A Rank Constrained LMI Algorithm for the Robust H$^\infty$ Control of an Uncertain System via a Stable Output Feedback Controller

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***Abstract*— The paper presents a numerical algorithm for constructing a stable output feedback controller for the robust H^∞ control of an uncertain system. The uncertain systems under consideration contain structured uncertainty described by integral quadratic constraints. The controller is designed to achieve absolute stabilization with a specified level of disturbance attenuation. The main result gives an algorithm for constructing the desired controller in terms of LMIs subject to rank constraints.**

I. Introduction

Output feedback control design problems for linear time-invariant (LTI) systems have been studied extensively in the literature. The three main classes of output feedback control design problems are static output feedback (SOF), full-order output feedback (FOOF) and reduced order output feedback (ROOF). For the LTI systems without uncertainties, it has been shown that SOF and ROOF involve LMI conditions with rank constraints [1], [2], and FOOF can be solved by Riccati or LMI methods [3]. When structured uncertainties are present in the system, the convexity of FOOF will be destroyed, leading to rank constraints on the scaling variables; e.g., see [4].

This paper considers the problem of robust H^∞ control via a full-order stable output feedback controller. It is well known that the use of stable controllers is preferable to the use of unstable feedback controllers in many practical control problems; e.g., see [5], [6]. Indeed, the use of unstable controllers can lead to problems with actuator and sensor failure, sensitivity to plant uncertainties and nonlinearities and implementation problems. Also, it is well known that issues of robustness and disturbance attenuation are important in control system design. This has motivated a number researchers to consider problems of H^∞ control via the use of stable feedback controllers; e.g., see [5]–[7].

The results of this paper build on the results in a recent paper [8] which considers a new approach to the problem of robust H^∞ control via a stable output feedback controller. As in [8], we consider a class of uncertain systems with structured uncertainty described by Integral Quadratic Constraints (IQCs); e.g., see [9], [10]. The key idea behind the approach of [8] is to begin with an uncertain system of the type considered in [9] and then add an additional uncertainty to form a new uncertain system. Solving the robust output feedback problem for the new system ensures that the resulting controller also absolutely stabilizes the original uncertain system with a specified level of disturbance attenuation, and simultaneously the controller is forced to be stable. This gives a procedure for constructing a stable output feedback controller solving a problem of absolute stabilization with a specified level of disturbance attenuation.

The algorithm proposed in [8] involves the solution of algebraic Riccati equations dependent on a set of scaling parameters. However, no indication is given as to how these unknown scaling parameters might be constructed. Indeed, the problem of finding a suitable solution to a pair of Riccati equations dependent on a set of scaling parameters is known to be a difficult numerical problem. In this paper, we relax and simplify the assumptions used in [8], and propose a numerical algorithm which will enable these scaling parameters to be constructed. This numerical algorithm involves the solution to a rank constrained LMI problem; e.g., see [11] and the references therein. Although such rank constrained LMI problems may in general be difficult to solve, some of the currently available algorithms, such as LMIRank [12], have been found to lead to solutions to this problem in many practical situations. The paper concludes with an example which illustrates the proposed algorithm.

II. Problem Statement

We consider an output feedback H^∞ control problem for an uncertain system of the following form:

$$\begin{aligned}
\dot{x}(t) &= Ax(t)+B_1w(t)+B_2u(t)+\sum_{s=1}^{k} D_s\xi_s(t);\\
z(t) &= C_1x(t)+D_{12}u(t);\\
\zeta_1(t) &= K_1x(t)+G_1u(t);\\
&\vdots\\
\zeta_k(t) &= K_kx(t)+G_ku(t);\\
y(t) &= C_2x(t)+D_{21}w(t) \qquad (1)
\end{aligned}$$

where $x(t)\in\mathbf{R}^n$ is the *state*, $w(t)\in\mathbf{R}^p$ is the *disturbance input*, $u(t)\in\mathbf{R}^m$ is the *control input*, $z(t)\in\mathbf{R}^q$ is the *error output*, $\zeta_1(t)\in\mathbf{R}^{h_1},\ldots,\zeta_k(t)\in\mathbf{R}^{h_k}$ are the *uncertainty outputs*, $\xi_1(t)\in\mathbf{R}^{r_1},\ldots,\xi_k(t)\in\mathbf{R}^{r_k}$ are the *uncertainty inputs* and $y(t)\in\mathbf{R}^l$ is the *measured output*. The uncertainty in this

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system is described by a set of equations of the form

$$\begin{aligned}\xi_1(t) &= \phi_1(t, \zeta_1(\cdot)|_0^t)\\ \xi_2(t) &= \phi_2(t, \zeta_2(\cdot)|_0^t)\\ &\vdots\\ \xi_k(t) &= \phi_k(t, \zeta_k(\cdot)|_0^t)\end{aligned} \tag{2}$$

where the following Integral Quadratic Constraint is satisfied.

Definition 1: (Integral Quadratic Constraint; see [9], [10].) An uncertainty of the form (2) is an admissible uncertainty for the system (1) if the following conditions hold: Given any locally square integrable control input $u(\cdot)$ and locally square integrable disturbance input $w(\cdot)$, and any corresponding solution to the system (1), (2), let $(0, t_*)$ be the interval on which this solution exists. Then there exist constants $d_1 \geq 0, \ldots, d_k \geq 0$ and a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \to t_*$, $t_i \geq 0$ and

$$\int_0^{t_i} \|\xi_s(t)\|^2 dt \leq \int_0^{t_i} \|\zeta_s(t)\|^2 dt + d_s \quad \forall i \quad \forall s = 1, \ldots, k. \tag{3}$$

Here $\|\cdot\|$ denotes the standard Euclidean norm and $\mathbf{L}_2[0,\infty)$ denotes the Hilbert space of square integrable vector valued functions defined on $[0,\infty)$. Note that t_i and $t_\star$ may be equal to infinity. The class of all such admissible uncertainties $\xi(\cdot) = [\xi_1(\cdot), \ldots, \xi_k(\cdot)]$ is denoted Ξ.

The above definitions expand the class of admissible uncertainties to include a broad class of uncertainties due to unknown nonlinearities and uncertainties arising from unmodeled dynamics. They also allow for effects of non-zero initial conditions of uncertain dynamics as well as norm bounded noises in the uncertainty channels. The terms d_s in the right hand side of the IQC conditions (3) correspond to bounds on these effects. Allowing for a wider class of uncertainties which satisfy the IQCs (3) enables us to apply the S-procedure for the system (1) to establish the equivalence between the robust control problem considered in this paper and some parameterized H^∞ control problems. Note that the norm bounded uncertainty is a special case of this IQC uncertainty model; see [10] for more details.

For the uncertain system (1), (3), we consider a problem of absolute stabilization with a specified level of disturbance attenuation. The class of controllers considered are full-order stable output feedback controllers of the form

$$\begin{aligned}\dot{x}_c(t) &= A_c x_c(t) + B_c y(t),\\ u(t) &= C_c x_c(t)\end{aligned} \tag{4}$$

where $x_c(t) \in \mathbf{R}^n$ and A_c is a Hurwitz matrix.

