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Essays on Market Design

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Certificate of original authorship

I, Hao Zhou, declare that this thesis is submitted in fulfillment of the requirements for the award of Doctor of Philosophy, in the Business School at the University of Technology Sydney. This thesis is wholly my own work unless otherwise indicated in the references or acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis. This document has not been submitted for qualifications at any other academic institution. This research was supported by the Australian Government Research Training Program.

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Abstract

This thesis consists of two independent essays. They are unified by the common theme of market design.

In the first essay, we consider a school choice program with the diversity constraints on the composition of the students by implementing affirmative actions. Students may belong to more than one disadvantaged types (e.g. a student may both belong to the "minority" and "financially distressed" group.) In addition, they may have (weak) preference orders over the channels through which they would possibly be enrolled (e.g. a student from a poor family may be granted a scholarship). We consider the case where the school tries to assign as many reserved seats as possible by sophisticatedly reshuffling the seats to the eligible students. We provide a choice rule that eliminates justified envy, satisfies the no swapping condition and is acceptant under this situation. We further show that the choice rule is bilaterally substitutable and thus the cumulative offering process can produce a stable outcome.

In the second essay, we establish the optimal auction when a seller can incentivize an existing buyer to refer a privately known potential buyer to compete for an object. We identify three optimal channels to provide referral incentives: discouraging non-referral, favoring referral, and providing informational rent for referral. While the first two channels always appear and are essential, the third one is supplementary and appears when the potential buyer is less likely to exist and stronger. We also provide conditions under which the optimal mechanism can be implemented by simple mechanisms. Finally, we show that the conventional resale mechanism is suboptimal.

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Chapter 1

Introduction

This thesis consists of two independent essays, unified by the common theme of market design.

Chapter 2 considers a controlled school choice problem where each student may have multiple disadvanged types. We investigate the case where some seats of the school are reserved for targeted groups and each student can take at most one reserved seat. We further assume that each student has weak preference over different type-specific seats which they are eligible to take. The school's objective is to assign the reserved seats as many as possible while taking students' preferences into account.

We propose three desirable properties of the choice function of the school. That is, no justified envy, no swapping condition and acceptance. A student i who is not enrolled has justified envy towards j if either of the following conditions hold. First, when i has a higher priority than j, the number of the reserved seats assigned should not reduce by replacing j with i. Second, when i has a lower priority than j, it should allow the school to assign strictly more reserved seats by enrolling i and rejecting j. No swapping condition says that there is no enrolled student who can be strictly better off by switch the type of the contract without hurting those who have higher priorities. Acceptance simply says that the school should enrolled the students up to its total capacity.

We propose the sequential reservation choice rule and show that it is the unique choice rule that satisfies these three properties. We then provide other properties of the sequential reservation choice rule such as strategy-proofness and bilaterally substitutable.

In Chapter 3, we establishe the optimal auction when a seller can incentivize an existing buyer to refer a privately known potential buyer to compete for an object. We characterize the optimal mechanism in this environment and we identify three optimal channels to provide referral incentives: discouraging non-referral, favoring referral, and providing informational rent for referral. We show that the first two channels always appear and are essential, while the third one is supplementary and appears when the potential buyer is less likely to exist and stronger. We also provide conditions under which the optimal mechanism can be implemented by simple mechanisms. Finally, we show that the conventional resale mechanism is suboptimal.

This thesis concludes with Chapter 4, where the key findings are summarized and ideas for future areas of research are outlined.

Chapter 2

Affirmative Actions with Multi-Dimensional Privileged Types

2.1 Introduction

Controlled school choice programs require considering students' priorities while maintaining the diversity constraints on the composition of the students. One typical aim is to provide underrepresented students with options to close the opportunity gap. This chapter investigates the controlled school choice problems where each student may have multiple disadvantaged types. For example, a students may both belong to the "minority" and the "financially distressed" group. Suppose each enrolled student can fill one reserved seat of one of her types, the school then is given the flexibility to decide which type of the reserved seat each enrolled student should take. For example, suppose the school has one reserved seat for the "minority" and one for the "financially distressed". Student A has both of the types and student B only belongs to the minority group. The school can assign both of the reserved seats by enrolling student A with financially distressed and student B with minority. It is a non-trivial problem for the school to assign the type-specific seats to the students if it do not want to "waste" the reserved seats. It is also not difficult to see that the way of assigning the reserved seats can have a significant effect on the outcome of the enrollment. In an important recent work, Sönmez and Yenmez (2021) propose a choice rule that allow the school to assign as many reserved seats as possible where they assume that each student is indifferent to the type they are enrolled.

This chapter generalizes Sönmez and Yenmez (2021) by considering the case where the students are not indifferent to the channels they could potentially be enrolled. Numerous examples show that this setting is relevant. Aygün and Turhan (2020) examine the admissions to Engineering schools (IITs) in India. They quote the survey showing that 56% of the students enrolled through under-privileged castes and tribes feel discriminated against in the school. This is because some opponents of the affirmative action policies believe that those who are enrolled through reserved seats decrease the average quality of schools. As a result, some students with high test scores who also belong to disadvantaged groups prefer not using their privileges. They would rather be enrolled as the general category instead. They also argue that the true preferences of students should be over program name-seat type pairs, not just program names. As another example, if a student belongs to the "financially distressed" category who uses her privilege and is assgined to a reserved seat then she may be granted a scholarship or a lower tuition fee. This may induce a preference for being admitted as a financially distressed student.

One can also think of a more general one-to-many matching model where students have preferences over "terms". Terms are only available to those who have certain qualifications. Our model can then serve as a special case if we assume that there is a one-to-one correspondence between terms and qualifications. From this view, our model can be applied to the fields other than affirmative action policies. For instance, China proposed the *Strengthening Basic Academic Disciplines Plan* in 2020 as a supplementary enrollment scheme to the *College Entrance Exam*, which focuses on students with special talents in math, physics, chemistry, biology, history, philosophy and ancient characters. According to the plan, universities will reserve a certain number of seats for majors of these basic academic disciplines. In addition, students who are enrolled through this plan are expected to enjoy a better learning environment and greater chances of admission to the graduate school. The students who wants to be enrolled through this plan should show the talent for the specific decipline. For example, they should rank high in the corresponding subject competitions. Assuming that students is not indifferent to the way they are enrolled is natural in this case. First, students are eligible for multiple deciplines and they naturally have preferences over these deciplines. Second, there are trade-offs on whether to apply for this plan in the first place. This is because these deciplines, especially humanities majors, are unpopular among students. It's harder for students from these majors to find good job opportunities after graduation and the switch of major is very restrictive if one is enrolled through the plan. [1]

We show that when taking into account the preferences of the students, the school's objective of assigning as many reserved seats as possible may incur conflict to fairness. We show that it could be possible that a student with a higher priority and strictly more disadvantaged types may take a less preferred type seat. We then propose three axioms that a desirable outcome in the weak preferences environment should satisfy. That is, no justified envy, no swapping condition and acceptence.

The concept of no justified envy defined in this chapter incorporates the usual spirit in the matching theory literature that a student with a higher priority should be enrolled prior than those who with lower priorities. In addition, the school should provide extra priori to those who have disadvantaged types. In particular, a student i who is not enrolled has justified envy towards an enrolled student j if either of the following two situations are satisfied. First, suppose i has a higher priority than j. By replacing j with i, there is still a way for the school to keep the same number of assigned reserved seats. Second, suppose i has a lower priority than j. The school can assign strictly more reserved seats by replacing j with i. No swapping condition requires that the enrolled students can ask for swap of the types of the reserved seats as long as the swapping will not make those who with higher priorities strictly worse off. Acceptant condition simply says that the school should choose the students up

[1] On the other hand, this plan can help solve the "admissions difficulty" for some graduate programs, so universities are quite active to promote it. This coincides with our assumption that the school wants to assign as many reserved seats as possible.

to its total capacity.

We then provide the sequential reservation choice rule that satisfies these properties. The sequential reservation choice rule not only specifies the set of the students which should be enrolled but also indicate which type of the seat an enrolled student should take. To do so, we introduce the strict and weak reassignment chain finding processes. The strict reassignment chain allows us to check whether enrolling a new student can let the school assign one more reserved seat and the weak reassignment chain allows us to determine whether enrolling a new student can let the school to keep the same cardinal number of reserved seats.

We further show that the sequential reservation choice rule is characterized by these three axioms. That is, it is the unique choice rule that satisfies these axioms. We also show that the sequential reservation choice rule is strategy proof and can be implemented in polynomial times.

Finally, we introduce the matching market where there are multiple schools. We show that the sequential reservation choice rule is bilaterally substitutable so that it can produce a stable outcome when embedded into the cumulative offering process.

Several papers on school choice with diversity constraints are related to our work. Kojima (2012) examines the model with two types of the students (namely, minority and majority), and shows that setting hard-bounds ceiling for the number of majority students may hurt the minority students. Hafalir, Yenmez, and Yildirim (2013) then propose the soft-bounds floors for the number of the minority students to overcome this problem. Ehlers, Hafalir, Yenmez, and Yildirim (2014) further extend the model of Hafalir et al. (2013) by relaxing the number of the privileged types. In Ehlers et al. (2014), there could be more than just one privileged type in total, but each student belongs to only one type.

Based on the seminal papers of affirmative actions where each student is assumed having only one type, controlled school choice models with students who may have multiple types are examined more recently. These papers are closely related to this work. Aygun and Bó (2021) study the implementation of the college admission rule in Brazil where public universities are mandated to use the affirmative policies for candidates from ethinical and income minorities, where the existing policy may reject high achieving minorities and accept low achieving majorities. The main difference between their paper and ours is that in their model, seats are reserved for the type combinations [2] while in this chapter the seats are reserved for each individual type. We argue that our setting has advantages. This is because with the number of type increases, the number of type combinations will increase exponentially. It is thus computationally difficult for the schools to set affirmative action goals. Kurata, Hamada, Iwasaki, and Yokoo (2017) consider a model in which students may have multiple types and have strict preferences over these types. Their work is diffetent from ours because they further assume that the school has a strict preference over the contracts. This rules out the flexibility of the school to assign the reserved seats and makes their setting quite different from ours.

A recent paper Sönmez and Yenmez (2021) is most related to ours. We introduce their model as the benchmark in section 3. The main difference between this chapter and theirs is that they assume students are indifferent to the privileged types and we show that in the weak preferences environment, it could cause unfairness. Sönmez and Yenmez (2021) also consider the one-to-many convention of the affirmative action policy, meaning that when a student is enrolled, she consumes one reserved seat for all of her types. We restrict our attention to the one-to-one convention, that is, when a student is enrolled, she consumes one reserved seat for one of her types. This is mainly because in the one-to-many convention, the preferences over types it is not meaningful.

There are also some papers in the computer science literature that study related problems. Aziz and Sun (2021) extend Sönmez and Yenmez (2021) such that the school has multi-ranked diversity goals, that is, there may be a requirement that some types should be filled only after other types are filled. Aziz, Gaspers, and Sun (2020) transfer the original policy targets on individual types to some artificial

[2]For example, the school may reserve 30 seats for the minorities, 30 seats for the financially distressed and 15 seats for the students who both belong to the minority and the financially distressed group.

targets on the type combinations. We do not treat the problem in this way because this setting may result in cases where the total number of the transferred reserved seats exceeds the total capacity.

This chapter is also related to the school choice problems with the distributional constraints (Goto, Kojima, Kurata, Tamura, and Yokoo (2017); Kamada and Kojima (2015, 2017)), where the regional maximum and minumum quotas are imposed. Finally, this chapter serves as an application of the general framework for matching with constrains (see Goto et al. (2017); Kojima, Tamura, and Yokoo (2018)).

The rest of the chapter is organized as follows. In section 2.2, we introduce the setting of our model. In section 2.3, we provide the results of Sönmez and Yenmez (2021) in one-to-one reservation convention as a benchmark. Section 2.4 introduces desirable properties of a choice rule in the weak preferences environment. We propose the sequential reservation choice rule in section 2.5. In section 2.6, we show our choice rule is characterized by the desired properties. Finally, in section 2.7, we extend our model to the multiple school environment and show that our choice rule leads to the the stable outcome in this environment.

2.2 Setup

We first introduce a single school environment where there is only one school endowed with several reserved seats for the privileged types, as well as normal seats. Each student has a weak preference over the type specific seats. We assume that each student can take at most one privileged type seat. The school enrolls a subset of students and decides what type of seat each enrolled student is assigned to.

A single school with reserved seats environment is denoted by $(\mathcal{S},\mathcal{T},\tau,\pi,q,(r_t)_{t\in\mathcal{T}},(\succeq_i^*)_{i\in\mathcal{S}})$. There is a finite set of students $\mathcal{S} = \{s_1,s_2,...,s_n\}$. $\mathcal{T} = \{t_1,...,t_m\}$ is the set of privileged types and we refer to t_0 as the general type. τ is a function mapping $\mathcal{S}$ to $2^{\mathcal{T}\cup\{t_0\}}$ specifying each student's types. We assume that $t_0 \in \tau(i)$ for all $i \in \mathcal{S}$. π is a strict priority order over $\mathcal{S}$ and q is the total capacity of the school. For each

$t_k \in \mathcal{T}$, denotes r_{t_k} the number of reserved positions for students with type t_k. We assume that the total number of the reserved seats is smaller than the capacity of the school. That is, $\sum_{t_k \in \mathcal{T}} r_{t_k} \leq q$. For each student i, $\succeq_i^*$ represents her weak preference over $\tau(i)$, where $t_k \succeq_i^* t_l$ means student i weakly prefers t_k over t_l. We denote the set of all possible preferences by Γ.

Throughout this chapter, we use the language of matching with contract proposed by Hatfield and Milgrom (2005). A contract $x = (i, t_k)$ means to enroll student i with one of her types t_k (note that a legitimate contract requires that $t_k \in \tau(i)$.) Let $\mathcal{X}$ be the set of all the legitimate contracts. We denote the null contract by $\emptyset$, which means a student is not enrolled. For $X \subseteq \mathcal{X}$, we define $X_i = \{(i, t_k) \in X | t_k \in \tau(i)\}$ and $X_{t_k} = \{(i, t_k) \in X | i \in \mathcal{S}\}$. In addition, for $S \subseteq \mathcal{S}$, $X_S = \cup_{i \in S} X_i$ and for $T \subseteq \mathcal{T}$, $X_T = \cup_{t_k \in T} X_{t_k}$. Given a set of contracts X, $s(X)$ refers to the set of students who have contracts in X, i.e. $s(X) = \{i \in \mathcal{S} | X_i \neq \varnothing\}$ and $\tau(i|X) = \{t_k \in \tau(i) | (i, t_k) \in X\}$ is the types of student i in X. Given a set of contracts X and student i's preference over her types $\succeq_i^*$, we define her induced preference over X_i such that $(i, t_k) \succeq_i (i, t_l)$ if and only if $t_k \succeq_i^* t_l$. We assume all the contracts (i, t_k) are acceptable to student i. Thus, $(i, t_k) \succeq_i \emptyset$ for all $(i, t_k) \in \mathcal{X}$.

A choice rule $Ch : 2^{\mathcal{X}} \times \Gamma^{|S|} \to 2^{\mathcal{X}}$ is a function that maps the Cartesian product of a considerate set of contracts X and students' preferences profile $\{\succeq_i^*\}_{i \in S}$ to a subset of X. This chosen set of contracts is called an outcome. For notational convenience, given a fixed preferences profile $\{\succeq_i^*\}_{i \in S}$, we simply write the choice function as $Ch(X)$ in stead of $Ch(X, \{\succeq_i^*\}_{i \in S})$ if no confusion arises.

Definition 2.1. *Given a set of contracts X, an outcome $Ch(X) \subseteq X$ is called **feasible** if it satisfies:*
1. $|Ch(X)| \leq q$, $|Ch(X)_{t_k}| \leq r_{t_k}$ for all $t_k \in \mathcal{T}$, and
2. $|Ch(X)_i| \leq 1$ for all $i \in \mathcal{S}$.

The first condition says that the school can sign at most r_{t_k} type t_k contracts and

q contracts in total. The second condition says that a student can sign at most one contract with the school. Note that by the definition of the feasible outcome, we only let each student be enrolled through a single type. Given X, $\mathcal{O}$ denotes the set of all feasible outcomes. Suppose Y is a feasible outcome. We sometimes write $(i,t_k) \succeq_i Y$ if one of the following is true: (1). $Y_i = (i,t_l)$ for some $t_l \in \tau(i)$ and $t_l \neq t_k$ such that $(i,t_k) \succeq_i (i,t_l)$, or (2). $Y_i = \emptyset$. We also write $Y \succeq_i Z$ if i is weakly better off under the outcome Y compared to the outcome Z.

Definition 2.2. *Given a set of contracts X, let $Y \in \mathcal{O}$ be a feasible outcome.* ***The number of assigned reserved seats*** *$\eta(Y)$ is the total number of contracts with privileged types. That is, $\eta(Y) = \sum_{t_k \in \mathcal{T}} |Y_{t_k}|$.*

Definition 2.3. *Given a set of contracts X, $\rho(s(X))$ is the* ***the maximum number of reserved seats*** *that can be assigned to the students in $s(X)$. That is*

$$\rho(s(X)) = \max_{\{Y \in \mathcal{O} | s(Y) \subseteq s(X)\}} \eta(Y)$$

We say that an outcome has ***maximum cardinality in reserve matching*** *if it assigns the maximum number of reserved seats to the students in $s(X)$ and $\mathcal{M}(s(X))$ denotes the set of all these outcomes. That is, $\mathcal{M}(s(X)) = argmax_{\{Y \in \mathcal{O} | s(Y) \subseteq s(X)\}} \eta(Y)$.*

Definition 2.4. *Given X and let $Y \in \mathcal{O}$ be a feasible outcome, a student $i \notin s(Y)$* ***increases reserve utilization with respect to*** *Y if there exists $Y' \in \mathcal{O}$ such that $s(Y') = s(Y) \cup \{i\}$ and $\rho(s(Y')) > \rho(s(Y))$.*

Our goal is to assign as many the reserved seats as possible given certain fairness conditions.

