Ψ-Type Multistability of Almost Periodic Solutions for Memristive Cohen-Grossberg Neural Networks

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Generalized-Type Multistability of Almost Periodic Solutions for Memristive Cohen-Grossberg Neural Networks

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***Abstract*—This paper investigates Ψ-type multistability of almost periodic solutions for memristive Cohen-Grossberg neural networks (MCGNNs). As the inevitable disturbances in biological neurons, almost periodic solutions are more common in nature than equilibrium points (EPs). They are also generalizations of EPs in mathematics. According to the concepts of almost periodic solutions and Ψ-type stability, this paper presents Ψ-type multistability definition of almost periodic solutions. Compared with exponential stability and polynomial stability, Ψ-type stability is more general stability, and generalizes the existing stability conclusions. The results show that $(K+1)^n$ Ψ-type stable almost periodic solutions can coexist in a MCGNNs with n neurons, where K is a parameter of the activation functions. The enlarged attraction basins are also estimated based on the original state space partition method. Two simulations are given to verify the theoretical results at the end of this paper.**

***Index Terms*—Memristive Cohen-Grossberg neural networks, Multiple Ψ-Type stability, Almost periodic solution, Memristor.**

I. Introduction

MEMRISTOR was first proposed by Chua in 1971 [1] according to the law of symmetry. In 2008, Strukov et al. successfully manufactured the first practical memristor device in HP Labs [2]. Soon after, Pershin and Ventra showed that the memristor have the characteristics of pinched hysteresis, which is similar to the neurons in the human brain have [3]. Based on the above reasons, the memristor can realize important functions of memory. Inspired by this important discovery, scientists constructed MCGNNs by using the memristors in conventional Cohen-Grossberg neural networks instead of resistors. Exploiting MCGNNs will be of great help to build a brain-like neural computer to realize the synapses in biological brains [4], [5]. It should be pointed out that the dynamical analysis of MCGNNs is of great significance in designing the MCGNNs. Scientists have been exploring this area in the last few years. Related researches about dynamical behaviors of MCGNNs are shown as following [6]–[9].

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Among many dynamic behaviors, multistability is one of the properties most closely related to the reproduction of human brain memory [10], [11], which refers to the property of having multiple stable EPs in neural networks system [12]. In practical applications, multistability is widely used in associative memory [13]–[17], deep learning [18], [19], pattern recognition [20], and other fields. For multistability, activation functions are important in deciding the number of EPs. In [21], regarding to the piecewise linear activation functions which have $2r$ corner points, $(r+1)^n$ local exponentially EPs can coexist in a n-neurons network. In [22], Guo et al. investigated multistability in the neural networks with sigmoid activation functions, they proved that 3^n EPs can exist in a n-neurons network at the same time. In [23], for neural networks which have Mexican-hat type activation functions, there are $2^{k_1}3^{k_2}$ locally exponentially stable EPs where k_1 and k_2 are parameters concerned with activation functions. In [24], the activation functions have $2m+1$ segments and are odd functions, Zeng and Zheng proved that $(m+1)^n$ stable EPs can coexist in one system.

In addition to the number of EPs, the convergence speed of each stable EP has always been the focus of scholars' research. According to the Lyapunov method, previous researchers have extensively studied polynomial stability, exponential stability and logarithmic stability [25], [26]. These different stabilities show the different decay speeds to the trivial solution. Then these stability concepts are generalized as the general decay stability. Wu and Hu proposed the definition of Ψ-type general decay stability, and successfully unified the previously stability types [26]. Zhang et al. discussed the multiple Ψ-type stability in [27] and [28]. However, these two papers did not take into account the almost periodic solutions caused by disturbance.

As mentioned above, disturbances can influence the multistability in neural networks. In [29], Lin and Shih pointed out that disturbances can lead to the formation of almost periodic solutions when considering dynamic behavior in neural networks. They successfully proved that there are 2^n stable attractors in the perturbed systems. Subsequently, Wang et al. studied the multistability of almost periodic solutions in neural networks which have different kind of activation functions, and they found that 2^n stable almost periodic solutions can coexist in a neural network with n neurons [30]. It should be emphasized that the almost periodic solutions are more general than EPs. Although the above studies have done valuable research on the almost periodic solutions, to our best knowledge, the research on the Ψ-type multistability of the almost periodic solutions is not thorough.

Based on the above discussions, we will investigate the Ψ-

type multistability of almost periodic solutions for MCGNNs. The main innovations are listed below.

1) Using the almost periodic solutions and Ψ-type stability concepts, this paper presents the definition about Ψ-type multistability of almost periodic solutions. Compare with [27], [28] and [31], this paper generalizes the Ψ-type multistability conclusions of EPs to the almost periodic solutions.
2) Ψ-type multistability is a more general concept of multistability, the decay rate conditions are weaken than [24], [32]. At the same time, Ψ-type stability can deduce the related conclusions, which shows that our conclusion is an extension of previous works [12], [16].
3) For almost periodic solutions, compare with [29], [30], it is not necessary for the activation functions to be neither piecewise linear nor monotonic. The multistability conclusions of MCGNNs can be specialized to the CGNNs in certain conditions [30], [31].

The remaining parts will be arranged as following. The model of MCGNNs, definitions and assumptions will be introduced in Section II. The positive invariant sets, and the coexistence of Ψ-type stable almost periodic state solutions are proved in Section III. Some simulations will be given to verify the criteria in Section IV. Conclusions and future outlooks are listed in section V.