Definition 2: The uncertain system (1), (3) is said to be *absolutely stabilizable with disturbance attenuation* γ via stable output feedback control if there exists a stable output feedback controller (4) and constants $c_1 > 0$ and $c_2 > 0$ such that the following conditions hold:

1) For any initial condition $[x(0), x_c(0)]$, any admissible uncertainty inputs $\xi(\cdot)$ and any disturbance input $w(\cdot) \in \mathbf{L}_2[0,\infty)$, then

$$[x(\cdot), x_c(\cdot), u(\cdot), \xi_1(\cdot), \ldots, \xi_k(\cdot)] \in \mathbf{L}_2[0,\infty)$$

(hence, $t_* = \infty$) and

$$\begin{aligned}&\|x(\cdot)\|_2^2 + \|x_c(\cdot)\|_2^2 + \|u(\cdot)\|_2^2 + \sum_{s=1}^{k} \|\xi_s(\cdot)\|_2^2\\ &\leq c_1[\|x(0)\|^2 + \|x_c(0)\|^2 + \|w(\cdot)\|_2^2 + \sum_{s=1}^{k} d_s].\end{aligned} \tag{5}$$

2) The following H^∞ norm bound condition is satisfied: If $x(0) = 0$ and $x_c(0) = 0$, then

$$J \triangleq \sup_{\substack{w(\cdot)\neq 0\\ w(\cdot)\in\mathbf{L}_2[0,\infty)}} \sup_{\xi(\cdot)\in\Xi} \frac{\|z(\cdot)\|_2^2 - c_2\sum_{s=1}^{k} d_s}{\|w(\cdot)\|_2^2} < \gamma^2. \tag{6}$$

Here, $\|q(\cdot)\|_2$ denotes the $\mathbf{L}_2[0,\infty)$ norm of a function $q(\cdot)$. That is, $\|q(\cdot)\|_2^2 \triangleq \int_0^\infty \|q(t)\|^2 dt$.

Let

$$\begin{aligned}\underline{B}_1 &= [B_1, D_1, \cdots, D_k], & \underline{D}_{21} &= [D_{21}, 0_{l\times r_1}, \cdots, 0_{l\times r_k}],\\ \underline{C}_1 &= [C_1', K_1', \cdots, K_k']', & \underline{D}_{12} &= [D_{12}', G_1', \cdots, G_k']'.\end{aligned}$$

We need the following assumptions about the uncertain system (1), (3) to derive the proposed robust control algorithm.

Assumption 1: The uncertain system (1), (3) will be assumed to satisfy the following conditions throughout the paper:

(i) $\underline{D}_{12}'\underline{D}_{12} > 0, D_{21}D_{21}' > 0$.

(ii) The matrix $\begin{bmatrix} A-\alpha I & B_2\\ \underline{C}_1 & \underline{D}_{12}\end{bmatrix}$ has full column rank for all $\alpha \in \mathbf{C}$ such that $Re(\alpha) \geq 0$.

(iii) The matrix $\begin{bmatrix} A-\alpha I & \underline{B}_1\\ C_2 & \underline{D}_{21}\end{bmatrix}$ has full row rank for all $\alpha \in \mathbf{C}$ such that $Re(\alpha) \geq 0$.

The above assumptions relax and simplify the assumptions used in [8]. They are standard technical assumptions commonly used in the Riccati approach to H^∞ control; e.g., see [13]. This will allow us to apply existing H^∞ control results to the class of uncertain systems under consideration. These assumptions will ensure that the related parameterized Riccati equations admit positive definite stabilizing solutions, so that the corresponding controllers can be derived. We will show in the next section how these assumptions lead to the resulting controllers; see Lemmas 3, 4, and 5 in the Appendix.

III. Controller Design

A new approach to the robust control of an uncertain system (1), (3) via a stable output feedback controller is presented in [8]. The algorithm proposed in [8] involves a two-step procedure. Firstly, a state feedback version of the approach of [9] is applied to the original uncertain system (1), (3). The resulting state feedback gain is then used to construct a new uncertain system for which the results of [9] is applied in order to obtain a stable controller which guarantees absolute stabilization with a specified level of disturbance attenuation. We briefly review this procedure in this section.

A. State feedback control of the original system

Let $\tau_1 > 0$, …, $\tau_k > 0$ be given constants and consider the algebraic Riccati equation

$$\begin{aligned}(A - B_2E_1^{-1}\hat{D}_{12}'\hat{C}_1)'X + X(A - B_2E_1^{-1}\hat{D}_{12}'\hat{C}_1)\\ +X(\hat{B}_1\hat{B}_1' - B_2E_1^{-1}B'_2)X\\ +\hat{C}_1'(I - \hat{D}_{12}E_1^{-1}\hat{D}_{12}')\hat{C}_1 = 0;\end{aligned} \tag{7}$$

where

$$\begin{aligned}\hat{C}_1 &= \begin{bmatrix} C_1 \\ \sqrt{\tau_1}K_1 \\ \vdots \\ \sqrt{\tau_k}K_k \end{bmatrix}; \hat{D}_{12} = \begin{bmatrix} D_{12} \\ \sqrt{\tau_1}G_1 \\ \vdots \\ \sqrt{\tau_k}G_k \end{bmatrix};\\ E_1 &= \hat{D}_{12}'\hat{D}_{12};\\ \hat{B}_1 &= \left[\ \gamma^{-1}B_1 \quad \sqrt{\tau_1}^{-1}D_1 \quad \dots \quad \sqrt{\tau_k}^{-1}D_k\ \right].\end{aligned} \tag{8}$$

Lemma 1: Suppose that constants $\tau_1 > 0, \dots, \tau_k > 0$ have been found such that the Riccati equation (7) has a solution $X > 0$ and let

$$K = -E_1^{-1}(B_2'X + \hat{D}_{12}'\hat{C}_1). \tag{9}$$

Then X is the stabilizing solution of (7), that is, $A + B_2K$ is Hurwitz.

Proof: From Lemma 3-(ii) in the Appendix, $\Big(A - B_2E_1^{-1}\hat{D}_{12}'\hat{C}_1,\ (I - \hat{D}_{12}E_1^{-1}\hat{D}_{12}')\hat{C}_1\Big)$ is detectable by Lemma 3.2.2 of Reference [10]. Therefore, X is the stabilizing solution of (7). ■

It has been shown in [8] that the uncertain system (1), (3) is absolutely stabilizable with disturbance attenuation γ via the state feedback controller $u(t) = Kx(t)$ where K is given in (9).

