: Tony.He1@uts.edu.au. Ph: (61 2) 9514 7726. Fax: (61 2) 9514 7722.

1. INTRODUCTION

The Sharpe-Lintner-Mossin (Sharpe (1964), Lintner (1965), Mossin (1966)) Capital Asset Pricing Model (CAPM) plays a central role in modern finance theory. It is founded on the paradigm of homogeneous beliefs and a rational representative agent. However, from a theoretical perspective this paradigm has been criticized on a number of grounds, in particular concerning its extreme assumptions about homogeneous beliefs, information about the economic environment, and the computational ability on the part of the rational representative economic agent. From an empirical perspective this paradigm has failed to explain a number of market anomalies including (i) the equity premium puzzle, (ii) excess volatility, and (iii) cross-sectional returns. Much of the CAPM literature addresses the first two empirical issues, however this paper is largely motivated by addressing the third issue and its relation to Miller's hypothesis. Miller (1977) proposes a direct relationship between a stock's *risk* and the *divergence of opinion* about the stock. He argues that "in practice, uncertainty, divergence of opinion about a security's return, and risk go together". Consequently, "the riskiest stocks are also those about which there is the greatest divergence of opinion", thus the market clearing price of a relatively high-risk stock will be greater than that for a relatively low-risk stock. Miller argues that the overvaluation of high-risk stocks is due to the short-sale constraints experienced by heterogeneous investors. The traditional CAPM with homogeneous beliefs cannot be used to explain Miller's hypothesis.

The impact of heterogeneous beliefs among investors on the market equilibrium price has been an important focus in the CAPM literature. A number of models with investors who have heterogeneous beliefs have been previously studied¹. A common finding in this strand of research is that heterogeneous beliefs can affect aggregate market returns. However, the question remains as to how exactly does the heterogeneity affect the market risk of risky assets? It is widely believed that society can benefit from a diversity of cultures. As an important part of society, financial markets might be expected to reflect this view. In other words, diversity in beliefs among investors in financial markets should reduce the market risk of assets in general. But, due to the complexity of financial markets, this issue has not been explored explicitly in the current literature. In much of this earlier work, the heterogeneous beliefs reflect either differences of opinion among the investors² or differences in information upon which investors are trying to learn by using some Bayesian updating rule³. Heterogeneity has been investigated in the context of either CAPM-like mean-variance models (for instance, Lintner (1969), Miller (1977), Williams (1977) and Mayshar (1982)) or Arrow-Debreu contingent claims models (as in Varian (1985), Abel (1989, 2002) and Calvet *et al.* (2004)).

¹See, for example, Lintner (1969), Williams (1977), Huang and Litzenberger (1988), Abel (1989), Detemple and Murthy (1994), Zapatero (1998) and Basak (2000)

²See, for example, Lintner (1969), Miller (1977), Mayshar (1982), Varian (1985), Abel (1989, 2002) and Cecchetti *et al.* (2000)

³Typical studies include Williams (1977), Detemple and Murthy (1994) and Zapatero (1998)

In most of the cited literature, the impact of heterogeneous beliefs is studied for the case of a portfolio of one risky asset and one risk-free asset (for example Abel (1989), Detemple and Murthy (1994), Zapatero (1998), Basak (2000) and Johnson (2004)). In those papers that do consider a portfolio of many risky assets and one risk-free asset, investors are assumed to be heterogeneous in their risk preferences and expected payoffs or returns of the risky assets (such as Williams (1977) and Varian (1985)), but not in their estimates of variances and covariances. The only exception seems to have been the early contribution of Lintner (1969) in which heterogeneity in both means and variances/covariances is investigated in a mean-variance portfolio context. The hypothesis of Miller (1977) highlights the fact that the variation of dispersion in the expected payoffs of risky assets among investors can be characterized by heterogeneous beliefs about the variance/covariance among investors. As suggested by the empirical study of Chan *et al.* (1999), while future variances and covariances are more easily predictable than expected future returns, the difficulties in doing so should not be understated. These authors argue that “while optimization (based on historical estimates of variances and covariances) leads to a reduction in volatility, the problem of forecasting covariance poses a challenge”. Variation in expectations among investors is characterized as the stock’s divergence of opinion. The early empirical study by Bart and Masse (1981) supports Miller’s hypothesis. Diether *et al.* (2002) provide empirical evidence that stocks with higher dispersion in analysts’ earnings forecasts earn lower future returns than otherwise similar stocks, in particular for small cap stocks and stocks that have performed poorly over the past year. Johnson (2004) offers a simple explanation for this phenomenon based on the interpretation of dispersion as a proxy for un-priced information risk arising when asset values are unobservable. Ang *et al.* (2006) examine the empirical relation between cross-sectional volatility and expected returns and find that stocks with high sensitivities to innovations in aggregate volatility have low average returns. Therefore, a theoretical understanding of the impact of heterogeneous beliefs in variances and covariances on equilibrium prices, volatility and cross-sectional expected returns is very important for a proper development of asset pricing theory. This paper is largely motivated by a re-reading of Lintner’s early work and the recent empirical studies related to Miller’s hypothesis. Although these earlier contributions discuss how to aggregate heterogeneous beliefs, the impact of heterogeneity on the market equilibrium price, risk premia and CAPM has never been fully explored.

Different from the above literature, heterogeneous agent models (HAMs) have been developed to characterize the dynamics of financial asset prices resulting from the interaction of heterogeneous agents with different attitudes towards risk and different expectations about the future evolution of asset prices. One of the key elements of this literature is the expectations feedback mechanism, see Brock and Hommes (1997, 1998). We refer the reader to Hommes (2006), LeBaron (2006) and Chiarella, Dieci and He (2009b) for surveys of recent literature on HAMs. This framework has successfully explained various aspects of market behaviour, such as the long-term swing of market prices from the fundamental price, asset bubbles and market crashes. It also shows

a potential to characterize and explain the stylized facts (for example, Gaunersdorfer and Hommes (2007) and Farmer *et al.* (2004)) and various power law behaviours (for instance Lux (2009), Alfarano *et al.* (2005) and He and Li (2007)) observed in financial markets. However, most of the HAMS analyzed in the literature involve a financial market with only one risky asset, except for some recent contributions by Böhm and Chiarella (2005), Wenzelburger (2004), Westerhoff (2004), Chiarella *et al.* (2005, 2007) and Westerhoff and Dieci (2006) showing that complex price dynamics may also result within a multi-asset market framework. In particular, Chiarella *et al.* (2007) demonstrate that heterogeneous beliefs about covariance between risky assets can have a complicated impact on portfolio diversification and asset pricing. More recently, Brock *et al.* (2009) consider a simple multi-asset pricing model with heterogeneous beliefs and show that more hedging instruments may destabilize markets when heterogeneous traders adapt their behaviour according to performance-based reinforcement learning. In general markets with many risky assets and heterogeneous investors, the impact of heterogeneity on the market equilibrium and standard portfolio theory remains a largely unexplored issue.

