Optimal Scaling Quantum Linear-Systems Solver via Discrete Adiabatic Theorem

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Recently, several approaches to solving linear systems on a quantum computer have been formulated in terms of the quantum adiabatic theorem for a continuously varying Hamiltonian. Such approaches have enabled near-linear scaling in the condition number κ of the linear system, without requiring a complicated variable-time amplitude amplification procedure. However, the most efficient of those procedures is still asymptotically suboptimal by a factor of $\log(\kappa)$. Here, we prove a rigorous form of the adiabatic theorem that bounds the error in terms of the spectral gap for intrinsically discrete-time evolutions. In combination with the qubitized quantum walk, our discrete adiabatic theorem gives a speed-up for all adiabatic algorithms. Here, we use this combination to develop a quantum algorithm for solving linear systems that is asymptotically optimal, in the sense that the complexity is strictly linear in κ, matching a known lower bound on the complexity. Our $\mathcal{O}[\kappa \log(1/\epsilon)]$ complexity is also optimal in terms of the combined scaling in κ and the precision ϵ. Compared to existing suboptimal methods, our algorithm is simpler and easier to implement. Moreover, we determine the constant factors in the algorithm, which would be suitable for determining the complexity in terms of gate counts for specific applications.

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I. INTRODUCTION

Finding the solution to a system of linear equations is a fundamental task that underlies many areas of science and technology. A classical linear-systems solver takes time proportional to the number of unknown variables even to write down the solution and thus has a prohibitive computational cost for solving large linear systems. However, a quantum computer with a suitable input access can produce a quantum state that encodes the problem solution much faster than its classical counterpart. The first quantum algorithm for the quantum linear-systems problem (QLSP) was proposed by Harrow, Hassidim, and Lloyd (HHL) [1] and has been subsequently refined by later work. Due to the ubiquitous nature of the problem, quantum algorithms for QLSP have found a variety of applications, such as computing electromagnetic scattering [2], solving differential equations [3,4], data fitting [5], machine learning [6,7], and more general solution of partial differential equations [8].

Specifically, the goal of QLSP is to produce a quantum state $|x\rangle$ proportional to the solution of linear system $Ax = b$, where A is an N-by-N matrix. The complexity of solving QLSP depends on various input parameters, such as the problem size N, the sparsity (for sparse linear systems), the norm of the coefficient matrix A, the condition number κ, and the error ϵ in the solution. To simplify the discussion, we assume that $\|A\| = 1$ and hence $\|A^{-1}\| = \kappa$, where $\|\cdot\|$ denotes the spectral norm. To further simplify the analysis, we assume that we have a block encoding of the coefficient matrix A and a given operation to prepare the target vector $|b\rangle$ and consider the number of queries to these oracles. One can also consider the complexity in terms of the number of calls to entries of a sparse matrix, as in Ref. [1], but there are standard methods to block encode sparse matrices [9], so our result can be easily applied to that case. These simplifications mean that the only relevant parameters on which our algorithm depends are κ and ϵ.

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The original algorithm proposed by HHL has a complexity scaling quadratically with the condition number κ and linearly with the inverse accuracy $1/\epsilon$ [1]. The scaling with the condition number was improved by Ambainis using "variable-time amplitude amplification" [10]; the resulting algorithm has a near-linear dependence on κ but a much worse dependence on $1/\epsilon$. A further improvement was provided in Ref. [11], which yields a complexity logarithmic in the allowable error ϵ using a linear combination of unitaries (LCU). Unfortunately, algorithms based on variable-time amplitude amplification [10,11] perform multiple rounds of recursive amplitude amplifications and can be challenging to implement in practice.

To address this, recent work has suggested alternative approaches based on adiabatic quantum computing (AQC). AQC is a universal model of quantum computing that has been shown to be polynomially equivalent to the standard gate model [12,13]. In AQC, one encodes the solution to a computational problem in the ground state of a Hamiltonian H_1. Then, one initializes a quantum system in the ground state of an easy-to-prepare Hamiltonian H_0 and slowly deforms from the ground state of H_0 to the ground state of H_1 under a time-dependent Hamiltonian that interpolates between the two, such as $H(s) = (1-s)H_0 + sH_1$. The advantage of using the adiabatic approach to solve QLSP as in Ref. [14] is that it naturally provides complexity close to linear in κ, without the highly complicated variable-time amplitude amplification procedure. That work has been further improved in Ref. [15] and then Ref. [16], which gives complexity logarithmic in ϵ by using eigenstate filtering. We summarize key developments reducing the complexity in Table I.

It is known that a quantum algorithm must make at least $\Omega[\kappa\log(1/\epsilon)]$ queries in general to solve the sparse QLSP problem [17]. Therefore, the method in Ref. [16] is already optimal in the scaling with solution accuracy ϵ. However, a question left open was: how can we achieve an optimal scaling with the condition number κ or is it possible to prove a lower bound ruling out this improvement? From the algorithmic perspective, finding a quantum algorithm with linear κ scaling is technically challenging. Previous fast linear systems solvers depend on polynomial approximations to implement the inverse function $1/x$ on $x \in [1/\kappa, 1]$ [11] or truncations of the Dyson series to implement the continuous adiabatic evolution [14–16]. In either case, an extra polylog(κ) factor is required to suppress the truncation or approximation error, resulting in a superlinear scaling with the condition number.

In this work, we develop a quantum algorithm for solving systems of linear equations with complexity $\mathcal{O}[\kappa\log(1/\epsilon)]$. That is, we achieve a strictly linear scaling with κ, while maintaining the logarithmic scaling with $1/\epsilon$ from the best previous algorithms. Combining with the lower bound of Ref. [17], we establish for the first time a quantum linear-systems algorithm with optimal scaling in the condition number. It is also optimal in the combined scaling with κ and ϵ, because one cannot, for example, reduce the scaling to $\mathcal{O}[\kappa + \log(1/\epsilon)]$. We formally state our result in Sec. V and preview it here.

Theorem: (QLSP with linear dependence on κ). *Let $Ax = b$ be a system of linear equations, where A is an N-by-N matrix with $\|A\| = 1$ and $\|A^{-1}\| = \kappa$. Given an oracle block encoding the operator A and an oracle preparing $|b\rangle$, there exists a quantum algorithm that produces the normalized state $|A^{-1}b\rangle$ to within error ϵ in terms of the ℓ^2 norm, using an average number*

$$\mathcal{O}[\kappa\log(1/\epsilon)] \tag{1}$$

of oracle calls.

Our algorithm is conceptually simple and easy to describe. All it requires is a sequence of quantum walk steps corresponding to a qubitized form of the Hamiltonian used in prior work [15,16]. It completely avoids the heavy mechanisms of variable-time amplitude amplification or the truncated Dyson-series subroutine from

TABLE I. The history of the lowest-scaling algorithms for solving linear systems of equations on a quantum computer. Specifically, the problem is to prepare the state $|x\rangle$ given oracular access to the matrix A and the ability to prepare the initial state $|b\rangle$ encoding a vector b with the relation $Ax = b$. Here, κ is the condition number of A and ϵ is the target precision to which we would like to prepare the state $|x\rangle$. However, the cost of a query for all classical algorithms is expected to scale polynomially in N (the dimension of the matrix A), whereas on a quantum computer it is possible to make queries in complexity scaling as $\mathcal{O}[\text{polylog}(N)]$ when A is a sparse matrix. The query complexity of $\Omega[\kappa\log(1/\epsilon)]$ is a known lower bound on the complexity.

Year	Reference	Primary innovation	Query complexity
2008	Harrow, Hassidim, and Lloyd [1]	First quantum approach	$\mathcal{O}(\kappa^2/\epsilon)$
2012	Ambainis [10]	Variable-time amplitude amplification	$\mathcal{O}(\kappa(\log(\kappa)/\epsilon)^3)$
2017	Childs, Kothari, and Somma [11]	Fourier or Chebyshev fitting using LCU	$\mathcal{O}[\kappa\ \text{polylog}(\kappa/\epsilon)]$
2018	Subasi, Somma, and Orsucci [14]	Adiabatic randomization method	$\mathcal{O}((\kappa\log\kappa)/\epsilon)$
2019	An and Lin [15]	Time-optimal adiabatic method	$\mathcal{O}[\kappa\ \text{polylog}(\kappa/\epsilon)]$
2019	Tong and Lin [16]	Zeno eigenstate filtering	$\mathcal{O}[\kappa\log(\kappa/\epsilon)]$
2022	This paper	Discrete adiabatic theorem	$\mathcal{O}[\kappa\log(1/\epsilon)]$

previous methods. Moreover, we provide a bound on the constant prefactor for our approach that allows estimation of the complexity in terms of the number of gates for specific applications. We expect that our estimate of the prefactor can be tightened and that our algorithm will be the most efficient for the early fault-tolerant regime of quantum computation as well as having the best asymptotic scaling for large-scale applications.

The new insight that allows us to establish the optimal κ scaling is the use of a discrete quantum adiabatic theorem, a result proved by Dranov, Kellendonk, and Seiler (DKS) [18]. Unlike the continuous version, the discrete adiabatic theorem is formulated based on a quantum walk operator $W(s)$. Provided that the steps of quantum walk vary sufficiently slowly, the eigenstates of the walk operator can be approximately preserved throughout the entire discrete adiabatic evolution. Indeed, DKS have shown that the error in the evolution scales as $\mathcal{O}(1/T)$ for T steps of the walk. However, their analysis overlooks the scaling with other parameters; in particular, the spectral-gap dependence. In the case of solving QLSP, the gap depends on κ, so the result of DKS is not sufficient to give the κ dependence of the algorithm. Here, we give a complete analysis of the discrete adiabatic theorem, keeping track of all the parameters of interest while fixing several minor mistakes in the original proof.

In developing our quantum linear-systems algorithm, we provide an improved method of filtering the final state that may be of independent interest. Prior methods have been based on singular-value processing [16], which requires a sequence of rotations to be found by a numerically demanding procedure [19–22].

Our method has two advantages; the sequence of operations needed is easily determined by an analytic formula and the efficiency is improved because an incorrect measurement result can be detected early. Thus, including the gap dependence together with the replacement of the asymptotic scaling by strict bounds over the total time in Ref. [18] allows us to use the qubitized quantum walk. Our discrete adiabatic theorem thus avoids the application of the truncated Dyson series for the time evolution, which is used in Refs. [15,16] and gives the extra logarithmic factor in the complexity. Finally, combining our discrete evolution with the improved eigenstate filtering gives our result on the solution of linear systems.

The remainder of the paper is organized as follows. In the following, we give more detailed background and summarize our result in Sec. II. Then, in Sec. III, we give a thorough proof of the discrete adiabatic theorem. We base our method on the approach of DKS but make many of the details rigorous and provide a strict bound on the error, including constant factors. We apply the discrete adiabatic theorem to the QLSP in Sec. IV. In Sec. V, we provide our general method of filtering, which is just as efficient as that based on singular-value processing.

II. DISCRETE ADIABATIC THEOREMS

A. Background

Before presenting our results, let us present the main ideas of the DKS bound on the error in discrete-time adiabatic evolution [18]. In this proposal, the model of the adiabatic evolution is based on a sequence of T walk operators $\{W(n/T) : n \in \mathbb{N}, 0 \le n \le T-1\}$. That is, the system is initially prepared in a state $|\psi_0\rangle$, then the sequence of unitary transformations $W(n/T)$ have the effect $|\psi_0\rangle \mapsto |\psi_1\rangle \mapsto \cdots$. To model this evolution, with $s = n/T$ we can write

$$U(s) = \prod_{m=0}^{sT-1} W(m/T) \tag{2}$$

and $U(0) \equiv I$, such that $|\psi_n\rangle = U(s)|\psi_0\rangle$. The adiabatic limit is then the limit $T \to \infty$. Alternatively, we can construct the total unitary evolution recursively as

$$U(s+1/T) = W(s)U(s), \quad U(0) = I. \tag{3}$$

The adiabatic limit is then the limit $T \to \infty$. For the purpose of quantum algorithm design, we are trying to choose U so that $\lim_{T\to\infty} U(1)|\psi_0\rangle = |\psi_{\text{target}}\rangle$, where $|\psi_{\text{target}}\rangle$ is a desired "target" state that enables us to solve a computational problem. In order for this to be an accurate adiabatic evolution yielding the target state, $U(n/T)|\psi_0\rangle$ should be approximately an eigenstate of $W(n/T)$ for all n.

We need to establish some terminology before we can present the statement of the result from Ref. [18]. For each $T \in \mathbb{R}$ and $n \in \mathbb{N}$, introduce a projector $P(s)$ (with $s \equiv n/T$ as before) called the *spectral projection*, which projects onto the eigenspace of interest. In addition, the *complementary spectral projection* $Q(s) = I - P(s)$ projects onto all eigenvectors orthogonal to the eigenspace of interest. An operator representing the ideal adiabatic evolution is denoted $U_A(s)$. The ideal adiabatic evolution is that where each eigenvector of the walk operator remains an eigenvector of the walk operator throughout the evolution. That implies

$$P(s) = U_A(s)P(0)U_A^\dagger(s). \tag{4}$$

That is, evolving the original eigenspace to step $n = sT$ under the ideal adiabatic evolution gives the corresponding eigenspace for the walk operator $W(s)$.

The adiabatic theorem is a statement about how close the evolution $U(s)$ is to the ideal adiabatic evolution $U_A(s)$ at a given time. Beginning with the initial state $|\phi(0)\rangle$ in the subspace of interest so $(P(0)|\phi(0)\rangle = |\psi(0)\rangle)$, the goal is to bound the *error* between U_T and its ideal evolution U_A

by an expression of the form

$$\|(U(s) - U_A(s))\,|\phi(0)\rangle\| \leq \|U(s) - U_A(s)\| \leq \frac{\theta}{T}, \quad (5)$$

where $\|\cdot\|$ is the spectral norm. Proving this result shows that increasing the number of steps reduces the error. The constant θ in Eq. (5) is a constant independent of the total time T but depends on the gap $\Delta(s)$ between the eigenspace of interest and the complementary eigenspace.

In Ref. [18], it has been shown that the error is $\mathcal{O}(1/T)$, which means that there exists some constant θ but that constant and its dependence on the gap have not been determined. That is a crucial difficulty in applying the result to the QLSP, because the gap in using the adiabatic approach to the QLSP depends on the condition number κ. Therefore, to determine the complexity of the algorithm in terms of κ, we need to know the dependence of the error on the gap. In particular, we show that the error scales as $\mathcal{O}(\kappa/T)$, which means that to obtain the solution to fixed error one can use $T = \mathcal{O}(\kappa)$ steps. Then, complexity linear in κ and logarithmic in $1/\epsilon$ can be obtained using filtering. To show this result, we cannot simply use the result as given [18] and we need to derive the bound for the error far more carefully in order to give the dependence on the gap.

B. Our result

Our main goal in this paper is to provide the explicit dependence on the gap in the discrete adiabatic theorem in order to improve the version given in Ref. [18]. In order to do this, we need to replace a number of initial assumptions that have just been given as order scalings in Ref. [18] and to properly account for the gap when using consecutive walk operators. We then work through the proof to give a strict bound on the error in the adiabatic evolution with all constant factors.

First, in Ref. [18] it has just been assumed that there is the general order scaling

$$W(s + 1/T) - W(s) \approx \mathcal{O}(T^{-1}). \quad (6)$$

We replace that with an upper bound with explicit schedule dependence,

$$\|W(s + 1/T) - W(s)\| \leq \frac{c(s)}{T}. \quad (7)$$

Implicit in this definition is the assumption that the behavior of $W(s)$ is sufficiently smooth that $c(s)$ can be chosen independently of T. This needs to be shown for the given applications. More generally, we need to consider higher-order differences, which result in values of $c_k(s)$ given in the following definition.

Definition 1: (multistep differences). For a positive integer k, the k^{th} difference of W is

$$D^{(k)}W(s) := D^{(k-1)}W\left(s + \frac{1}{T}\right) - D^{(k-1)}W(s),$$
$$D^{(1)}W(s) := DW(s) = W\left(s + \frac{1}{T}\right) - W(s). \quad (8)$$

For $T > 0$, we define the function $c_k(s)$, which is implicitly dependent of T, such that

$$\left\|D^{(k)}W(s)\right\| \leq \frac{c_k(s)}{T^k}. \quad (9)$$

We then define the $\hat{c}_k(s)$ taking into account neighboring steps as

$$\hat{c}_k(s) = \max_{s' \in \{s-1/T, s, s+1/T\} \cap [0, 1-k/T]} c_k(s'). \quad (10)$$

The principle of the gap is that it separates the eigenvalues of $W(s)$ into two groups that depend on the time parameter s. Since $W(s)$ is unitary, these are groups on the unit circle in the complex plane. Because it is on the unit circle, we need to separate these groups of eigenvalues with gaps in two locations. We denote one set of eigenvalues as $\sigma_P(s)$ and the other as $\sigma_Q(s)$, with corresponding projectors $P(s)$ and $Q(s)$, respectively. That is, $P(s)$ projects onto the subspace where the eigenvalues of $W(s)$ are in the set $\sigma_P(s)$. We call $\sigma_P(s)$ the "spectrum of interest" because we are concentrating on applications where we attempt to maintain a state within this subspace.

We also need to account for the gaps for successive operators $W(s)$ and $W(s + 1/T)$. That is, there needs to be a gap between $\sigma_P(s) \cup \sigma_P(s + 1/T)$ and $\sigma_Q(s) \cup \sigma_Q(s + 1/T)$. Moreover, we need to ensure that these regions are noninterleaved. The reason is that in order to place a bound on the error, we need to place bounds on the norms of a long sequence of operators, which depend on the difference of projection operators $P(s)$ at successive steps. This difference can be bounded using a contour integral but the contour must simultaneously pass through the gaps for two successive walk operators.

To ensure that the regions are not interleaved, we define arcs that contain the eigenvalues, such that

$$\sigma_P^{(1)} \supseteq \sigma_P(s) \cup \sigma_P(s + 1/T),$$
$$\sigma_Q^{(1)} \supseteq \sigma_Q(s) \cup \sigma_Q(s + 1/T). \quad (11)$$

To rule out interleaved regions, these arcs cannot intersect and we consider the gap between these arcs. We are interested in the case where this only has a small effect on the gap. In turn, this means that T should not be too large, so we introduce a lower bound T^* on the values of T allowed. We therefore define the multistep gaps as follows.

Definition 2: (multistep gap). For $T \in \mathbb{N}$ and k a nonnegative integer, $\Delta_k(s)$ is defined to be the minimum angular distance between arcs $\sigma_P^{(k)}$ and $\sigma_Q^{(k)}$, which satisfy

$$\sigma_P^{(k)} \supseteq \bigcup_{l=0}^{k} \sigma_P(s+l/T), \qquad \sigma_Q^{(k)} \supseteq \bigcup_{l=0}^{k} \sigma_Q(s+l/T). \tag{12}$$

The gap $\Delta(s)$, which is also implicitly dependent on T, is then in most cases the minimum gap for three successive steps, except in the cases at the boundaries:

$$\Delta(s) = \begin{cases} \Delta_2(s), & 0 \le s \le 1-2/T, \\ \Delta_1(s), & s = 1-1/T, \\ \Delta_0(s), & s = 1. \end{cases} \tag{13}$$

Finally, $\check{\Delta}(s)$ is an adjustment for $\Delta(s)$ at neighboring points:

$$\check{\Delta}(s) = \min_{s' \in \{s-1/T, s, s+1/T\} \cap [0,1]} \Delta(s'). \tag{14}$$

Note that we have freedom to choose larger arcs than necessary, so these are lower bounds on the gap, though we often call them the "gap" for convenience. Also, given $\Delta_2(s)$, one can always choose arcs $\sigma_P^{(k)}$ and $\sigma_Q^{(k)}$ for $k = 0, 1$ such that $\Delta_k(s) \ge \Delta_2(s)$. This means that in Eq. (13), we can simply use $\Delta_2(s)$, rather than taking the minimum of $\Delta_k(s)$ for $k \in \{0, 1, 2\}$.

We prove two forms of the discrete adiabatic theorem. One is highly complicated, so we give it explicitly later in Sec. III B. Here, we instead give a simplified but looser form of the discrete adiabatic theorem.

Theorem 3: (the second discrete adiabatic theorem). *Suppose that the operators $W(s)$ satisfy $\left\|D^{(k)}W(s)\right\| \le c_k(s)/T^k$ for $k = 1, 2$, as per Eq. (9), and $T \ge \max_{s\in[0,1]} [4\hat{c}_1(s)/\check{\Delta}(s)]$. Then, for any time s, such that $sT \in \mathbb{N}$, we have*

$$\begin{aligned} \|U(s) - U_A(s)\| \le{}& \frac{12\hat{c}_1(0)}{T\check{\Delta}(0)^2} + \frac{12\hat{c}_1(s)}{T\check{\Delta}(s)^2} + \frac{6\hat{c}_1(s)}{T\check{\Delta}(s)} \\ &+ 305 \sum_{n=1}^{sT-1} \frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^3} \\ &+ 44 \sum_{n=0}^{sT-1} \frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^2} \\ &+ 32 \sum_{n=1}^{sT-1} \frac{\hat{c}_2(n/T)}{T^2\check{\Delta}(n/T)^2}, \end{aligned} \tag{15}$$

where $\hat{c}_k(s)$ and $\check{\Delta}(s)$ are defined in Definitions 1 and 2, respectively.

Note that this theorem depends on the first and second differences, described by $\hat{c}_1(s)$ and $\hat{c}_2(s)$, respectively. These are analogous to the first and second derivatives in the continuous form of the adiabatic theorem, so we can see that these results are analogous. We have three single terms with $1/T$ scaling and three sums with $1/T^2$ scaling, which gives overall scaling of the complexity as $\mathcal{O}(1/T)$. We also have a cubic dependence in the inverse gap $1/\Delta$ in the first sum given. In choosing the quantum walk, one would aim to schedule the variation of W such that they vary more slowly where the gap is small, making $\hat{c}_1$ smaller. The condition $T \ge \max_{s\in[0,1]}[4\hat{c}_1(s)/\check{\Delta}(s)]$ is implicit, because the definitions of $\hat{c}_1(s)$ and $\check{\Delta}(s)$ depend on T. In practice, the right side is only weakly dependent on T and we ensure that this condition is satisfied where we use this theorem.

III. THE ADIABATIC THEOREMS

In this section, we prove our first form of the discrete adiabatic theorem, given later as Theorem 7, and then use it to prove Theorem 3. Following the general method and notation of Ref. [18], we use the *wave operator*

$$\Omega(s) := U_A^\dagger(s)U(s). \tag{16}$$

The aim of the discrete adiabatic theorem, the first and the second, is to prove that $\Omega(s)$ is close to the identity because

$$\|U(s) - U_A(s)\| = \left\|U_A^\dagger(s)U(s) - I\right\| = \|\Omega(s) - I\|. \tag{17}$$

In Ref. [18], it has been shown that $\Omega(s) = I + \mathcal{O}(1/T)$ but we instead aim to provide the explicit bounds dependent on the gap.

To prove the bound, one can define a *kernel function* $K(s)$ as well, which corresponds to the difference of a single step of $\Omega(s)$ from the identity. The wave operator at step n is then given by

$$\Omega(n/T) = I - \frac{1}{T}\sum_{m=0}^{n-1} K(m/T)\Omega(m/T). \tag{18}$$

The goal is then to show that the sum is small. This is done with a summation-by-parts formula. The general principle of a summation-by-parts formula is that it transforms a difference in one function in the sum into a difference in the other function. Here, $K(m/T)$ is essentially in the form of a difference and our summation-by-parts formula gives a sum where the difference is in $\Omega(m/T)$ instead. Although the expression is still in the form of a sum, the difference in $\Omega(m/T)$ is small enough that it is possible to usefully bound the overall sum by bounding every term and using the triangle inequality. That then yields our first discrete adiabatic theorem.

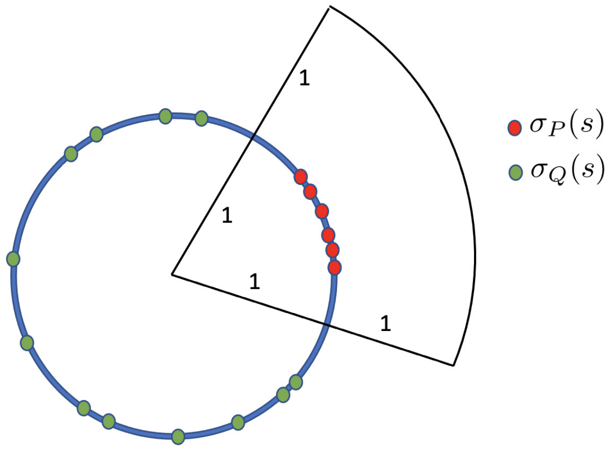

FIG. 1. An illustration of the choice of the contour $\Gamma(s,k)$ for $k=0$. That is, we consider the eigenvalues for only a single step of the walk. The red dots indicate the spectrum of interest, which will often just be a single eigenvalue; for example, for a ground state. The contour around the spectrum of interest is used to obtain a projector onto the spectrum of interest. For the illustration, we use a contour with radius 2 but in practice, we take the limit that the radius goes to infinity.

A. Operator definitions

We next define the operators that are needed to understand the proof. Let $\Gamma(s)$ be a sector contour enclosing the spectrum of interest (see, e.g., Fig. 1). Then, the spectral projection $P(s)$ onto the spectrum of interest is given by the integral

$$P(s) = \frac{1}{2\pi i}\oint_{\Gamma(s)} R(s,z)dz, \tag{19}$$

where

$$R(s,z) := (W(s) - zI)^{-1}, \tag{20}$$

is the resolvent of $W(s)$. Let

$$S(s,s') := P(s)P(s') + Q(s)Q(s'), \tag{21}$$

$$v(s,s') := \sqrt{S(s,s')S^\dagger(s,s')} = \sqrt{I - (P(s) - P(s'))^2}, \tag{22}$$

and

$$V(s,s') := v(s,s')^{-1}S(s,s'), \tag{23}$$

which is the unitary of the left polar decomposition of $S(s,s')$ (see Eq. (11) of Ref. [18]).

The general principle is that $S(s,s')$ uses $P(s)P(s')$ to map the eigenspace of interest for $W(s')$ to that for $W(s)$ and similarly uses $Q(s)Q(s')$ to map the orthogonal eigenspaces. This operator is not unitary but by applying $v(s,s')^{-1}$ we can obtain the unitary operation $V(s,s')$ that also performs this exact mapping of eigenspaces. This provides a unitary operator with the effect of exactly mapping the eigenspaces like the ideal adiabatic evolution but does not apply phase factors like the walk operator $W(s)$. If we apply $V(s+1/T,s)W(s)$, then we obtain the phase factors from $W(s)$ and then $V(s+1/T,s)$ maps the eigenspaces to those for $W(s+1/T)$. This is then a good description of an ideal adiabatic step, denoted $W_A(s)$, that has eigenvalues similar to $W(s)$ but perfectly maps the eigenspaces. The ideal adiabatic evolution $U_A(s)$ can then be constructed from a sequence of these walk steps. Note that this ideal adiabatic evolution is never applied in reality; it is a purely theoretical construct to quantify how close the actual evolution is to adiabatic.

We use the shorthand notations $S(s) = S(s+1/T,s)$, $v(s) = v(s+1/T,s)$, $V(s) = V(s+1/T,s)$, and define (see Eqs. (7) and (10) of Ref. [18])

$$W_A(s) := V(s)W(s), \tag{24}$$

$$U_A(s+1/n) := W_A(s)U_A(s), \tag{25}$$

$$U_A(0) := I. \tag{26}$$

It can be checked from the definition that $V(s)$ is a unitary operator and thus W_A and U_A are unitary. In fact, W_A is exactly the adiabatic walk operator and the corresponding U_A is the corresponding adiabatic evolution operator.

To describe the proof, we use the *wave operator*

$$\Omega(s) := U_A^\dagger(s)U(s), \tag{27}$$

which describes the difference between the actual evolution given by $U(s)$ and the ideal adiabatic evolution $U_A(s)$. To demonstrate that the evolution is close to adiabatic, we should have $\Omega(s)$ close to I. The *ripple operator* is defined as

$$\Theta(s) := \Omega(s+1/T)\Omega^\dagger(s), \tag{28}$$

so the wave operator is a product of ripple operators for each time step. The kernel function is defined as

$$K(s) := T[I - \Theta(s)] \tag{29}$$

and should be close to zero for the evolution to be close to adiabatic. It can be seen here that $K(s)$ is in the form of a difference.

Now, we provide some properties of the operators from Ref. [18]. But first, we give properties of the adiabatic operators and the projectors onto the subspaces, with the proofs presented in Appendix B.

Proposition 4: *For any integers T, n, and m, and the corresponding discrete times $s = n/T$ and $s' = m/T$, we have* *that $W_A(s)$ and $U_A(s)$ are unitary and*

$$P(s+1/T)W_A(s) = W_A(s)P(s), \tag{30}$$

$$U_A(s)P(0) = P(s)U_A(s), \tag{31}$$

$$P(s+1/T)W(s)P(s) = P(s+1/T)v(s)W_A(s)P(s), \tag{32}$$

$$Q(s+1/T)W(s)Q(s) = Q(s+1/T)v(s)W_A(s)Q(s). \tag{33}$$

Here, Eqs. (30) and (31) are the key properties showing that $W_A(s)$ is a true adiabatic walk step. That is, projecting onto the desired subspace at the beginning is the same as projecting onto the desired subspace at the end after the walk, so the subspace must have been preserved.

Next, we consider properties of the *wave operator* $\Omega(s)$, the *ripple operator* $\Theta(s)$, and the kernel function $K(s)$. One can simply prove that the ripple operator is a rotation of the operator V and Ω satisfies a discrete form of the Volterra equation. The key results are as in the following proposition, which is equivalent to Eqs. (19) and (20) from Ref. [18], and proofs are also given in Appendix B.

Proposition 5: *For any integers T and n and the discrete time $s = n/T$, we have*

$$\Theta(s) = U_A^\dagger(s+1/T)V^\dagger(s)U_A(s+1/T) \tag{34}$$

and the Volterra equation

$$\Omega(n/T) = I - \frac{1}{T}\sum_{m=0}^{n-1} K(m/T)\Omega(m/T). \tag{35}$$

B. Summation by parts and discrete adiabatic theorem

In order to show that the evolution is close to adiabatic, we aim to show that $\Omega(n/T)$ is close to the identity, and to do that we use the expression given in Eq. (35). In the sum, we substitute the identity being equal to the sum of projections onto the desired subspace and the orthogonal subspace. That gives us four sums. Two of these are "diagonal" sums with two projections onto the same subspace and two are "off-diagonal" sums with two different projections.

The diagonal sums are relatively easily bounded, whereas for the off-diagonal sums are more difficult. For those, we use the "summation-by-parts formula" given below in Theorem 6.

Theorem 6: (summation-by-parts formula). *Let $W(s)$, $s \in \mathbb{Z}/T$, be a sequence of unitaries, let $U_A(s)$ be the corresponding ideal adiabatic evolution, let $P(s)$ be a projection onto an eigenspace of $W(s)$, let $\Gamma(s)$ be a contour around the eigenvalues corresponding to $P(s)$, $Q(s) = I - P(s)$, and suppose that $X(s)$ and $Y(s)$ are sequences of operators. Then,*

$$\sum_{n=1}^{l} Q_0 U_A^\dagger\left(\tfrac{n}{T}\right) X\left(\tfrac{n}{T}\right) U_A\left(\tfrac{n}{T}\right) P_0 Y\left(\tfrac{n}{T}\right) = \mathcal{B} - \frac{1}{T}\mathcal{S}, \tag{36}$$

where $P_0 = P(0)$ and $Q_0 = Q(0)$,

$$\begin{aligned}\mathcal{B} = {} & Q_0 U_A^\dagger\left(\tfrac{l}{T}\right) \tilde{X}\left(\tfrac{l_+}{T}\right) U_A\left(\tfrac{l_+}{T}\right) P_0 Y\left(\tfrac{l_+}{T}\right) \\ & - Q_0 U_A^\dagger(0)\tilde{X}\left(\tfrac{1}{T}\right) U_A\left(\tfrac{1}{T}\right) P_0 Y\left(\tfrac{1}{T}\right)\end{aligned} \tag{37}$$

is the boundary term,

$$\begin{aligned}\mathcal{S} = {} & \sum_{n=1}^{l} Q_0 U_A^\dagger\left(\tfrac{n}{T}\right) \left(Z\left(\tfrac{n}{T}\right) U_A\left(\tfrac{n}{T}\right) P_0 Y\left(\tfrac{n}{T}\right)\right. \\ & \left. + \tilde{X}\left(\tfrac{n_+}{T}\right) W_A\left(\tfrac{n}{T}\right) U_A\left(\tfrac{n}{T}\right) P_0 T\, DY\left(\tfrac{n}{T}\right)\right)\end{aligned} \tag{38}$$

is the sum, and

$$\tilde{X}(s) := -\frac{1}{2\pi i}\oint_{\Gamma(s)} R(s,z)X(s)R(s,z)dz, \tag{39}$$

$$A(s) := W(s) - W_A(s) = \left(V^\dagger(s) - I\right) W_A(s), \tag{40}$$

$$B(s) := D\tilde{X}(s)W_A(s) + DW_A(s-1/T)\tilde{X}(s), \tag{41}$$

$$Z(s) := T\left(\left[A(s), \tilde{X}(s)\right] + B(s)\right). \tag{42}$$

This formula is given in Theorem 1 of Ref. [18], with a typographical error in the sign of both operators $\mathcal{S}$ and $\mathcal{B}$. Here, we correct the sign slightly differently for the two quantities, taking $\mathcal{B}$ to be the negative of the $\mathcal{B}$ defined in Ref. [18] and taking $\mathcal{S}$ to be the same but placing a minus sign in the statement of the theorem (so there is $\mathcal{B} - \mathcal{S}/T$). Throughout the lemma and its proof, we encounter slight shifts of the discrete time very frequently. To simplify the notation, for any positive integer n, we define $n_+ = n + 1$ and $n_- = n - 1$. As we are making a correction to the theorem and it is quite lengthy, we give a proof in Appendix C.