2.3 The Benchmark

Sönmez and Yenmez (2021) consider the situation where all students are indifferent to the type they are enrolled. They find a choice rule that maximally complies with the reservations, eliminates justified envy and is non-wasteful. Maximally complying with the reservation means that the school chooses the set of students that can take the maximum cardinality of the reserved seats. Eliminating justified envy means if a student with lower priority is enrolled while a high priority student is not, this only happens when enrolling the low priority student can increase the reserve utilization. Non-wastefulness means that the school always chooses the students up to its total capacity.

They call the choice rule the horizontal envelope choice rule (denoted as $\tilde{Ch}$) and it works as follows:

Consider the students in the order of their priorities one by one.
Round 1: Choose a student in this round if and only if she can increase reserve utilization with respect to the currently enrolled pool of the students.
Round 2: Choose the remaining students with the highest priority until all the positions are filled or there is no unchosen students remaining.

Note that, since the students are indifferent between types, the outcome of the horizontal envelope choice rule is a set of students rather than a set of contracts. They further show that the chosen set of the students that satisfies the maximally complying with the reservations, eliminating justified envy then non-wastefulness is unique, implying that these three properties fully characterized the choice rule.

2.4 The weak preferences environment

We first provide some motivative examples which show that when students have weak preferences over the types, directly employing the horizontal envelope choice rule can be problematic since the preferences are not taken into consideration.

Example 2.1. *(Unfairness in the weak preferences environment)*

- $\mathcal{S} = \{s_1,s_2,s_3,s_4\}$ *with* $s_1 \pi s_2 \pi s_3 \pi s_4$
- $\mathcal{T} = \{t_1,t_2\}$, $r_{t_1} = 2$, $r_{t_2} = 1$, $q = 3$
- $\tau(s_1) = \{t_0,t_1,t_2\}$; $\tau(s_2) = \{t_0,t_2\}$; $\tau(s_3) = \{t_0,t_1\}$; $\tau(s_4) = \{t_0,t_2\}$

- $t_2 \succ_{s_1} t_1 \succ_{s_1} t_0$; $t_2 \succ_{s_2} t_0$; $t_1 \succ_{s_3} t_0$; $t_2 \succ_{s_4} t_0$

In this case, we can show that $\tilde{Ch}(\mathcal{S}) = \{s_1, s_2, s_3\}$. *In addition, the only possible way to accomodate* s_1, s_2 *and* s_3 *in the horizontal envelope choice rule is to match* s_1 *with* t_1, s_2 *with* t_2 *and* s_3 *with* t_1.

In Example 2.1, even though s_1 has a higher priority than s_2 and s_2's types is a proper subset of s_1's, s_1 will be enrolled through a less preferred type (i.e., t_1). If being admitted through t_2 means being provided with a more generous scholarship, $\tilde{Ch}$ can cause unfairness in the sense we illustrate in this example.

A more appropriate outcome is to enroll s_1 with t_2, s_2 with t_0 and s_3 with t_1. The reasoning is as follows. First, since s_1 has the highest priority and the richest type space. s_1 should be enrolled with his top choice, namely t_2. Now the school is facing a sub-problem. There are three candidates, s_2, s_3 and s_4; the total capacity for the school is 2 and there are 2 reserved seats for t_1. To enroll s_3 with type t_1 and s_2 with t_0 is the straightforward solution to this sub-problem.

Example 2.2. *(Paternalism in the weak preferences environment)*

- $\mathcal{S} = \{s_1, s_2, s_3\}$ *with* $s_1 \pi s_2 \pi s_3$
- $\mathcal{T} = \{t_1\}$, $r_{t_1} = 1$, $q = 2$
- $\tau(s_1) = \{t_0, t_1\}$; $\tau(s_2) = \tau(s_3) = \{t_0\}$
- $t_0 \succ_{s_1} t_1$

In this environment, we can confirm that $\tilde{Ch}(\mathcal{S}) = \{s_1, s_2\}$ *with* s_1 *matched to* t_1 *and* s_2 *matched to* t_0.

In this example, student s_1 has the highest priotity and is the only one who has the privileged type. Thus, s_1 would have been enrolled even if she doesn't have the privileged type. In the example, s_1 prefers to be enrolled as the normal type than the privileged type t_1. This can be justified by interpreting that being enrolled as t_1 will incur some obligations. A more appropriate outcome is to enroll s_1 with t_0 and s_2 with t_0.

In the remaining of this section, we formalize the properties that a desirable choice rule should have. We first introduce the concept of no justified envy.

Definition 2.5. *Given a feasible outcome Y, a student i* ***has justified envy*** *towards student j where $i \notin s(Y)$ and $j \in s(Y)$, if either of the following condition holds: There exists a feasible outcome Z such that $s(Z) = (s(Y) \cup \{i\}) \setminus \{j\}$, $\eta(Z) \geq \eta(Y)$ and $Z \succeq_m Y$ for all $m\pi i$ except for j. In addition, if $j\pi i$, the inequality should be strict.*

In words, a student i who is not enrolled has justified envy towords an enrolled student j if either of the following two conditions hold. First, suppose i has higher priority than j. By enrolling i and rejecting j, there is still a way for the school to keep the same number of assigned reserved seats. Second, suppose i has lower priority than j. The school can assign strictly more reserved seats by replacing j with i. Such assignment may involve reassigning other students' seats. We impose the restriction on the reassignment that those who has higher priority than i will not be strictly worse off.

Note that this definition of no justified envy provide extra priori for those who can let the school increase reserve utilization. Thus, the spirit that the school should not waste the reservation seats is inherantly embedded in this concept of no justified envy. On the other hand, the school cannot assign as many reserved seats as it wishes. The restriction that the reassignment of reserved seats should not hurt those who has higher priority than i can prevent undesired situations in Example 2.1 and 2.2 and this makes the definition different from the concept proposed by Sönmez and Yenmez (2021).

Definition 2.6. *An outcome Y* ***exhibits no justified envy*** *if no student has justified envy. A choice rule* ***has no justified envy*** *if it always chooses outcomes that exhibit no justified envy.*

Example 2.3. *(No justified envy)*

- $\mathcal{S} = \{s_1, s_2, s_3\}$ *with* $s_1 \pi s_2 \pi s_3$

- $\mathcal{T} = \{t_1\}$, $r_{t_1} = 1$, $q = 2$
- $\tau(s_1) = \tau(s_3) = \{t_0,t_1\}$; $\tau(s_2) = \{t_0\}$
- $t_1 \sim_{s_1} t_0$; $t_1 \sim_{s_3} t_0$

Consider the outcome $Y_1 = \{(s_1,t_0),(s_3,t_1)\}$. *In* Y_1, s_2 *has justified envy towards* s_3. *To see this, let* $Z_1 = \{(s_1,t_1),(s_2,t_0)\}$. s_1 *is indifferent between* Y_1 *and* Z_1 *while* s_2 *prefers* Z_1 *to* Y_1 *and the total number of the privileged contracts are the same. So* s_2 *has justified envy. On the other hand,* $Y_2 = \{(s_1,t_1),(s_2,t_0)\}$ *has no justified envy.*

The concept of no justified envy describes if a student is not enrolled in a particular outcome, in what condition she can block this outcome. In our setting, students care about the types of seats, as is shown in Example 2.1 and 2.2, so it is possible that even if a student is enrolled, she still want to block the outcome. We introduce the concept of no swapping condition indicating under what circumstances a enrolled student can block the outcome.

Definition 2.7. *Given a feasible outcome* Y, *a student* i *who has a contract in* Y *can* ***ask for swapping by a privileged contract*** *if:*

- *There exists* (i, t_k) *such that* $(i, t_k) \succ_i Y$, *and*
- *There exists a feasible outcome* Z *with* $(i, t_k) \in Z$ *such that* $Z \succeq_m Y$ *for all* $m \pi i$.

In words, a student i can ask for swapping by a privileged contract (i, t_k) if the school can reassign the reserved seats to the enrolled students and all the students who have a higher priority than i are not worse off because of the reassignment. Note that such reassignment may sacrifice the number of reserved seats assigned, in order to satisfy fairness.

Definition 2.8. *Given a feasible outcome* Y, *a student* i *who has a contract* (i,t_k) *in* Y ***can ask for swapping by a normal contract*** *if:*

- $(i,t_0) \succ_i (i,t_k)$; *and*
- *There exists a feasible outcome* Z *with* $(i,t_0) \in Z$ *such that* $Z \succeq_m Y$ *for all*

$m \pi i$.

- *No student has justified envy towards i under Z.*

The only difference between asking for swapping by a privileged contract and a normal one is that when a student wants to replace her contract by a normal one, a further requirement which says no one will have justified envy towards i under the new outcome should be satisfied. This requirement is needed because in our setting, the school gives the students with privileged types extra prioiri and if a student want to be enrolled as a normal type, that means she should give up the extra priori. The next example makes this point clear.

Example 2.4. *(Asking for swapping by a normal contract)*

- $\mathcal{S} = \{s_1,s_2,s_3\}$ *with* $s_1 \pi s_2 \pi s_3$
- $\mathcal{T} = \{t_1\}$, $r_{t_1} = 1$, $q = 2$
- $\tau(s_3) = \{t_0,t_1\}$; $\tau(s_1) = \tau(s_2) = \{t_0\}$
- $t_0 \succ_{s_3} t_1$

Let $Y = \{(s_1,t_0),(s_3,t_1)\}$. Here, s_3 cannot exchange his contract by (s_3,t_0) since s_3 should not be enrolled if she give up the extra priority of being belong to the t_1 group.

Definition 2.9. *We say that an outcome satisfies the* ***no swapping condition*** *if no student can ask for swapping by a privileged contract or a normal contract.*

Example 2.1(continued) The outcome $\{(s_1,t_1),(s_2,t_2),(s_3,t_1)\}$ does not satisfies the no swapping condition because s_1 ask for swapping by a privileged contract (s_1,t_2).

Example 2.2(continued) The outcome $\{(s_1, t_1),(s_2,t_0)\}$ does not satisfies the no swapping condition because s_1 can ask for swapping by a normal contract (s_1,t_0).

Lastly, we provide another commonly used property of choice rule.

Definition 2.10. *A choice rule is* ***acceptant*** *if for every $X \subseteq \mathcal{X}$, $|Ch(X)| = \min\{|s(X)|, q\}$.*

Acceptance says that the school enrolls the students up to its capacity. This condition is commonly required in the matching literature and we will see in section 6, it is crucial to characterize the choice rule that we propose in the next section.

Note that there are two channels that the concept of no justified envy condition and no swapping condition can potentially prevent the maximum utilization of the reserved seats. First, by the requirement of no justified envy conditon, a student i may not has justfied envy towards another student j even if replacing j with i can allow the school to assign one more reserved seat. This is because such reassignment may hurt some other students with a higher priority. For instance, in example 2.1, in the proposed outcome $\{(s_1,t_2),(s_3,t_1),(s_2,t_0)\}$, s_4 does not have justified envy towards s_2, even if the outcome $\{(s_1,t_1),(s_3,t_1),(s_4,t_2)\}$ assigns one more reserved seat. Second, no swapping condition allows some enrolled students to ask for swapping by a a normal contract. This again reduces the utilization of reserved seats. In example 2.2, s_1 is entitled to choose to be enrolled with normal type.

2.5 The Sequential Reservation Choice Rule

Now we propose the sequential reservation choice rule that has no justified envy, satisfies the no swapping condition and is acceptant in the weak preference environment.

We first propose the static reservation choice rule which is modified from the horizontal envelope choice rule proposed by Sönmez and Yenmez (2021). The static reservation choice rule provides us wth a systematic way to determine whether a new student can increase the reservation utilization and explicitly tells us the type-specific seat each enrolled student will be assigned. To do this, we first introduce a concept called the "rearrangement chain".

2.5.1 The Rearrangement Chain

We specify two kinds of rearrangement chain. We use the strict rearrangement chain to determine whether adding a new student with one of her privileged contracts is feasible given the existing pool of contracts and if it is, we can reassign the type seats to the students according to the rearrangement chain. By doing so, the school can strictly enlarge the utilization of the reserved seats by one. We also use the weak rearrangement chain to add a new student without decreasing the cardinal number of assigned reserved seats. Both of them are crucial to defining the choice rule we shall propose.

Definition 2.11. *Given a feasible outcome $Y \in \mathcal{O}$, we call the following sequence of contracts:*
$(s^1,t^1),(s^2,t^1),(s^2,t^2),(s^3,t^2),...,(s^k,t^k)$ *where* $s^i \neq s^j$, $t^i \in \mathcal{T}$, $t^i \neq t^j$ *for all* $i,j \in \{1,2,...,k\}$ *a* ***strict rearrangement chain****, if it satisfies:*

1. *the* $2n^{th}$ *contracts belong to* Y*, where* $n \in \{0,1,...,k-1\}$*;*
2. *the* $(2n+1)^{th}$ *contracts do not belong to* Y*, where* $n \in \{0,1,...,k-1\}$*;*
3. $|Y_{t^k}| < r_{t^k}$*.*

Definition 2.12. *Given a feasible outcome $Y \in \mathcal{O}$, we call the following sequence of contracts:*
$(s^1,t^1),(s^2,t^1),(s^2,t^2),(s^3,t^2),...,(s^k,t^k)$ *where* $s^i \neq s^j$, $t^i \in \mathcal{T}$, $t^i \neq t^j$ *for all* $i,j \in \{1,2,...,k-1\}$ *a* ***weak rearrangement chain****, if it satisfies:*

1. $2n^{th}$ *contracts belong to* Y*, where* $n \in \{0,1,...,k-1\}$*;*
2. $(2n+1)^{th}$ *contracts do not belong to* Y*, where* $n \in \{0,1,...,k-1\}$*;*
3. $t^k = t_0$ *and* $|Y| < q$*.*

Note that each rearrangement chain contains odd number of contracts. Each $2n^{th}$ contract has the same type as the contract immediately prior to it and each $(2n+1)^{th}$ has the same student as the contract immediately prior to it. The only difference of the strict and weak rearrangement chain is that, in a strict rearrangement chain, all the contracts are privileged contracts and we require that the number of contracts in the current pool which share the same type with the last contract of the chain (namely, (s^k,t^k)) is strictly less than the number of the seats reserved for that type while in a weak rearrangement chain, the last contract of the chain is a normal contract and we only require that the number of overall contracts in the current pool is strictly less than the capacity of the school. We now provide an example of the strict rearrangement chain to illustrate the idea.

Example 2.5. *(The strict rearrangement chain)*

- $\mathcal{S} = \{s_1,s_2,s_3,s_4\}$ *with* $s_1 \pi s_2 \pi s_3 \pi s_4$
- $\mathcal{T} = \{t_1,t_2,t_3\}$, $r_{t_1} = r_{t_2} = 1$, $r_{t_3} = 2$
- $\tau(s_1) = \{t_0,t_2,t_3\}$; $\tau(s_2) = \{t_0,t_1,t_3\}$; $\tau(s_3) = \{t_0,t_1,t_3\}$; $\tau(s_4) = \{t_0,t_1\}$
- $Y = \{(s_2,t_1),(s_1,t_3),(s_3,t_3)\}$
- *A strict rearrangement chain is:* $(s_4,t_1),(s_2,t_1),(s_2,t_3),(s_1,t_3),(s_1,t_2)$.

The idea of rearrangement chain is that we can add the first contact into Y and replace $2n^{th}$ contracts by $(2n+1)^{th}$ contracts and since the number of the contracts with the type in the last contract of the chain does not hit the type-specific upper bound, (i.e., $|Y_{t_k}| < r_{t_k}$) in the strict rearrangement chain or the total capacity, (i,e, $|Y| < q$) in the weak rearrangement chain, the resulting outcome is still feasible. In addition, by doing the replacement according to the strict rearrangment chain, the school can assign one more reserved seat and by doing the replacement according to the weak rearrangement, the school can assign the same number of the reserved seats as the original outcome.

Thus, in Example 2.5, under Y, the school assigns three reserved seats in total. Starting with Y, according to the rearrangement chain, the school can reassign the t_1 seat which is previously assigned to s_2 under Y to s_4, where s_2 now in turn takes s_1's t_3 seat and s_1 takes the t_2 seat. The existence of the untaken t_1 seat is assured by the last condition of the strict reassignment chain. After being operated by the rearrangement chain, we now get another feasible outcome $Y' = \{(s_4,t_1),(s_2,t_3),(s_1,t_2),(s_3,t_3)\}$, where the school assigns four reserved seats to the students.

Also note that both strict and weak rearrangement chain can consist of a single contract where we simply add this contract to the current pool and in that case no "rearrangment" occurs.

Example 2.5 shows that starting with a particular outcome Y, if we can find a strict rearrangement chain starting with a new student, we can assign one more reserved seat. The next lemma shows that if we can assign one more reserved seat with respect to one particular feasible outcome Y, then we can do the same thing with respect to every feasible outcomes containing the same set of students as Y.

