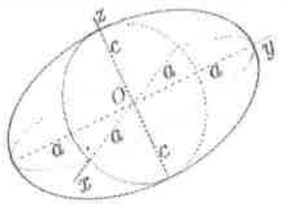
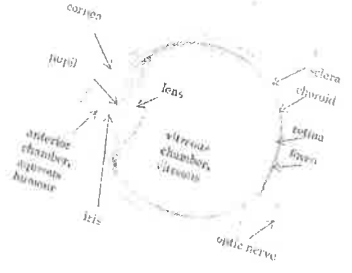
REFLECTIONS

Vol. 47, No. 3, 2022

Journal of
The Mathematical Association
of New South Wales Inc.



$$\begin{aligned}\frac{dr}{dt} &= \frac{dV}{dt}\Big/\frac{dV}{dr}\\ &= \frac{C.4\pi r^2}{4\pi r^2}\\ &= C.\end{aligned}$$



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REFLECTIONS
Journal of The Mathematical Association of New South Wales Inc.

Promoting quality mathematics education for all

- Professional development for teachers
- Quality resources for teachers and students
- Opportunities for the exchange of ideas
- Activities to enrich student learning
- A representative voice for mathematics educators

Reflections is designed to aid communication of news, ideas, instructional techniques, and matters of interest to mathematics educators. It is published four times a year and is devoted to promoting teachers' ideas and strategies for improving the quality of mathematics learning.

Manuscripts for articles, contributions to features, books and other material for review, and general editorial correspondence should be sent to MANSW at the address below.

Persons, schools, and other institutions interested in mathematics or mathematics education are invited to join the Association.The details of membership categories and entitlements and a membership form can be found on the MANSW website www.mansw.nsw.edu.au.

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④# *Reflections*, Vol. 47, No. 3, 2022

How to submit an article for *Reflections*

Each year there are four issues of *Reflections*. The last issue is dedicated to papers from the previous annual conference. We invite authors to submit papers for publication in the other three issues. Copy should be sent to the editor, Anne Prescott, by email anne.e.prescott@gmail.com by 31 January, 30 April, and 31 July so that members of the editorial panel can read and evaluate the material before printing in the subsequent issue.

Authors should submit their work as a Word document using the template at: https://www.mansw.nsw.edu.au/about-us/reflections-template. Clear, black diagrams and tables need to be labelled and referred to in the paper. All references must be given. References to a journal paper or monograph should follow this format:

Kissane, B., Bradley, J., & Kemp, M. (1994). Graphic calculators, equity, and assessment. Australian Senior Mathematics Journal, 8(2), 31–43.
Heid, M. K. (1995). Algebra in a technological world. Reston, Virginia: National Council of Teachers of Mathematics.

We particularly encourage teachers to write about successful classroom experiences. Areas of interest include:

- the use of technology;
- alternative assessment strategies;
- teaching strategies;
- approaches to the new syllabus;
- ideas for gifted and talented students;
- thematic approaches;
- mathematical ideas;
- ideas to encourage reluctant learners; and
- alternative ways to present senior topic areas.

Teachers often produce and regularly use excellent worksheets. In each issue we would like to reproduce some of these. Worksheets must be original as we cannot reprint material without consulting and acknowledging the initial publisher. Please send a copy of your favourite worksheet to anne.e.prescott@gmail.com.

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THIRTEEN THINGS ABOUT EYES AND MATHEMATICS, AND WHY I FEEL LUCKY TO BE ABLE TO SEE AT ALL

Mary Coupland
UTS School of Mathematical and Physical Sciences

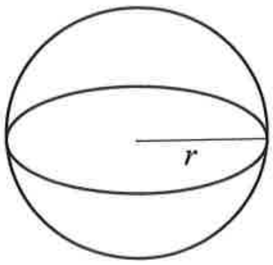
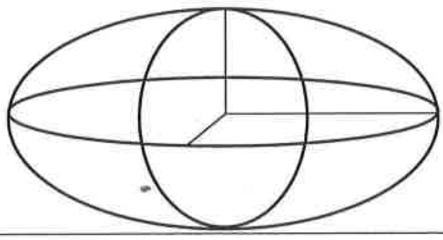
Some time ago, in 2013 in fact, I was unlucky enough to endure a retinal detachment. Along the way to recovery I found that there were many things about eyes, vision, and surgery that mathematics helps to understand. This paper addresses some of these interesting things.