We consider a portfolio of one risk-free asset and many risky assets and extend the mean-variance model to allow for heterogeneity not only in the means but also in the variances/covariances across investors. The heterogeneous beliefs are considered as given. They reflect either differences of opinion among the investors or differences in information. By introducing the concept of a *consensus belief*, we show that the consensus belief can be constructed as a weighted average of the heterogeneous beliefs so that the market aggregate behaviour is in principle a weighted average of heterogeneous individual behaviours. In particular, we show that the market aggregate expected payoff of the risky assets is a weighted average of the heterogeneous expected payoffs of the risky assets across the investors, where the weights are given by the inverse covariance matrices adjusted by the risk tolerance of heterogeneous investors; the market equilibrium price is a weighted average of the equilibrium prices under the separate beliefs of each agent. As a consequence, we establish an equilibrium relation between the market aggregate expected payoff of the risky assets and the market portfolio's expected payoff, leading to a CAPM-like relation under heterogeneous beliefs. In particular, we give a positive answer to our earlier question by considering some special structure in heterogeneous beliefs. This means that, together with the standard diversification effect in portfolio theory, a portfolio of beliefs also has a diversification effect, in particular the diversity of heterogeneous beliefs can reduce the market risk of risky assets. In addition, we provide an explanation of Miller's hypothesis on the relation between a stock's risk and the divergence of opinions. Intuitively, we show that, without short-sale constraints, the overvaluation of high-risk stocks is due to the diversification of the aggregate risk among heterogeneous investors. It is the diversification effect of the aggregate risk under heterogeneous beliefs that plays an important role in our analysis.

The paper is organized as follows. Heterogeneous beliefs are introduced and the standard mean-variance analysis is conducted in Section 2. In Section 3, we first introduce a consensus belief, and show how the consensus belief can be constructed from heterogeneous beliefs. We then derive the market equilibrium price of risky assets based on the consensus belief and extend the traditional CAPM under homogeneous belief to the one under heterogeneous beliefs. Aggregation properties and the impact of heterogeneity are examined in Section 4. Section 5 concludes. Mathematical proofs are given in the appendix.

2. MEAN-VARIANCE ANALYSIS UNDER HETEROGENEOUS BELIEFS

The single period mean-variance model considered in this section is standard except that we allow the investors to have different risk preferences, as well as subjective means, variances and covariances. Consider a market with one risk-free asset and $K (\geq 1)$ risky assets. Let the current price of the risk-free asset be 1 and its payoff be $R_f = 1 + r_f$. Let $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_K)^T$ be the payoff vector of the risky assets, where $\tilde{x}_k = \tilde{p}_k + \tilde{d}_k$ ($k = 1, \dots, K$) corresponds to the cum-prices with \tilde{p}_k being the end of period price of asset k and \tilde{d}_k its dividend.

Assume that there are I investors in the market indexed by $i = 1, 2, \dots, I$. The heterogeneous (subjective) belief of investor i , referred to as \mathcal{B}_i , is defined with respect to the means, variances and covariances of the payoffs of the risky assets and are written

$$\mathbf{y}_i = \mathbb{E}_i(\tilde{\mathbf{x}}) = (y_{i,1}, y_{i,2}, \dots, y_{i,K})^T, \quad \Omega_i = (\sigma_{i,kl})_{K \times K},$$

where

$$y_{i,k} = \mathbb{E}_i[\tilde{x}_k], \quad \sigma_{i,kl} = \text{Cov}_i(\tilde{x}_k, \tilde{x}_l), \quad i = 1, 2, \dots, I; \quad k, l = 1, 2, \dots, K. \quad (2.1)$$

In the following we use, in particular, the notation $\sigma_{i,j}^2 := \sigma_{i,jj}$, $j = 1, 2, \dots, K$ to denote agent i 's estimate of the variance of asset j . Let $z_{i,o}$ and $\bar{z}_{i,o}$ be the amount to be invested and the endowment of investor i in the risk-free asset, respectively, and

$$\mathbf{z}_i = (z_{i,1}, z_{i,2}, \dots, z_{i,K})^T \quad \text{and} \quad \bar{\mathbf{z}}_i = (\bar{z}_{i,1}, \bar{z}_{i,2}, \dots, \bar{z}_{i,K})^T$$

be the selected risky portfolio and the portfolio endowment, respectively, of investor i in the risky assets. Then the end-of-period wealth of the portfolio for investor i is

$$\tilde{W}_i = R_f z_{i,o} + \tilde{\mathbf{x}}^T \mathbf{z}_i.$$

Then, under the belief \mathcal{B}_i , the expected value and variance of portfolio wealth \tilde{W}_i are given, respectively, by

$$\mathbb{E}_i(\tilde{W}_i) = R_f z_{i,o} + \mathbf{y}_i^T \mathbf{z}_i, \quad \sigma_i^2(\tilde{W}_i) = \mathbf{z}_i^T \Omega_i \mathbf{z}_i. \quad (2.2)$$

We now make the following standard assumptions under the mean-variance framework.

(H1) Assume the expected utility of the wealth generated from the portfolio $(z_{i,o}, \mathbf{z}_i)$ of investor i has the form $V_i(\mathbb{E}_i(\tilde{W}_i), \sigma_i^2(\tilde{W}_i))$, where $V_i(x, y)$ is continuously differentiable and satisfies $V_{i1}(x, y) := \partial V_i(x, y)/\partial x > 0$ and $V_{i2}(x, y) := \partial V_i(x, y)/\partial y < 0$.

(H2) Assume $-2V_{i2}(x, y)/V_{i1}(x, y)$ to be a constant θ_i for all (x, y) , i.e.

$$\theta_i = \frac{-2V_{i2}(x, y)}{V_{i1}(x, y)} = \text{const.}$$

Assumption (H1) is in particular consistent with the constant absolute risk aversion (CARA) utility function $U_i(w) = -e^{-A_i w}$ with normally distributed w . Here $A_i > 0$ corresponds to the constant absolute risk aversion (CARA) coefficient. In this case, investor- i 's optimal investment portfolio is obtained by maximizing the certainty-equivalent of his/her future wealth, $C_i(\tilde{W}_i) = \mathbb{E}_i(\tilde{W}_i) - \frac{A_i}{2} \text{Var}_i(\tilde{W}_i)$, and therefore $V_i(x, y) = x - \frac{A_i}{2}y$. Under assumption (H2), $\theta_i = A_i$, which is the CARA coefficient of investor i . Based on this, we refer to θ_i as the risk aversion measure and $\tau_i = 1/\theta_i$ as the risk tolerance of investor i .

Under (H1), the optimal portfolio of investor- i of risky assets \mathbf{z}_i^* and risk-free asset $z_{i,o}^*$ is determined by

$$\max_{z_{i,o}, \mathbf{z}_i} V_i(\mathbb{E}_i(\tilde{W}_i), \sigma_i^2(\tilde{W}_i))$$

subject to the budget constraint

$$z_{i,o} + \mathbf{p}_o^T \mathbf{z}_i = \bar{z}_{i,o} + \mathbf{p}_o^T \bar{\mathbf{z}}_i, \quad (2.3)$$

where $\mathbf{p}_o = (p_{1o}, p_{2o}, \dots, p_{Ko})^T$ is the vector of market equilibrium prices of the risky assets, which is to be determined. We can then obtain the following Lemma 2.1 for the optimal demand of investor i in equilibrium.