Lemma 2.1. *Given a feasible outcome $Y \in \mathcal{O}$, if a student $i \notin s(Y)$ increases reserve utilization with respect to Y then, for all feasible Y' such that $s(Y') = s(Y)$, i increases reserve utilization with respect to Y'.*

Equipped with lemma 2.1, it is without loss of generality for us to say that a student can increase reserve utilization with respect to a set of existing students without referring to the particular outcome.

Proposition 2.1. *Given a feasible outcome $Y \in \mathcal{O}$, a student $i \notin s(Y)$ increases reserve utilization with respect to Y if and only if we can find a strict rearrangement chain starting with (i, t_k) for some $t_k \in \tau(i) \setminus t_0$.*

As we can see in Example 2.5, suppose we can find a strict rearrangement chain, then the school can assign one more reserved seats to the students by replacing all the even number indexed contracts with the odd number indexed contracts. Proposition 2.1 says that the opposite is also true. Suppose there is an approach that the school can increase the reserve utilization by including a new student, then there is a strict rearrangement chain starting with one of the privileged contract of that student.

2.5.2 The Rearrangement Chain Finding Process

From Proposition 2.1 we can see that the rearrangement chain is important to determine whether adding a new student can make the school increase reserve utilization. The question now is that how to find a rearrangement chain given the current pool of the contracts and we answer this question in this subsection.

Given a consideration set X and a feasible outcome $Y \in \mathcal{O}$ and a student $i \notin s(Y)$

with a contract (i,t^1), we refer to the following process as **a strict rearrangement chain finding process**:

Step 1: Let A^1 be the students with type t^1 in Y, (i.e., $A^1 = s(Y_{t^1})$), and let $T^1 = \{t^1\}$. **1.1**: Suppose $|A^1| < r_{t^1}$, then call $Y' := \{(i,t^1)\}$ a strict rearrangement chain and stop.

1.2: Suppose $|A^1| = r_{t^1}$ and $\forall s \in A^1$, $\tau(s|X) \subseteq \{t_0\} \cup T^1$, then (i,t^1) is not chosen.

1.3: Suppose $|A^1| = r_{t^1}$ and $\exists s \in A^1$, $\tau(s|X) \nsubseteq \{t_0\} \cup T^1$, then define $T^2 = \{t_k | \exists s \in A^1 \text{ s.t. } t_k \in \tau(s|X) \text{ and } t_k \notin \{t_0\} \cup T^1\}$ and $A^2 = \cup_{t_k \in T^2} s(Y_{t_k})$. Go to the next step.

Step 2: Given A^2 and T^2.

2.1: Suppose $|A^2| < \sum_{t_k \in T^2} r_{t_k}$, then we can find $t^2 \in T^2$ such that $|Y_{t^2}| < r_{t^2}$ and $s^1 \in A^1$ with $t^2 \in \tau(s^1|X)$. Call $Y' := \{(i,t^1),(s^1,t^1),(s^1,t^2)\}$ a strict rearrangement chain and stop.

2.2: Suppose $|A^2| = \sum_{t_k \in T^2} r_{t_k}$ and $\forall s \in A^2$, $\tau(s|X) \subseteq \{t_0\} \cup T^1 \cup T^2$, then (i,t^1) is not chosen.

2.3: Suppose $|A^2| = \sum_{t_k \in T^2} r_{t_k}$ and $\exists s \in A^2$, $\tau(s|X) \nsubseteq \{t_0\} \cup T^1 \cup T^2$, then define $T^3 = \{t_k | \exists s \in A^2 \text{ s.t. } t_k \in \tau(s|X) \text{ and } t_k \notin \{t_0\} \cup T^1 \cup T^2\}$ and $A^3 = \cup_{t_k \in T^3} s(Y_{t_k})$. Go to the next step.

In general, step m: Given A^m and T^m.

m.1: Suppose $|A^m| < \sum_{t_k \in T^m} r_{t_k}$, then we can find $t^m \in T^m$ such that $|Y_{t^m}| < r_{t^m}$ and $s^{m-1} \in A^{m-1}$ with $t^m \in \tau(s^{m-1}|X)$. Call $Y' := \{(i,t^1),(s^1,t^1),(s^1,t^2),...,(s^{m-1},t^m)\}$ where $(s^j,t^j) \in Y$, $s^j \in A^j$ and $t^j \in T^j$ for all $1 \leq j \leq m$, a strict rearrangement chain and stop.

m.2: Suppose $|A^m| = \sum_{t_k \in T^m} r_{t_k}$ and $\forall s \in A^m$, $\tau(s|X) \subseteq \cup_{1 \leq j \leq m} T^j \cup \{t_0\}$, then (i,t^1) is not chosen.

m.3: Suppose $|A^m| = \sum_{t_k \in T^m} r_{t_k}$ and $\exists s \in A^m$, $\tau(s|X) \nsubseteq \cup_{1 \leq j \leq m} T^j \cup \{t_0\}$, then define $T^{m+1} = \{t_k | \exists s \in A^m \text{ s.t. } t_k \in \tau(s|X) \text{ and } t_k \notin \cup_{1 \leq j \leq m} T^j \cup \{t_0\}\}$ and $A^{m+1} = \cup_{t_k \in T^{m+1}} s(Y_{t_k})$. Go to the next step.

The process must terminate since $\cup T^j$ is strictly increasing. This process either produces a strict rearrangement chain Y' or does not choose (i,t^1).

In words, the strict rearragement chain finding process can be described as follows: First search for all the students who has type t^1 contracts in Y. We refer to all these students as A^1 and the type t^1 as T^1. If there is a vacant t^1 seat, then (i, t^1) is a strict rearrangement chain and we are done. If not, check whether the students in A^1 have privileged types other than the types in T^1. We refer to all these types as T^2. If T^2 is empty, then there is no strict rearrangement chain towards Y starting with (i,t^1). If it is nonempty, then we call all the students who has contracts in Y with the type in T^2 as A^2. We continue to check whether there is a vacant seat in T^2. If there is, then we are done and if not, we continue to check whether the students in A^2 has types which do not belong to $T^1 \cup T^2$. And so on.

Note that Given a feasible outcome Y and a strict rearrangement chain Y', we can define a new outcome $Y'' := (Y \cup Y') \setminus (Y \cap Y')$. This outcome is still feasible and the school enrolls one more student and assigns one more reserved seat compared to the original outcome Y.

Given the consideration set X, a feasible outcome Y with $|Y| < q$ and a student $i \notin s(Y)$ with a contract (i,t^1), we now provide the **weak rearrangement chain finding process** as follows:

Step 1: Let A^1 be the students with type t^1 in Y, (i.e., $A^1 = s(Y_{t^1})$), and let $T^1 = \{t^1\}$. **1.1:** Suppose $t^1 = t_0$, the call $Y' := \{(i,t^1)\}$ a weak rearrangement chain and stop.

1.2: Suppose $t^1 \neq t_0$ and $\forall s \in A^1$, $\tau(s|X) \subseteq T^1$, then (i, t^1) is not chosen.

1.3: Suppose $t^1 \neq t_0$ and $\exists s \in A^1$, such that $\tau(s|X) \nsubseteq T^1$, then define $T^2 = \{t_k | \exists s \in A^1 \text{ s.t. } t_k \in \tau(s|X) \text{ and } t_k \notin T^1\}$ and $A^2 = \cup_{t_k \in T^2} s(Y_{t_k})$. Go to the next step.

Step 2: Given A^2 and T^2.

2.1: Suppose $t_0 \in T^2$, then we can find $s^1 \in A^1$ with $t_0 \in \tau(s^1)$. Call $Y' := \{(i,t^1),(s^1,t^1),(s^1,t_0)\}$ a weak rearrangement chain and stop.

1.2: Suppose $t_0 \nsubseteq T^2$ and $\forall s \in A^2$, $\tau(s|X) \subseteq T^1 \cup T^2$, then (i, t^1) is not chosen.

1.3: Suppose $t^0 \nsubseteq T^2$ and $\exists s \in A^2$, $\tau(s|X) \nsubseteq T^1 \cup T^2$, then define $T^3 = \{t_k|\exists s \in A^2$ s.t. $t_k \in \tau(s|X)$ and $t_k \notin T^1 \cup T^2\}$ and $A^3 = \cup_{t_k \in T^3} s(Y_{t_k})$. Go to the next step.

In general, step m: Given A^m and T^m.

2.1: Suppose $t_0 \in T^m$, then we can find $s^{m-1} \in A^{m-1}$ with $t_0 \in \tau(s^{m-1|X})$. Call $Y' := \{(i,t^1),(s^1,t^1),(s^1,t^2),...,(s^{m-1},t_0)\}$ a weak rearrangement chain and stop.

1.2: Suppose $t_0 \nsubseteq T^m$ and $\forall s \in A^m$, $\tau(s|X) \subseteq \cup_{1\leq j\leq m} T^j$, then (i, t^1) is not chosen.

1.3: Suppose $t^0 \nsubseteq T^m$ and $\exists s \in A^m$, $\tau(s|X) \nsubseteq \cup_{1\leq j\leq m} T^j$, then define $T^{m+1} = \{t_k|\exists s \in A^m$ s.t. $t_k \in \tau(s|X)$ and $t_k \notin \cup_{1\leq j\leq m} T^j\}$ and $A^{m+1} = \cup_{t_k \in T^{m+1}} s(Y_{t_k})$. Go to the next step.

The process must terminate since $\cup T^j$ is strictly increasing. This process either produces a weak rearrangement chain Y' or does not choose (i,t^1).

In words, the weak rearrangement chain finding process work as follows: we first refer to all the students who has type t^1 contracts in Y as A^1 and the set of type t^1 as T^1. If the normal contract belongs to T^1, then $\{(i,t^1)\}$ is a weak rearrangement chain and we are done. Otherwise, check whether the students in A^1 have types other than the types in T^1. We refer to all these types as T^2. If T^2 is empty, then there is not weak rearrangement chain starting with (i,s^1). If it is nonempty, then we call all the students who has contract in Y with the type in T^2 as A^2. We continue to check whether the normal type belongs to T^2. If it is, when we can find the weak rearrangement chain and if not, we continue to check whether the students in A^2 have the types which do not belong to $T^1 \cup T^2$. And so on.

Note that given a feasible outcome Y and a weak rearrangement chain Y', we can define a new outcome $Y'' := (Y \cup Y') \setminus (Y \cap Y')$. This outcome is still feasible. In the new outcome, the school enrolls one more student and the total number of the reserved seats assigned remains the same.

2.5.3 The Reservation Choice Rule

The Static Reservation Choice Rule

Equipped with the rearrangement chain finding process, we can define the **static reservation choice rule** Ch^* as follows:

Consider the students in the order of their priorities one by one. Without loss of generality, we denote them by $S = \{s_1,s_2,...,s_l\}$ such that $s_1 \pi s_2 \pi s_3 \pi ... \pi s_l$.

Round 1: In this round, only the contracts with privileged types are considered.
Step 1: Set $Y^0 = \emptyset$. Consider s_1's privileged types (if any) one by one starting with the smallest index, and run the strict rearrangement chain finding process. Whenever we find a strict rearrangement chain Y_1, set $Y^1 = (Y^0 \cup Y_1) \setminus (Y^0 \cap Y_1)$ and go to the next step. Otherwise, s_1 will not be chosen in this step and set $Y^1 = Y^0$.
In general, step k: Given Y^{k-1}. Consider s_k's privileged types (if any) one by one starting with the smallest index, and run the strict rearrangement chain finding process. Whenever we find a strict rearrangement chain Y_k, set $Y^k = (Y^{k-1} \cup Y_k) \setminus (Y^{k-1} \cap Y_k)$ and go to the next step. Otherwise, s_k will not be chosen in this step and set $Y^k = Y^{k-1}$.
Round 2: The outcome of Round 1 is Y_l. Since the sum of the reserved seats is less than the total capacity, there are still vacant seats under Y_l. Consider the students who are not chosen in round 1 one by one based on their priorities. We denote them by we denote them by $S = \{s_a,s_b,...,s_j\}$ such that $s_a \pi s_b \pi s_c \pi ... \pi s_j$.
Step 1: Consider s_a's contracts in the order of the index of the types. Starting with the smallest index, run the weak rearrangement chain finding process. Whenever

we find a weak rearrangement chain Y_a, set $Y^a = (Y^l \cup Y_a) \setminus (Y^l \cap Y_a)$. Otherwise, s_a will not be chosen and set $Y^a = Y^l$. Suppose $|Y_a| < q$, go to the next step. Otherwise, Y_a is the outcome of the static reservation choice rule.

Continue this process until all positions are filled or there is no unchosen students remaining. The resulting outcome is the outcome chosen by the static reservation choice rule.

Roughly speaking, in the first round of the static reservation choice rule, the students are considered one by one according to their priorities. The school will enroll a new student if including her can strictly increase the reserve utilization. The type specific seat each enrolled student will get is determined by the strict rearrangement chains. In the second round, the students who are not chosen in the first round are considered. The school will enroll a new student if there are still some vacant seats and including her will not decrease the utilization of the reservation. The type specific seat each enrolled student (including those who are chosen in the first round) will get is determined by the weak rearrangement chains.

Also note that when searching for a rearrangement chain for a student, we start with the her type of the smallest index. This is without loss of generality. The order of searching for the rearragement will not affect the set of students being enrolled and this can be assured by Lemma 1.

The Sequential Reservation Choice Rule

The sequential reservation choice rule takes students' preferences into account. Before describing the choice rule, we first need to introduce the concept of ordered indiffence classes of the types.

Given a student's preference $\succsim_i$, we can partition $\tau(s_i)$ into several **indifference classes** according to $\sim_i$. That is, if two types are indifferent with each other, they are in the same indifference class. We can then rank these indifference classes according to $\succ_i$ to form **ordered indifference classes**.

For example, $(s_i,t_1) \sim (s_i,t_2) \succ (s_i,t_3) \sim (s_i,t_0)$. We refer to $\{(s_i,t_1),(s_i,t_2)\}$ as first indifference class and $\{(s_i,t_3),(s_i,t_0)\}$ as second indifference class. In general, we can denote i's ordered indifference classes by $J_{i1},J_{i2},...$, and we will denote corresponding contracts by $X_{i1},X_{i2},...$

The **sequential reservation choice rule** $Ch^{\#}$ works as follows:

Step 1: Each student's first indifference class contracts will be considered. Let $X_1 = \cup_{i\in\mathcal{S}} X_{i1}$. Find $Ch^*(X_1)$. Denote $S^1 = \mathcal{S} \setminus s(Ch^*(X_1))$.

Step 2: Each student in S^1's next indifference class contracts will be included into the considerate set, if any. Let $X_2 = \cup_{i\in S^1} X_{i2} \cup X_1$ Find $Ch^*(X_2)$. Denote $S^2 = \mathcal{S} \setminus s(Ch^*(X_2))$.

In general, Step k: Each student in S^{k-1}'s next indifference class contracts will be included into the considerate set, if any. Let X_k be the union of $\cup_{1\leq i\leq k-1} X_i$ and all the contracts newly included in this step. Find $Ch^*(X_k)$. Denote $S^k = \mathcal{S} \setminus s(Ch^*(X_k))$.

The process will terminate when each student who hasn't a contract been chosen at the last step doesn't have any further contracts to propose. Suppose the process stops at step t, the final outcome of $Ch^{\#}$ is $Ch^*(X^t)$.

2.5.4 An Example of the Sequential Reservation Choice Rule

Now we provide a comprehensive example to show how does the sequential reservation choice rule work.

Example 2.6. *(The Sequential Reservation Choice Rule)*

- $\mathcal{S} = \{s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8\}$ *with* $s_1 \pi s_2 \pi s_3 \pi s_4 \pi s_5 \pi s_6 \pi s_7 \pi s_8$
- $\mathcal{T} = \{t_1,t_2,t_3\}$, $r_{t_1} = r_{t_3} = 1$, $r_{t_2} = 2$, $q = 5$
- *The types and preferences of the students are summarized in table 1.*

Table 1

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
First indifference class	t_1, t_2	t_1	t_3, t_0	t_1	t_3	t_0	t_1, t_3	t_2, t_3
Second indifference class	t_0	t_0		t_0	t_2		t_0	t_0
Third indifference class					t_0			

The sequential reservation choice rule work as follows: In the first round, only the first indifference class contracts will be considered. So the considerate set is $X_1 = \{(s_1,t_1),(s_1,t_2),$ $(s_2,t_1),(s_3,t_3),(s_3,t_0),(s_4,t_1),(s_5,t_3),(s_6,t_0),(s_7,t_1),(s_7,t_3),(s_8,t_2),(s_8,t_3)\}$.

In step 1, (s_1,t_1) *will first be chosen. now consider* (s_2,t_1)*, we can find the following rearrangement chain* $\{(s_2,t_1),(s_1,t_1),(s_1,t_2)\}$*, so* (s_2,t_1) *will be chosen and* s_1 *will be reassigned with* (s_1,t_2)*. Then,* (s_3,t_3) *will be chosen,* s_4*,*s_5*,*s_6 *and* s_7 *will not be chosen in this step because they can not strictly increase the utilization of the reserved seats. Finally,* s_8 *will be enrolled as* t_2*.*

In step 2, those who have not been chosen in step 1 (in this example, they are s_4*,*s_5*,*s_6*,*s_7*) will be considered in priority order. In this step,* (s_5,t_3) *will be chosen and* s_3 *will be reassigned with* t_0*. Thus, the result in this round is* $Y_1 = \{(s_1,t_2),(s_2,t_1),(s_3,t_0),(s_8,t_2),(s_5,t_3)\}$.