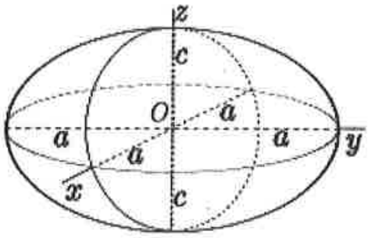
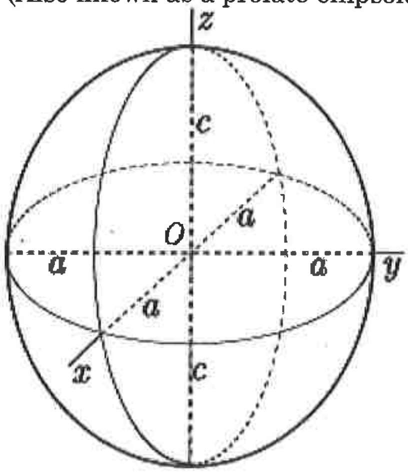
Introduction

Eye problems and diseases happen to all kinds of people for many different reasons. When I had this particular eye problem I found that at each stage of diagnosis, treatment planning, operations, and recovery it was helpful to talk to people who had been on a similar journey, and of course extremely important to research the situation by reading hospital leaflets, asking questions of the doctors, and searching the internet. I tried to use mathematics where I could to make sense of the facts that I found out, and I was intrigued that there was so much maths in this story. In keeping with the MANSW 2013 conference theme 'Thirteen—unlucky for sum?' I decided that thirteen snippets of eye-related mathematical ideas and facts might make an interesting talk. The talk was well received, and now at last here it is written up for *Reflections*.

Let's begin with the anatomy of the human eye. Is the eyeball a sphere or an ellipsoid? What is an ellipsoid anyway? Why is that important to the story about retinal detachments?

MATHS ITEM 1. Spheres, ellipsoids, volume, and surface area

Sphere, r is the radius.  	**Ellipsoid**: the general case, unequal semi-axes of lengths a, b, c. 
Volume $V = \frac{4}{3}\pi r^3$,	Volume $V = \frac{4}{3}\pi abc$
Surface area $S = 4\pi r^2$	The surface area involves elliptic integrals and is not expressible in terms of elementary functions, unless two of the semi-axes are of the same length and the ellipsoid is an 'ellipsoid of revolution'.

<table>
<tr><th colspan="2">Special cases of ellipsoids</th></tr>
<tr><td>An oblate ellipsoid of revolution
(Also known as an oblate ellipsoid)

Volume $V=\frac{4}{3}\pi a^2 c, \quad a > c$.
Surface Area is
$$S=2\pi a^2\left(1+\frac{1-e^2}{e}\tanh^{-1}e\right)$$
Where e is the ellipticity and $e^2=1-\frac{a^2}{c^2}$.</td>
<td>A prolate ellipsoid of revolution
(Also known as a prolate ellipsoid)

Volume $V=\frac{4}{3}\pi a^2 c, \quad a < c$.
Surface Area is
$$S=2\pi a^2\left(1+\frac{c}{ae}\sin^{-1}e\right)$$
Where e is the ellipticity and $e^2=1-\frac{a^2}{c^2}$.</td></tr>
<tr><td>Examples: The Earth, persimmon, Smarties.</td><td>Examples: Some kinds of footballs, watermelon.</td></tr>
<tr><td>$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, with $a=b>c$</td><td>$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, with $a=b<c$</td></tr>
<tr><td colspan="2">A third special case is the sphere, whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, with $a=b=c=r$.</td></tr>
</table>

So what shape is the eye? It could be described as consisting of two fused ellipsoids. The main chamber is roughly spherical with a radius of about 12 mm in adults, growing from about 8 mm radius at birth. The front chamber is like a dome of radius 8 mm in adults. The cornea is the main feature of the front or anterior chamber, and is a clear structure that does about two-thirds of the eye's focusing. If the cornea is not formed in the shape of part of a perfect sphere, the eye is unable to focus rays of light to one spot. The results in an *astigmatism*. Note the structure of the word—*a* as a prefix often means 'not' and so this word means 'not coming to a mark or spot'. This was carefully explained to me by my ophthalmologist and made me think about other words that begin with *a*, perhaps in mathematics, where the *a* as a prefix means 'not'.

MATHS ITEM 2. Words that begin with *a*

For example, asymmetric, means not symmetric, asymptotic means not falling together.

To continue the story, we now need to consider the rest of the eye.