The summation-by-parts formula is given for arbitrary operator sequences $X(s)$ and $Y(s)$, but when applied to the proof of the discrete adiabatic theorem these are taken to be $T(I - V^\dagger)$ and Ω, respectively. Note from Eq. (34) that $X(s)$ is unitarily related to $K(s)$ and is in the form of a difference. Moreover, we use Ω for $Y(s)$ in Eq. (38) and there it is given in the form of a difference. The form of the summation-by-parts formula is somewhat more subtle than this, though, because we also have the first term in Eq. (38), which does not have this intuitive interpretation.

In our proof, we find that $DY(s)$ is then sufficiently small that we can usefully bound the entire sum using the triangle inequality. To bound the norms of the terms, we need to prove a sequence of bounds on the operators and their differences at successive steps, starting from the simplest, such as P, and working toward the most complicated, such as $DY(s)$ and Z. The core feature of these bounds is that they rely upon taking a contour integral between the groups of eigenvalues for successive steps of the walk. Bounding these operators then enables us to provide the complete explicit form of the discrete adiabatic theorem.

Theorem 7: (the first discrete adiabatic theorem). *Let $U(s)=\prod_{l=0}^{sT-1}W(l/T)$ for $s\in\mathbb{Z}/T$ be a product of unitary operators $W(l/T)$ as per Eq. (2) and let $U_A(s)$ be the corresponding ideal adiabatic evolution that maps an eigenstate of $W(0)$ to the corresponding eigenstate of $W(s)$. Suppose further that the operators $W(s)$ satisfy $\left\|D^{(k)}W(s)\right\|\le c_k(s)/T^k$ for $k=1,2$, as per Definition 1, we consider the gaps $\Delta_k(s)$ as defined in Definition 2, and $T\ge\max_{s\in[0,1]}[2c_1(s)/\Delta_1(s)]$. Then, for any time s, we have*

$$
\begin{aligned}
&\|U(s)-U_A(s)\|\\
&\le\frac{4}{\Delta_0(1/T)}\mathcal{D}_2\left(\frac{2c_1(0)}{T\Delta_1(0)}\right)+\frac{4}{\Delta_0(s)}\mathcal{D}_2\left(\frac{2c_1(s-1/T)}{T\Delta_1(s-1/T)}\right)+2\mathcal{D}_2\left(\frac{2c_1(s-1/T)}{T\Delta_1(s-1/T)}\right)\\
&\quad+\sum_{n=1}^{sT-1}4\left(\frac{1}{\Delta_0(n_+/T)}+\frac{2}{\Delta_0(n/T)}\right)\mathcal{D}_2\left(\frac{2c_1(n/T)}{T\Delta_1(n/T)}\right)\mathcal{D}_2\left(\frac{2c_1(n_-/T)}{T\Delta_1(n_-/T)}\right)\\
&\quad+\sum_{n=1}^{sT-1}\frac{4\mathcal{G}_3(n_-/T)}{T^2\Delta_1(n/T)}+\sum_{n=1}^{sT-1}\frac{4c_1(n/T)}{\pi T[1-\cos(\Delta_1(n/T)/2)]}\mathcal{D}_2\left(\frac{2c_1(n_-/T)}{T\Delta_1(n_-/T)}\right)\\
&\quad+\sum_{n=1}^{sT-1}\frac{4\mathcal{G}_4(n_-/T)}{T\Delta_0(n/T)}\mathcal{D}_2\left(\frac{2c_1(n_-/T)}{T\Delta_1(n_-/T)}\right)+\sum_{n=0}^{sT-1}\frac{24c_1(n/T)^2}{T^2\Delta_1(n/T)^2}+\sum_{n=0}^{sT-1}\frac{4c_1(n/T)^2}{T^2\Delta_1(n/T)^2}\left(1-\frac{2c_1(n/T)}{T\Delta_1(n/T)}\right)^{-1},
\end{aligned}
\tag{43}
$$

where

$$
\begin{aligned}
\mathcal{D}_1(z)&:=\frac{1}{\sqrt{1-z^2}},\quad \mathcal{D}_2(z):=\sqrt{\frac{1+z}{1-z}}-1,\\
\mathcal{D}_3(z)&:=\frac{z}{(1-z^2)^{3/2}},
\end{aligned}
\tag{44}
$$

$$
\mathcal{G}_1(s):=\frac{c_1(s)^2+c_1(s)c_1(s_+)}{\pi[1-\cos(\Delta_2(s)/2)]}+\frac{2c_2(s)}{\Delta_2(s)},
\tag{45}
$$

$$
\mathcal{G}_2(s):=\mathcal{G}_1(s)\mathcal{D}_3\left(\max\left(\frac{2c_1(s+1/T)}{T\Delta_1(s+1/T)},\frac{2c_1(s)}{T\Delta_1(s)}\right)\right),
\tag{46}
$$

$$
\begin{aligned}
\mathcal{G}_3(s)&:=\mathcal{G}_2(s)\left(1+\frac{2c_1(s)}{T\Delta_1(s)}\right)+\mathcal{D}_1\left(\frac{2c_1(s)}{T\Delta_1(s)}\right)\\
&\quad\times\left(\mathcal{G}_1(s)+\frac{8c_1(s)^2}{\Delta_1(s)^2}\right),
\end{aligned}
\tag{47}
$$

$$
\mathcal{G}_4(s):=\frac{\mathcal{G}_3(s)}{T}+c_1(s).
\tag{48}
$$

Proof. Starting from the definition of K and Proposition 5, for any discrete time s,

$$
\begin{aligned}
\|U(s)-U_A(s)\|&=\|\Omega(s)-I\|\\
&=\left\|\frac{1}{T}\sum_{n=0}^{sT-1}K\left(\tfrac{n}{T}\right)\Omega\left(\tfrac{n}{T}\right)\right\|\\
&=\left\|\sum_{n=0}^{sT-1}\left(I-\Theta\left(\tfrac{n}{T}\right)\right)\Omega\left(\tfrac{n}{T}\right)\right\|\\
&=\left\|\sum_{n=1}^{sT}U_A^\dagger\left(\tfrac{n}{T}\right)\left(I-V^\dagger\left(\tfrac{n_-}{T}\right)\right)U_A\left(\tfrac{n}{T}\right)\Omega\left(\tfrac{n_-}{T}\right)\right\|.
\end{aligned}
\tag{49}
$$

We now split the sum into "diagonal" and "off-diagonal" terms, where the "diagonal" ones are those where two projectors of the same type are used and the "off-diagonal" ones are those where two different projectors are used. In the summation-by-parts formula, only the "off-diagonal" term is considered; we use that formula to bound that more difficult term. The splitting of the sum gives

$$
\begin{aligned}
\|U(s)-U_A(s)\|=\Bigg\|&\sum_{n=1}^{sT}(P_0+Q_0){U_A}^\dagger\left(\tfrac{n}{T}\right)\left(I-V^\dagger\left(\tfrac{n_-}{T}\right)\right)\\
&\times U_A\left(\tfrac{n}{T}\right)(P_0+Q_0)\Omega\left(\tfrac{n_-}{T}\right)\Bigg\|
\end{aligned}
\tag{50}
$$

$$\leq \left\| \sum_{n=1}^{sT} P_0 U_A^\dagger\left(\tfrac{n}{T}\right) \left(I - V^\dagger\left(\tfrac{n-}{T}\right)\right) U_A\left(\tfrac{n}{T}\right) P_0 \Omega\left(\tfrac{n-}{T}\right) \right\| \tag{51}$$

$$+ \left\| \sum_{n=1}^{sT} Q_0 U_A^\dagger\left(\tfrac{n}{T}\right) \left(I - V^\dagger\left(\tfrac{n-}{T}\right)\right) U_A\left(\tfrac{n}{T}\right) Q_0 \Omega\left(\tfrac{n-}{T}\right) \right\| \tag{52}$$

$$+ \left\| \sum_{n=1}^{sT} Q_0 U_A^\dagger\left(\tfrac{n}{T}\right) \left(I - V^\dagger\left(\tfrac{n-}{T}\right)\right) U_A\left(\tfrac{n}{T}\right) P_0 \Omega\left(\tfrac{n-}{T}\right) \right\| \tag{53}$$

$$+ \left\| \sum_{n=1}^{sT} P_0 U_A^\dagger\left(\tfrac{n}{T}\right) \left(I - V^\dagger\left(\tfrac{n-}{T}\right)\right) U_A\left(\tfrac{n}{T}\right) Q_0 \Omega\left(\tfrac{n-}{T}\right) \right\|, \tag{54}$$

where Eqs. (51) and (52) are the diagonal and Eqs. (53) and (54) are the off-diagonal components. For the "diagonal" term, it is possible to show that

$$\left\| \sum_{n=1}^{sT} P_0 U_A^\dagger\left(\tfrac{n}{T}\right) \left(I - V^\dagger\left(\tfrac{n-}{T}\right)\right) U_A\left(\tfrac{n}{T}\right) P_0 \Omega\left(\tfrac{n-}{T}\right) \right\|$$
$$\leq \sum_{n=0}^{sT-1} \left\|I - \mathcal{F}\left(\tfrac{n}{T}\right)\right\| \left(1 + \left\|DP\left(\tfrac{n}{T}\right)\right\|\right) + 3\sum_{n=0}^{sT-1} \left\|DP\left(\tfrac{n}{T}\right)\right\|^2. \tag{55}$$

This is shown in Appendix E 1, where the result is given in Eq. (E5). The reasoning for the term with Q_0 is identical and gives the same result. Using Lemma 13, one can show that

$$\|\mathcal{F}(s) - I\| \leq \mathcal{D}_1\left(\frac{2c_1(s)}{T\Delta_1(s)}\right) - 1. \tag{56}$$

The steps for deriving the above bound are given in Eq. (E6) and the function $\mathcal{D}_1$ is defined in Eq. (44). Therefore, one obtains the following bound for the "diagonal" term:

$$\left\| \sum_{n=1}^{sT} P_0 U_A^\dagger\left(\tfrac{n}{T}\right) \left(I - V^\dagger\left(\tfrac{n-}{T}\right)\right) U_A\left(\tfrac{n}{T}\right) P_0 \Omega\left(\tfrac{n-}{T}\right) \right\|$$
$$\leq \sum_{n=0}^{sT-1} \left(\mathcal{D}_1\left(\frac{2c_1(n/T)}{T\Delta_1(n/T)}\right) - 1\right)\left(1 + \frac{2c_1(n/T)}{T\Delta_1(n/T)}\right) + \sum_{n=0}^{sT-1} \frac{12c_1(n/T)^2}{T^2\Delta_1(n/T)^2}$$
$$\leq \sum_{n=0}^{sT-1} \frac{2c_1(n/T)^2}{T^2\Delta_1(n/T)^2}\left(1 - \frac{2c_1(n/T)}{T\Delta_1(n/T)}\right)^{-1} + \sum_{n=0}^{sT-1} \frac{12c_1(n/T)^2}{T^2\Delta_1(n/T)^2}, \tag{57}$$

where in the last inequality we use $[(1-z^2)^{-1/2} - 1](1+z) \leq z^2/[2(1-z)]$ for all $0 \leq z < 1$. The exact same bound holds for the second diagonal term with Q_0 in Eq. (52). The reason is we are treating the eigenspace of interest and the complementary eigenspace completely symmetrically. Therefore, exactly the same bounds hold with P replaced with Q and the above bound must continue to hold.

For the "off-diagonal" term, we can similarly consider only the term with Q_0 on the left and P_0 on the right, as in Eq. (53), and exactly the same bound holds for the other off-diagonal term in Eq. (54). Using Theorem 6 with $X(s) = T[I - V^\dagger(s - 1/T)]$ and $Y(s) = \Omega(s - 1/T)$ (note the slight shift in time), it is possible to show that

$$\left\| \sum_{n=1}^{sT} Q_0 U_A^\dagger\left(\tfrac{n}{T}\right) \left(I - V^\dagger\left(\tfrac{n-}{T}\right)\right) U_A\left(\tfrac{n}{T}\right) P_0 \Omega\left(\tfrac{n-}{T}\right) \right\|$$
$$\leq \frac{1}{T}\left\|\tilde{X}\left(\tfrac{1}{T}\right)\right\| + \frac{1}{T}\left\|\tilde{X}(s)\right\| + \frac{1}{T}\|X(s)\| + \frac{1}{T^2}\sum_{n=1}^{sT-1}\left\|Z\left(\tfrac{n}{T}\right)\right\| + \frac{1}{T}\sum_{n=1}^{sT-1}\left\|\tilde{X}\left(\tfrac{n+}{T}\right)\right\|\left\|DY\left(\tfrac{n}{T}\right)\right\|. \tag{58}$$

See Appendix E 2 for the derivation and see the result in Eq. (E7). We then can derive a sequence of lemmas in order to bound the norms of the operators, which are proven in Appendix D:

(a) In Lemma 12, we bound the norms $\|DP\|$ and $\|D^{(2)}P\|$. The quantity $DP(s)$ is the difference in P at successive time steps and $D^{(2)}P$ is the difference in DP. These quantities are bounded in terms of the bounds on DW and $D^{(2)}W$.
(b) Lemma 13 gives V in terms of P and a new operator $\mathcal{F}$.
(c) Lemma 14 uses $\mathcal{F}$ to place an upper bound on the norm of $V - I$. Because X is $T(I - V^\dagger)$, that enables us to place an upper bound on X.
(d) Lemma 15 provides an upper bound on the norm of $D\mathcal{F}$, which enables us to place an upper bound on DV. This uses the upper bounds on $\|DP\|$ and $\|D^{(2)}P\|$ from Lemma 12.
(e) Lemma 16 places an upper bound on $\|DW_A\|$ using the upper bound on DV from Lemma 15. Recall that W_A is the ideal step for adiabatic evolution.
(f) Lemma 17 places an upper bound on $\|D\Omega\|$ using the upper bound on $\|V - I\|$ from Lemma 14.
(g) Lemma 18 places upper bounds on $\|\tilde{X}\|$ and $\|D\tilde{X}\|$ in terms of $\|X\|$ and $\|DX\|$. Recall that X corresponds to $T(I - V^\dagger)$.
(h) Lemma 19 places upper bounds on the norms of the A, B, and Z operators. The bound on $\|A\|$ uses the bound on $\|V - I\|$ from Lemma 14. The bounds on

$\|B\|$ and $\|Z\|$ use the bounds on $\|\tilde{X}\|$ and $\|D\tilde{X}\|$ from Lemma 18 as well as the bound on $\|DW_A\|$ from Lemma 16. By using these lemmas, we can show that

$$\left\|X\left(\tfrac{n}{T}\right)\right\| \le T\mathcal{D}_2\left(\frac{2c_1(n_-/T)}{T\Delta_1(n_-/T)}\right), \tag{59}$$

$$\left\|\tilde{X}\left(\tfrac{n}{T}\right)\right\| \le \frac{2T}{\Delta_0(n/T)}\mathcal{D}_2\left(\frac{2c_1(n_-/T)}{T\Delta_1(n_-/T)}\right), \tag{60}$$

$$\begin{aligned}\left\|Z\left(\tfrac{n}{T}\right)\right\| \le{}& \frac{4T^2}{\Delta_0(n/T)}\mathcal{D}_2\left(\frac{2c_1(n/T)}{\Delta_1(n/T)}\right)\mathcal{D}_2\left(\frac{2c_1(n_-/T)}{\Delta_1(n_-/T)}\right) + \frac{2Tc_1(n/T)}{\pi(1-\cos(\Delta_1(n/T)/2))}\mathcal{D}_2\left(\frac{2c_1(n_-/T)}{T\Delta_1(n_-/T)}\right)\\ &+ \frac{2T\mathcal{G}_4(n_-/T)}{\Delta_0(n/T)}\mathcal{D}_2\left(\frac{2c_1(n_-/T)}{T\Delta_1(n_-/T)}\right) + \frac{2\mathcal{G}_3(n_-/T)}{\Delta_1(n/T)},\end{aligned} \tag{61}$$

$$\left\|DY\left(\tfrac{n}{T}\right)\right\| \le \mathcal{D}_2\left(\frac{2c_1(n_-/T)}{T\Delta_1(n_-/T)}\right), \tag{62}$$

with $\mathcal{D}_2(x)$, $\mathcal{G}_3(n/T)$, and $\mathcal{G}_4(n/T)$ given in Eqs. (44), (47), and (48), respectively. The details of how to give these expressions are given in Appendix E 2; see Eqs. (E11), (E12), (E20), and (E21).

Therefore,

$$\begin{aligned}&\left\|\sum_{n=1}^{sT} Q_0 U_A^\dagger\left(\tfrac{n}{T}\right)\left(I - V^\dagger\left(\tfrac{n_-}{T}\right)\right)U_A\left(\tfrac{n}{T}\right)P_0\Omega\left(\tfrac{n_-}{T}\right)\right\|\\ &\le \frac{2}{\Delta_0(1/T)}\mathcal{D}_2\left(\frac{2c_1(0)}{T\Delta_1(0)}\right) + \frac{2}{\Delta_0(s)}\mathcal{D}_2\left(\frac{2c_1(s-1/T)}{T\Delta_1(s-1/T)}\right) + \mathcal{D}_2\left(\frac{2c_1(s-1/T)}{T\Delta_1(s-1/T)}\right)\\ &+\sum_{n=1}^{sT-1} 2\left(\frac{1}{\Delta_0(n_+/T)} + \frac{2}{\Delta_0(n/T)}\right)\mathcal{D}_2\left(\frac{2c_1(n/T)}{T\Delta_1(n/T)}\right)\mathcal{D}_2\left(\frac{2c_1(n_-/T)}{T\Delta_1(n_-/T)}\right)\\ &+\sum_{n=1}^{sT-1}\frac{2\mathcal{G}_3(n_-/T)}{T^2\Delta_1(n/T)} + \sum_{n=1}^{sT-1}\frac{2c_1(n/T)}{\pi T[1-\cos(\Delta_1(n/T)/2)]}\mathcal{D}_2\left(\frac{2c_1(n_-/T)}{T\Delta_1(n_-/T)}\right) + \sum_{n=1}^{sT-1}\frac{2\mathcal{G}_4(n_-/T)}{T\Delta_0(n/T)}\mathcal{D}_2\left(\frac{2c_1(n_-/T)}{T\Delta_1(n_-/T)}\right).\end{aligned} \tag{63}$$

Finally, by using Eqs. (57) and (63) in Eq. (50), we obtain the required overall bound in Eq. (43). ■

Because the first form of the discrete adiabatic theorem is quite complicated, we give a simplified but looser form in Theorem 3. The key ideas to obtain Theorem 3 from Theorem 7 are as follows: replace the functions $c_1(s)$ and $c_2(s)$ by Eq. (10), which take into account neighboring steps; replace the gaps $\Delta_k(s)$ by $\check{\Delta}(s)$ as defined in Eq. (14), which takes into account the minimum gap in neighboring steps; and bound the higher-order terms by lower-order terms with a slightly more strict assumption on T, that it is no less than $\max_s[4\hat{c}_1(s)/\check{\Delta}(s)]$.

By using that bound, we are able to restrict to a regime where $\mathcal{D}_1$ is upper bounded by a constant, whereas $\mathcal{D}_2$ and $\mathcal{D}_3$ are upper bounded by linear functions in z. As a result, the expressions on the right-hand side of Eq. (43) simplify as follows:

(a) The first three terms in the first line give the first three terms in Theorem 3.
(b) The sum in the second line simplifies to a sum over $\hat{c}_1(s)^2/[T^2\check{\Delta}(s)^3]$, contributing to the first sum in Theorem 3.
(c) The first sum in the third line gives both $\hat{c}_2(s)/[T^2\check{\Delta}(s)^2]$, contributing to the last sum in Theorem 3, as well as $\hat{c}_1(s)^2/[T^2\check{\Delta}(s)^3]$, contributing to the first sum in Theorem 3.
(d) The second sum in the third line gives $\hat{c}_1(s)^2/[T^2\check{\Delta}(s)^3]$ again.
(e) The first sum in the last line gives $\hat{c}_1(s)^2/[T^2\check{\Delta}(s)^3]$, $\hat{c}_2(s)/[T^2\check{\Delta}(s)^2]$, and $\hat{c}_1(s)^2/[T^2\check{\Delta}(s)^2]$ after some simplifications using the bound on T.
(f) The second and third sums in the last line both give $\hat{c}_1(s)^2/[T^2\check{\Delta}(s)^2]$.

The complete proof of the second adiabatic theorem is given in Appendix E 3.

IV. APPLICATION: SOLVING LINEAR SYSTEMS

A high-level description of the algorithm is as follows:

(1) The matrix A is described by a block encoding and the vector b is described by a unitary operation to prepare $|b\rangle$.
(2) From those descriptions, we construct a block encoding of a Hamiltonian $H(s)$, where an eigenstate of $H(0)$ is $|b\rangle$ and an eigenstate of $H(1)$ is the solution $|A^{-1}b\rangle$.
(3) The unitary operator for the block encoding of $H(s)$, together with a reflection on the control ancilla, gives the (unitary) qubitized walk operator $W(s)$.
(4) The discrete sequence of walk operators $W(s)$ for s varying from 0 to 1 is shown to give a good overlap with the solution $|A^{-1}b\rangle$ via the discrete adiabatic theorem.
(5) At the end, we apply filtering via a linear combination of powers of $W(1)$ in order to obtain $|A^{-1}b\rangle$ with precision ϵ. If there is failure of the filtering, then the adiabatic evolution is repeated.

The error in the discrete walk is shown to scale as a constant times κ/T, so one can take T proportional to κ to obtain reasonable overlap with the solution. The use of a walk here, instead of Hamiltonian evolution as in prior work, gives complexity that is strictly linear in κ. It is the final filtering that gives the solution to accuracy ϵ and gives the multiplicative factor of $\log(1/\epsilon)$ in the complexity. The filtering method is described in Sec. V and improves over prior work by using a more efficient form of linear combinations of unitaries.

A. Preparing the walker

In this section we apply Theorem 7, about adiabatic evolution in the discrete setting, to solve the quantum linear-systems problem. The key feature is to transform the Hamiltonian into a discrete quantum walk. This is done via qubitization [23,24]. That is, when there is a block encoding of the Hamiltonian, one can simply combine the unitary operation that is used for the block encoding with a reflection on the ancilla qubits used and one obtains a step of a walk with eigenvalues related to those of the Hamiltonian. We show that this discrete walk can be used for adiabatic evolution in a similar way as the continuous Hamiltonian evolution.

In adiabatic quantum computation, one usually uses a Hamiltonian that is a combination of two Hamiltonians, as

$$H(s) = [1 - f(s)]H_0 + f(s)H_1, \tag{64}$$

where the function $f(s) : [0, 1] \to [0, 1]$ is called the schedule function. Normally, H_0 is the Hamiltonian where the ground state is easy to prepare and H_1 is the one where the ground state encodes the solution of the problem that we are trying to determine. For the case of linear-systems solvers, the ground state of $H(1)$ should encode the normalized solution for a linear system. In other words, for $A \in \mathbb{C}^{N\times N}$ an invertible matrix with $\|A\| = 1$ and a normalized vector $|b\rangle \in \mathbb{C}^N$, the goal is to prepare a normalized state $|\tilde{x}\rangle$ that is an approximation of $|x\rangle = A^{-1}|b\rangle / \|A^{-1}|b\rangle\|$. For precision ϵ of the approximation, we require $\| |\tilde{x}\rangle - |x\rangle \| \le \epsilon$. One can also bound the error in terms of $\| |\tilde{x}\rangle\langle\tilde{x}| - |x\rangle\langle x| \|$, as has been done in some prior work [14,15], which is asymptotically equal (for small error). Translating this problem to our theorem for the adiabatic evolution, $|\tilde{x}\rangle$ would be the state achieved from the steps of the walk and $|x\rangle$ would be obtained from the ideal adiabatic evolution.

Beginning with the simplest case, where A is Hermitian and positive definite, one takes the Hamiltonians [15]

$$H_0 := \begin{pmatrix} 0 & Q_b \\ Q_b & 0 \end{pmatrix} \tag{65}$$

and

$$H_1 := \begin{pmatrix} 0 & AQ_b \\ Q_bA & 0 \end{pmatrix}, \tag{66}$$

where $Q_b = I_N - |b\rangle\langle b|$. The state $|0, b\rangle$ is an eigenstate of H_0 with eigenvalue 0 and one would aim for this to evolve adiabatically to eigenstate $|0, A^{-1}b\rangle$ of H_1. There is also an eigenstate $|1, b\rangle$ for both H_0 and H_1 with the same eigenvalue 0 but it is orthogonal and we show that there is no crossover in the ideal adiabatic evolution using the walk.

Denoting the condition number of the matrix as κ, a lower bound for the gap of $H(s)$ is [15]

$$\Delta_0(s) = 1 - f(s) + f(s)/\kappa. \tag{67}$$

Note that according to Definition 2, $\Delta_0(s)$ is a lower bound on the exact gap between the eigenvalues, so we use an equality here rather than an inequality.

Since the goal is to obtain a schedule function that slows down the evolution as the gap becomes small, a standard condition for the schedule is [25]

$$\dot{f}(s) = d_p \Delta_0^p(s), \tag{68}$$

where $f(0) = 0$, $p > 0$ and $d_p = \int_0^1 \Delta_0^{-p}(u)\, du$ is a normalization constant chosen so that $f(1) = 1$. It is possible to show that [15]

$$f(s) = \frac{\kappa}{\kappa - 1}\left[1 - \left(1 + s\left(\kappa^{p-1} - 1\right)\right)^{\frac{1}{1-p}}\right] \tag{69}$$

satisfies Eq. (68) but with $\Delta_0(s)$ replaced with the lower bound on the gap from Eq. (67). This schedule function has two properties that have useful applications in estimating the upper bounds for the difference between consecutive walker operators, namely that $f(s)$ is monotonic increasing and that $\dot{f}(s)$ is monotonic decreasing.

Distinct from the continuous version of the adiabatic theorem, in our discrete version of the theorem, we have to take into account the gap between the different groups of eigenvalues of $W(s)$ for s, $s+1/T$, and $s+2/T$, as described in Eq. (12). From the property that the gap function is monotonically increasing, we have

$$\Delta_k(s) = 1 - f(s+k/T) + f(s+k/T)/\kappa, \quad k = 0,1,2. \tag{70}$$

To address the case where A is not positive definite or Hermitian, we take a different approach than Ref. [14]. We take the Hamiltonian

$$H(s) = \begin{pmatrix} 0 & A[f(s)]Q_{\mathbf{b}} \\ Q_{\mathbf{b}}A[f(s)] & 0 \end{pmatrix}, \tag{71}$$

where

$$A(f) := (1-f)\sigma_z \otimes I_N + f\,\mathbf{A} = \begin{pmatrix} (1-f)I & fA \\ fA^\dagger & -(1-f)I \end{pmatrix} \tag{72}$$

with

$$\mathbf{A} := \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}, \tag{73}$$

and $Q_{\mathbf{b}} = I_{2N} - |1,b\rangle\langle 1,b|$ is a projection. This is equivalent to taking $H(s) = [1-f(s)]H_0 + f(s)H_1$ with

$$H_0 = \sigma_+ \otimes [(\sigma_z \otimes I_N)Q_{\mathbf{b}}] + \sigma_- \otimes [Q_{\mathbf{b}}(\sigma_z \otimes I_N)] \tag{74}$$

$$H_1 = \sigma_+ \otimes [\mathbf{A}Q_{\mathbf{b}}] + \sigma_- \otimes [Q_{\mathbf{b}}\mathbf{A}], \tag{75}$$

where $\sigma_+ = |1\rangle\langle 0|$, $\sigma_- = |0\rangle\langle 1|$.

Then, it is found that

$$H^2(s) = \begin{pmatrix} A[f(s)]Q_{\mathbf{b}}A[f(s)] & 0 \\ 0 & Q_{\mathbf{b}}A^2[f(s)]Q_{\mathbf{b}} \end{pmatrix} \tag{76}$$

As per the analysis in the Supplemental Material of Ref. [14], the spectra of $A[f(s)]Q_{\mathbf{b}}A[f(s)]$ and $Q_{\mathbf{b}}A^2[f(s)]Q_{\mathbf{b}}$ are identical. Moreover, following that analysis, the gap of $A[f(s)]Q_{\mathbf{b}}A[f(s)]$ is lower bounded by the minimum eigenvalue of $A^2[f(s)]$. In this case, since

$$A^2(f) = \begin{pmatrix} (1-f)^2 I + f^2 AA^\dagger & 0 \\ 0 & (1-f)^2 I + f^2 A^\dagger A \end{pmatrix}, \tag{77}$$

the minimum eigenvalue is $(1-f)^2 + (f/\kappa)^2$. This translates to a minimum gap of $H(s)$ of $\sqrt{[1-f(s)]^2 + (f(s)/\kappa)^2}$. To avoid the need to use this formula, one can use the relation that for $0 \le f(s) \le 1$,

$$\sqrt{[1-f(s)]^2 + (f(s)/\kappa)^2} \ge (1 - f(s) + f(s)/\kappa)/\sqrt{2}. \tag{78}$$

By using the qubitized quantum walk for the implementation of W, we can avoid the logarithmic factor in the complexity that arises from using the Dyson series to simulate continuous Hamiltonian evolution.

In order to block encode the Hamiltonian $H(s)$, one can use block encodings of both H_0 and H_1, supplemented with an ancilla qubit that is rotated to select between H_0 and H_1. The rotation is given by

$$R(s) = \frac{1}{\sqrt{(1-f(s))^2 + f(s)^2}} \begin{pmatrix} 1-f(s) & f(s) \\ f(s) & -[1-f(s)] \end{pmatrix}. \tag{79}$$

To block encode $A[f(s)]$, instead of using symmetric rotations, we use the initial rotation $R(s)$, then apply the controlled operations

$$\mathrm{SEL} = |0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1, \tag{80}$$

where U_0 and U_1 are unitaries used for the block encodings of $\sigma_z \otimes I_N$ and $\mathbf{A}$. Then, after this operation, instead of applying the inverse of $R(s)$, we simply perform a Hadamard. This means that, instead of block encoding $A[f(s)]$, we have block encoded

$$\frac{1}{\sqrt{2[[1-f(s)]^2 + f(s)^2]}} A[f(s)]. \tag{81}$$

This prefactor is between $1/\sqrt{2}$ and 1 and it reduces the gap. Thus the overall gap is reduced by a maximum factor of 2. That is, when we consider the gap of the block encoding when A is not positive definite, one can keep using the same schedule function from Eq. (69) but the spectral gaps can instead be lower bounded by

$$\Delta'_k(s) = [1 - f(s+k/T) + f(s+k/T)/\kappa]/2, \quad k = 0,1,2. \tag{82}$$

In order for the walk operator constructed as in Refs. [23,24] to have the simple relation with the eigenvalues of the Hamiltonian, the unitary used for the block encoding of the Hamiltonian needs to be self-inverse. That can be achieved by constructing a symmetric sequence of operators such that applying the unitary twice in succession gives the identity. For the complete qubitization of $H(s)$ as in Eq. (71), the principle is to apply $Q_{\mathbf{b}}$ in a controlled way before and after $A[f(s)]$, making the sequence symmetric.

Similarly, instead of applying $R(s)$ at the beginning and the Hadamard at the end, we apply it in a controlled way each time, reversing the order between the two blocks.

The qubitized operator $W(s)$ is then obtained by combining the block encoding of $H(s)$ with a reflection on the control qubits. For a complete description of the procedure for block encoding the Hamiltonian, see Appendix F.

B. Choosing values for $c_1(s)$ and $c_2(s)$

Now, to apply Theorem 3 for the QLSP, two things that should be estimated are the functions $c_1(s)$ and $c_2(s)$, which in turn require upper bounds for $DW(s)$ and $D^{(2)}W(s)$. In order to bound the difference in $W(s)$, we can use the fact that the only way $W(s)$ is dependent on s is through $R(s)$. The key feature of this operation is that it has $R(s)$ in two cross-diagonal blocks (in the matrix representation). As a result, the spectral norm of the difference of operators is equal to the spectral norm of the difference of $R(s)$. Note that this feature is identical regardless of the specific way of encoding H in terms of A from the preceding subsection. All that is required is that H_0 and H_1 are combined using the rotation $R(s)$. The result can be described as in the following lemma.

