A NOTE ON BROUSSEAU'S SUMMATION PROBLEM

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Abstract. This paper takes a historical view of some long-standing problems associated with the development of sums of Fibonacci numbers in which the latter have powers of integers as coefficients. The sequences of coefficients of these polynomials are arrayed in matrices with links to *The On-Line Encyclopedia of Integer Sequences.* This is an extension of previous work on the summation problem of Ledin because Brousseau introduced some elegant techniques for contracting the summations and the papers of both authors link with some interesting matrices.

1. Introduction

Brousseau [1] and Ledin [4] studied Fibonacci sums of the form

$$S\,(m,n)=\sum_{k=1}^{n}k^{m}F_{k} \tag{1.1}$$

for integer $m,n\geq 0$ and where F_k is the k^{th} Fibonacci number, and generalized the calculations of them up to a point. As an example, $S(2,2)=F_1+4F_2=5$. Shannon and Ollerton [6] discussed some further generalisations and developed a recurrence relation with associated conjectures. It is the purpose of this note to extend from Brousseau's paper some number theoretic techniques for further exploration and to revisit the recurrence relation of [6] using matrix methods.

2. Using Brousseau's Approach

Ledin showed that equation (1.1) can be expressed in the form

$$S\,(m,n)=P_1\,(m,n)\,F_n+P_2\,(m,n)\,F_{n+1}+C\,(m) \tag{2.1}$$

in which $C(m)$ is a constant depending only on m, and $P_1(m,n)$ and $P_2(m,n)$ are polynomials in n of degree m of the form

$$P_i\,(m,n)=\sum_{j=0}^{m}(-1)^j\binom{m}{j}M_{i,j}n^{m-j},\quad i=1,2, \tag{2.2}$$

where $M_{1,j}$ and $M_{2,j}$ are integers dependent only on j.

In this section, we provide additional results for the $S(m,n)$, $P_i(m,n)$ and $C(m)$ functions. The following are examples of equation (2.1) for $m=4,5$ to make the subsequent discussion clearer:

We thank the participants of the 19^{th} International Fibonacci Conference for useful comments on an earlier version.

$$S(4,n) = \left(n^4 - 4n^3 + 30n^2 - 124n + 257\right) F_n$$
$$+ \left(n^4 - 8n^3 + 48n^2 - 200n + 416\right) F_{n+1} - 416,$$

$$S\,(5,n) = \left(n^5 - 5n^4 + 50n^3 - 310n^2 + 1285n - 2671\right) F_n$$
$$+ \left(n^5 - 10n^4 + 80n^3 - 500n^2 + 2080n - 4322\right) F_n + 4322.$$

It was also shown previously [6] that $M_{1,j}$ and $M_{2,j}$ follow the patterns set out in Table 1. *The On-Line Encyclopedia of Integer Sequences* (OEIS) [7] notes that sequence A000556 gives the coefficients of the expansion of $e^{-x}/(1 - e^x + e^{-x})$.

TABLE 1. $M_{1,j}$ and $M_{2,j}$

j	0	1	2	3	4	5	6	7	OEIS
$M_{1,j}$	1	1	5	31	257	2671	33305	484471	A000556
$M_{2,j}$	1	2	8	50	416	4322	53888	783890	A000557

We start with the P functions themselves since they are part of the main result and the referee of the previous paper [6] raised a question about the polynomial nature of $P_1(m, n)$. Suppose we consider a real variable x in equation (2.2), then

$$\begin{aligned}\frac{d}{dx} P_1\,(m+1, x) &= \sum_{j=0}^{m+1} (-1)^j\,(m+1-j) \binom{m+1}{j} M_{1,j} x^{m-j} \\ &= \sum_{j=0}^{m} (-1)^j\,(m+1)\,\frac{m!}{(m-j)\,!\,j!} M_{1,j} x^{m-j} \\ &= (m+1) \sum_{j=0}^{m} (-1)^j \binom{m}{j} M_{1,j} x^{m-j} \\ &= (m+1)\,P_1(m, x).\end{aligned}$$