Anatomy of the eye—basic structures

This is easily researched and there are plenty of diagrams on the internet. Mindful of copyright, I have made my own sketch and I do not claim that it is entirely accurate, as any sketch is a simplification. I will concentrate here on the features that make a difference to my story, and this is only an outline. The tough white covering is called the *sclera* and helps maintain the shape of the eyeball. There is a mucous membrane called the *conjunctiva* covering everything and lining the inside of the eyelids. The personalized part of the eye is the coloured *iris*, which is muscular and widens and narrows a hole (the *pupil*) to allow more or less light into the eye. Behind the pupil the crystalline lens, about 5 mm thick with diameter of about 9 mm, completes the job of focusing the incoming rays of light onto the retina, where the light is changed into electric signals and carried to the optic nerve and then to the brain, where they are interpreted and we 'see'. The lens is suspended by fibres and the muscles around the lens change its curvature and thus allow us to focus on objects that are near or far away. Between the lens and the retina is a gel called the *vitreous humour*, or simply the vitreous. Between the sclera and the retina is a layer called the *choroid*, which mainly contains blood vessels that nourish parts of the eye.

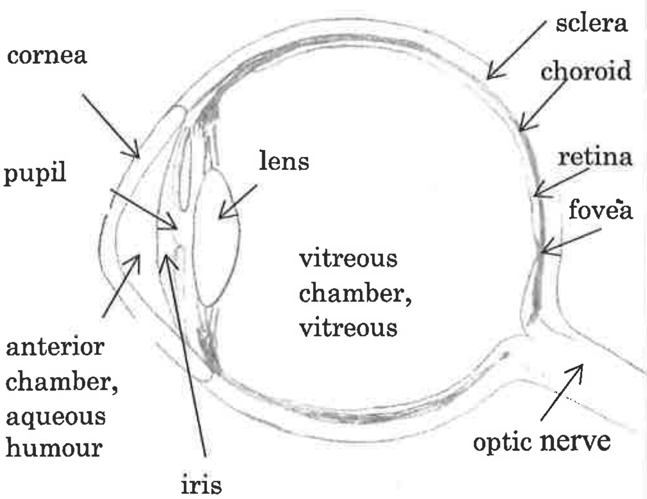
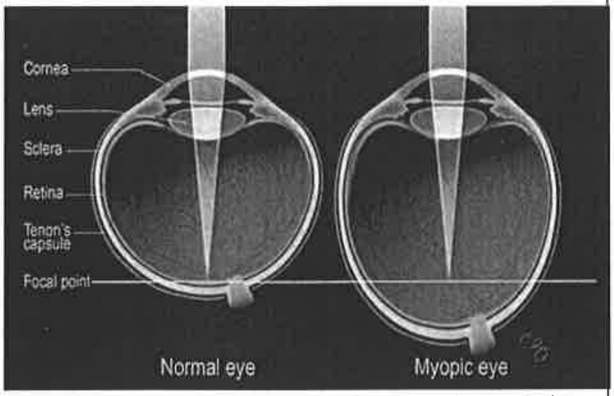


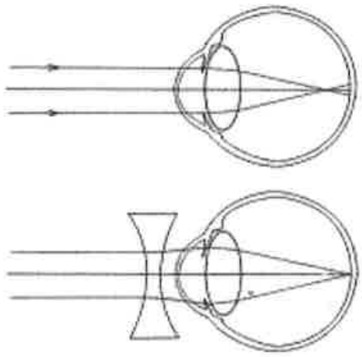
FIG. 1. SIMPLIFIED SKETCH OF THE HUMAN EYE

MATHS ITEM 3. What happens if the eye is more like an ellipsoid than a sphere?

This is mathematical but also biological: in some people the eye is elongated front to back and the rays of light don't focus where they should, giving a blurred image on the retina. This is called being short-sighted, or *myopic*, although there are other causes of myopia.



http://www.improveeyesightfast.com/myopia/



From Wikipedia, showing how concave lenses in glasses are used to correct myopia.

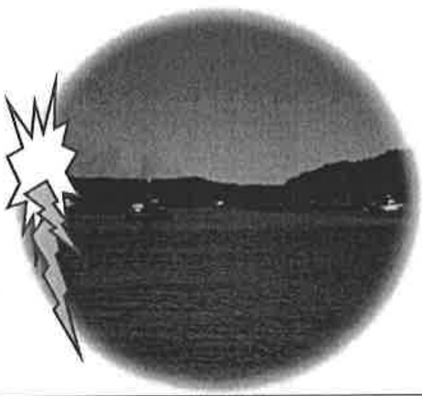
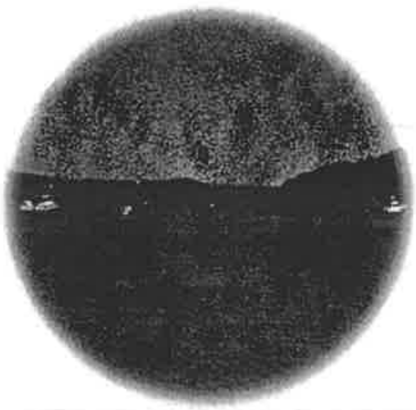
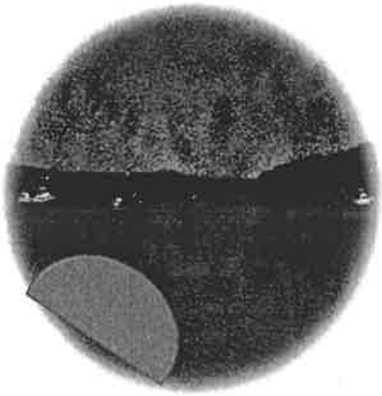
MATHS ITEM 4. How to investigate the result of elongation on the inner surface area of the eye