In the second round, The considerate set will expand which will include the second indifference class contracts for those who has not been chosen in the first round. Thus, the considerate set $X_2 = X_1 \cup \{(s_4,t_0),(s_7,t_0),(s_8,t_0)\}$*. Then we do the same exercise as round 1 and this will lead to the outcome in this round* $Y_2 = \{(s_1,t_2),(s_2,t_1),(s_3,t_3),(s_8,t_2),(s_4,t_0)\}$.

In the third round the considerate set further expands to include s_5*'s second indifference class contracts. So* $X_3 = X_2 \cup \{(s_5,t_2)\}$*. The final outcome of the sequential reservation choice rule is* $Y_3 = \{(s_1,t_2),(s_2,t_1),(s_3,t_3),(s_5,t_2),(s_4,t_0)\}$.

2.6 The properties of the sequential reservation choice rule

In this section, we show several properties of the sequential reservation choice rule.

Proposition 2.2. *The sequential reservation choice rule has no justified envy, satisfies the no swapping condition and is acceptant.*

To prove Proposition 2.2, we need the following lemma.

Lemma 2.2. *For any contract* $(i,t_k) \in X_{il}$*, if no contact in* X_{il} *is chosen in a particular step of the sequential reservation choice rule* $Ch^{\#}$*, then* (i,t_k) *will not be chosen by any subsequent steps of* $Ch^{\#}$*.*

Note that Lemma 2.2 is different from the substitutes condition of the horizontal envelope choice rule $\tilde{Ch}$. The substitutes condition of $\tilde{Ch}$ requires that if a student is not chosen, then she will still be rejected when some new students are available to the school. By comparison, Lemma 2.2 involves adding new contracts of the existing students. In our setting, the expanding of consideration set may potentially incur the choice rule to revisit the formerly rejected contracts. As a simple example, suppose there are two students (say, s_1 and s_2 with s_1 has the higher priority) who both have the type t_1 but there is only one reserved seat for t_1. Then (s_2, t_1) is rejected. But suppose now (s_1, t_2) is available and there are vacant seats for t_2, then the school can reassign the reserved seat for t_2 to s_1 so that (s_2, t_1) can be chosen. Thus in this sense, (s_1,t_2) and (s_2,t_1) are complementary to each other. However, lemma 2.2 rules out this possibility. Intuitively speaking, this is because the consideration set cannot expand arbitarily. A new constract of an existing student will be considered only when none of the existing constracts of this student is chosen in the current step.

Proposition 2.3. *Let* Y *be the outcome of running* $Ch^{\#}$ *on* X *and* Z *be another outcome that eliminates justified envy, satisfies the no swapping condition and is non-wasteful. Then for each student* i*,* i *is indifferent between* Y *and* Z*. In addition,* Z *can be obtained by first reordering the index of the types and then running the sequential reservation choice rule.*

Proposition 2.3 establishes that, the sequential reservation choice rule is unique. That is, eliminating justified envy, satisfying the no swapping condition and non-wastefulness characterizes the sequential reservation choice rule. This result is also a generalization of Sönmez and Yenmez (2021)'s result that when all the student is indifferent to the way they are enrolled, the chosen set of the students that satisfies maximally filling the reserved seats, eliminates justified envy and is acceptant is unique.

Proposition 2.4. *The sequential reservation choice rule can be implemented in polynomial time.*

This proposition shows that the sequential reservation choice rule is computationally efficient.

The next proposition shows that the sequential choice rule is strategy-proof. That is, it is a weakly domiant strategy for each student to reveal her own preference truthfully. Note that this property is not satisfied if the school simply want to make the maximum utilization of the reserved seat. In Example 2.1, if s_1 hides her type t_1, then she will be enrolled through t_2, which make her strictly better off. In Example 2.2, if s_1 hides her privileged type t_1, she will be enrolled through t_0 which she strictly prefers.

Proposition 2.5. *For each student i and for each fixed X, $Ch^{\#}(X, \succsim_{-i}, \succsim_i) \succsim_i Ch^{\#}(X, \succsim_{-i}, \succsim_i')$, for all $\succsim_i'$ in Γ.*

2.7 Multiple school environment

In this section, we explore the environment where there are multiple schools. Our analysis in the previous sections is useful in the decentralized market where each school's enrollment decision is independent of others, while in the centralized market, we need to consider a mechanism that can lead to the stable outcome. We first introduce some notations.

The set of the school is denoted by $\mathcal{C} = \{c_1, c_2, ..., c_p\}$. π_a is the strict priority order over the set of the students in school a and q_a is the total capacity of school a. We denote the the quota of the reserved seats for type t_k in school a by $r^a_{t_k}$. A contract $x : (i, a, t_k)$ now means student i is enrolled by school a with one of her type t_k. The set of all the legitimate contracts is denoted by $\mathcal{X}$ and all the legitimate contracts associated with student i is denoted by $\mathcal{X}_i$ and we assume that each student i has a weak preference over $\mathcal{X}_i$ with the restriction that $a \neq b$ implies that (i, a, t_k) is not indifferent with (i, b, t_l) for all t_k and t_l in $\tau(i)$. Let $Ch_{\mathcal{C}}(\cdot) = \cup_{a \in \mathcal{C}} Ch_a(\cdot)$. Given $X \in \mathcal{X}$, for each student i, let $Ch_i(X) = \max_{\succeq_i} X_i$ and let $Ch_{\mathcal{S}}(X) = \cup_{i \in \mathcal{S}} Ch_i(X)$.

We now introduce the concept of the stability which is commonly used in the matching literature.

Definition 2.13. *A feasible outcome Y is a* ***stable outcome*** *if: (i) $Ch_{\mathcal{C}}(Y) = Y = Ch_{\mathcal{S}}(Y)$ and (ii) there is no school a and $Y' \neq Ch_a(Y)$ such that $Y' = Ch_a(Y' \cup Y) \subseteq Ch_{\mathcal{S}}(Y' \cup Y)$.*

Condition (i) is the individual rationality condition. If (i) fails, some student or school blocks the outcome. If conditon (2) fails, there is an alternative set of contracts that a school strictly prefers and that its corresponding students weakly prefer. To achieve the stable outcome, we only need to check whether the sequential reservation choice rule satisfies the substitute condition. We first show that the sequential reservation choice rule violates the substitutability condition introduced by Hatfield and Milgrom (2005) and the unilateral substitutability condition introduced by Kamada and Kojima (2018).

Definition 2.14. *A choice rule $Ch(\cdot)$ is* ***substitutable*** *if for all z,$z' \in X$ and $Y \subseteq X$, $z \notin Ch(Y \cup \{z\})$ implies $z \notin Ch(Y \cup \{z,z'\})$.*

Definition 2.15. *A choice rule $Ch(\cdot)$ is* ***unilaterally substitutable*** *if for all z,$z' \in X$ and $Y \subseteq X$ for which $s(z) \notin s(Y)$, $z \notin Ch(Y \cup \{z\})$ implies $z \notin Ch(Y \cup \{z,z'\})$.*

It's clear that substitutability condition is stronger than the unilateral substitutability condition. The following example shows that the sequential reservation choice rule is not unilaterally substitutable.

Example 2.7. *(The sequential reservation choice rule is not substitutable.)*

- $\mathcal{S} = \{s_1,s_2,s_3\}$ *with* $s_1 \pi s_2 \pi s_3$
- $\mathcal{T} = \{t_1,t_2\}$, $r_{t_1} = r_{t_2} = 1$, $q = 2$

- $Y = \{(s_1,t_1),(s_2,t_0)\}$, $z = (s_3,t_1)$ *and* $z' = (s_1, t_2)$.

In this example, we have $Ch(Y \cup \{z\}) = \{(s_1,t_1); (s_2,t_0)\}$ *and* $Ch(Y \cup \{z,z'\}) = \{(s_1,t_2); (s_3,t_1)\}$. *Since* $z \notin Ch(Y \cup \{z\})$ *but* $z \in Ch(Y \cup \{z,z'\})$ *sequential reservation choice rule is not substitutable.*

The following lemma shows that the sequential reservation choice rule satisfies the bilateral substitutability condition, which is weaker than unilateral substitutatility.

Definition 2.16. *A choice rule* $Ch(\cdot)$ *is* ***bilaterally substitutable*** *if for all* $z,z' \in X$ *and* $Y \subseteq X$ *for which* $s(z), s(z') \notin s(Y)$, $z \notin Ch(Y \cup \{z\})$ *implies* $z \notin Ch(Y \cup \{z,z'\})$.

Lemma 2.3. *The sequential reservation choice rule is bilaterally substitutable.*

Armed with Lemma 2.3, by Kamada and Kojima (2018), we can show that when we embed the sequential reservation choice rule into the cumulative offering process, it can result in a stable outcome. We put this result in the next proposition.

Proposition 2.6. *The cumulative offering process with the choice function to be the sequential reservation choice rule results in a stable outcome.*

2.8 Conclusion

In this chapter, we examine the model of the controlled school choice programs with affirmative actions, in which a student may belong to multiple privileged types and have a weak preference over these types. We first provide examples that the pursuit of maximally complying with the reservations could be problematic. We then propose the properties that a desirable choice function in the weak preferences environment should satisy: having no justified envy, satisfying no swapping condition and non-wastefulness. Next, we propose the sequential reservation choice rule that satisfies these properties. Furthermore, we show that the sequential reservation choice rule is characterized by these three properties.

There are many directions our analysis can be extended. For example, it would be interesting to consider maximum quotas in additon to minimum quotas (reserves).

Second, Kojima et al. (2018) provide a general framework to design matching mechanisms under constraints. They show that if the preferences of the schools can be represented by an $M^\natural$-concave function, then the generalized deferred acceptence algorithm can produce student-optimal stable matchings and is strategy-proof for students. While we do not use their framework in this chapter and it is tricky to check whether our fairness requirements satisfy the sufficient condition provided in their paper for $M^\natural$-concavity, it is interesting to search for a weaker sufficient condition to incorporate more situations in their framework. The model in this chapter may serve as a good starting example. These are left for future research.

Appendix

Proof of Lemma 2.1: Suppose Y assigns k reserved seats to the students and including a new student can let the school assign $k+1$ reserved seat. Now, starting with Y' which involve the same set of the students as Y. The school can can first replace all the contracts so that the resulting outcome is exactly Y and then adding the new student. ■

Lemma 2.4. *Let $S \in \mathcal{S}$ to be a set of students. Suppose there is a way to assign a reserved seat to each student in S (i.e., $\rho(S) = |S|$), then for any subset A of S, we must have: $|A| \leq \sum_{t_k \in T} r_{t_k}$ where $T = \{t_k | \exists i \in A \quad s.t. \quad t_k \in \tau(i) \setminus t_0\}$.*

Proof of Lemma 2.4: Suppose there exists a set A of sutdents such that the number of students is strictly larger than the sum of reserved seats for which the students in A are eligible. Then it is clear that at least one of the student in A cannot be assgined to a reserved seat anyway. This is contradictory to the assumption that each of the student in S is assigned to a reserved seat. ■

Proof of Proposition 2.1:

The "if" part: if there is a strict rearrangement chain starting with (i, t_k), then we can add this contact into Y and replace $2n^{th}$ contracts by $(2n+1)^{th}$ contracts and since the number of the contracts with the type in the last contract of the chain hasn't hit the upper bound, (i.e., $|Y_{t_k}| < r_{t_k}$), the resulting outcome is still feasible.

The "only if" part: suppose i increases reserve utilization with respect to Y, we will use a constructive way to find a strict rearrangement sequence.

Let A denote the set of students among $s(Y)$ who are assigned the reserved seats in Y. Suppose i increases reserve utilization with respect to A through (i,t). Without loss of generality, we set $t := t^1$. Now there are two cases.

Case 1: $|(Y_A)_{t^1}| < r_{t^1}$. In this case, (i,t^1) is the strict required rearrangement chain and we are done.

Case 2: $|(Y_A)_{t^1}| = r_{t^1}$. In this case, there must exists $s' \in A$ such that $(s',t^1) \in Y_A$ and $\exists t' \in \tau(s') \setminus t_0$ and $t' \neq t^1$. Let T^2 be the set of all the types that students who are currently assigned t^1 have, excluding $\{t_1\}$. Note that the nonemptiness of T^2 is guaranteed by Lemma 2.4. Suppose otherwise, let A' be the set of students in A having type t^1 and none of them has other privileged types. Then $|A' \cap \{i\}| > r_{t^1}$, contradicting to the assumption that i can increase reserve utilization through (i,t^1).

Suppose we have already found T^2. Now there are still two cases.

Case 1: there exists $s^1 \in s(Y_{t^1})$, $t^2 \in (\tau(s^1) \setminus t_0)$ and $t^2 \in T^2$ such that $|(Y_A)_{t^2}| < r_{t^2}$. If this is the case, then (i,t^1),(s^1,t^1),(s^1,t^2) is the required strict rearrangement chain.

Case 2: for all $s \in s(Y_{t^1})$ and $t \in (\tau(s) \setminus t_0) \subseteq T^2$ such that $|(Y_A)_t| = r_t$. In this case, there must exists $s'' \in Y_{t'}$ for some $t' \in T^2$ and $\exists t'' \in \tau(s'') \setminus t_0$ such that $t'' \notin T^2 \cup \{t^1\}$. Let T^3 be the set of all such types. That is, T_3 is the set of privileged types the students currently assigned reserved seat in T^2 have, excluding $T^2 \cup \{t^1\}$. Again, the nonemptiness of and T^3 is guaranteed by Lemma 2.4. Suppose otherwise, let A'' to be the set of students in A having types in $\{t_1\} \cup T^2$, then $|A'' \cup \{i\}| > \sum_{t_i \in \{t^1\} \cup T^2} r_{t_i}$, contradicting to the assumption that i can increase reserve utilization through (i,t^1).

We can proceed the process in the above way. Since the number of privileged types $|\mathcal{T}|$ is finite, we can always find a required rearrangement chain. ■

Proof of Lemma 2.2: We differentiate two ways for the contract (i,t_k) to be chosen at some step of $Ch^{\#}$. The first one is that there is a strict or weak rearrangement chain starting at (i,t_k). In the second situation, (i,t_k) belongs to a stict or weak rearrangement chain but not the starting contract. We first show that if no contract in X_{il} is chosen in step m of $Ch^{\#}$, then there is no strict or weak rearrangement starting with (i,t_k) in any sebsequent steps of $Ch^{\#}$.

Let Y to be the set of chosen contracts in step h, round 1 before i is considered. Since (i,t_k) is not chosen in step h of the $Ch^{\#}$, then in step h, there is no strict rearrangement chain with respect to Y starting at (i,t_k). According to the strict rearrangement chain finding process, this means that there is no rearrangement chain start with (i,t_i) with respect to Y means there exists an integer n such that

$$\sum_{1\leq m\leq n} A^m = \sum_{t_k\in\cup_{1\leq m\leq n}T^m} r_{t_k} \text{and for all} \quad s\in\cup_{1\leq m\leq n}A^m, \tau(s)\subseteq\{t_0\}\cup_{1\leq m\leq n}T_m \tag{2.1}$$

Where A^m and T^m are defined in the strict rearrangement chain finding process. The only possible case to find a strict rearrangement chain start with (i,t_k) in the subsequent steps is that some students in $\cup_{1\leq m\leq n}A^m$ are rejected. Consider the first step where a student (say, s_a) in $\cup_{1\leq m\leq n}A^m$ is rejected. There must exists a student s_b such that no contract of s_b is chosen at step h, s_b has a higher priority than s_a and s_b has a privileged contract being chosen in this step. In addition, for $t_j\in\tau(b)\nsubseteq\{t_0\}\cup_{1\leq m\leq n}T_m$, we cannot find a strict rearrangement chain that contains (b,t_j) because otherwise we can find a strict rearrangement chain starting with one of s_a's contract and s_a will not be rejected. Thus, the condition (2.1) still holds with s_b taking the place of s_a and therefore (i,t_k) will not be chosen in this step. We can repeatedly use this argument and conclude that (i,t_k) will not be the starting contract of a strict rearrangement chain in any subsequent steps of $Ch^{\#}$.

We then show that there is no weak rearrangement chain starting with (i,t_k) in step h of $Ch^{\#}$. Let $\sigma(X_h)$ denote the number of privileged contracts chosen in the first round of step h. Denote $A(h)$ the set of the students who are qualified to be enrolled in round 2 of step h and have higher priorities than i. According to the weak rearrangement chain finding process, we have

$$|A(h)| > q - |\sigma(X_h)| \tag{2.2}$$

Consider a subsequent step k. Note that $|X_h| \subset |X_k|$ and this implies $|\sigma(X_h)| < |\sigma(X_k)|$. Thus the remaining seats for round 2 of step k (i.e.,$q - |\sigma(X_h)|$) is even less. Thus, (i,t_k) will not be the starting contract of a weak rearrangement chain in any subsequent steps of $Ch^{\#}$.

Finally, if (i,t_k) belongs to a strict or weak rearrangement chain but not the starting contract in some subsequent step, we can truncate this rearrangement chain so that (i,t_k) is the starting contract. This contradicts to the results be have discussed above. ∎

We now propose an immediate corollary of lemma which says that as the sequential choice rule processes, all the students are weakly worse off.

Corollary 2.1 Let Y be the outcome of running $Ch^{\#}$ on X. Let X^t be the contracts available to the school in the final step of $Ch^{\#}$. Then for $X_i \cap Y = \emptyset$, $X_i = X_i^t$. For $X_i \cap Y \neq \emptyset$, for all ($i$,$t_k$) $\in X_i^t$, we have (i,t_k) $\succeq_i Y$.