Similarly,

$$\frac{d}{dx} P_2\,(m+1, x) = (m+1)\,P_2\,(m, x)\,. \tag{2.3}$$

Ledin arrived at similar results though in a less direct manner (and [4] contains typographical errors in the statement of the P_2 case). They do suggest though that rather than using a differential operator we try using a difference operator:

$$\Delta y_n = y_{n+1} - y_n. \tag{2.4}$$

Following Brousseau, we consider a quantity of the form $f[n, F_{(n)}]$, a function of n and Fibonacci numbers involving n in their subscripts, and define the adapted finite difference relation as follows:

$$\Delta f\left[n, F_{(n)}\right] = f\left[n+1, F_{(n+1)}\right] - f\left[n, F_{(n)}\right]. \tag{2.5}$$

For example:

$$\begin{aligned}\Delta\left(n^2F_n\right) &= (n+1)^2F_{n+1} - n^2F_n \\ &= n^2F_{n-1} + (2n+1)\,F_{n+1}.\end{aligned}$$

Similarly, the corresponding inverse operator Δ^{-1} is defined so that

$$\Delta^{-1}\left(f\left[(n+1)\,,F_{(n+1)}\right] - f\left[n, F_{(n)}\right]\right) = f\left[n, F_{(n)}\right] + C_1, \tag{2.6}$$

where C_1 is an arbitrary summation constant independent of n. Pond [5] has made use of forms of these two operators to establish several Horadam sequence summations [3] which can also be related to this present work through their well-known connections with the Fibonacci sequence.

Following Brousseau, if we let $S\,(m,n) = \sum_{k=1}^{n} k^mF_k$ be denoted by $\phi\left[(n+1)\,,F_{(n+1)}\right]$, then

$$S\,(m,n) = \phi\left[n, F_{(n)}\right] + n^mF_n, \tag{2.7}$$

with the difference relations

$$\begin{aligned}\Delta\left(\phi\left[n, F_{(n)}\right]\right) &= \sum_{k=1}^{n} k^mF_k - \sum_{k=1}^{n-1} k^mF_k \\ &= n^mF_n\end{aligned}$$

and

$$\begin{aligned}\Delta^{-1}\left(n^mF_n\right) &= \phi\left[n, F_{(n)}\right] + C_1\,(m) \\ &= S\,(m, n-1) + C_1\,(m)\end{aligned}$$

where the dependency of the summation constant on m in this case is noted. For example:

$$\begin{aligned}\Delta\left(nF_{n+1}\right) &= (n+1)\,F_{n+2} - nF_{n+1} \\ &= (n+1)\,F_{n+2} - n\left(F_{n+2} - F_n\right) \\ &= nF_n + ((n+1) - n)\,F_{n+2} \\ &= nF_n + \Delta\,(n)\,F_{n+2}\end{aligned}$$

and

$$\begin{aligned}\Delta\left(n^2F_{n+1}\right) &= (n+1)^2\,F_{n+2} - n^2F_{n+1} \\ &= n^2F_n + (2n+1)\,F_{n+2} \\ &= n^2F_n + \Delta\left(n^2\right)F_{n+2}.\end{aligned}$$

In general,

$$\Delta\left(n^mF_{n+1}\right) = n^mF_n + \Delta\left(n^m\right)F_{n+2}. \tag{2.8}$$

Similarly, with Δ^t denoting repeated applications of the difference operator, it can be shown that

$$\Delta^{-1}\left(n^mF_n\right) = \sum_{t=0}^{m} (-1)^t\Delta^t\left(n^m\right)F_{n+2t+1} + C_2\,(m) \tag{2.9}$$

for summation constant $C_2(m)$, which can be verified by calculating $\Delta(\Delta^{-1}(n^mF_n))$, the left hand side of which is obviously n^mF_n, and noting the contractions that occur on the right hand side when adding terms of the form

$$\Delta\left((-1)^t\,\Delta^t\left(n^m\right)F_{n+2t+1}\right)=(-1)^t\left(\Delta^t\left(n^m\right)F_{n+2t}+\Delta^{t+1}\left(n^m\right)F_{n+2(t+1)}\right),\qquad t=0,1,\ldots,m. \tag{2.10}$$

Thus,

$$S\left(m,n-1\right)=\sum_{t=0}^{m}(-1)^t\Delta^t\left(n^m\right)F_{n+2t+1}+C\left(m\right) \tag{2.11}$$

where the summation constant $C(m)$ is determined by use of $S(m,0)=0$ in equation (2.1).