Surface area and volume now reappear in this story. My understanding is that we are all born with roughly the same size eyes and in each eye the retina covers about 70% of the inside surface of the eye. For various reasons, in some unlucky people the eye becomes elongated in the front-to-back orientation: it becomes a prolate spheroid. This stretches the retina, which becomes even more fragile (it has been said to have the consistency of wet tissue paper), and prone to tears and holes. This is significant as it contributes to retinal detachments, so there is an interesting mathematical question—if an ellipsoid of fixed volume changes from a sphere to a prolate ellipsoid, keeping the same volume, what is the change in the surface area? As the formulae for calculating the surface area of an ellipsoid are so complicated, this is quite a lengthy investigation. However we can get an approximation by considering a sphere of radius 12 mm, with an interior lining of 0·5 mm which is approximately the thickness of the retina. The total volume of that lining is then $\frac{4}{3}\pi(12^3 - 11{\cdot}5^3) = 867{\cdot}603\ \text{mm}^2$. Now if we take that 'quantity' of lining and stretch it on the inside of a sphere 13 mm in radius, my calculations show that the thickness is reduced to 0·42 mm. That is quite a reduction when expressed as a percentage.

What causes a retinal detachment?

An injury such as a blow to the head can cause a retinal detachment. Sports injuries can cause retinal detachments, as can certain recreational pursuits such as bungee jumping (Hassan et al., 2012).

Retinal detachment can also occur simply as a result of aging. As humans age, the consistency of the vitreous changes from a thick egg-white or even jelly-like substance to a more liquefied form. As this happens, the vitreous pulls on the retina, causing a signal that the brain interprets as a flash of light. The vitreous may become detached from the retina. As the vitreous comes away from the retina, bits of gel may be released and are seen as 'floaters' in the field of vision. This is a normal event and may last from two weeks to six months, and is called a *posterior vitreous detachment* (PVD). In very unlucky cases, as the vitreous tugs on the retina it may cause tears or holes. If fluid gets behind the retina through these tears, the retina itself can peel away or detach from the choroid layer. In my case this is what happened and I have illustrated the effect this had on my vision in this sequence of doctored pictures.

<table>
<tr><th colspan="2">The visual disturbances that indicated that I had a retinal detachment</th></tr>
<tr><td>First sign of a PVD—Flashes of light as the vitreous tugs on the retina
</td><td>Secondly, one big floater appeared.
</td></tr>
<tr><td>Thirdly, a shower of floaters, mainly red and brown, quite pale and only seen when looking at a bright background like the sky.
</td><td>Most dangerous—loss of vision in the lower left of my field of vision.
</td></tr>
</table>

MATHS ITEM 5. The image on the retina is inverted

Although the grey shadow that I saw was in the lower left of my field of vision, because of the way that the lens system in the eye works the detachment was in fact at the top of my eye.

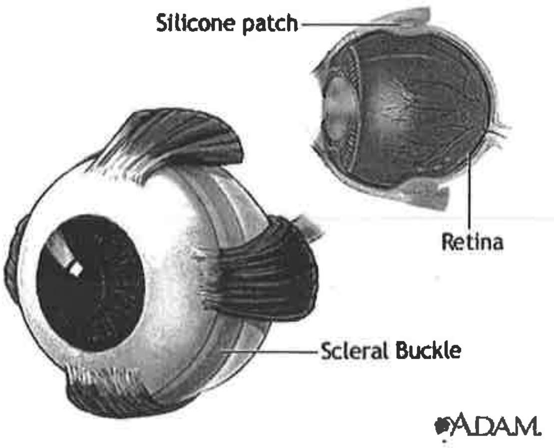
Can a retinal detachment be mended?

