A non-singular version of the Kingman ergodic theorem.

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Abstract. Kingman's subadditive ergodic theorem is traditionally proved in the setting of a measure-preserving invertible transformation T of a measure space (X, μ). We show how to extend the theorem to the setting of a non-singular transformation, that is under the assumption that μ and $\mu \circ T$ have the same null sets. Using this, we are able to produce non-singular versions of the Furstenburg-Kestern Theorem and the Oseledeč ergodic theorem for products of random matrices.

1. *Introduction*

The study of ergodic theorems is an important bridge between functional analysis and probability theory. Originally proved by von Neumann and Birkhoff [**4**] in 1931, the Birkhoff ergodic theorem has become the fundamental theorem in the study of measure-preserving transformations of a measure space. The subadditive ergodic theorem,obtained by Kingman [**8**] in 1968 is an important extension of this fundamental result, which has found many applications. One important application is the Furstenberg-Kesten theorem [**6**], on the structure of multiplicative cocycles from a measure-preserving transformation T of a measure space (X, μ), with values in $GL(d, \mathbb{R})$. The Furstenberg-Kestern theorem has been extended and refined by the well-known Oseledeč ergodic theorem on the products of randomly chosen matrices [**9**].

A statement of Kingman's theorem is as follows:

Theorem 1 (The sub-additive ergodic theorem) *Let $(X, \mathcal{B}, \mu)$ be a probability space, and $T : X \to X$ an invertible and measure preserving transformation. Let $f_n \in L^1$ be a sequence of function satisfying the* ***subadditivity condition****: $f_{m+n}(x) \leq f_n(x) + f_m(T^n x)$ for almost all $x \in X$, then*

$$\lim_{n\to\infty} \frac{f_n(x)}{n} = f(x) < \infty$$

exists μ a.e. Furthermore, $f(x)$ is a T- invariant measurable function over $(X, \mathcal{B}, \mu)$.

There have been several proofs of this theorem since Kingman's original version. See [**1**] for survey of these. Most of these have made the assumption that the measure μ is invariant under the transformation T. However, [**12**] Theorem 3.4 is a version of the subadditive ergodic theorem under the assumption that T is a Markovian transformation of $(X, \mathcal{B}, \mu)$.

Note that the Theorem generalises the following result in elementary analysis, which we recover in the case where the f_n are all constant functions:

LEMMA 1. *If (f_n) is a subadditive sequence then*

$$\lim_n \frac{f_n}{n} = \inf_n \frac{f_n}{n} < \infty$$

The aim of this paper is to extend the Kingman theorem, the Furstenberg-Kestern Theorem and the Oceledeč theorem to the setting of non-singular transformations. The key idea is to define subaddditive sequences by

$$f_{m+n}(x) \leq f_n(x) + \omega_n(x) f_m(T^n x)$$

where $\omega_n(x) = \frac{d\mu \circ T^n}{d\mu}$ is the Radon-Nikodým derivative.

Our version of the Kingman theorem (Theorem 4) then concludes that

$$\lim_{n\to\infty} \frac{f_n(x)}{\sum_{i=0}^{n-1} \omega_i(x)} = f_*(x) < \infty$$

exists μ a.e.

After some preliminary remarks and definitions in section 2, we state and prove our version of the Kingman theorem in section 3. In section 4, we state and prove non-singular versions of the Furstenberg-Kestern theorem (Theorem 5) and the Oseledeč ergodic theorem (Theorem 6).

We expect these results to lead to new applications of these theorems in the non-singular setting. One key application of the Oseledeč theorem in the measure-preserving case, is the calculation of Lyapunov exponents for Random Dynamical Systems, see [**2**]. In future work, we will extend this construction to non-singular random dynamical systems.

We have been influenced by Raghunathan's elegant proof of the Oseledeč theorem [**10**]. We would like to thank Anthony Quas for drawing our attention to this paper, and for useful discussions.

2. *Preliminaries*

The dynamical system $(X, \mathcal{B}, \mu, T)$ is said to be **non-singular** if the map $T : X \to X$ is a non-singular transformation of (X, μ), that is for any $N \in \mathcal{B}$, $\mu(T^{-1}N) = 0$ if and only if $\mu(N) = 0$. Recall that the system is measure preserving if $\mu(A) = \mu(T^{-1}A)$ for all $A \in \mathcal{B}$. By the Poincaré recurrence lemma, measure preserving transformations are conservative.

The structure of non-singular transformations is given by the Hopf Decomposition Theorem, a proof of which can be found in [**1**].

THEOREM 2. *[Hopf Decomposition] Let T be a non-singular transformation. There exist disjoint invariant sets $C, D \in \mathcal{B}$ such that $X = C \sqcup D$, T restricted to C is conservative, and $D = \sqcup_{n=-\infty}^{\infty} T^n W$, where W is a wandering set. If $f \in L^1(X,\mu)$, $f > 0$, then $C = \{x : \sum_{i=1}^{n-1} f(T^i x)\omega_i(x) = \infty \text{ a.e.}\}$ and $D = \{x : \sum_{i=1}^{n-1} f(T^i x)\omega_i(x) < \infty \text{ a.e.}\}$*

The set C is called the **conservative part** of T. We suppose further that the non-singular transformation is invertible, so that T and its inverse T^{-1} are measurable. If T is non-singular then we have both $\mu \circ T^{-1} \sim \mu$ and $\mu \circ T \sim \mu$.

We will denote the Radon Nikodým-derivative $\frac{d(\mu \circ T^i)}{d\mu}$ by ω_i. Note that the Radon-Nikodým derivatives must satisfy the cocycle identity:

$$\omega_{i+j}(x) = \omega_i(x)\omega_j(T^i x)$$

for a.e. x and for every $i, j \in \mathbb{Z}$. Clearly, T is measure preserving if and only if $\omega_i(x) = 1$ for a.e. x for all i.