B. Output feedback control of the new system

Now, the state feedback gain matrix K defined in (9) is used to define a new uncertain system as follows:

$$\begin{aligned}\dot{x}(t) &= \tilde{A}x(t) + B_1w(t) + \tilde{B}_2u(t) + \sum_{s=1}^{k+1} D_s\xi_s(t);\\ z(t) &= \tilde{C}_1x(t) + J\xi_{k+1} + \tilde{D}_{12}u(t);\\ \zeta_1(t) &= \tilde{K}_1x(t) + F_1\xi_{k+1} + \tilde{G}_1u(t);\\ &\vdots\\ \zeta_k(t) &= \tilde{K}_kx(t) + F_k\xi_{k+1} + \tilde{G}_ku(t);\\ \zeta_{k+1}(t) &= \tilde{K}_{k+1}x(t) + \tilde{G}_{k+1}u(t);\\ y(t) &= C_2x(t) + D_{21}w(t)\end{aligned} \tag{10}$$

where

$$\begin{aligned}\tilde{A} &= A + \frac{1}{2}B_2K;\quad \tilde{B}_2 = \frac{1}{2}B_2;\quad D_{k+1} = B_2;\\ \tilde{C}_1 &= C_1 + \frac{1}{2}D_{12}K;\quad J = D_{12};\quad \tilde{D}_{12} = \frac{1}{2}D_{12};\\ \tilde{K}_1 &= K_1 + \frac{1}{2}G_1K;\quad F_1 = G_1;\quad \tilde{G}_1 = \frac{1}{2}G_1;\\ &\vdots\\ \tilde{K}_k &= K_k + \frac{1}{2}G_kK;\quad F_k = G_k;\quad \tilde{G}_k = \frac{1}{2}G_k;\\ \tilde{K}_{k+1} &= \frac{1}{2}K;\quad \tilde{G}_{k+1} = -\frac{1}{2}I_{m\times m}.\end{aligned} \tag{11}$$

Also, the IQCs (3) are extended to include the additional uncertainty input ξ_{k+1}:

$$\int_0^{t_i} \|\xi_s(t)\|^2dt \le \int_0^{t_i} \|\zeta_s(t)\|^2dt + d_s \quad \forall i \quad \forall s = 1,\dots,k+1. \tag{12}$$

Here d_{k+1} is any positive constant.

Remark The additional uncertainty ξ_{k+1} in the new system (10) has the property that for one specific value of the uncertainty, the new uncertain system reduces to the original uncertain system and thus any suitable controller for the new uncertain system will also solve the problem of absolute stabilization with a specified level of disturbance attenuation for the original system. Also, for a different value of the new uncertainty, the new uncertain system reduces to a certain open loop system in such a way that the controller is forced to be stable. The reader is referred to [8] for more details.

The Riccati equations under consideration are defined as follows: Let $\tilde{\tau}_1 > 0$, …, $\tilde{\tau}_{k+1} > 0$ be given constants and consider the following algebraic Riccati equations and spectral radius condition:

$$\begin{aligned}(\check{A} - \check{B}_2\check{E}_1^{-1}\check{D}_{12}'\check{C}_1)'\check{X} + \check{X}(\check{A} - \check{B}_2\check{E}_1^{-1}\check{D}_{12}'\check{C}_1)\\ +\check{X}(\check{B}_1\check{B}_1' - \check{B}_2\check{E}_1^{-1}\check{B}_2')\check{X}\\ +\check{C}_1'(I - \check{D}_{12}\check{E}_1^{-1}\check{D}_{12}')\check{C}_1 = 0;\end{aligned} \tag{13}$$

$$\begin{aligned}(\check{A} - \check{B}_1\check{D}_{21}'\check{E}_2^{-1}\check{C}_2)\check{Y} + \check{Y}(\check{A} - \check{B}_1\check{D}_{21}'\check{E}_2^{-1}\check{C}_2)'\\ +\check{Y}(\check{C}_1'\check{C}_1 - \check{C}_2'\check{E}_2^{-1}\check{C}_2)\check{Y}\\ +\check{B}_1(I - \check{D}_{21}'\check{E}_2^{-1}\check{D}_{21})\check{B}_1' = 0;\end{aligned} \tag{14}$$

$$\rho(\check{X}\check{Y}) < 1 \tag{15}$$

where

$$
\begin{aligned}
\breve{A} &= \tilde{A}+\bar{B}_1\bar{D}_{11}'\left(I_{\tilde{q}\times\tilde{q}}-\bar{D}_{11}\bar{D}_{11}'\right)^{-1}\bar{C}_1;\\
\breve{B}_2 &= \tilde{B}_2+\bar{B}_1\bar{D}_{11}'\left(I_{\tilde{q}\times\tilde{q}}-\bar{D}_{11}\bar{D}_{11}'\right)^{-1}\bar{D}_{12};\\
\breve{C}_2 &= C_2+\bar{D}_{21}\bar{D}_{11}'\left(I_{\tilde{q}\times\tilde{q}}-\bar{D}_{11}\bar{D}_{11}'\right)^{-1}\bar{C}_1;\\
\breve{D}_{22} &= \bar{D}_{21}\bar{D}_{11}'\left(I_{\tilde{q}\times\tilde{q}}-\bar{D}_{11}\bar{D}_{11}'\right)^{-1}\bar{D}_{12};\\
\breve{B}_1 &= \bar{B}_1\left(I_{\tilde{p}\times\tilde{p}}-\bar{D}_{11}'\bar{D}_{11}\right)^{-\frac{1}{2}};\\
\breve{D}_{21} &= \bar{D}_{21}\left(I_{\tilde{p}\times\tilde{p}}-\bar{D}_{11}'\bar{D}_{11}\right)^{-\frac{1}{2}};\\
\breve{C}_1 &= \left(I_{\tilde{q}\times\tilde{q}}-\bar{D}_{11}\bar{D}_{11}'\right)^{-\frac{1}{2}}\bar{C}_1;\\
\breve{D}_{12} &= \left(I_{\tilde{q}\times\tilde{q}}-\bar{D}_{11}\bar{D}_{11}'\right)^{-\frac{1}{2}}\bar{D}_{12};\\
\breve{E}_1 &= \breve{D}_{12}'\breve{D}_{12};\quad \breve{E}_2=\breve{D}_{21}\breve{D}_{21}';\\
\bar{B}_1 &= \begin{bmatrix}\gamma^{-1}B_1 & \sqrt{\tilde{\tau}_1}^{-1}D_1 & \dots & \sqrt{\tilde{\tau}_{k+1}}^{-1}D_{k+1}\end{bmatrix};\\
\bar{C}_1 &= \begin{bmatrix}\tilde{C}_1\\ \sqrt{\tilde{\tau}_1}\tilde{K}_1\\ \vdots\\ \sqrt{\tilde{\tau}_{k+1}}\tilde{K}_{k+1}\end{bmatrix};\\
\bar{D}_{11} &= \begin{bmatrix}
0_{q\times p} & 0_{q\times r_1} & \dots & 0_{q\times r_k} & \frac{1}{\sqrt{\tilde{\tau}_{k+1}}}J\\
0_{h_1\times p} & 0_{h_1\times r_1} & \dots & 0_{h_1\times r_k} & \frac{\sqrt{\tilde{\tau}_1}}{\sqrt{\tilde{\tau}_{k+1}}}F_1\\
\vdots & & & & \\
0_{h_k\times p} & 0_{h_k\times r_1} & \dots & 0_{h_k\times r_k} & \frac{\sqrt{\tilde{\tau}_k}}{\sqrt{\tilde{\tau}_{k+1}}}F_k\\
0_{m\times p} & 0_{m\times r_1} & \dots & 0_{m\times r_k} & 0_{m\times m}
\end{bmatrix};\\
\bar{D}_{12} &= \begin{bmatrix}\tilde{D}_{12}\\ \sqrt{\tilde{\tau}_1}\tilde{G}_1\\ \vdots\\ \sqrt{\tilde{\tau}_{k+1}}\tilde{G}_{k+1}\end{bmatrix};\\
\bar{D}_{21} &= \begin{bmatrix}\gamma^{-1}D_{21} & 0_{l\times r_1} & \dots & 0_{l\times r_k} & 0_{l\times m}\end{bmatrix}. \quad (16)
\end{aligned}
$$

Here $\tilde{q}=q+h_1\ldots+h_k+m$ and $\tilde{p}=p+r_1\ldots+r_k+m$.