Lemma 2.1. *Under assumptions (H1) and (H2) and the heterogeneous belief $\mathcal{B}_i = (\tau_i, \mathbb{E}_i(\tilde{\mathbf{x}}), \Omega_i)$, investor- i 's optimal risky portfolio \mathbf{z}_i^* at the market equilibrium is given by*

$$\mathbf{z}_i^* = \tau_i \Omega_i^{-1} [\mathbf{y}_i - R_f \mathbf{p}_o]. \quad (2.4)$$

Lemma 2.1 shows that the optimal demand of investor- i is determined by his/her risk tolerance and his/her belief about the expected payoffs and variance/covariance matrix of the risky assets' payoffs. We will see that, in the market equilibrium, the optimal demand depends on the dispersion of expected payoffs of investor- i from the aggregate expected market payoff.

3. CONSENSUS BELIEF, EQUILIBRIUM ASSET PRICES AND CAPM

In this section, we first define a consensus belief. By construction, we show the existence and uniqueness of the consensus belief. The market equilibrium prices of risky assets are then derived by using the consensus belief.

A market **equilibrium** is a vector of asset prices \mathbf{p}_o determined by the individual demands (2.4) together with the market aggregation condition

$$\sum_{i=1}^I \mathbf{z}_i^* = \sum_{i=1}^I \bar{\mathbf{z}}_i := \mathbf{z}_m, \quad (3.1)$$

which defines the **market portfolio** \mathbf{z}_m . To characterize the market equilibrium, we introduce the following definition of **consensus belief**.

Definition 3.1. A belief $\mathcal{B}_a = (\mathbb{E}_a(\tilde{\mathbf{x}}), \Omega_a)$, defined by the expected payoff of the risky assets $\mathbb{E}_a(\tilde{\mathbf{x}})$ and the variance and covariance matrix of the risky asset payoffs Ω_a , is called a **consensus belief** if and only if the equilibrium price under the heterogeneous beliefs $\mathcal{B} = \{\mathcal{B}_i\}_{i=1}^I$ is also the equilibrium price under the homogeneous belief \mathcal{B}_a .

The following Proposition 3.2 indicates that such a consensus belief can be uniquely constructed and the market equilibrium price can be characterized by the consensus belief.

Proposition 3.2. Under assumptions (H1) and (H2), let $\tau_a = \sum_{i=1}^I \tau_i$ be the aggregate risk tolerance. Then

(i) the consensus belief $\mathcal{B}_a = (\mathbb{E}_a(\tilde{\mathbf{x}}), \Omega_a)$ is given by

$$\Omega_a^{-1} = \sum_{i=1}^I \frac{\tau_i}{\tau_a} \Omega_i^{-1}, \quad (3.2)$$

$$\mathbf{y}_a = \mathbb{E}_a(\tilde{\mathbf{x}}) = \Omega_a \sum_{i=1}^I \frac{\tau_i}{\tau_a} \Omega_i^{-1} \mathbf{y}_i; \quad (3.3)$$

(ii) the market equilibrium price \mathbf{p}_o is determined by

$$\mathbf{p}_o = \frac{1}{R_f} \left[\mathbf{y}_a - \frac{1}{\tau_a} \Omega_a \mathbf{z}_m \right]; \quad (3.4)$$

(iii) the equilibrium optimal portfolio of agent i is given by

$$\mathbf{z}_i^* = \tau_i \Omega_i^{-1} \left[(\mathbf{y}_i - \mathbf{y}_a) + \frac{1}{\tau_a} \Omega_a \mathbf{z}_m \right]. \quad (3.5)$$

The results of Proposition 3.2 are very intuitive. First, the market risk tolerance τ_a is an aggregate risk tolerance of all the investors. Secondly, if we treat the heterogeneous beliefs as a probability state space and the weights $\{\tau_i/\tau_a\}_{i=1}^I$ as a probability measure of the heterogeneous beliefs, then the consensus belief can be interpreted as the expectation of the heterogeneous beliefs under this risk tolerance measure. More precisely, the inverse covariance matrix under the consensus belief is the expectation of the inverse covariance matrices of the heterogeneous beliefs under the risk tolerance measure, and the expected payoff under the consensus belief is the expectation of the expected payoffs of the heterogeneous beliefs under the risk tolerance measure. This interpretation is particularly useful when we come to examine the diversification effect

under aggregation in the next section. It will also be very useful to think of the beliefs as random variables following certain distributions.

The equilibrium price determined in Proposition 3.2 can be used to establish a CAPM relationship in either the prices or returns of the risky assets. In fact, the value of the market portfolio \mathbf{z}_m in the market equilibrium is given by $W_{m,o} = \mathbf{z}_m^T \mathbf{p}_o$ and its future payoff is $\tilde{W}_m = \tilde{\mathbf{x}}^T \mathbf{z}_m$. Hence, under the consensus belief \mathcal{B}_a ,

$$W_m := \mathbb{E}_a(\tilde{W}_m) = \mathbb{E}_a(\tilde{\mathbf{x}})^T \mathbf{z}_m, \quad \sigma_m^2 := \text{Var}_a(\tilde{W}_m) = \mathbf{z}_m^T \Omega_a \mathbf{z}_m. \quad (3.6)$$

Based on Proposition 3.2 and the above observation, we obtain the following CAPM-like price relation under heterogeneous beliefs. We shall call this relationship the *Heterogeneous CAPM (HCAPM)* in price.

Corollary 3.3. *In equilibrium the market aggregate expected payoff of the risky assets are related to the expected payoff of the market portfolio \mathbf{z}_m by the CAPM-like price relation*

$$\mathbb{E}_a(\tilde{\mathbf{x}}) - R_f \mathbf{p}_o = \frac{1}{\sigma_m^2} \Omega_a \mathbf{z}_m [\mathbb{E}_a(\tilde{W}_m) - R_f W_{m,o}], \quad (3.7)$$

or equivalently,

$$\mathbb{E}_a(\tilde{x}_k) - R_f p_{k,o} = \frac{\sigma(\tilde{W}_m, \tilde{x}_k)}{\sigma_m^2} [\mathbb{E}_a(\tilde{W}_m) - R_f W_{m,o}], \quad k = 1, 2, \dots, K, \quad (3.8)$$

where $\Omega_a = (\sigma_{kj})_{K \times K}$ and $\sigma(\tilde{W}_m, \tilde{x}_k) = \sum_{j=1}^K z_{m,j} \sigma_{kj}$ for $k = 1, \dots, K$ corresponds to the covariance, under the consensus belief, between the market payoff of the risky asset k and the aggregate market portfolio payoff \tilde{W}_m .

The HCAPM price relation (3.7) can be converted to the standard CAPM-like return relation. Define the returns

$$\tilde{r}_j = \frac{\tilde{x}_j}{p_{j,o}} - 1, \quad \tilde{r}_m = \frac{\tilde{W}_m}{W_{m,o}} - 1,$$

from which

$$\mathbb{E}_a(\tilde{r}_j) = \frac{\mathbb{E}_a(\tilde{x}_j)}{p_{j,o}} - 1, \quad \mathbb{E}_a(\tilde{r}_m) = \frac{\mathbb{E}_a(\tilde{W}_m)}{W_{m,o}} - 1.$$

With these notations, we can obtain from (3.7) the following HCAPM relation between returns of risky assets and the market portfolio.