It follows that for every $f \in L^1(X,\mu)$

$$\int_X f(x)d\mu(x) = \int_X f(Tx)\omega_1(x)d\mu(x) = \int_X f(T^n x)\omega_n(x)d\mu(x)$$

If $f_n = \sum_{i=0}^{n-1} f(T^i x)\omega_i(x), n \geq 1$, where $\omega_0(x) = 1$. It is easy to show that $f_{m+n}(x) = f_n(x) + \omega_n(x) f_m(T^n x)$. The Hurewicz ergodic theorem [**7**] is a generalization of the Birkhoff Ergodic Theorem to the setting of non-singular conservative transformations.

THEOREM 3. *[Hurewicz ergodic theorem] Let $(X, \mathcal{B}, \mu)$ be a probability space, and $T : X \to X$ an invertible, non-singular and conservative transformation. If $f \in L^1(\mu)$, then*

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} f(T^i x)\omega_i(x)}{\sum_{i=0}^{n-1} \omega_i(x)} = f_*(x)$$

exists μ a.e. Furthermore, $f_*(x)$ is T- invariant and

$$\int_X f(x)d\mu(x) = \int_X f_*(x)d\mu(x)$$

Note that if T is measure-preserving, the left hand side becomes $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)$, and so we recover the Birkhoff Theorem.

Our aim is to use the Hurewicz theorem to formulate and prove a version of Theorem 4 which holds for non-singular conservative systems. We shall do that in the next section.

3. *Non-singular Kingman Theorem*

Thus, let T be a conservative non-singular transformation of a measure space $(X, \mathcal{B}, \mu)$, and denote by ω_i the Radon-Nikodým derivative $\frac{d\mu \circ T^i}{d\mu}$.

DEFINITION 1. *We say that $\{f_n\}$ in $L^1(X,\mu)$ is a **subadditive sequence** for T if for all integers m and n*

$$f_{m+n}(x) \leq f_n(x) + \omega_n(x) f_m(T^n x)$$

It is easy to see that if f is integrable, then

$$f_n(x) = \sum_{i=0}^{n-1} f(T^i x)\omega_i(x).$$

is subadditive.

Similarly, we say that $\{f_n\}$ in $L^1(X,\mu)$ is **superadditive** for T if for all integers m and n

$$f_{m+n}(x) \geq f_n(x) + \omega_n(x) f_m(T^n x)$$

Observe that f_n is a superadditive sequence if and only if $-f_n$ is subadditive sequence.

The goal of this section is to give a proof of the following:

THEOREM 4. *[Non-singular Kingman ergodic theorem] Let $(X,\mathcal{B},\mu)$ be a probability space, and $T : X \to X$ an invertible, non-singular and conservative transformation. Let $f_n \in L^1$ be a sequence of functions satisfying the subadditivity relation $f_{m+n}(x) \leq f_n(x) + \omega_n(x) f_m(T^n x)$ for almost all $x \in X$, then*

$$\lim_{n\to\infty} \frac{f_n(x)}{\sum_{i=0}^{n-1}\omega_i(x)} = f_*(x) < \infty$$

exists μ a.e. Furthermore, $f_*(x)$ is T-invariant and

$$\int_X f(x)d\mu(x) = \int_X f_*(x)d\mu(x)$$

The essential idea of the proof is to show that

$$\int_X \limsup_{n\to\infty} \frac{f_n}{\sum_{i=0}^{n-1}\omega_i(x)} d\mu \leq \int L d\mu \leq \int_X \liminf_{n\to\infty} \frac{f_n}{\sum_{i=0}^{n-1}\omega_i(x)} d\mu$$

where L is

$$\lim_{n\to\infty} \frac{f_n}{\sum_{i=0}^{n-1}\omega_i(x)}.$$

Let $\overline{h}(x) = \limsup_{n\to\infty} \frac{f_n(x)}{\sum_{i=0}^{n-1}\omega_i(x)}$ and $\underline{h} = \liminf_{n\to\infty} \frac{f_n(x)}{\sum_{i=0}^{n-1}\omega_i(x)}$.

LEMMA 2. *Let $(X,\mathcal{B},\mu)$ be a probability space, and $T : X \to X$ an invertible, non-singular and conservative transformation.Let f_n be a superadditive sequence. Then $\underline{h} = \underline{h} \circ T$ almost everywhere.*

PROOF 1. *We have*

$$\frac{f_{n+1}}{\sum_{i=0}^{n}\omega_i(x)} \geq \frac{f_1 + f_n(Tx)\omega_1}{\omega_0 + \sum_{i=0}^{n-1}\omega_i(Tx)\omega_1}$$

taking the $\liminf$ *of each side,*

$$\underline{h} \geq \underline{h} \circ T.$$

We set $\alpha(x) = \frac{\underline{h}(x)}{1+\underline{h}(x)} \in [0,1]$ and the set $A = \{x : \underline{h}(x) > \underline{h} \circ T(x)\} = \{\alpha(x) > \alpha \circ T(x)\}$. Let $\phi(x) = \alpha(x) - \alpha \circ T(x)$, by Theorem 2, $\sum_{i=1}^{n-1}\phi(T^i x)\omega_i = \infty$ almost everywhere on A. However, $\sum_{i=1}^{n-1}\phi(T^i x)\omega_i = \alpha - \alpha \circ T^{n-1}\omega_n \in [-1,1]$ everywhere, so we must have $\mu(A) = 0$.

For ease of notation, we put $\Omega_j = \sum_{i=0}^{j-1} \omega_i$.

LEMMA 3. $\int L d\mu = \int_X \liminf_{n\to\infty} \frac{f_n}{\sum_{i=0}^{n-1}\omega_i(x)} d\mu$.