II. Problem formulation

A. Memristive Cohen-Grossberg Neural Networks Model

The parameters of the MCGNNs will vary with the states, which makes the MCGNNs models different from the common Cohen-Grossberg neural networks models. According to [12], here we introduce the following MCGNNs model:

$$\begin{aligned}\dot{x}_i(t) =& A_i(x_i(t))\Bigg[- a_i(x_i(t))x_i(t) + \sum_{j=1}^{n} b_{ij}(t) \\ & \times f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}(t)g_j(x_j(t-\tau_{ij})) + I_i(t) \\ & + \sum_{j=1}^{n} d_{ij}(t)\int_0^{\infty} h_j(x_j(t-s))l_j(s)ds\Bigg], \end{aligned} \tag{1}$$

where $i,j \in \{1,2,\cdots,n\}$, $t \geq 0$. More precisely, $x_i(t)$ represents the state of the i-th neuron; $A_i(x_i(t))$ denotes the amplification function, $a_i(x_i(t))$ stands for the inhibition rate of the i-th neuron, $B(t) = [b_{ij}(t)]_{nn}$, $C(t) = [c_{ij}(t)]_{nn}$ and $D(t) = [d_{ij}(t)]_{nn}$ are connection weight matrices, $f_j(\cdot), g_j(\cdot)$ and $h_j(\cdot)$ are activation functions, $I_i(t)$ is external input, τ_{ij} denotes the bounded continuous delay and $\tau \triangleq \max_{i,j}\{\tau_{ij}\}$. $\int_0^{\infty} h_j(x_j(t-s))l_j(s)ds$ is the distributed delay and there is $\int_0^{\infty} e^{\theta s}l_j(s)ds < L_j$, where $\theta \in (0,1)$ is a positive constant and $L_j > 0$. It should be noted that in the MCGNNs model, the inhibition rates depend on the neural state, which means that

$$a_i(x_i(t)) = \begin{cases} \acute{a}_i(t), & x_i(t) > 0, \\ (\acute{a}_i(t) + \grave{a}_i(t))/2, & x_i(t) = 0, \\ \grave{a}_i(t), & x_i(t) < 0, \end{cases}$$

where $\acute{a}_i(t)$, $\grave{a}_i(t)$ are bounded positive functions, the switching threshold is 0. According to the above state switching formula, $a_i(x_i(t))$ can be converted into the following form

$$a_i(x_i(t)) = \frac{\acute{a}_i(t) + \grave{a}_i(t)}{2} + \mathrm{sign}(x_i(t))\frac{\acute{a}_i(t) - \grave{a}_i(t)}{2},$$

where $\mathrm{sign}(\cdot)$ is the sign function. For convenience, denote $\mathcal{A}_{i+}(t) = [\acute{a}_i(t) + \grave{a}_i(t)]/2$, $\mathcal{A}_{i-}(t) = [\acute{a}_i(t) - \grave{a}_i(t)]/2$ and $\underline{\mathcal{A}}_{i+} = \inf_t \mathcal{A}_{i+}(t)$. So the MCGNNs (1) can be transformed like following

$$\begin{aligned}\dot{x}_i(t) =& A_i(x_i(t))\Bigg\{ - [\mathcal{A}_{i+}(t) + \mathrm{sign}(x_i(t))\mathcal{A}_{i-}(t)]x_i(t) \\ & + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}(t)g_j(x_j(t-\tau_{ij})) \\ & + \sum_{j=1}^{n} d_{ij}(t)\int_0^{\infty} h_j(x_j(t-s))l_j(s)ds + I_i(t)\Bigg\}, \\ =& A_i(x_i(t)) \times \Bigg\{ - \mathcal{A}_{i+}(t)x_i(t) - \mathcal{A}_{i-}(t)|x_i(t)| \\ & + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}(t)g_j(x_j(t-\tau_{ij})) \\ & + \sum_{j=1}^{n} d_{ij}(t)\int_0^{\infty} h_j(x_j(t-s))l_j(s)ds + I_i(t)\Bigg\}. \end{aligned} \tag{2}$$

Remark 1: If we choose $\acute{a}_i(t) = \grave{a}_i(t)$, the MCGNNs (1) can be treated as common Cohen-Grossberg neural networks. Thus this model is a generalization of Cohen-Grossberg neural networks [30], [31].

For the initial conditions $\phi(s)$, let $C([-\tau,0],R^n)$ be the Banach space of continuous functions. Suppose

$$x_i(s) = \phi_i(s), \quad s \in [-\tau, 0], \quad i = 1,2,\cdots,n, \tag{3}$$

where $\phi(s) = (\phi_1(s), \phi_2(s), \cdots, \phi_n(s)) \in C([-\tau,0],R^n)$.

B. Ψ-Type Stability of Almost Periodic Solution

Based on the previous works, we need some preliminaries. Also, a new definition about Ψ-type stability of almost periodic solution is given.

Definition 1 [26]: Function $\psi : R_+ \to (0,+\infty)$ is Ψ-type function if the following conditions are satisfied:

1) $\psi(t)$ is nondecreasing and differentiable;
2) $\psi(0) = 1$ and $\psi(+\infty) = +\infty$;
3) Denote $\dot{\psi}(t)$ as the derivative of $\psi(t)$, then $\bar{\psi}(t) = \frac{\dot{\psi}(t)}{\psi(t)}$ is a nonincreasing function;
4) For $\forall t, s \geq 0$, there is $\psi(t+s) \leq \psi(t)\psi(s)$.

Definition 2 [30]: Denote $x^*(t) : R_+ \to R^n$ as the solution of system (1), $x^*(t)$ is an almost periodic solution if the following conditions are met:

1) $x^*(t)$ is a continuous function;
2) For $\forall \epsilon > 0$, in arbitrary intervals with length l, there exists a constant $\omega = \omega(\epsilon)$ satisfying

$$|x^*(t+\omega) - x^*(t)| \leq \epsilon, \quad \forall t \in R.$$

Assume $x^*(t)$ is an almost periodic solution in MCGNNs (1). For the error networks $e_i(t) = x_i(t) - x_i^*(t)$, the definition about Ψ-type stability of almost periodic solution is presented as following.

Definition 3: The error networks is Ψ-type stable with respect to almost periodic solution if there is a positive constant γ and

$$\limsup_{t\to+\infty} \frac{\ln ||e(t)||}{\ln \psi(t)} \leq -\gamma,$$

where $e(t) = (e_1(t), e_2(t), \cdots, e_n(t))^T$, $\psi(t)$ is a Ψ-type function. Furthermore, γ stands for the convergence rate.