Corollary 3.4. *In equilibrium, the HCAPM price relation (3.7) can be expressed in terms of returns as*

$$\mathbb{E}_a[\tilde{\mathbf{r}}] - r_f \mathbf{1} = \boldsymbol{\beta} [\mathbb{E}_a(\tilde{r}_m) - r_f], \quad (3.9)$$

where

$$\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_K)^T, \quad \beta_k = \frac{\text{cov}_a(\tilde{r}_m, \tilde{r}_k)}{\sigma_a^2(\tilde{r}_m)}, \quad k = 1, \dots, K,$$

and the mean and variance/covariance of returns under the consensus belief \mathcal{B}_a are defined similarly in terms of returns.

The equilibrium relation (3.9) is the standard CAPM except that the mean and variance/covariance are calculated based on the consensus belief \mathcal{B}_a . The β coefficients of risky assets depend upon not only the covariance between the market returns and asset returns, but also the aggregation of the heterogeneous beliefs. We have thus shown that a CAPM-like relationship still holds under heterogeneous beliefs.

The main obstacle in dealing with heterogeneity is the complexity and heavy notation involved when the number of assets and the dimension of the heterogeneity increase, as illustrated in Lintner (1969). Lintner (1969) is the first paper to consider the problem of market equilibrium in the same setup as in this paper. Surprisingly, this significant contribution from Lintner has not received much attention until recent years. This might be due to the notational obstacle mentioned above, which makes the paper not easy to follow, and renders rather difficult the analysis of the impact of heterogeneity on the market equilibrium price. Proposition 3.2 shows not only the existence of the consensus belief but also how it can be constructed explicitly from the heterogeneous beliefs. The equilibrium asset pricing formula is the standard one under the consensus belief. It is the explicit construction of the consensus belief in Proposition 3.2 that overcomes the notational obstacle of Lintner's contribution and makes it possible to examine the impact of heterogeneity on the market. The following section is devoted to examining the impact of different aspects of heterogeneity on the market equilibrium price, equity risk premium and the market aggregation in general.

4. THE IMPACT OF HETEROGENEITY IN BELIEFS

In this paper, heterogeneity is characterized by the diversity of risk aversion coefficients and investor beliefs in expected payoffs and variance/covariance matrices of the payoffs of the risky assets. Understanding the impact of different aspects of heterogeneity when the market is in equilibrium is important for a proper understanding of asset pricing theory.

4.1. The impact of heterogeneity on the market equilibrium price. Consider first the relationship between the market equilibrium prices under heterogeneous beliefs and the prices in a series of markets populated by homogeneous investors with the beliefs of the respective investors in the original heterogeneous belief market. Note that the market equilibrium price (3.4) in Proposition 3.2 is exactly the same as the traditional equilibrium price for a representative agent holding the consensus belief \mathcal{B}_a . If we define $\mathbf{p}_{i,o}$ as the equilibrium price vector of the risky assets for investor i as if he/she were the only investor in the market, then we would have

$$\mathbf{p}_{i,o} = \frac{1}{R_f} [\mathbb{E}_i(\tilde{\mathbf{x}}) - \frac{1}{\tau_i} \Omega_i \tilde{z}_i] = \frac{1}{R_f} [\mathbf{y}_i - \frac{1}{\tau_i} \Omega_i \tilde{z}_i].$$

Equation (3.4) can then be rewritten as

$$\begin{aligned} \mathbf{p}_o &= \frac{1}{R_f} \left[\mathbb{E}_a(\tilde{\mathbf{x}}) - \frac{1}{\tau_a} \Omega_a \mathbf{z}_m \right] = \frac{1}{\tau_a} \Omega_a \frac{1}{R_f} \left[\sum_{i=1}^I \tau_i \Omega_i^{-1} \mathbb{E}_i(\tilde{\mathbf{x}}) - \sum_{i=1}^I \tilde{z}_i \right] \\ &= \frac{1}{\tau_a} \Omega_a \sum_{i=1}^I \tau_i \Omega_i^{-1} \frac{1}{R_f} [\mathbf{y}_i - \frac{1}{\tau_i} \Omega_i \tilde{z}_i] = \frac{1}{\tau_a} \Omega_a \sum_{i=1}^I \tau_i \Omega_i^{-1} \mathbf{p}_{i,o}. \end{aligned} \quad (4.1)$$

Using the interpretation of expectation under the τ measure introduced in the previous section, equation (4.1) can be written as

$$\mathbf{p}_o = \frac{1}{\tau_a} \Omega_a \sum_{i=1}^I \tau_i \Omega_i^{-1} \mathbf{p}_{i,o}.$$

This means that the aggregate market equilibrium price is the expectation of the equilibrium prices of all investors under their beliefs, weighted by the inverse covariance matrices. This relationship between the market equilibrium price under heterogeneous beliefs and prices under alternative homogeneous beliefs indicates that the market equilibrium price reflects a weighted average of the prices under individual beliefs. For given market risk tolerance (τ_a) and covariance matrix (Ω_a), the market equilibrium price is bounded above by the price determined by the investor who is less risk averse and optimistic, and below by the price determined by the investor who is more risk averse and pessimistic. In general, the equilibrium price is dominated by investors who are less risk averse and optimistic about the expected payoff. This observation is very intuitive. Consistent with Miller's argument, the market price may reflect the expectations of only the most optimistic (but less risk averse) minority, as long as this minority can absorb the entire supply of stock.

4.2. Equity risk premium and diversification effect. We first examine the equity risk premium. It follows from the equilibrium price equation (3.4) that the equity risk premium, $\Omega_a \mathbf{z}_m / \tau_a$, is negatively related to the aggregate risk tolerance τ_a but positively related to the covariance between the risky assets and the market portfolio $\Omega_a \mathbf{z}_m$. Note that, for a given market portfolio \mathbf{z}_m and risk tolerance τ_a , when investors have homogeneous beliefs about covariance matrices so that $\Omega_i = \Omega_o$, then $\Omega_a = \Omega_o$ and hence the risk premium is unchanged when investors have heterogeneous beliefs about expected payoffs. However, when investors have heterogeneous beliefs of covariance matrices, then the heterogeneity does affect the risk premium. What we would like to know is, how exactly does heterogeneity or dispersion of beliefs among heterogeneous investors affect the risk premium? This is an important but difficult question to answer in general. In the following discussion, we try to tackle this question from different perspectives by considering some special cases.

To simplify our analysis, consider the special case where the risky asset payoffs are uncorrelated. In this case, the covariance matrix Ω_i becomes a diagonal matrix.

Consequently the covariance matrix under the consensus belief is given by

$$\Omega_a = \text{diag}(\sigma_{a,1}^2, \sigma_{a,2}^2, \dots, \sigma_{a,K}^2), \quad (\sigma_{a,j}^2)^{-1} = \sum_{i=1}^I \frac{\tau_i}{\tau_a} (\sigma_{i,j}^2)^{-1} \quad (4.2)$$

and the market equilibrium prices are given by

$$p_{o,j} = \frac{1}{R_f} [y_j - \frac{1}{\tau_a} \sigma_{a,j}^2 z_{m,j}], \quad j = 1, \dots, N. \quad (4.3)$$

To examine the effect of the market aggregation in terms of the covariance matrix, we consider two cases.