Lemma 8: *For any $0 \le s \le 1 - 1/T$, with $W(s)$ encoded using the block encoding of $H(s)$ where H_0 and H_1 are combined using the rotation $R(s)$ given in Eq. (79), it is consistent with Definition 1 to choose*

$$c_1(s) = 2T[f(s+1/T) - f(s)] \tag{83}$$

and

$$c_2(s) = \begin{cases} 2\max_{\tau\in\{s,s+1/T,s+2/T\}}\left(2|f'(\tau)|^2 + |f''(\tau)|\right), & 0 \le s \le 1-2/T, \\ 2\max_{\tau\in\{s,s+1/T\}}\left(2|f'(\tau)|^2 + |f''(\tau)|\right), & s = 1-1/T. \end{cases} \tag{84}$$

Proof. As discussed above, to bound the difference in $W(s)$, we can use the difference in the rotation operation $R(s)$, which can be upper bounded as

$$\begin{aligned} \|R(s+1/T) - R(s)\| &= \left\| \int_s^{s+1/T} \frac{dR}{ds} ds \right\| \\ &= \left\| \int_s^{s+1/T} \frac{dR}{df}\frac{df}{ds} ds \right\| \\ &\le \int_s^{s+1/T} \left\| \frac{dR}{df}\frac{df}{ds} \right\| ds \\ &\le 2\int_s^{s+1/T} \left| \frac{df}{ds} \right| ds \\ &= 2\left(f(s+1/T) - f(s)\right). \end{aligned} \tag{85}$$

Here, we upper bound the norm of dR/df by 2. To show this, we take the derivative of $R(s)$ with respect to f,

$$\frac{dR}{df} = \frac{1}{[(1-f(s))^2 + f(s)^2]^{3/2}} \begin{pmatrix} -f(s) & 1-f(s) \\ 1-f(s) & f(s) \end{pmatrix}, \tag{86}$$

so the norm of dR/df is $1/[[1-f(s)]^2 + f(s)^2]$, which varies between 1 and 2. We also use the fact that $df/ds > 0$. Since $\|W(s+1/T) - W(s)\| = \|R(1+1/T) - R(s)\|$ and $c_1(s)$ required in Definition 1 to satisfy

$$\|W(s+1/T) - W(s)\| \le \frac{c_1(s)}{T}, \tag{87}$$

we can take $c_1(s)$ as in Eq. (83).

Now, for the second difference of the walk operator, we use Taylor's theorem in two directions to give

$$\begin{aligned} W(s+2/T) = W(s+1/T) + \frac{1}{T}\frac{dW(s+1/T)}{ds} \\ + \int_{s+1/T}^{s+2/T} (s+2/T-\tau)\frac{d^2W(\tau)}{d\tau^2} d\tau \end{aligned} \tag{88}$$

and

$$\begin{aligned} W(s) = W(s+1/T) - \frac{1}{T}\frac{dW(s+1/T)}{ds} \\ + \int_s^{s+1/T} (\tau - s)\frac{d^2W(\tau)}{d\tau^2} d\tau. \end{aligned} \tag{89}$$

That gives

$$\begin{aligned} \|D^{(2)}R(s)\| &= \left\| \int_{s+1/T}^{s+2/T} (s+2/T-\tau)R''(\tau)d\tau \right. \\ &\quad \left. + \int_s^{s+1/T} (\tau - s)R''(\tau)d\tau \right\| \\ &\le \frac{1}{T^2} \max_{\tau\in[s,s+2/T]} \|R''(\tau)\|. \end{aligned} \tag{90}$$

Moving to the second derivative of $R(s)$, we have

$$\frac{d^2R(s)}{ds^2} = \frac{d^2R}{df^2}\left(\frac{df(s)}{ds}\right)^2 + \frac{dR}{df}\frac{df^2(s)}{ds^2}, \tag{91}$$

where

$$\frac{d^2R}{df^2} = \frac{1}{[(1-f(s))^2+f(s)^2]^{5/2}} \times \begin{pmatrix} (4f(s)-1)f(s)-1 & (4f(s)-7)f(s)+2 \\ (4f(s)-7)f(s)+2 & [1-4f(s)]f(s)+1 \end{pmatrix}, \tag{92}$$

and its norm is

$$\sqrt{\frac{16(f(s)-1)f(s)+5}{[2(f(s)-1)f(s)+1]^4}}, \tag{93}$$

which varies between $\sqrt{5}$ and 4. Then, we conclude that

$$\left\|D^{(2)}R(s)\right\| \le \frac{2}{T^2}\max_{\tau\in\{s,s+1/T,s+2/T\}}\left(2|f'(\tau)|^2+|f''(\tau)|\right). \tag{94}$$

Now, we have $\|D^2W(s)\| = \|D^2R(s)\|$ and Definition 1 requires that $\|D^2W(s)\| \le c_2(s)/T^2$, so we can take $c_2(s)$ as in Eq. (84). ■

C. Linear κ for $p = 3/2$

Our next step is to show the strict linear dependence in κ for the QLSP based on our discrete adiabatic theorem. In the continuous case, it has been shown in Ref. [15] that for all $1 < p < 2$, the corresponding AQC-based linear-systems solver can achieve κ/ϵ scaling. This suggests that taking p as the midpoint of $3/2$ will give high efficiency. Here, we consider this case to estimate the constant factors in the algorithm.

Theorem 9: (strict linear dependence in κ). *Consider solving the QLSP $Ax = b$ for a normalized state $|A^{-1}b\rangle$, where $\|A\| = 1$ and $\|A^{-1}\| = \kappa$. By using $T \ge \max(\kappa, 39\sqrt{\kappa})$ steps of a quantum walk and the schedule function of Eq. (69) with $p = 3/2$, the error in the solution may be bounded as*

$$\|U(s) - U_A(s)\| \le 44864\frac{\kappa}{T} + \mathcal{O}\left(\frac{\sqrt{\kappa}}{T}\right), \tag{95}$$

using the encoding of $H(s)$ as in Eq. (71).

Proof. In Theorem 3, there are six terms to bound, three which are individual terms and three of which are sums. The details of the derivations of bounds on these are given in Appendix G. Namely, for the individual terms, it is shown in Appendix G 1 [Eqs. (G5), (G10) and (G11)] that

$$\frac{\hat{c}_1(0)}{T\check{\Delta}(0)^2} = \frac{4\sqrt{\kappa}}{T} + \mathcal{O}\left(\frac{\kappa}{T^2}\right), \tag{96}$$

$$\frac{\hat{c}_1(1)}{T\check{\Delta}(1)^2} = \frac{4\kappa}{T} + \mathcal{O}\left(\frac{\kappa}{T^2}\right), \tag{97}$$

$$\frac{\hat{c}_1(1)}{T\check{\Delta}(1)} = \frac{4}{T} + \mathcal{O}\left(\frac{1}{T^2}\right), \tag{98}$$

and for the sums with $c_1(s)$, it is shown in Appendix G 2 that

$$\sum_{n=1}^{T-1}\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^3} = \frac{16\kappa}{T} + \mathcal{O}\left(\frac{\kappa^{3/2}}{T^2}\right), \tag{99}$$

$$\sum_{n=0}^{T-1}\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^2} \le \frac{16\kappa}{T} + \mathcal{O}\left(\frac{\sqrt{\kappa}}{T}\right). \tag{100}$$

Finally, for the sum with $c_2(s)$, it is shown in Appendix G 3 that

$$\sum_{n=1}^{T-1}\frac{\hat{c}_2(n/T)}{T^2\check{\Delta}(n/T)^2} \le \frac{22\kappa}{T} + \mathcal{O}\left(\frac{\kappa^{3/2}}{T^2}\right). \tag{101}$$

These results are for the case where A is positive definite and Hermitian. However, we are interested in the general case of A with the qubitized implementation of W, which has the gap given by Eq. (82). This means that the upper bounds on the terms with $\check{\Delta}(s)$, $\check{\Delta}(s)^2$ and $\check{\Delta}(s)^3$ in the denominator may be multiplied by 2, 4, and 8, respectively. Then, by adding all the inequalities above, after the respective corrections of the gaps of the general case, and including the constant factors in Theorem 3, we obtain the total upper bound in Eq. (95). Note that we only include the leading term proportional to κ/T and terms with scalings such as $\kappa^{3/2}/T^2$ are order $\sqrt{\kappa}/T$ due to the requirement that $T > \kappa$.

There are two further subtleties in using the adiabatic algorithm for the solution. One is that the zero eigenvalue of the Hamiltonian is degenerate, with one giving the solution and the other just the state $|b\rangle$ but with a bit flip in an ancilla. Because these eigenstates are orthogonal (due to the bit flip), there is no crossover between them in the adiabatic evolution. This means that the degeneracy has no effect on the quality of the solution (see Appendix H).

A further subtlety is that the qubitized quantum walk yields two eigenstates for each eigenstate of the Hamiltonian. For the case here, the eigenvalue of the Hamiltonian we are interested in is 0, which gives eigenvalues ± 1 of the walk operator. We may use the discrete adiabatic theorem separately on each of these eigenvalues to show that the eigenstate is preserved in the discrete adiabatic evolution. The problem is that we need to have a positive superposition of these two eigenstates, which means that there should be no relative phase factor introduced in the adiabatic evolution. It is shown that there is no relative phase factor in Appendix H. Therefore, neither of these subtleties has an effect on the solution and no adjustment to the bounds is required. ■

D. General *p*

In this subsection, we show that the κ/ϵ scaling also holds for all $1 < p < 2$ in the discrete setting. This result is more general but due to a number of approximations will not be as tight an estimate as that for the specific case of $p = 3/2$. Since here we do not assume a specific value of p, the direct-computation approach in proving Theorem 9 is not applicable. Instead, we approximate the upper bound of the discrete error by some continuous integrals and then bound both the integrals and the approximation errors. More precisely, we first note that in Theorem 3, the dominant terms are the last three terms, the summations over equidistant discrete time steps. These summations are exactly in the form of Riemann sums and approximate some integrals. Then, the dominant part of the discrete adiabatic errors can be bounded by some integrals plus the difference between the integrals and corresponding Riemann sums. Similar to what has been shown in Ref. [15], the integrals scale exactly $\mathcal{O}(\kappa/T)$. The difference between the integrals and Riemann sums is indeed of higher order according to the error bound of the first-order quadrature formula. Combining all these together, we can prove that the discrete adiabatic error for general $1 < p < 2$ also scales as $\mathcal{O}(\kappa/T)$, which further implies an $\mathcal{O}(\kappa/\epsilon)$ complexity of the discrete AQC-based algorithm to solve the linear-systems problem within ϵ error.

We summarize the main result in the following theorem. A detailed proof is given in Appendix I.

Theorem 10: (linear dependence on κ for general p). *Consider using T steps of the discrete adiabatic evolution with the schedule function defined in Eq. (69) for $1 < p < 2$ to solve the QLSP with general matrix A. Then:*

(1) For any $\kappa > 2$ and $T \geq 38 d_p = \mathcal{O}(\kappa^{p-1})$, with $d_p = \int_0^1 \Delta_0^{-p}(u)\, du$, there exists a positive constant C_p, which only depends on p, such that the difference between the discrete adiabatic evolution and the solution of the linear-systems problem can be bounded by

$$C_p \left(\frac{\kappa}{T} + \frac{\kappa^{p-1}}{T} + \frac{\kappa}{T^2} + \frac{1}{T} \right). \tag{102}$$

(2) In order to prepare an ϵ approximation of the solution of the linear-systems problem, it suffices to choose

$$T = \mathcal{O}\left(\frac{\kappa}{\epsilon}\right). \tag{103}$$

We give an explicit formula for C_p in Eq. (I50) in Appendix I. We remark that Theorem 10 only guarantees the asymptotic performance of the discrete AQC-based solvers and the preconstant C_p is much larger than we obtained in Theorem 9 for $p = 3/2$ and for what we observe numerically for the other values. This is because in Theorem 10 we use a general proof strategy, which is applicable for all $1 < p < 2$ at a sacrifice of using potentially unnecessary inequalities to simplify the analysis.

E. Numerical results

We first report the numerical results for the case where A is a Hermitian and positive-definite matrix. In order to compute these results, we use the bound in Theorem 7 (rather than any specific examples of A). Theorem 7 can give much tighter bounds than Theorem 9 if we do not use the approximations made in deriving that theorem but requires computation for example values of κ, p, and T. Rather than using the upper bounds for the first and second differences of the walk operator from Lemma 8, we exactly compute the norm of DR and $D^{(2)}R$ in order to give values of $c_1(s)$ and $c_2(s)$ in Theorem 7. We also account for the fact that the actual gap for the quantum walk operator is the arcsin of that in Eq. (82) and solve the differential equation for the scheduling function with the actual gap.

In Fig. 2, we show the numerical results for the (upper bound on the) error as a function of the condition number κ of the matrix A. That is, we replace the arbitrary gap dependence in the upper bound of Theorem 7 by the gap of the QLSP given in Eq. (67) and then we use a fixed number of steps $T = 10^5$ and three different values of p. In each case, it can be seen that the error is approximately linear

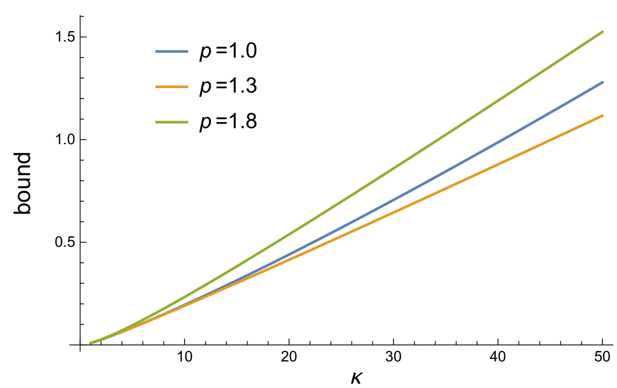

FIG. 2. The upper bound on the error in the adiabatic evolution versus the condition number κ for a range of values of p used in the scheduling function $f(s)$. The upper bound on the error is computed using Theorem 7 and in all cases the number of steps of the walk is $T = 10^5$.

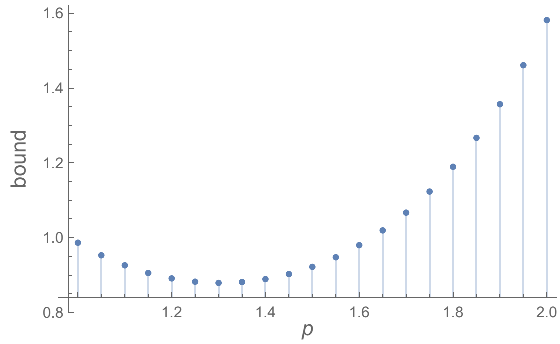

FIG. 3. The upper bound on the error in the adiabatic evolution as a function of p used in the scheduling function $f(s)$. In this plot, we use constant values $\kappa = 40$ and $T = 10^5$.

in κ, which is what results in an overall complexity that is linear in κ. The different values of p result in different scaling constants with values close to 1 or 2 giving poorer scaling, which is as expected, since we require $1 < p < 2$.

To more clearly see the dependence of the error in p, in Fig. 3 we show the error as a function of p for constant κ (of 40). In this case, it turns out that the smallest error is for $p = 1.3$, which is on the lower side of the range $(1, 2)$ and smaller than the value $p = 3/2$ chosen for Theorem 11. From Fig. 3, we can also estimate the constant factors for the κ/T scaling of the error. In the case with $p = 3/2$, for instance, as used in Theorem 9, we have $\|U(s) - U_A(s)\| \lesssim 2305\kappa/T$. The estimate of the constant factor in Theorem 9 is around 20 times larger. This is not unreasonable considering the many approximations made, though it indicates that the constant factor in the analysis can be improved by a more careful analysis.

V. FILTERING FOR SOLVING LINEAR EQUATIONS

To provide a solution to linear equations using the adiabatic method, one can use the approach of Ref. [16], where the initial adiabatic algorithm is used to find the solution to some constant error (independent of ϵ) and then the solution can be filtered. The approach used in Ref. [16] is to apply filtering by singular-value processing (similar to quantum signal processing), which is efficient and only needs one ancilla qubit but has the drawback that it requires a highly complicated procedure for finding the correct rotation angles. Here, we provide a method using a linear combination of unitaries with similar efficiency and only requiring two ancilla qubits (one more than singular-value processing). This has the advantage that it is much simpler to determine the sequence of gates needed.

For filtering by a linear combination of unitaries with weights w_j, we would initially prepare the control register in the state

$$\frac{1}{\sqrt{\sum_j w_j}} \sum_j \sqrt{w_j}\, |j\rangle . \tag{104}$$

Given that we are performing j steps of the walk and the input system state is an eigenvector of the walk with eigenvalue $e^{i\phi}$, the resulting state is

$$\frac{1}{\sqrt{\sum_j w_j}} \sum_j \sqrt{w_j} e^{ij\phi}\, |j\rangle . \tag{105}$$

The inner product with the state in Eq. (104) gives

$$\frac{1}{\sum_j w_j} \sum_j w_j e^{ij\phi}. \tag{106}$$

In practice, the adiabatic walk prepares the target register in a superposition of the eigenstates

$$|\psi\rangle = \sum_k \psi_k \,|k\rangle\,, \tag{107}$$

$$
\Qcircuit @C=1em @R=1em {
\lstick{|0\rangle} & \gate{R_0} & \ctrl{1} & \qw & \qw & \ctrl{4} & \qw & \qw & \qw & \qw & \qw & \ctrl{1} & \gate{R_0^\dagger} & \rstick{\langle 0|} \\
\lstick{|0\rangle} & \qw & \gate{R_1} & \ctrl{1} & \qw & \qw & \ctrl{3} & \qw & \qw & \qw & \ctrl{1} & \gate{R_1^\dagger} & \qw & \rstick{\langle 0|} \\
\lstick{|0\rangle} & \qw & \qw & \gate{R_2} & \ctrl{1} & \qw & \qw & \ctrl{2} & \qw & \ctrl{1} & \gate{R_2^\dagger} & \qw & \qw & \rstick{\langle 0|} \\
\lstick{|0\rangle} & \qw & \qw & \qw & \gate{R_3} & \qw & \qw & \qw & \ctrl{1} & \gate{R_3^\dagger} & \qw & \qw & \qw & \rstick{\langle 0|} \\
\lstick{|\psi\rangle} & \qw & \qw & \qw & \qw & \gate{W} & \gate{W} & \gate{W} & \gate{W} & \qw & \qw & \qw & \qw & \qw
}
$$

FIG. 4. A linear combination of steps using control registers prepared in unary using a linear sequence of controlled rotations. The state $|\psi\rangle$ would be the state output by the adiabatic walk and W is used to indicate the final walk step $W(1)$ from the adiabatic walk. In this example, the four control qubits would encode $|j\rangle$ in unary and the rotations would prepare the amplitudes proportional to $\sqrt{w_j}$. The rotations on the right invert this preparation. Given that the control qubits are measured in the state $|0\rangle$, the target will be in a state proportional to that given in Eq. (108).

where we are using $|k\rangle$ to indicate the eigenstate of $W(1)$ corresponding to eigenvalue ϕ_k. The state after applying the linear combination of unitaries is then

$$\frac{1}{\sum_j w_j} \sum_{j,k} w_j \psi_k e^{ij\phi_k} |k\rangle = \sum_k \tilde{w}(\phi_k)\psi_k |k\rangle, \tag{108}$$

where

$$\tilde{w}(\phi) = \frac{1}{\sum_j w_j} \sum_j w_j e^{ij\phi}. \tag{109}$$

Note that the state is not normalized, with the norm giving the probability of the success of this linear combination of unitaries. We aim to have $\tilde{w}(\phi) = 1$ for ϕ in the spectrum of interest, which is $\{0, \pi\}$ for the solution of linear equations. Now, let us assume that the initial probability of the state on the spectrum of interest is at least $1/2$. One can then show that the resulting normalized state obtained after the filtering has error, as quantified by the norm of the difference of states, upper bounded by

$$\max_{k\in\perp} \tilde{w}(\phi_k), \tag{110}$$

where $\perp$ is the set of k such that ϕ_k is not in the spectrum of interest (so $e^{i\phi_k} \notin \sigma_P$) (for the proof, see Appendix J).

The result of this reasoning is that to bound the error in the filtering, we need to bound the maximum of $\tilde{w}(\phi_k)$, which is minimized by the Dolph-Chebyshev window. This is obtained by taking the discrete Fourier transform of the Chebyshev polynomials, so that $\tilde{w}(\phi)$ is given by Chebyshev polynomials in a similar way as for Ref. [16]. In particular, the Fourier transform of the Dolph-Chebyshev window is given by

$$\tilde{w}(\phi) = \epsilon T_\ell\left(\beta \cos\left(\phi\right)\right) \tag{111}$$

for ϕ taking discrete values $\pi k/\ell$ for k from $-\ell$ to ℓ and where $\beta = \cosh[\frac{1}{\ell}\cosh^{-1}(1/\epsilon)]$. Taking the discrete Fourier transform of these values gives the window and the Fourier transform simply yields the formula for $\tilde{w}(\phi)$ in terms of Chebyshev polynomials. One obtains powers of $e^{2i\phi}$ from $-\ell/2$ to $+\ell/2$, which means that we need a maximum power of $e^{i\phi}$ of ℓ. One can obtain the positive and negative powers simultaneously with negligible cost by simply controlling whether the reflection is performed in the qubitization. As a result, the cost in terms of calls to the block-encoded matrix is ℓ, as compared to 2ℓ for the singular-value-processing approach.

The peak for $\tilde{w}(\phi)$ will be at 0 and π, which is what is needed because the qubitized operator produces duplicate eigenvalues at 0 and π. The width of the operator can be found by noting that the peak is for the argument of the Chebyshev polynomial equal to β and the width is where the argument is 1, so $\beta\cos(\phi) = 1$. This gives us

$$\cosh[\tfrac{1}{\ell}\cosh^{-1}(1/\epsilon)]\cos(\phi) = 1. \tag{112}$$

Now, because the width of the peak should be equal to the gap and the gap is $1/\kappa$, we can replace ϕ with $1/\kappa$ and solving for ℓ gives

$$\ell = \frac{\cosh^{-1}(1/\epsilon)}{\cosh^{-1}[1/\cos(1/\kappa)]} \leq \kappa \ln(2/\epsilon). \tag{113}$$

Note that Eq. (112) is for finding the width given an integer ℓ but solving for ℓ with a width of $1/\kappa$, we should round ℓ up to the nearest integer to provide a width no larger than κ.

In comparison, in Ref. [16] the error is given as $2e^{-\sqrt{2}\ell\Delta}$, which would imply that one can take $\ell \approx \sqrt{1/2}\kappa\ln(2/\epsilon)$. Since the order of the polynomial is 2ℓ, which is also the number of applications of the block encoding needed, this would imply a cost of $\sqrt{2}\kappa\ln(2/\epsilon)$, which is greater than what we have here by a factor of $\sqrt{2}$. However, it turns out that the scaling given in Ref. [16] is overly conservative and the actual scaling is $2e^{-2\ell\Delta}$, which then gives the same complexity as we have here.

Next, we consider how to apply the linear combination of unitaries with minimum ancilla qubits. To do this, we first represent the control registers in unary. That is, for each of the ℓ controlled operations, we use a single qubit that is 1 or 0 depending on whether or not this operation is to be performed. It may seem counterproductive to expand the size of the ancilla in this way but it has the advantage that it has a simple state-preparation procedure, where an initial qubit is rotated and then the following qubits are prepared by controlled rotations. When doing this procedure, we can apply a just-in-time preparation procedure, where each qubit is prepared just as it is needed to be used as a control. An example of this is shown in Fig. 4.

Then, in inverting the preparation, one could simply perform the reverse of all the controlled rotations as in Fig. 4. However, the trick is that the sequential state-preparation procedure for the unary can be performed from either end. The preparation could be achieved by performing rotations starting from the last qubit and working back to the first. We do not do that for the preparation but we do the reverse of that for the inverse preparation. An example of this is shown in Fig. 5. Then, the operations are performed in a sequence from the first qubit to the last, the same as for the preparation. That means that we only need to use two ancillas at once, by rearranging the operations as shown in Fig. 6. See Appendix J for a more explicit description of the sequence of rotations.

The major advantage of this procedure over singular-value processing or quantum signal processing is that there

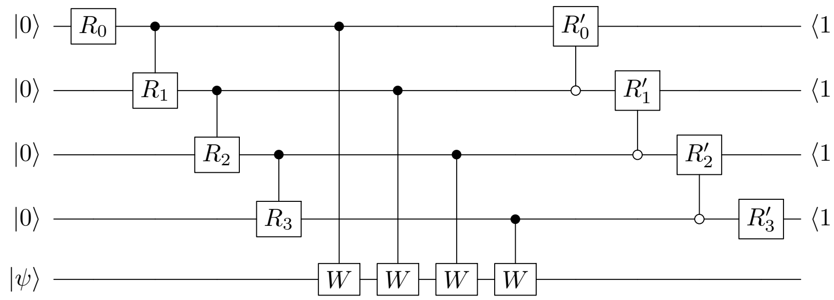

FIG. 5. A linear combination of steps using control registers prepared in unary using a linear sequence of controlled rotations but with the inverse preparation performed with the linear sequence in the reverse order.

is a very simple prescription for finding the sequence of operations. A second advantage is that, instead of the measurement being performed at the end, the measurements are performed sequentially and a failure (the incorrect measurement result) can be flagged early. That means that in cases where there is a failure, it will on average be flagged halfway through, with the result that half the number of operations are needed, since one can discard the state and start again.

Combining our result for the solution of QLSP with the filtering, we find that the overall complexity of the QLSP algorithm can be given as $\mathcal{O}[\kappa\log(1/\epsilon)]$. In particular, the result is as follows.

Theorem 11: (QLSP with linear dependence on κ). *Let $Ax=b$ be a system of linear equations, where A is an N-by-N matrix with $\|A\|=1$ and $\|A^{-1}\|=\kappa$. Given an oracle block encoding the operator A and an oracle preparing $|b\rangle$, there exists a quantum algorithm that produces the normalized state $|A^{-1}b\rangle$ to within error ϵ in terms of the ℓ^2 norm, using an average number*

$$\mathcal{O}[\kappa\log(1/\epsilon)] \tag{114}$$

of oracle calls.

Proof. In this theorem, we use standard assumptions that access to the oracles includes forward, reverse, and controlled uses. We initially apply the oracle for preparing $|b\rangle$ to prepare the initial state. This preparation is also used to construct the projection operator $Q_{\mathbf{b}}$. Together with the oracle for block encoding A, we can construct the operator for block encoding $H(s)$ as described in detail in Appendix F. A reflection on the ancillas yields the walk operator.

Now use the discrete adiabatic theorem for the QLSP as given in Theorem 10 for fixed precision, such as $1/2$. That step has complexity $\mathcal{O}(\kappa)$ and the only error is the overlap with other states that are not the solution. Next, use the filtering as described above, which has complexity $\mathcal{O}[\kappa\log(1/\epsilon)]$. In the case of success, one has produced the state $|A^{-1}b\rangle$ to within norm-distance ϵ. In the case of failure of the filtering, repeat the procedure. Since the probability of success may be made at least $1/2$ by suitably choosing the fixed precision for the adiabatic procedure, the adiabatic and filtering steps need only be applied 2 times on average before success. This gives a factor of 2 to the total complexity of $\mathcal{O}(\kappa)$ plus $\mathcal{O}[\kappa\log(1/\epsilon)]$. The total complexity is therefore $\mathcal{O}[\kappa\log(1/\epsilon)]$ as claimed. ■

Perhaps surprisingly, in this complexity, the largest asymptotic complexity is for the filtering step, because it

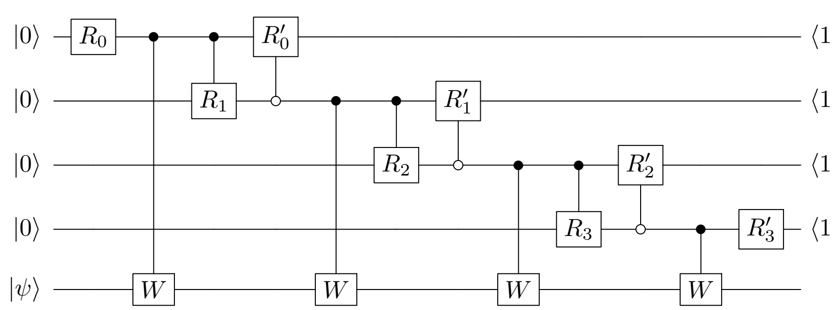

FIG. 6. A linear combination of steps using control registers prepared in unary but with the order of the operations changed so that we only need to use two ancilla qubits at a time.

has a factor of $\log(1/\epsilon)$, which is absent from the adiabatic step. In practice, we find quite large constant factors for the adiabatic evolution, so it is likely that the adiabatic step will still be the most costly part of the algorithm for realistic values of the parameters. In particular, for the numerical calculation of the upper bound, it is found that the scaling constant is about 638, so to obtain our requirement of an initial probability on the spectrum of interest of at least $1/2$ (corresponding to needing to repeat the algorithm twice on average), we would need about 834κ steps of the adiabatic evolution. In contrast, the $\ln(2/\epsilon)$ factor is only about 20 for ϵ as one part in 10^9.

VI. CONCLUSIONS

In this work, we show the first QLSP algorithm that scales optimally in terms of the condition number. We achieve this by adapting prior algorithms for the QLSP based on adiabatic evolutions so that they do not require the additional overhead of the Dyson-series algorithm for precisely evolving under time-dependent Hamiltonians on a gate-model quantum computer. Instead, we show that one can directly discretize the time evolution using quantum walks and that the error in this procedure can be obtained using a discrete adiabatic theorem. We also obtain rigorous new error bounds on the performance of those discrete adiabatic theorems.

While this improvement is "only" by a log factor, the fact that we can asymptotically match the lower bound is of fundamental interest. Furthermore, there is widespread anticipation that compelling practical applications of the QLSP may eventually be found and that error-corrected quantum computers capable of realizing those applications may eventually be realized. Should this occur, then it will be crucial to program those devices using the best possible scaling versions of these algorithms in order to have the fastest implementations requiring the least overhead due to error correction. Our expectation is that the QLSP approach described in this paper would be more performant than any other approach in the literature both in terms of asymptotic scaling and also in terms of the constant factors associated with realizing finite instances. Thus, we also foresee practical value in these results.

As well as scaling optimally in the condition number, our algorithm scales optimally in terms of the combination of the condition number and the precision ϵ. As has recently been proven, a lower bound to the complexity is $\mathcal{O}[\kappa \log(1/\epsilon)]$ [17]. Our result matches this lower bound, showing that it is optimal. It is interesting that the complexity is multiplicative between κ and $\log(1/\epsilon)$, in contrast to Hamiltonian simulation, which is additive between the time and $\log(1/\epsilon)$. In this approach to solving linear equations, the $\log(1/\epsilon)$ factor only comes from the filtering step, which in practice would have lower complexity than the initial adiabatic step.

Another question is the scaling with the sparsity in the case where the matrix is sparse and given by oracles for positions of nonzero entries. In this work, we give the complexity in terms of calls to a block encoding of the matrix, rather than those more fundamental oracles. The lower bound in terms of those oracles has a multiplicative factor of $\sqrt{d}$ in the sparsity d. One could obtain such a scaling if there was a way of block encoding the matrix with complexity $\sqrt{d}$ but the standard methods are linear. It is shown in Ref. [9] how to simulate a Hamiltonian with complexity $\sqrt{d}$ up to logarithmic factors using a nested-interaction-picture approach. One could use that combined with the adiabatic approach to obtain this scaling with sparsity but it would reintroduce logarithmic factors, so the complexity would no longer be strictly linear in κ.

More generally, we expect that other quantum algorithms based on continuous time evolutions might benefit from using discrete-time adiabatic algorithms. For example, there are quantum algorithms for optimization that use adiabatic evolution. There has been some analysis demonstrating that discrete adiabatic evolution could be used in Ref. [26] but our analysis here is far tighter. There has also been recent work showing that digital adiabatic simulation based on Trotter-type formulas is robust against discretization [27], whereas our approach does not introduce any discretization error, since we directly invoke the discrete adiabatic theorem. Our analysis here could be tightened further in terms of the constant factors. There is over an order of magnitude difference between the numerical results and the analytically proven scaling constants. A more careful accounting for the inequalities could tighten this difference, but we do not do that in this work because our analysis is already very lengthy.

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APPENDIX A: LIST OF VARIABLES

Here, we give a list of variable names with links to their definitions.