Remark 2: As we mentioned before, Ψ-type stability can be treated as the generalization of many different kinds of stability. For example, If $\psi(t)$ takes the functions e^t, $1+\ln(1+t)$, and $1+t$, the solution is stable with exponential, logarithmic, and polynomial decay rate, respectively.

C. Assumptions

For the existence of solution and analysis of multistability, the following assumptions are needed.

Assumption 1: $A_i(\cdot)$ is bounded and continuous, which means there exist positive constants $\underline{A_i}$, $\bar{A}_i$ and $\underline{A_i} \leq A_i(\cdot) \leq \bar{A}_i$,where $i = 1, 2, \cdots, n$.

Assumption 2: The activation functions $f_i(\cdot), g(\cdot), h(\cdot)$ are bounded and continuous, which means there are positive constants m_{fi}, m_{gi}, m_{hi} and M_{fj}, M_{gi}, M_{hi} so that

$$m_{fi} \leq f_i(\cdot) \leq M_{fi}, \quad m_{gi} \leq g_i(\cdot) \leq M_{gi},$$
$$m_{hi} \leq h_i(\cdot) \leq M_{hi},$$

where $i = 1, 2, \cdots, n$.

Assumption 3: $\mathcal{A}_{i+}(t)$, $\mathcal{A}_{i-}(t)$, $\mathcal{B}_{ij+}(t)$, $\mathcal{B}_{ij-}(t)$, $\mathcal{C}_{ij+}(t)$, $\mathcal{C}_{ij-}(t)$, $\mathcal{D}_{ij+}(t)$, $\mathcal{D}_{ij-}(t)$ $I_i(t)$ are almost periodic, so for $\forall\epsilon > 0, \exists l = l(\epsilon) > 0$, for $\forall t$, there is $\omega = \omega(\epsilon)$ in any intervals with length l satisfying

$$|\mathcal{A}_{i+}(t+\omega) - \mathcal{A}_{i+}(t)| < \epsilon, \quad |\mathcal{A}_{i-}(t+\omega) - \mathcal{A}_{i-}(t)| < \epsilon,$$
$$|b_{ij}(t+\omega) - b_{ij}(t)| < \epsilon, \qquad |c_{ij}(t+\omega) - c_{ij}(t)| < \epsilon,$$
$$|d_{ij}(t+\omega) - d_{ij}(t)| < \epsilon, \qquad |I(t+\omega) - I(t)| < \epsilon.$$

Moreover, the connection weight need to be bounded, and

$$\underline{\mathcal{A}}_{i+} = \inf_t |A_{i+}(t)|, \quad \bar{\mathcal{A}}_{i-} = \sup_t |A_{i-}(t)|,$$
$$\bar{\mathcal{B}}_{ij} = \sup_t |b_{ij}(t)|, \quad \bar{\mathcal{C}}_{ij} = \sup_t |c_{ij}(t)|, \quad \bar{\mathcal{D}}_{ij} = \sup_t |d_{ij}(t)|.$$

Assumption 4: Given a positive integer K. For $i \in \{1, 2, \cdots, n\}$ and $k \in \{1, 2, \cdots, K\}$, there are p_i^k and q_i^k such that

$$-\infty < p_i^{(0)} < q_i^{(0)} < p_i^{(1)} < q_i^{(1)}$$
$$< \cdots < p_i^{(K-1)} < q_i^{(K-1)} < p_i^{(K)} < q_i^{(K)} < +\infty.$$

What is more, assume that there exist $\lambda_{fi}^{(k)}$, $\lambda_{gi}^{(k)}$, $\lambda_{hi}^{(k)}$, $\mu_{fi}^{(k)}$, $\mu_{gi}^{(k)}$, $\mu_{hi}^{(k)}$, such that for $\forall y, z \in [q_i^{(k-1)}, p_i^{(k)}]$, $y \neq z$,

$$\lambda_{fi}^{(k)} \leq \tfrac{f_i(y)-f_i(z)}{y-z} \leq \mu_{fi}^{(k)}, \quad \lambda_{gi}^{(k)} \leq \tfrac{g_i(y)-g_i(z)}{y-z} \leq \mu_{gi}^{(k)},$$
$$\lambda_{hi}^{(k)} \leq \tfrac{h_i(y)-h_i(z)}{y-z} \leq \mu_{hi}^{(k)}.$$

D. State Space Partition

Suppose I is an interval, we denote that $I^0 = \emptyset$, $I^1 = I$. Furthermore, according to Assumption 4, there are

$$\begin{aligned}[q_i^{(k-1)}, p_i^{(k)}] =& (p_i^{(0)}, q_i^{(0)})^0 \cup \cdots \cup [q_i^{(k-1)}, p_i^{(k)}]^1 \\ & (p_i^{(k)}, q_i^{(k)})^0 \cup \cdots \cup (p_i^{(K)}, q_i^{(K)})^0, \\ (p_i^{(k)}, q_i^{(k)}) =& (p_i^{(0)}, q_i^{(0)})^0 \cup \cdots \cup [q_i^{(k-1)}, p_i^{(k)}]^0 \\ & (p_i^{(k)}, q_i^{(k)})^1 \cup \cdots \cup (p_i^{(K)}, q_i^{(K)})^0,\end{aligned}$$

where $k = 1, 2, \cdots, K$. So there are $(2K+1)^n$ subsets in $\prod_{i=1}^n (p_i^{(0)}, q_i^{(K)})$. Given a positive integer N, the definition of set Δ are shown as following

$$\Delta(N) \triangleq \left\{ (\delta_1, \delta_2, \cdots, \delta_N) \middle| \sum_{i=1}^{N} \delta_i = 1, \delta_i = 0 \quad \text{or} \quad \delta_i = 1 \right\}.$$