Case 1: In the first case, we compare the variances of any portfolio under both the aggregate covariance matrix and the average belief of the heterogeneous covariance matrix defined under the probability measure $\{\tau_i/\tau_a\}_{i=1}^I$. The first diversification effect of the aggregate beliefs of covariance is given by the following corollary.

Corollary 4.1. *Assume the payoffs of risky assets are uncorrelated, and define⁴.*

$$\sigma_a^2(\mathbf{z}) = \mathbf{z}^T \Omega_a \mathbf{z}, \quad \bar{\sigma}^2(\mathbf{z}) = \mathbf{z}^T \bar{\Omega} \mathbf{z}, \quad \bar{\Omega} = \sum_{i=1}^I \frac{\tau_i}{\tau_a} \Omega_i.$$

Then for any given portfolio \mathbf{z} we have $\sigma_a^2(\mathbf{z}) \leq \bar{\sigma}^2(\mathbf{z})$, and $\sigma_a^2(\mathbf{z}) = \bar{\sigma}^2(\mathbf{z})$ only if all Ω_i are the same.

Note that when $\Omega_i = \Omega_o$ for all i , then $\Omega_a = \bar{\Omega} = \Omega_o$. Corollary 4.1 implies that, when asset payoffs are uncorrelated, if we treat the risk tolerance weighted heterogeneous variances of risky assets as a benchmark, the aggregate (or market consensus) belief of the covariance Ω_a leads to smaller portfolio risk. That is, the market aggregation lowers the asset portfolio risk, leading to lower equity risk premium, higher equilibrium price, and lower expected return. This diversification effect of risk due to the market aggregation in equilibrium helps us to understand one aspect of the impact of heterogeneity on the risk premium.

Case 2: In the second case, we consider the dispersion of beliefs among investors. The investors may have the same belief of the expected payoffs of the risky assets, but different levels of confidence in their beliefs. The different levels of confidence can be measured by the heterogeneous beliefs of variances. In the following, we introduce the concept of a mean preserving spread⁵ to compare the heterogeneity in variances among different assets and examine the impact of the mean preserving spread on the risk premium. The basic intuition is best understood in the special case of two investors and two risky assets, for which we first introduce the following definition.

⁴The reader should contrast the definition of $\bar{\Omega}$ with that of Ω_a in (3.2).

⁵The mean preserving spread introduced here is different from the one used in the stochastic dominance, but shares the same underlying idea.

Definition 4.2. Let $\sigma_j^2 = (\sigma_{1,j}^2, \sigma_{2,j}^2)$ be the heterogeneous beliefs of the variance of asset $j = 1, 2$ among the two investors and $\omega = (\omega_1, \omega_2)$ be a positive weighting vector satisfying $0 < \omega_i < 1$ and $\omega_1 + \omega_2 = 1$. We define the weighted mean of the variances of the assets by

$$\bar{\sigma}_j^2 = \omega_1 \sigma_{1,j}^2 + \omega_2 \sigma_{2,j}^2, \quad j = 1, 2.$$

Assume $\sigma_{1,1}^2 \leq \sigma_{2,1}^2$ for asset 1. Then a combination of beliefs of variances for asset 2 is called a mean preserving spread of that for asset 1 if

$$\sigma_{1,2}^2 \leq \sigma_{1,1}^2, \quad \sigma_{2,2}^2 \geq \sigma_{2,1}^2$$

and $\bar{\sigma}_2^2 = \bar{\sigma}_1^2$.

Thus this mean preserving spread has the property that the weighted mean of the variances of the two assets are the same but agent one has greater confidence in the payoff of asset 2 while agent two has greater confidence in the payoff of asset 1. With this definition, we can measure belief dispersion among two assets and examine the impact of heterogeneity in variances on the equity risk premia. We obtain the following result on the diversification effect of the mean preserving spread of beliefs of variances.

Corollary 4.3. The belief of the variances for asset 2 defined by

$$\{\sigma_{1,2}^2, \sigma_{2,2}^2\} = \{\sigma_{1,1}^2(1 - \varepsilon), \sigma_{2,1}^2(1 + \delta)\}$$

with $1 > \varepsilon > 0$ and $\delta = \varepsilon \frac{\omega_1 \sigma_{1,1}^2}{\omega_2 \sigma_{2,1}^2}$, is a mean preserving spread of beliefs of variance for asset 1. Also $\sigma_{a,2}^2 \leq \sigma_{a,1}^2$ if and only if

$$\frac{\tau_1}{\tau_2} \left(1 + \varepsilon \frac{\omega_1 \sigma_{1,1}^2}{\omega_2 \sigma_{2,1}^2} \right) \geq \left(\frac{\sigma_{1,1}^2}{\sigma_{2,1}^2} \right)^2 \frac{\omega_1}{\omega_2} (1 - \varepsilon). \quad (4.4)$$

Corollary 4.3 provides a condition under which a mean preserving spread of the beliefs in variances results in a reduction of aggregate market risk. Note from Definition 4.2 that $\sigma_{1,1}^2 \leq \sigma_{2,1}^2$. It follows from the conditions for a mean preserving spread that $\sigma_{1,2}^2 \leq \sigma_{1,1}^2 \leq \sigma_{2,1}^2 \leq \sigma_{2,2}^2$. Thus agent 2 is ‘less confident’ than agent 1 in his/her belief about the expected payoffs of both assets, in the sense that $\sigma_{1,j}^2 \leq \sigma_{2,j}^2$ for $j = 1, 2$. Before rewriting (4.4) in a slightly different way and discussing its implications, we consider two particular cases.

In the first case, let ω_i be defined by the risk tolerance $\omega_i = \tau_i / \tau_a$. Then

$$\bar{\sigma}_j^2 = \frac{\tau_1 \sigma_{1,j}^2 + \tau_2 \sigma_{2,j}^2}{\tau_1 + \tau_2}, \quad j = 1, 2,$$

defines a risk-tolerance weighted average of variances. Substituting $\omega_i = \tau_i / \tau_a$ into (4.4), we obtain that $\sigma_{a,2}^2 \leq \sigma_{a,1}^2$ if and only if

$$\left(1 + \varepsilon \frac{\tau_1 \sigma_{1,1}^2}{\tau_2 \sigma_{2,1}^2} \right) \geq \left(\frac{\sigma_{1,1}^2}{\sigma_{2,1}^2} \right)^2 (1 - \varepsilon) \quad (4.5)$$

which is always true because $\sigma_{1,1}^2 \leq \sigma_{2,1}^2$ by assumption. This is not surprising, because we are comparing a harmonic mean and an arithmetic mean using the same weights, see also the previous Case 1. This implies that, when the weights are defined by the risk tolerance, a mean preserving spread of beliefs in variances always reduces the market risk, leading to a lower equity risk premium, a higher market price and a lower expected return.