1. List of variables presented in Sec. II

(a) $W(s)$—The discrete-walk operator
(b) n—An integer index used for the discrete-walk operators, so $s = n/T$
(c) $U(s)$—The product of walk operators up to s
(d) $P(s)$—The projector onto the spectrum of interest
(e) $Q(s)$—The projector onto the complementary spectrum
(f) $U_A(s)$—The ideal adiabatic evolution, in contrast to $U(s)$ given by the actual walk operators
(g) $W_A(s)$—Ideal adiabatic walk operators that exactly preserve eigenstates
(h) s—A variable used to index adiabatic evolution, starting from 0 and ending at 1
(i) T—An integer corresponding to the number of discrete-walk operators in discrete adiabatic evolution
(j) D—A difference operator; so, for example, $DW(s) = W(s+1/T) - W(s)$
(k) $D^{(k)}$—The iterated difference operator
(l) $c_k(s)$—A bound on the norm of $D^{(k)}W(s)$ as in Definition 1
(m) $\hat{c}_k(s)$—The maximum of $c_k(s)$ over neighboring time steps, as in Eq. (10)
(n) $\sigma_P(s)$—The spectrum of interest
(o) $\sigma_Q(s)$—The complementary spectrum
(p) $\sigma_P^{(k)}$—An arc including the spectrum $\sigma_P(s)$ at $k+1$ successive steps, as in Eq. (12)
(q) $\sigma_Q^{(k)}$—Similar to $\sigma_P^{(k)}$ but for the complementary spectrum
(r) $\Delta_k(s)$—The gap between the spectra accounting for $k+1$ successive steps; see Definition 2
(s) $\Delta(s)$—The gap accounting for up to three successive steps as defined in Eq. (13)
(t) $\check{\Delta}(s)$—The maximum of $\Delta(s)$ accounting for neighboring steps; see Eq. (14)

2. List of variables presented in Sec. III

(a) $R(s,z)$—The resolvent of $W(s)$; see Eq. (20)
(b) $S(s,s')$—The operator exactly mapping from the spectrum at step s' to s; see Eq. (21). We also use $S(s) = S(s+1/T, s)$
(c) $V(s,s')$—The unitary obtained from a polar decomposition of $S(s,s')$; see Eq. (23). We also use $V(s) = V(s+1/T, s)$
(d) $V(s,s')$—The correction to obtain $V(s,s')$ from $S(s,s')$. We also use $V(s) = V(s+1/T, s)$
(e) $\Omega(s)$—The wave operator, accounting for the difference between the ideal and adiabatic walk; see Eq. (16)
(f) $\Theta(s)$—The ripple operator, corresponding to a step of $\Omega(s)$; see Eq. (28)
(g) $K(s)$—The kernel function; see Eq. (29)
(h) $X(s)$—This operator is arbitrary in the summation-by-parts formula in Theorem 6 but then is used as $T[I - V^\dagger(s-1/T)]$ in the proof of the adiabatic theorem
(i) $Y(s)$—Again, this operator is arbitrary in the summation-by-parts formula but then it is used as $\Omega(s-1/T)$ in the proof of the adiabatic theorem
(j) $\tilde{X}(s)$—Obtained from a contour integral of $X(s)$ as in Eq. (39)
(k) $A(s)$—A variable used in the proof of the discrete adiabatic theorem; see Eq. (40)
(l) $B(s)$—Used in the proof of the discrete adiabatic theorem; see Eq. (41)
(m) $Z(s)$—Used in the proof of the discrete adiabatic theorem; see Eq. (42)
(n) $\Gamma(s)$—A contour that encloses the spectrum of interest
(o) $\Gamma(s,k)$—A contour that encloses the spectrum of interest for $k+1$ successive steps of the walk
(p) $\mathcal{F}(s)$—A function of $DP(s)$ used for expressing $V(s)$; see Eq. (D18)
(q) $\mathcal{B}$—The boundary term used in Theorem 6
(r) $\mathcal{S}$—The sum used in Theorem 6
(s) $n_\pm$—We use $n_+ = n+1$ and $n_- = n-1$; we also use this notation for l
(t) $s_\pm, s_{++}$—We use $s_+ = s + 1/T$, $s_{++} = s + 2/T$ and $s_- = s - 1/T$
(u) P_0—The initial projector onto the spectrum of interest, $P_0 = P(0)$
(v) Q_0—Similarly for the complementary spectrum $Q_0 = Q(0)$
(w) $\mathcal{D}_j(x)$—The simple scalar functions $\mathcal{D}_1(x)$, $\mathcal{D}_2(x)$, and $\mathcal{D}_3(x)$ are defined in Eq. (44)
(x) ξ_j—Constants used for upper bounds on $\mathcal{D}_j(x)$ as in Eq. (E23)
(y) $\mathcal{G}_{T,j}(s)$—These functions for $j = 1,2,3,4$ are defined in Eqs. (45)–(48)

3. List of variables presented in Sec. IV

(a) A—The matrix in the QLSP $Ax = b$
(b) b—The vector in the QLSP
(c) x—This is usually used as the solution vector in $Ax = b$ but in Appendix D as a real variable of integration
(d) N—The dimension of the QLSP
(e) κ—The condition number of A
(f) ϵ—The allowable error in the solution
(g) H_0—The initial Hamiltonian in adiabatic evolution
(h) H_1—The final Hamiltonian in adiabatic evolution
(i) $|b\rangle$—The state with amplitudes proportional to the entries of b
(j) Q_b—The projector eliminating $|b\rangle$, given as $I_N - |b\rangle\langle b|$

(k) $f(s)$—Used for the scheduling function, which we take as in Eq. (69)
(l) d_p—A constant used in constructing $f(s)$; see Eq. (68)
(m) p—An adjustable parameter used in the scheduling function, taking values in the range $(1, 2]$
(n) $\mathbf{A}$—A matrix constructed from A so as to be Hermitian; see Eq. (73)
(o) $A(f)$—The intermediate value of A used in the adiabatic evolution; see Eq. (72)
(p) $H(s)$—The Hamiltonian constructed from $A(f)$; see Eq. (71)
(q) $R(s)$—A rotation used in block encoding $H(s)$; see Eq. (79)

4. List of variables presented in Sec. V

(a) w_j—Weights used for the linear combination of unitaries for filtering
(b) ϕ_k—Used to label eigenvalues of the walk operator, so the eigenvalue is $e^{i\phi_k}$
(c) $\tilde{w}(\phi)$—A Fourier transform of w_j as in Eq. (109)
(d) $\perp$—A set of k such that ϕ_k is not in the spectrum of interest
(e) T_ℓ—The Chebyshev polynomial of the first kind
(f) ℓ—The order of the Chebyshev polynomial

APPENDIX B: PROOF OF PROPOSITIONS

1. Proof of Proposition 4

In order to simplify the notation in the calculations, we define $s_+ := s + 1/T$, $s_{++} := s + 2/T$, and $s_- := s - 1/T$. First, we find that

$$\begin{aligned}P(s_+)\left(P(s) - P(s_+)\right)^2 &= P(s_+)\left(P(s) + P(s_+)\right.\\&\quad\left.- P(s_+)P(s) - P(s)P(s_+)\right)\\&= P(s_+)P(s) + P(s_+)\\&\quad - P(s_+)P(s) - P(s_+)P(s)P(s_+)\\&= P(s_+) - P(s_+)P(s)P(s_+)\\&= \left(P(s) + P(s_+) - P(s_+)P(s) - P(s)P(s_+)\right)P(s_+)\\&= \left(P(s) - P(s_+)\right)^2 P(s_+).\end{aligned} \tag{B1}$$

We can write $v(s_+) = v(s_+, s)$ explicitly in terms of P to give

$$\begin{aligned}v(s_+, s) &= \sqrt{I - \left(P(s) - P(s_+)\right)^2},\\&= I - \frac{1}{2}\left(P(s) - P(s_+)\right)^2\\&\quad - \frac{1}{8}\left(P(s) - P(s_+)\right)^4 + \cdots.\end{aligned} \tag{B2}$$

Therefore, we see that

$$P(s_+)\, v(s_+, s) = v(s_+, s)\, P(s_+), \tag{B3}$$

which implies that

$$P(s_+)\, v(s_+, s)^{-1} = v(s_+, s)^{-1}\, P(s_+). \tag{B4}$$

Using this relation, we obtain

$$\begin{aligned}P(s_+)\, W_A(s) &= P(s_+)\, v(s_+, s)^{-1}\, S(s_+, s)\, W(s)\\&= v(s_+, s)^{-1}\, P(s_+)\, S(s_+, s)\, W(s)\\&= v(s_+, s)^{-1}\, P(s_+)\, P(s) W(s)\\&= v(s_+, s)^{-1}\, P(s_+)\, P(s) P(s) W(s)\\&= v(s_+, s)^{-1}\, P(s_+)\, P(s) W(s) P(s)\\&= W_A(s) P(s).\end{aligned} \tag{B5}$$

This is the equality of Eq. (30) in Proposition 4.

Next, we use this relation to show that

$$\begin{aligned}U_A(s)P(0) &= W_A(s_-)W_A(s - 2/T)\cdots W_A(1/T)W_A(0)P(0)\\&= W_A(s)W_A(s_-)\cdots W_A(1/T)P(1/T)W_A(0)\\&= P(s)W_A(s_-)W_A(s - 2/T)\cdots W_A(1/T)W_A(0)\\&= P(s)U_A(s).\end{aligned} \tag{B6}$$

This is Eq. (31) from Proposition 4.

Next, we use

$$\begin{aligned}P(s_+)v(s)W_A(s)P(s) &= P(s_+)v(s)v(s)^{-1}S(s_+, s)W(s)P(s)\\&= P(s_+)\left(P(s_+)P(s)\right.\\&\quad\left. + Q(s_+)Q(s)\right)W(s)P(s)\\&= P(s_+)P(s)W(s)P(s)\\&= P(s_+)W(s)P(s)\end{aligned} \tag{B7}$$

where in the last row we use the fact that $W(s)P(s) = P(s)W(s)$. This gives Eq. (32) in Proposition 4. The same steps can be performed to prove Eq. (33), where the projectors involved are $Q(s_+)$ and $Q(s)$ rather than those for P.

2. Proof of Proposition 5

From the definition of the ripple operator, given in Eq. (28), and the wave operator, given in Eq. (27), we have

$$\Theta(s) = U_A^\dagger(s_+)U(s_+)U^\dagger(s)U_A(s). \tag{B8}$$

Now, since $W(s)U(s) = U(s_+)$, the above equality becomes

$$\Theta(s) = U_A^\dagger(s_+)W(s)U_A(s). \tag{B9}$$

By inserting $I = V^\dagger(s)V(s)$ and from the definition of the adiabatic walk, i.e., $W_A(s) = V(s)W(s)$, the first equation of the proposition can be proved:

$$\begin{aligned}\Theta(s) &= U_A^\dagger(s_+)V^\dagger(s)V(s)W(s)U_A(s)\\ &= U_A^\dagger(s_+)V^\dagger(s)W_A(s)U_A(s)\\ &= U_A^\dagger(s_+)V^\dagger(s)U_A(s_+).\end{aligned}\tag{B10}$$

For the Volterra equation, the idea is even simpler. From the fact that the Ω operator is unitary and that $\Omega(0) = I$, we have

$$\begin{aligned}\Omega(n/T) &= I - (\Omega(0) - \Omega(n/T))\\ &= I - \sum_{m=0}^{n-1}(\Omega(m/T) - \Omega(m+1/T))\\ &= I - \sum_{m=0}^{n-1}\left(I - \Omega(m+1/T)\Omega^\dagger(m/T)\right)\Omega(m/T)\\ &= I - \sum_{m=0}^{n-1}(I - \Theta(m/T))\,\Omega(m/T)\end{aligned}\tag{B11}$$

where in the last row, we use the definition of the ripple operator as given in Eq. (28). Therefore, the Volterra equation can be concluded by replacing the kernel operator given in Eq. (29) on the right-hand side in the last equality above.

APPENDIX C: PROOF THEOREM 6

Our initial point is to note the following identity:

$$Q(s)X(s)P(s) = -Q(s)[W(s),\tilde{X}(s)]P(s),\tag{C1}$$

which follows from

$$\begin{aligned}[W(s),\tilde{X}(s)] &= -\frac{1}{2\pi i}\oint_{\Gamma(s)}[W(s),R(s,z)X(s)R(s,z)]dz\\ &= -\frac{1}{2\pi i}\oint_{\Gamma(s)}[W(s)-zI,R(s,z)\\ &\quad\times X(s)R(s,z)]dz\\ &= -\frac{1}{2\pi i}\oint_{\Gamma(s)}[X(s)R(s,z)-R(s,z)X(s)]dz\\ &= [P(s),X(s)].\end{aligned}\tag{C2}$$

Now, using the definition Eq. (24) for W_A, we obtain $W(s) = V^\dagger(s)W_A(s)$. Substituting into Eq. (C1), we obtain

$$\begin{aligned}Q(s)X(s)P(s) &= -Q(s)[W_A(s),\tilde{X}(s)]P(s)\\ &\quad - Q(s)[A(s),\tilde{X}(s)]P(s),\end{aligned}\tag{C3}$$

where $A(s)$ is given in Eq. (40). Now, we can use $P(s) = U_A(s)P_0U_A^\dagger(s)$ and $Q(s) = U_A(s)Q_0U_A^\dagger(s)$ in Eq. (C3) to obtain

$$\begin{aligned}Q_0U_A^\dagger(s)X(s)U_A(s)P_0 &= -Q_0U_A^\dagger(s)[W_A(s),\tilde{X}(s)]U_A(s)P_0\\ &\quad - Q_0U_A^\dagger(s)[A(s),\tilde{X}(s)]U_A(s)P_0;\end{aligned}\tag{C4}$$

then,

$$\begin{aligned}&U_A^\dagger(s)\left[W_A(s),\tilde{X}(s)\right]U_A(s)\\ &= U_A^\dagger(s)W_A(s)\tilde{X}(s)U_A(s) - U_A^\dagger(s)\tilde{X}(s)W_A(s)U_A(s)\\ &= U_A^\dagger(s)W_A(s)\tilde{X}(s)U_A(s) - U_A^\dagger(s)\tilde{X}(s)U_A(s_+)\\ &= U_A^\dagger(s)W_A(s_-)\tilde{X}(s)U_A(s) - U_A^\dagger(s)\tilde{X}(s)U_A(s_+) + U_A^\dagger(s)\left(W_A(s) - W^A(s_-)\right)\tilde{X}(s)U_A(s)\\ &= U_A^\dagger(s_-)\tilde{X}(s)U_A(s) - U_A^\dagger(s)\tilde{X}(s)U_A(s_+) + U_A^\dagger(s)DW_A(s_-)\tilde{X}(s)U_A(s)\\ &= U_A^\dagger(s_-)\tilde{X}(s)U_A(s) - U_A^\dagger(s)\tilde{X}(s_+)U_A(s_+) + U_A^\dagger(s)DW_A(s_-)\tilde{X}(s)U_A(s)\\ &\quad + U_A^\dagger(s)D\tilde{X}(s)U_A(s_+)\\ &= U_A^\dagger(s_-)\tilde{X}(s)U_A(s) - U_A^\dagger(s)\tilde{X}(s_+)U_A(s_+) + U_A^\dagger(s)B(s)U_A(s),\end{aligned}\tag{C5}$$

where $B(s)$ is given in Eq. (41). To complete this proof, we have to multiply by $Y(s)$ on the right-hand side of Eq. (C4) and then do a sum from $1/T$ to l/T. First, let us look to the boundary term, which is derived from the first part on the right-hand side of Eq. (C4), i.e.,

$$
\begin{aligned}
&- \sum_{n=1}^{l} Q_0 U_A^{\dagger}\left(\tfrac{n}{T}\right)\left[W_A\left(\tfrac{n}{T}\right), \tilde{X}\left(\tfrac{n}{T}\right)\right] U_A\left(\tfrac{n}{T}\right) P_0 Y\left(\tfrac{n}{T}\right) \\
&\quad = \sum_{n=1}^{l} Q_0 U_A^{\dagger}\left(\tfrac{n}{T}\right) \tilde{X}\left(\tfrac{n_+}{T}\right) U_A\left(\tfrac{n_+}{T}\right) P_0 Y\left(\tfrac{n}{T}\right) - \sum_{n=1}^{l} Q_0 U_A^{\dagger}\left(\tfrac{n_-}{T}\right) \tilde{X}\left(\tfrac{n}{T}\right) U_A\left(\tfrac{n}{T}\right) P_0 Y\left(\tfrac{n}{T}\right) \\
&\qquad - \sum_{n=1}^{l} Q_0 U^{A\dagger}\left(\tfrac{n}{T}\right) B\left(\tfrac{n}{T}\right) U_A\left(\tfrac{n}{T}\right) P_0 Y\left(\tfrac{n}{T}\right) \\
&\quad = \sum_{n=1}^{l} Q_0 U^{A\dagger}\left(\tfrac{n}{T}\right) \tilde{X}\left(\tfrac{n_+}{T}\right) U_A\left(\tfrac{n_+}{T}\right) P_0 Y\left(\tfrac{n}{T}\right) - \sum_{n=0}^{l-1} Q_0 U_A^{\dagger}\left(\tfrac{n}{T}\right) \tilde{X}\left(\tfrac{n_+}{T}\right) U_A\left(\tfrac{n_+}{T}\right) P_0 Y\left(\tfrac{n_+}{T}\right) \\
&\qquad - \sum_{n=1}^{l} Q_0 U_A^{\dagger}\left(\tfrac{n}{T}\right) B\left(\tfrac{n}{T}\right) U_A\left(\tfrac{n}{T}\right) P_0 Y\left(\tfrac{n}{T}\right) \\
&\quad = \mathcal{B} - \sum_{n=1}^{l} Q_0 U_A^{\dagger}\left(\tfrac{n}{T}\right) \tilde{X}\left(\tfrac{n_+}{T}\right) U_A\left(\tfrac{n_+}{T}\right) P_0\, DY\left(\tfrac{n}{T}\right) - \sum_{n=1}^{l} Q_0 U_A^{\dagger}\left(\tfrac{n}{T}\right) B\left(\tfrac{n}{T}\right) U_A\left(\tfrac{n}{T}\right) P_0 Y\left(\tfrac{n}{T}\right). \qquad \text{(C6)}
\end{aligned}
$$

Here, we combine two sums using $Y(s) = -DY(s) + Y(s_+)$, and $\mathcal{B}$ is a correction accounting for the extra term at $n = 0$ and the missing term at $n = l$. It is given by

$$
\begin{aligned}
\mathcal{B} &= Q_0\, U_A^{\dagger}\left(\tfrac{l}{T}\right) \tilde{X}\left(\tfrac{l_+}{T}\right) U_A\left(\tfrac{l_+}{T}\right) P_0 Y\left(\tfrac{l_+}{T}\right) \\
&\quad - Q_0\, U_A^{\dagger}(0) \tilde{X}\left(\tfrac{1}{T}\right) U_A\left(\tfrac{1}{T}\right) P_0\, Y\left(\tfrac{1}{T}\right). \qquad \text{(C7)}
\end{aligned}
$$

We then insert the above result into Eq. (C4), to give

$$
\begin{aligned}
&\sum_{n=1}^{l} Q_0 U_A^{\dagger}\left(\tfrac{n}{T}\right) X\left(\tfrac{n}{T}\right) U_A\left(\tfrac{n}{T}\right) P_0 Y\left(\tfrac{n}{T}\right) \\
&\quad = \mathcal{B} - \sum_{n=1}^{l} \Big\{ Q_0 U_A^{\dagger}\left(\tfrac{n}{T}\right) \tilde{X}\left(\tfrac{n_+}{T}\right) U_A\left(\tfrac{n_+}{T}\right) P_0 DY\left(\tfrac{n}{T}\right) \\
&\qquad + Q_0\, U_A^{\dagger}\left(\tfrac{n}{T}\right) \left(\left[A\left(\tfrac{n}{T}\right), \tilde{X}\left(\tfrac{n}{T}\right)\right]\right. \\
&\qquad + \left. B\left(\tfrac{n}{T}\right)\right) U_A\left(\tfrac{n}{T}\right) P_0\, Y\left(\tfrac{n}{T}\right) \Big\}, \\
&\quad = \mathcal{B} - \frac{1}{T}\mathcal{S}, \qquad \text{(C8)}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{S} = \sum_{n=1}^{l} \Big\{ &Q_0\, U^{A\dagger}\left(\tfrac{n}{T}\right) \tilde{X}\left(\tfrac{n_+}{T}\right) U_A\left(\tfrac{n_+}{T}\right) P_0 T\, DY\left(\tfrac{n}{T}\right) \\
&+ T\, Q_0\, U^{A\dagger}\left(\tfrac{n}{T}\right) \left(\left[A\left(\tfrac{n}{T}\right), \tilde{X}\left(\tfrac{n}{T}\right)\right]\right. \\
&+ \left. B\left(\tfrac{n}{T}\right)\right) U_A\left(\tfrac{n}{T}\right) P_0\, Y\left(\tfrac{n}{T}\right) \Big\}. \qquad \text{(C9)}
\end{aligned}
$$

APPENDIX D: BOUNDING THE OPERATORS

Here, we show the bounds for operators of interest with explicit dependence in terms of the gap. A key part of the method is that we need to consider a contour $\Gamma(s)$ that encloses the spectrum of interest for successive steps of the walk. In particular, we use the notation $\Gamma(s,k)$ to indicate a contour that encloses the spectrum of interest for $k+1$ successive steps of the walk.

In particular, it encloses the spectrum of interest for steps $W(s)$, $W(s+1/T)$ up to $W(s+k/T)$; that is, it encloses the set $\bigcup_{j=0}^{k} \sigma_P(s+j/T)$. Moreover, it does not enclose any eigenvalues in the complementary spectra $\bigcup_{j=0}^{k} \sigma_Q(s+j/T)$.

In order to bound the operators, we choose a specific contour for $\Gamma(s,k)$ that passes in straight lines from the center through the gaps in the spectrum, as shown in Figs. 1 and 7. Those figures indicate that the contour is closed by an arc at radius 2. We take the closure of the contour to be at a distance that approaches infinity for the contours $\Gamma(s,k)$. The results can be obtained by taking the closure at a finite radius and then taking that radius to infinity but for simplicity of the explanation, we do not give that limit explicitly except for one illustrative example. Note that we only take this limit when the integrand approaches zero more quickly than $1/|z|$. That will be true for all the contour integrals that we consider except that for $P(s)$.

We start with the bounds for DP and $D^{(2)}P$, which can be obtained by direct calculations from the definitions.

Lemma 12: *For any integer T and n and the corresponding discrete time $s = n/T$, we have*

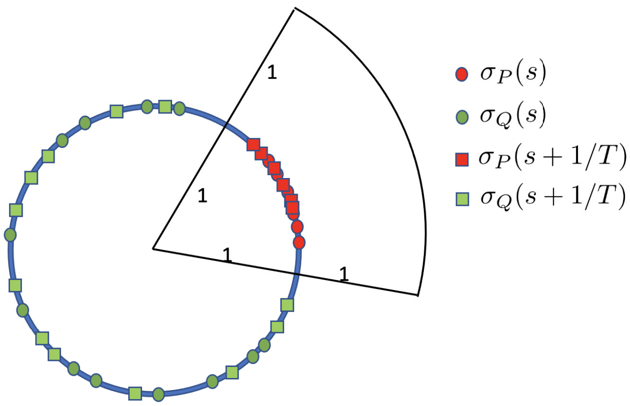

FIG. 7. An illustration of the choice of the contour $\Gamma(s,k)$ for $k=1$. That is, there are two successive steps of the walk and we would need to consider the spectrum for both. We need to be able to use a contour that separates out the spectrum of interest for both steps of the walk. This ensures that we have projectors onto the spectrum of interest that are consistent for both steps, with a gap between the contour and the eigenvalues. We do not allow eigenvalues of interest to cross the gap between one step and the next. Again, we show a contour with radius 2 but we would take the infinite limit of the radius.

$$\|DP(s)\| \leq \frac{2c_1(s)}{T\Delta_1(s)} \tag{D1}$$

and

$$\|D^{(2)}P(s)\| \leq \frac{\mathcal{G}_1(s)}{T^2}, \tag{D2}$$

with

$$\mathcal{G}_1(s) := \frac{c_1(s)^2 + c_1(s)c_1(s_+)}{\pi[1-\cos(\Delta_2(s)/2)]} + \frac{2c_2(s)}{\Delta_2(s)}. \tag{D3}$$

Proof. In order to bound $DP(s)$, we first rewrite $DR(s,z)$ as

$$\begin{aligned} DR(s,z) &= (W(s_+)-zI)^{-1} - (W(s)-zI)^{-1} \\ &= (W(s_+)-zI)^{-1}(W(s)-zI)(W(s)-zI)^{-1} \\ &\quad - (W(s_+)-zI)^{-1}(W(s_+)-zI)(W(s)-zI)^{-1} \\ &= (W(s_+)-zI)^{-1}(W(s)-W(s_+))(W(s)-zI)^{-1} \\ &= -R(s_+,z)DW(s)R(s,z). \end{aligned} \tag{D4}$$

Using this expression, we can then express $DP(s)$ in terms of a contour integral as

$$\begin{aligned} DP(s) &= \frac{1}{2\pi i}\oint_{\Gamma(s,1)} [R(s_+,z) - R(s,z)]\,dz \\ &= -\frac{1}{2\pi i}\oint_{\Gamma(s,1)} R(s_+,z)DW(s)R(s,z)dz. \end{aligned} \tag{D5}$$

When we consider $P(s)$, the integrand drops off as $1/|z|$, so the contour must be at a finite distance as illustrated in Fig. 8. For $DP(s)$, we can use the same contour for both $P(s)$ and $P(s_+)$. The principle now is that the integrand falls off as $1/|z|^2$, so the contribution from the arc falls to zero for large radius. We denote the contour as $\Gamma(s,1,a)$, which is a sector of radius $(a+1)$ for some real number a, and we take the limit $a\to\infty$. Then, we have

$$\begin{aligned} \|DP(s)\| &= \frac{1}{2\pi}\left\|\oint_{\Gamma(s,1,a)} R(s_+,z)DW(s)R(s,z)dz\right\| \\ &\leq \frac{1}{2\pi}\oint_{\Gamma(s,1,a)} \|R(s_+,z)\|\,\|DW(s)\|\|R(s,z)\||dz| \\ &\leq \frac{c_1(s)}{T}\frac{1}{2\pi}\oint_{\Gamma(s,1,a)} \|R(s_+,z)\|\,\|R(s,z)\||dz|, \end{aligned} \tag{D6}$$

where in the last line we use the bound from Eq. (9).

Since $R(s,z)$ is the resolvent of the unitary operator $W(s)$, we know that

$$\|R(s,z)\| = \left\|(W(s)-zI)^{-1}\right\| = \frac{1}{d\left(\sigma[W(s)],z\right)}, \tag{D7}$$

where $d\left(\sigma[W(s)],z\right)$ is the distance between the spectra of W and z. Therefore, by separating the contour integral into three parts, two of them along the radius and one along the arc, we have that

$$\begin{aligned} &\oint_{\Gamma(s,1,a)} \|R(s_+,z)\|\,\|R(s,z)\|\,|dz| \\ &\quad\leq 2\int_0^{a+1} \frac{dx}{[x-\cos(\Delta_1(s)/2)]^2 + [\sin(\Delta_1(s)/2)]^2} \\ &\qquad + \int_{\arg[g_{1,1}(s)]}^{\arg[g_{2,1}(s)]} \frac{1}{a^2}(a+1)d\theta \\ &\quad\leq 2\int_0^{a+1} \frac{dx}{[x-\cos(\Delta_1(s)/2)]^2 + [\sin(\Delta_1(s)/2)]^2} \\ &\qquad + \frac{2\pi(a+1)}{a^2}. \end{aligned} \tag{D8}$$

Here, we denote the complex numbers in the centers of the gaps by $g_{1,1}(s)$ and $g_{2,1}(s)$. By taking the limit $a\to\infty$, we have

$$\begin{aligned}\lim_{a\to\infty}\oint_{\Gamma'(s,1,a)} &\|R(s_+,z)\|\,\|R(s,z)\|\,|dz| \\ &\leq 2\int_0^\infty \frac{dx}{(x-\cos(\Delta_1(s)/2))^2+\sin(\Delta_1(s)/2)^2} \\ &= \frac{2\pi-\Delta_1(s)}{\sin(\Delta_1(s)/2)} \\ &\leq \frac{4\pi}{\Delta_1(s)},\end{aligned} \tag{D9}$$

where in the last line we use $(\pi - x)/\sin x \leq \pi/x$ for $0 < x \leq \pi/2$. Note that taking the limit of $a\to\infty$, the contribution from the arc completely vanishes and we have integrals to infinity along the two straight lines for the contour. That gives a bound on $\|DP(s)\|$ as

$$\|DP(s)\| \leq \frac{2c_1(s)}{T\Delta_1(s)}. \tag{D10}$$

A number of other integrals can be obtained in a similar way. In exactly the same way, we have

$$\begin{aligned}\oint_{\Gamma(s,1)} &\|R(s_+,z)\|^2\,\|R(s,z)\|\,|dz| \\ &\leq 2\int_0^\infty \frac{dx}{[(x-\cos(\Delta_1(s)/2))^2+\sin(\Delta_1(s)/2)^2]^{3/2}} \\ &= \frac{2}{1-\cos(\Delta_1(s)/2)}.\end{aligned} \tag{D11}$$

The same bound holds for similar products of three terms. Here, we write the integral as for the contour $\Gamma(s,1)$. This contour can be regarded as the limit as $a\to\infty$ of the contour $\Gamma(s,1,a)$ but from now on we omit the explicit procedure in taking the limit.

Now, we move on to the $D^{(2)}P$. Since we are dealing with the second-order difference, the contour should be chosen to be $\Gamma(s,2)$, which passes through the eigenvalue gap for three consecutive steps. The above reasoning for the contour integrals is unchanged, except that the gap $\Delta_1(s)$ is changed to $\Delta_2(s)$ for three consecutive steps. We therefore have

$$\begin{aligned}D^{(2)}P(s) &= DP(s_+) - DP(s) \\ &= -\frac{1}{2\pi i}\oint_{\Gamma(s,2)}[R(s_{++},z)\,DW(s_+)\,R(s_+,z) - R(s_+,z)DW(s)R(s,z)]\,dz \\ &= -\frac{1}{2\pi i}\oint_{\Gamma(s,2)}[R(s_{++},z)-R(s_+,z)]\,DW(s_+)\,R(s_+,z)dz - \frac{1}{2\pi i}\oint_{\Gamma(s,2)} R(s_+,z)D^{(2)}W(s)R(s_+,z)dz \\ &\quad - \frac{1}{2\pi i}\oint_{\Gamma(s,2)} R(s_+,z)DW(s)\,[R(s_+,z)-R(s,z)]\,dz \\ &= \frac{1}{2\pi i}\oint_{\Gamma(s,2)} R(s_{++},z)\,DW(s)R(s_+,z)DW(s_+)\,R(s_+,z)dz - \frac{1}{2\pi i}\oint_{\Gamma(s,2)} R(s_+,z)D^{(2)}W(s)R(s_+,z)dz \\ &\quad - \frac{1}{2\pi i}\oint_{\Gamma(s,2)} R(s_+,z)DW(s)R(s_+,z)DW(s)R(s,z)dz.\end{aligned} \tag{D12}$$

We can bound the first term as

$$\begin{aligned}&\left\|\frac{1}{2\pi i}\oint_{\Gamma(s,2)} R(s_{++},z)\,DW(s)R(s_+,z)DW(s_+)\,R(s_+,z)dz\right\| \\ &\quad\leq \frac{1}{2\pi}\oint_{\Gamma(s,2)}\|R(s_{++},z)\|\,\|DW(s)\|\,\|R(s_+,z)\|\,\|DW(s_+)\|\,\|R(s_+,z)\|\,|dz| \\ &\quad\leq \frac{c_1(s)c_1(s_+)}{T^2}\frac{1}{2\pi}\oint_{\Gamma(s,2)}\|R(s_{++},z)\|\,\|R(s_+,z)\|\,\|R(s_+,z)\|\,|dz| \\ &\quad\leq \frac{c_1(s)c_1(s_+)}{T^2}\frac{1}{\pi}\int_0^\infty \frac{dx}{[[x-\cos(\Delta_2(s)/2)]^2+\sin[\Delta_2(s)]/2)^2]^{3/2}} \\ &\quad= \frac{c_1(s)c_1(s_+)}{\pi T^2}\frac{1}{1-\cos(\Delta_2(s)/2)}.\end{aligned} \tag{D13}$$

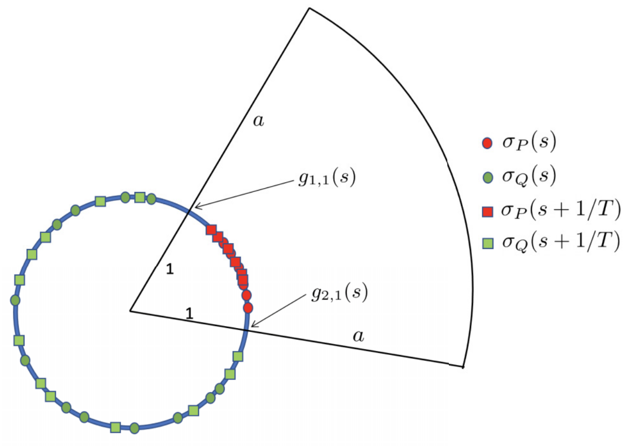

FIG. 8. The contour $\Gamma(s,1,a)$ that passes through the two gaps and has a closure of the contour via an arc at radius $a+1$. The centers of the two gaps are denoted $g_{1,1}(s)$ and $g_{2,1}(s)$ and the contour is taken to have two straight lines that are multiples of these complex numbers.