Brousseau's approach leads to some contractions in the summations. For example, with repeated use of the connections between adjacent Fibonacci numbers in the case of $m=3$, equation (2.9) gives:

$$\begin{aligned}\Delta^{-1}\left(n^3F_n\right)&=n^3F_{n+1}-\Delta\left(n^3\right)F_{n+3}+\Delta^2\left(n^3\right)F_{n+5}-\Delta^3\left(n^3\right)F_{n+7}+C_2\left(3\right)\\&=n^3F_{n+1}-\left(3n^2+3n+1\right)F_{n+3}+\left(6n+6\right)F_{n+5}-6F_{n+7}+C_2\left(3\right)\\&=\left(n^3-3n^2+9n-19\right)F_{n+3}-\left(n^3-6n+12\right)F_{n+2}+C_2\left(3\right).\end{aligned}$$

Another example with the difference operators applied to Brousseau's method is from (2.11) with $m=5$ and summation terms evaluated at $n=50$:

$$\begin{aligned}S\left(5,49\right)&=\sum_{k=1}^{49}k^5F_k\\&=[n^5F_{51}-\Delta\left(n^5\right)F_{53}+\Delta^2\left(n^5\right)F_{55}-\Delta^3\left(n^5\right)F_{57}\\&\quad+\Delta^4\left(n^5\right)F_{59}-\Delta^5\left(n^5\right)F_{61}]_{n\to 50}+C\left(5\right).\end{aligned}$$

Some of the terms are calculated as in Table 2. That is,

TABLE 2. $\left[\Delta^t\left(n^5\right)\right]_{n\to 50}$, $t=0,1,\ldots,5$

t	$\left[\Delta^t\left(n^5\right)\right]_{n\to 50}$	Value
0	50^5	312500000
1	51^5-50^5	32525251
2	$52^5-2\cdot 51^5+50^5$	2653530
3	$53^5-3\cdot 52^5+3\cdot 51^5-50^5$	159150
4	$54^5-4\cdot 53^5+6\cdot 52^5-4\cdot 51^5+50^5$	6240
5	$55^5-5\cdot 54^5+10\cdot 53^5-10\cdot 52^5+5\cdot 51^5-50^5$	120

$$\sum_{k=1}^{49}k^5F_k=312500000F_{51}-32525251F_{53}+2653530F_{55}-159150F_{57}+6240F_{59}-120F_{61}+C\left(5\right), \tag{2.12}$$

in which the constant $C(5)$ can be found from equations (2.1) and (2.2) with Table 1 as

$$C(5)=-P_2\left(5,0\right)=M_{2,5}=4322,$$

or by applying Brousseau's approach with $n = 1$:

$$\begin{aligned} 0 &= S(5,0) \\ &= \sum_{k=1}^{0} k^5 F_k \\ &= \left[\Delta^{-1}\left(n^5 F_n\right)\right]_{n\to 1} \\ &= [n^5 F_2 - \Delta\left(n^5\right) F_4 + \Delta^2\left(n^5\right) F_6 - \Delta^3\left(n^5\right) F_8 \\ &\quad + \Delta^4\left(n^5\right) F_{10} - \Delta^5\left(n^5\right) F_{12}]_{n\to 1} + C(5) \\ &= -4322 + C(5) \end{aligned}$$

after calculating $\left[\Delta^t\left(n^5\right)\right]_{n\to 1}$ in similar manner to Table 2. Thus,

$$\begin{aligned} \sum_{k=1}^{49} k^5 F_k &= 312500000 F_{51} - 32525251 F_{53} + 2653530 F_{55} \\ &\quad - 159150 F_{57} + 6240 F_{59} - 120 F_{61} + 4322 \\ &= 4947840524712253969. \end{aligned}$$

The preceding example also motivates the following formula for the $C(m)$ terms of Ledin's form for $S(m,n) = \sum_{k=1}^{n} k^m F_k = P_1(m,n) F_n + P_2(m,n) F_{n+1} + C(m)$ using Brousseau's approach:

$$\begin{aligned} C(m) &= -P_2(m,0) \\ &= (-1)^{m+1} M_{2,m} \\ &= -\sum_{t=0}^{m} (-1)^t \left[\Delta^t\left(n^m\right)\right]_{n\to 1} F_{2+2t}. \end{aligned} \tag{2.13}$$