PROOF 2. *Fix $\varepsilon > 0$ and define, for each $k \in \mathbb{N}$,*

$$B_k = \{x \in X : f_j(x) \le \Omega_j(\underline{h}(x) + \varepsilon), j \in \{1, ..., k\}\},$$

where $\underline{h} > -\infty$. It is clear that $\cup_k B_k = X$. Define

$$F_k = \begin{cases} \underline{h}(x) + \varepsilon & x \in B_k \\ f_1(x) & x \in B_k^c \end{cases}$$

by the definition of B_k, $f_1 > \underline{h}$ if $x \in B_k^c$. We define two sequences m_i and n_i such that:

$$0 = m_0 \le n_1 \le m_1 \le n_2 \le .....$$

where n_j is the smallest integer greater than or equal to m_{j-1} such that $T^{n_j}x \in B_k$. By the definition of B_k, we may choose $1 \le m_j - n_j$ such that

$$\frac{f_{m_j - n_j}(T^{n_j}x)}{\Omega_{m_j-n_j}(T^{n_j}x)} \le \underline{h} + \varepsilon.$$

Now, given $n \ge k$, let $l \ge 0$ be the largest integer such that $m_l \le n$. By subadditivity,

$$\frac{f_{n_j - m_{j-1}}(T^{m_{j-1}}x)}{\Omega_{n_j-m_{j-1}}(T^{m_{j-1}}x)} \le f_1(T^i x)\omega_i.$$

Thus,

$$\frac{f_n}{\Omega_n} \le \sum f_1(T^i x)\omega_i \Omega_{n_j - m_{j-1}}(T^{m_{j-1}}x) + \sum_{j=1}^{l} \frac{f_{m_j-n_j}(T^{n_j}x)}{\Omega_{m_j-n_j}(T^{n_j}x)},$$

where the first sum $f_1(T^i x) = F_k(T^i x)$ for every $i \in \cup_{j=1}^{l}[m_{j-1}, n_j) \cup [m_l, \min\{n_{l+1}, n\})$. On the other hand, $\underline{h}$ is constant on the orbits, and $F_k \ge \underline{h} + \varepsilon$, so

$$\frac{f_{m_j-n_j}(T^{n_j}x)}{\Omega_{m_j-n_j}(T^{n_j}x)} \le \underline{h} + \varepsilon \le F_k(T^{n_j}x).$$

It follows that

$$\frac{f_n}{\Omega_n} \le \sum_{i=1}^{\min\{n_{l+1},n\}-1} F_k(T^i x)\Omega_{m_j-n_j}(T^{n_j}x) + \sum_{i=n_{l+1}}^{n-1} f_1(T^i x)\omega_i \Omega_{n_j-m_{j-1}}(T^{m_{j-1}}x).$$

Integrating each side, we obtain

$$\int \frac{f_n}{\Omega_n} d\mu \le (n-k)\int F_k d\mu + k\int (F_k \vee f_1) d\mu.$$

Letting $n \to \infty$, we obtain $\int L d\mu \le \int F_k d\mu$. Then letting $k \to \infty$, we see that $\int L d\mu \le \int \underline{h} d\mu + \varepsilon$. We see that there exists $C \in \mathbb{R}^+$, such that $\frac{f_n}{\Omega_n} \ge -C$. By Fatou's lemma, $\underline{h}$ is integrable, with $\int \underline{h} \le L$. Hence the lemma holds.

LEMMA 4. *For any fixed k,*

$$\limsup_n \frac{f_{kn}}{\Omega_{kn}} = \limsup_n \frac{f_n}{\Omega_n}.$$

PROOF 3. *Since $\frac{f_{kn}}{\Omega_{kn}}$ is a subsequence of $\frac{f_n}{\Omega_n}$, the statement is true with $=$ replaced by $\leq$. Fix k, and write $n = kq + r$ with $r \in \{1, ..., k\}$. By subadditivity,*

$$f_n \leq f_{kq} + f_r(T^{kq}x)\omega_{kq}(x) \leq f_{kq} + \psi(T^{kq}x)\omega_{kq}(x),$$

where $\psi = \max\{f_1^+, ...f_k^+\}$. Since k is fixed, $q \to \infty$ as $n \to \infty$. Taking the lim sup *as $n \to \infty$, we see that*

$$\limsup_n \frac{f_n}{n} \leq \limsup_n \frac{f_{kq}}{n} + \limsup_n \frac{\psi(T^{kq}x)\omega_{kq}(x)}{n} = \frac{1}{k}\limsup_q \frac{f_{kq}}{q},$$

as $n \to \infty$ with $\psi \in L^1$, and $\psi(T^{kq}x)\omega_{kq}(x)$ converges to zero. Dividing both sides by $\frac{1}{n}\Omega_n$, we obtain

$$\limsup_n \frac{f_n}{\Omega_n} \leq \limsup_n \frac{f_{kq}}{\Omega_n},$$

as stated in the lemma.

LEMMA 5. $\int_X \limsup_{n\to\infty} \frac{f_n}{\sum_{i=0}^{n-1}\omega_i(x)} d\mu \leq \int L d\mu$

PROOF 4. *Let $\theta_n = -\sum_{j=0}^{n-1} f_k(T^{jk}x)\omega_{jk}$. By Theorem 3,*

$$\int \frac{\theta_n}{\Omega_n} d\mu = -\int f_k d\mu.$$

Let $\underline{\theta} = \liminf \frac{\theta_n}{\Omega_n}$. Since the sequence f_n is subadditive, $\theta_n \leq -f_{kn}$ for every n. By Lemma 3,

$$\underline{\theta} = \liminf_n \frac{\theta_n}{\Omega_n} \leq -\limsup_n \frac{f_{kn}}{\Omega_{kn}} = -\limsup_n \frac{f_n}{\Omega_n} = -\overline{h}.$$

Thus $\int \overline{h} \leq -\int \underline{\theta} \leq \int f_k$. Hence $\int \overline{h} \leq L$.

Note that in the measure-preserving case, we have $\Omega_n(x) = n$, which gives us back the standard Kingman Theorem, Theorem 4.

Theorem 3.4 of [**12**] states that if $\{f_n\}$ is subadditive and $\{g_n\}$ is superadditive, the limit

$$\lim_{n\to\infty} \frac{f_n}{g_n} = \frac{\lim_{n\to\infty}(1/n)\mathbb{E}_{h\mu}[f_n/h|\mathcal{I}]}{\lim_{n\to\infty}(1/n)\mathbb{E}_{h\mu}[g_n/h|\mathcal{I}]},$$

for any positive μ-integrable function h (where $\mathcal{I}$ is the invariant σ-algebra). Our theorem shows how to replace the quantity $1/n$ with $\frac{1}{\sum_{i=0}^{n-1}\omega_i(x)}$, which is the key to proving the multiplicative ergodic theorem below.