For the second term, we have the upper bound

$$\begin{aligned}
&\left\|\frac{1}{2\pi i}\oint_{\Gamma(s,2)} R(s_+,z)D^{(2)}W(s)R(s_+,z)dz\right\| \\
&\quad\le \frac{1}{2\pi}\oint_{\Gamma(s,2)} \|R(s_+,z)\|\left\|D^{(2)}W(s)\right\|\|R(s_+,z)\|\,|dz| \\
&\quad\le \frac{c_2(s)}{T^2}\frac{1}{2\pi}\oint_{\Gamma(s,2)} \|R(s_+,z)\|\,\|R(s_+,z)\|\,|dz| \\
&\quad\le \frac{c_2(s)}{\pi T^2}\int_0^\infty \frac{dx}{[x-\cos(\Delta_2(s)/2)]^2+\sin(\Delta_2(s)/2)^2} \\
&\quad= \frac{c_2(s)}{\pi T^2}\frac{\pi-\Delta_2(s)/2}{\sin(\Delta_2(s)/2)} \\
&\quad\le \frac{2c_2(s)}{T^2}\frac{1}{\Delta_2(s)}.
\end{aligned} \tag{D14}$$

For the third term, we have identical reasoning as for the first term, except that $DW(s_+)$ is replaced with $DW(s)$. This gives an upper bound

$$\frac{c_1^2(s)}{\pi T^2}\frac{1}{1-\cos(\Delta_2(s)/2)}. \tag{D15}$$

The three bounds together give us

$$\|D^{(2)}P(s)\| \le \frac{c_1(s)^2+c_1(s)c_1(s_+)}{\pi T^2[1-\cos(\Delta_2(s)/2)]}+\frac{2c_2(s)}{T^2\Delta_2(s)}. \tag{D16}$$

■

Now, we move on to bounding the finite difference of the kernel function and the adiabatic walk operator. The key here is to express and bound the operator V, because it is related to both the kernel and the adiabatic walk operator. First, we reexpress V in terms of P.

Lemma 13: *For a discrete time s, we have*

$$V(s) = \mathcal{F}(s)\left[I+DP(s)(2P(s)-I)\right], \tag{D17}$$

where

$$\mathcal{F}(s) := \left[I-(DP(s))^2\right]^{-1/2}. \tag{D18}$$

Proof. By the definition of $V(s)$ in Eq. (23),

$$\begin{aligned}
V(s) &= \left[I-(P(s)-P(s_+))^2\right]^{-1/2} S(s_+,s) \\
&= \left[I-(P(s)-P(s_+))^2\right]^{-1/2}[I-P(s)-P(s_+) \\
&\quad +2P(s_+)P(s)] \\
&= \left[I-(P(s)-P(s_+))^2\right]^{-1/2}[I+(P(s_+) \\
&\quad -P(s))(2P(s)-I)] \\
&= \mathcal{F}(s)\left[I+DP(s)(2P(s)-I)\right],
\end{aligned} \tag{D19}$$

where in the second equality above we use the definition of $S(s,s')$ in Eq. (21) with $Q(s)=I-P(s)$. ■

This enables us to bound the difference of V from the identity.

Lemma 14: *For a discrete time s,*

$$\|V(s)-I\| \le \|\mathcal{F}(s)-I\| + \|DP(s)\|\|\mathcal{F}(s)\|. \tag{D20}$$

Proof. From Lemma 13, we have

$$V(s)-I = \mathcal{F}(s)+\mathcal{F}(s)[DP(s)(2P(s)-I)]-I. \tag{D21}$$

Then, the triangle inequality gives

$$\begin{aligned}
\|V(s)-I\| &\le \|\mathcal{F}(s)-I\| + \|\mathcal{F}(s)[DP(s)(2P(s)-I)]\| \\
&= \|\mathcal{F}(s)-I\| + \|\mathcal{F}(s)DP(s)\|,
\end{aligned} \tag{D22}$$

where in the second line we use the fact that $2P(s)-I$ is a unitary reflection operator. The inequality $\|\mathcal{F}(s)DP(s)\| \le \|\mathcal{F}(s)\|\,\|DP(s)\|$ then gives the required bound. ■

Next, we bound the change in V.

Lemma 15: *For a discrete time s,*

$$\begin{aligned}
\|DV(s)\| &\le (1+\|DP(s_+)\|)\left\|D^{(2)}P(s)\right\| \\
&\quad\times \mathcal{D}_3\left(\max(\|DP(s_+)\|,\|DP(s)\|)\right) \\
&\quad+\|\mathcal{F}(s)\|\left(\left\|D^{(2)}P(s)\right\|+2\|DP(s)\|^2\right),
\end{aligned} \tag{D23}$$

with

$$\mathcal{D}_3(z) := \frac{z}{(1-z^2)^{3/2}}. \tag{D24}$$

Proof. For the difference operator D, there is the product rule $D[X(s)Y(s)] = DX(s)Y(s_+) + X(s)DY(s)$ for any two operators $X(s)$ and $Y(s)$. Then, from Eq. (D19), we have

$$\begin{aligned}
DV(s) &= D\mathcal{F}(s)[I + DP(s_+)(2P(s_+) - I)] + \mathcal{F}(s)D\,[DP(s)(2P(s) - I)] \\
&= D\mathcal{F}(s)[I + DP(s_+)(2P(s_+) - I)] + \mathcal{F}(s)\left[D^{(2)}P(s)(2P(s_+) - I)\right] + \mathcal{F}(s)DP(s)D\,[2[P(s)] - I] \\
&= D\mathcal{F}(s)[I + DP(s_+)(2P(s_+) - I)] + \mathcal{F}(s)\left[D^{(2)}P(s)(2P(s_+) - I) + 2[DP(s)]^2\right].
\end{aligned} \tag{D25}$$

By the triangle inequality and using the fact that the reflection operator is unitary, we obtain

$$\|DV(s)\| \le \|D\mathcal{F}(s)\|\,(1 + \|DP(s_+)\|) + \|\mathcal{F}(s)\|\left(\|D^{(2)}P(s)\| + 2\|DP(s)\|^2\right). \tag{D26}$$

Now, from the Taylor expansion of $\mathcal{F}$,

$$\mathcal{F}(s) = I + \sum_{k=1}^{\infty}\frac{\Pi_{j=1}^{k}(2j-1)}{2^k k!}[DP(s)]^{2k}, \tag{D27}$$

the bound of $D\mathcal{F}$ in terms of P can be computed, i.e.,

$$\begin{aligned}
\|\mathcal{F}(s_+) - \mathcal{F}(s)\| &= \left\|\sum_{k=1}^{\infty}\frac{\Pi_{i=1}^{k}(2i-1)}{2^k k!}\left[(DP(s_+))^{2k} - (DP(s))^{2k}\right]\right\| \\
&= \left\|\sum_{k=1}^{\infty}\frac{\Pi_{i=1}^{k}(2i-1)}{2^k k!}\sum_{j=0}^{2k-1}(DP(s_+))^{j}\,[DP(s_+) - DP(s)]\,(DP(s))^{2k-1-j}\right\| \\
&\le \sum_{k=1}^{\infty}\frac{\Pi_{i=1}^{k}(2i-1)}{2^k k!}\left\|D^{(2)}P(s)\right\|\sum_{j=0}^{2k-1}\|DP(s_+)\|^{j}\,\|DP(s)\|^{2k-1-j} \\
&\le \left\|D^{(2)}P(s)\right\|\sum_{k=1}^{\infty}\frac{\Pi_{i=1}^{k}(2i-1)}{2^k k!}(2k)\,[\max(\|DP(s_+)\|, \|DP(s)\|)]^{2k-1} \\
&= \left\|D^{(2)}P(s)\right\|\mathcal{D}_3\left(\max(\|DP(s_+)\|, \|DP(s)\|)\right),
\end{aligned} \tag{D28}$$

where we use the Taylor expansion of the function $\mathcal{D}_3$. Substituting this into Eq. (D26) gives the required bound. ■

Lemma 16: *For any discrete time s with $W_A(s)$ defined as in Eq. (24), we have*

$$\|DW_A(s)\| \le \frac{c_1(s)}{T} + \|DV(s)\|. \tag{D29}$$

Proof. According to the definition of $W_A(s)$ as $V(s)W(s)$,

$$DW_A(s) = DV(s)W(s_+) + V(s)DW(s). \tag{D30}$$

Since W and V are unitary and using the triangle inequality, we have

$$\|DW_A(s)\| \le \|DV(s)\| + \|DW(s)\|.$$

Use of Eq. (9) with $k = 1$ for $\|DW(s)\|$ then gives Eq. (D29). ■

Lemma 17: *For any discrete time s, Ω as defined in Eq. (27), and $\mathcal{F}$ defined as in Eq. (D18), we have the upper bound on $D\Omega(s)$,*

$$\|D\Omega(s)\| \le \|\mathcal{F}(s) - I\| + \|DP(s)\|\|\mathcal{F}(s)\|. \tag{D31}$$

Proof. Using the definition of the ripple operator $\Theta(s)$ in Eq. (28), we have

$$\begin{aligned} D\Omega(s) &= \Omega(s_+) - \Omega(s) \\ &= (\Theta(s) - I)\Omega(s) \\ &= U_A(s_+)^\dagger (V(s) - I)^\dagger U_A(s_+)\Omega(s). \end{aligned} \tag{D32}$$

In the third line, we use Eq. (34) for $\Theta(s)$. Since U_A and Ω are unitary, we have

$$\|D\Omega(s)\| = \|(V(s) - I)^\dagger\| = \|V(s) - I\|. \tag{D33}$$

Then, the desired bound follows from Lemma 14. ∎

Finally, we summarize the bounds for the operators related to the summation-by-parts formula.

Lemma 18: *For any discrete time s in $\tilde{X}(s)$ as defined in Eq. (39) and any bounded operator $X(s)$, we have*

$$\left\|\tilde{X}(s)\right\| \le \frac{2}{\Delta_0(s)}\|X(s)\| \tag{D34}$$

and

$$\left\|D\tilde{X}(s)\right\| \le \frac{2}{\Delta_1(s)}\|DX(s)\| + \frac{2c_1(s)}{\pi T[1-\cos(\Delta_1(s)/2)]}\|X(s)\|. \tag{D35}$$

Proof. The bound for $\tilde{X}$ follows directly from the definition in Eq. (39) and choosing an appropriate contour $\Gamma(s,0)$. As shown in Fig. 1, the contour passes in a straight line from the center through both gaps and has a circular arc of radius 2 between these two straight lines. That is, Eq. (39) gives

$$\begin{aligned} \|\tilde{X}(s)\| &\le \frac{1}{2\pi}\oint_{\Gamma(s,0)} \|R(s,z)\|^2\|X(s)\|\,|dz| \\ &\le \frac{1}{2\pi}\|X(s)\|\frac{4\pi}{\Delta_0(s)} = \frac{2}{\Delta_0(s)}\|X(s)\|, \end{aligned} \tag{D36}$$

where we use Eq. (D9) but replace $\Delta_1(s)$ with $\Delta_0(s)$ because we need only consider the eigenvalues for a single step of the walk.

For $D\tilde{X}(s)$, using $\Gamma(s,1)$ (for two consecutive steps of the walk) and using Eq. (39), we have

$$\begin{aligned} D\tilde{X}(s) &= -\frac{1}{2\pi i}\oint_{\Gamma(s,1)} (R(s_+,z)X(s_+)R(s_+,z) - R(s,z)X(s)R(s,z))\,dz \\ &= -\frac{1}{2\pi i}\oint_{\Gamma(s,1)} R(s_+,z)DX(s)R(s_+,z)dz \\ &\quad - \frac{1}{2\pi i}\left(\oint_{\Gamma(s,1)} R(s,z)X(s)DR(s,z)dz + \oint_{\Gamma(s,1)} DR(s,z)X(s)R(s_+,z)dz\right). \end{aligned} \tag{D37}$$

Using Eq. (D4), we have

$$DR(s,z) = -R(s_+,z)DW(s)R(s,z), \tag{D38}$$

so

$$\begin{aligned}\|DR(s,z)\| &\leq \|R(s_+,z)\|\,\|DW(s)\|\,\|R(s,z)\| \\ &\leq \frac{c_1(s)}{T}\,\|R(s_+,z)\|\,\|R(s,z)\|\,. \end{aligned} \tag{D39}$$

We can therefore write an upper bound as

$$\begin{aligned}\|D\tilde{X}(s)\| \leq{}& \frac{\|DX(s)\|}{2\pi}\oint_{\Gamma(s,1)}\|R(s_+,z)\|^2\,|dz| \\ &+ \frac{c_1(s)}{2\pi T}\,\|X(s)\|\left(\oint_{\Gamma(s,1)}\|R(s,z)\|^2\,\|R(s_+,z)\|\,|dz| + \oint_{\Gamma(s,1)}\|R(s,z)\|\,\|R(s_+,z)\|^2\,|dz|\right). \end{aligned} \tag{D40}$$

Using the bounds on the contour integrals given in Eqs. (D9) and (D11), we then obtain

$$\begin{aligned}\left\|D\tilde{X}(s)\right\| &\leq \frac{1}{2\pi}\,\|DX(s)\|\,\frac{4\pi}{\Delta_1(s)} + \frac{1}{\pi}\|X(s)\|\frac{c_1(s)}{T}\,\frac{2}{1-\cos(\Delta_1(s)/2)} \\ &= \frac{2}{\Delta_1(s)}\,\|DX(s)\| + \frac{2c_1(s)}{\pi T[1-\cos(\Delta_1(s)/2)]}\|X(s)\|. \end{aligned} \tag{D41}$$

■

Lemma 19: *For a discrete time s in $A(s)$, $B(s)$ and $Z(s)$ defined in Eqs. (40), (41), and (42), respectively, and any bounded operator $X(s)$, we have*

$$\|A(s)\| \leq \|\mathcal{F}(s)-I\| + \|DP(s)\|\|\mathcal{F}(s)\|, \tag{D42}$$

$$\|B(s)\| \leq \frac{2}{\Delta_1(s)}\,\|DX(s)\| + \frac{2c_1(s)}{\pi T[1-\cos(\Delta_1(s)/2)]}\|X(s)\| \tag{D43}$$

$$+ \frac{2}{\Delta_0(s)}\left(\frac{c_1(s_-)}{T} + \|DV(s_-)\|\right)\|X(s)\|, \tag{D44}$$

and

$$\begin{aligned}\|Z(s)\| \leq{}& \frac{4T}{\Delta_0(s)}\left(\|\mathcal{F}(s)-I\| + \|DP(s)\|\|\mathcal{F}(s)\|\right)\|X(s)\| + \frac{2T}{\Delta_1(s)}\,\|DX(s)\| \\ &+ \frac{2c_1(s)}{\pi[1-\cos(\Delta_1(s)/2)]}\|X(s)\| + \frac{2}{\Delta_0(s)}\left(c_1(s_-) + T\|DV(s_-)\|\right)\|X(s)\|. \end{aligned} \tag{D45}$$

Proof. From the definition of A in Eq. (40), we have

$$\|A(s)\| = \left\|\left(V(s)^\dagger - I\right)W_A(s)\right\| \leq \left\|\left(V(s)^\dagger - I\right)\right\|\,\|W_A(s)\| = \left\|\left(V(s)^\dagger - I\right)\right\|. \tag{D46}$$

The bound for $\|A\|$ follows from Lemma 14.

For B, from Eq. (41), we have that

$$\|B(s)\| \leq \left\|D\tilde{X}(s)W_A(s)\right\| + \left\|DW_A(s_-)\tilde{X}(s)\right\| \leq \left\|D\tilde{X}(s)\right\| + \|DW_A(s_-)\|\,\left\|\tilde{X}(s)\right\|. \tag{D47}$$

By inserting the bounds previously computed for $D\tilde{X}$ (in Lemma 18) and DW_A (in Lemma 16), the desired bound for B is established.

Finally, for Z, the definition in Eq. (42) immediately gives

$$\|Z(s)\| \leq T\left(2\,\|A(s)\|\,\left\|\tilde{X}(s)\right\| + \|B(s)\|\right). \tag{D48}$$

The bounds previously computed in Eqs. (D43), (D34) and (D42) then give the required upper bound. ∎

APPENDIX E: DETAILS FOR THE PROOF OF THE DISCRETE ADIABATIC THEOREMS

1. Diagonal term

Here, we bound the "diagonal" term in Eq. (51). For this term (without loss of generality, we only consider the term projected on P_0), we have

$$\begin{aligned}\left\|\sum_{n=1}^{sT} P_0 U_A^{\dagger}\left(\tfrac{n}{T}\right)\left(I - V^{\dagger}\left(\tfrac{n-}{T}\right)\right) U_A\left(\tfrac{n}{T}\right) P_0 \Omega\left(\tfrac{n-}{T}\right)\right\| &= \left\|\sum_{n=1}^{sT} U_A^{\dagger}\left(\tfrac{n}{T}\right) P\left(\tfrac{n}{T}\right)\left(I - V^{\dagger}\left(\tfrac{n-}{T}\right)\right) P\left(\tfrac{n}{T}\right) U_A\left(\tfrac{n}{T}\right) \Omega\left(\tfrac{n-}{T}\right)\right\| \\ &\leq \sum_{n=1}^{sT}\left\|P\left(\tfrac{n}{T}\right)\left(I - V^{\dagger}\left(\tfrac{n-}{T}\right)\right) P\left(\tfrac{n}{T}\right)\right\|\left\|\Omega\left(\tfrac{n-}{T}\right)\right\| \\ &= \sum_{n=1}^{sT}\left\|P\left(\tfrac{n}{T}\right)\left(I - V^{\dagger}\left(\tfrac{n-}{T}\right)\right) P\left(\tfrac{n}{T}\right)\right\|. \end{aligned} \tag{E1}$$

In the second line, we use Eq. (31). Using Lemma 13, we have

$$\begin{aligned} &\sum_{n=1}^{sT}\left\|P\left(\tfrac{n}{T}\right)\left(I - V^{\dagger}\left(\tfrac{n-}{T}\right)\right) P\left(\tfrac{n}{T}\right)\right\| \\ &\quad= \sum_{n=1}^{sT}\left\|P\left(\tfrac{n}{T}\right)\left(I - \mathcal{F}\left(\tfrac{n-}{T}\right) + \left(2P\left(\tfrac{n-}{T}\right) - I\right) DP\left(\tfrac{n-}{T}\right)\mathcal{F}\left(\tfrac{n-}{T}\right)\right) P\left(\tfrac{n}{T}\right)\right\| \\ &\quad\leq \sum_{n=1}^{sT}\left\|P\left(\tfrac{n}{T}\right)\left(I - \mathcal{F}\left(\tfrac{n-}{T}\right)\right) P\left(\tfrac{n}{T}\right)\right\| + \sum_{n=1}^{sT}\left\|P\left(\tfrac{n}{T}\right)\left(2P\left(\tfrac{n-}{T}\right) - I\right) DP\left(\tfrac{n-}{T}\right)\mathcal{F}\left(\tfrac{n-}{T}\right) P\left(\tfrac{n}{T}\right)\right\| \\ &\quad\leq \sum_{n=0}^{sT-1}\left\|I - \mathcal{F}\left(\tfrac{n}{T}\right)\right\| + \sum_{n=1}^{sT}\left\|P\left(\tfrac{n}{T}\right)\left(2P\left(\tfrac{n-}{T}\right) - I\right) DP\left(\tfrac{n-}{T}\right)\left(\mathcal{F}\left(\tfrac{n-}{T}\right) - I\right) P\left(\tfrac{n}{T}\right)\right\| \\ &\qquad+ \sum_{n=1}^{sT}\left\|P\left(\tfrac{n}{T}\right)\left(2P\left(\tfrac{n-}{T}\right) - I\right) DP\left(\tfrac{n-}{T}\right) P\left(\tfrac{n}{T}\right)\right\| \end{aligned} \tag{E2}$$

From the second term in the last inequality, we have

$$\begin{aligned}\sum_{n=1}^{sT}\left\|P\left(\tfrac{n}{T}\right)\left(2P\left(\tfrac{n-}{T}\right) - I\right) DP\left(\tfrac{n-}{T}\right)\left(\mathcal{F}\left(\tfrac{n-}{T}\right) - I\right) P\left(\tfrac{n}{T}\right)\right\| &\leq \sum_{n=1}^{sT}\left\|DP\left(\tfrac{n-}{T}\right)\left(\mathcal{F}\left(\tfrac{n-}{T}\right) - I\right)\right\| \\ &= \sum_{n=0}^{sT-1}\left\|I - \mathcal{F}\left(\tfrac{n}{T}\right)\right\|\left\|DP\left(\tfrac{n}{T}\right)\right\|. \end{aligned} \tag{E3}$$

Now, if we replace $P(\frac{n_-}{T}) = P(\frac{n}{T}) - DP(\frac{n_-}{T})$ in the last term of Eq. (E2), then we obtain

$$\begin{aligned}
\left\|P\left(\tfrac{n}{T}\right)\left(2P\left(\tfrac{n_-}{T}\right)-I\right)DP\left(\tfrac{n_-}{T}\right)P\left(\tfrac{n}{T}\right)\right\| &= \left\|P\left(\tfrac{n}{T}\right)\left[2\left(P\left(\tfrac{n}{T}\right)-DP\left(\tfrac{n_-}{T}\right)\right)-I\right]DP\left(\tfrac{n_-}{T}\right)P\left(\tfrac{n}{T}\right)\right\| \\
&\le \left\|P\left(\tfrac{n}{T}\right)\left(2P\left(\tfrac{n}{T}\right)-I\right)DP\left(\tfrac{n_-}{T}\right)P\left(\tfrac{n}{T}\right)\right\| + 2\left\|P\left(\tfrac{n}{T}\right)DP\left(\tfrac{n_-}{T}\right)^2P\left(\tfrac{n}{T}\right)\right\| \\
&= \left\|P\left(\tfrac{n}{T}\right)DP\left(\tfrac{n_-}{T}\right)P\left(\tfrac{n}{T}\right)\right\| + 2\left\|P\left(\tfrac{n}{T}\right)DP\left(\tfrac{n_-}{T}\right)^2P\left(\tfrac{n}{T}\right)\right\| \\
&= 3\left\|P\left(\tfrac{n}{T}\right)DP\left(\tfrac{n_-}{T}\right)^2P\left(\tfrac{n}{T}\right)\right\|.
\end{aligned} \tag{E4}$$

In the last calculation above, we use the equality $p(p-q)p = p(p-q)^2p$ when we have p and q as any two projections. Thus,

$$\begin{aligned}
\sum_{n=1}^{sT}\left\|P\left(\tfrac{n}{T}\right)\left(I-V^\dagger\left(\tfrac{n_-}{T}\right)\right)P\left(\tfrac{n}{T}\right)\right\| &\le \sum_{n=0}^{sT-1}\left\|I-\mathcal{F}\left(\tfrac{n}{T}\right)\right\|\left(1+\left\|DP\left(\tfrac{n}{T}\right)\right\|\right) + 3\sum_{n=1}^{sT}\left\|P\left(\tfrac{n}{T}\right)DP\left(\tfrac{n_-}{T}\right)^2P\left(\tfrac{n}{T}\right)\right\| \\
&= \sum_{n=0}^{sT-1}\left\|I-\mathcal{F}\left(\tfrac{n}{T}\right)\right\|\left(1+\left\|DP\left(\tfrac{n}{T}\right)\right\|\right) + 3\sum_{n=1}^{sT}\left\|P\left(\tfrac{n}{T}\right)(DP\left(\tfrac{n_-}{T}\right)^2P\left(\tfrac{n}{T}\right)\right\| \\
&\le \sum_{n=0}^{sT-1}\left\|I-\mathcal{F}\left(\tfrac{n}{T}\right)\right\|\left(1+\left\|DP\left(\tfrac{n}{T}\right)\right\|\right) + 3\sum_{n=0}^{sT-1}\left\|DP\left(\tfrac{n}{T}\right)\right\|^2.
\end{aligned} \tag{E5}$$

To bound $\|\mathcal{F}(s) - I\|$, we can use Lemma 12 and the definition of $\mathcal{F}(s)$ as follows:

$$\begin{aligned}
\|\mathcal{F}(s) - I\| &\le \sum_{k=1}^{\infty}\frac{\Pi_{j=1}^k(2j-1)}{2^kk!}\|DP(s)\|^{2k} \\
&\le \sum_{k=1}^{\infty}\frac{\Pi_{j=1}^k(2j-1)}{2^kk!}\left(\frac{2c_1(s)}{T\Delta_1(s)}\right)^{2k} \\
&= \left(1-\frac{4c_1(s)^2}{T^2\Delta_1(s)^2}\right)^{-1/2} - 1 \\
&= \mathcal{D}_1\left(\frac{2c_1(s)}{T\Delta_1(s)}\right) - 1.
\end{aligned} \tag{E6}$$

2. Off-diagonal term

For the "off-diagonal" term in Eq. (53) used for Theorem 7, we have

$$\begin{aligned}
&\left\|\sum_{n=1}^{sT}Q_0U_A^\dagger\left(\tfrac{n}{T}\right)\left(I-V^\dagger\left(\tfrac{n_-}{T}\right)\right)U_A\left(\tfrac{n}{T}\right)P_0\Omega\left(\tfrac{n_-}{T}\right)\right\| \\
&\le \frac{1}{T}\left\|\sum_{n=1}^{sT-1}Q_0U_A^\dagger\left(\tfrac{n}{T}\right)X\left(\tfrac{n}{T}\right)U_A\left(\tfrac{n}{T}\right)P_0Y\left(\tfrac{n}{T}\right)\right\| + \frac{1}{T}\left\|Q_0U_A^\dagger(s)X(s)U_A(s)P_0Y(s)\right\| \\
&\le \frac{1}{T}\|\mathcal{B}\| + \frac{1}{T^2}\|\mathcal{S}\| + \frac{1}{T}\|X(s)\|\,\|Y(s)\| \\
&\le \frac{1}{T}\left\|\tilde{X}\left(\tfrac{1}{T}\right)\right\|\left\|Y\left(\tfrac{1}{T}\right)\right\| + \frac{1}{T}\left\|\tilde{X}(s)\right\|\|Y(s)\| + \frac{1}{T^2}\sum_{n=1}^{sT-1}\left\|Z\left(\tfrac{n}{T}\right)\right\|\left\|Y\left(\tfrac{n}{T}\right)\right\| + \frac{1}{T}\sum_{n=1}^{sT-1}\left\|\tilde{X}\left(\tfrac{n_+}{T}\right)\right\|\left\|DY\left(\tfrac{n}{T}\right)\right\| + \frac{1}{T}\|X(s)\|\,\|Y(s)\| \\
&= \frac{1}{T}\left\|\tilde{X}\left(\tfrac{1}{T}\right)\right\| + \frac{1}{T}\left\|\tilde{X}(s)\right\| + \frac{1}{T^2}\sum_{n=1}^{sT-1}\left\|Z\left(\tfrac{n}{T}\right)\right\| + \frac{1}{T}\sum_{n=1}^{sT-1}\left\|\tilde{X}\left(\tfrac{n_+}{T}\right)\right\|\left\|DY\left(\tfrac{n}{T}\right)\right\| + \frac{1}{T}\|X(s)\|.
\end{aligned} \tag{E7}$$

In the third line, we use the summation-by-parts result in Theorem 6 with $l = sT - 1$. In the fourth line, we use the product rule for norms of products and the fact that the spectral norms of projectors and unitary operators are 1. In the last line, we use the fact that the choice of Y is unitary. Next, we use the previously derived lemmas to provide bounds for the individual operators, $X(s)$, $\tilde{X}(s)$, $Z(s)$, and $DY(s)$. Starting with $X(s)$, we have, from Lemma 14 combined with Lemma 12, that

$$\|X(s)\| = T\|V(s_-) - I\| \leq T\|\mathcal{F}(s_-) - I\| + T\frac{2c_1(s_-)}{T\Delta_1(s_-)}\|\mathcal{F}(s_-)\|. \tag{E8}$$

Now, we use the upper bound from Eq. (E6) to provide a bound on $\mathcal{F}(s_-)$ as

$$\|\mathcal{F}(s_-)\| \leq 1 + \sum_{k=1}^{\infty}\frac{\Pi_{j=1}^{k}(2j-1)}{2^k k!}\left(\frac{2c_1(s_-)}{T\Delta_1(s_-)}\right)^{2k} = \left(1 - \frac{4c_1(s_-)^2}{T^2\Delta_1(s_-)^2}\right)^{-1/2}, \tag{E9}$$

and, similarly,

$$\|\mathcal{F}(s_-) - I\| \leq \left(1 - \frac{4c_1(s_-)^2}{T^2\Delta_1(s_-)^2}\right)^{-1/2} - 1. \tag{E10}$$

That gives the following upper bound for $X(s)$:

$$\begin{aligned}\|X(s)\| &\leq T\left[\left(1 + \frac{2c_1(s_-)}{T\Delta_1(s_-)}\right)\left(1 - \frac{4c_1(s_-)^2}{T^2\Delta_1(s_-)^2}\right)^{-1/2} - 1\right]\\ &= T\left[\left(1 + \frac{2c_1(s_-)}{T\Delta_1(s_-)}\right)^{1/2}\left(1 - \frac{2c_1(s_-)}{T\Delta_1(s_-)}\right)^{-1/2} - 1\right]\\ &= T\mathcal{D}_2\left(\frac{2c_1(s_-)}{T\Delta_1(s_-)}\right).\end{aligned} \tag{E11}$$

Now, for $\tilde{X}(s)$, we can use the bound from Lemma 18, which gives

$$\left\|\tilde{X}(s)\right\| \leq \frac{2T}{\Delta_0(s)}\mathcal{D}_2\left(\frac{2c_1(s_-)}{T\Delta_1(s_-)}\right). \tag{E12}$$

For the bound on $Z(s)$, we can use Eq. (D44) from Lemma 19, but first we need bounds for $DX(s)$ and $DV(s_-)$, which can be obtained using Lemma 15 in combination with Lemma 12 as follows:

$$\|DX(s)\| = T\|DV(s_-)\| \tag{E13}$$

$$\begin{aligned}&\leq T(1 + \|DP(s)\|)\left\|D^{(2)}P(s_-)\right\|\mathcal{D}_3\left(\max(\|DP(s)\|, \|DP(s_-)\|)\right) + T\|\mathcal{F}(s_-)\|\left(\left\|D^{(2)}P(s_-)\right\| + 2\|[DP(s_-)]\|^2\right)\\ &\leq \left(1 + \frac{2c_1(s)}{T\Delta_1(s)}\right)\frac{\mathcal{G}_1(s_-)}{T}\mathcal{D}_3\left(\max\left(\frac{2c_1(s)}{T\Delta_1(s)}, \frac{2c_1(s_-)}{T\Delta_1(s_-)}\right)\right) + \mathcal{D}_1\left(\frac{2c_1(s_-)}{T\Delta_1(s_-)}\right)\left(\frac{\mathcal{G}_1(s_-)}{T} + \frac{8c_1(s_-)^2}{T\Delta_1(s_-)^2}\right).\end{aligned} \tag{E14}$$

In the first line, we use $X(s) = T[I - V^{\dagger}(s_-)]$; in the second line, we use Lemma 15; and at the end, we use Lemma 12 in combination with the fact that the functions $\mathcal{D}_1$ and $\mathcal{D}_3$ are monotonically increasing. Now, the functions $\mathcal{G}_2$ and $\mathcal{G}_3$ from

Eqs. (46) and (47) can be used in the last expression above, to give

$$\begin{aligned} \|DX(s)\| &\le \left(1+\frac{2c_1(s)}{T\Delta_1(s)}\right)\frac{\mathcal{G}_2(s_-)}{T}+\mathcal{D}_1\left(\frac{2c_1(s_-)}{T\Delta_1(s_-)}\right)\left(\frac{\mathcal{G}_1(s_-)}{T}+\frac{8c_1(s_-)^2}{T\Delta_1(s_-)^2}\right)\\ &= \frac{\mathcal{G}_3(s_-)}{T}. \end{aligned} \tag{E15}$$

Then, from Eq. (E13), we have

$$\|DV(s_-)\| \le \frac{\mathcal{G}_3(s_-)}{T^2}. \tag{E16}$$

Now that we have these bounds, we proceed to bound $Z(s)$ using Eq. (D44). Starting with the replacement of the bound of $DV(s_-)$, we can make use of the function $\mathcal{G}_4(s)$ as defined in Eq. (48):

$$\begin{aligned} \|Z(s)\| \le{}& \frac{4T}{\Delta_0(s)}\left(\|\mathcal{F}(s)-I\|+\|DP(s)\|\,\|\mathcal{F}(s)\|\right)\|X(s)\|+\frac{2T}{\Delta_1(s)}\left\|DX\left(\tfrac{n}{T}\right)\right\|\\ &+\frac{2c_1(s)}{\pi[1-\cos(\Delta_1(s)/2)]}\left\|X\left(\tfrac{n}{T}\right)\right\|+\frac{2\mathcal{G}_4(s_-)}{\Delta_0(s)}\|X(s)\|. \end{aligned} \tag{E17}$$

Our next step is the replacement of the bound of $X(s)$,

$$\begin{aligned} \|Z(s)\| \le{}& \frac{4T^2}{\Delta_0(s)}\left(\|\mathcal{F}(s)-I\|+\|DP(s)\|\,\|\mathcal{F}(s)\|\right)\mathcal{D}_2\left(\frac{2c_1(s_-)}{\Delta_1(s_-)}\right)+\frac{2T}{\Delta_1(s)}\|DX(s)\|\\ &+\frac{2Tc_1(s)}{\pi[1-\cos(\Delta_1(s)/2)]}\mathcal{D}_2\left(\frac{2c_1(s_-)}{T\Delta_1(s_-)}\right)+\frac{2T\mathcal{G}_4(s_-)}{\Delta_0(s)}\mathcal{D}_2\left(\frac{2c_1(s_-)}{T\Delta_1(s_-)}\right). \end{aligned} \tag{E18}$$

Now, we use

$$\|\mathcal{F}(s)-I\|+\|DP(s)\|\,\|\mathcal{F}(s)\| \le \mathcal{D}_2\left(\frac{2c_1(s)}{T\Delta_1(s)}\right) \tag{E19}$$

and the bound derived above for $DX(s)$ to yield

$$\begin{aligned} \|Z(s)\| \le{}& \frac{4T^2}{\Delta_0(s)}\mathcal{D}_2\left(\frac{2c_1(s)}{\Delta_1(s)}\right)\mathcal{D}_2\left(\frac{2c_1(s_-)}{\Delta_1(s_-)}\right)+\frac{2\mathcal{G}_3(s_-)}{\Delta_1(s)}\\ &+\frac{2Tc_1(s)}{\pi[1-\cos(\Delta_1(s)/2)]}\mathcal{D}_2\left(\frac{2c_1(s_-)}{T\Delta_1(s_-)}\right)+\frac{2T\mathcal{G}_4(s_-)}{\Delta_0(s)}\mathcal{D}_2\left(\frac{2c_1(s_-)}{T\Delta_1(s_-)}\right). \end{aligned} \tag{E20}$$

Finally, for the upper bound of $DY(s)$, first note that $Y(s)=\Omega(s_-)$, so $D\Omega(s_-)=DY(s)$. Therefore, using Lemma 17 and our bound in Eq. (E19), we obtain

$$\|DY(s)\| \le \mathcal{D}_2\left(\frac{2c_1(s_-)}{T\Delta_1(s_-)}\right). \tag{E21}$$

3. Proof of the second adiabatic theorem

Proof. We first bound the functions $\mathcal{D}_k$ with simpler expressions. Recall that the definitions of $\mathcal{D}_k$ are

$$\mathcal{D}_1(z)=\frac{1}{\sqrt{1-z^2}}, \qquad \mathcal{D}_2(z)=\sqrt{\frac{1+z}{1-z}}-1, \qquad \mathcal{D}_3(z)=\frac{z}{(1-z^2)^{3/2}}. \tag{E22}$$

Note that in Theorem 7, all the arguments in $\mathcal{D}_k$ are in the form of $2c_1/(T\Delta_1)$; then, under the assumption on T, we are only interested in the case $0\le z\le 1/2$. Therefore, we have

$$\mathcal{D}_1(z)\le \xi_1, \qquad \mathcal{D}_2(z)\le \xi_2 z, \qquad \mathcal{D}_3(z)\le \xi_3 z, \tag{E23}$$

with constants $\xi_1=2/\sqrt{3}$, $\xi_2=2\sqrt{3}-2$, and $\xi_3=8/(3\sqrt{3})$.