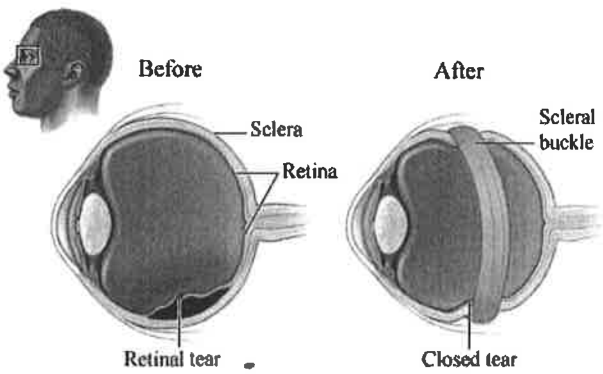
In earlier times, blindness was the inevitable consequence of a retinal detachment. However we are indeed fortunate that procedures have been invented to repair them. One such procedure we owe to Charles Schepens, (1912–2006), a Belgian-Amercian ophthalmologist. Schepens finished medical school in Belgium, then trained in ophthalmology in London. During the war he served as a medical officer in the Belgian air force and also joined the French Resistance. He migrated to the USA in 1947 and worked at the Harvard Medical School, eventually starting The Retina Foundation in 1950 which became the Schepens Eye Research Institute, a world famous research centre affiliated with the Harvard Medical School.

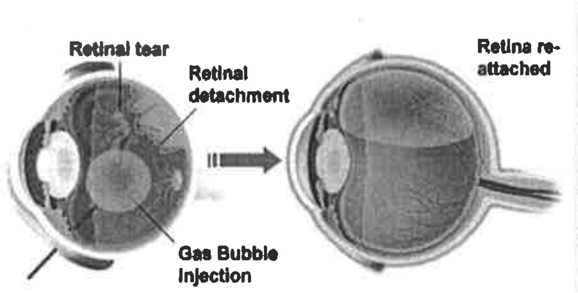
Photo from http://www.schepens.harvard.edu/

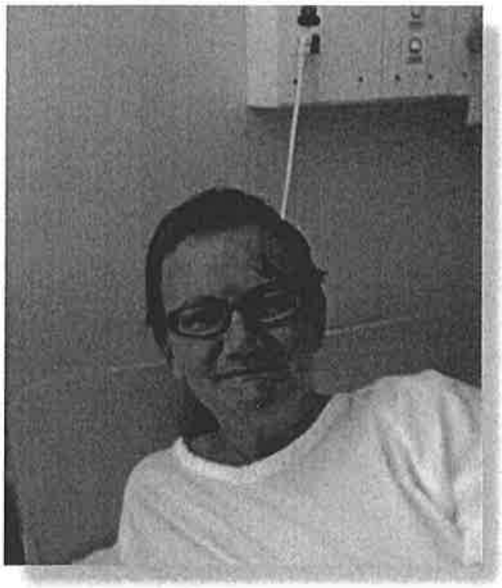
MATHS ITEM 6. Dr Charles Schepens originally studied mathematics before becoming an ophthalmologist

In my case, the procedure that the surgeons at Sydney Eye Hospital recommended was one pioneered by Schepens: the scleral buckle technique. Avoiding the more gruesome details, I will just say that it involves stitching a band around the outside of the eyeball so that the surface is flattened, and moved closer to the detached retina. A bubble of gas was inserted also, to push the detached retina into place. Note that the band has to be stitched on below the muscles that hold the eye in place.









The photos show 'before' and 'after' images from hospital. Note in the 'before' image, an arrow that a medical assistant drew on my forehead while I was waiting for the surgery, just to be clear about which eye needed the operation!

The operation took about 50 minutes, and although the patient is initially sedated, it is not a full anaesthetic and I can remember being conscious but very relaxed on the operating table, talking to the surgeons, and not feeling much at all. Overnight in hospital, then a week resting at home completed the recovery. However, I still had very 'muddy' vision in that eye. Even more unlucky for some is the fact that sometimes the choroid layer is pierced and a vitreous haemorrhage can result in blood being released into the vitreous. Unlucky for some indeed!

MATHS ITEM 7. How large is a gas molecule?

In my case, the bubble of gas inserted was sulphur hexafluoride. This is chosen as it is an inert gas and the molecules are 'large', so that it dissipates slowly. Other similar procedures can involve other gases and even silicon oil. But how do you measure the size of a single molecule? Something to ask the Science teachers one day.

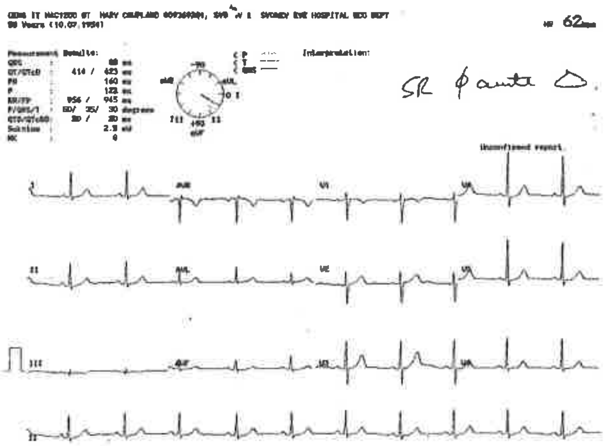
This is what I saw as the gas bubble disappeared:

There were many floaters still left behind in the eye.