In the case where the measure μ is non-singular and has critical dimension $\alpha \in [0, 1]$, see [**5**], we have $\frac{\Omega_n(x)}{n^\alpha}$ is non-zero a.e., and the conclusion of Theorem 5 is equivalent to $\lim_{n\to\infty} \frac{f_n(x)}{n^\alpha} = f(x) < \infty$.

4. *The multiplicative ergodic theorem*
We now introduce the notion of cocycles with values in $GL(d)$ of a non-singular transformation T of $(X, \mathcal{B}, \mu)$, see [**11**]. A **cocycle** with respect to the action of T is a function $\Phi : \mathbb{N} \times X \to GL(d)$ satisfying $\Phi(n+m, x) = \Phi(n, x)\Phi(m, T^{n-1}x)$.

Cocycles can be generated by choosing a (random) $d \times d$-matrix, $A(x)$ for each $x \in X$, and defining

$$\Phi(n, x) = A(x) \times A(Tx) \times A(T^2x)....... \times A(T^{n-1}x).$$

It is easy to see that this formula defines a cocycle. We will say that $A(x)$ is the **generator** of Φ.

The operator norm of a square matrix A of dimension d is defined as follows:

$$\|A\| = \sup\left\{\frac{\|Av\|}{\|v\|} : v \in \mathbb{R}^d \backslash \{0\}\right\}.$$

It follows directly from the definition that the norm of the product of two matrices is less than or equal to the product of the norms of those matrices. Thus

$$\|\Phi(n, x)\| \leq \|A(x)\|\|A(Tx)\|......\|A(T^{n-1}x)\|.$$

If T is measure preserving, the Furstenberg-Kesten theorem is an application of the Kingman subadditive ergodic theorem, and the subadditive sequence is defined as follows:

$$\log \|\Phi(n+m, x)\| \leq \log \|\Phi(n, x)\| + \log \|\Phi(m, T^m x)\|.$$

In the non-singular case, we define the non-singular subadditive sequence by:

$$\log \|\Phi(n, x)\| \leq \sum_{i=0}^{n-1} \omega_i \log \|A(T^i x)\|.$$

We are going to define singular values and exterior powers before we introduce the theorem.

Definition 2 (Exterior power) *Let V be a vector space with dimension r, for $1 < k < r$, the k-fold Exterior power of V is $\wedge^k V$, which is the vector space of alternating k-linear forms on the dual space. The k-fold Exterior power of a matrix A is $\wedge^k A$, which has following property:*
(i) $(AB)^{\wedge k} = A^{\wedge k} B^{\wedge k}$
(ii) $(A^{\wedge k})^{-1} = (A^{-1})^{\wedge k}$
(iii) $(cA)^{\wedge k} = c^{\wedge k} A^{\wedge k}$, *where* $c \in \mathbb{R}$.
The singular valued decomposition of Exterior Powers is $\wedge^k A = \wedge^k V \wedge^k D \wedge^k U$, where $\wedge^k D$ is a diagonal matrix with entries$\{\delta_{i_1}\delta_{i_2}....\delta_{i_k}, 1 \leq i_1 \leq ... \leq i_k \leq r\}$. The largest singular value is $\delta_{r-k+1}....\delta_r$, the smallest value is $\delta_1....\delta_k$. The norm of $\wedge^k A$ is the largest singular value.

Theorem 5. *(Non-singular Furstenberg-Kestern Theorem)*

Let Φ be a linear cocycle with one side in discrete time over the non-singular dynamical system $(\Omega, \mathcal{F}, \mu, T)$. Assume the generator $A : X \to Gl(d, \mathbb{R})$ of Φ satisfies

$$\log^+ \|A\| \in L^1$$

Then the following statements hold:
(1) *For each* $k = 1, ..., d$ *the sequence*

$$f_n^k(x) = \log \| \wedge^k \Phi(n,x)\|, \; n \in \mathbb{N}$$

is subadditive and $f_1^{k+} \in L^1(X, \mathcal{F}, \mu)$*. That is*

$$f_{n+m}^k(x) \leq f_m^k(x) + f_n^k(T^m x)\omega_m$$

(2) *There is an invariant set* $\bar{\Omega}$ *of full measure and measurable functions* $\gamma^k : X \to \mathbb{R}$ *with* $\gamma^{k+} \in L^1(X, \mathcal{F}, \mu)$

$$\gamma^k(x) = \lim_{n\to\infty} \frac{\log \| \wedge^k \Phi(n,x)\|}{\sum_{i=0}^{n-1} \omega_i(x)}$$

and

$$\gamma^k(Tx) = \gamma^k(x), \gamma^{k+l}(x) \leq \gamma^k(x) + \gamma^l(x).$$

Let Λ_k *be the function defined by* $\Lambda_k = \gamma^{k+1} - \gamma^k$*, and let* δ_k *be the corresponding singular value of* $\Phi(n,x)$*. then*

$$\Lambda_k = \lim_{n\to\infty} \frac{\log \delta_k(\Phi(n,x))}{\sum_{i=0}^{n-1} \omega_i(x)}$$

PROOF 5. *Note that* $A(x) = \Phi(1,x)$*. For all* k

$$f_n^k(x) = \log \| \wedge^k \Phi(n,x)\|$$

is a subadditive sequence.

$$f_{n+1}^k(x) = \log \| \wedge^k \Phi(n+1,x)\| \leq f_n^k(Tx)\omega_1(x) + \log \| \wedge^k A(x)\|.$$

Hence subadditivity of $f_n^k(x)$ *follows. By theorem 4, we have*

$$\gamma^k(x) = \lim_{n\to\infty} \frac{\log \| \wedge^k \Phi(n,x)\|}{\sum_{i=0}^{n-1} \omega_i(x)}.$$

Since $\| \wedge^{k+l} \Phi(n,x)\| \leq \| \wedge^k \Phi(n,x)\| \| \wedge^l \Phi(n,x)\|$*,* γ^k *is a subadditive sequence. For* $k = 1, ..., d$*:*

$$\log \| \wedge^k \Phi(n,x)\| = \sum_{i=1}^{k} \log \delta_i(\Phi(n,x))$$

where δ_i *is the corresponding singular value of* $\Phi(n,x)$

Now we consider the behaviour of $\|\Phi(n,x)v\|$ for $v \in \mathbb{R}^d$ as $n \to \infty$. If $A \in M_d(\mathbb{R})$ with transpose A^*, both A^*A and AA^* are symmetric and positive semidefinite. Any positive semidefinite and symmetric matrix S may be written as the form as

$$S = C^{-1}DC$$

where D is diagonal with non-negative entries in non-deceasing order and C is orthogonal.