Now, we move on to the functions $\mathcal{G}_k$. For any positive integer n, from Theorem 7 we need to bound $\mathcal{G}_3(n_-/T)$ and $\mathcal{G}_4(n_-/T)$, which in turn depend on $\mathcal{G}_1(n_-/T)$ and $\mathcal{G}_2(n_-/T)$. Using the inequality $1-\cos(\theta/2) = 2\sin^2(\theta/4) \geq \theta^2/\pi^2$ for all $0 \leq \theta \leq \pi$, it is possible to show that (using s to indicate n/T)

$$\begin{aligned}\mathcal{G}_1(s_-) &= \frac{c_1(s_-)^2 + c_1(s_-)c_1(s)}{\pi[1-\cos(\Delta_2(s_-)/2)]} + \frac{2c_2(s_-)}{\Delta_2(s_-)}\\ &\leq \frac{\pi c_1(s_-)^2 + \pi c_1(s_-)c_1(s)}{\Delta_2(s_-)^2} + \frac{2c_2(s_-)}{\Delta_2(s_-)}\\ &\leq \frac{2\pi\hat{c}_1(s)^2}{\check{\Delta}(s)^2} + \frac{2\hat{c}_2(s)}{\check{\Delta}(s)}.\end{aligned} \tag{E24}$$

Use of that and the assumption that $T \geq \max[4\hat{c}_1(s)/\check{\Delta}(s)]$ gives the bound

$$\mathcal{G}_2(s_-) \leq \xi_3 \frac{2\hat{c}_1(s)}{T\check{\Delta}(s)}\left(\frac{2\pi\hat{c}_1(s)^2}{\check{\Delta}(s)^2} + \frac{2\hat{c}_2(s)}{\check{\Delta}(s)}\right) = \frac{4\pi\xi_3\hat{c}_1(s)^3}{T\check{\Delta}(s)^3} + \frac{4\xi_3\hat{c}_1(s)\hat{c}_2(s)}{T\check{\Delta}(s)^2}. \tag{E25}$$

Note that the assumption $T \geq \max[4\hat{c}_1(s)/\check{\Delta}(s)]$ is required for the function $\mathcal{D}_3(z)$ used in $\mathcal{G}_2(s_-)$ to be appropriately bounded. This is why there is a factor of 4 in this condition for the second discrete adiabatic theorem, whereas there is a factor of 2 in the condition in the first discrete adiabatic theorem. For the remaining functions, we obtain the bounds

$$\begin{aligned}\mathcal{G}_3(s_-) &\leq \frac{3}{2}\left(\frac{4\pi\xi_3\hat{c}_1(s)^3}{T\check{\Delta}(s)^3} + \frac{4\xi_3\hat{c}_1(s)\hat{c}_2(s)}{T\check{\Delta}(s)^2}\right) + \xi_1\left(\frac{2\pi\hat{c}_1(s)^2}{\check{\Delta}(s)^2} + \frac{2\hat{c}_2(s)}{\check{\Delta}(s)} + \frac{8\hat{c}_1(s)^2}{\check{\Delta}(s)^2}\right)\\ &\leq \frac{3}{2}\left(\frac{\pi\xi_3\hat{c}_1(s)^2}{\check{\Delta}(s)^2} + \frac{\xi_3\hat{c}_2(s)}{\check{\Delta}(s)}\right) + \xi_1\left(\frac{(2\pi+8)\hat{c}_1(s)^2}{\check{\Delta}(s)^2} + \frac{2\hat{c}_2(s)}{\check{\Delta}(s)}\right)\\ &\leq (3\pi\xi_3/2 + (2\pi+8)\xi_1)\frac{\hat{c}_1(s)^2}{\check{\Delta}(s)^2} + (3\xi_3/2 + 2\xi_1)\frac{\hat{c}_2(s)}{\check{\Delta}(s)}\end{aligned} \tag{E26}$$

and

$$\mathcal{G}_4(s_-) \leq (3\pi\xi_3/2 + (2\pi+8)\xi_1)\frac{\hat{c}_1(s)^2}{T\check{\Delta}(s)^2} + (3\xi_3/2 + 2\xi_1)\frac{\hat{c}_2(s)}{T\check{\Delta}(s)} + \hat{c}_1(s). \tag{E27}$$

Inserting all these bounds back into Theorem 7 and using $1-\cos(\theta/2) \geq \theta^2/\pi^2$ again, we have

$$\begin{aligned}&\|U(s) - U_A(s)\|\\ &\leq \frac{8\xi_2\hat{c}_1(0)}{T\check{\Delta}(0)^2} + \frac{8\xi_2\hat{c}_1(s)}{T\check{\Delta}(s)^2} + \frac{4\xi_2\hat{c}_1(s)}{T\check{\Delta}(s)} + \sum_{n=1}^{sT-1}\frac{12}{\check{\Delta}(n/T)}\xi_2^2\left(\frac{2\hat{c}_1(n/T)}{T\check{\Delta}(n/T)}\right)^2\\ &\quad + \sum_{n=1}^{sT-1}(6\pi\xi_3 + (8\pi+32)\xi_1)\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^3} + \sum_{n=1}^{sT-1}(6\xi_3 + 8\xi_1)\frac{\hat{c}_2(n/T)}{T^2\check{\Delta}(n/T)^2} + \sum_{n=1}^{sT-1}\frac{4\pi\hat{c}_1(n/T)}{T\check{\Delta}(n/T)^2}\xi_2\frac{2\hat{c}_1(n/T)}{T\check{\Delta}(n/T)}\\ &\quad + \sum_{n=1}^{sT-1}(6\pi\xi_3 + (8\pi+32)\xi_1)\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^3}\left(\xi_2\frac{2\hat{c}_1(n/T)}{T\check{\Delta}(n/T)}\right) + \sum_{n=1}^{sT-1}(6\xi_3 + 8\xi_1)\frac{\hat{c}_2(n/T)}{T^2\check{\Delta}(n/T)^2}\left(\xi_2\frac{2\hat{c}_1(n/T)}{T\check{\Delta}(n/T)}\right)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{sT-1}\frac{4\hat{c}_1(n/T)}{T\check{\Delta}(n/T)}\left(\xi_2\frac{2\hat{c}_1(n/T)}{T\check{\Delta}(n/T)}\right)+\sum_{n=0}^{sT-1}\frac{24\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^2}+\sum_{n=0}^{sT-1}\frac{8\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^2}\\
\le{}&\frac{8\xi_2\hat{c}_1(0)}{T\check{\Delta}(0)^2}+\frac{8\xi_2\hat{c}_1(s)}{T\check{\Delta}(s)^2}+\frac{4\xi_2\hat{c}_1(s)}{T\check{\Delta}(s)}+\sum_{n=1}^{sT-1}\frac{48\xi_2^2\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^3}\\
&+\sum_{n=1}^{sT-1}(6\pi\xi_3+(8\pi+32)\xi_1)\,\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^3}+\sum_{n=1}^{sT-1}(6\xi_3+8\xi_1)\frac{\hat{c}_2(n/T)}{T^2\check{\Delta}(n/T)^2}+\sum_{n=1}^{sT-1}\frac{8\pi\xi_2\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^3}\\
&+\sum_{n=1}^{sT-1}(12\pi\xi_2\xi_3+(16\pi+64)\xi_1\xi_2)\,\frac{\hat{c}_1(n/T)^3}{T^3\check{\Delta}(n/T)^4}+\sum_{n=1}^{sT-1}(12\xi_2\xi_3+16\xi_1\xi_2)\,\frac{\hat{c}_1(n/T)\hat{c}_2(n/T)}{T^3\check{\Delta}(n/T)^3}\\
&+\sum_{n=1}^{sT-1}\frac{8\xi_2\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^2}+\sum_{n=0}^{sT-1}\frac{24\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^2}+\sum_{n=0}^{sT-1}\frac{8\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^2}\\
\le{}&\frac{8\xi_2\hat{c}_1(0)}{T\check{\Delta}(0)^2}+\frac{8\xi_2\hat{c}_1(s)}{T\check{\Delta}(s)^2}+\frac{4\xi_2\hat{c}_1(s)}{T\check{\Delta}(s)}+\sum_{n=1}^{sT-1}\left(48\xi_2^2+6\pi\xi_3+(8\pi+32)\xi_1+8\pi\xi_2\right)\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^3}\\
&+\sum_{n=1}^{sT-1}(6\xi_3+8\xi_1)\frac{\hat{c}_2(n/T)}{T^2\check{\Delta}(n/T)^2}+\sum_{n=0}^{sT-1}\frac{(32+8\xi_2)\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^2}\\
&+\sum_{n=1}^{sT-1}(12\pi\xi_2\xi_3+(16\pi+64)\xi_1\xi_2)\,\frac{\hat{c}_1(n/T)^3}{T^3\check{\Delta}(n/T)^4}+\sum_{n=1}^{sT-1}(12\xi_2\xi_3+16\xi_1\xi_2)\,\frac{\hat{c}_1(n/T)\hat{c}_2(n/T)}{T^3\check{\Delta}(n/T)^3}.
\end{aligned}\tag{E28}
$$

Finally, for a clear representation in terms of the gap, we slightly modify some terms with T^3 on the denominator to T^2 by using the bounds $\hat{c}_1(s)/[T\check{\Delta}(s)]\le 1/4$. Then,

$$
\begin{aligned}
\|U(s)-U_A(s)\|\le{}&\frac{8\xi_2\hat{c}_1(0)}{T\check{\Delta}(0)^2}+\frac{8\xi_2\hat{c}_1(s)}{T\check{\Delta}(s)^2}+\frac{4\xi_2\hat{c}_1(s)}{T\check{\Delta}(s)}+\sum_{n=1}^{sT-1}\left[48\xi_2^2+6\pi\xi_3+(8\pi+32)\xi_1+8\pi\xi_2\right]\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^3}\\
&+\sum_{n=1}^{sT-1}(6\xi_3+8\xi_1)\frac{\hat{c}_2(n/T)}{T^2\check{\Delta}(n/T)^2}+\sum_{n=0}^{sT-1}\frac{(32+8\xi_2)\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^2}\\
&+\sum_{n=1}^{sT-1}(3\pi\xi_2\xi_3+(4\pi+16)\xi_1\xi_2)\,\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^3}+\sum_{n=1}^{sT-1}(3\xi_2\xi_3+4\xi_1\xi_2)\,\frac{\hat{c}_2(n/T)}{T^2\check{\Delta}(n/T)^2}\\
={}&\frac{8\xi_2\hat{c}_1(0)}{T\check{\Delta}(0)^2}+\frac{8\xi_2\hat{c}_1(s)}{T\check{\Delta}(s)^2}+\frac{4\xi_2\hat{c}_1(s)}{T\check{\Delta}(s)}\\
&+\left[48\xi_2^2+6\pi\xi_3+(8\pi+32)\xi_1+8\pi\xi_2+3\pi\xi_2\xi_3+(4\pi+16)\xi_1\xi_2\right]\sum_{n=1}^{sT-1}\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^3}\\
&+(32+8\xi_2)\sum_{n=0}^{sT-1}\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^2}+(6\xi_3+8\xi_1+3\xi_2\xi_3+4\xi_1\xi_2)\sum_{n=1}^{sT-1}\frac{\hat{c}_2(n/T)}{T^2\check{\Delta}(n/T)^2}\\
\le{}&\frac{12\hat{c}_1(0)}{T\check{\Delta}(0)^2}+\frac{12\hat{c}_1(s)}{T\check{\Delta}(s)^2}+\frac{6\hat{c}_1(s)}{T\check{\Delta}(s)}\\
&+305\sum_{n=1}^{sT-1}\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^3}+44\sum_{n=0}^{sT-1}\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^2}+32\sum_{n=1}^{sT-1}\frac{\hat{c}_2(n/T)}{T^2\check{\Delta}(n/T)^2},
\end{aligned}\tag{E29}
$$

where the last inequality is derived by inserting the concrete values of ξ_k into the bound and rounding the resulting constants to the closest integers greater than or equal to them. ■

APPENDIX F: BLOCK ENCODING OF $H(s)$

Here, we describe how to perform the block encoding of $H(s)$ as given in Eq. (71). We denote the unitary for the block encoding of A as U_A, which acts on an ancilla denoted with subscript a and the system such that

$$ {}_a\langle 0|\, U_A\, |0\rangle_a = A. \tag{F1}$$

We also denote the unitary oracle for preparing $|b\rangle$ as U_b such that

$$U_b\, |0\rangle = |b\rangle\,. \tag{F2}$$

As well as the ancilla system used for the block encoding of A, we use four ancilla qubits. These ancilla qubits are used as follows:

(1) The first selects between the blocks in $A(f)$.
(2) The next is used for preparing the combination of $\sigma_z \otimes I$ and $\mathbf{A}$.
(3) The third is used in implementing $Q_{\mathbf{b}}$.
(4) The fourth selects between the blocks in $H(s)$.

These four qubits are denoted with subscripts a_1–a_4.

First consider $A(f)$, which can be written as

$$\begin{aligned} A(f) = (1-f)\sigma^z_{a_1} \otimes I_N + f\,(|0\rangle\langle 1|_{a_1} \otimes A \\ + |1\rangle\langle 0|_{a_1} \otimes A^\dagger). \end{aligned} \tag{F3}$$

Note that the first operators in the tensor products here, σ^z and $|0\rangle\langle 1|$ or $|1\rangle\langle 0|$, act upon the ancilla denoted a_1. To block encode the operation using ancilla a_2, we can use the select operation

$$\begin{aligned} U_{A(f)} = |0\rangle\langle 0|_{a_2} \otimes \sigma^z_{a_1} \otimes I_N \otimes I_a + |1\rangle\langle 1|_{a_2} \\ \otimes (|0\rangle\langle 1|_{a_1} \otimes U_A + |1\rangle\langle 0|_{a_1} \otimes U_A^\dagger). \end{aligned} \tag{F4}$$

Here, we include I_a to indicate that the operation is acting as the identity on the ancilla system used for the block encoding of A. Note that we require the ability to apply the oracle U_A in a selected way, where we either perform U_A, $U_A^\dagger$ or the identity.

Next consider $Q_{\mathbf{b}}$, which is given by

$$Q_{\mathbf{b}} = I_{a_1} \otimes I_N - |1\rangle\langle 1|_{a_1} \otimes |b\rangle\langle b|. \tag{F5}$$

Here, we use the ancilla a_1 to account for using $\mathbf{b}$, which is encoded as $|1\rangle_{a_1} \otimes |b\rangle$. We can construct this projector using

$$(I_{a_1} \otimes U_b^\dagger)\left[I_{a_1} \otimes I_N - |1\rangle\langle 1|_{a_1} \otimes |0\rangle\langle 0|_N\right](I_{a_1} \otimes U_b), \tag{F6}$$

where we are using subscript N on $|0\rangle\langle 0|$ to indicate that it is on the system. We can block encode this projector using the ancilla a_3. We simply need to create this ancilla in an equal superposition and use the unitary operation

$$\begin{aligned} U_{Qb} = (I_{a_3} \otimes I_{a_1} \otimes U_b^\dagger)\big[|0\rangle\langle 0|_{a_3} \otimes I_{a_1} \otimes I_N + |1\rangle\langle 1|_{a_3} \\ \otimes (I_{a_1} \otimes I_N - 2|1\rangle\langle 1|_{a_1} \otimes |0\rangle\langle 0|_N)\big] \\ (I_{a_3} \otimes I_{a_1} \otimes U_b). \end{aligned} \tag{F7}$$

That gives the projector as a linear combination of the identity and a reflection. Finally, we are prepared to describe the unitary to block encode $H(s)$, which can be written as

$$H(s) = |0\rangle\langle 1|_{a_4} \otimes A[f(s)]Q_{\mathbf{b}} + |1\rangle\langle 0|_{a_4} \otimes Q_{\mathbf{b}}A[f(s)]. \tag{F8}$$

In order to select between $A[f(s)]Q_{\mathbf{b}}$ and $Q_{\mathbf{b}}A[f(s)]$, we apply $Q_{\mathbf{b}}$ in a controlled way before and after $A[f(s)]$. We denote the controlled unitary for $Q_{\mathbf{b}}$, as controlled on 0 or 1, by CU^0_{Qb} or CU^1_{Qb}, respectively. We may make $Q_{\mathbf{b}}$ controlled simply by making the reflection $2I_{a_1} \otimes I_N - |1\rangle\langle 1|_{a_1} \otimes |0\rangle\langle 0|_N$ controlled and we do not need to make the oracle U_b controlled. We can therefore apply CU^1_{Qb} as

$$\begin{aligned} CU^1_{Qb} &= |0\rangle\langle 0|_{a_4} \otimes I_{a_3} \otimes I_{a_1} \otimes I_N + |1\rangle\langle 1|_{a_4} \otimes U_{Qb} \\ &= (I_{a_3} \otimes I_{a_1} \otimes U_b^\dagger)\big[I_{a_4} \otimes |0\rangle\langle 0|_{a_3} \otimes I_{a_1} \otimes I_N \\ &\quad + |0\rangle\langle 0|_{a_4} \otimes |1\rangle\langle 1|_{a_3} \otimes I_{a_1} \otimes I_N \\ &\quad + |1\rangle\langle 1|_{a_4} \otimes |1\rangle\langle 1|_{a_3} \otimes (I_{a_1} \otimes I_N \\ &\quad - 2|1\rangle\langle 1|_{a_1} \otimes |0\rangle\langle 0|_N)\big](I_{a_3} \otimes I_{a_1} \otimes U_b) \end{aligned} \tag{F9}$$

and similarly for CU^0_{Qb}.

We also need to perform the rotation $R(s)$ before or after these operations controlled on the ancilla a_4. That is, we perform at the beginning

$$CR^0(s) = |0\rangle\langle 0|_{a_4} \otimes R(s)_{a_2} + |1\rangle\langle 1|_{a_4} \otimes \mathcal{H}_{a_2}, \tag{F10}$$

where $\mathcal{H}$ denotes the Hadamard operation. Then, at the end, we perform the controlled operation

$$CR^1(s) = |1\rangle\langle 1|_{a_4} \otimes R(s)_{a_2} + |0\rangle\langle 0|_{a_4} \otimes \mathcal{H}_{a_2}, \tag{F11}$$

We are finally ready to provide the complete sequence of operations to block encode $H(s)$. In the following, we use the various operations defined above on subsets of the ancillas, with the convention that they act as the identity on any ancillas on which their action has not been described:

(1) Apply the Hadamard on a_3 to provide the linear combination needed for $Q_{\mathbf{b}}$.
(2) Next, apply CU^1_{Qb} for controlled implementation of $Q_{\mathbf{b}}$ before $A[f(s)]$.
(3) Apply $CR^0(s)$ to provide the rotation on ancilla a_2.
(4) Apply $U_{A(f)}$ for the block encoding of $A[f(s)]$.
(5) Apply $CR^1(s)$ to provide the symmetric form of the rotation on ancilla a_2.
(6) Apply CU^0_{Qb} for controlled implementation of $Q_{\mathbf{b}}$ after $A[f(s)]$.
(7) Apply the Hadamard on a_3 again.
(8) Finally, apply σ^x on a_4 to flip that bit.

A further simplification can be made by noting that steps 2 and 3 can be made the same as steps 6 and 5 (respectively) by placing the bit flip on a_4 in between these operations. That is, we have $CU^0_{Qb}\sigma^x_{a_4} = \sigma^x_{a_4}CU^1_{Qb}$ and $CR^1(s)\sigma^x_{a_4} = \sigma^x_{a_4}CR^0(s)$. Then, the complete circuit diagram can be given as in Fig. 9.

Now recall that we require the unitary operation in the block encoding to be self-inverse for the qubitization. This can be seen fairly easily from Fig. 9 but it can also be shown explicitly using

$$
\begin{aligned}
&\left[\sigma^x_{a_4}\,\mathcal{H}_{a_3}\,CU^0_{Qb}\,CR^1(s)\,U_{A(f)}\,CR^0(s)\,CU^1_{Qb}\,\mathcal{H}_{a_3}\right]\left[\sigma^x_{a_4}\,\mathcal{H}_{a_3}\,CU^0_{Qb}\,CR^1(s)\,U_{A(f)}\,CR^0(s)\,CU^1_{Qb}\,\mathcal{H}_{a_3}\right]\\
&\quad=\sigma^x_{a_4}\,\mathcal{H}_{a_3}\,CU^0_{Qb}\,CR^1(s)\,U_{A(f)}\,CR^0(s)\,CU^1_{Qb}\,\sigma^x_{a_4}\,CU^0_{Qb}\,CR^1(s)\,U_{A(f)}\,CR^0(s)\,CU^1_{Qb}\,\mathcal{H}_{a_3}\\
&\quad=\sigma^x_{a_4}\,\mathcal{H}_{a_3}\,CU^0_{Qb}\,CR^1(s)\,U_{A(f)}\,CR^0(s)\,\sigma^x_{a_4}\,CU^0_{Qb}\,CU^0_{Qb}\,CR^1(s)\,U_{A(f)}\,CR^0(s)\,CU^1_{Qb}\,\mathcal{H}_{a_3}\\
&\quad=\sigma^x_{a_4}\,\mathcal{H}_{a_3}\,CU^0_{Qb}\,CR^1(s)\,U_{A(f)}\,CR^0(s)\,\sigma^x_{a_4}\,CR^1(s)\,U_{A(f)}\,CR^0(s)\,CU^1_{Qb}\,\mathcal{H}_{a_3}\\
&\quad=\sigma^x_{a_4}\,\mathcal{H}_{a_3}\,CU^0_{Qb}\,CR^1(s)\,U_{A(f)}\,\sigma^x_{a_4}\,CR^1(s)\,CR^1(s)\,U_{A(f)}\,CR^0(s)\,CU^1_{Qb}\,\mathcal{H}_{a_3}\\
&\quad=\sigma^x_{a_4}\,\mathcal{H}_{a_3}\,CU^0_{Qb}\,CR^1(s)\,U_{A(f)}\,\sigma^x_{a_4}\,U_{A(f)}\,CR^0(s)\,CU^1_{Qb}\,\mathcal{H}_{a_3}\\
&\quad=\sigma^x_{a_4}\,\mathcal{H}_{a_3}\,CU^0_{Qb}\,CR^1(s)\,\sigma^x_{a_4}\,U_{A(f)}\,U_{A(f)}\,CR^0(s)\,CU^1_{Qb}\,\mathcal{H}_{a_3}\\
&\quad=\sigma^x_{a_4}\,\mathcal{H}_{a_3}\,CU^0_{Qb}\,CR^1(s)\,\sigma^x_{a_4}\,CR^0(s)\,CU^1_{Qb}\,\mathcal{H}_{a_3}\\
&\quad=\sigma^x_{a_4}\,\mathcal{H}_{a_3}\,CU^0_{Qb}\,\sigma^x_{a_4}\,CR^0(s)\,CR^0(s)\,CU^1_{Qb}\,\mathcal{H}_{a_3}\\
&\quad=\sigma^x_{a_4}\,\mathcal{H}_{a_3}\,CU^0_{Qb}\,\sigma^x_{a_4}\,CU^1_{Qb}\,\mathcal{H}_{a_3}\\
&\quad=\sigma^x_{a_4}\,\mathcal{H}_{a_3}\,\sigma^x_{a_4}\,CU^1_{Qb}\,CU^1_{Qb}\,\mathcal{H}_{a_3}\\
&\quad=\sigma^x_{a_4}\,\mathcal{H}_{a_3}\,\sigma^x_{a_4}\,\mathcal{H}_{a_3}\\
&\quad=I.
\end{aligned}
\tag{F12}
$$

Here, we repeatedly commute $\sigma^x_{a_4}$ through operators and use the property that operators are self-inverse to cancel them. This shows that our sequence of operations is self-inverse as required.

In our block encoding, qubits a_1 and a_4 are the two extra qubits used as part of the system upon which $H(s)$ acts, whereas a_2, a_3, and a are used as the registers for the block encoding. For the walk step $W(s)$ that is used in our algorithm, the block encoding of $H(s)$ needs to be supplemented with a reflection on the registers used for this block encoding, which are a_2, a_3, and a.

APPENDIX G: UPPER BOUNDS OF THEOREM 3 WITH $p = 3/2$

We split the proof of Theorem 9 into three parts: the upper bounds for the three terms without sums, the sums with $\hat{c}_1$, and the summation term with $\hat{c}_2$. Before we proceed with each calculation, first we note that in Theorem 3 the three gaps are replaced by the minimum one, Eq. (14), which is

$$
\check{\Delta}(s) = \min_{s' \in \{s-1/T, s, s_+\} \cap [0,1]} \Delta(s'). \tag{G1}
$$

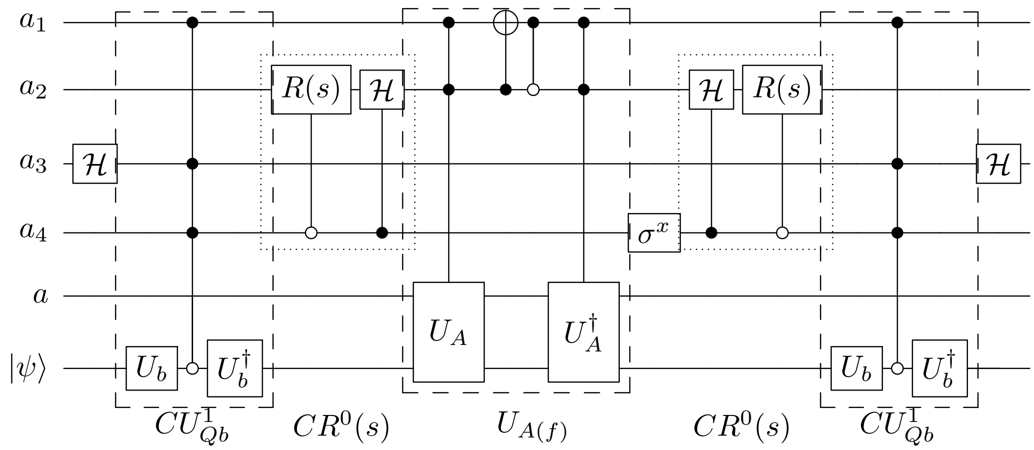

FIG. 9. The block encoding of the Hamiltonian $H(s)$, where the target system is labeled as $|\psi\rangle$ and the ancillas are labeled a_1–a_4 and a (where a is the ancilla needed in the block encoding of A). The dashed boxes show CU^1_{Qb} and $U_{A(f)}$, whereas the dotted boxes show $CR^0(s)$. The σ^x on ancilla a_4 has been moved to the middle, so the operations CU^0_{Qb} and $CR^1(s)$ are implemented via CU^1_{Qb} and $CR^0(s)$.

We have Eq. (70) using the fact that the gap is monotonically decreasing, so then the fact that f is monotonically increasing gives us

$$\check{\Delta}(s) = \begin{cases} (1 - f(s+3/T) + f(s+3/T)/\kappa), & 0 \le s \le 1 - 3/T, \\ 1/\kappa, & s = 1 - 2/T, 1 - 1/T, 1. \end{cases} \tag{G2}$$

For simplicity of the analysis, we first analyze the $\check{\Delta}$ corresponding to the gap with a Hermitian and positive-definite A. Later, we introduce a factor of 2 to account for the gap with general A and a block encoding.

Next, in Theorem 3, we have the functions $\hat{c}_1(s)$ and $\hat{c}_2(s)$ as defined in Eq. (10). Choices for the functions $c_1(s)$ and $c_2(s)$ are given in Lemma 8. Using the monotonicity properties of the function f, we find that

$$\hat{c}_1(s) = \begin{cases} 2Tf(1/T), & s = 0, \\ 2T[f(s) - f(s-1/T)], & 1/T \le s \le 1, \end{cases} \tag{G3}$$

and

$$\hat{c}_2(s) = 2\left(2|f'(s)|^2 + |f''(s)|\right). \tag{G4}$$

For $\hat{c}_1(s)$, we use the fact that $f'(s)$ is monotonically decreasing, so a larger difference will be obtained for a smaller value of s. For $\hat{c}_2(s)$, we also use the fact that $|f''(s)|$ is monotonically decreasing, so again larger values will be obtained for smaller values of s. The monotonicity properties of f are easily checked by checking expressions for the derivatives; $f'(s)$ is positive, $f''(s)$ is negative, and $f'''(s)$ is positive.

In the block encoding, we need to account for how the gap in $H(s)$ is translated to the gap in the walk operators. The solution state has eigenvalue 0, which is translated to the eigenvalues ± 1 for the walk operator. The eigenvalues λ of H are generally translated to $\pm e^{\pm i \arcsin \lambda}$, which means that the gap for the walk operator is increased to the arcsine of the gap of the Hamiltonian. Since the arcsine can only increase the gap, the lower bounds on the gap for $H(s)$ also apply to the walk operator.