Overnight in the hospital I woke up in some pain and not feeling at all well. Among the tests I was put through was an electrocardiogram (ECG)—and did you know that . . .

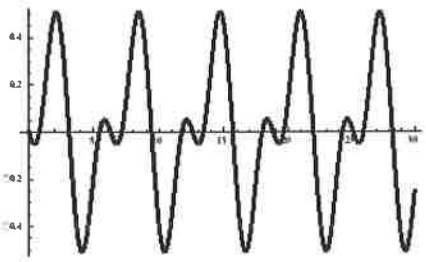
MATHS ITEM 8. Medical students are taught about vectors to help them learn how to read an ECG

This takes a bit of researching but I recommend you chat to your doctor about the maths they need to know: you will be surprised!

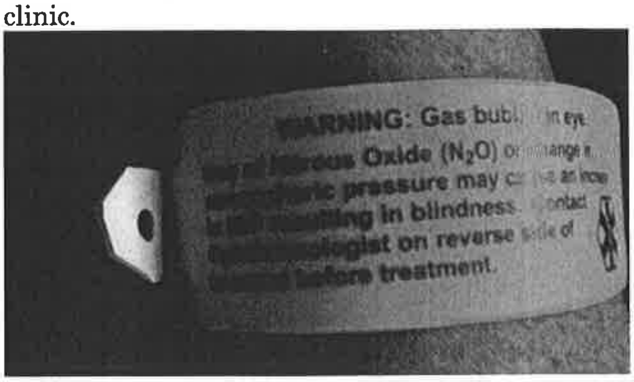


It was fun later to try to reproduce some of these wave shapes . . .

Plot[(1/3)*Sin[t]-(1/4)*Sin[2t], {t,0,30}]



In the recovery bay after the surgery, and again back on the ward, it was emphasized that there were restrictions I must be aware of as a result of having a bubble of gas in my eye. In case I might have forgotten, I was given a bright green plastic bracelet and told not to remove it myself—that it needed to stay on until my visit a week later to the clinic.



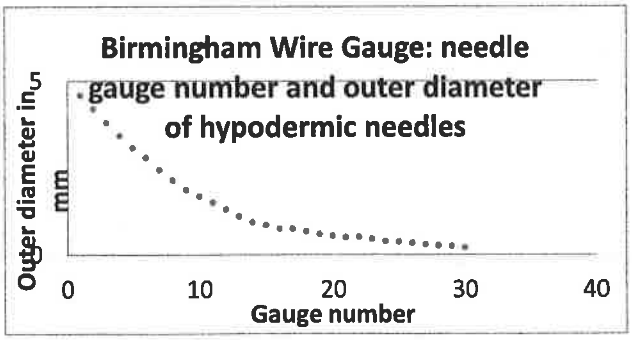
MATHS ITEM 9. Why you must not fly in an aeroplane if you have a bubble of gas in your eye

For a fixed quantity of gas at constant temperature, pressure is inversely proportional to volume. If you go to a place of low pressure with a gas bubble in your eye, the volume of the bubble will increase, with painful and possibly disastrous consequences!

The story continues now, as I could not see very well at all in my left eye. The clinic doctors said to wait two weeks to see if the blood in the vitreous would disperse of its own accord. The need for a possible second operation to remove the blood was discussed. While waiting for those two weeks I decided to seek some other opinions and did some research into vitrectomies. Quite soon I discovered that several ophthalmic surgeons in Sydney, and indeed many more all over the globe, offered this procedure, and a key thing being advertised was 'We use the new 23 gauge needles'. This is significant because the vitrectomy procedure involves inserting several needles into the eyeball, and if these are thinner, recovery is swifter and there is less need for stitching.