In the second case, let $\omega_i = \theta_i / (\theta_1 + \theta_2)$, $i = 1, 2$. Then

$$\bar{\sigma}_j^2 = \frac{\theta_1 \sigma_{1,j}^2 + \theta_2 \sigma_{2,j}^2}{\theta_1 + \theta_2}$$

defines a risk-aversion weighted average of variances. Substitute $\omega_i = \theta_i / (\theta_1 + \theta_2)$ into (4.4), we obtain that $\sigma_{a,2}^2 \leq \sigma_{a,1}^2$ if and only if

$$\theta_2^2 (\sigma_{2,1}^2)^2 + \varepsilon \theta_1 \theta_2 \sigma_{1,1}^2 \sigma_{2,1}^2 \geq \theta_1^2 (\sigma_{1,1}^2)^2 (1 - \varepsilon),$$

which after a few more computations reduces to

$$\frac{\theta_2}{\theta_1} \geq \frac{\sigma_{1,1}^2}{\sigma_{2,1}^2} (1 - \varepsilon). \quad (4.6)$$

Also in this case a mean preserving spread of variance beliefs is able to reduce market risk, for a wide set of combinations of parameters, provided that the risk aversion coefficient θ_2 of agent 2 is large enough. For a general weighting vector (ω_1, ω_2) , the condition (4.4) can be rewritten as

$$\tau_1 \omega_2 (\sigma_{2,1}^2)^2 - \tau_2 \omega_1 (\sigma_{1,1}^2)^2 + \varepsilon \omega_1 \sigma_{1,1}^2 (\tau_1 \sigma_{2,1}^2 + \tau_2 \sigma_{1,1}^2) \geq 0.$$

It follows that a *sufficient* condition for $\sigma_{a,2}^2 \leq \sigma_{a,1}^2$ is

$$\tau_1 \omega_2 (\sigma_{2,1}^2)^2 - \tau_2 \omega_1 (\sigma_{1,1}^2)^2 \geq 0,$$

Using $\theta_i = 1/\tau_i$, this is equivalent to

$$\sqrt{\frac{\theta_2}{\theta_1}} \geq \sqrt{\frac{\omega_1}{\omega_2} \frac{\sigma_{1,1}^2}{\sigma_{2,1}^2}}. \quad (4.7)$$

This is obviously much more general than (4.6) but it can be interpreted along similar lines. In particular, when $\omega_i = \theta_i / (\theta_1 + \theta_2)$ for $i = 1, 2$, the sufficient condition (4.7) becomes

$$\frac{\theta_2}{\theta_1} \geq \frac{\sigma_{1,1}^2}{\sigma_{2,1}^2}. \quad (4.8)$$

Both conditions (4.7) and (4.8) implies that a mean preserving spread of beliefs in variances reduces market risk when the risk aversion coefficient of the less confident agent (that is the one with higher variance belief, being agent 2 in our case) is large enough. If a higher variance can be interpreted as a measure of the doubt of an investor with respect to the expected payoff of a risky asset, then the diversification effect due to a mean preserving spread of beliefs in variances holds when the doubt and risk aversion

of investors are positively correlated. Intuitively, a positive correlation between doubt and risk aversion makes the less risk averse and more confident investors dominate the market, which in turn results in a higher market equilibrium price and a lower equity risk premium for the asset.

4.3. Miller's hypothesis. We now discuss Miller's hypothesis within our framework. Based on the construction of the consensus belief in Proposition 3.2, one can see that the heterogeneity of beliefs in covariance affects the market expected payoff only when the beliefs of investors are heterogeneous in expected payoffs, too. In fact, if investors agree on the expected payoff, $\mathbb{E}_i(\tilde{\mathbf{x}}) = \mathbb{E}_o(\tilde{\mathbf{x}})$, then it follows from (3.3) that $\mathbb{E}_a(\tilde{\mathbf{x}}) = \mathbb{E}_o(\tilde{\mathbf{x}})$, even though they may disagree on their risk preferences, variances and covariances. If investors agree on the variance and covariance, then

$$\mathbb{E}_a(\tilde{\mathbf{x}}) = \sum_{i=1}^I \frac{\tau_i}{\tau_a} \mathbb{E}_i(\tilde{\mathbf{x}}), \quad (4.9)$$

which reflects a risk tolerance weighted average opinion of the market on the expected payoffs of risky assets. In this case, the expected market payoff is dominated by investors who are less (more) risk averse and optimistic (pessimistic) about the expected payoff, as we would expect when market prices move up (down), although such dominance may be asymmetric in different market conditions. Alternatively, the aggregate expected payoff may be unchanged even though investors have divergent opinions on their expected payoffs, this would be the case if their risk tolerance weighted expectations are *balanced*.

According to Miller, if investors' average opinion about future payoff is the same for two firms, the divergence of opinion among investors reflects disagreement and the firm about which a greater divergence of opinion will have more extreme optimistic investors than the one about which less divergence of opinion. Hence the market price should be higher for the firm with a greater divergence of opinion, which in turn generates lower expected return. Diether et al. (2002) and Ang et al. (2006) find empirical evidence that stocks with higher dispersion in analysts' earnings forecasts earn lower future returns than otherwise similar stocks. This evidence provides empirical support for Miller's hypothesis. Within the framework of this paper, we are able to provide an explanation and condition for Miller's hypothesis to hold. Following the discussion in the previous part for the two investors and two assets case, we assume that both assets 1 and 2 have the same expected payoffs, but investors disagree on the dispersion of the payoffs of the two assets. That is, we assume that both investors have homogeneous beliefs about the expected payoffs of risky assets 1 and 2 but have different risk aversion coefficients and heterogeneous beliefs about variances of the assets. The homogeneous beliefs in the expected payoff reflect the same opinion on the average expected payoff for the two assets and heterogeneous beliefs in the variances reflect divergence of opinion in the dispersion of the payoffs. In this case, the market expected payoffs for the two assets are the same. If we assume that the belief in variance of

asset 2 is a mean-preserving spread of variance beliefs about asset 1, and if investor-2 is more risk averse than investor-1 (in the sense of condition (4.7)) or (4.8)), then it follows from the discussion in the previous subsection that the aggregate variance of asset 2 is less than that of asset 1. Thus, from the equilibrium price equation (4.3), the equilibrium price for asset 2 is higher than the equilibrium price for asset 1. This in turn implies that asset 2 has lower expected return than asset 1. In other words, stocks with higher dispersion in expected payoffs have higher market clearing prices and earn lower future expected returns than otherwise similar stocks. This result is consistent with Miller's hypothesis that divergence of opinion and risk "go together". It is also interesting to note that this kind of argument cannot hold when investors have homogeneous beliefs.