Proof of Proposition 2.2: We first show that $Ch^{\#}$ has no justified envy. Let $Y = Ch^{\#}(X)$ and let X^t be the consideration set for the school in the final step of $Ch^{\#}$. Suppose there exists $i \notin s(Y)$ who has justifed envy towards $j \in s(Y)$. Then we can find a rearrangement chain with respect to Y (strict if $j\pi i$, weak if $i\pi j$), starting with one of i's privileged contract (denote it by (i,t_k)). Since $i \notin s(Y)$, by corollary 2.1, (i,t_k) $\in X^t$, which means it has been rejected in some earlier step. By lemma 2.2, we cannot find a rearrangement chain in the final step, which creates a contradiction.

We then show that no student can ask for swapping by a privileged contract. Suppose i can ask for swapping by a privileged contract (i,t_k). Since $(i,t_k) \succ_i Y$, (i,t_k) is in the consideration contract in some previous step and is not chosen. By lemma 2.2, we cannot find a rearrangement chain starting with (i,t_k). The second condition for asking for a swapping by a privileged contract cannot hold which is a contradiction.

Suppose a student i who has a contract $(i,t_k) \in Y$ can ask for swapping by a normal contract. Since $(i,t_0) \succ_i (i,t_k)$, (i,t_0) is in the consideration set in some previous step (say, step l) and being rejected. Then there exists another student j who has contract being chosen in step l but not in the final step. j has justified envy towards i with respect to $Y \setminus \{(i,t_k)\} \cup \{(i,t_0)\}$, otherwise there is a weak rearrangement chain starting with (i,t_0) and this contradicts to Lemma 2.2.

Finally, the choice rule fills its seats up to its total capacity, so it is acceptant. ■

Proof of Proposition 2.3 Suppose that there exists students who are not indifferent between Y and Z. Let i be the student who has the highest priority among them according to π. Without loss of generality, we can assume that $Y \succ_i Z$. We denote the set of the students who have higher priorities than i by $\overline{p}(i)$ and who has lower priority than i by $\underline{p}(i)$. Then for $j \in \overline{p}(i)$, we have $Y \sim_j Z$. There are four cases for this to happen, we show that all of them result in contradiction.

In the first case, suppose $(i, t_k) \in Y$ where $t_k \neq t_0$ and $(i, t_l) \in Z$. The school can re-sign the contracts in Z with all the students in $\overline{p}(i)$ such that the re-signed contracts are the same as in Y and $\underline{p}(i)$ to be the normal contracts. Such reassignment of the seats is feasible and will not make those who in $\overline{p}(i)$ strictly worse off. Thus, Z does not satisfies the no swapping condition.

Secondly, suppose $(i, t_0) \in Y$ and $(i, t_l) \in Z$. We can do the same reassignment as in the first case. Since there is no justified envy towards i in Y, there is no justified envy towards i in the outcome re-signed from Z as well. Thus, Z does not satisfies the no swapping condition.

In the third case, suppose $(i,t_k) \in Y$ where $t_k \neq t_0$ but $i \notin s(Z)$. Suppose students in $\overline{p}(i)$ take weakly more reserved seats in Y than in Z. Starting from Z, we can reassign the seats in the following way. For each student in $\overline{p}(i)$, assign the seat just the same as in Y. If a type of privileged contracts, say t_s, exceeds its maximal number by n because of this process, then reassign those who have the lowest n priorities of type t_s to the normal contracts. Note that this reassignment weakly signs more privileged contracts than Z and it's thus easy to see that i has justified envy towards any of the student who has lower priority than her. This contradicts to the assumption that Z eliminates justified envy. Suppose students in $\overline{p}(i)$ take strictly less reserved seats in Y than in Z. Then under Z either i can increase the reserve utilization through (i,t_k) or we can including (i,t_k), let any one of the student in $\overline{p}(i)$ who is assigned a privileged seat to be reassigned to a normal seat and reject any one of the student in $\underline{p}(i) \cap s(Z)$. Otherwise, Y is not feasible.

In the last case, suppose $(i,t_0) \in Y$ but $i \notin s(Z)$. Suppose students in $\overline{p}(i)$ take strictly more reserved seats in Y than in Z. Starting from Z, for each student in $\overline{p}(i)$, assign the seat just the same as in Y. If the resulting outcome is still feasible, then i has justified envy towards any of the student in $\underline{p}(i) \cap s(Z)$. If a type of privileged contracts, say t_s, exceeds its maximal number, then i has justified envy towards a student in $\underline{p}(i) \cap s(Z)$ who takes the type t_s seat in Z. Suppose students in $\overline{p}(i)$ take weekly less reserved seats in Y than in Z. We can also assume that under Z, all the students in $\underline{p}(i) \cap s(Z)$ take the privileged seats, for otherwise we can easily find the justified envy. Denote j as the student who has the highest priority in $s(Z) \setminus s(Y)$. By the construction, we know that $i \pi j$, so j must take the privileged seat under Z. Denote all the students who has higher priority than j but has lower priority than i by $p(i,j)$. By the same argument as in the first and the second cases, we can show that all the students in $p(i,j) \cap s(Z)$ must weakly prefer Z to Y. Finally we claim that j has justified envy under Y. Starting from Y, we reassign the seats of the students in $\overline{p}(i) \cup p(i,j)$ such that they are the same as Z. If a type of privileged contracts, say t_s, exceeds its maximal number by n because of this process, then reassign those who have the lowest n priorities of type t_s to the normal contracts. Note that this process assigns weekly more privileged seats compared to Y. Suppose there exists a student m in $s(Y)$ who has lower priority than j, then j has justified envy towards m. Suppose each student in $s(Y)$ has higher priority than j, then j has justified envy towards i. ■

Proof of Proposition 2.4 Firstly, the running time for exploring a rearrangement chain is $O(\#|X|)$ since the total length of the rearrangement chain cannot exceed $\#|X|$. In each round of the choice rule, the number of implementing the rearrangement chain finding process is less than $2\#|X|$. We have at most $\#|\mathcal{T}|$. So the sequential reservation choice rule can be implemented in $O(\#|X|^2)$. ■

Proof of Proposition 2.5 Suppose s_1's true preference is $\succeq_i$ and her reported preference is $\succeq_i'$. If s_1 is not enrolled under $\succeq_i'$, then the proposition 5 holds immediately since we assume that every contract is acceptable by the studets. Suppose the contract s_1 get by reporting $\succeq_1'$ is (s_1,t'). First note that by only reporting (s_1,t'), s_1 can also obtain (s_1,t'). Next, define $\succeq_i''$ in the following way: trancating the $\succeq_i$ so that (s_1,t') it is easy to see that s_1 will still get (s_1,t') under $\succeq_i''$. Finally, suppse (s_1,t) is outcome of s_1 by reporting the true preference. Then $(s_1,t) \succeq_i (s_1,t')$. This is because otherwise by corollary 1, (s_1,t') is in the consideration set in the last step of the sequential reservation choice rule when reporting the true preference and it is (s_1,t') rather than (s_1,t) that will be chosen and this is a contradiction. ■

Proof of Lemma 2.3 Suppose $z' \notin Ch(Y \cup \{z,z'\})$, then it is clear that $z \notin Ch(Y \cup \{z,z'\})$. Suppose $z' \in Ch(Y \cup \{z,z'\})$, if the type associated with z' is not null type, then we still cannot find the rearrangement chain starting with z, thus z will not be chosen at the first round. In addition, the number of the previliged contracts will weakly increase, so that z will not be chosen at the second round either. If the type associated with z' is null type. then both z and z' will not be chosen at the first round and they are competitive in the second round so that z will not be chosen at the second round. ■

Chapter 3

Optimal Auction Design with Referral

3.1 Introduction

There are ample situations where designers can access certain players only through existing players' referral, and designers need to provide the right incentives. Suppose you are an owner of a house and would like to sell your property. While it is possible for you to sell it on your own, it is often in your interest to pay and hire an real-estate agent since he/she can make your property known to much more potential buyers.[1] InnoCentive, one of the leading crowdsourcing platforms for solving challenges, recently launches a new program called "finder's fee" which pays solvers who refer asuitable candidate.[2] It is intuitive that more buyers and solvers benefit sellers and seekers. It is now well-known that more buyers in auctions benefit the seller: a famous result in Bulow and Klemperer (1999) shows that adding one more bidder to an auction is better than running an optimal auction among existing pool

[1] Here, we can treat the real-estate agent as a buyer with zero value. Similar arguments hold for selling antiques through the Sotheby's, renting short-term accommodation through Airbnb.com, etc.

[2] For example, in a recent call for "Seeking Commercially Available Zinc-Rich Materials", it is noted that "Proposals from Solvers who have the ability to work directly as the collaboration partner or supplier will be considered first. The Seeker will also consider proposals from other Solvers who refer a suitable candidate as a collaboration partner. A finder's fee of $2,000 USD will be paid to a Solver who can refer the Seeker to a candidate."

of bidders. In particular, running an auction between two bidders without a reserve price is better than the optimal negotiation with one bidder. One effective and straightforward way to attract more bidders is through the networks of the existing bidders by inducing them to refer potential buyers they may privately know. However, it may not be in the interests of the bidders to refer others to the auctions since this induces more competition and may hurt themselves in return. This leads to challenges for the design of optimal auctions: (1) Would it be beneficial to induce bidders to refer? (2) If yes, what is the optimal way to provide such referral incentives?

In this chapter, a seller has one unit of indivisible object to sell. We frame our problem as a selling problem, but the procurement problem can be reinterpreted accordingly by changing the terms seller to seeker, buyer to solver, object to project, valuation to productivity, and bid to score. There is one existing buyer who privately knows whether there exists another potential buyer. The seller can incentivize the existing buyer to refer the potential buyer to participate and compete for the object. The private valuations of the buyers are drawn independently from asymmetric distributions. The challenge is the misalignment of interests between the seller and the existing buyer toward the potential buyer's appearance. We fully characterize the optimal mechanism and reveal three optimal channels to provide referral incentive: discouraging non-referral, favoring referral, and providing money transfer for referral. If the existence of the potential buyer is public information, then the seller can simply run Myerson's optimal mechanism (that is, allocating the object to the buyer with the highest virtue valuation.) depending on the number of the buyers. Since we assume the existence of the potential buyer is the private information of the exsiting buyer, the seller need to modify Myerson's optimal mechanism to satisfy the referral incentive constraint. We show that distorting the allocation rule to disfavor the non-referral and favor the referral is less costly at the beginning of the modification process and thus the first two channels always appear, and are thus essential. At some point, the cost of changing the allocation rule exceeds providing the money transfer and thus the third channel arises.

We also conduct some comparative statics. When the potential buyer is more likely to exist, the existing buyer is more disfavored with non-referral, is less favored with referral, and is provided with a lower informational rent for referral. The existing buyer is worse off regardless of his valuation. In contrast, when the potential buyer becomes stronger, the existing buyer is more disfavored with non-referral, is more favored with referral, and is provided with a higher informational rent for referral. The existing buyer with the highest (lowest) valuation is worse (better) off. In particular, the supplementary channel is needed when the potential buyer is less likely to exist and becomes stronger.

We also investigate when the optimal mechanism can be implemented by simple mechanisms. With uniform distributions, the optimal mechanism can be implemented by a second-price auction with a reserve and a constant unconditional coupon. This mechanism is simple and easy to use since the unconditional coupon is a fixed number and independent of bids. When the potential buyer is strong, the optimal mechanism can be implemented by a sequential selling: the seller first makes a take-or-leave-it offer to the existing buyer; if refused, the seller pays the existing buyer a constant cash reward for referral and makes a take-it-or-leave-it offer to the potential buyer. One may ask why not the seller simply sells the object to the existing buyer who then can resell to the potential buyer. We show that this conventional resale mechanism is never optimal, and it is always better for the seller to directly include the potential buyer in the mechanism when he exists.

Despite of the significance and relevance of referral in auctions, there is little investigation from the optimal mechanism design approach. Nevertheless, there exist an emerging literature studying auctions in social network. The approach of this literature in quite different from ours. They allow a general network structure among buyers and assume that buyers can only communicate with neighbors. Their aim is to design a mechanism with good properties including full diffusion of information, strategy-proofness, individual rationality and weakly budget-balance. In some sense, this literature studies the problem from an axiomatic approach as in the vast matching literature pioneered by Gale and Shapley (1962) and Shapley and Scarf

(1974). A straightforward attempted mechanism is the VCG mechanism. However, Li, Hao, Zhao, and Zhou (2017) show that it satisfies all properties except weak budget-balance, implying that the seller may not have incentive to adopt it. Various mechanisms are then proposed to improve upon VCG mechanisms in terms of *ex-post* revenue: information diffusion mechanism in Li et al. (2017), closest winner of Myerson's and all winners of Myerson's, potential winner's mechanisms in Zhang and Zhao (2021), critical diffusion mechanism in Li, Hao, Zhao, and Yokoo (2019), multilevel mechanism in Lee (2017), groupwise-pivotal referral mechanism Jeong and Lee (2020).[3] Since the implementation of the mechanisms depends on the structure of social network, another important question in this literature is computation using algorithms, which makes this strand of literature more related to computer science.

We take the traditional approach of mechanism design following Myerson (1981).[4] Instead of looking for strategic-proof mechanisms to improve the *ex-post* revenue, we are interested in the optimal Bayesian incentive compatible mechanism to maximize the seller's *ex-ante* revenue. To our best knowledge, we are the first in the literature to study the optimal referral incentives in auctions. It can be verified that none of the mechanisms above maximizes the seller's *ex-ante* revenue in our simple framework. One obvious difference is that when the potential buyer does not exist, all these mechanisms allocate the object to the existing buyer in Myerson's fashion, while the optimal mechanism needs to distort the allocation.

Several papers on referral in the market places are related to our work. Lobel, Sadler, and Varshney (2017) study how to optimally attract new buyers by using a referral program. They find the optimal nonlinear referral payment function for the seller and argue that a linear referral payment program with a threshold bonus can approximately implement the optimal payment function. Leduc, Jackson, and Johari (2017) study the case where the seller attracts part of the buyers to try a new product

[3]One major extension of the this literature is to multiple unit auctions including Kawasaki, Barrot, Takanashi, Todo, and Yokoo (2020) and Takanashi, Kawasaki, Todo, and Yokoo (2019). Li, Hao, and Zhao (2020) provide a necessary and sufficient condition for strategy-proofness. They also propose a class of natural monotonic allocation policies and obtain the corresponding optimal payment policy that maximizes the seller's revenue.

[4]See Budish (2012) for a detailed discussion on the difference between the "axiomatic approach" and the mechanism design approach.

and then refer other potential buyers. They establish the condition under which the referral reward program is better than the inter-temporal price discrimination. In these models, the seller sells multiple units of homogeneous products, therefore, the new buyer and the existing buyer do not have to compete after the referral. In contrast, in our paper, referring a new buyer induces more competition for existing buyers, which makes the referral incentive challenging to design. Technically, we examine the issue from the mechanism design approach which enables us to examine all possible referral incentives instead of assuming particular forms.[5]

Our paper builds on the vast literature on optimal mechanism design pioneered by Myerson (1981), and is related mostly to studies with multidimensional private information. While the canonical one-dimensional mechanism design model is well understood, its natural multidimensional analogue turns out to be much less tractable. Rochet (1985) shows that a mechanism is incentive-compatible if, and only if, the buyer's utility induced by the mechanism is convex, as in one-dimensional environments. Laffont, Maskin, and Rochet (1987) consider a model with a single good, but where consumers are differentiated by a two-dimensional parameter. McAfee and McMillan (1988) extends Laffont et al. (1987) by proposing a generalized single crossing property. Wilson (1993) derives first order conditions for the optimality of a mechanism. Since this approach does not generally yield a description of the optimal mechanism, he also uses computational methods to obtain particular solutions. Rochet and Choné (1998) analyze a general multi-dimensional screening model and show that bunching will be a prominent feature of the optimal mechanism. In our model, the existing buyer's private information is two-dimensional: his own valuation and the existence of the potential buyer. Since the second dimension is binary, we are able to obtain closed-form solutions.

This chapter is organized as follows. In Section 2, we provide the setup of the model. In Section 3, we present a benchmark model. In Section 4, we solve the seller's optimization problem. In Section 5, we derive the optimal mechanism in

[5]The term referral has been used in the literature on intermediaries as well. For instance, Arbatskaya and Konishi (2012), Park (2005), and Garicano and Santos (2004) study matching between buyers and sellers can be improved with referrals. Condorelli, Galeotti, and Skreta (2018) endogenous intermediaries' choice to operate in a referral mode and a merchant mode.

several special cases. In Section 6, we conclude. An appendix contains the technical proofs.

3.2 The Model

A seller has one unit of indivisible object. There is one existing buyer A with private valuation v_A. It is commonly known that with probability $x \in (0,1)$ buyer A potentially knows buyer B whose private valuation is v_B.[6] We assume that the seller can access buyer B only through buyer A's referral, and the existence of buyer B is buyer A's private information.[7] It is also commonly known that v_A and v_B are independently drawn from atomless $c.d.fs$ $F_A(v_A)$ and $F_B(v_B)$ with $p.d.fs$ $f_A(v_A)$ and $f_B(v_B)$ on supports $[\underline{a},\overline{a}]$ and $[\underline{b},\overline{b}]$, respectively. We assume that the seller can verify whether buyer B indeed has a distribution $F_B(v_B)$ through eligibility checks to rule out the possibility for buyer A to refer a random person. The seller's valuation is normalized to be zero, and all players are risk neutral.