The following assumption is needed in the next theorem which, (similar to Assumption 1) is a standard technical assumption used in the Riccati equation solution to the H^∞ control problem.

Assumption 2: The uncertain system (1), (3) will be assumed to satisfy the following condition for any $\tilde{\tau}_1>0$, ..., $\tilde{\tau}_{k+1}>0$:

$$\bar{D}_{11}\bar{D}_{11}'<I.$$

Theorem 1: [8] Suppose that the uncertain system (1), (3) satisfies Assumptions 1-2 and that there exist constants $\tau_1>0,\ldots,\tau_k>0$ such that the Riccati equation (7) has a solution $X>0$ and let

$$K=-E_1^{-1}(B'_2X+\hat{D}_{12}'\hat{C}_1). \quad (17)$$

Furthermore, suppose there exist constants $\tilde{\tau}_1>0,\ldots,\tilde{\tau}_{k+1}>0$ such that the Riccati equations (13) and (14) have solutions $\breve{X}>0$ and $\breve{Y}>0$ and such that the spectral radius condition (15) holds. Then the uncertain system (1), (3) is absolutely stabilizable with disturbance attenuation γ via a stable linear controller of the form (4) where

$$
\begin{aligned}
A_c &= \breve{A}_c-B_c\breve{D}_{22}C_c\\
\breve{A}_c &= \breve{A}+\breve{B}_2C_c-B_c\breve{C}_2+(\breve{B}_1-B_c\breve{D}_{21})\breve{B}_1'\breve{X}\\
B_c &= (I-\breve{Y}\breve{X})^{-1}(\breve{Y}\breve{C}_2'+\breve{B}_1\breve{D}_{21}')\breve{E}_2^{-1}\\
C_c &= -\breve{E}_1^{-1}(\breve{B}_2'\breve{X}+\breve{D}_{12}'\breve{C}_1). \quad (18)
\end{aligned}
$$

IV. A RANK CONSTRAINED LMI APPROACH

As shown in Theorem 1, the proposed stable output feedback controller design involves solving a pair of parameterized game-type Riccati equations. Generally, it is difficult to provide a systematic solution to such a problem. In this section, we discuss one possible numerical approach to address this difficulty. Similar to the technique in [14], the idea is to replace the Riccati equations with an equivalent feasibility problem involving rank constrained LMIs. First we introduce a related rank constrained LMI feasibility problem. Next, we prove the equivalence between the two problems.

Corresponding to the Riccati equations (13), (14) and the spectral radius condition (15) arising in Theorem 1, consider the following matrix inequalities and the spectral radius condition,

$$
\begin{aligned}
&(\breve{A}-\breve{B}_2\breve{E}_1^{-1}\breve{D}_{12}'\breve{C}_1)'\breve{X}+\breve{X}(\breve{A}-\breve{B}_2\breve{E}_1^{-1}\breve{D}_{12}'\breve{C}_1)\\
&\quad+\breve{X}(\breve{B}_1\breve{B}_1'-\breve{B}_2\breve{E}_1^{-1}\breve{B}_2')\breve{X}\\
&\quad+\breve{C}_1'(I-\breve{D}_{12}\breve{E}_1^{-1}\breve{D}_{12}')\breve{C}_1<0, \quad (19)
\end{aligned}
$$

$$
\begin{aligned}
&(\breve{A}-\breve{B}_1\breve{D}_{21}'\breve{E}_2^{-1}\breve{C}_2)\breve{Y}+\breve{Y}(\breve{A}-\breve{B}_1\breve{D}_{21}'\breve{E}_2^{-1}\breve{C}_2)'\\
&\quad+\breve{Y}(\breve{C}_1'\breve{C}_1-\breve{C}_2'\breve{E}_2^{-1}\breve{C}_2)\breve{Y}\\
&\quad+\breve{B}_1(I-\breve{D}_{21}'\breve{E}_2^{-1}\breve{D}_{21})\breve{B}_1'<0, \quad (20)
\end{aligned}
$$

$$\rho(\breve{X}\breve{Y})<1. \quad (21)$$

Note that the feasibility of (19-21) is equivalent to that of (13-15) under certain assumptions, see Theorem 2 given below.

Rewrite (19) as

$$
\begin{aligned}
&\breve{A}'\breve{X}+\breve{X}\breve{A}+\breve{X}\breve{B}_1\breve{B}_1'\breve{X}+\breve{C}_1'\breve{C}_1\\
&\quad-(\breve{X}\breve{B}_2+\breve{C}_1'\breve{D}_{12})\breve{E}_1^{-1}(\breve{X}\breve{B}_2+\breve{C}_1'\breve{D}_{12})'<0. \quad (22)
\end{aligned}
$$

By left and right multiplying (22) with $\hat{X}=\breve{X}^{-1}$, we obtain

$$
\begin{aligned}
&\hat{X}\breve{A}'+\breve{A}\hat{X}+\breve{B}_1\breve{B}_1'+\hat{X}\breve{C}_1'\breve{C}_1\hat{X}\\
&\quad-(\breve{B}_2+\hat{X}\breve{C}_1'\breve{D}_{12})\breve{E}_1^{-1}(\breve{B}_2+\hat{X}\breve{C}_1'\breve{D}_{12})'<0. \quad (23)
\end{aligned}
$$

Introducing a matrix variable $\hat{F}\in\mathbf{R}^{m\times n}$, without changing the feasibility of (23), we add a quadratic term involving $\hat{F}$ to the left-hand side of (23) as follows:

$$
\begin{aligned}
&\hat{X}\breve{A}'+\breve{A}\hat{X}+\breve{B}_1\breve{B}_1'+\hat{X}\breve{C}_1'\breve{C}_1\hat{X}\\
&+[\hat{F}'+(\breve{B}_2+\hat{X}\breve{C}_1'\breve{D}_{12})\breve{E}_1^{-1}]\breve{E}_1[\hat{F}'+(\breve{B}_2+\hat{X}\breve{C}_1'\breve{D}_{12})\breve{E}_1^{-1}]'\\
&-(\breve{B}_2+\hat{X}\breve{C}_1'\breve{D}_{12})\breve{E}_1^{-1}(\breve{B}_2+\hat{X}\breve{C}_1'\breve{D}_{12})'<0,
\end{aligned}
$$

which is

$$\hat{X}\breve{A}' + \breve{A}\hat{X} + \breve{B}_1\breve{B}_1' + \breve{B}_2\hat{F} + \hat{F}'\breve{B}_2' + (\breve{C}_1\hat{X} + \breve{D}_{12}\hat{F})'(\breve{C}_1\hat{X} + \breve{D}_{12}\hat{F}) < 0. \quad (24)$$