The polar decomposition of a matrix A is

$$A = C(AA^*)^{\frac{1}{2}}C' = C''(A^*A)^{\frac{1}{2}}$$

where C', C'' are orthogonal matrices. Applying the polar decomposition to $\Phi(n,x)$ in the theorem, we obtain:

$$\Phi(n,x) \approx C''_n A^n(x)$$

for some orthogonal matrix C''_n. Since orthogonal matrices are isometries, we have $\|C''_n v\| = \|v\|$. Thus

$$\|\Phi(n,x)v\| = \|A^n(x)v\|.$$

Returning to the symmetric matrix $\Phi(n,x)^*\Phi(n,x)$, we know $\Phi(n,x)^*\Phi(n,x) = C_n^* D_n C_n$, and $\Phi(n,x)) = L_n(D_n)^{\frac{1}{2}}C_n$, and hence

$$\lim_{n\to\infty}(\Phi(n,x)^*\Phi(n,x))^{\frac{1}{2n}} = \lim_{n\to\infty} C_n^*(D_n)^{\frac{1}{2n}}C_n.$$

If the limit exists, then there are an orthogonal matrix $C = \lim C_n$ and a diagonal matrix $D = \lim D_n^{\frac{1}{2n}}$. By theorem 5, we see that $\lim_{n\to\infty} \frac{\log \|\wedge^k \Phi(n,x)\|}{\sum_{i=0}^{n-1}\omega_i(x)}$ converges for all k to $-\infty$ or a finite limit. Hence $\lim_{n\to\infty} \frac{\log \delta_i(\Phi(n,x))}{\sum_{i=0}^{n-1}\omega_i(x)}$ converges to a finite limit for every i. Now, we can assume that $D_n^{\frac{1}{2n}}$ converges as $n \to \infty$.

By the monotonicity of Λ^i,

$$\Lambda^r(x) \geq ...... \geq \Lambda^1(x)$$

There is a unique partition I, given by:

$$I = \{1 = i_2 < i_1 < ..... < i_p < i_{p+1} = r+1\}$$

such that $\Lambda^{i_q} = \Lambda^{i_{q+1}-1} < \Lambda^{i_{q+1}}$. This partition splits $\{1, 2, \cdots, r\}$ into finitely many intervals $[i_q, i_{q+1} - 1]$. If $\Lambda^i = \Lambda^j$, then they belong to same interval. Let $\Sigma(I,q)(x)$ be vector subspace of $\mathbb{R}^r$, it is a union of the zero vector 0 and the set of all eigenvectors corresponding to eigenvalues smaller than or equal to $\Lambda^{i_{q+1}-1}$. It is easily seen that $\Sigma(I,0)(x)$ is $\{0\}$ and $\Sigma(I,p)(x)$ is $\mathbb{R}^r$. We see that $C_n^{-1}e_i$ is an eigenvector of $(\Phi(n,x)^*\Phi(n,x))^{\frac{1}{2}}$ with eigenvalue δ_i^2. We know $D_n^{\frac{1}{2n}}$ converges, but the question is whether the vector space spanned by $C_n^{-1}e_i$ converges. We will formulate a one-sided multiplicative ergodic theorem which is based on Raghunathan's version [**10**].

Theorem 6. *(Non-singular Oseledeč theorem) Let (X,S,m) be a probability space. Suppose that T is a non-singular transformation and $u : Z \times X \to M(r,\mathbb{R})$ is a measurable cocycle over T such that* $\log^+ \|\Phi(1,\dot{})\| \in L^1(X,S,m)$. *We set*

$$B = \{(x,v) \in X \times \mathbb{R}^r : \frac{\log \|\Phi(n,x)v\|}{\sum_{i=0}^{n-1}\omega_i(x)} \text{ tends to a finite limit or } -\infty\}$$

and

$$X' = \{x \in X | (x,v) \in B \text{ forall} v \in \mathbb{R}^r\}.$$

Then there is a subset Y of X' which has full measure and a sequence of functions $\Lambda^1(x) \leq ...... \leq \Lambda^r(x)$ (taking values in $\mathbb{R} \cup -\infty$) such that:

(i) *Let* $I = \{1 = i_1 < i_2 < ..... < i_p < i_{p+1} = r+1\}$ *be n+1 tuples of integers. We define*
$Y(I) = \{x \in X' | \Lambda^i(x) = \Lambda^j(x)$ *for* $i_q \leq i, j < i_{q+1}$ *and* $\Lambda^{i_q}(x) < \Lambda^{i_{q+1}}(x)$ *for all* q *with* $1 < q < p\}$.
Then for all $x \in Y(I), 1 < q < p$

$$\Sigma(I,q)(x) = \{v \in \mathbb{R}^r | \lim_{n\to\infty} \frac{\log \|\Phi(n,x)v\|}{\sum_{i=0}^{n-1} \omega_i(x)} \leq \Lambda^{i_q}(x)\}$$

is a vector subspace of $\mathbb{R}^r$ *with dimension* $i^{q+1} - 1$.