By the revelation principle, it is without loss of generality to focus on direct mechanisms. When buyer B does not exist, the seller allocates the object to buyer A with probability $q_A^{\alpha}(v_A)$ for an expected payment of $t_A^{\alpha}(v_A)$. We refer this sub-mechanism as $\alpha - mechanism$. When buyer B exists, the seller allocates the object to buyer A and buyer B with probabilities $q_A^{\beta}(v_A,v_B)$ and $q_B^{\beta}(v_A,v_B)$ for expected payments $t_A^{\beta}(v_A)$ and $t_B^{\beta}(v_B)$, respectively. We refer this sub-mechanism as $\beta - mechanism$.[8]

Buyer A's payoff by reporting $\tilde{v}_A$ and nonexistence of B is given by

$$u_A^{\alpha}(v_A,\tilde{v}_A) = v_A q_A^{\alpha}(\tilde{v}_A) - t_A^{\alpha}(\tilde{v}_A) \tag{3.1}$$

[6] Here we exclude $x = 0$ and 1 to avoid triviality of the analysis. Furthermore, the optimal mechanism in these extreme cases can be treated as the limit of our optimal mechanism.

[7] If the existence of the potential buyer is known by the seller but can only be accessed through referral from the existing buyer, the seller can achieve the same outcome as if she could access the potential buyer directly by committing not selling to buyer A directly.

[8] Here, we implicitly assume that the seller always induces buyer A to refer buyer B whenever buyer B exists. This is without loss of generality since it is costless for buyer A to refer buyer B. Furthermore, it is without loss of generality to assume that all payments are in expectation due to risk neutrality.

Buyer A's payoff by reporting $\tilde{v}_A$ and existence of B is given by

$$u_A^\beta(v_A,\tilde{v}_A) = v_A \int_{\underline{b}}^{\overline{b}} q_A^\beta(\tilde{v}_A,v_B)dF_B(v_B) - t_A^\beta(\tilde{v}_A). \tag{3.2}$$

Buyer B's payoff by reporting $\tilde{v}_B$ is given by

$$u_B^\beta(v_B,\tilde{v}_B) = v_B \int_{\underline{a}}^{\overline{a}} q_B^\beta(v_A,\tilde{v}_B)dF_A(v_A) - t_B^\beta(\tilde{v}_B). \tag{3.3}$$

The incentive compatibility (IC) constraints and individual rationality (IR) constraints are defined as follows.

Definition 3.1. *A direct mechanism satisfies IC and IR if and only if:*

$$u_A^\alpha(v_A,v_A) \geq u_A^\alpha(v_A,\tilde{v}_A), \quad \forall v_A, \tilde{v}_A \tag{3.4}$$

$$u_A^\beta(v_A,v_A) \geq u_A^\beta(v_A,\tilde{v}_A), \quad \forall v_A, \tilde{v}_A \tag{3.5}$$

$$u_A^\beta(v_A,v_A) \geq u_A^\alpha(v_A, v_A), \quad \forall v_A, \tag{3.6}$$

$$u_B^\beta(v_B,v_B) \geq u_B^\beta(v_B,\tilde{v}_B), \quad \forall v_B, \tilde{v}_B, \tag{3.7}$$

$$u_A^\alpha(v_A,v_A) \geq 0, \qquad \forall v_A, \tag{3.8}$$

$$u_A^\beta(v_A,v_A) \geq 0, \qquad \forall v_A, \tag{3.9}$$

$$u_B^\beta(v_B,v_B) \geq 0, \qquad \forall v_B. \tag{3.10}$$

(3.4) ensures that when B does not exist, A has no incentive to lie about his valuation. (3.5) ensures that when B exists, A has no incentive to lie about his valuation. (3.6) ensures that when B exists, A has no incentive to lie about B's existence. Note that in (3.6) potentially buyer A can also lie about his valuation at the same time, but this is not profitable due to (3.4). Also note that when B does not exist, A cannot report that B exists. (3.7) ensures that B has no incentive to lie about his valuation. IR constraints (3.8)-(3.10) are standard and mean that participating in the mechanism is better than one's outside option which is normalized to be zero. Note that buyer A's private information is two-dimensional: valuation and knowledge about the existence of buyer B. Denote

$U_A^\alpha(v_A) = u_A^\alpha(v_A,v_A), U_A^\beta(v_A) = u_A^\beta(v_A,v_A), U_B^\beta(v_B) = u_B^\beta(v_B,v_B)$ as the truthful payoffs of the buyers.

Given that all buyers are truthful, the seller's expected revenue from a direct mechanism is

$$(1-x)\int_{\underline{a}}^{\overline{a}} t_A^\alpha(v_A)dF_A(v_A) + x\int_{\underline{b}}^{\overline{b}}\int_{\underline{a}}^{\overline{a}}\left[t_A^\beta(v_A) + t_B^\beta(v_B)\right]dF_A(v_A)dF_B(v_B).$$

The seller's problem is choosing a direct mechanism to maximize her expected revenue subject to IC and IR constraints described in Definition 1.

Applying the envelope theorem on (3.4), (3.5) and (3.7) yields

$$U_A^\alpha(v_A) = \int_{\underline{a}}^{v_A} q_A^\alpha(\xi)d\xi + U_A^\alpha(\underline{a}) \tag{3.11}$$

$$U_A^\beta(v_A) = \int_{\underline{a}}^{v_A}\int_{\underline{b}}^{\overline{b}} q_A^\beta(\xi,v_B)dF_B(v_B)d\xi + U_A^\beta(\underline{a}) \tag{3.12}$$

$$U_B^\beta(v_B) = \int_{\underline{b}}^{v_B}\int_{\underline{a}}^{\overline{a}} q_B^\beta(v_A,\xi)dF_A(v_A)d\xi + U_B^\beta(\underline{b}) \tag{3.13}$$

By similar arguments as in Myerson (1981) and utilizing the above equalities, we have a revenue equivalency result: the expected revenue for the seller is not affected by the detailed payment rules since they will be determined by the allocation rules. We thus can rewrite the original problem equivalently as

$$\max_{q^\alpha,q^\beta,U_A^\alpha(\underline{a}),U_B^\beta(\underline{b}),U_A^\beta(\underline{a})} (1-x)\left[\int_{\underline{a}}^{\overline{a}} q_A^\alpha(v_A)J_A(v_A)dF_A(v_A) - U_A^\alpha(\underline{a})\right] - x\left[U_A^\beta(\underline{a}) + U_B^\beta(\underline{b})\right]$$
$$+ x\int_{\underline{b}}^{\overline{b}}\int_{\underline{a}}^{\overline{a}}\left[J_A(v_A)q_A^\beta(v_A,v_B) + J_B(v_B)q_B^\beta(v_A,v_B)\right]dF_A(v_A)dF_B(v_B)$$

subject to:

$$q_A^\alpha(v_A), \int_{\underline{a}}^{\overline{a}} q_B^\beta(v_A,v_B)dF_A(v_A), \int_{\underline{b}}^{\overline{b}} q_A^\beta(v_A,v_B)dF_B(v_B) \text{ are nondecreasing}, \tag{3.14}$$

$$\int_{\underline{a}}^{v_A}\int_{\underline{b}}^{\overline{b}} q_A^\beta(\xi,v_B)dF_B(v_B)d\xi + U_A^\beta(\underline{a}) \geq \int_{\underline{a}}^{v_A} q^\alpha(\xi)d\xi, \quad \forall v_A, \tag{3.15}$$

$$U_A^\alpha(\underline{a}) \geq 0, U_A^\beta(\underline{a}) \geq 0, U_B^\beta(\underline{b}) \geq 0 \tag{3.16}$$

where $J_i(v_i) \equiv v_i - \frac{1-F_i(v_i)}{f_i(v_i)}, i \in \{A,B\}$, denotes the virtual valuation. (3.14) and (3.16) are standard, resulting from the second order conditions of the IC constraints and IR constraints for the lowest valuations. (3.15) rewrites the referral constraint (3.6) by replacing expected payments with the allocation rules.

3.3 A benchmark

One important benchmark is when the seller observes whether buyer B exists or not, and does not rely on buyer A's referral. This is the the same as when (3.15) does not appear. The following proposition characterizes the optimal mechanism in this benchmark model and shows that it is not feasible in the original problem. The proof is standard and thus omitted.

Proposition 3.1. *In the benchmark model where the seller knows whether or not buyer B exists, the optimal mechanism is as follows:*

$$q^{\alpha^B}(v_A) = \begin{cases} 1 & if \quad J_A(v_A) \geqslant 0 \\ 0 & if \quad J_A(v_A) < 0 \end{cases} \tag{3.17}$$

$$q_A^{\beta^B}(v_A, v_B) = \begin{cases} 1 & if\ J_A(v_A) \geqslant \max\{J_B(v_B), 0\} \\ 0 & otherwise \end{cases} \tag{3.18}$$

$$q_B^{\beta^B}(v_A, v_B) = \begin{cases} 1 & if\ J_B(v_B) \geqslant \max\{J_A(v_A), 0\} \\ 0 & otherwise \end{cases} \tag{3.19}$$

$$U_A^{\alpha^B}(\underline{a}) = U_A^{\beta}(\underline{a}) = U_B^{\beta}(\underline{b}) = 0 \tag{3.20}$$

This mechanism violates (3.15), and thus can never be optimal in the original problem.

The solution to the benchmark is simple: the seller simply maximizes her expected revenue separately conditional on whether or not buyer B exists. In the $\alpha - mechanism$, she sells whenever buyer A's virtual valuation is greater than zero. In the $\beta - mechanism$, she sells to the buyer with the highest positive virtual valuations. It is also optimal to leave no informational rents to buyer A with the lowest valuation regardless whether or not buyer B exists. With the presence of buyer B, buyer A is less likely to win and he has no incentive to refer buyer B, i.e., the above optimal mechanism violates the referral constraint (3.15). As a result, in the original problem, the fact that the existence of buyer B is buyer A's private information will strictly hurt the seller. This implies that in order to solve the original problem, we need to take (3.15) into consideration directly.

3.4 The original problem

The following lemma characterizes some properties of the optimal mechanism in the original problem.

Lemma 3.1. *In the optimal mechanism, we must have* $U_A^\alpha(\underline{a}) = 0, U_B^\beta(\underline{b}) = 0$.

As is common in the mechanism design literature and consistent with the benchmark model, the above lemma shows that it is still optimal not to leave any informational rents to buyer A with the lowest valuation when buyer B does not exist, and to buyer B with the lowest valuation. This lemma is straightforward since otherwise the seller can achieve a strictly higher revenue by reducing these information rents. However, we cannot conclude $U_A^\beta(\underline{a}) = 0$, since lowering $U_A^\beta(\underline{a})$ could potentially trigger the referral incentive (3.15). Note that $U_A^\beta(\underline{a})$ stands for the informational rent for buyer A with the lowest valuation when buyer B exists. This is the minimum amount of informational rent that has to be given out to buyer A when buyer B exists, we can call it informational rent for referral.

Obviously, the seller's problem cannot be solved using the standard point-wise maximization due to the referral constraint (3.15). Our approach is as follows. We first obtain a relaxed problem by ignoring the monotonicity constraints (3.14) and relaxing (3.15). We then fully characterize the optimal mechanism in the relaxed problem and show that it also satisfies all the constraints in the original problem, and thus is optimal in the original problem. The challenge is how to relax (3.15). The key observation is that buyer A with the highest valuation seems to have the least incentive to refer buyer B. Therefore, instead of imposing the referral constraint (3.15) for all valuations of buyer A, we only require it for the highest valuation. Readers with general interests are advised to skip the next subsection, which contains technical details for deriving the optimal mechanism, and jump to the characterization of the optimal mechanism directly.

3.4.1 The relaxed problem

We thus obtain the following relaxed problem:

$$\begin{aligned}\max_{q^\alpha, q^\beta, U_A^\beta(\underline{a})} \quad & (1-x)\int_{\underline{a}}^{\overline{a}} J_A(v_A) q_A^\alpha(v_A) dF_A(v_A) - xU_A^\beta(\underline{a}) \\ + \quad & x\int_{\underline{b}}^{\bar{b}}\int_{\underline{a}}^{\overline{a}} \Big[J_A(v_A) q_A^\beta(v_A, v_B) + J_B(v_B)\, q_B^\beta(v_A, v_B)\Big] dF_B(v_B)\end{aligned}$$

subject to:

$$\int_{\underline{a}}^{\overline{a}}\int_{\underline{b}}^{\bar{b}} q_A^\beta(v_A,v_B) dF_B(v_B) dv_A + U_A^\beta(\underline{a}) \geq \int_{\underline{a}}^{\overline{a}} q_A^\alpha(v_A) dv_A \tag{3.21}$$

$$U_A^\beta(\underline{a}) \geq 0 \tag{3.22}$$

Constraint (3.21) must be binding, otherwise the problem is the same as the benchmark model, and according to Proposition 3.1, its solution would violate (3.21). Thus, we must have

$$U_A^\beta(\underline{\alpha}) = \int_{\underline{a}}^{\overline{a}}\int_{\underline{b}}^{\bar{b}} q_A^\beta(v_A,v_B) dF_B(v_B) dv_A - \int_{\underline{a}}^{\overline{a}} q_A^\alpha(v_A) dv_A, \tag{3.23}$$

which pins downs the informational rent for referral with allocation rules. Define $J_A^\alpha(v_A) = J_A(v_A) - \frac{x}{(1-x)f_A(v_A)}$ and $J_A^\beta(v_A) = J_A(v_A) + \frac{1}{f_A(v_A)}$, which are buyer A's adjust virtual valuations in the $\alpha - \mathit{mechanism}$ and $\beta - \mathit{mechanism}$, respectively. As a result, the relaxed problem is reduced to choosing the optimal allocation rules only:

$$\begin{aligned}\max_{q^\alpha,q^\beta} &= (1-x)\int_{\underline{a}}^{\overline{a}} J_A^\alpha(v_A)q_A^\alpha(v_A)dF_A(v_A)\\ &+x\int_{\underline{b}}^{\bar{b}}\int_{\underline{a}}^{\overline{a}}\left\{J_A^\alpha(v_A)q_A^\beta\left(v_A,v_B\right)+J_B\left(v_B\right)q_B^\beta\left(v_A,v_B\right)\right\}dF_A(v_A)dF_B(v_B),\end{aligned}$$

s.t.

$$\int_{\underline{a}}^{\overline{a}} q_A^\alpha(v_A)dv_A - \int_{\underline{a}}^{\overline{a}}\int_{\underline{b}}^{\bar{b}} q_A^\beta(v_A,v_B)dF_B(v_B)dv_A \geq 0. \tag{3.24}$$

Consider the unconstraint optimal solution by ignoring the constraint (3.24), which is obviously:

$$q^\alpha\left(v_A\right) = \begin{cases} 1 & \text{if} \quad J_A^\alpha\left(v_A\right)\geqslant 0\\ 0 & \text{if} \quad J_A^\alpha\left(v_A\right)<0\end{cases} \tag{3.25}$$

$$q_A^\beta\left(v_A,v_B\right) = \begin{cases} 1 & \text{if } J_A^\beta\left(v_A\right)\geqslant \max\left\{J_B\left(v_B\right),0\right\}\\ 0 & \text{otherwise}\end{cases} \tag{3.26}$$

$$q_B^\beta\left(v_A,v_B\right) = \begin{cases} 1 & \text{if } J_B\left(v_B\right)\geqslant \max\left\{J_A^\beta\left(v_A\right),0\right\}\\ 0 & \text{otherwise}\end{cases} \tag{3.27}$$

The LHS of (3.24) under this unconstraint optimal solution can be written as

$$[\overline{a} - J_A^{\alpha^{-1}}(0)] - \int_{J_A^{\beta^{-1}}(0)}^{\bar{a}} F_B\left(J_B^{-1}\circ J_A^\beta\left(v_A\right)\right)dv_A \tag{3.28}$$

If this term is greater than zero, then the unconstraint optimal solution solves the relaxed problem. Otherwise, (3.24) must be binding and we can take the continuous linear programming approach and construct the Lagrangian:

$$\begin{aligned}\mathbb{L} &= (1-x)\int_{\underline{a}}^{\overline{a}} J_A^\alpha(v_A;\lambda)q_A^\alpha(v_A)dF_A(v_A)\\ &+x\int_{\underline{b}}^{\bar{b}}\int_{\underline{a}}^{\overline{a}}\left\{J_A^\beta(v_A;\lambda)q_A^\beta\left(v_A,v_B\right)+J_B\left(v_B\right)q_B^\beta\left(v_A,v_B\right)\right\}dF_A(v_A)dF_B(v_B),\end{aligned}$$

where $J_A^\alpha(v_A;\lambda) = J_A(v_A) - \frac{x-\lambda}{(1-x)f_A(v_A)}$ and $J_A^\beta(v_A;\lambda) = J_A(v_A) + \frac{x-\lambda}{xf_A(v_A)}$ are buyer A's adjust virtual valuations in the $\alpha-mechanism$ and $\beta-mechanism$, respectively. Note that $J_A^\alpha(v_A;0) = J_A^\alpha(v_A)$ and $J_A^\beta(v_A;0) = J_A^\beta(v_A)$. Obviously, given $\lambda = \lambda^*$, the solution is

$$q^{\alpha^*}(v_A) = \begin{cases} 1 & \text{if} \quad J_A^\alpha(v_A;\lambda^*) \geqslant 0 \\ 0 & \text{if} \quad J_A^\alpha(v_A;\lambda^*) < 0 \end{cases} \tag{3.29}$$

$$q_A^{\beta^*}(v_A, v_B) = \begin{cases} 1 & \text{if } J_A^\beta(v_A;\lambda^*) \geqslant \max\{J_B(v_B), 0\} \\ 0 & \text{otherwise} \end{cases} \tag{3.30}$$

$$q_B^{\beta^*}(v_A, v_B) = \begin{cases} 1 & \text{if } J_B(v_B) \geqslant \max\left\{J_A^\beta(v_A;\lambda^*), 0\right\} \\ 0 & \text{otherwise} \end{cases} . \tag{3.31}$$

Plugging this solution into (3.24) and making it binding determine the value of λ^*:

$$\int_{\underline{a}}^{\overline{a}} q_A^{\alpha^*}(v_A)dv_A - \int_{\underline{a}}^{\overline{a}} \int_{\underline{b}}^{\overline{b}} q_A^{\beta^*}(v_A,v_B)dF_B(v_B)dv_A \geq 0, \tag{3.32}$$

which can be rewritten equivalently in terms of primitives:

$$[\overline{a} - J_A^{\alpha^{-1}}(0;\lambda^*)] - \int_{J_A^{\beta^{-1}}(0;\lambda^*)}^{\bar{a}} F_B\left(J_B^{-1} \circ J_A^\beta(v_A;\lambda^*)\right) dv_A = 0. \tag{3.33}$$

Now we have fully solved the relaxed problem. We summarize the optimal mechanism is the next subsection.