The summation constants from repeated use of (2.9) or (2.11), as set out below:

$$S(0,n) = (1) F_n + (1) F_{n+1} - (0F_2 + 1F_1) = (1) F_n + (1) F_{n+1} - 1,$$

$$\begin{aligned} S(1,n) &= (n-1) F_n + (n-2) F_{n+1} + (1F_3 + 0F_2) \\ &= (n-1) F_n + (n-2) F_{n+1} + 2, \end{aligned}$$

$$\begin{aligned} S(2,n) &= \left(n^2 - 2n + 5\right) F_n + \left(n^2 - 4n + 8\right) F_{n+1} - (2F_4 + 1F_3) \\ &= \left(n^2 - 2n + 5\right) F_n + \left(n^2 - 4n + 8\right) F_{n+1} - 8, \end{aligned}$$

$$\begin{aligned} S(3,n) &= \left(n^3 - 3n^2 + 15n - 31\right) F_n + \left(n^3 - 6n^2 + 24n - 50\right) F_{n+1} + (7F_5 + 5F_4) \\ &= \left(n^3 - 3n^2 + 15n - 31\right) F_n + \left(n^3 - 6n^2 + 24n - 50\right) F_{n+1} + 50, \end{aligned}$$

and shown further in Table 3, confirm that the absolute values of the summation constants $C(m)$ do belong to the sequence $M_{2,j}$. The Fibonacci expressions are not unique and the coefficients do not seem to belong to known OIES sequences, so interested readers may like to explore these expressions further.

Table 3. Summation constants (unsigned, c.f. Table 1)

m	Fibonacci Expression	Value
0	$0F_2 + 1F_1$	1
1	$1F_3 + 0F_2$	2
2	$2F_4 + 1F_3$	8
3	$7F_5 + 5F_4$	50
4	$37F_6 + 24F_5$	416
5	$242F_7 + 147F_6$	4322
6	$1861F_8 + 1139F_7$	53888
7	$16679F_9 + 10324F_8$	783890
8	$171362F_{10} + 106089F_9$	13031936

3. Using Matrix Methods

In this section, matrix methods will be used to explore the $S(m,n) = \sum_{k=1}^{n} k^m F_k$ function from two perspectives: a previously derived recurrence relation for $S(m,n)$, and Brousseau's approach described above.

Recurrence Relation. In [6], the authors of the present note used algebraic methods to show that $S(m,n)$ satisfies a linear recurrence relation in m, as follows. For integer $m, n \geq 0$, let

$$p(m,n) = \sum_{k=1}^{n} k^m = \sum_{j=0}^{m+1} a_{m+1,j} n^j \tag{3.1}$$

where

$$a_{m+1,j} = \frac{1}{(m+1)} \binom{m+1}{j} B^{+}_{m+1-j} \qquad (j > 0) \tag{3.2}$$

and B_i^+ are the appropriately signed Bernoulli numbers, and let

$$b_{m+1,j} = \sum_{r=j}^{m+1} \binom{r}{j} a_{m+1,r} \tag{3.3}$$

for $j \neq m$; in the case $j = m$, let $b_{1,0} = 2$ and $b_{m+1,m} = 5/2$ for $m > 0$. Then,

$$\frac{1}{m+1} S(m+1,n) = p(m,n+1) F_n + p(m,n) F_{n+1} - \sum_{j=0}^{m} b_{m+1,j} S(j,n) \tag{3.4}$$

with initial condition $S(0,n) = F_n + F_{n+1} - 1$.

We note that the recurrence relation (3.4) reflects the form of Ledin's result in the polynomial and Fibonacci product terms but his P_1 and P_2 polynomials do not appear explicitly.

To develop some matrix results for the recurrence relation's $b_{m+1,j}$ coefficients, we begin with the matrix form of formulas for sums of powers of integers $p\,(m,n) = \sum_{k=1}^{n} k^m$. At first glance, the coefficients of the sums of powers of integers formulas look uninspiring:

$$\begin{pmatrix} \sum_{k=1}^{n} k^0 \\ \sum_{k=1}^{n} k^1 \\ \sum_{k=1}^{n} k^2 \\ \sum_{k=1}^{n} k^3 \\ \sum_{k=1}^{n} k^4 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \\ 1/2 & 1/2 & 0 & 0 & 0 & \\ 1/6 & 1/2 & 1/3 & 0 & 0 & \cdots \\ 0 & 1/4 & 1/2 & 1/4 & 0 & \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & \\ & & \vdots & & & \ddots \end{pmatrix} \begin{pmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ \vdots \end{pmatrix}. \tag{3.5}$$

However, as is well known, row-sums equal one, alternating-signed row-sums are zero for rows $r \geq 1$, and the coefficients depend only on those in the first column which are the (suitably signed) Bernoulli numbers B_j^+. Further, the inverse matrix is immediately recognised as a Pascal-type matrix ([2]):