(ii) *If* $\Sigma(I,0) = \{0\}$ *then*

$$\lim_{n\to\infty} \frac{\log \|\Phi(n,x)v\|}{\sum_{i=0}^{n-1} \omega_i(x)} = \Lambda^{i_q}(x)$$

for any vector $v \in \Sigma(I,q)(x) - \Sigma(I,q-1)(x)$

(iii) *for* $x \in Y$ *the sequence*

$$A(n,x) = (\Phi(n,x)^*\Phi(n,x))^{\frac{1}{2n}}$$

converges to a matrix $A(x) \in M(r,\mathbb{R})$. *The eigenspace of* $A(x)$ *is the orthogonal complement of* $\Sigma(I,q)(x)$ *in* $\Sigma(I,q+1)(x)$ *corresponding to the eigenvalue* $\exp \Lambda^{i_{q+1}}$

LEMMA 6. *Suppose that* $\log^+ \|\Phi(1, \dot{})\|$ *is a measurable function and* T *a non-singular transformation. There is a set* $Y \subset X$ *of full measure such that for every* $x \in Y$,

(i) *The sequence* $S_n = \frac{\sum_{0\leq q<n} \log^+ \|\Phi(1,T^q(x))\|}{\sum_{i=0}^{n-1} \omega_i(x)}$ *converges to a limit a.e.*

(ii) $\lim_{n\to\infty} \|\Phi(1,T^n(x))\| = 0$.

PROOF 6. (i) *This follows directly from the Hurewicz Ergodic Theorem.*

(ii) *By* (i), *the sequence* S_n *converges to a limit,*

$S_n = \frac{\sum_{0\leq q<n} \log^+ \|\Phi(1,T^q(x))\|}{\sum_{i=0}^{n-1} \omega_i(x)}$

$S_{n-1} = \frac{\sum_{0\leq q<n-1} \log^+ \|\Phi(1,T^q(x))\|}{\sum_{i=0}^{n-2} \omega_i(x)}$

$S_n = \frac{\sum_{i=0}^{n-2} \omega_i(x)}{\sum_{i=0}^{n-1} \omega_i(x) \sum_{i=0}^{n-2} \omega_i(x)} \sum_{0\leq q<n-1} \log^+ \|\Phi(1,T^q(x))\| + \frac{\log^+ \|\Phi(1,T^n(x))\|}{\sum_{i=0}^{n-1} \omega_i(x)}$.

Since $n \to \infty$ *and* $|S_n - S_{n-1}| \to 0$, *we can conclude that*

$$\frac{\log^+ \|\Phi(1,T^n(x))\|}{\sum_{i=0}^{n-1} \omega_i(x)} \to 0$$

for all $x \in Y$.

Now, given $\varepsilon > 0$, there exists $N(\varepsilon, x)$ such that for all $n > N$

$$\|\Phi(1,T^n(x))\| < \exp \sum_{i=0}^{n-1} \omega_i(x)\varepsilon$$

$\|\Phi(n,x)\|$ satisfies the cocycle identity: $\Phi(n+1,x) = \Phi(1,T^n(x))\Phi(n,x)$.
For a unit vector $v \in \Sigma(I,q,n)(x)$,

$\|\Phi(n+1,x)v\| \leq \|\Phi(1,T^n(x))\|\|\Phi(n,x)v\|$

$\|\Phi(n+1,x))v\| \leq \exp(\sum_{i=0}^{n-1} \omega_i(x)\varepsilon)\ \delta_{i_q}(\Phi(n,(x))$

$\|\Phi(n+1,x)v\| \leq \exp(\sum_{i=0}^{n-1} \omega_i(x)\varepsilon)\ \exp(\sum_{i=0}^{n-1} \omega_i(x)(\Lambda^{i_q} + \varepsilon))$

$\|T(n+1,x)v\| \leq \exp(\sum_{i=0}^{n-1} \omega_i(x)(\Lambda^{i_q}+2\varepsilon))$
Choose a unit vector $v \in \Sigma(I,q,n)(x)$, and let $v' \in \Sigma(I,q,n+1)(x)$ be the orthogonal projection of v on $\Sigma(I,q,n+1)(x)$. The orthogonal complement of v' in $\Sigma(I,q,n+1)(x)$ has the form $C_{n+1}^{-1}\sum_{i\geq q+1-1} b_i e_i$.

LEMMA 7. *Given $\varepsilon > 0$ there exists $N(\varepsilon,x), x \in Y$ with the following property. There is a unit vector $v \in \Sigma(I,q,n)(x)$, for some number $b_i \in \mathbb{R}$ and $v' \in \Sigma(I,q,n+1)(x)$*

$$v = v' + C_{n+1}^{-1}\sum_{i\geq i_{q+1}} b_i e_i$$

Then $|b_i| < \exp\{-\sum_{i=0}^{n-1}\omega_i(x)(\Lambda^i - \Lambda^{i_q} - \varepsilon\}$ for $n \geq N$.

PROOF 7. *We have $\|\Phi(n+1,x)v\| \leq exp\sum_{i=0}^{n-1}\omega_i(x)(\Lambda^{i_q}+2\varepsilon)$ by Lemma 6. Notice that*

$$\|\Phi(n+1,x)v\| \geq \|\sum_{i\geq i_{q+1}} |b_i|\Phi(n+1,x)C_{n+1}^{-1}e_i\|.$$

Now $\|b_i\Phi(n+1,x)C_{n+1}^{-1}e_i\| \leq \|\sum_{i\geq i_{q+1}} b_i\Phi(n+1,x)C_{n+1}^{-1}e_i\|$ as Λ^i is non-decreasing.

$$\text{Hence } \|\Phi(n+1,x)v\| \geq \|b_i\Phi(n+1,x)C_{n+1}^{-1}e_i\|.$$

Now $\|\Phi(n+1,x)C_{n+1}^{-1}e_i\|$ is the i-th eigenvalue, since

$$\|b_i\Phi(n+1,x)C_{n+1}^{-1}e_i\| = |b_i|\|L_{n+1}(D_{n+1})^{\frac{1}{2}}C_{n+1}C_{n+1}^{-1}e_i\| = |b_i|\delta_i(\Phi_{n+1}(x)).$$