Then denote $\gamma^{(i)} = (\gamma_1^{(i)}, \cdots, \gamma_{2K+1}^{(i)}) \in \Delta(2K+1)$ and for $i = 1, 2, \cdots, n$,

$$\Gamma_{\gamma^{(i)}} = \left(\bigcup_{k=1}^{K} [p_i^{(k-1)}, q_i^{(k)}]^{\gamma_{2k}^{(i)}} \right) \cup \left(\bigcup_{k=0}^{K} (p_i^{(k)}, q_i^{(k)})^{\gamma_{2k+1}^{(i)}} \right).$$

Next, suppose

$$\Omega = \left\{ \Gamma_\gamma = \prod_{i=1}^{n} \Gamma_{\gamma^{(i)}} \middle| \gamma = (\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(n)}), \right.$$
$$\left. \gamma^{(i)} = (\gamma_1^{(i)}, \cdots, \gamma_{2K+1}^{(i)}) \in \Delta(2K+1) \right\}.$$

Then it is obvious that there exist $(2K+1)^n$ elements in Ω. What is more, in order to distinguish the existence space of stable solution, suppose

$$\begin{aligned}\Omega_1 =& \left\{ \Gamma_\gamma^{'} = \prod_{i=1}^{n} \left(\bigcup_{k=0}^{K} (p_i^{(k)}, q_i^{(k)})^{\gamma_{k+1}^{(i)}} \right) \right| \\ & \gamma = (\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(n)}), \\ & \left. \gamma^{(i)} = (\gamma_1^{(i)}, \cdots, \gamma_{K+1}^{(i)}) \in \Delta(K+1) \right\}, \\ \Omega_2 =& \Omega - \Omega_1.\end{aligned}$$

Easy to see, in Ω_1, there exist $(K+1)^n$ elements, and in Ω_2, there are $(2K+1)^n - (K+1)^n$ elements . Here we finish the state space partition.

III. Main Results

The main results are proved in three parts. Firstly, for each $\Gamma_\gamma^{'} \in \Omega_1$, this paper will prove that it is the positive invariance set as shown in Lemma 1-2; Secondly, the Ψ-type almost periodic solutions characters, existence, multistability will be shown in Theorem 1-3 respectively; Thirdly, the enlarged attraction basins will be estimated in Theorem 4. The proofs will be given in turn below.

A. Positive Invariance

For convenience, denote that

$$\mathcal{F}_{i+}(t, \alpha) = -\mathcal{A}_{i+}(t)\alpha - \mathcal{A}_{i-}(t)|\alpha| + b_{ii}(t)f_i(\alpha)$$

$$
\begin{aligned}
&+\sum_{j=1,j\neq i}^{n}\max\{b_{ij}(t)m_{fj},b_{ij}(t)M_{fj}\}\\
&+\sum_{j=1}^{n}\max\{c_{ij}(t)m_{gj},c_{ij}(t)M_{gj}\}+I_i(t)\\
&+\sum_{j=1}^{n}\max\{d_{ij}(t)L_jm_{hj},d_{ij}(t)L_jM_{hj}\}, \quad (4)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_{i-}(t,\alpha)=&-\mathcal{A}_{i+}(t)\alpha-\mathcal{A}_{i-}(t)|\alpha|+b_{ii}(t)f_i(\alpha)\\
&+\sum_{j=1,j\neq i}^{n}\min\{b_{ij}(t)m_{fj},b_{ij}(t)M_{fj}\}\\
&+\sum_{j=1}^{n}\min\{c_{ij}(t)m_{gj},c_{ij}(t)M_{gj}\}+I_i(t)\\
&+\sum_{j=1}^{n}\min\{d_{ij}(t)L_jm_{hj},d_{ij}(t)L_jM_{hj}\}. \quad (5)
\end{aligned}
$$

Now, we give the result of positive invariance as Lemma 1.

Lemma 1: Under Assumptions 1, 2, and 4, if the following conditions established,

$$
\mathcal{F}_{i+}(t,q_i^{(k)})<0, \quad \mathcal{F}_{i-}(t,p_i^{(k)})>0, \quad (6)
$$

where $k=1,2,\cdots,K, i=1,2,\cdots,n$ and $t>0$, then each set $\Gamma_\gamma^{'}$ is a positive invariance.

Proof: According to the definition of Ω_1, it is obvious that

$$
\Gamma_\gamma^{'}=\prod_{i=1}^{n}\bigg(\bigcup_{k=0}^{K}(p_i^{(k)},q_i^{(k)})^{\gamma_{K+1}^{(i)}}\bigg).
$$

Suppose $\phi(s)\in C([-\tau,0],R)$ is the initial condition of MCGNNs (1), and $x(t)$ is the state solution with $x(s)=\phi(s)$ where $s\in[-\tau,0]$. Then we will show that, once $\phi(0)\in\Gamma_\gamma^{'}$, $x(t)$ will stay in this set for $\forall t\geq 0$. Here we use the contradiction method to illustrate.

If there exist a $k\in\{1,2,\cdots,K\},\beta\in\{1,2,\cdots,n\}$ and time t_1, such that $x_\beta(t)$ reaches q_β^k at time t_1, which means

$$
\begin{cases}
x_\beta(t_1)=q_\beta^{(k)},\\
p_\beta^{(k)}<x_\beta(t)<q_\beta^{(k)}, \quad t\in[0,t_1),\\
\dot{x}_\beta(t_1)\geq 0.
\end{cases}
$$

However, from MCGNNs (1), we know that

$$
\begin{aligned}
\dot{x}_\beta(t_1)=&A_i(q_\beta^{(k)})\times[-a_i(q_\beta^{(k)})q_\beta^{(k)}+\sum_{j=1}^{n}b_{ij}(t_1)f_j(x_j(t_1))\\
&+\sum_{j=1}^{n}c_{ij}(t_1)g_j(x_j(t_1-\tau_{ij}))+I_i(t_1)\\
&+\sum_{j=1}^{n}d_{ij}(t_1)\int_0^\infty h_j(x_j(t_1-s))l_j(s)ds]\\
\leq&A_i(q_\beta^{(k)})\times\mathcal{F}_{i+}(t_1,q_\beta^{(k)})<0.
\end{aligned}
$$

So far, we have completed the proof of the contradiction part. And under the condition that $x_\beta(t)$ reaches $p_\beta^{(k)}$, relevant conclusions can be similarly proved, here we omit it.