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \\ 1/2 & 1/2 & 0 & 0 & 0 & \\ 1/6 & 1/2 & 1/3 & 0 & 0 & \cdots \\ 0 & 1/4 & 1/2 & 1/4 & 0 & \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & \\ & & \vdots & & & \ddots \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \\ -1 & 2 & 0 & 0 & 0 & \\ 1 & -3 & 3 & 0 & 0 & \cdots \\ -1 & 4 & -6 & 4 & 0 & \\ 1 & -5 & 10 & -10 & 5 & \\ & & \vdots & & & \ddots \end{pmatrix} \tag{3.6}$$

so there is a more fundamental connection between sums of powers of integers and Fibonacci numbers than is immediately evident in the formulas mentioned so far for $S\,(m,n) = \sum_{k=1}^{n} k^m F_k$.

The algebraic proof of the recurrence relation for $S(m,n)$ given in [6] involves the following constructions to obtain the $b_{m+1,j}$ coefficients:

$$a_{m+1,j} = \frac{1}{m+1} \binom{m+1}{j} B_{m+1-j}^+ \tag{3.7}$$

for $m > 0$, with $a_{0,0} = 1$ and $a_{0,j} = 0$ for all $j > 0$;

$$a_{m+1,j}^* = \sum_{r=j}^{m+1} a_{m+1,r} \binom{r}{j} \tag{3.8}$$

for all j; and

$$b_{m+1,j} = a_{m+1,j}^* \tag{3.9}$$

for $j \neq m$, with $b_{m+1,m} = a_{m+1,m}^* + 1$.

Noting that $\{a_{m+1,j}\} = P^{*-1}$ where

$$P^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & \\ 0 & -1 & 2 & 0 & 0 & \cdots \\ 0 & 1 & -3 & 3 & 0 & \\ 0 & -1 & 4 & -6 & 4 & \\ & & \vdots & & & \ddots \end{pmatrix}, \tag{3.10}$$

and letting

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \\ 1 & 4 & 6 & 4 & 1 & \\ & & \vdots & & & \ddots \end{pmatrix} \tag{3.11}$$

and

$$I^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ & & \vdots & & & \ddots \end{pmatrix}, \tag{3.12}$$

the algebraic manipulations displayed above indicate that the recurrence relation coefficients in (3.4) are related directly to Pascal-type matrices:

$$\{b_{m+1,j}\} = B = P^{*-1}P + I^*. \tag{3.13}$$

The resulting recurrence relation coefficients also seem somewhat uninspiring:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 2 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 5/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 1 & 13/6 & 5/2 & 1/3 & 0 & 0 & \\ 1 & 3 & 13/4 & 5/2 & 1/4 & 0 & \\ 1 & 119/30 & 6 & 13/3 & 5/2 & 1/5 & \\ & & \vdots & & & & \ddots \end{pmatrix}, \tag{3.14}$$

even after noticing that row-sums are $2^{r-1}+2$ for rows $r \geq 1$ and suitably alternating-signed row-sums are unity. However, taking the hint given by the sums of powers of integers coefficient matrix and its invere, we observe that:

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ -2 & 1 & 0 & 0 & 0 & 0 & \\ 8 & -5 & 2 & 0 & 0 & 0 & \cdots \\ -50 & 31 & -15 & 3 & 0 & 0 & \\ 416 & -257 & 124 & -30 & 4 & 0 & \\ -4322 & 2671 & -1285 & 310 & -50 & 5 & \\ & & \vdots & & & & \ddots \end{pmatrix} \tag{3.15}$$

in which the first two columns are the (signed) M_2 and M_1 sequences, respectively (see Tables 1 and 3 above), fundamental to the functions in Ledin's form for $S(m,n)$ in equation (2.1). It is then clear that Ledin's functions are encoded in the recurrence relation coefficients matrix B in quite a straightforward manner.