MATHS ITEM 10. Trying to understand the numbers in needle gauges

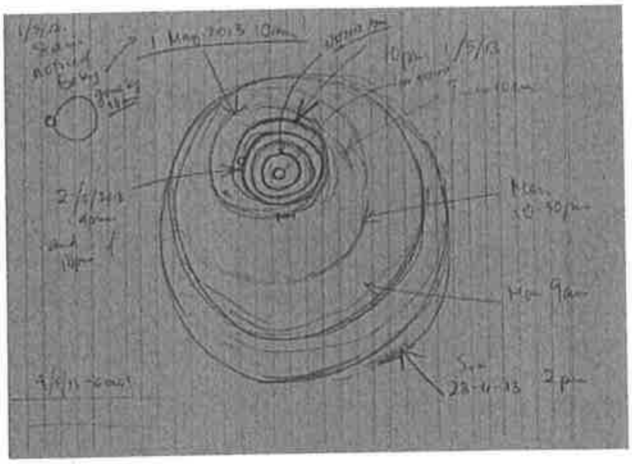
The system for indicating the size of hypodermic needles is based on the pre-metric Birmingham Wire Gauge. There does not seem to be a simple formula relating the diameter of the needle to the gauge number. This would make an interesting investigation, along with knitting needle gauges and the American Wire Gauge.



Eventually I did indeed have a vitrectomy, again at the Sydney Eye Hospital, on 24 April. Finally I could see! It was a wonderful feeling. There was a big black disk in my field of vision, which was another gas bubble. This time it was inserted to replace the vitreous, which had to be removed. It kept the shape of the eye, and over time it shrank as the eye made more fluid to replace the vitreous.

MATHS ITEM 11. The second bubble story

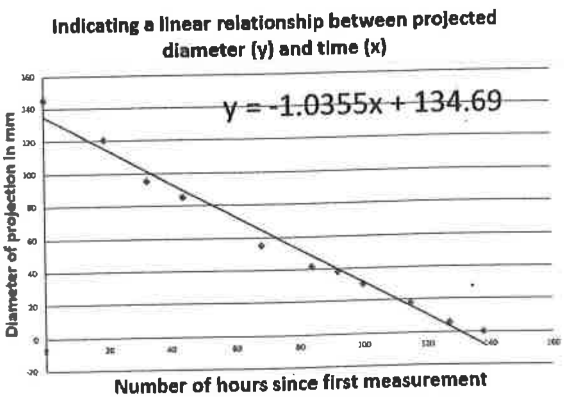
After the second operation, I was expecting the bubble and decided to measure it, as best I could. I put a sheet of paper on a desk, held my face a set distance from the table and traced around the edge of the big black bubble as I saw it in my field of vision. It disappeared after about six days. This is what I recorded:



This table shows the decreasing diameter of the projection, with x the number of hours after the first measurement:

Date	*Time*	x	*Diameter of projection in mm*
28 Apr	14:00	0	145
29 Apr	9:00	19	120
29 Apr	22:30	32.5	95
30 Apr	10:00	44	85
1 May	10:00	68	55
1 May	22:00	84	42
2 May	14:00	92	38
2 May	22:00	100	30
3 May	13:00	115	18
4 May	1:00	127	5
4 May	12:00	138	0

I plotted a line of best fit with Excel, and the equation turned out to be $y = -1{\cdot}0355x + 134{\cdot}69$.



MATHS ITEM 12. Calculus and bubbles.

Let's write

$$y = -1{\cdot}0355x + 134{\cdot}69$$

as $y = -1{\cdot}0355t + 134{\cdot}69$. Then dy/dt is the rate of change of the projected diameter with time, and $dy/dt = -1{\cdot}0355$. Since the projected diameter of the bubble is some multiple of the actual radius, we can say that, if r is the radius of the bubble, dr/dt = a constant.

Is this what we expect? Can we use related rates to investigate this? Let V be the volume of the bubble, assumed to be a sphere.

$$\frac{dr}{dt} = \frac{dV}{dt} \Big/ \frac{dV}{dr} \text{ (from chain rule),}$$

and we now need to work out dV/dt and dV/dr.

We could assume that the rate of change of the volume with time, dV/dt, is proportional to the surface area of the bubble, as the bubble decreases in size because of its contact with the surrounding liquid. So

$$\frac{dV}{dt} = C.4\pi r^2,$$

where C is a constant.

Then, we have $V = \frac{4}{3}\pi r^3$, so $dV/dr = 4\pi r^2$.

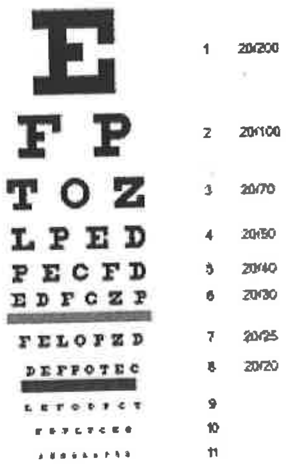
Putting this together,

$$\begin{aligned}\frac{dr}{dt} &= \frac{dV}{dt} \Big/ \frac{dV}{dr}\\ &= \frac{C.4\pi r^2}{4\pi r^2}\\ &= C.\end{aligned}$$

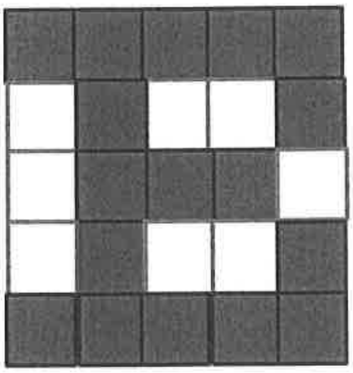
This is what was found!