So, for all $t\geq 0$, $x(t)$ will never get out of $\Gamma_\gamma^{'}$ once $\phi(0)\in\Gamma_\gamma^{'}$, and $\Gamma_\gamma^{'}$ is a positive invariant set.

Remark 3: From Lemma 1, we can see that only $\phi(0)$ is in the positively invariant needed to be guaranteed. Thus it directly reduces the conservativeness required for initial conditions.

Lemma 2: If the conditions in Lemma 1 are met, then there are at least $(2K+1)^n$ EPs in MCGNNs (1).

Proof: According to Lemma 1, it is obviously to get that $\mathcal{F}_{i-}(t,x_i(t))\leq\dot{x}_i(t)\leq\mathcal{F}_{i+}(t,x_i(t))$. Denote $\mathcal{F}(x_i(t))\triangleq\dot{x}_i(t)$, we have

$$
\begin{aligned}
&\mathcal{F}(q_i^{(k)})<\mathcal{F}_{i+}(t,q_i^{(k)})<0,\\
&\mathcal{F}(p_i^{(k)})>\mathcal{F}_{i-}(t,q_i^{(k)})>0.
\end{aligned}
$$

It should be noted that $\mathcal{F}$ is a continuous function. By the intermediate value theorem, for each element $\Gamma_\gamma\in\Omega$, there is one point $\bar{x}_i\in\Gamma_\gamma$, so that $\mathcal{F}(\bar{x}_i)=0$ holds. Let set $\bar{\Gamma}_\gamma$ be the closure of the set Γ_γ, where $\bar{\Gamma}_\gamma$ is a compact convex set. Construct the following mapping function relationship: $\mathcal{G}:\bar{\Gamma}_\gamma\to\bar{\Gamma}_\gamma$ and $\mathcal{G}(x_1,x_2,\cdots,x_n)\to(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n)$. According to Brouwer's fixed point theory, there must exist a fixed point $x^*=(x_1^*,x_2^*,\cdots,x_n^*)\in\bar{\Gamma}_\gamma$ so that $\mathcal{G}(x^*)=x^*$, which follows that $x^*\in\bar{\Gamma}_\gamma$ is an EP.

Furthermore, as there are $(2K+1)^n$ elements in Ω, so there exist at least $(2K+1)^n$ EPs in MCGNNs (1). Here we finish our proof.

B. Existence and Multiple Ψ-type Stability of Almost Periodic Solutions

In this part, it will be shown that there are $(K+1)^n$ Ψ-type stable almost periodic solutions in MCGNNs (1).

Theorem 1: Under the conditions of Assumption 1-4, if there exist Ψ-type function $\psi(\cdot)$ and positive constants $M_x,k_1,k_2,\cdots,k_n$, such that

$$
\begin{aligned}
&\beta\psi(0)\psi(t_0)\frac{1}{\underline{A}_i}-\underline{\mathcal{A}}_{i+}+\sum_{j=1}^{n}\bar{\mathcal{B}}_{ij}\max\{|\lambda_{fj}^{(k)}|,|\mu_{fj}^{(k)}|\}\\
&+\psi(\tau_{ij})^\beta\sum_{j=1}^{n}[\bar{\mathcal{C}}_{ij}\max\{|\lambda_{gj}^{(k)}|,|\mu_{gj}^{(k)}|\}\\
&+L_j\bar{\mathcal{D}}_{ij}\max\{|\lambda_{hj}^{(k)}|,|\mu_{hj}^{(k)}|\}]<0, \quad (7)
\end{aligned}
$$

$$
\frac{1}{k_i}(\frac{\psi(t_0)}{\psi(t)})^\beta\int_0^t\sum_{j=1}^{n}k_j(\frac{\psi_j(s)}{\psi_j(t_0)})^\beta\Theta_jds<1, \quad (8)
$$

where

$$
\begin{aligned}
\Theta_j=&\sum_{j=1}^{n}(\max\{|m_{fj}|,|M_{fj}|\}+\max\{|m_{gj}|,|M_{gj}|\}\\
&+\max\{|m_{hj}|,|M_{hj}|\}L_j)+1+2M_x.
\end{aligned}
$$

Then there exists time T, $\forall t>T$, neuron state $x(t)$ in Ω_1 are almost periodic.

Proof: Firstly, we define the following Lyapunov function

$$
V(x(t),t)=V_1(x(t),t)+V_2(x(t),t). \quad (9)
$$

To be more precise, there is $\beta>0$ satisfying

$$
V_1(x(t),t)\triangleq\sum_{i=1}^{n}k_i(\frac{\psi(t)}{\psi(t_0)})^\beta\bigg|\int_{x_i(t)}^{x_i(t+\omega)}\frac{1}{A_i(s)}ds\bigg|, \quad (10)
$$

and

$$\begin{aligned}&V_2(x(t),t)\\&\triangleq\sum_{i=1}^{n}\Bigg[\sum_{j=1}^{n}k_i\bar{\mathcal{C}}_{ij}\int_{t-\tau_{ij}}^{t}(\frac{\psi(s+\tau_{ij})}{\psi(t_0)})^{\beta}|g_j(x_j(s+\omega))\\&-g_j(x_j(s))|ds+\sum_{j=1}^{n}k_i\bar{\mathcal{D}}_{ij}\int_{0}^{\infty}l_j(s)\int_{t-s}^{t}(\frac{\psi(u+s)}{\psi(t_0)})^{\beta}\\&\times|h_j(x_j(u+\omega))-h_j(x_j(u))|duds\Bigg].\end{aligned}\tag{11}$$