Further, just as the Pascal triangle coefficients produce the P^{*-1} matrix coefficients (including the Bernoulli numbers) which provide formulas for $\sum_{k=1}^{n} k^m$, i.e.,

$$P^{*-1}\begin{pmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} k^0 \\ \sum_{k=1}^{n} k^1 \\ \sum_{k=1}^{n} k^2 \\ \sum_{k=1}^{n} k^3 \\ \sum_{k=1}^{n} k^4 \\ \vdots \end{pmatrix}, \tag{3.16}$$

the M_1 and M_2 sequences underpinning the $b_{m+1,j}$ coefficients of the recurrence relation for $S(m,n) = \sum_{k=1}^{n} k^m F_k$ provide the following formulas:

$$B\begin{pmatrix} 1 \\ n \\ n^2 \\ n^3 \\ n^4 \\ \vdots \end{pmatrix} = \left(P^{*-1}P + I^*\right)\begin{pmatrix} 1 \\ n \\ n^2 \\ n^3 \\ n^4 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ \sum_{k=1}^{n+1} k^0 + 1 \\ \sum_{k=1}^{n+1} k^1 + n \\ \sum_{k=1}^{n+1} k^2 + n^2 \\ \sum_{k=1}^{n+1} k^3 + n^3 \\ \vdots \end{pmatrix}. \tag{3.17}$$

Brousseau's Approach. In this subsection, we observe that equations (2.11) and (2.13) derived using Brousseau's approach can also be written in terms of matrices. Noting that

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \\ 1 & 4 & 6 & 4 & 1 & \\ & & \vdots & & & \ddots \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \\ -1 & 1 & 0 & 0 & 0 & \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ -1 & 3 & -3 & 1 & 0 & \\ 1 & -4 & 6 & -4 & 1 & \\ & & \vdots & & & \ddots \end{pmatrix}, \tag{3.18}$$

for $(m+1)\times(m+1)$ sized P^{-1} matrices we have

$$\begin{aligned} S(m,n-1) &= \sum_{t=0}^{m} (-1)^t \Delta^t (n^m) F_{n+2t+1} + C(m) \\ &= F(n) P^{-1} N(m,n) - F(1) P^{-1} N(m,1) \end{aligned}$$

for vectors

$$F(n) = \begin{pmatrix} F_{n+1} & -F_{n+3} & \cdots & (-1)^{m+1} F_{n+2m+1} \end{pmatrix} \tag{3.19}$$

and

$$N(m,n) = \begin{pmatrix} n^m & (n+1)^m & \cdots & (n+m)^m \end{pmatrix}^T. \tag{3.20}$$

In particular, Ledin's constant term is

$$\begin{aligned} C(m) &= -\sum_{t=0}^{m} (-1)^t \left[\Delta^t (n^m)\right]_{n\to 1} F_{2+2t} \\ &= F(1) P^{-1} N(m,1) \end{aligned}$$

where

$$F(1) = \begin{pmatrix} F_2 & -F_4 & \cdots & (-1)^{m+1} F_{2(m+1)} \end{pmatrix} \tag{3.21}$$

and

$$N(m,1) = \begin{pmatrix} 1^m & 2^m & \cdots & (m+1)^m \end{pmatrix}^T. \tag{3.22}$$

For example:

$$C(5) = -\begin{pmatrix} F_2 & -F_4 & F_6 & -F_8 & F_{10} & -F_{12} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ -1 & 5 & -10 & 10 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1^5 \\ 2^5 \\ 3^5 \\ 4^5 \\ 5^5 \\ 6^5 \end{pmatrix} = 4322 \tag{3.23}$$

as expected in $\{M_{2,j}\} = \{1,\ 2,\ 8,\ 50,\ 416,\ \mathbf{4322},\ 53888,\ 783890, \ldots\}$, and which displays previously unobserved structure of the M_2 sequence terms.

4. Concluding Comments

Analysing a problem from different perspectives is fundamental to mathematical research. The application of both Brousseau's approach and matrix methods to the analysis of $S(m,n) = \sum_{k=1}^{n} k^m F_k$ provides opportunities for deeper understanding of this function. Both approaches are clearly applicable to a wider range of related problems.

Areas of further research include:

- The OEIS notes that the M_1 sequence (A000556) gives the coefficients of the expansion of $e^{-x}/(1-e^x+e^{-x})$. How is this function related to Ledin's form and/or the recurrence relation? N.B. $e^{-x}/(1-e^x+e^{-x}) = 1/(1+x-x^2)|_{x\to e^x}$ which involves the characteristic polynomial of the Fibonacci sequence. Also, the OEIS does not provide a similar function for the M_2 sequence (A000557).
- What can be said more generally about $\sum_{k=1}^{n} f(m,k) F_k$?
- Our paper [6] contains some conjectures which are also related to the material in this note.

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