Let Λ^i be the limit of $\frac{\log\delta_i(\Phi_{n+1}(x))}{\sum_{i=0}^{n}\omega_i(x)}$. It follows from the above that

$$\|\Phi(n+1,x)v\| \geq |b_i|\ exp\ ((\sum_{i=0}^{n}\omega_i(x))(\Lambda^i - \varepsilon)).$$

Thus Λ^i is in a bounded interval which is greater than Λ^{i_q}, $0 < \varepsilon < 1$. For large n, we may assume

$$\exp((\sum_{i=0}^{n}\omega_i(x))(\Lambda^i - \varepsilon)) \geq \exp((\sum_{i=0}^{n-1}\omega_i(x))(\Lambda^i - 2\varepsilon)).$$

Thus

$$|b_i| \leq exp\ -(\sum_{i=0}^{n-1}\omega_i(x))(\Lambda^i - 4\varepsilon - \Lambda^{i_q})$$

which completes the proof of the Lemma.

Lemma 7 shows that a vector in $\Sigma(I,q,n)(x)$ can be combined with the projection on $\Sigma(I,q,n+1)(x)$ and the orthogonal complement of $\Sigma(I,q,n+1)(x)$, that is

$$C_n^{-1}\sum_{i=1}^{i_{q+1}-1} Ke_i = C_{n+1}^{-1}\sum_{j=1}^{r} b_j e_j \text{ and } v_n = v'_{m+1} + w_{m+1}.$$

Now v'_{m+1} is the orthogonal projection of v_n onto $\Sigma(I,q,n+1)(x)$ and the norm of v'_{m+1} is given by:

$$\|v'_{m+1}\| = \|v_n - C_{n+1}^{-1}\sum_{i\geq i_{q+1}} b_i e_i\|.$$

Since v_n is a unit vector, we have the following upper and lower bounds for $\|v'_{m+1}\|$:

$$1 - \|C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i\| \leq \|v'_{m+1}\| \leq 1 + \|C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i\|.$$

We want to show $\|v_{n+1} - v_n\| \leq 2r \exp\{-(\sum_{i=0}^{n-1})\omega_i(x)(\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon)\}$

LEMMA 8. *If $v_{n+1} \in \Sigma(I, q, n{+}1)(x)$ is a unit vector and $v_n \in \Sigma(I, q, n)(x)$ satisfies the conditions of Lemma 2, then*

$$\|v_{n+1} - v_n\| \leq 2r \exp\{-(\sum_{i=0}^{n-1} \omega_i(x))(\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon)\}$$

PROOF 8. $\|v_{n+1} - v_n\| = \|v_n - v_{n+1}\| = \|v_n - v'_{n+1} + v'_{n+1} - v_{n+1}\|$, *and* $\|v_n - v'_{n+1} + v'_{n+1} - v_{n+1}\| \leq \|v_n - v'_{n+1}\| + \|v'_{n+1} - v_{n+1}\|$. *It follows that* $\|v_n - v'_{n+1}\| = \|C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i\|$.
On the other hand, let $v'_{n+1} = a v_{n+1}$, *and we have* $1 - \|C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i\| \leq \|v'_{m+1}\| \leq 1 + \|C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i\|$. *Thus*

$$1 - \|C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i\| \leq a \leq 1 + \|C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i\|. \text{ Now}$$

$\|v'_{n+1} - v_{n+1}\| = \|(a-1)v_{n+1}\|$, *which is smaller than* $\|C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i\|$. *We thus have*

$$\|v_{n+1} - v_n\| \leq 2\|C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i\|.$$

It follows that

$$\|C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i\| \leq \sum_{i \geq i_{q+1}} \|b_i e_i\| \leq r|b_{i_{q+1}}| \leq r \exp\{-(\sum_{i=0}^{n-1} \omega_i(x))(\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon)\}.$$

This completes the proof.

We will show that $\|v_{n+k} - v_{n+l}\|$ is a Cauchy sequence. In fact, the sequence $\sum_{i=l}^{\infty} r$ $\exp\{-(n+i)(\Lambda^{i_{q+i}} - \Lambda^{i_q} - \varepsilon)\}$ is the sum of a geometric series. Thus

$$\begin{aligned}
\|v_{n+k} - v_{n+l}\| &\leq \sum_{i=l}^{\infty} \|v_{n+k} - v_{n+l}\| \\
&= 2r \frac{1}{1 - \exp\{-(\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon)} \exp\{-(\sum_{i=0}^{n-1} \omega_i(x) + l)(\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon)\} \\
&= C \exp\{-(\sum_{i=0}^{n-1} \omega_i(x) + l)(\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon)\}.
\end{aligned}$$

This shows that v_n is a Cauchy sequence, as claimed.

LEMMA 9. *Let $\{v_n^1, v_n^2, \cdots, v_n^i\}$ be a collection of vectors which is a basis for $\Sigma(I, q, n)(x)$, where $0 \leq i < i_{q+1}$. The sequence $\{\Sigma(I, q, n)(x)\}$ has limit $\Sigma(I, q)(x)$.*

Furthermore,

$$\|v_{n+k} - v_{n+l}\| \leq \sum_{i=l}^{k} \|v_{n+k} - v_{n+l}\| \leq \sum_{i=l}^{\infty} \|v_{n+k} - v_{n+l}\|.$$

PROOF 9. *By Lemma 8, we see easily that*

$$\|v^i_{n+k} - v^i_{n+l}\| \le C\ \exp\{-(\textstyle\sum_{i=0}^{n-1}\omega_i(x) + \max(k,l))(\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon)\}$$

for $i_q \le i < i_{q+1}$. *Moreover*

$$\|v^i_n - v^i\| \le C\ \exp\{-(\textstyle\sum_{i=0}^{n-1}\omega_i(x))(\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon)\}$$

where v^i *is the limit of* v^i_n. *The sequence* $\{v^i_n\}$ *converges to a linearly independent set of vectors as* $n \to \infty$. $\Sigma(I,q,n)(x)$ *is the space spanned by* $\{v^1_n, v^2_n, \cdots, v^i_n\}$. *Thus* $\Sigma(I,q,n)(x) \to \Sigma(I,q)(x)$ *as* $n \to \infty$.