Furthermore, for convenience, we define Δ_{i+} as following:

$$\begin{aligned}\Delta_{i+}=&-(\mathcal{A}_{i+}(t+\omega)-\mathcal{A}_{i+}(t))x_i(t+\omega)-\mathcal{A}_{i+}(t)\\&\times(x_i(t+\omega)-x_i(t))-\mathcal{A}_{i-}(t+\omega)|x_i(t+\omega)|\\&+\mathcal{A}_{i-}(t)|x_i(t)|+\sum_{j=1}^{n}((b_{ij}(t+\omega)-b_{ij}(t))\\&\times f_j(x_j(t+\omega))+b_{ij}(t)(f_j(x_j(t+\omega))-f_j(x_j(t))))\\&+\sum_{j=1}^{n}((c_{ij}(t+\omega)-c_{ij}(t))g_j(x_j(t+\omega-\tau_{ij}))\\&+c_{ij}(t)(g_j(x_j(t+\omega-\tau_{ij}))-g_j(x_j(t-\tau_{ij}))))\\&+\sum_{j=1}^{n}((d_{ij}(t+\omega)-d_{ij}(t))\int_0^{\infty}h_j(x_j(t+\omega-s))\\&\times l_j(s)ds+d_{ij}(t)\int_0^{\infty}l_j(s)(h_j(x_j(t+\omega-s))\\&-h_j(x_j(t-s)))ds)+I_i(t+\omega)-I_i(t).\end{aligned}\tag{12}$$

Then, we have the Dini-Derivative of $V_1(x(t),t)$

$$\begin{aligned}&D^{+}V_1(x(t),t)\\&=\sum_{i=1}^{n}k_i\Bigg\{\beta\bigg(\frac{\psi(t)}{\psi(t_0)}\bigg)^{\beta-1}\dot{\psi}(t)\bigg|\int_{x_i(t)}^{x_i(t+\omega)}\frac{1}{A_i(s)}ds\bigg|\\&+\bigg(\frac{\psi(t)}{\psi(t_0)}\bigg)^{\beta}\operatorname{sign}(x_i(t+\omega)-x_i(t))\\&\times\bigg(\frac{\dot{x}_i(t+\omega)}{A_i(x_i(t+\omega))}-\frac{\dot{x}_i(t)}{A_i(x_i(t))}\bigg)\Bigg\}\\&=\sum_{i=1}^{n}k_i\Bigg\{\beta\bigg(\frac{\psi(t)}{\psi(t_0)}\bigg)^{\beta-1}\dot{\psi}(t)\bigg|\int_{x_i(t)}^{x_i(t+\omega)}\frac{1}{A_i(s)}ds\bigg|\\&+\bigg(\frac{\psi(t)}{\psi(t_0)}\bigg)^{\beta}\operatorname{sign}(x_i(t+\omega)-x_i(t))\Delta_{i+}\Bigg\}.\end{aligned}\tag{13}$$

In $D^{+}V_1(x(t),t)$, according to Assumption 1, we find that

$$\bigg|\int_{x_i(t)}^{x_i(t+\omega)}\frac{1}{A_i(s)}ds\bigg|\le\frac{1}{\underline{A_i}}\bigg|x_i(t+\omega)-x_i(t)\bigg|.\tag{14}$$

For Δ_{i+}, combine with Assumption 3, the following inequations are established

$$\begin{aligned}\Delta_{i+}\le&-(\mathcal{A}_{i+}(t+\omega)-\mathcal{A}_{i+}(t))x_i(t+\omega)\\&-\mathcal{A}_{i+}(t)(x_i(t+\omega)-x_i(t))\\&-\mathcal{A}_{i-}(t+\omega)|x_i(t+\omega)|+\mathcal{A}_{i-}(t)|x_i(t)|\\&+\sum_{j=1}^{n}(\epsilon\max\{|m_{fj}|,|M_{fj}|\}\\&+\mathcal{B}_{ij}(t)|f_j(x_j(t+\omega))-f_j(x_j(t))|\\&+\sum_{j=1}^{n}(\epsilon\max\{|m_{gj}|,|M_{gj}|\}\\&+\mathcal{C}_{ij}(t)|g_j(x_j(t+\omega-\tau_{ij}))-g_j(x_j(t-\tau_{ij}))|\\&+\sum_{j=1}^{n}(\epsilon\max\{|m_{hj}|,|M_{hj}|\}L_j\\&+\mathcal{D}_{ij}(t)\int_0^{\infty}|l_j(s)||(h_j(x_j(t+\omega-s))\\&-h_j(x_j(t-s)))|ds)+\epsilon\\=&\epsilon\sum_{j=1}^{n}(\max\{|m_{fj}|,|M_{fj}|\}+\max\{|m_{gj}|,|M_{gj}|\}\\&+\max\{|m_{hj}|,|M_{hj}|\}L_j+1)\\&-(\mathcal{A}_{i+}(t+\omega)-\mathcal{A}_{i+}(t))x_i(t+\omega)\\&-\mathcal{A}_{i+}(t)(x_i(t+\omega)-x_i(t))\\&-\mathcal{A}_{i-}(t+\omega)|x_i(t+\omega)|+\mathcal{A}_{i-}(t)|x_i(t)|\\&+\sum_{j=1}^{n}[\mathcal{B}_{ij}(t)|f_j(x_j(t+\omega))-f_j(x_j(t))|\\&+\mathcal{C}_{ij}(t)|g_j(x_j(t+\omega-\tau_{ij}))-g_j(x_j(t-\tau_{ij}))|\\&+\mathcal{D}_{ij}(t)\int_0^{\infty}|l_j(s)||h_j(x_j(t+\omega-s))\\&-h_j(x_j(t-s))|ds].\end{aligned}\tag{15}$$