MATHS ITEM 13. How is visual acuity measured?

In time I was fortunate enough to make a full recovery. New glasses were required, so I visited the optometrist to be measured for new lenses. You may be familiar with this eye chart, called a *Snellen Chart*. It was invented by the Dutch ophthalmologist Hermann Snellen in 1862. This illustration is from Wikipedia. The letters on the chart are called *optotypes* and are carefully designed on a 5 by 5 grid.



The thickness of the lines is one-fifth of the size of the character. The optotype can be recognized only if you can discriminate the pattern. 'Standard vision' is defined as being able to recognize an optotype when it subtends 5 minutes of arc at your eye.



More recently, problems with the Snellen Chart have been identified (McGraw et al., 1995), and improvements made. That is a story for another time perhaps!!

References

Hassan HM, Mariatos G, Papanikolaou T, Ranganath A, Hassan H. (2012). Ocular complications of bungee jumping. Clin Ophthalmol. 6:1619-22. doi: 10.2147/OPTH.S33169. Epub 2012 Oct 4. PMID: 23055687; PMCID: PMC3468282.

McGraw, P., Winn, B., & Whitaker, D. (1995). Reliability of the Snellen chart. Bmj, 310(6993), 1481-1482.

BOOK REVIEWS

Math Curse (1995)

By Jon Scieszka and Lane Smith
(Viking: New York)

Reviewed by Tricia Forrester
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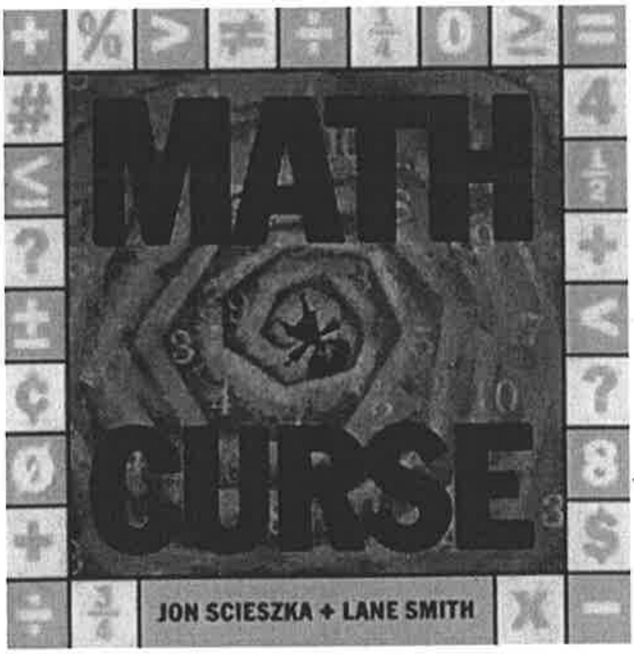

https://www.booktopia.com.au/math-curse-jon-scieszka/book/9780670861941.html

This engaging picture-book is about a girl in a humorously manic grip of a maths curse put on her when her teacher, Mrs Fibonacci, says, 'YOU KNOW, you can think of almost everything as a maths problem'. Indeed, try as she might, she can't think of anything that isn't a maths problem! Waking up, getting ready for school, eating breakfast, catching the bus, the classroom, her classmates, recess, English, social studies, lunch, phys ed. art, birthday cupcakes . . . there are maths problems everywhere! The heroine is becoming a raving maths lunatic.

Can the curse be broken?

To find out, read it to your students Stages 2–4. It's a great introduction to identifying mathematics in their lives and as a basis for mathematical investigation in all strands (but if you haven't got that excuse, it's good enough to read for your own enjoyment). The only drawbacks are those associated with its target audience being American, but none of those are insurmountable and might well offer opportunities for further investigations.

WARNING: The maths curse has taken hold of the book itself and the reader runs the risk of becoming a maths lunatic too! I found myself reading all the details of the publication that usually go unnoticed: The date of publication ('1990 + 5'), its target audience ('for ages > 6 and < 99'), its price tag "' $'[(\$3{\cdot}75+\$1.75)\times 2]-\$0.01=\$9.99\text{ USA}'$, and even its dedications ('If the sum of my nieces and nephews equals 15, and their product equals 54, and I have more nephews than nieces, how many nephews and how many nieces is this book dedicated to?').