Substituting (16) into (24) and using the property

$$I + \bar{D}_{11}'(I - \bar{D}_{11}\bar{D}_{11}')^{-1}\bar{D}_{11} = (I - \bar{D}_{11}'\bar{D}_{11})^{-1},$$

we have

$$\begin{aligned} &\hat{X}\tilde{A}' + \tilde{A}\hat{X} + \bar{B}_1\bar{B}_1' + \tilde{B}_2\hat{F} + \hat{F}'\tilde{B}_2' \\ &\quad + (\bar{D}_{11}\bar{B}_1' + \bar{C}_1\hat{X} + \bar{D}_{12}\hat{F})'(I - \bar{D}_{11}\bar{D}_{11}')^{-1} \\ &\qquad \times (\bar{D}_{11}\bar{B}_1' + \bar{C}_1\hat{X} + \bar{D}_{12}\hat{F}) < 0, \end{aligned}$$

which, by Schur complement, is equivalent to

$$\begin{bmatrix} \mathcal{M}_{11} & (\bar{D}_{11}\bar{B}_1' + \bar{C}_1\hat{X} + \bar{D}_{12}\hat{F})'\tilde{\Gamma}^{-\frac{1}{2}} \\ \star & -\tilde{\Gamma}^{-1} + \tilde{\Gamma}^{-\frac{1}{2}}\bar{D}_{11}\bar{D}_{11}'\tilde{\Gamma}^{-\frac{1}{2}} \end{bmatrix} < 0, \quad (25)$$

where

$$\begin{aligned} \mathcal{M}_{11} &= \hat{X}\tilde{A}' + \tilde{A}\hat{X} + \bar{B}_1\bar{B}_1' + \tilde{B}_2\hat{F} + \hat{F}'\tilde{B}_2', \\ \tilde{\Gamma} &= \mathrm{diag}(I_q, \tilde{\tau}_1 I_{h_1}, \cdots, \tilde{\tau}_k I_{h_k}, \tilde{\tau}_{k+1} I_m). \end{aligned}$$

Here the notation $\star$ in the above matrix indicates that the corresponding elements in the matrix are such that the overall matrix is symmetric. Define $\hat{\tau}_i = \tilde{\tau}_i^{-1}, i = 1, \cdots, k+1$, and note that by (16)

$$\begin{aligned} \bar{B}_1\bar{B}_1' &= \gamma^{-2}B_1B_1' + \hat{\tau}_1 D_1 D_1' + \cdots + \hat{\tau}_{k+1} D_{k+1} D_{k+1}', \\ \bar{B}_1\bar{D}_{11}' &= [\hat{\tau}_{k+1} D_{k+1} J', \hat{\tau}_{k+1} D_{k+1} F_1', \\ &\qquad \cdots, \hat{\tau}_{k+1} D_{k+1} F_k', 0]\tilde{\Gamma}^{\frac{1}{2}}, \\ \hat{X}\bar{C}_1' &= [\hat{X}\tilde{C}_1', \hat{X}\tilde{K}_1', \cdots, \hat{X}\tilde{K}_{k+1}']\tilde{\Gamma}^{\frac{1}{2}}, \\ \hat{F}'\bar{D}_{12}' &= [\hat{F}'\tilde{D}_{12}', \hat{F}'\tilde{G}_1', \cdots, \hat{F}'\tilde{G}_{k+1}']\tilde{\Gamma}^{\frac{1}{2}}, \\ \tilde{\Gamma}^{-1} &= \mathrm{diag}(I_q, \hat{\tau}_1 I_{h_1}, \cdots, \hat{\tau}_k I_{h_k}, \hat{\tau}_{k+1} I_m), \\ \tilde{\Gamma}^{-\frac{1}{2}}\bar{D}_{11}\bar{D}_{11}'\tilde{\Gamma}^{-\frac{1}{2}} &= \hat{\tau}_{k+1} M_{JF} M_{JF}', \end{aligned}$$

where

$$M_{JF} = \begin{bmatrix} 0_{q\times p} & 0_{q\times r_1} & \dots & 0_{q\times r_k} & J \\ 0_{h_1\times p} & 0_{h_1\times r_1} & \dots & 0_{h_1\times r_k} & F_1 \\ \vdots & & & & \\ 0_{h_k\times p} & 0_{h_k\times r_1} & \dots & 0_{h_k\times r_k} & F_k \\ 0_{m\times p} & 0_{m\times r_1} & \dots & 0_{m\times r_k} & 0_{m\times m} \end{bmatrix}.$$

Then (25) is transformed into

$$\begin{bmatrix} \mathcal{M}_o & \mathcal{Q}_o \\ \star & \mathcal{S}_o \end{bmatrix} < 0, \quad (26)$$

where

$$\begin{aligned} \mathcal{M}_o &= \hat{X}\tilde{A}' + \tilde{A}\hat{X} + \tilde{B}_2\hat{F} + \hat{F}'\tilde{B}_2' + \gamma^{-2}B_1B_1' + \sum_{i=1}^{k+1}\hat{\tau}_i D_i D_i', \\ \mathcal{Q}_o &= [\hat{\tau}_{k+1} D_{k+1} J' + \hat{X}\tilde{C}_1' + \hat{F}'\tilde{D}_{12}', \\ &\qquad \hat{\tau}_{k+1} D_{k+1} F_1' + \hat{X}\tilde{K}_1' + \hat{F}'\tilde{G}_1', \cdots, \\ &\qquad \hat{\tau}_{k+1} D_{k+1} F_k' + \hat{X}\tilde{K}_k' + \hat{F}'\tilde{G}_k', \hat{X}\tilde{K}_{k+1}' + \hat{F}'\tilde{G}_{k+1}'], \\ \mathcal{S}_o &= -\hat{\Gamma} + \hat{\tau}_{k+1} M_{JF} M_{JF}', \\ \hat{\Gamma} &= \mathrm{diag}(I_q, \hat{\tau}_1 I_{h_1}, \cdots, \hat{\tau}_k I_{h_k}, \hat{\tau}_{k+1} I_m). \end{aligned}$$

Similarly, by defining $\hat{Y} = \check{Y}^{-1}$ and $\hat{L} \in \mathbf{R}^{n\times l}$, (20) is transformed into

$$\begin{bmatrix} \mathcal{N}_o & \mathcal{P}_o \\ \star & \mathcal{T}_o \end{bmatrix} < 0, \quad (27)$$

where

$$\begin{aligned} \mathcal{N}_o &= \tilde{A}'\hat{Y} + \hat{Y}\tilde{A} + \hat{L}C_2 + C_2'\hat{L}' + \tilde{C}_1'\tilde{C}_1 + \sum_{i=1}^{k+1}\tilde{\tau}_i\tilde{K}_i'\tilde{K}_i, \\ \mathcal{P}_o &= [\hat{Y}B_1 + \hat{L}D_{21}, \hat{Y}D_1, \cdots, \hat{Y}D_k, \\ &\qquad \hat{Y}D_{k+1} + \tilde{C}_1'J + \sum_{i=1}^{k}\tilde{\tau}_i\tilde{K}_i'F_i], \\ \mathcal{T}_o &= -\tilde{\Gamma}_\gamma + M_{JF}' * \tilde{\Gamma} * M_{JF}, \\ \tilde{\Gamma}_\gamma &= \mathrm{diag}(\gamma^2 I_p, \tilde{\tau}_1 I_{r_1}, \cdots, \tilde{\tau}_k I_{r_k}, \tilde{\tau}_{k+1} I_m), \\ \tilde{\Gamma} &= \mathrm{diag}(I_q, \tilde{\tau}_1 I_{h_1}, \cdots, \tilde{\tau}_k I_{h_k}, \tilde{\tau}_{k+1} I_m). \end{aligned}$$

The spectral radius condition (21) is equivalent to

$$\begin{bmatrix} \hat{X} & I_n \\ \star & \hat{Y} \end{bmatrix} > 0. \quad (28)$$

Furthermore, the conditions $\hat{\tau}_i\tilde{\tau}_i = 1, \tilde{\tau}_i > 0, i = 1, \cdots, k+1$ are equivalent to

$$\begin{bmatrix} \hat{\tau}_i & 1 \\ 1 & \tilde{\tau}_i \end{bmatrix} \geq 0, \quad rank \begin{bmatrix} \hat{\tau}_i & 1 \\ 1 & \tilde{\tau}_i \end{bmatrix} \leq 1, \quad i = 1, \cdots, k+1. \quad (29)$$

Our main result shows that the feasibility problem (13), (14), (15) in the variables $\check{X}, \check{Y}, \tilde{\tau}_i, i = 1, \cdots, k+1$ is equivalent to the rank constrained LMIs (26), (27), (28), (29) in the variables $\hat{X}, \hat{F}, \hat{Y}, \hat{L}, \hat{\tau}_i, \tilde{\tau}_i, i = 1, \cdots, k+1$. We need the following lemma to prove the equivalence between these problems.