4.4. The impact on the optimal demands and trading volume. Finally, we consider the impact of heterogeneity on the optimal portfolios and trading volume. Proposition 3.2 (iii) implies that the equilibrium demand of an individual investor has two components. The first term $\tau_i \Omega_i^{-1} [\mathbb{E}_i(\tilde{\mathbf{x}}) - \mathbb{E}_a(\tilde{\mathbf{x}})]$ corresponds to the standard myopic demand. It reflects the dispersion of the investor's expected payoff from the market expected payoff. This implies that optimistic (pessimistic) investors will take long (short) positions in risky assets. When short selling is not allowed, the market price would be higher than the equilibrium price, in particular this is the case when the optimistic investors are less risk averse, as Miller has argued. The second term $(\tau_i/\tau_a) \Omega_a \Omega_i^{-1} \mathbf{z}_m$ corresponds to the so-called hedging component. It reflects the covariance of the payoffs between risky assets and the market portfolio, weighted by investor's risk tolerance and beliefs. The hedging position of investors is long (short) when the payoffs of the risky asset and the market portfolio are positively (negatively) correlated. When an investor's expected payoff is the same as the aggregate expected payoff, that is, $\mathbb{E}_i(\tilde{\mathbf{x}}) = \mathbb{E}_a(\tilde{\mathbf{x}})$, the investor's demand is simply determined by the second component.

When investors are homogeneous in the covariance matrix $\Omega_i = \Omega_o$, the second component reduces to $(\tau_i/\tau_a) \mathbf{z}_m$, which is a risk tolerance weighted average share of the market portfolio. In this case, the equilibrium demand of investor i reduces to

$$\mathbf{z}_i^* = \tau_i \Omega_o^{-1} [\mathbb{E}_i(\tilde{\mathbf{x}}) - \mathbb{E}_a(\tilde{\mathbf{x}})] + (\tau_i/\tau_a) \mathbf{z}_m, \quad (4.10)$$

and the market equilibrium price reduces to

$$\mathbf{p}_o = \frac{1}{R_f} [\mathbb{E}_a(\tilde{\mathbf{x}}) - \frac{1}{\tau_a} \Omega_o \mathbf{z}_m], \quad \text{where} \quad \mathbb{E}_a(\tilde{\mathbf{x}}) = \sum_{i=1}^I \frac{\tau_i}{\tau_a} \mathbb{E}_i(\tilde{\mathbf{x}}). \quad (4.11)$$

From (4.10), one can see that the optimal portfolio of investor i is different from the market portfolio unless the investor's belief is the same as the market aggregate belief. From (4.10) and (4.11), it follows that a risk-tolerance weighted mean-preserving spread in the distribution of the expected payoffs among investors will not change the equilibrium price, but will spread optimal demands among investors around the average market portfolio, this in turn will increase the trading volume in the market.

This implies that a high trading volume due to diversified beliefs about asset expected payoffs may not necessarily lead to high asset prices. If the deviation of investors' expected payoffs from the average expected payoff does not change, investors demands will not change. However a high average of the expected payoffs will lead to a high market equilibrium asset price. This suggests that a higher (or lower) market price due to a higher (or lower) average expected payoff may not necessarily lead to higher trading volume.

5. CONCLUSION

This paper provides an aggregation procedure for the construction of a market consensus belief from the heterogeneous beliefs of different investors. This allows us to characterize the market equilibrium in the traditional mean-variance model under the consensus belief. Various impacts of heterogeneity are discussed. In particular, the impact of diversity of heterogeneous beliefs is examined. We show that the market aggregate behaviour is a weighted average of heterogeneous individual behaviour, a very intuitive result. The weights are proportional to the individual risk tolerances and the covariance matrix. For example, the market equilibrium price reflects a weighted average of the individuals' equilibrium prices under their own beliefs. We have established an equilibrium relation between the market aggregate expected payoff of the risky assets and the market portfolio's expected payoff, which leads to the CAPM-like relationship under heterogeneous beliefs. By considering some special structure of heterogeneity in beliefs, we show that diversity of beliefs reduces the market risk. Our results also provide a simple explanation for Miller's hypothesis on cross-sectional expected returns which is supported by some recent empirical studies.

This paper provides a simple framework for dealing with heterogeneous beliefs and aggregation. The framework has been extended recently along several directions. First, by imposing a mean-preserving spread distribution on heterogeneous beliefs and conducting a statistical analysis based on Monte Carlo simulations, He and Shi (2010) show that diversity in heterogeneous beliefs among investors can improve the Sharpe and Treynor ratios of the market portfolio and the optimal portfolios of investors, demonstrating further the diversification effect of heterogeneous beliefs. Secondly, when the heterogeneous beliefs are formed in the rates of returns rather than payoffs of risky assets, Chiarella *et al.* (2009a) construct a consensus belief (with respect to the means and covariances of the risky asset returns) to represent the aggregate market belief when the market is in equilibrium, extend the analysis further to a repeated one-period set-up, and establish a framework for a dynamic CAPM using a market fraction model in which agents are grouped according to their beliefs. The exact relation between heterogeneous beliefs, the market equilibrium returns and the ex-ante beta-coefficients is obtained. By incorporating the three most popular types of investors, fundamentalists, chartists and noise traders, into the model, Chiarella *et al.* (2010) further show that, independent of the fundamentals, there is a systematic change in the market portfolio, risk-return relationships, and time varying betas when investors

change their behaviour, such as the chartists acting as momentum traders. In particular, it demonstrates the stochastic nature of time-varying betas and shows that the commonly used rolling window estimates of time-varying betas may not be consistent with the *ex-ante* betas implied by the equilibrium model. The results provide insights into an understanding of time-varying beta. Thirdly, when there is no riskless asset, He and Shi (2009b) extend the framework in this paper and Chiarella *et al.* (2009a) and the standard Black's zero-beta CAPM to incorporate heterogeneous beliefs. They show that in general the subjective optimal portfolios of heterogeneous investors are mean-variance inefficient and the traditional geometric relation of the mean variance frontiers with and without the riskless asset under homogeneous beliefs does not hold under heterogeneous beliefs. He and Shi (2009a) further examine the impact of the disagreement among agents on the market equilibrium and equity premium by considering a market of two risky assets (with correlated returns) and two agents who have different preference and heterogeneous beliefs in the mean and variance/covariance of the asset returns. When the differences in opinion are formed as mean preserving spreads of a benchmark homogeneous belief, they examine explicitly the impact on the market equilibrium and equity premium, showing that the risk tolerance, optimism/pessimism and confidence/doubt jointly can generate a high equity premium and a low riskfree rate. Overall, the framework developed in this paper provides a solid foundation for the characterization of the impact of heterogeneous beliefs on the market equilibrium.

APPENDIX A. PROOF OF LEMMA 2.1

Proof. Let λ_i be the Lagrange multiplier and set

$$L(z_{i,o}, \mathbf{z}_i, \lambda_i) := V_i(\mathbb{E}_i(\tilde{W}_i), \sigma_i^2(\tilde{W}_i)) + \lambda_i[(\bar{z}_{i,o} + \mathbf{p}_o^T \bar{\mathbf{z}}_i) - (z_{i,o} + \mathbf{p}_o^T \mathbf{z}_i)].$$

Then the optimal portfolio of agent i is determined by the first order conditions

$$V_{i1} \frac{\partial \mathbb{E}_i(\tilde{W}_i)}{\partial z_{i,o}} = \lambda_i, \quad (\text{A.1})$$

$$V_{i1} \frac{\partial \mathbb{E}_i(\tilde{W}_i)}{\partial z_{i,k}} + V_{i2} \frac{\partial \sigma_i^2(\tilde{W}_i)}{\partial z_{i,k}} = \lambda_i p_{ko}, \quad k = 1, 2, \dots, K. \quad (\text{A.2})$$