At the same time,

$$\begin{aligned}&D^{+}V_2(x(t),t)\\&=\sum_{i=1}^{n}\Bigg\{\sum_{j=1}^{n}k_i\bar{\mathcal{C}}_{ij}[(\frac{\psi(t+\tau_{ij})}{\psi(t_0)})^{\beta}|g_j(x_j(t+\omega))-g_j(x_j(t))|\\&-(\frac{\psi(t)}{\psi(t_0)})^{\beta}|g_j(x_j(t+\omega-\tau_{ij}))-g_j(x_j(t-\tau_{ij}))|]\\&+\sum_{i=1}^{n}k_i\bar{\mathcal{D}}_{ij}\int_0^{\infty}l_j(s)[(\frac{\psi(t+s)}{\psi(t_0)})^{\beta}|h_j(x_j(t+\omega))\\&-h_j(x_j(t))|-(\frac{\psi(t)}{\psi(t_0)})^{\beta}|h_j(x_j(t+\omega-s))\\&-h_j(x_j(t-s))|]ds\Bigg\}.\end{aligned}\tag{16}$$

Combine $(10)-(16)$, denote $e_{\omega i}\triangleq|x_i(t+\omega)-x_i(t)|$,

$$\begin{aligned}&D^{+}V(x(t),t)\\&\le\sum_{i=1}^{n}k_i\Bigg\{\beta(\frac{\psi(t)}{\psi(t_0)})^{\beta}\frac{\dot{\psi}(t)}{\psi(t)}\psi(t_0)\frac{1}{\underline{A}_i}e_{\omega i}\\&+(\frac{\psi(t)}{\psi(t_0)})^{\beta}\Bigg[\epsilon\sum_{j=1}^{n}(\max\{|m_{fj}|,|M_{fj}|\}\\&+\max\{|m_{gj}|,|M_{gj}|\}+\max\{|m_{hj}|,|M_{hj}|\}L_j\\&+1)-(\mathcal{A}_{i+}(t+\omega)-\mathcal{A}_{i+}(t))x_i(t+\omega)\\&-\mathcal{A}_{i+}(t)e_{\omega i}-\mathcal{A}_{i-}(t+\omega)|x_i(t+\omega)|\\&+\mathcal{A}_{i-}(t)|x_i(t)|+\sum_{j=1}^{n}\mathcal{B}_{ij}(t)|f_j(x_j(t+\omega))\end{aligned}$$

$$- f_j(x_j(t))|\Big] + (\frac{\psi(t+\tau_{ij})}{\psi(t_0)})^\beta \sum_{j=1}^{n} \Big[\bar{\mathcal{C}}_{ij} \times |g_j(x_j(t+\omega)) - g_j(x_j(t))| + \bar{\mathcal{D}}_{ij} \times \int_0^\infty l_j(s)|h_j(x_j(t+\omega)) - h_j(x_j(t))|ds\Big]\Big\}. \quad (17)$$

Since $x_i(t) \in \Omega_1$, so there is $x_i(t) \le M_x$ holds, where $M_x > 0$ is a constant. From Assumption 4, we have

$$\begin{aligned} D^+V(x(t),t) &\le \sum_{i=1}^{n} k_i \Big\{ (\frac{\psi(t)}{\psi(t_0)})^\beta \Big[\beta \frac{\dot{\psi}_i(t)}{\psi(t)} \psi(t_0) \frac{1}{\underline{A}_i} e_{\omega i} - \mathcal{A}_{i+}(t) e_{\omega i} \\ &+ [\epsilon \sum_{j=1}^{n} (\max\{|m_{fj}|, |M_{fj}|\} + \max\{|m_{gj}|, |M_{gj}|\} \\ &+ \max\{|m_{hj}|, |M_{hj}|\} L_j + 1)] + 2\epsilon M_x \\ &+ \sum_{j=1}^{n} \bar{\mathcal{B}}_{ij} \max\{|\lambda_{fj}^{(k)}|, |\mu_{fj}^{(k)}|\} e_{\omega i}\Big] + \\ & e_{\omega i} (\frac{\psi(t+\tau_{ij})}{\psi(t_0)})^\beta \sum_{j=1}^{n} \Big[\bar{\mathcal{C}}_{ij} \max\{|\lambda_{gj}^{(k)}|, |\mu_{gj}^{(k)}|\} \\ &+ L_j \bar{\mathcal{D}}_{ij} \max\{|\lambda_{hj}^{(k)}|, |\mu_{hj}^{(k)}|\}\Big]\Big\}. \end{aligned}$$

Moreover, from the property of Ψ-type function, there are $(\frac{\psi(t+\tau_{ij})}{\psi(t_0)})^\beta \le (\frac{\psi(t)\psi(\tau_{ij})}{\psi(t_0)})^\beta = (\frac{\psi(t)}{\psi(t_0)})^\beta \psi(\tau_{ij})^\beta$ and $\frac{\dot{\psi}_i(t)}{\psi(t)} \le \dot{\psi}_i(0)$, so

$$\begin{aligned} D^+V(x(t),t) &\le \sum_{i=1}^{n} k_i \Big\{ (\frac{\psi(t)}{\psi(t_0)})^\beta \Big[e_{\omega i} \{\beta \psi(0) \psi(t_0) \frac{1}{\underline{A}_i} - \underline{\mathcal{A}}_{i+} \\ &+ \sum_{j=1}^{n} \bar{\mathcal{B}}_{ij} \max\{|\lambda_{fj}^{(k)}|, |\mu_{fj}^{(k)}|\} \\ &+ \psi(\tau_{ij})^\beta \sum_{j=1}^{n} [\bar{\mathcal{C}}_{ij} \max\{|\lambda_{gj}^{(k)}|, |\mu_{gj}^{(k)}|\} \\ &+ L_j \bar{\mathcal{D}}_{ij} \max\{|\lambda_{hj}^{(k)}|, |\mu_{hj}^{(k)}|\}]\} \\ &+ \epsilon (\sum_{j=1}^{n} (\max\{|m_{fj}|, |M_{fj}|\} + \max\{|m_{gj}|, |M_{gj}|\} \\ &+ \max\{|m_{hj}|, |M_{hj}|\} L_j) + 1 + 2M_x)\Big]\Big\}. \end{aligned} \quad (18)$$