Lemma 2: For any $\tilde{\tau}_1 > 0, \ldots, \tilde{\tau}_{k+1} > 0$,

(i) $\left(\check{A} - \check{B}_2\check{E}_1^{-1}\check{D}_{12}'\check{C}_1, (I - \check{D}_{12}\check{E}_1^{-1}\check{D}_{12}')\check{C}_1\right)$ is detectable.

(ii) $\left(\check{A} - \check{B}_1\check{D}_{21}'\check{E}_2^{-1}\check{C}_2, (I - \check{D}_{21}'\check{E}_2^{-1}\check{D}_{21})\check{B}_1'\right)$ is stabilizable.

Proof: This result follows directly from Lemma 5 in the Appendix and Lemma 3.2.2 of Reference [10]. ■

Theorem 2: Under Assumptions 1-2, the following statements hold.

(i) If the rank constrained LMIs (26), (27), (28), (29) admit solutions $\hat{X}, \hat{F}, \hat{Y}, \hat{L}, \hat{\tau}_i, \tilde{\tau}_i, i = 1, \cdots, k+1$, then there exist $\check{X}_R > 0, \check{Y}_R > 0$, such that (13), (14), (15) hold for $\check{X}_R, \check{Y}_R, \tilde{\tau}_i, i = 1, \cdots, k+1$.

(ii) If (13), (14), (15) admit solutions $\check{X}_R > 0, \check{Y}_R > 0, \tilde{\tau}_i > 0, i = 1, \cdots, k+1$, then there exist $\hat{X}, \hat{Y}$, such that the rank constrained LMIs (26), (27), (28), (29) hold for $\hat{X}, \hat{F} = -\check{E}_1^{-1}(\check{B}_2 + \hat{X}\check{C}_1'\check{D}_{12})', \hat{Y}, \hat{L} = -(\check{C}_2' + \hat{Y}\check{B}_1\check{D}_{21}')\check{E}_2^{-1}, \hat{\tau}_i = \tilde{\tau}_i^{-1}, \tilde{\tau}_i, i = 1, \cdots, k+1$.

The proof follows a similar line to that in [14] (see also [4]), and thus is omitted here. Note that both problems, if feasible, admit the same $\tilde{\tau}_i, i = 1, \cdots, k+1$, as seen in Theorem 2. As mentioned in the introduction, to solve this problem in our numerical experiments, we use the rank constrained LMI solver LMIRank [12].

Remark In contrast to the ROOF and SOF problems in which rank constraints are imposed on the Lyapunov variables [1], [2], the rank constraints (29) are imposed on the auxiliary Lagrange multipliers. We note here that these rank constraints are arising from the characterization of output feedback control and the structured uncertainties in the new system (10), (12). Similar non-convex conditions can also be found in [4], [14].

We should keep in mind that until now, the state feedback gain K is unknown. Using a similar argument to the above, we can show that the existence of a solution to the Riccati equation (7) is equivalent to the following convex optimization problem in the variables $W, \bar{X}, \bar{F}, \bar{\tau}_i, i = 1, \cdots, k$:

$$\begin{aligned} &\min \operatorname{tr}(W) \quad \text{subject to:} \\ &\begin{bmatrix} \mathcal{M}_s & \mathcal{Q}_s \\ \star & \mathcal{S}_s \end{bmatrix} < 0, \\ &\begin{bmatrix} W & I_n \\ \star & \bar{X} \end{bmatrix} > 0, \end{aligned} \tag{30}$$

where

$$\begin{aligned} \mathcal{M}_s &= \bar{X}A' + A\bar{X} + B_2\bar{F} + \bar{F}'B_2' + \gamma^{-2}B_1B_1' + \sum_{i=1}^{k} \bar{\tau}_i D_i D_i', \\ \mathcal{Q}_s &= [(C_1\bar{X} + D_{12}\bar{F})', (K_1\bar{X} + G_1\bar{F})', \cdots, (K_k\bar{X} + G_k\bar{F})'], \\ \mathcal{S}_s &= -diag(I_q, \bar{\tau}_1 I_{h_1}, \cdots, \bar{\tau}_k I_{h_k}). \end{aligned}$$

Letting $X = \bar{X}^{-1}, \tau_i = \bar{\tau}_i^{-1}, i = 1, \cdots, k$, then the state feedback gain K is obtained from (9).

We summarize the proposed control design algorithm as follows.

- Solve the convex optimization problem (30) to obtain $\bar{X}, \bar{\tau}_i, i = 1, \cdots, k$.
- Letting $X = \bar{X}^{-1}, \tau_i = \bar{\tau}_i^{-1}, i = 1, \cdots, k$, calculate K from (9).
- Construct new matrices in (11) with the resulting K.
- Solve the rank constrained LMIs (26), (27), (28), (29) to obtain a feasible solution $\hat{X}, \hat{F}, \hat{Y}, \hat{L}, \hat{\tau}_i, \tilde{\tau}_i, i = 1, \cdots, k+1$.
- Substitute the constants $\tilde{\tau}_i, i = 1, \cdots, k+1$ which have been found into the Riccati equations (26), (27) and solve them to obtain $\check{X} = \check{X}_R, \check{Y} = \check{Y}_R$.
- Construct the controller (18) using the parameters $\tilde{\tau}_i$, $\check{X}_R$ and $\check{Y}_R$ which have been found.