3.4.2 The optimal mechanism and its properties

Here we restate some key notations so that this subsection is self-contained. Let $J_A^\alpha(v_A;\lambda) = J_A(v_A) - \frac{x-\lambda}{(1-x)f_A(v_A)}$ and $J_A^\beta(v_A;\lambda) = J_A(v_A) + \frac{x-\lambda}{xf_A(v_A)}$ be buyer A's adjust virtual valuations in the $\alpha - mechanism$ and $\beta - mechanism$, respectively. Also let $J_A^\alpha(v_A) = J_A^\alpha(v_A;0)$ and $J_A^\beta(v_A) = J_A^\beta(v_A;0)$ The following proposition summarizes the optimal solution in the relaxed problem and shows that it is also feasible in the original problem and thus solves the original problem.

Proposition 3.2. *The optimal mechanism in the original problem is as follows.*

$$q^{\alpha^*}(v_A) = \begin{cases} 1 & \text{if} \quad J_A^{\alpha}(v_A;\lambda^*) \geqslant 0 \\ 0 & \text{if} \quad J_A^{\alpha}(v_A;\lambda^*) < 0 \end{cases} \tag{3.34}$$

$$q_A^{\beta^*}(v_A, v_B) = \begin{cases} 1 & \text{if } J_A^{\beta}(v_A;\lambda^*) \geqslant \max\{J_B(v_B), 0\} \\ 0 & \text{otherwise} \end{cases} \tag{3.35}$$

$$q_B^{\beta^*}(v_A, v_B) = \begin{cases} 1 & \text{if } J_B(v_B) \geqslant \max\left\{J_A^{\beta^*}(v_A;\lambda^*), 0\right\} \\ 0 & \text{otherwise} \end{cases} \tag{3.36}$$

$$U_A^{\beta^*}(\underline{a}) = G(0) \tag{3.37}$$

where λ^* *is the unique solution to*

$$G(\lambda^*) \equiv [\bar{a} - J_A^{\alpha^{-1}}(0;\lambda^*)] - \int_{J_A^{\beta^{-1}}(0;\lambda^*)}^{\bar{a}} F_B\left(J_B^{-1} \circ J_A^{\beta}(v_A;\lambda^*)\right) dv_A = 0, \tag{3.38}$$

if $G(0) \leq 0$*, and* $\lambda^* = 0$ *if* $G(0) > 0$.

The optimal mechanism can be interpreted in a similar way as that for the benchmark model.[9] In the $\alpha - mechanism$, the seller should sell to buyer A whenever his adjusted virtual valuation is greater than zero; in the $\beta - mechanism$, the seller allocates the object to the buyer with a higher positive adjusted virtual valuation. The most important issue is how to adjust the virtual valuations. Note that the adjusted virtual valuations are indexed by λ^*. There are several important features of the optimal mechanism. If we compare buyer A's adjusted virtual valuation with the standard virtual valuation in the benchmark, we obtain the following result.

Corollary 3.1. *We have* $0 \leq \lambda^* < x$. *Thus, compared to the benchmark model, buyer A is strictly disfavored in the* $\alpha - mechanism$, *and is strictly favored in the* $\beta - mechanism$.

This is intuitive. Due to the existence of buyer B, the seller needs to distort the allocation rule to induce buyer A to refer buyer B. This can be achieved by either

[9] Note that $U_A^{\alpha^*}(\underline{a}) = 0, U_B^{\beta^*}(\underline{b}) = 0$, and the expected transfer functions are determined uniquely by the IC constraints.

reducing the payoff for non-referral in the $\alpha-mechanism$ or increasing it for referral in the $\beta - mechanism$. Note that $\frac{x-\lambda^*}{(1-x)f_A(v_A)}$ measures the magnitude of disfavor in the $\alpha - mechanism$, and $\frac{x-\lambda^*}{xf_A(v_A)}$ measures the magnitude of favor in the $\beta - mechanism$.

In the benchmark model, there is no need to set $U_A^\beta(\underline{a})$ strictly positive. The following corollary illustrates that it could be optimal to set the informational rent for referral strictly positive to provide referral incentive.

Corollary 3.2. $U_A^{\beta^*}(\underline{a}) > 0$ *if and only if* $[\overline{a}-J_A^{\alpha^{-1}}(0)]-\int_{J_A^{\beta-1}(0)}^{\bar{a}} F_B\left(J_B^{-1}\circ J_A^\beta(v_A)\right)dv_A > 0$.

Note that when $\lambda = x$, the allocation rule reduces to the optimal allcoation rule for the benchmark problem proposed by proposition 3.1. Also note that the only difference of the benchmark problem is that it ignores the referral constraint. These two observations provide us the idea that if the seller can carefully modify the optimal mechanism for the benchmark problem to satisfy the referral constraint, the resulting mechanism is then optimal. Corollary 3.2 gives the insight of the modification process. It is always better to change the allocation rule first by reducing λ while keeping the unconditional payment $U_A^{\beta^*}(\underline{a})$ to be 0. If λ hits the lower bar 0 and the corresponding mechanism still violates referral constraint, then the seller should then provide the unconditional payment.

The above two corollaries fully spell out the difference between the optimal mechanism with referral and the benchmark model by revealing three different channels to provide incentive for buyer A to refer buyer B: discouraging non-referral by disfavoring the existing buyer in the $\alpha - mechansim$, favoring referral by favoring the existing buyer in the $\beta - mechanism$, and providing informational rent for referral. Corollary 3.1 implies that it is necessary to distort the allocation rule and the first two channels always arise. Corollary 3.2 implies that the third channel may be needed. We thus conclude that

Corollary 3.3. *To provide referral incentive, discouraging non-referral and favoring referral are nacessary, and providing informational rent for referral is supplemen-*

tary.

3.4.3 The effect of potential buyer's existence and strength

It is of particular interest to investigate how the probability of the existence of buyer B would affect the optimal mechanism. This is summarized in the following proposition.

Proposition 3.3. *When buyer B is more likely to exist,*

(1) buyer A is more disfavored in the $\alpha-$ mechanism and less favored in the $\beta-$ mechanism.

(2) a lower informational rent for referral is needed.

(3) buyer A with any valuation is worse off, and more so for higher valuation.

Here is the intuition. When buyer B is more likely to exist, the seller is less likely to sell to buyer A directly and will disfavor him more in the $\alpha-$ *mechanism*. This reduces buyer A's payoff from not referring buyer B. Therefore, there is less need to provide referral incentives by either favoring him in the $\beta-$*mechanism* or providing informational rent for referral. These explain the first two results. For the third result, it is because buyer B will be in direct competition with buyer A and the presence of buyer B hurts buyer A.

Another interesting comparative statistic to conduct is on the effect of how attractive buyer B is, which is summarized in the following proposition.

Proposition 3.4. *When buyer B becomes stronger in term of first order stochastic dominance,*

(1) buyer A is more disfavored in the $\alpha-$ mechanism and more favored in the $\beta-$ mechanism.

(2) a higher informational rent for referral is needed.

(3) buyer A with the highest valuation is worse off, while buyer A with the lowest valuation is better off.

When buyer B is stronger, the seller is less likely to sell to buyer A directly and will disfavor him more in the $\alpha - mechanism$. The not so intuitive parts are for the distortion in $\beta - mecahanism$ and the informational rent for referring. On one hand, less referral incentive is needed due to a lower payoff in the $\alpha - mechanism$ from not referring.

On the other hand, a stronger B makes buyer A worse off in the $\beta{-}mechanism$. This reduces buyer A's payoff from referring buyer B, which implies more need for referral incentives. It turns out that the impact on $\beta - mechansim$ is of higher order since buyer B does not appear in the $\alpha - mechansim$ but enters in the $\beta - mechansim$ directly. As a result, overall, more referral incentive is needed. These explain the first two results. For the third result, a stronger buyer B has different implication for buyer A with different valuations. A stronger buyer B is more attractive to the seller, and she is willing to provide higher incentive for buyer A to refer; at the same time a stronger B reduces buyer A's chance of winning. The first effect benefits buyer A while the second hurts him. For buyer A with low valuation, winning is unlikely anyway and thus the first effect dominates. For buyer A with high valuation, winning is the priority and the second effect dominates.

Since informational rent for referral is supplementary, it is interesting to ask when one would expect the seller to use it. This is a direct implication of Propositions 3.3 and 3.4.

Corollary 3.4. *A higher informational rent for referral is needed when buyer B is less likely to exist and becomes stronger.*

3.4.4 Implementation with simple mechanisms

In this section, we investigate when the optimal mechanism can be implemented by simple mechanisms. Just as standard auctions with reserve prices cannot implement Myerson's optimal auction in general, we need further assumptions on primitives.

Suppose both buyers' valuations are uniform distributed on [0,1]. This is the most natural example to start with. The corresponding virtual valuations are:

$$J_A(v_A) = 2v_A - 1, \tag{3.39}$$

$$J_A^{\alpha}(v_A;\lambda) = 2v_A + \tfrac{\lambda-1}{1-x}, \tag{3.40}$$

$$J_A^{\beta}(v_A;\lambda) = 2v_A - \tfrac{\lambda}{x}, \tag{3.41}$$

$$J_B(v_B) = 2v_B - 1. \tag{3.42}$$

Plug these into the G function, we can get

$$G(\lambda) = \tfrac{x^2+3x-4\lambda}{8x(x-1)} \tag{3.43}$$

$$G(0) = \tfrac{x+3}{8(x-1)} < 0 \tag{3.44}$$

$$\lambda^* = \tfrac{x^2+3x}{4} \tag{3.45}$$

We thus have the following proposition.

Corollary 3.5. *Suppose both buyer A and buyer B's valuations follow uniform distribution on* [0,1]*, then the optimal mechanism is:*

$$q^{\alpha}(v_A) = \begin{cases} 1 & if \quad v_A \geq \frac{1}{2} + \frac{x}{8} \\ 0 & otherwise \end{cases}; \tag{3.46}$$

$$q_A^{\beta}(v_A,v_B) = \begin{cases} 1 & if \quad v_A + \frac{1-x}{8} > max\{v_B,\frac{1}{2}\} \\ 0 & otherwise \end{cases}; \tag{3.47}$$

$$q_B^{\beta}(v_A,v_B) = \begin{cases} 1 & if \quad v_B \geq max\{v_A + \frac{1-x}{8},\frac{1}{2}\} \\ 0 & otherwise \end{cases}; \tag{3.48}$$

$$U_A^{\beta}(0) = 0 \tag{3.49}$$

In this case, it is expectable that one can implement the optimal mechanism by extending standard auctions. This is confirmed by the following proposition.

Proposition 3.5. *With symmetric uniform buyers, the following modified second-price auction with conditional coupon implements the optimal mechanism. If buyer*

A does not refer Buyer B, then set a take-it-or-leave-it price at $\frac{1}{2} + \frac{x}{8}$ *to buyer A. If buyer A refers buyer B, then run a second-price auction with reserve price* $\frac{1}{2}$ *and conditional coupon equal to* $\frac{1}{8}(1-x)$ *to buyer A if he wins.*

This mechanism is simple and easy to use since the conditional coupon is a constant and independent of bids. Compared with the benchmark, a higher price for non-referral and a conditional coupon are used to distort the allocation as so to provide referral incentive. Note that with uniform distributions, there is no need to provide informational rent for referral. One remark is that when x converges to 1, the take-it-or-leave-it price converges to $\frac{5}{8}$. However this does mean that buyer A will buy directly when his valuation is higher than $\frac{5}{8}$. In fact, in this case, buyer A never buys directly.

When buyer B is strong, it seems reasonable not to sell to buyer A when he refers, and the optimal mechanism may be able to implemented by sequential selling. The following proposition provides such a situation.

Proposition 3.6. *When* $J_B(\underline{b}) \geq J_A^{\beta}(\overline{a},\lambda^*)$*, the following sequential selling mechanism implements the optimal mechanism. First, make a take-it-or-leave-it price at* $J_A^{\alpha-1}(0)$ *to buyer A; if he does not buy, then provide him with a cash bonus* $G(0)$ *to refer buyer B and make a take-it-or-leave-it price at* $J_B^{-1}(0)$ *to buyer B only.*

This mechanism is simple and easy to use since the cash bonus is a constant. Note that real-estate agents and other intermediaries fit into the assumption of proposition 3.6. Therefore our model is general enough to incorporate these real world situations.

3.4.5 Would resale mechanism be optimal?

Calzolari and Pavan (2006) consider a model where a seller sells an indivisible good to a buyer with the consideration that the buyer may resell the object to another buyer in the secondary market. This is particularly relevant when the good is durable such as real estates, artwork and antiques. In our environment, one natural question is why not the seller simply sells the object to buyer A who then can resell the object to buyer B? Let us consider the following natural resale mechanism: the seller sets an optimal take-it-or-leave-it price to buyer A knowing that buyer A will set an optimal take-it-or-leave-it price to resell the item to buyer B if exists. We have the following proposition.

Proposition 3.7. *The resale mechanism is never optimal when referral is possible.*

Here is the intuition. In the resale mechanism, the seller does not communicate with buyer B directly. When buyer A communicates with buyer B, buyer A's new knowledge about B becomes another piece of private information of buyer A and some extra information rent has to be provided to him. This proposition implies that whenever the seller has the chance to communicate with buyer B directly, it is always optimal to do so.

3.4.6 Multiple Buyers

While our model of one existing representative buyer and one potential third party are general enough to incorporate our motivative examples such as real-estate agents and crowdsourcing platforms, it is still worthwhile to investigate how our model can be extended to have multiple agents. First note that the insights our analysis provides above still hold where the existing buyer knows more than one potiential buyers. The only difference is how to adjust the virtue valuation.

We then examine the case where there are multiple existing buyers and they have one common potential buyer. [10] We claim that the following two-step mechanism is optimal. The seller first asks all the existing buyers to refer the potential buyer. If all the existing buyers do not refer, then the mechanism goes to the second step. If at least one of the existing buyer refers, then the seller imposes sufficiently large punishments on those who don't refer. In the second step, the seller runs Myerson's optimal mechanism correspondingly (i.e., to allocate the good to the buyer who has the highest positive virtue valuation). To see why this mechansim is optimal, first note that if we ignore the referral incentive constraints, Myerson's optimal mechaism can achieve the highest revenue. In addition, the first step of the mehanism can

[10] Actually, if any potential buyer is not exclusive to one of the existing buers, (i,e, at least two existing buyers know him), the analysis here still holds.

make all the existing buyers truthfully reveal the information. Unlike what we have discussed in the main paragraph, in this case the seller do not need to provide any information rent to induce referral. Note that here the private information regarding whether the potiential buyer exists is correlated among the existing buyers. Thus, according to Cremer and McLean (1988), the seller can fully extract the surplus of this piece of information.

3.5 Conclusion

While more bidders benefit the seller, it is not in the interest of a bidder to invite more bidders to the auction since it reduces his chance of winning. This apparent conflict of interests makes the design of optimal auction nontrivial in providing referral incentive. In this chapter, we provide a first step toward the design problem involved. We are able to identify three optimal channels. It is essential to distort the allocation by discouraging non-referral and favoring referral. When the potential buyer is less likely to exist and is stronger, the essential channels become too costly for the seller. In this case, a supplementary channel is to provide informational rent for referral. We also identify two situations in which the optimal mechanism can be implemented by simple mechanisms: either a second price auction with constant conditional coupon or sequential selling. Finally, we show that the conventional resale mechanism is suboptimal.

There are many directions in which our analysis can be extended. For instance, we can assume that the existing buyer must incur a strictly positive cost to refer the potential buyer. In this case, we only need to compare the revenue difference between the following two senarios. First, running the optimal mechanism derived by this chapter but compensating the existing buyer for the cost of referral. Second, if the referral cost is too high, simply ignoring the potential buyer and setting a take-it-or-leave-it price to the existing buyer. The optimal mechanism is the better one of the two. It is compelling to extend the analysis to allow general social network as usually allowed in the literature with the axiomatic approach. While it is possible to consider some specific network structures, a general treatment is challenging but rewarding. Finally, the referral incentive is not only relevant for auction design but also for any mechanism design problem. For example, we may compare the following three scenarios. First, referring is strategically substitutable to the existing agents. Examples include auction as including a new bidder intensifies the competition among the current bidders, which is the main topic of this chapter. Second, strategically neutral. The popular referral refunding program serves as a typical example. Introducing a new friend to the seller will not cause strategic behaviors among the buyers. The last scenario is strategically complementary. For instance, in the public good provision case, an additional agent will reduce the average cost of providing the public good. This chapter thus opens the possibility to consider optimal referral incentive for a general mechanism design problem. These will be pursued by the authors in future work.