1. Single components

Beginning with the first term from the bound in Theorem 3, using the expression for $\check{\Delta}(0)$ from Eq. (G2), for $\hat{c}_1(0)$ from Eq. (G3), and using $f(s)$ from Eq. (69), we obtain

$$\begin{aligned}
\frac{\hat{c}_1(0)}{T\check{\Delta}(0)^2} &= 2\frac{f(1/T)}{(1-f(3/T)+f(3/T)/\kappa)^2} \\
&= \frac{2}{T^4}\frac{\kappa}{\sqrt{\kappa}+1}\frac{\left(3\sqrt{\kappa}-3+T\right)^4\left(\sqrt{\kappa}-1+2T\right)}{(\sqrt{\kappa}-1+T)^2} \\
&= \frac{4}{T}\frac{\kappa}{\sqrt{\kappa}+1}\frac{(1+2\alpha_1)^4(1-\alpha_1/2)}{(1-\alpha_1)^3} \\
&= \frac{4}{T}\frac{\kappa}{\sqrt{\kappa}+1}[1+\mathcal{O}(\alpha_1)] \\
&= \frac{4\sqrt{\kappa}}{T}+\mathcal{O}\left(\frac{\kappa}{T^2}\right),
\end{aligned} \tag{G5}$$

where

$$\alpha_n := \frac{\sqrt{\kappa}-1}{T+n(\sqrt{\kappa}-1)}, \tag{G6}$$

so $\alpha_n = \mathcal{O}(\sqrt{\kappa}/T)$ and we use $T > \kappa$. This result is given in Eq. (96) of the body.

We next show Eqs. (97) and (98). This time, we use $\hat{c}_1(s)$ and $\check{\Delta}(s)$ for $s = 1$; by Eq. (G3), we obtain $\hat{c}_1(1) = 2[1 - f(1-1/T)]$ and from Eq. (G2) we have $\check{\Delta}(1) = 1/\kappa$. Therefore,

$$\begin{aligned}
\frac{\hat{c}_1(1)}{T\check{\Delta}(1)^2} &= 2\kappa^2[1-f(1-1/T)] \\
&= 2\kappa^2\left[1+\frac{\kappa}{1-\kappa}\left(1-\frac{1}{(1+[\sqrt{\kappa}-1)(1-1/T)]^2}\right)\right].
\end{aligned} \tag{G7}$$

Now, we simplify the terms inside the square brackets to give

$$\begin{aligned}
1+\frac{\kappa}{1-\kappa}\left[1-\frac{T^2}{[T+(\sqrt{\kappa}-1)(T-1)]^2}\right] &= \frac{(1-\kappa)\left(1+\sqrt{\kappa}(T-1)\right)^2+\kappa\left(1+\sqrt{\kappa}(T-1)\right)^2-T^2\kappa}{(1-\kappa)\left(1+\sqrt{\kappa}(T-1)\right)^2} \\
&= \frac{\left(1+\sqrt{\kappa}(T-1)\right)^2-T^2\kappa}{(1-\kappa)\left(1+\sqrt{\kappa}(T-1)\right)^2} \\
&= \frac{\sqrt{\kappa}(2T-1)+1}{(\sqrt{\kappa}+1)[1+\sqrt{\kappa}(T-1)]^2} \\
&= \frac{2}{T(\kappa+\sqrt{\kappa})}\frac{1-\beta/2}{(1-\beta)^2} \\
&= \frac{2}{T(\kappa+\sqrt{\kappa})}[1+\mathcal{O}(\beta)] \\
&= \frac{2}{\kappa T}+\mathcal{O}\left(\frac{1}{\kappa T^2}\right),
\end{aligned} \tag{G8}$$

with

$$\beta = \frac{1-1/\sqrt{\kappa}}{T}, \tag{G9}$$

so $\beta = \mathcal{O}(1/T)$. Therefore, we can conclude that

$$\frac{\hat{c}_1(1)}{T\check{\Delta}(1)^2} = \frac{4\kappa}{T} + \mathcal{O}\left(\frac{\kappa}{T^2}\right). \tag{G10}$$

This is the result given in Eq. (97). For the other upper bound, we have $\check{\Delta}(1)$ instead of $\check{\Delta}(1)^2$, so for the upper bound shown in Eq. (98), we obtain

$$\frac{\hat{c}_1(1)}{T\check{\Delta}(1)} = \frac{4}{T} + \mathcal{O}\left(\frac{1}{T^2}\right). \tag{G11}$$

2. $c_1(s)$ summations

We start by considering the sum of $\hat{c}_1(s)^2/(T^2\check{\Delta}(s)^3)$ for $1/T \leq s \leq 1 - 3/T$. In this range, we obtain

$$\begin{aligned}
\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^3} &= 4\frac{[f(n/T) - f((n-1)/T)]^2}{(1 - f((n+3)/T) + f((n+3)/T)/\kappa)^3} \\
&= \frac{16\kappa^2}{(\sqrt{\kappa}+1)^2T^2}\frac{\left[(3+n)(\sqrt{\kappa}-1)+T\right]^6\left[(n-1/2)(\sqrt{\kappa}-1)+T\right]^2}{\left[n(\sqrt{\kappa}-1)+T\right]^4\left[(n-1)(\sqrt{\kappa}-1)+T\right]^4} \\
&= \frac{16\kappa^2}{(\sqrt{\kappa}+1)^2T^2}\frac{(1+3\alpha_n)^6(1-\alpha_n/2)^2}{(1-\alpha_n)^4} \\
&= \frac{16\kappa^2}{(\sqrt{\kappa}+1)^2T^2}[1+\mathcal{O}(\alpha_n)] \\
&= \frac{16\kappa}{T^2} + \mathcal{O}\left(\frac{\kappa^{3/2}}{T^3}\right).
\end{aligned} \tag{G12}$$

Now, for the last two elements of the sum, we have

$$\begin{aligned}
\frac{\hat{c}_1(1-2/T)^2}{T^2\check{\Delta}(1-2/T)^3} &= 4\kappa^3(f(1-2/T) - f(1-3/T))^2 \\
&= \frac{16\kappa^2}{T^2(1+\sqrt{\kappa})^2}\frac{[1-5/(2T)+5/(2T\sqrt{\kappa})]^2}{[1+6/T^2-5/T+6/(T^2\kappa)-12/(T^2\sqrt{\kappa})+5/(T\sqrt{\kappa})]^4} \\
&= \frac{16\kappa^2}{T^2(1+\sqrt{\kappa})^2}\left[1+\mathcal{O}\left(\frac{1}{T}\right)\right] \\
&= \frac{16\kappa}{T^2} + \mathcal{O}\left(\frac{\kappa}{T^3}\right)
\end{aligned} \tag{G13}$$

and

$$\begin{aligned}
\frac{\hat{c}_1(1-1/T)^2}{T^2\check{\Delta}(1-1/T)^3} &= 4\kappa^3(f(1-1/T) - f(1-2/T))^2 \\
&= \frac{16\kappa^2}{T^2(1+\sqrt{\kappa})^2}\frac{[1-3/(2T)+3/(2T\sqrt{\kappa})]^2}{[1+2/T^2-3/T+2/(T^2\kappa)-4/(T^2\sqrt{\kappa})+3/(T\sqrt{\kappa})]^4} \\
&= \frac{16\kappa^2}{T^2(1+\sqrt{\kappa})^2}\left[1+\mathcal{O}\left(\frac{1}{T}\right)\right] \\
&= \frac{16\kappa}{T^2} + \mathcal{O}\left(\frac{\kappa}{T^3}\right).
\end{aligned} \tag{G14}$$

Therefore, for all n in the sum, we have an upper bound of $16\kappa/T^2$ up to leading order. The total upper bound for the sum of $\hat{c}_1(n/T)^2/(T^2\check{\Delta}(n/T)^3)$ from $n=1$ to $T-1$ is therefore

$$\sum_{n=1}^{T-1}\hat{c}_1(n/T)^2/(T^2\check{\Delta}(n/T)^3)=\frac{16\kappa}{T}+\mathcal{O}\left(\frac{\kappa^{3/2}}{T^2}\right), \tag{G15}$$

which is given in Eq. (99).

Next, we show the upper bound for the sum of the elements $\hat{c}_1(s)^2/(T^2\check{\Delta}(s)^2)$, which is given in Eq. (100) above. When $1/T\leq s\leq 1-3/T$, we have

$$\begin{aligned}\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^2}&=4\frac{[f(n/T)-f((n-1)/T)]^2}{(1-f((n+3)/T)+f((n+3)/T)/\kappa)^2}\\&=\frac{4\kappa^2}{(\sqrt{\kappa}+1)^2}\frac{\left[(3+n)(\sqrt{\kappa}-1)+T\right]^4\left[(2n-1)\left(\sqrt{\kappa}-1\right)+2T\right]^2}{\left[n(\sqrt{\kappa}-1)+T\right]^4\left[(n-1)(\sqrt{\kappa}-1)+T\right]^4}\\&=\frac{16\kappa^2}{[T+n(\sqrt{\kappa}-1)]^2(\sqrt{\kappa}+1)^2}\frac{(1-\alpha_n/2)^2(1+3\alpha_n)^4}{(1-\alpha_n)^4}\\&=\frac{16\kappa^2}{[T+n(\sqrt{\kappa}-1)]^2(\sqrt{\kappa}+1)^2}\left[1+\mathcal{O}(\alpha_n)\right]\\&\leq\frac{16\kappa}{T^2}+\mathcal{O}\left(\frac{\kappa^{3/2}}{T^3}\right).\end{aligned} \tag{G16}$$

Because the sum starts from $n=0$, we need the following upper bound:

$$\begin{aligned}\frac{\hat{c}_1(0)^2}{T^2\check{\Delta}(0)^2}&=4\frac{f(1/T)^2}{(1-f(3/T)+f(3/T)/\kappa)^2}\\&=\frac{4\kappa^2}{(\sqrt{\kappa}+1)^2}\frac{\left(3(\sqrt{\kappa}-1)+T\right)^4\left(\sqrt{\kappa}-1+2T\right)^2}{T^4\left(\sqrt{\kappa}-1+T\right)^4}\\&=\frac{16\kappa^2}{(\sqrt{\kappa}+1)^2T^2}\frac{(1+\alpha_0/2)^2(1+3\alpha_0)^4}{(1+\alpha_0)^4}\\&=\frac{16\kappa^2}{(\sqrt{\kappa}+1)^2T^2}\left[1+\mathcal{O}(\alpha_0)\right]\\&=\frac{16\kappa}{T^2}+\mathcal{O}\left(\frac{\kappa^{3/2}}{T^3}\right).\end{aligned} \tag{G17}$$

We also have to upper bound the cases where $s=1-1/T$ and $s=1-2/T$. This upper bound is the same as we have in Eqs. (G13) and (G14) but now with $1/\kappa^2$ in the denominator rather than $1/\kappa^3$, so we obtain

$$\frac{\hat{c}_1(1-2/T)^2}{T^2\check{\Delta}(1-2/T)^2}=4\kappa^2(f(1-2/T)-f(1-3/T))^2=\frac{16}{T^2}+\mathcal{O}\left(\frac{1}{T^3}\right) \tag{G18}$$

and

$$\frac{\hat{c}_1(1-1/T)^2}{T^2\check{\Delta}(1-1/T)^2}=4\kappa^2(f(1-1/T)-f(1-2/T))^2=\frac{16}{T^2}+\mathcal{O}\left(\frac{1}{T^3}\right). \tag{G19}$$

There are T terms in the sum and each is upper bounded by $16/T^2$ to leading order except that at $n=0$.

We therefore obtain

$$\sum_{n=0}^{T-1}\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^2}\leq\frac{16\kappa}{T}+\frac{16\kappa}{T^2}+\mathcal{O}\left(\frac{\sqrt{\kappa}}{T}\right)$$
$$=\frac{16\kappa}{T}+\mathcal{O}\left(\frac{\sqrt{\kappa}}{T}\right). \tag{G20}$$

This is the result given as Eq. (100) above.

So far, we have used $\check{\Delta}$, which is the gap for the special case of Hermitian and positive-definite A, and we use a factor of 2 to account for general A and the block encoding, so the actual gap is $\check{\Delta}'\geq\check{\Delta}/2$. This means that we need $T\geq 8\hat{c}_1(n/T)/\check{\Delta}(n/T)$ to use the second discrete adiabatic theorem. We show that choosing $T\geq 39\sqrt{\kappa}$ ensures that inequality holds. The general principle is that $\hat{c}_1(n/T)$ involves a maximum over neighboring steps separated by $1/T$ and so will decrease with T. The quantity $\check{\Delta}(n/T)$ involves a *minimum* over neighboring steps and increases with T. As a result, the ratio $\hat{c}_1(n/T)/\check{\Delta}(n/T)$ decreases with T and so to upper bound the ratio one can simply use the lower bound on T to compute it. We then find that the ratio approaches its upper bound for large κ.

For the case $1\leq n\leq T-3$, note that

$$\frac{(1-\alpha_n/2)(1+3\alpha_n)^2}{(1-\alpha_n)^2} \tag{G21}$$

is monotonically increasing for $\alpha_n\in[0,1]$. Next, if we require $T\geq 39\sqrt{\kappa}$ and $n\geq 1$, then the *maximum* value of α_n is for $T=39\sqrt{\kappa}$ and $n=1$, which gives $\alpha_n=(\sqrt{\kappa}-1)/(40\sqrt{\kappa}-1)$. If we take this value of α_n then we obtain

$$\frac{(1-\alpha_n/2)(1+3\alpha_n)^2}{(1-\alpha_n)^2}=\frac{(43\sqrt{\kappa}-4)^2(79\sqrt{\kappa}-1)}{3042\kappa(40\sqrt{\kappa}-1)}. \tag{G22}$$

This is monotonically increasing with κ for $\kappa\geq 1$ and has a limiting value of $146071/121680<39/32$. Therefore, we have

$$\frac{\hat{c}_1(n/T)}{\check{\Delta}(n/T)}=\frac{4\kappa}{[1+n(\sqrt{\kappa}-1)/T](\sqrt{\kappa}+1)}\times\frac{(1-\alpha_n/2)(1+3\alpha_n)^2}{(1-\alpha_n)^2}$$
$$<\frac{39\kappa}{8[1+n(\sqrt{\kappa}-1)/T](\sqrt{\kappa}+1)}$$
$$<\frac{39}{8}\sqrt{\kappa}. \tag{G23}$$

Therefore, we can ensure that $T>8\hat{c}_1(n/T)/\check{\Delta}(n/T)$ for $1\leq n\leq T-3$.

Next, for the case $n=0$, we note that

$$\frac{(1+\alpha_0/2)(1+3\alpha_0)^2}{(1+\alpha_0)^2} \tag{G24}$$

is monotonically increasing for $\alpha_0\in[0,1]$. Then, taking $T\geq 39\sqrt{\kappa}$, we have a maximum value for α_0 of $(\sqrt{\kappa}-1)/(39\sqrt{\kappa})$, which gives

$$\frac{(1+\alpha_0/2)(1+3\alpha_0)^2}{(1+\alpha_0)^2}=\frac{3(14\sqrt{\kappa}-1)^2(79\sqrt{\kappa}-1)}{26\sqrt{\kappa}(40\sqrt{\kappa}-1)^2}. \tag{G25}$$

This is monotonically increasing with κ for $\kappa\geq 1$ and has a limiting value of $11613/10400$, which is less than $5/4$. That then gives

$$\frac{\hat{c}_1(0/T)}{\check{\Delta}(0/T)}=\frac{4\kappa}{\sqrt{\kappa}+1}\frac{(1+\alpha_0/2)(1+3\alpha_0)^2}{(1+\alpha_0)^2}$$
$$<\frac{5\kappa}{\sqrt{\kappa}+1}$$
$$<5\sqrt{\kappa}. \tag{G26}$$

Next, for $n=T-2$, we obtain

$$\frac{\hat{c}_1(1-2/T)}{\check{\Delta}(1-2/T)}=2\kappa T(f(1-2/T)-f(1-3/T))$$
$$=\frac{2\kappa}{(1+\sqrt{\kappa})}\frac{T^3\kappa[\sqrt{\kappa}(2T-5)+3]}{[\kappa(T-2)(T-3)+\sqrt{\kappa}(5T-12)+6]^2}. \tag{G27}$$

This is monotonically decreasing with T, so we can maximize it by taking the minimum value $T=39\sqrt{\kappa}$, which gives

$$\frac{\hat{c}_1(1-2/T)}{\check{\Delta}(1-2/T)}\leq\frac{2\kappa}{(1+\sqrt{\kappa})}\frac{6591\kappa^{5/2}(78\kappa-5\sqrt{\kappa}+5)}{(13\kappa-\sqrt{\kappa}+1)^2(39\kappa-2\sqrt{\kappa}+2)^2}$$
$$<4. \tag{G28}$$

The function of κ is monotonically increasing and approaches its supremum of 4 in the limit $\kappa \to \infty$. Hence $T \geq 39\sqrt{\kappa}$ guarantees $T > 8\hat{c}_1(1-2/T)/\check{\Delta}(1-2/T)$.

Next, for $n = T-1$ and $n = T$, we obtain

$$\begin{aligned}\frac{\hat{c}_1(1-1/T)}{\check{\Delta}(1-1/T)} &= 2\kappa T(f(1-1/T) - f(1-2/T)),\\ \frac{\hat{c}_1(1)}{\check{\Delta}(1)} &= 2\kappa T(f(1) - f(1-1/T)).\end{aligned} \tag{G29}$$

However, it is easily shown that f is a convex and monotonically increasing function, so the derivative is decreasing and the differences in f must be decreasing. That is,

$$\frac{\hat{c}_1(1)}{\check{\Delta}(1)} \leq \frac{\hat{c}_1(1-1/T)}{\check{\Delta}(1-1/T)} \leq \frac{\hat{c}_1(1-2/T)}{\check{\Delta}(1-2/T)} < 4. \tag{G30}$$

Hence $T \geq 39\sqrt{\kappa}$ guarantees $T > 8\hat{c}_1(1-n/T)/\check{\Delta}(1-n/T)$ for these two cases as well.

3. $c_2(s)$ summation

Next, we show the upper bound given in Eq. (101). Using Eqs. (G4) and (G2) for $1 \leq s \leq 1-3/T$, we have

$$\begin{aligned}\frac{\hat{c}_2(n/T)}{T^2\check{\Delta}(n/T)^2} &= \frac{1}{T^2}\frac{4|f'(n/T)|^2 + |f''(n/T)|}{(1-f((n+3)/T)+f((n+3)/T)/\kappa)^2}\\ &= \frac{2\kappa}{T^2(\sqrt{\kappa}+1)^2}(1+3a_n)^4((3+8\gamma^2)\kappa - 3)\\ &\leq \frac{22\kappa}{T^2(\sqrt{\kappa}+1)^2}(1+3a_n)^4\\ &= \frac{22\kappa}{T^2} + \mathcal{O}\left(\frac{\kappa^{3/2}}{T^3}\right),\end{aligned} \tag{G31}$$

where

$$\gamma = \frac{T}{T+n(\sqrt{\kappa}-1)} < 1. \tag{G32}$$

We need to separately consider the case $s = 1-1/T$, which gives

$$\begin{aligned}\frac{\hat{c}_2(1-1/T)}{T^2\check{\Delta}(1-1/T)^2} &= \frac{\kappa^2}{T^2}\left(4|f'(1-1/T)|^2 + |f''(1-1/T)|\right)\\ &= \frac{2T^2\kappa^3}{(\sqrt{\kappa}+1)^2(\sqrt{\kappa}(T-1)+1)^4}\\ &\quad\times(3\kappa+5+16\delta+8\delta^2)\\ &= \frac{6\kappa}{T^2} + \mathcal{O}\left(\frac{1}{T^2}\right),\end{aligned} \tag{G33}$$

where

$$\delta = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}(T-1)+1} = \mathcal{O}(1/T). \tag{G34}$$

Similarly, we obtain

$$\begin{aligned}\frac{\hat{c}_2(1-2/T)}{T^2\check{\Delta}(1-2/T)^2} &= \frac{\kappa^2}{T^2}\left(4|f'(1-2/T)|^2 + |f''(1-2/T)|\right)\\ &= \frac{2T^2\kappa^3}{(\sqrt{\kappa}+1)^2(\sqrt{\kappa}(T-2)+2)^4}\\ &\quad\times(3\kappa+5+32\delta+32\delta^2)\\ &= \frac{6\kappa}{T^2} + \mathcal{O}\left(\frac{1}{T^2}\right),\end{aligned} \tag{G35}$$

where this time

$$\delta = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}(T-2)+2} = \mathcal{O}(1/T). \tag{G36}$$

Finally, since there are $T-1$ terms in the sum and each is upper bounded by $22\kappa/T^2$ to leading order, we obtain

$$\sum_{n=1}^{T-1}\frac{\hat{c}_2(n/T)}{T^2\check{\Delta}(n/T)^2} \leq \frac{22\kappa}{T} + \mathcal{O}\left(\frac{\kappa^{3/2}}{T^2}\right). \tag{G37}$$

This is the bound given in Eq. (101).

APPENDIX H: PHASE FACTORS IN THE ADIABATIC EVOLUTION

One would normally consider the eigenspace of interest in a single group for the adiabatic theorem. In contrast, here the eigenspace of interest is separated in two parts, corresponding to ± 1, and the remaining eigenspace is separated by a gap in two parts in the upper and lower halves of the complex plane. In order to address this, one can instead consider just the eigenvalue 1 as the eigenspace of interest, which is then in a single group. Then, the discrete adiabatic theorem can be applied unchanged to show that the state is correctly mapped to the final eigenstate. Similarly, one can just consider the adiabatic theorem with -1. Since using the adiabatic theorem separately on each eigenstate shows that it properly evolves to the final state, the superposition of the two eigenstates must also do so.

To be more specific, as discussed in Eqs. (10)–(13) of Ref. [24], the eigenvectors of the walk operator are of the form (correcting a missing i in the reference)

$$\frac{1}{\sqrt{2}}\left(|0\rangle_a|k\rangle_s \pm i\,|0k^{\perp}\rangle_{as}\right), \tag{H1}$$

where $|0\rangle_a$ is the zero state on the ancilla, $|k\rangle_s$ is the eigenstate of H of energy E_k on the system, and $|0k^{\perp}\rangle_{as}$

is a state orthogonal to $|0\rangle_a$ on the ancilla. In our case, the target eigenvalue of the Hamiltonian is $E_k = 0$, which yields eigenvalues of ± 1 of the walk operator with these two eigenstates. When we have a positive superposition of the two eigenstates, then the resulting state is the solution given by $|0\rangle_a\,|k\rangle_s$. In contrast, if we have a negative superposition of the two eigenstates, then the result is the nonsolution state $|0k^{\perp}\rangle_{as}$. In the adiabatic evolution, we start with the positive superposition and we must maintain that positive superposition at the end in order to obtain the solution. Therefore, we should show that there is no phase factor introduced by the adiabatic evolution.

To show this, it is again sufficient to consider the evolution of each eigenvalue on its own. To obtain the phase factor, it is sufficient to consider the exact adiabatic evolution given by the adiabatic walk operators

$$\begin{aligned} W_A(s) &= V(s)W(s)\\ &= V(s',s)^{-1}S(s',s)W(s)\\ &= [S(s',s)S^{\dagger}(s',s)]^{-1/2}S(s',s)W(s)\\ &= [P(s')P(s)P(s') + Q(s')Q(s)Q(s')]^{-1/2}\\ &\quad\times [P(s')P(s) + Q(s')Q(s)]W(s), \end{aligned} \tag{H2}$$

where $s' = s_+$. (We are swapping the s' and s from the way S and V were given originally.) For the case in which we are interested, there may be multiple states within the spectrum of interest but they are orthogonal. More specifically, there is the solution state (ground state of the Hamiltonian)

$$\begin{pmatrix} A[f(s)]^{-1}\mathbf{b} \\ 0 \end{pmatrix}, \tag{H3}$$

as well as a nonsolution state of the form

$$\begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix}. \tag{H4}$$

In that case, the product of projectors is of the form

$$\begin{aligned} P(s')P(s) &= \sum_{j,j'} |\lambda_j(s')\rangle\,\langle\lambda_j(s')|\lambda_{j'}(s)\rangle\,\langle\lambda_{j'}(s)|\\ &= \sum_j \langle\lambda_j(s')|\lambda_j(s)\rangle\,|\lambda_j(s')\rangle\,\langle\lambda_j(s)|, \end{aligned} \tag{H5}$$

and, similarly,

$$P(s')P(s)P(s') = \sum_j |\,\langle\lambda_j(s')|\lambda_j(s)\rangle\,|^2\,|\lambda_j(s')\rangle\,\langle\lambda_j(s')|. \tag{H6}$$

We use $|\lambda_j(s)\rangle$ to indicate eigenstates, including for degenerate eigenvalues. What this means is that we cannot flip between orthogonal eigenstates in the spectrum of interest during the (exact) adiabatic evolution and we do not have the solution state leaking into the nonsolution state.

For the eigenstate $|\lambda_j(s)\rangle$ in the spectrum of interest for $W(s)$, we have

$$\begin{aligned} &[P(s')P(s)P(s') + Q(s')Q(s)Q(s')]^{-1/2}[P(s')P(s) + Q(s')Q(s)]W(s)\,|\lambda_j(s)\rangle\\ &\quad= \lambda_j(s)[P(s')P(s)P(s') + Q(s')Q(s)Q(s')]^{-1/2}[P(s')P(s) + Q(s')Q(s)]\,|\lambda_j(s)\rangle\\ &\quad= \lambda_j(s)\,\langle\lambda_j(s')|\lambda_j(s)\rangle\,[P(s')P(s)P(s') + Q(s')Q(s)Q(s')]^{-1/2}\,|\lambda_j(s')\rangle\\ &\quad= \lambda_j(s)\,\langle\lambda_j(s')|\lambda_j(s)\rangle\,[|\,\langle\lambda_j(s')|\lambda_j(s)\rangle\,|^2\,|\lambda_j(s')\rangle\,\langle\lambda_j(s')|]^{-1/2}\,|\lambda_j(s')\rangle\\ &\quad= \lambda_j(s)\frac{\langle\lambda_j(s')|\lambda_j(s)\rangle}{|\,\langle\lambda_j(s')|\lambda_j(s)\rangle\,|}\,|\lambda_j(s')\rangle. \end{aligned} \tag{H7}$$

In the second line, applying $W(s)$ gives the eigenvalue. If this eigenstate is $|\lambda_j(s)\rangle$, then applying $P(s')P(s) + Q(s')Q(s)$ gives the updated state $|\lambda_j(s')\rangle$ times $\langle\lambda_j(s')|\lambda_j(s)\rangle$. Then, applying $V(s',s)^{-1}$ cancels the magnitude of $\langle\lambda_j(s')|\lambda_j(s)\rangle$ and we only have its phase.

For our application, the eigenvalues of $W(s)$ are ± 1 but provided that the total number of steps of the walk is even, then this sign flip cancels out. Ideally, we would show that $\langle\lambda_j(s')|\lambda_j(s)\rangle$ is real in order to show that there are no spurious phase factors. However, it is sufficient to just show that any phase factor from $\langle\lambda_j(s')|\lambda_j(s)\rangle$ is the same between the ± 1 eigenvectors of $W(s)$. The eigenstates of $W(s)$ are given as in Eq. (H1). In order to describe the inner products between the eigenstates at successive time steps, let us use k_1 and k_2. Then, the inner product of eigenstates at successive steps is

$$\begin{aligned} &\frac{1}{2}\left({}_a\langle 0|\,{}_s\langle k_1| \mp i\,{}_{as}\langle 0k_1^{\perp}|\right)\left(|0\rangle_a\,|k_2\rangle_s \pm i\,|0k_2^{\perp}\rangle_{as}\right)\\ &\quad= \frac{1}{2}\left({}_s\langle k_1|k_2\rangle_s + {}_{as}\langle 0k_1^{\perp}|0k_2^{\perp}\rangle_{as}\right). \end{aligned} \tag{H8}$$

Here, we use $\pm$ to indicate that we are using $+1$ for both steps or -1 for both steps. The crucial result here is that the inner product does *not* depend on whether we are considering the $+1$ or -1 eigenstates. This means that there may be a phase factor but it will be the *same* between the ± 1 eigenstates. The one caveat is that we need to use an even number of steps to avoid a -1 factor but it is always possible to slightly adjust the schedule so that there is an even number of steps and that does not change the asymptotic scaling of the number of steps needed.

The net result of this is that the adiabatic walk with the qubitized walk operator still works, despite there being separated eigenvalues at ± 1. Here, we do not need to use any special properties of the Hamiltonian other than that the eigenvalue of interest (for the Hamiltonian) is 0, so this result may be used for applications other than solving linear equations. If there was a nonzero eigenvalue that was *known*, then it would be possible to add a multiple of the identity to rezero that eigenvalue and the above method would again work.

APPENDIX I: UPPER BOUNDS OF THEOREM 10 WITH $1 < p < 2$

In this appendix, we prove Theorem 10 for general possibly non-Hermitian matrix A. We use consistent notation, as follows. The time-dependent Hamiltonian $H(s)$ is given in Eq. (71), where the schedule function $f(s)$ follows from the definition in Eq. (68). The spectral gaps of $H(s)$ are lower bounded by $\Delta_k'(s)$ as defined in Eq. (82). We remark that in the proof we also use the notation $\Delta_0(s)$ defined in Eq. (67) but this is only for the quantities related to the schedule function $f(s)$ and the spectral gap of $H(s)$ in the general non-Hermitian case should be $\Delta_0'(s)$.

In order to show the linear dependence in T on κ, we use Theorem 3 and calculate the scaling of each term. There are two main difficulties: estimates of finite differences of the walk operator and different discrete time points used in Theorem 3. To overcome the first difficulty, we establish a connection between the discrete finite-difference coefficients $c_k(s)$ and the corresponding continuous derivatives of the schedule function. Then, according to the definition of the schedule function in Eq. (68), this can be directly related to the spectrum gap and canceled with the denominators in the error bound. For the second difficulty, we use the continuity and monotonicity of the spectrum gap in the linear-systems problem to unify the time points, at the sacrifice of larger preconstants.

We first reformulate the coefficients c_1 and c_2, which were previously given in Lemma 8. Here, we use a slightly different version with continuous time values, that

$$c_1(s) = 2 \max_{\tau\in[s,s+1/T]\cap[0,1]} |f'(\tau)| \tag{I1}$$

and

$$c_2(s) = 2 \max_{\tau\in[s,s+2/T]\cap[0,1]} (2|f'(\tau)|^2 + |f''(\tau)|). \tag{I2}$$

Note that the choices of c_1 and c_2 here are even larger than those in Lemma 8. Then, we can use the definition of the schedule function to establish the connection between $c_k(s)$ and the spectrum gap.

Lemma 20: *Consider solving linear-systems problems using discrete adiabatic evolution with the schedule function defined in Eq. (68). Then, the walk operators satisfy the following:*

(1) For any $0 \le s \le 1 - 1/T$, we have

$$c_1(s) = 2d_p\Delta_0(s)^p. \tag{I3}$$

(2) For any $0 \le s \le 1 - 2/T$, we have

$$c_2(s) = 4d_p^2\Delta_0(s)^{2p} + 2d_p^2 p(1-1/\kappa)\Delta_0(s)^{2p-1}. \tag{I4}$$

Proof. According to Lemma 8, we only need to compute the derivatives of the schedule function. The first-order derivative directly comes from the definition of the schedule function, according to which

$$f'(\tau) = d_p\Delta_0(\tau)^p. \tag{I5}$$

For the second-order derivative, we have

$$\begin{aligned} f''(\tau) &= \frac{d}{d\tau}\left(d_p\,(1 - f(\tau) + f(\tau)/\kappa)^p\right) \\ &= d_p p\,(1 - f(\tau) + f(\tau)/\kappa)^{p-1}\,(-1 + 1/\kappa)f'(\tau) \\ &= d_p^2 p(-1 + 1/\kappa)\Delta_0(\tau)^{2p-1}. \end{aligned} \tag{I6}$$

The proof is completed using the monotonicity of Δ_0. ■

In the error estimate in Theorem 3, we encounter taking the maximum or minimum of several consequent time steps, which poses a technical difficulty in calculating the scaling of the error. In the following lemma, we show how to resolve the different-time-point issue.