V. Illustrative Example

In this section, we consider a problem of absolute stabilization with a specified level disturbance attenuation in order to illustrate the algorithm developed above. We consider a system of the form (1), where $k = 0$ and

$$A = \begin{bmatrix} 0 & 1.00 & 0 & 0 & 0 & 0 & 0 \\ -0.61 & -0.12 & 0 & 0 & 0.01 & 0 & 0.01 \\ 0 & 0 & 0 & 1.00 & 0 & 0 & 0 \\ 0 & 0 & -0.45 & -0.10 & 0.01 & 0 & 0.01 \\ 0 & 0 & 0 & 0 & 0 & 1.00 & 0 \\ 0.01 & 0 & 0.01 & 0 & -0.61 & -0.12 & 0.01 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.98 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.12 & -0.07 & 0 & -0.06 & 0 & -0.06 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.07 & 0.10 & -0.07 & 0 & -0.07 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.06 & 0 & -0.05 & 0.12 & -0.06 & 0 \\ 0 & -0.12 & 0 & -0.12 & 0 & -0.12 & 10.00 \end{bmatrix},$$

$$\begin{aligned} B_1 &= \operatorname{diag}(B_{1,1}, B_{1,1}, B_{1,1}, B_{1,4}), \\ C_1 &= \operatorname{diag}(C_{z,1}, C_{z,1}, C_{z,1}, C_{z,4}), \\ D_{12} &= \operatorname{diag}(D_{z,1}, D_{z,1}, D_{z,1}, D_{z,4}), \\ C_2 &= \operatorname{diag}(C_{y,1}, C_{y,1}, C_{y,1}, C_{y,4}), \\ D_{21} &= \operatorname{diag}(D_{y,1}, D_{y,1}, D_{y,1}, D_{y,1}), \\ B_{1,1} &= \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix}, \quad B_{1,4} = \begin{bmatrix} 0.1 & 0 \end{bmatrix}, \\ C_{z,1} &= 0.1 \times \begin{bmatrix} I_{2\times 2} \\ 0_{2\times 2} \end{bmatrix}, \quad C_{z,4} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \\ D_{z,1} &= 0.1 \times \begin{bmatrix} 0_{2\times 2} \\ I_{2\times 2} \end{bmatrix}, \quad D_{z,4} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \\ C_{y,1} &= \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad C_{y,4} = 1, \quad D_{y,1} = \begin{bmatrix} 0 & 0.091 \end{bmatrix}. \end{aligned} \tag{31}$$

This example is a modification of the example given in [14]. Note that in this example, we are considering the special case in which the original uncertain system contains no uncertainty and so we are looking at a H^∞ strong stabilization problem; e.g., [5]–[7]. The standard H^∞ central controller (e.g., see [13]) for this system (corresponding to $\gamma = 1$) is unstable and has eigenvalues $s = -804.86,\ 28.84,\ -98.88,\ -118.54,\ -113.14,\ -0.24,\ -0.86$. Also, the corresponding state feedback gain matrix is

$$K = \begin{bmatrix} -0.10 & -0.81 & 0.03 & 0.13 & 0.00 & 0.08 & -0.00 \\ 0.08 & 0.35 & 0.04 & 0.47 & 0.06 & 0.28 & 0.01 \\ -0.03 & 0.10 & -0.10 & -0.85 & -0.03 & 0.09 & -0.00 \\ 0.07 & 0.30 & 0.05 & 0.53 & 0.06 & 0.28 & 0.01 \\ -0.00 & 0.08 & 0.02 & 0.11 & -0.09 & -0.83 & -0.00 \\ 0.07 & 0.31 & 0.05 & 0.47 & 0.07 & 0.32 & 0.01 \\ -0.01 & -0.05 & -0.00 & -0.08 & -0.01 & -0.05 & -0.88 \end{bmatrix}.$$

We now apply the algorithm outlined in Section IV to this system. For $\tilde{\tau}_1 = 1.0$, we find that the conditions of Theorem 1 are satisfied and we construct the corresponding controller of form (4) where

$$\begin{aligned} A_c &= \begin{bmatrix} 0.00 & 1.49 & -0.00 & -0.02 & -0.00 & -0.00 & 0.00 \\ -0.64 & -2.34 & -0.00 & -0.42 & -0.01 & -0.31 & -0.02 \\ -0.00 & 0.02 & 0.00 & 1.49 & -0.00 & 0.02 & 0.00 \\ -0.02 & -0.39 & -0.47 & -2.18 & -0.01 & -0.34 & -0.02 \\ -0.00 & 0.00 & -0.00 & -0.01 & 0.00 & 1.49 & 0.00 \\ -0.01 & -0.31 & 0.01 & -0.36 & -0.63 & -2.20 & -0.02 \\ -0.12 & -0.69 & -0.06 & -1.01 & -0.10 & -0.63 & -115.15 \end{bmatrix}, \\ B_c &= \begin{bmatrix} -0.49 & 0.02 & 0.00 & -0.00 \\ 2.06 & 0.34 & 0.27 & 0.03 \\ -0.02 & -0.49 & -0.02 & -0.00 \\ 0.34 & 1.91 & 0.29 & 0.03 \\ -0.00 & 0.01 & -0.49 & -0.00 \\ 0.27 & 0.29 & 1.94 & 0.03 \\ 0.03 & 0.03 & 0.03 & 105.33 \end{bmatrix}, \\ C_c &= \begin{bmatrix} -0.10 & -0.81 & 0.03 & 0.13 & 0.00 & 0.08 & -0.00 \\ 0.08 & 0.35 & 0.04 & 0.47 & 0.06 & 0.28 & 0.01 \\ -0.03 & 0.10 & -0.10 & -0.85 & -0.03 & 0.09 & -0.00 \\ 0.07 & 0.30 & 0.05 & 0.53 & 0.06 & 0.28 & 0.01 \\ -0.00 & 0.08 & 0.02 & 0.11 & -0.09 & -0.83 & -0.00 \\ 0.07 & 0.31 & 0.05 & 0.47 & 0.07 & 0.32 & 0.01 \\ -0.01 & -0.05 & -0.00 & -0.08 & -0.01 & -0.05 & -0.88 \end{bmatrix}. \end{aligned} \tag{32}$$

This system is stable and has poles at $s = -115.15,\ -2.63,\ -0.32,\ -0.72,\ -1.16,\ -0.92,\ -0.97$. Furthermore, when the controller (32) is applied to the system (31), the resulting closed loop system has H^∞-norm 0.12. From this we can see that the stable controller (32) does indeed solve the H^∞ strong stabilization problem under consideration.

VI. Conclusions

In this paper we have presented a numerical algorithm for the problem of absolute stabilization with a specified level of disturbance attenuation via the use of a stable output feedback controller presented in [8]. The key idea of our algorithm is to reformulate the parameterized algebraic Riccati equation in terms of rank constrained LMIs which would be solved to construct the parameters on which the main result depends.

APPENDIX

Lemma 3: For any $\tau_1 > 0, \ldots, \tau_k > 0$,

(i) $E_1 > 0$.

(ii) The matrix $\begin{bmatrix} A-\alpha I & B_2 \\ \hat{C}_1 & \hat{D}_{12} \end{bmatrix}$ has full column rank for all $\alpha \in \mathbf{C}$ such that $Re(\alpha) \geq 0$.

Proof: Let $\Gamma = \mathrm{diag}(I_q, \sqrt{\tau_1} I_{h_1}, \cdots, \sqrt{\tau_k} I_{h_k})$.

(i) $E_1 = \underline{D}_{12}' \Gamma^2 \underline{D}_{12} \geq \min(1, \tau_1, \cdots, \tau_k) * \underline{D}_{12}' \underline{D}_{12} > 0$ by Assumption 1-(i).

(ii) It is obvious from Assumption 1-(iii) and the fact that $\hat{C}_1 = \Gamma \underline{C}_1, \hat{D}_{12} = \Gamma \underline{D}_{12}$. ■

We define the following notation, which will be used in the next lemma.

$$\begin{aligned} \underline{\tilde{B}}_1 &= [\underline{B}_1, D_{k+1}], & \underline{\tilde{D}}_{21} &= [\underline{D}_{21}, 0_{l\times m}], \\ \underline{\tilde{C}}_1 &= [\tilde{C}_1', \tilde{K}_1', \cdots, \tilde{K}_{k+1}']', & \underline{\tilde{D}}_{12} &= [\tilde{D}_{12}', \tilde{G}_1', \cdots, \tilde{G}_{k+1}']'. \end{aligned}$$

Lemma 4: Consider the uncertain system (1), (3). Let the constants $\tau_1 > 0, \ldots, \tau_k > 0$ be given as in Lemma 1 and consider the matrices defined in (11). Then the following conditions are satisfied.