3.6 Appendix

Proof for Proposition 3.2

We need to establish three results. Result (A), the optimal mechanism satisfies the monotonicity conditions in the original problem (3.14). Result (B), the optimal mechanism satisfies the referral incentive constraint in the original problem (3.15). Result (C), if $G(0) \leq 0$, there exists a unique solution to (3.55). Result (A) is obvious, and thus the proof is omitted.

Result(B). By replacing $U_A^{\beta}(\underline{a})$ with $G(0)$, constraint (3.15) becomes:

$$\int_{v_A}^{\bar{a}} q^{\alpha}(\xi)\, d\xi \geqslant \int_{v_A}^{\bar{a}} Q^{\alpha}(\xi)\, d\xi, \forall v_A \tag{3.50}$$

For $v_A \geqslant J_A^{\alpha^{-1}}(0)$, in the $\alpha-$mechanism buyer A obtains the object for sure, and thus have

$$LHS = \overline{a} - v_A \geq RHS. \tag{3.51}$$

For $v_A < J_A^{\alpha^{-1}}(0)$, we have

$$LHS = \int_{\underline{a}}^{\bar{a}} q^{\alpha}(\xi)\, d\xi \geq \int_{\underline{a}}^{\bar{a}} Q^{\alpha}(\xi)\, d\xi \geq RHS \tag{3.52}$$

Result (C). We first show that $G(\lambda)$ is strictly increasing. $\forall \lambda > \lambda' \in [0,x]$, we have $J_A^{\alpha}(v_A;\lambda) > J_A^{\alpha}(v_A;\lambda')$ for all v_A. Thus, $\int_{\underline{a}}^{\overline{a}} q^{\alpha}(v_A;\lambda)\, dv_A > \int_{\underline{a}}^{\overline{a}} q^{\alpha}(v_A;\lambda')\, dv_A$. Similarly, since $J_A^{\beta}(v_A;\lambda) < J_A^{\beta}(v_A;\lambda')$ for all v_A, we obtain

$$\int_{\underline{a}}^{\bar{a}} \int_{\underline{b}}^{\bar{b}} q_A^{\beta}(v_A, v_B;\lambda)\, f_B(v_B)\, dv_B dv_A < \int_{\underline{a}}^{\bar{a}} \int_{\underline{b}}^{\bar{b}} q_A^{\beta}(v_A, v_B;\lambda')\, f_B(v_B)\, dv_B dv_A. \tag{3.53}$$

By the definition of $G(\lambda)$, it is strictly increasing.

Second, we show that $G(x) > 0$. This is because when $\lambda = x$, we have $J_A^{\alpha}(v_A) = J_A^{\beta}(v_A) = J_A(v_A)$. Therefore, $\int_{\underline{a}}^{\overline{a}} q^{\alpha}(v_A;x)\, dv_A > \int_{\underline{a}}^{\overline{a}} Q_A^{\beta}(v_A;x)\, dv_A$, which implies that $G(x) > 0$.

Therefore, if $G(0) \leq 0$, there exists a unique solution such that $G(\lambda^*) = 0$. ■

Proof of proposition 3.3

It is convenient to introduce a new notation $w = \frac{\lambda}{x}$. We then index the modified virtual valuations for buyer A using w instead of λ as: $J_A^{\alpha}(v_A;w) = J_A(v_A) + \frac{x}{1-x}\frac{w-1}{f_A(v_A)}$ and $J_A^{\beta}(v_A;w) = J_A(v_A) + \frac{1-w}{f_A(v_A)}$.

Lemma 3.2. *w strictly increases in x.*

Proof: Note that ω is determined by

$$[\overline{a} - J_A^{\alpha^{-1}}(0;w)] - \int_{J_A^{\beta^{-1}}(0;w)}^{\bar{a}} F_B\left(J_B^{-1} \circ J_A^{\beta}(v_A;w)\right) dv_A = 0. \tag{3.54}$$

Take the total derivative of $J_A^{\alpha^{-1}}(0;w)$ with respect to x:

$$-\underbrace{\frac{\partial[J_A^{\alpha^{-1}}(0;w)]}{\partial x}}_{K_1} - \underbrace{\frac{\partial[J_A^{\alpha^{-1}}(0;w)]}{\partial w}}_{K_2}\frac{dw}{dx} - \underbrace{\frac{\partial \int_{J_A^{\beta^{-1}}(0;w)}^{\bar{a}} F_B\left(J_B^{-1} \circ J_A^{\beta}(v_A;w)\right) dv_A}{\partial w}}_{K_3}\frac{dw}{dx} = 0 \tag{3.55}$$

We thus obtain

$$\frac{dw}{dx} = -\frac{K_1}{K_2 + K_3}. \tag{3.56}$$

$$K_1 := \frac{\partial \left[J_A^{\alpha-1}(0;w)\right]}{\partial x} = \frac{-(w-1)}{(1-x)^2 f_A\left(J_A^{\alpha-1}(0)\right)} \frac{1}{J_A^{\alpha\prime}\left(J_A^{\alpha-1}(0)\right)} > 0, \tag{3.57}$$

$$K_2 := \frac{\partial \left[J_A^{\alpha-1}(0;w)\right]}{\partial w} = \frac{-x}{(1-x) f_A\left(J_A^{\alpha-1}(0)\right)} \frac{1}{J_A^{\alpha\prime}\left(J_A^{\alpha-1}(0)\right)} < 0, \tag{3.58}$$

and

$$\begin{aligned} K_3 : &= \left\{ -F_B\left[J_B^{-1}(0)\right] \frac{d\left[J_A^{\beta-1}(0;w)\right]}{dw} + \int_{J^{\beta}{}_A^{-1}(0;w)}^{\bar{a}} \frac{\partial}{\partial w} F_B\left(J_B^{-1} \circ J_A^{\beta}(v_A;w)\right) dv_A \right\} \\ &= \left\{ -\frac{F_B\left[J_B^{-1}(0)\right]}{f_A\left(J_A^{\alpha-1}(0)\right) J_A^{\beta\prime}\left(J_A^{\beta-1}(0)\right)} + \int_{J^{\beta}{}_A^{-1}(0;w)}^{\bar{a}} f_B(\cdot) J_A^{\beta\prime}(\cdot)\left(-\frac{1}{f_A(v_A)}\right) dv_A \right\} \\ &< 0. \end{aligned}$$

Therefore,

$$\frac{dw}{dx} = -\frac{K_1}{K_2 + K_3} > 0. \tag{3.59}$$

■

Now we are ready to show the proposition.

Part (1). We have

$$\frac{d\left[J_A^{\alpha-1}(0;w)\right]}{dx} = K_1 + K_2\frac{dw}{dx} = K_1 - K_2\frac{K_1}{K_2+K_3} = \frac{K_1K_3}{K_2+K_3} > 0; \tag{3.60}$$

$$\frac{dJ_A^{\beta}(v_A)}{dx} = -\frac{1}{f_A(v_A)}\frac{dw}{dx} < 0. \tag{3.61}$$

Part (2). Note that $U_A^{\beta^*}(\underline{a}) = G(0)$, thus:

$$\frac{\partial G(0)}{\partial x} = \frac{\partial \left[\bar{a} - J_A^{\alpha^{-1}}(0,0)\right]}{\partial x} \tag{3.62}$$

$$= -\frac{f\left(J_A^{-1}(0,0)\right)(1-x)^2}{J_A'\left(J_A^{\alpha^{-1}}(0,0)\right)} < 0 \tag{3.63}$$

Part (3). We know that $U_A^{\beta^*}(v_A) = \int_{\underline{a}}^{v_A}\int_{\underline{b}}^{\bar{b}} q_A^{\beta^*}(\xi,v_B)dF_B(v_B)d\xi + U_A^{\beta^*}(\underline{a})$. We have already shown that $U_A^{\beta^*}(\underline{a})$ decreases in x. Thus it suffices to show that $\int_{\underline{a}}^{v_A}\int_{\underline{b}}^{\bar{b}} q_A^{\beta^*}(\xi,v_B)dF_B(v_B)d\xi$ decreases in x for all v_A. But then it is sufficient to show that $\int_{\underline{b}}^{\bar{b}} q_A^{\beta^*}(v_A,v_B)\, f_B(v_B)\, dv_B$ decreases in x:

$$\frac{d\left(\int_{\underline{b}}^{\bar{b}} q_A^{\beta^*}(v_A,v_B)\, f_B(v_B)\, dv_B\right)}{dx} = K_3\frac{dw}{dx} < 0. \tag{3.64}$$

Similarly, we can show that buyer A is also worse off in the $\alpha-$mechanism regardless of his type. ■

Proof for proposition 3.4

In this proof it is convenient to index the terms with distributions. We first establish how λ changes:

Lemma 3.3. *When $f_B(v_B)$ increases in terms of first order stochastic dominance, λ increases.*

Proof: Note that λ is determined by:

$$\int_{\underline{a}}^{\bar{a}} q^{\alpha^*}(v_A)\, dv_A = \int_{\underline{a}}^{\bar{a}}\int_{\underline{b}}^{\bar{b}} q_A^{\beta^*}(v_A,v_B)\, f(v_B)\, dv_B dv_A. \tag{3.65}$$

Suppose g_B first order stochastically dominates f_B. We need to show that $\lambda(f_B) >$

$\lambda(g_B)$. Suppose not and $\lambda(f_B) \leq \lambda(g_B)$. We then have

$$\int_{a}^{\bar{a}} \int_{\underline{b}}^{\bar{b}} q_A^{\beta^*}(v_A, v_B; \lambda(f_B)) f_B(v_B) dv_B dv_A \quad (3.66)$$

$$= \int_{a}^{\bar{a}} q^{\alpha^*}(v_A; \lambda(f_B)) dv_A \quad (3.67)$$

$$\leq \int_{a}^{\bar{a}} q^{\alpha^*}(v_A; \lambda(g_B)) dv_A \quad (3.68)$$

$$= \int_{a}^{\bar{a}} \int_{\underline{b}}^{\bar{b}} q_A^{\beta^*}(v_A, v_B; \lambda(g_B)) g_B(v_B) dv_B dv_A, \quad (3.69)$$

which is a contraception since a higher virtual valuation of B and a higher λ both decrease buyer A's probability of winning the object. ■

Now we are ready to show the proposition.

Part (1).

$$\frac{dJ_A^{\alpha}(v_A)}{d\lambda} = \frac{1}{(1-x)} \cdot \frac{1}{f_A(v_A)} > 0, \quad (3.70)$$

$$\frac{dJ_A^{\beta}(v_A)}{d\lambda} = -\frac{1}{x f_A(v_A)} < 0. \quad (3.71)$$

The result then follows Lemma 3.3.

Part (2). It suffices to show that $G(0; f_B) > 0$ implies $G(0; g_B) \geq G(0; f_B)$. To see this, notice that for any given v_A:

$$\int_{\underline{b}}^{\bar{b}} q_A^{\beta^*}(v_A, v_B; 0; f_B) f_B(v_B) dv_B \quad (3.72)$$

$$= Prob\{J_A^{\beta}(v_A, v_B; 0) > \max\{0, J_B(v_B, 0; f_B)\}\} \quad (3.73)$$

$$= Prob\{J_A^{\beta}(v_A, v_B; 0) > 0 \text{ and } J_A^{\beta}(v_A; 0) > J_B(v_B; 0; f_B)\}\} \quad (3.74)$$

$$\geq Prob\{J_A^{\beta}(v_A; 0) > 0 \text{ and } J_A^{\beta}(v_A; 0) > J_B(v_B; 0; g_B)\}\} \quad (3.75)$$

$$= \int_{\underline{b}}^{\bar{b}} q_A^{\beta^*}(v_A, 0; g_B) f_B(v_B) dv_B \quad (3.76)$$

Since the distribution of buyer B will not affect $q^{\alpha^*}(v_A; 0)$, we have $G(g_B; 0) \geq G(f_B; 0) > 0$.

Part (3). Since buyer A with the lowest valuation obtains a payoff equal to the informational rent for referral, the result in Part (2) already implies he is better off. We only need to show that buyer A with the highest valuation is worse off. Note that $U_A^{\beta^*}(\overline{a}) = \int_{\underline{a}}^{\bar{a}} \hat{q}^{\alpha}(v_A; 0)\, dv_A)$. Thus,

$$\frac{dU_A^{\beta^*}(\overline{a})}{d\lambda} = \frac{1}{(1-x)} \frac{1}{f_A\left(J_A^{\alpha-1}(0;\lambda)\right)} / J_A^{\alpha\prime}\left[J_A^{\alpha-1}(0,\lambda)\right] > 0. \tag{3.77}$$

The desired result then follows Lemma 3.3. ■

Proof for proposition 3.5

It is routine to show that it is a dominant strategy for buyer A to bid $v_A + \frac{1}{8}(1-x)$ while it is a dominant strategy for buyer B to bid his own value v_B. Thus, it is clear that the allocation rules are the same as the optimal mechanism. Since buyer A's payoff is zero if he does not refer buyer B, we must have $U_A^{\beta}(0) = 0$. This completes the proof. ■

Proof for proposition 3.6

Suppose $J_B(\underline{b}) \geq J_A^{\beta}(\bar{a}, \lambda^*)$, in the optimal $\beta-$mechanism buyer A can never the item and buyer B obtains the item if his value is larger than $J_B^{-1}(0)$ with payment $J_B^{-1}(0)$. This is effectively a take-it-or-leave-it price for buyer B at the price $J_B^{-1}(0)$. In addition, it is easy to see that the optimal $\alpha-$mechanism is a take-it-or-leave-it price for buyer A at price $J_A^{\alpha-1}(0)$. Finally, the unconditional payment $G(0)$ can induce buyer A to refer buyer B whenever he exists. ■

Proof for proposition 3.7

In the resale stage, buyer A with valuation v_A chooses the resale price r to maximize his payoff:

$$\max_r F_B(r) v_A + [1 - F_B(r)]\, r \tag{3.78}$$

FOC yields:

$$f_B(r)v_A + [1 - F_B(r)] - f_B(r)r = 0 \tag{3.79}$$

$$\Rightarrow \qquad r = J_B^{-1}(v_A) \tag{3.80}$$

From the point view of the seller, the possibility of resale modifies buyer A's valuation to $F_B[J_B^{-1}(v_A)]v_A + \{1 - F_B[J_B^{-1}(v_A)]\}J_B^{-1}(v_A)$. Denote this modified valuation as w with *c.d.f* $K(w)$ and *p.d.f* $k(w)$. The seller's problem is to choose a price p to maximize her revenue:

$$\max_p (1 - x) \cdot \{[1 - K(p)]\, p] + x\, \{[1 - K(p)]\, p\} \tag{3.81}$$

The first order condition is that :

$$(1 - x)\, \{[1 - K(p)] - f_A(p)p\} + x\, \{[1 - K(p)] - k(p) \cdot p\} = 0 \tag{3.82}$$

This implies that the optimal take-or-leave-it price for the seller satisfies:

$$\left\{p - \frac{1 - F_A(p)}{f_A(p)}\right\} = \frac{x}{1 - x}\left[\frac{1 - K(p)}{f_A(p)} - \frac{k(p)p}{f_A(p)}\right] \tag{3.83}$$

It suffices to show that in the $\beta - mechanism$ the allocation between buyer A and buyer B is different from that in the optimal mechanism. In the optimal resale mechanism the allocation is determined by the comparison between v_A and $J_B(v_B)$; in our optimal mechanism the allocation is determined by the comparison between v_A and $J_B(v_B)$. Since we assume atomless distribution, $v_A = J_A(v_A)$ cannot hold for all v_A, from which the proposition follows. ■

Chapter 4

Conclusion

In Chapter 2, we propose a desirable choice rule in the environment where each student may have multiple privileged types and have weak priorities over these types. We further show that the proposed sequential reservation choice rule is strategy-proof, can be implemented in polynomial times and is bilaterrally substitutable. One interesting direction for the future exploring is to compare the minimum quotas (reserves) to other kinds of affirmative action policies in the multi-dimensional type case. For example, it is popular in some area of the world that if a student belongs to a particular privileged group, she will have bonus points. Suppose a student belongs to multiple privileged group, one intuitive way to implement the affirmative action policy is simply to sum up the bonus points of each privileged type. It is interesting to explore who will be better off and who will be worse off in this implementation compared to in the sequential reservation choice rule.

In Chapter 3, we investigate the optimal mechanism for a seller who wants to sell one indivisible good. She can incentivize the exisiting buyer to refer his privately known potential buyers to participate in the competition for winning the good. We identify the optimal way to incentivize the existing buyer to refer. Based on this chapter, we can think of a general framework of mechanism deisn of referral. We can categorize all the mechanism design problems involving referral into three scenarios. First, referring is strategically substitutable to the existing agents. Examples include auction as including a new bidder intensifies the competition among the current bidders. Second, strategically neutral. The popular referral refunding program serves as a typical example. Introducing a new friend to the seller will not cause strategic behavior among the buyers. The last scenario is strategically complementary. For instance, in the public good provision case, an additional agent will reduce the average cost of providing the public good. We can compare the optimal mechods to induce referral among these three scenarios. Exploring the results in this general mechanism setting can be an interesting direction for the future research.

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