Writing the matrix $B = C_n C^{-1}_{n+k}$, a vector in the space $\Sigma(I,q,n)(x)$ has the form $C^{-1}_n e_i$ which we can split into an orthogonal projection in $\Sigma(I,q,n+1)(x)$ and its orthogonal complement in $\Sigma(I,q,n+1)(x)^{\perp}$. By lemma 7, $C^{-1}_n e_i = C^{-1}_{n+1}\sum_{j=1}^{r} b_j e_j$. Hence $|C_{n+1}C^{-1}_n| = |b_j|$ as e_i is the standard basis. The inverse of $C_{n+1}C^{-1}_n$ is $C_n C^{-1}_{n+1}$. Thus we have a similar decomposition of a vector in $\Sigma(I,q,n+1)(x)$, viz.

$$v = v' + C^{-1}_n \textstyle\sum_{i \ge_{q+1}} b_{i*} e_i,$$

where $v' \in \Sigma(I,q,n)(x)$ and $C^{-1}_n \sum_{i\ge_{q+1}} b_{i*}e_i \in \Sigma(I,q,n)(x)^{\perp}$. We set $a_i = \exp(\Lambda^i - i\varepsilon)$, and note that $|C_n C^{-1}_{n+1}| = |b_{i*}|$. It then follows that

$$|b_{i*}| \le C\ \exp\ -(\textstyle\sum_{i=0}^{n-1}\omega_i(x))(\Lambda^i - \Lambda^{i_q} - r\varepsilon),$$

since we have a cycle of length r.

LEMMA 10. *If* $v^i \in \Sigma(I,q)(x)$, *then* $\limsup \frac{1}{n}\log\|T(n,x)(v^i)\| \le \lambda^{i_q}(x)$.

PROOF 10. *Firstly, we can see that* $\limsup \frac{\log\|\Phi(n,x)(v^i_n)\|}{\Omega(x)} = \lim_{n\to\infty}\frac{\log\delta_i(\Phi(n,x))}{\Omega(x)} = \Lambda^i$. *If* $i < i_{q+1}$, *we have* $\limsup \frac{\log\|\Phi(n,x)(v^i_n)\|}{\Omega(x)} \le \Lambda^{i_q}$.

On the other hand, $v^i - v^i_n = w + C^{-1}_n \sum_{i\ge_{q+1}} b_{i*}e_i$. *For* $w \in \Sigma(I,q,n)(x)$, $\limsup \frac{\log\|\Phi(n,x)(w)\|}{\Omega(x)} \le \Lambda^{i_q}$. *Hence*

$$\begin{aligned}&\limsup \frac{\log\|\Phi(n,x)C^{-1}_n\sum_{i\ge i_{q+1}} b_{i*}e_i\|}{\sum_{i=0}^{n-1}\omega_i(x)}\\ &\le \limsup \frac{\log\{C\exp-(\Omega(x))(\Lambda^i - \Lambda^{i_q} - r\epsilon)\times\delta_i(\Phi(n,x))\}}{\Omega(x)}\\ &= -\Lambda^i + \Lambda^{i_q} + r\epsilon + \Lambda^i = \Lambda^{i_q} + r\epsilon\end{aligned}$$

for $i \ge i_{q+1}$.

The triangle inequality implies that

$$\|\Phi(n,x)(v^i)\| \le \|\Phi(n,x)(v^i_n)\| + \|\Phi(n,x)(v^i - v^i_n)\| \le 2\exp(\sum_{i=0}^{n-1}\omega_i(x))(\Lambda^{i_q} + \varepsilon),$$

and thus $\limsup \frac{\log\|\Phi(n,x)(v^i)\|}{\sum_{i=0}^{n-1}\omega_i(x)} \le \Lambda^{i_q}$.

LEMMA 11. *If the vector v^i is not in $\Sigma(I, q-1)(x)$, for large n, the projection $v^{i\prime}$ belongs to $\Sigma(I, q-1, n)(x)$ with $\|v^{i\prime}\| > c > 0$. Then*

$$\liminf \frac{\log \|T(n,x)(v^i)\|}{\sum_{i=0}^{n-1} \omega_i(x)} \geq \Lambda^{i_q}(x)$$

PROOF 11. *This proof is quite straightforward. We take a unit vector v^i which is not in $\Sigma(I, q-1)(x)$. There is a $\delta \in V$ such that $v^i + \delta \in \Sigma(I, q-1)(x)$. When n is large enough, the vector v_n^i has projection $v^{i\prime}$ in $\Sigma(I, q-1, n)(x)$ and orthogonal complement $v^{i\prime\prime} \in \Sigma(I, q-1, n)(x)^{\perp}$. We take the difference $\|v^i - v^{i\prime}\| \geq \frac{\delta}{2}$, obtaining*

$$\begin{aligned}
\|\Phi(n,x)(v^i)\| &\approx \|\Phi(n,x)v_n^i\| \\
&\geq \|\Phi(n,x)(v^i - v^{i\prime})\| \\
&\geq \frac{\delta}{2} \exp(\sum_{i=0}^{n-1} \omega_i(x))(\Lambda^{i_q}(x) - \varepsilon)
\end{aligned}$$

Combining Lemmas 10 and 11, we can conclude that $\lim \frac{\log \|\Phi(n,x)(v^i)\|}{\sum_{i=0}^{n-1} \omega_i(x)} = \Lambda^{i_q}(x)$, for $v_i \in \Sigma(I, q)(x) \backslash \Sigma(I, q-1)(x)$. Now we see that the eigenspace is $C_n^{-1} e_i \to C^{-1} e_i$ and the eigenvalue is $\Lambda^k(x) = \lim_{n\to\infty} \frac{\log \delta_k(\Phi(n,x))}{\sum_{i=0}^{n-1} \omega_i(x)}$, so that the limit matrix $A(x) = \lim_{n\to\infty} (\Phi(n,x)^* \Phi(n,x))^{\frac{1}{2n}}$ exists.

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