(i) The matrix $\begin{bmatrix} \tilde{A}-\alpha I & \tilde{B}_2 \\ \underline{\tilde{C}}_1 & \underline{\tilde{D}}_{12} \end{bmatrix}$ has full column rank for all $\alpha \in \mathbf{C}$ such that $Re(\alpha) \geq 0$.

(ii) The matrix $\begin{bmatrix} \tilde{A}-\alpha I & \underline{\tilde{B}}_1 \\ C_2 & \underline{\tilde{D}}_{21} \end{bmatrix}$ has full row rank for all $\alpha \in \mathbf{C}$ such that $Re(\alpha) \geq 0$.

Proof: (i) Suppose $\begin{bmatrix} \tilde{A}-\alpha I & \tilde{B}_2 \\ \underline{\tilde{C}}_1 & \underline{\tilde{D}}_{12} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$ for some $\alpha \in \mathbf{C}$ such that $Re(\alpha) \geq 0$, then

$$\begin{cases} \tilde{K}_{k+1} x + \tilde{G}_{k+1} y = 0 \quad \Rightarrow \quad y = Kx; \\ (\tilde{A} - \alpha I)x + \tilde{B}_2 y = 0 \quad \Rightarrow \quad (A + B_2 K - \alpha I)x = 0. \end{cases}$$

Therefore $x = 0, y = 0$ since $A + B_2 K$ is Hurwitz from Lemma 1.

(ii) Suppose $\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \tilde{A}-\alpha I & \underline{\tilde{B}}_1 \\ C_2 & \underline{\tilde{D}}_{21} \end{bmatrix} = 0$ for some $\alpha \in \mathbf{C}$ such that $Re(\alpha) \geq 0$, then

$$\begin{cases} x'(\tilde{A} - \alpha I) + y' C_2 = 0; \\ x' \underline{B}_1 + y' \underline{D}_{21} = 0; \\ x' D_{k+1} = 0 \quad \Rightarrow \quad x' B_2 = 0. \end{cases}$$

Therefore $\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} A-\alpha I & \underline{B}_1 \\ C_2 & \underline{D}_{21} \end{bmatrix} = 0$. Then $x = 0, y = 0$ from Assumption 1-(iii). ■

Lemma 5: For any $\tilde{\tau}_1 > 0, \ldots, \tilde{\tau}_{k+1} > 0$,

(i) $\check{E}_1 > 0, \check{E}_2 > 0$.

(ii) The matrix $\begin{bmatrix} \check{A}-\alpha I & \check{B}_2 \\ \check{C}_1 & \check{D}_{12} \end{bmatrix}$ has full column rank for all $\alpha \in \mathbf{C}$ such that $Re(\alpha) \geq 0$.

(iii) The matrix $\begin{bmatrix} \check{A}-\alpha I & \check{B}_1 \\ \check{C}_2 & \check{D}_{21} \end{bmatrix}$ has full row rank for all $\alpha \in \mathbf{C}$ such that $Re(\alpha) \geq 0$.

Proof: (i) $\bar{D}_{12}$ has full column rank since $\tilde{G}_{k+1} = -\frac{1}{2} I_{m\times m}$, therefore, $\check{E}_1 = \bar{D}_{12}'(I - \bar{D}_{11}\bar{D}_{11}')^{-1}\bar{D}_{12} > 0$. Similarly, $\bar{D}_{21}$ has full row rank since $\bar{D}_{21}\bar{D}_{21}' = \gamma^{-2} \underline{D}_{21} \underline{D}_{21}' \geq 0$ by Assumption 1-(i), therefore, $\check{E}_2 = \bar{D}_{21}(I - \bar{D}_{11}'\bar{D}_{11})^{-1}\bar{D}_{21}' > 0$.

(ii)

$$\begin{aligned} &\begin{bmatrix} \check{A}-\alpha I & \check{B}_2 \\ \check{C}_1 & \check{D}_{12} \end{bmatrix} \\ &= \begin{bmatrix} I & \bar{B}_1 \bar{D}_{11}'(I - \bar{D}_{11}\bar{D}_{11}')^{-1} \\ 0 & (I - \bar{D}_{11}\bar{D}_{11}')^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \tilde{A}-\alpha I & \underline{\tilde{B}}_2 \\ \bar{C}_1 & \bar{D}_{12} \end{bmatrix} \\ &= \begin{bmatrix} I & \bar{B}_1 \bar{D}_{11}'(I - \bar{D}_{11}\bar{D}_{11}')^{-1} \\ 0 & (I - \bar{D}_{11}\bar{D}_{11}')^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \tilde{\Gamma}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \tilde{A}-\alpha I & \tilde{B}_2 \\ \underline{\tilde{C}}_1 & \underline{\tilde{D}}_{12} \end{bmatrix}, \end{aligned}$$

where $\tilde{\Gamma} = \mathrm{diag}(I_q, \tilde{\tau}_1 I_{h_1}, \cdots, \tilde{\tau}_k I_{h_k}, \tilde{\tau}_{k+1} I_m)$. From Lemma 4, $\begin{bmatrix} \check{A}-\alpha I & \check{B}_2 \\ \check{C}_1 & \check{D}_{12} \end{bmatrix}$ has full column rank.

(iii)

$$\begin{aligned} &\begin{bmatrix} \check{A}-\alpha I & \check{B}_1 \\ \check{C}_2 & \check{D}_{21} \end{bmatrix} \\ &= \begin{bmatrix} \tilde{A}-\alpha I & \bar{B}_1 \\ C_2 & \bar{D}_{21} \end{bmatrix} \begin{bmatrix} I & 0 \\ \bar{D}_{11}'(I - \bar{D}_{11}\bar{D}_{11}')^{-1}\bar{C}_1 & (I - \bar{D}_{11}'\bar{D}_{11})^{-\frac{1}{2}} \end{bmatrix} \\ &= \begin{bmatrix} \tilde{A}-\alpha I & \underline{\tilde{B}}_1 \\ C_2 & \underline{\tilde{D}}_{21} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \tilde{\Gamma}_\gamma^{-\frac{1}{2}} \end{bmatrix} \\ &\quad \times \begin{bmatrix} I & 0 \\ \bar{D}_{11}'(I - \bar{D}_{11}\bar{D}_{11}')^{-1}\bar{C}_1 & (I - \bar{D}_{11}'\bar{D}_{11})^{-\frac{1}{2}} \end{bmatrix}, \end{aligned}$$

where $\tilde{\Gamma}_\gamma = \mathrm{diag}(\gamma^2 I_p, \tilde{\tau}_1 I_{r_1}, \cdots, \tilde{\tau}_k I_{r_k}, \tilde{\tau}_{k+1} I_m)$. From Lemma 4, $\begin{bmatrix} \check{A}-\alpha I & \check{B}_1 \\ \check{C}_2 & \check{D}_{21} \end{bmatrix}$ has full column rank. ■

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