From equation (2.2) we have

$$\frac{\partial \mathbb{E}_i(\tilde{W}_i)}{\partial z_{i,o}} = R_f, \quad \frac{\partial \mathbb{E}_i(\tilde{W}_i)}{\partial z_{i,k}} = y_{i,k}, \quad \frac{\partial \sigma_i^2(\tilde{W}_i)}{\partial z_{i,k}} = 2 \sum_{l=1}^K \sigma_{i,kl} z_{i,l}$$

for $k = 1, 2, \dots, K$. Then (A.1) and (A.2) become

$$V_{i1} R_f = \lambda_i, \quad (\text{A.3})$$

$$V_{i1} y_{i,k} + 2V_{i2} \sum_{l=1}^K \sigma_{i,kl} z_{i,l} = \lambda_i p_{ko}, \quad k = 1, 2, \dots, K. \quad (\text{A.4})$$

Substituting (A.3) into (A.4) leads to

$$V_{i1}[y_{i,k} - R_f p_{ko}] + 2V_{i2} \sum_{l=1}^K \sigma_{i,kl} z_{i,l} = 0, \quad k = 1, 2, \dots, K, \quad (\text{A.5})$$

which in matrix notation can be written as

$$V_{i1}[\mathbf{y}_i - R_f \mathbf{p}_o] + 2V_{i2} \Omega_i \mathbf{z}_i = 0.$$

This, together with assumption (H2), leads to the optimal portfolio (2.4) of investor i at the market equilibrium. \square

APPENDIX B. PROOF OF PROPOSITION 3.2

Proof. It follows from the individuals demand (2.4) and the market clearing condition (3.1) that

$$\mathbf{z}_m = \sum_{i=1}^I \bar{\mathbf{z}}_i = \sum_{i=1}^I \mathbf{z}_i^* = \sum_{i=1}^I \tau_i \Omega_i^{-1} [\mathbf{y}_i - R_f \mathbf{p}_o]. \quad (\text{B.1})$$

Under the definitions (3.2) and (3.3), equation (B.1) can be rewritten as

$$\begin{aligned} \mathbf{z}_m &= \sum_{i=1}^I \tau_i \Omega_i^{-1} \mathbf{y}_i - R_f \left(\sum_{i=1}^I \tau_i \Omega_i^{-1} \right) \mathbf{p}_o \\ &= \tau_a \Omega_a^{-1} \left[\Omega_a \sum_{i=1}^I \frac{\tau_i}{\tau_a} \Omega_i^{-1} \mathbf{y}_i - R_f \mathbf{p}_o \right] \\ &= \tau_a \Omega_a^{-1} [\mathbb{E}_a(\tilde{\mathbf{x}}) - R_f \mathbf{p}_o]. \end{aligned} \quad (\text{B.2})$$

This leads to the market equilibrium price (3.4). Inserting (3.4) into the optimal demand function of investor- i in (2.4) we obtain the equilibrium demand (3.5) of investor- i for the risky assets. The uniqueness of the consensus belief follows from the uniqueness of the equilibrium price and the construction. \square

APPENDIX C. PROOF OF COROLLARY 3.3

Proof. From (B.2) and (3.6),

$$0 < \sigma_m^2 = \tau_a [\mathbb{E}_a(\tilde{W}_m) - R_f W_{m,o}]$$

and hence

$$\mathbb{E}_a(\tilde{W}_m) - R_f W_{m,o} = \sigma_m^2 / \tau_a. \quad (\text{C.1})$$

On the other hand, from (3.4),

$$\mathbb{E}_a(\tilde{\mathbf{x}}) - R_f \mathbf{p}_o = \Omega_a \mathbf{z}_m / \tau_a.$$

This last equation, together with (C.1), lead to the CAPM-like price relation (3.7) under heterogeneous beliefs in vector form. \square

APPENDIX D. PROOF OF COROLLARY 3.4

Proof. We divide throughout by $p_{k,o}$ on both sides of (3.8), then

$$[\mathbb{E}_a(\tilde{r}_k)+1]-[r_f+1] = \frac{W_{m,o}\sigma(\tilde{W}_m, \tilde{x}_k)}{p_{k,o}\sigma_m^2} [(\mathbb{E}_a(\tilde{r}_m)+1)-(r_f+1)], \quad k = 1, 2, \dots, K.$$

That is,

$$\mathbb{E}_a(\tilde{r}_k) - r_f = \beta_k [\mathbb{E}_a(\tilde{r}_m) - r_f], \quad k = 1, 2, \dots, K,$$

where

$$\beta_k = \frac{W_{m,o}}{p_{k,o}} \frac{\sigma(\tilde{W}_m, \tilde{x}_k)}{\sigma_m^2} = \frac{\text{cov}_a(\tilde{x}_k/p_{k,o}, \tilde{W}_m/W_{m,o})}{\text{Var}_a(\tilde{W}_m/W_{m,o})} = \frac{\text{cov}_a(\tilde{r}_m, \tilde{r}_k)}{\sigma_a^2(\tilde{r}_m)}.$$

□

APPENDIX E. PROOF OF COROLLARY 4.1

Proof. In fact, in this case, it follows from (4.2) that the aggregate variance of asset j , $\sigma_{a,j}^2$, is a weighted harmonic mean of the beliefs in variance. It is well known that, for any continuous convex function $f(x)$, $f(\sum_{i=1}^n \alpha_i x_i) \leq \sum_{i=1}^n \alpha_i f(x_i)$ holds for $\alpha_i > 0$ satisfying $\sum_{i=1}^n \alpha_i = 1$. The equality holds if and only if all x_i are the same. Using this fact by taking $f(x) = 1/x$, we see that

$$\sigma_{a,j}^2 \leq \sum_{i=1}^I \frac{\tau_i}{\tau_a} \sigma_{i,j}^2 = \bar{\sigma}_j^2. \quad (\text{E.1})$$

□

APPENDIX F. PROOF OF COROLLARY 4.3

Proof. For $\epsilon > 0$, we can verify that both assets have the same average $\bar{\sigma}_2^2 = \bar{\sigma}_1^2$ for the chosen $\delta > 0$. Now we compute the consensus variances (that is the variances determined by market aggregation) $\sigma_{a,1}^2$ and $\sigma_{a,2}^2$, for asset 1 and 2. We obtain

$$(\sigma_{a,1}^2)^{-1} = \frac{\tau_1 \sigma_{2,1}^2 + \tau_2 \sigma_{1,1}^2}{\tau_a \sigma_{1,1}^2 \sigma_{2,1}^2},$$

and

$$(\sigma_{a,2}^2)^{-1} = \frac{\tau_1 \sigma_{2,1}^2 (1 + \delta) + \tau_2 \sigma_{1,1}^2 (1 - \epsilon)}{\tau_a \sigma_{1,1}^2 (1 - \epsilon) \sigma_{2,1}^2 (1 + \delta)},$$

respectively, where $\tau_a := \tau_1 + \tau_2$, and $\delta = \epsilon \frac{\omega_1 \sigma_{1,1}^2}{\omega_2 \sigma_{2,1}^2}$. The proof then follows from imposing $\sigma_{a,2}^2 \leq \sigma_{a,1}^2$. □

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