Lemma 21: *Let $\Delta(s)$ denote the quantity defined in Eq. (67) and let $\Delta_0'(s)$ denote the spectral gap of the Hamiltonian $H(s)$. Assume that $T \ge 16(\sqrt{2})^p\left(\frac{\kappa^{p-1}-1}{p-1}\right) = \mathcal{O}(\kappa^{p-1})$. Then, for any $s \le s' \le s + 4/T$, we have*

$$\Delta_0(s) \le \frac{4}{3}\Delta_0(s'), \quad \Delta_0'(s) \le \frac{4}{3}\Delta_0'(s'). \tag{I7}$$

Proof. Since $\Delta_0'(s) = \Delta_0(s)/2$, it suffices only to prove that $\Delta_0(s) \leq \frac{4}{3}\Delta_0(s')$. We define $\Delta_{\text{linear}}(y) = 1 - y + y/\kappa$. Then, $\Delta_0(s) = \Delta_{\text{linear}}(f(s))$. Since $\Delta(s)$ is a monotonically decreasing function, it suffices to prove that $\Delta_0(s)/\Delta_0(s+4/T) \leq 4/3$. We first compute the derivative of the gap,

$$\frac{d}{ds}\Delta_0(s) = \frac{d}{ds}\left(\Delta_{\text{linear}}(f(s))\right) = \frac{d\Delta_{\text{linear}}(f(s))}{df}f'(s) = (-1+1/\kappa)d_p\Delta_0(s)^p. \tag{I8}$$

Then, for any $0 \leq s \leq 1 - 4/T$,

$$|\Delta_0(s) - \Delta_0(s+4/T)| \leq \frac{4}{T}\max_{s'\in[s,s+4/T]}|d\Delta_0(s')/ds'| = \frac{4}{T}(1-1/\kappa)d_p\Delta_0(s)^p \tag{I9}$$

and thus

$$\begin{aligned}\frac{\Delta_0(s)}{\Delta_0(s+4/T)} &= 1 + \frac{\Delta_0(s) - \Delta_0(s+4/T)}{\Delta_0(s+4/T)} \\ &\leq 1 + \frac{4(1-1/\kappa)d_p\Delta_0(s)^{p-1}}{T}\frac{\Delta_0(s)}{\Delta_0(s+4/T)} \\ &\leq 1 + \frac{4(1-1/\kappa)d_p}{T}\frac{\Delta_0(s)}{\Delta_0(s+4/T)}.\end{aligned} \tag{I10}$$

It has been computed in Ref. [15] that $d_p = \frac{2^{p/2}}{p-1}\frac{\kappa}{\kappa-1}(\kappa^{p-1}-1)$. Together with the assumption that $T \geq 16(\sqrt{2})^p\left(\frac{\kappa^{p-1}-1}{p-1}\right)$, we have

$$\frac{4(1-1/\kappa)d_p}{T} = \frac{2^{p/2+2}}{T(p-1)}(\kappa^{p-1}-1) \leq \frac{1}{4} \tag{I11}$$

and thus

$$\frac{\Delta_0(s)}{\Delta_0(s+4/T)} \leq 1 + \frac{1}{4}\frac{\Delta_0(s)}{\Delta_0(s+4/T)}, \tag{I12}$$

which implies $\Delta_0(s)/\Delta_0(s+4/T) \leq 4/3$. ■

Now we are ready to prove Theorem 10, the complexity estimate of using discrete adiabatic evolution to solve linear-systems problems.

Proof of Theorem 10. Let $\Delta'(s)$ be the multistep gap of $H(s)$, i.e.,

$$\Delta'(s) = \begin{cases}\Delta_2'(s), & 0 \leq s \leq 1 - 2/T, \\ \Delta_1'(s), & s = 1 - 1/T, \\ \Delta_0'(s), & s = 1,\end{cases} \tag{I13}$$

and let $\check{\Delta}'(s)$ be an adjustment for $\Delta'(s)$ at neighboring points:

$$\check{\Delta}'(s) = \min_{s'\in\{s-1/T,s,s+1/T\}\cap[0,1]}\Delta'(s'). \tag{I14}$$

The proof is organized as follows. First, to simplify further computation, we unify the time in the hat and check notations by applying Lemma 21. Then, we verify that the assumptions in Theorem 3 are satisfied. This is followed by the majority of the proof, in which we estimate each term in the error bound in Theorem 3.

To simplify the computation, we first unify the time in the hat and check notations by applying Lemma 21. More precisely,

$$\hat{c}_1(s) = \max_{s' \in \{s-1/T, s, s+1/T\} \cap [0,1-1/T]} c_1(s') = \begin{cases} 2d_p \Delta_0(0)^p, & s = 0, \\ 2d_p \Delta_0(s-1/T)^p, & 1/T \leq s \leq 1. \end{cases} \tag{I15}$$

Applying Lemma 21 to change all the discrete times to s, we have, for all $0 \leq s \leq 1$,

$$\hat{c}_1(s) \leq \frac{2^{2p+1}}{3^p} d_p \Delta_0(s)^p. \tag{I16}$$

Similarly,

$$\hat{c}_2(s) = \begin{cases} 4d_p^2 \Delta_0(0)^{2p} + 2pd_p^2(1-1/\kappa)\Delta_0(0)^{2p-1}, & s = 0 \\ 4d_p^2 \Delta_0(s-1/T)^{2p} + 2pd_p^2(1-1/\kappa)\Delta_0(s-1/T)^{2p-1}, & 1/T \leq s \leq 1-1/T, \end{cases} \tag{I17}$$

and for all $0 \leq s \leq 1 - 1/T$,

$$\hat{c}_2(s) \leq \frac{2^{4p+2}}{3^{2p}} d_p^2 \Delta_0(s)^{2p} + \frac{2^{4p-1}}{3^{2p-1}} pd_p^2 (1-1/\kappa)\Delta_0(s)^{2p-1}. \tag{I18}$$

For the spectrum gap, by Eq. (G2) and Lemma 21, we have

$$\check{\Delta}'(s) \geq \frac{3}{4}\Delta_0'(s) = \frac{3}{2^3}\Delta_0(s). \tag{I19}$$

Note that, by Eqs. (I16) and (I19),

$$4\frac{\hat{c}_1(s)}{\check{\Delta}'(s)} \leq 4\frac{2^{2p+1}}{3^p} d_p \Delta_0(s)^p \frac{2^3}{3\Delta_0(s)} = \frac{2^{2p+6} d_p \Delta_0(s)^{p-1}}{3^{p+1}} \leq 38 d_p. \tag{I20}$$

Therefore, the assumption that $T \geq 38d_p$ ensures that the assumption in Theorem 3 is satisfied.

Combining Eqs. (I16), (I18) and (I19) and the fact that

$$d_p = \frac{2^{p/2}}{p-1}\frac{\kappa}{\kappa-1}(\kappa^{p-1}-1) \leq \frac{2^{1+p/2}}{p-1}\kappa^{p-1}, \tag{I21}$$

we are now ready to bound each term in the error bound in Theorem 3. The first three terms (i.e., the boundary terms) in Theorem 3 can be bounded as follows:

$$\frac{\hat{c}_1(0)}{T\check{\Delta}'(0)^2} \leq \frac{2^{2p+1}}{3^p} d_p \Delta_0(0)^p \frac{2^6}{3^2 T \Delta_0(0)^2} = \frac{2^{2p+7} d_p}{3^{p+2} T} \leq \frac{2^{8+5p/2}}{3^{p+2}(p-1)} \frac{\kappa^{p-1}}{T}, \tag{I22}$$

$$\frac{\hat{c}_1(1)}{T\check{\Delta}'(1)^2} \leq \frac{2^{2p+1}}{3^p} d_p \Delta_0(1)^p \frac{2^6}{3^2 T \Delta_0(1)^2} = \frac{2^{2p+7} d_p}{3^{p+2} T \Delta_0(1)^{2-p}} \leq \frac{2^{8+5p/2}}{3^{p+2}(p-1)} \frac{\kappa}{T}, \tag{I23}$$

and

$$\frac{\hat{c}_1(1)}{T\check{\Delta}'(1)} \leq \frac{2^{2p+1}}{3^p} d_p \Delta_0(1)^p \frac{2^3}{3T\Delta_0(1)} = \frac{2^{2p+4} d_p}{3^{p+1} T}\Delta_0(1)^{p-1} \leq \frac{2^{5+5p/2}}{3^{p+1}(p-1)}\frac{1}{T}. \tag{I24}$$

Again by Eqs. (I16), (I18) and (I19), the last three terms in Theorem 3 can be bounded as

$$\sum_{n=1}^{T-1} \frac{\hat{c}_1(n/T)^2}{T^2 \check{\Delta}'(n/T)^3} \leq \sum_{n=1}^{T-1} \frac{2^{4p+2}}{3^{2p}} d_p^2 \Delta_0(n/T)^{2p} \frac{1}{T^2} \frac{2^9}{3^3 \Delta_0(n/T)^3} = \frac{2^{4p+11} d_p^2}{3^{2p+3} T^2} \sum_{n=1}^{T-1} \Delta_0(n/T)^{2p-3}, \tag{I25}$$

$$\sum_{n=0}^{T-1} \frac{\hat{c}_1(n/T)^2}{T^2 \check{\Delta}'(n/T)^2} \leq \sum_{n=0}^{T-1} \frac{2^{4p+2}}{3^{2p}} d_p^2 \Delta_0(n/T)^{2p} \frac{1}{T^2} \frac{2^6}{3^2 \Delta_0(n/T)^2} = \frac{2^{4p+8} d_p^2}{3^{2p+2} T^2} \sum_{n=0}^{T-1} \Delta_0(n/T)^{2p-2}, \tag{I26}$$

and

$$\begin{aligned}\sum_{n=1}^{T-1} \frac{\hat{c}_2(n/T)}{T^2 \check{\Delta}'(n/T)^2} &\leq \sum_{n=1}^{T-1} \frac{2^{4p+2}}{3^{2p}} d_p^2 \Delta_0(n/T)^{2p} \frac{1}{T^2} \frac{2^6}{3^2 \Delta_0(n/T)^2} \\ &\quad + \sum_{n=1}^{T-1} \frac{2^{4p-1}}{3^{2p-1}} p d_p^2 (1-1/\kappa) \Delta_0(n/T)^{2p-1} \frac{1}{T^2} \frac{2^6}{3^2 \Delta_0(n/T)^2} \\ &= \frac{2^{4p+8} d_p^2}{3^{2p+2} T^2} \sum_{n=1}^{T-1} \Delta_0(n/T)^{2p-2} + \frac{2^{4p+5} p d_p^2}{3^{2p+1} T^2} (1-1/\kappa) \sum_{n=1}^{T-1} \Delta_0(n/T)^{2p-3}. \end{aligned} \tag{I27}$$

To proceed, we need to bound the summations $\frac{1}{T}\sum_{n=0}^{T-1} \Delta_0(n/T)^{2p-2}$ and $\frac{1}{T}\sum_{n=0}^{T-1} \Delta_0(n/T)^{2p-3}$. Note that the summations are in the Riemann-sum form. The idea is then to approximate the summations by corresponding integrals and to bound both the integrals and the difference terms. More precisely, according to Ref. [28], for any continuously differentiable $g(t)$ on the interval $[a, b]$, we have

$$\left| \int_a^b g(t)dt - (b-a)g(a) \right| \leq \frac{(b-a)^2}{2} \max_{t \in [a,b]} \left| g'(t) \right|. \tag{I28}$$

This implies that

$$\left| \int_0^1 g(t)dt - \frac{1}{T} \sum_{n=0}^{T-1} g(n/T) \right| \leq \frac{1}{2T^2} \sum_{n=0}^{T-1} \max_{t \in [n/T, (n+1)/T]} \left| g'(t) \right|. \tag{I29}$$

If we further assume $g(t) > 0$ for all t, then

$$\frac{1}{T} \sum_{n=0}^{T-1} g(n/T) \leq \int_0^1 g(t)dt + \frac{1}{2T^2} \sum_{n=0}^{T-1} \max_{t \in [n/T, (n+1)/T]} \left| g'(t) \right|. \tag{I30}$$

By taking the function $g(t)$ to be $\Delta_0(t)^{2p-2}$ and $\Delta_0(t)^{2p-3}$, respectively, we can bound the desired summations.

We start with the summation of Δ_0^{2p-2}. By change of variable $x = f(t)$, the integral can be computed as

$$\begin{aligned}\int_0^1 \Delta_0(t)^{2p-2} dt &= \int_0^1 (1 - f(t) + f(t)/\kappa)^{2p-2} dt \\ &= \int_0^1 (1 - f + f/\kappa)^{2p-2} \frac{1}{d_p (1 - f + f/\kappa)^p} df \\ &= \frac{1}{d_p} \int_0^1 (1 - f + f/\kappa)^{p-2} df \\ &= \frac{1}{d_p} \frac{\kappa^{2-p}}{p-1} \frac{\kappa^{p-1} - 1}{\kappa - 1}. \end{aligned} \tag{I31}$$

The derivative can be computed as

$$\frac{d}{dt} \Delta_0(t)^{2p-2} = (2p-2)(-1+1/\kappa) d_p (1 - f(t) + f(t)/\kappa)^{3p-3} = (2p-2)(-1+1/\kappa) d_p \Delta_0(t)^{3p-3}. \tag{I32}$$

Therefore, according to Eq. (I30) and the fact that $\Delta_0(t)$ is bounded by 1, we have

$$\begin{aligned}\frac{1}{T}\sum_{n=0}^{T-1}\Delta_0(n/T)^{2p-2} &\leq \frac{1}{d_p}\frac{\kappa^{2-p}}{p-1}\frac{\kappa^{p-1}-1}{\kappa-1} + \frac{(p-1)d_p}{T^2}\frac{\kappa-1}{\kappa}\sum_{n=0}^{T-1}\Delta_0(n/T)^{3p-3} \\ &\leq \frac{1}{d_p}\frac{\kappa^{2-p}}{p-1}\frac{\kappa^{p-1}-1}{\kappa-1} + \frac{(p-1)d_p}{T^2}\frac{\kappa-1}{\kappa}\sum_{n=0}^{T-1}\Delta_0(n/T)^{2p-2} \\ &\leq \frac{1}{d_p}\frac{\kappa^{2-p}}{p-1}\frac{\kappa^{p-1}-1}{\kappa-1} + \frac{d_p}{T}\frac{1}{T}\sum_{n=0}^{T-1}\Delta_0(n/T)^{2p-2}. \end{aligned} \tag{I33}$$

By the assumption that $T > 38d_p$, we have $d_p/T \leq 1/38$ and thus

$$\frac{1}{T}\sum_{n=0}^{T-1}\Delta_0(n/T)^{2p-2} \leq \frac{1}{d_p}\frac{\kappa^{2-p}}{p-1}\frac{\kappa^{p-1}-1}{\kappa-1} + \frac{1}{38}\frac{1}{T}\sum_{n=0}^{T-1}\Delta_0(n/T)^{2p-2}. \tag{I34}$$

Solving the summation from the above inequality leads to

$$\begin{aligned}\frac{1}{T}\sum_{n=0}^{T-1}\Delta_0(n/T)^{2p-2} &\leq \frac{38}{37}\frac{1}{d_p}\frac{\kappa^{2-p}}{p-1}\frac{\kappa^{p-1}-1}{\kappa-1} \\ &\leq \frac{38}{37}(p-1)\frac{1}{d_p}, \end{aligned} \tag{I35}$$

where the second inequality follows from $\kappa^{2-p}\frac{\kappa^{p-1}-1}{\kappa-1} \leq 1$.

The summation of Δ_0^{2p-3} can be bounded similarly but requires some more delicate computations. We first assume that $p \neq 1.5$ such that $2p-3 \neq 0$. Again, the integral and the derivative can be computed as

$$\begin{aligned}\int_0^1 \Delta_0(t)^{2p-3}dt &= \int_0^1 (1-f(t)+f(t)/\kappa)^{2p-3}dt \\ &= \int_0^1 (1-x+x/\kappa)^{2p-3}\frac{1}{d_p(1-x+x/\kappa)^p}dx \\ &= \frac{1}{d_p}\int_0^1 (1-x+x/\kappa)^{p-3}dx \\ &= \frac{1}{d_p}\frac{1}{2-p}\frac{\kappa}{\kappa-1}(\kappa^{2-p}-1) \end{aligned} \tag{I36}$$

and

$$\frac{d}{dt}\Delta_0(t)^{2p-3} = (2p-3)(-1+1/\kappa)d_p(1-f(t)+f(t)/\kappa)^{3p-4} = (2p-3)(-1+1/\kappa)d_p\Delta_0(t)^{3p-4}. \tag{I37}$$

According to Eq. (I30), we have

$$\begin{aligned}\frac{1}{T}\sum_{n=0}^{T-1}\Delta_0(n/T)^{2p-3} &\leq \frac{1}{d_p}\frac{1}{2-p}\frac{\kappa}{\kappa-1}(\kappa^{2-p}-1) + \frac{|2p-3|d_p}{2T^2}\frac{\kappa-1}{\kappa}\sum_{n=0}^{T-1}\max_{t\in[n/t,(n+1)/T]}\Delta_0(t)^{3p-4} \\ &\leq \frac{1}{d_p}\frac{1}{2-p}\frac{\kappa}{\kappa-1}(\kappa^{2-p}-1) + \frac{|2p-3|d_p}{2T^2}\frac{\kappa-1}{\kappa}\sum_{n=0}^{T-1}\max_{t\in[n/t,(n+1)/T]}\Delta_0(t)^{2p-3}. \end{aligned} \tag{I38}$$

Since $\Delta_0(t)^{2p-3}$ is always monotonic, $\max_{t\in[n/T,(n+1)/T]}\Delta_0(t)^{2p-3}$ becomes either $\Delta_0(n/T)^{2p-3}$ or $\Delta_0((n+1)/T)^{2p-3}$. The corresponding summation is then bounded by either $\sum_{n=0}^{T-1}\Delta_0(n/T)^{2p-3}$ or $\sum_{n=0}^{T-1}\Delta_0((n+1)/T)^{2p-3}$, both of which can be bounded by $\sum_{n=0}^{T} \Delta_0(n/T)^{2p-3}$. Then,

$$\begin{aligned}\frac{1}{T}\sum_{n=0}^{T-1}\Delta_0(n/T)^{2p-3} &\leq \frac{1}{d_p}\frac{1}{2-p}\frac{\kappa}{\kappa-1}(\kappa^{2-p}-1)+\frac{|2p-3|d_p}{2T^2}\frac{\kappa-1}{\kappa}\sum_{n=0}^{T}\Delta_0(n/T)^{2p-3}\\ &\leq \frac{1}{d_p}\frac{1}{2-p}\frac{\kappa}{\kappa-1}(\kappa^{2-p}-1)+\frac{d_p}{2T^2}\sum_{n=0}^{T}\Delta_0(n/T)^{2p-3}.\end{aligned}\tag{I39}$$

Again using the fact that $d_p/T\leq 1/38$ and separating the term with $n=T$ in the summation on the right-hand side, we obtain

$$\begin{aligned}\frac{1}{T}\sum_{n=0}^{T-1}\Delta_0(n/T)^{2p-3} &\leq \frac{1}{d_p}\frac{1}{2-p}\frac{\kappa}{\kappa-1}(\kappa^{2-p}-1)+\frac{1}{38}\frac{1}{T}\sum_{n=0}^{T}\Delta_0(n/T)^{2p-3}\\ &\leq \frac{1}{d_p}\frac{1}{2-p}\frac{\kappa}{\kappa-1}(\kappa^{2-p}-1)+\frac{1}{38}\frac{1}{T}\sum_{n=0}^{T-1}\Delta_0(n/T)^{2p-3}+\frac{1}{38}\frac{\kappa^{3-2p}}{T}.\end{aligned}\tag{I40}$$

Solving the summation gives

$$\begin{aligned}\frac{1}{T}\sum_{n=0}^{T-1}\Delta_0(n/T)^{2p-3} &\leq \frac{38}{37}\frac{1}{d_p}\frac{1}{2-p}\frac{\kappa}{\kappa-1}(\kappa^{2-p}-1)+\frac{1}{37}\frac{\kappa^{3-2p}}{T}\\ &\leq \frac{38}{37d_p(2-p)}\kappa^{2-p}+\frac{1}{37}\frac{\kappa^{3-2p}}{T},\end{aligned}\tag{I41}$$

where the second inequality follows from $\kappa^{p-1}\frac{\kappa^{2-p}-1}{\kappa-1}\leq 1$. Note that the above estimate also holds for $p=1.5$ since when $p=1.5$, the left-hand side is a constant 1 and the right-hand side is always larger than 1.

We are now ready to bound the last three terms in Theorem 3. By inserting Eqs. (I35) and (I41) back into Eqs. (I25)–(I27) and using the representation of d_p in Eq. (I21), we have

$$\begin{aligned}\sum_{n=1}^{T-1}\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}'(n/T)^3} &\leq \frac{2^{4p+11}d_p}{3^{2p+3}T}\frac{38}{37}\frac{1}{2-p}\kappa^{2-p}+\frac{2^{4p+11}d_p^2}{3^{2p+3}T}\frac{1}{37}\frac{\kappa^{3-2p}}{T}\\ &= \frac{2^{4p+12}19d_p}{3^{2p+3}37(2-p)T}\kappa^{2-p}+\frac{2^{4p+11}d_p^2}{3^{2p+3}37T}\frac{\kappa^{3-2p}}{T}\\ &\leq \frac{2^{13+9p/2}19}{3^{2p+3}37(2-p)(p-1)}\frac{\kappa}{T}+\frac{2^{5p+13}}{3^{2p+3}37(p-1)^2}\frac{\kappa}{T^2},\end{aligned}\tag{I42}$$

$$\sum_{n=0}^{T-1}\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}'(n/T)^2}\leq\frac{2^{4p+8}d_p^2}{3^{2p+2}T}\frac{38}{37(p-1)}\frac{1}{d_p}\leq\frac{2^{10+9p/2}19}{3^{2p+2}37(p-1)^2}\frac{\kappa^{p-1}}{T}\tag{I43}$$

and

$$\begin{aligned}\sum_{n=1}^{T-1}\frac{\hat{c}_2(n/T)}{T^2\check{\Delta}'(n/T)^2} &\leq \frac{2^{4p+8}d_p^2}{3^{2p+2}T}\frac{38}{37(p-1)}\frac{1}{d_p}+\frac{2^{4p+5}pd_p^2}{3^{2p+1}T}\frac{\kappa-1}{\kappa}\left(\frac{38}{37}\frac{1}{d_p}\frac{1}{2-p}\frac{\kappa}{\kappa-1}(\kappa^{2-p}-1)+\frac{1}{37}\frac{\kappa^{3-2p}}{T}\right)\\ &= \frac{2^{4p+9}19d_p}{3^{2p+2}37(p-1)T}+\frac{2^{4p+6}19pd_p(\kappa^{2-p}-1)}{3^{2p+1}37(2-p)T}+\frac{2^{4p+5}pd_p^2}{3^{2p+1}37T}\frac{\kappa-1}{\kappa}\frac{\kappa^{3-2p}}{T}\\ &\leq \frac{2^{10+9p/2}19}{3^{2p+2}37(p-1)^2}\frac{\kappa^{p-1}}{T}+\frac{2^{7+9p/2}19p}{3^{2p+1}37(2-p)(p-1)}\frac{\kappa}{T}+\frac{2^{5p+7}p}{3^{2p+1}37(p-1)^2}.\end{aligned}\tag{I44}$$

Finally, inserting Eqs. (I22)–(I24) and Eqs.(I42)–(I44) into the estimate in Theorem 3, the adiabatic error can be bounded by

$$
\begin{aligned}
&12\frac{2^{8+5p/2}}{3^{p+2}(p-1)}\frac{\kappa^{p-1}}{T}+12\frac{2^{8+5p/2}}{3^{p+2}(p-1)}\frac{\kappa}{T}+6\frac{2^{5+5p/2}}{3^{p+1}(p-1)}\frac{1}{T}\\
&+305\frac{2^{13+9p/2}19}{3^{2p+3}37(2-p)(p-1)}\frac{\kappa}{T}+305\frac{2^{5p+13}}{3^{2p+3}37(p-1)^2}\frac{\kappa}{T^2}+44\frac{2^{10+9p/2}19}{3^{2p+2}37(p-1)^2}\frac{\kappa^{p-1}}{T}\\
&+32\frac{2^{10+9p/2}19}{3^{2p+2}37(p-1)^2}\frac{\kappa^{p-1}}{T}+32\frac{2^{7+9p/2}19p}{3^{2p+1}37(2-p)(p-1)}\frac{\kappa}{T}+32\frac{2^{5p+7}p}{3^{2p+1}37(p-1)^2}\frac{\kappa}{T^2}\\
&\leq C_p^{(1)}\frac{\kappa}{T}+C_p^{(2)}\frac{\kappa^{p-1}}{T}+C_p^{(3)}\frac{\kappa}{T^2}+C_p^{(4)}\frac{1}{T},
\end{aligned}
\tag{I45}
$$

where

$$C_p^{(1)}:=12\frac{2^{8+5p/2}}{3^{p+2}(p-1)}+305\frac{2^{13+9p/2}19}{3^{2p+3}37(2-p)(p-1)}+32\frac{2^{7+9p/2}19p}{3^{2p+1}37(2-p)(p-1)},\tag{I46}$$

$$C_p^{(2)}:=12\frac{2^{8+5p/2}}{3^{p+2}(p-1)}+44\frac{2^{10+9p/2}19}{3^{2p+2}37(p-1)^2}+32\frac{2^{10+9p/2}19}{3^{2p+2}37(p-1)^2},\tag{I47}$$

$$C_p^{(3)}:=305\frac{2^{5p+13}}{3^{2p+3}37(p-1)^2}+32\frac{2^{5p+7}p}{3^{2p+1}37(p-1)^2},\tag{I48}$$

$$C_p^{(4)}:=6\frac{2^{5+5p/2}}{3^{p+1}(p-1)}.\tag{I49}$$

This completes the proof of the first part by defining C_p to be the largest constant factor in Eq. (I45):

$$C_p:=\max_j C_p^{(j)}.\tag{I50}$$

The second part of Theorem 10, which is $T=\mathcal{O}(\kappa/\epsilon)$, follows directly from this bound by noting that each term of the adiabatic error in the first part can be bounded by $\mathcal{O}(\kappa/T)$. ■

APPENDIX J: ADDITIONAL DETAILS FOR FILTERING

Here, we give a proof of the upper bound on the norm of the difference of states for filtering. We are assuming that $\tilde{w}(\phi)=0$ for the desired part of the spectrum and that the initial probability for the desired part of the spectrum is at least $1/2$. Then, the squared norm for the undesired part of the state is

$$P(\perp)=\left\|\sum_{k\in\perp}\tilde{w}(\phi_k)\psi_k\left|k\right\rangle\right\|^2\leq\left(\max_{k\in\{\perp\}}\tilde{w}(\phi_k)\right)^2\left(\sum_{k\in\perp}|\psi_k|^2\right),\tag{J1}$$

where we are using $\perp$ to denote the set of undesired states. Recall that this is part of a state that is not normalized. The squared norm for the desired part of the spectrum is

$$P(\not\perp)=\sum_{k\in\not\perp}|\psi_k|^2,\tag{J2}$$

where $\not\perp$ indicates the desired part of the spectrum. As a result, the normalized probability for the desired part is lower bounded by

$$\frac{P(\not\perp)}{P(\not\perp)+P(\perp)}\geq\frac{\sum_{k\in\not\perp}|\psi_k|^2}{\sum_{k\in\not\perp}|\psi_k|^2+(\max_{k\in\perp}\tilde{w}(\phi_k))^2\left(\sum_{k\in\perp}|\psi_k|^2\right)}\geq\frac{1}{1+(\max_{k\in\perp}\tilde{w}(\phi_k))^2},\tag{J3}$$

where the second inequality comes from assuming that the initial probability for the desired part of the spectrum is at least $1/2$. Given this probability, the norm of the difference from the desired state is

$$\sqrt{2-\frac{2}{1+(\max_{k\in\perp}\tilde{w}(\phi_k))^2}} \le \max_{k\in\perp}\tilde{w}(\phi_k). \tag{J4}$$

Next, we give a more explicit description of the sequence of rotations needed for the filtering. The first rotation prepares the state

$$\frac{1}{\sqrt{\sum_j w_j}}\left(\sqrt{w_0}\,|0\rangle + \sqrt{\sum_{j>0} w_j}\,|1\rangle\right). \tag{J5}$$

The first controlled rotation gives

$$\sqrt{\sum_{j>0} w_j}\,|10\rangle \mapsto \sqrt{w_1}\,|10\rangle + \sqrt{\sum_{j>1} w_j}\,|11\rangle\,. \tag{J6}$$

In general, the controlled rotation with qubit k as control and $k+1$ as target maps

$$\sqrt{\sum_{j\ge k} w_j}\,|10\rangle \mapsto \sqrt{w_k}\,|10\rangle + \sqrt{\sum_{j>k} w_j}\,|11\rangle\,. \tag{J7}$$

If one were to perform the rotations for the preparation in the reverse order, one would use a rotation on the last qubit to take zero to

$$\frac{1}{\sqrt{\sum_j w_j}}\left(\sqrt{\sum_{j=0}^{\ell-1} w_j}\,|0\rangle + \sqrt{w_\ell}\,|1\rangle\right). \tag{J8}$$

Then, the controlled rotation would take

$$\sqrt{\sum_{j=0}^{\ell-1} w_j}\,|10\rangle \mapsto \sqrt{\sum_{j=0}^{\ell-2} w_j}\,|00\rangle + \sqrt{w_{\ell-1}}\,|10\rangle\,. \tag{J9}$$

Inverting this rotation gives

$$\sqrt{\sum_{j=0}^{\ell-1} w_j}\,|00\rangle \mapsto \sqrt{\sum_{j=0}^{\ell-2} w_j}\,|10\rangle - \sqrt{w_{\ell-1}}\,|00\rangle\,, \tag{J10}$$

$$\sqrt{\sum_{j=0}^{\ell-1} w_j}\,|10\rangle \mapsto \sqrt{w_{\ell-1}}\,|10\rangle + \sqrt{\sum_{j=0}^{\ell-2} w_j}\,|00\rangle\,. \tag{J11}$$

More generally, the rotation with qubit $k+1$ as control and k as target gives

$$\sqrt{\sum_{j=0}^{k} w_j}\,|00\rangle \mapsto \sqrt{\sum_{j=0}^{k-1} w_j}\,|10\rangle - \sqrt{w_k}\,|00\rangle\,, \tag{J12}$$

$$\sqrt{\sum_{j=0}^{k} w_j}\,|10\rangle \mapsto \sqrt{w_k}\,|10\rangle + \sqrt{\sum_{j=0}^{k-1} w_j}\,|00\rangle . \tag{J13}$$

This means that the sequence of two controlled rotations gives

$$\begin{aligned}\sqrt{\sum_{j\geq k} w_j}\,|1\rangle &\mapsto \sqrt{w_k}\,|10\rangle + \sqrt{\sum_{j>k} w_j}\,|11\rangle \\ &\mapsto \frac{\sqrt{w_k}}{\sqrt{\sum_{j=0}^{k} w_j}}\left(\sqrt{w_k}\,|10\rangle + \sqrt{\sum_{j=0}^{k-1} w_j}\,|00\rangle\right) + \sqrt{\sum_{j>k} w_j}\,|11\rangle\end{aligned} \tag{J14}$$

and

$$|0\rangle \mapsto \frac{1}{\sqrt{\sum_{j=0}^{k} w_j}}\left(\sqrt{\sum_{j=0}^{k-1} w_j}\,|10\rangle - \sqrt{w_k}\,|00\rangle\right). \tag{J15}$$

Projecting onto $|1\rangle$ on the first qubit then gives the mapping

$$|1\rangle \mapsto \frac{1}{\sqrt{\sum_{j\geq k} w_j}}\left(\frac{w_k}{\sqrt{\sum_{j=0}^{k} w_j}}\,|0\rangle + \sqrt{\sum_{j>k} w_j}\,|1\rangle\right) \tag{J16}$$

$$|0\rangle \mapsto \frac{\sqrt{\sum_{j=0}^{k-1} w_j}}{\sqrt{\sum_{j=0}^{k} w_j}}\,|0\rangle . \tag{J17}$$

To see the effect of this, let us consider $k = 1$, so that we are considering the operation immediately after the qubit rotation and controlled W on the target system. Assuming that the target system is in an eigenstate with eigenvalue $e^{i\phi}$, the state at this point will be

$$\frac{1}{\sqrt{\sum_j w_j}}\left(\sqrt{w_0}\,|0\rangle + \sqrt{\sum_{j>0} w_j e^{i\phi}}\,|1\rangle\right). \tag{J18}$$

The above mapping then gives

$$\frac{1}{\sqrt{\sum_j w_j}}\left(\frac{w_0}{\sqrt{\sum_{j=0}^{1} w_j}}\,|0\rangle + e^{i\phi}\left(\frac{w_1}{\sqrt{\sum_{j=0}^{1} w_j}}\,|0\rangle + \sqrt{\sum_{j>1} w_j}\,|1\rangle\right)\right). \tag{J19}$$

This can be written as

$$\frac{1}{\sqrt{\sum_j w_j}}\left(\frac{w_0 + e^{i\phi} w_1}{\sqrt{\sum_{j=0}^{1} w_j}}\,|0\rangle + e^{i\phi}\sqrt{\sum_{j>1} w_j}\,|1\rangle\right). \tag{J20}$$

Thus we can see that we have the desired weights w_0 and w_1 on the $|0\rangle$ state and that the $|1\rangle$ state is flagging the remainder of the linear combination still to be obtained. More generally, after performing the controlled rotations between qubits k and $k+1$ and the projection onto $|1\rangle$ on the ancilla qubit, the state will be of the form

$$\frac{1}{\sqrt{\sum_j w_j}}\left(\frac{\sum_{j=0}^{k} e^{ij\phi} w_j}{\sqrt{\sum_{j=0}^{k} w_j}}\,|0\rangle + e^{ik\phi}\sqrt{\sum_{j>k} w_j}\,|1\rangle\right). \tag{J21}$$

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