Clearly,

$$\begin{aligned} \Xi_i =& \beta \psi(0) \psi(t_0) \frac{1}{\underline{A}_i} - \underline{\mathcal{A}}_{i+} + \sum_{j=1}^{n} \bar{\mathcal{B}}_{ij} \max\{|\lambda_{fj}^{(k)}|, |\mu_{fj}^{(k)}|\} \\ &+ \psi(\tau_{ij})^\beta \sum_{j=1}^{n} [\bar{\mathcal{C}}_{ij} \max\{|\lambda_{gj}^{(k)}|, |\mu_{gj}^{(k)}|\} \\ &+ L_j \bar{\mathcal{D}}_{ij} \max\{|\lambda_{hj}^{(k)}|, |\mu_{hj}^{(k)}|\}] < 0, \\ \Theta_i =& \sum_{j=1}^{n} (\max\{|m_{fj}|, |M_{fj}|\} + \max\{|m_{gj}|, |M_{gj}|\} \\ &+ \max\{|m_{hj}|, |M_{hj}|\} L_j) + 1 + 2M_x, \end{aligned}$$

are constants. Rewrite (18) as following

$$\begin{aligned} D^+V(x(t),t) &\le \sum_{i=1}^{n} k_i (\frac{\psi(t)}{\psi(t_0)})^\beta [e_{\omega i} \Xi_i + \epsilon \Theta_i] \\ &\le \sum_{i=1}^{n} k_i (\frac{\psi(t)}{\psi(t_0)})^\beta \epsilon \Theta_i. \end{aligned} \quad (19)$$

So there are

$$V(x(t),t) \le V(x_i(0),0) + \int_0^t \sum_{i=1}^{n} k_i (\frac{\psi(t)}{\psi(t_0)})^\beta \epsilon \Theta_i ds.$$

Meanwhile, for $\forall i \in \{1, 2, \cdots, n\}$

$$\begin{aligned} k_i (\frac{\psi(t)}{\psi(t_0)})^\beta \frac{e_{\omega i}}{\bar{A}_i} &\le \sum_{i=1}^{n} k_i (\frac{\psi(t)}{\psi(t_0)})^\beta \frac{e_{\omega i}}{\bar{A}_i} \\ &\le V_1(x(t),t) \le V(x(t),t). \end{aligned}$$

Thus there is

$$\begin{aligned} e_{\omega i} =& |x_i(t+\omega) - x_i(t)| \le \frac{\bar{A}_i}{k_i} (\frac{\psi(t_0)}{\psi(t)})^\beta V(x_i(0),0) \\ &+ \epsilon \frac{\bar{A}_i}{k_i} (\frac{\psi(t_0)}{\psi(t)})^\beta \int_0^t \sum_{j=1}^{n} k_j (\frac{\psi_j(s)}{\psi_j(t_0)})^\beta \Theta_j ds. \end{aligned} \quad (20)$$

According to (8), it should be pointed that there exists some moment T satisfying

$$\begin{aligned} &\frac{\bar{A}_i}{k_i} (\frac{\psi(t_0)}{\psi(T)})^\beta V(x_i(0),0) \\ &\le \epsilon \Big(1 - \frac{\bar{A}_i}{k_i} (\frac{\psi(t_0)}{\psi(T)})^\beta \int_0^T \sum_{j=1}^{n} k_j (\frac{\psi(s)}{\psi(t_0)})^\beta \Theta_j ds\Big). \end{aligned}$$

Thus we have $|x_i(t+\omega) - x_i(t)| \le \epsilon$ holds after time T, and the neuron states in Ω_1 are almost periodic solutions for MCGNNs (1). Here we finish our proof.

Theorem 1 shows that if there exist solutions in Ω_1, they must be almost periodic solutions. Then in Theorem 2, we state that there exists at least one solution in each element of Ω_1 for MCGNNs (1).

Theorem 2: If the conditions in Theorem 1 hold, there exists at least one almost periodic solution in each element of Ω_1.

Proof: Denote

$$\begin{aligned} \mathcal{S}_{i,k}(t) =& -(\mathcal{A}_{i+}(t+\mathcal{T}_k) - \mathcal{A}_{i+}(t)) x_i(t+\mathcal{T}_k) \\ &- (\mathcal{A}_{i-}(t+\mathcal{T}_k) - \mathcal{A}_{i-}(t)) |x_i(t+\mathcal{T}_k)| \\ &+ \sum_{j=1}^{n} (b_{ij}(t+\mathcal{T}_k) - b_{ij}(t)) f_j(x_j(t+\mathcal{T}_k)) \\ &+ \sum_{j=1}^{n} (c_{ij}(t+\mathcal{T}_k) - c_{ij}(t)) g_j(x_j(t+\mathcal{T}_k - \tau_{ij})) \\ &+ \sum_{j=1}^{n} (d_{ij}(t+\mathcal{T}_k) - d_{ij}(t)) \int_0^\infty l_j(s) \\ &\times h_j(x_j(t+\mathcal{T}_k - s)) ds + I_i(t+\mathcal{T}_k) - I_i(t). \end{aligned}$$

For $x(t) \in \Omega_1$ and the connection weights are almost periodic, a sequence $\{\mathcal{T}_k\}$ can be selected so that

$$\begin{cases} \lim_{k \to +\infty} \mathcal{T}_k = +\infty, \\ |\mathcal{S}_{i,k}(t)| \le \frac{1}{k}, \quad \forall t > 0. \end{cases}$$

Furthermore, by diagonal selection principle and Arzela-Ascoli theorem, there is a subsequence $\mathcal{T}_{k_j}$ such that $x(t+\mathcal{T}_{k_j})$ is uniformly convergent to a continuous function $x^*(t)$ on any compact set of R^n. Then, according to Lebesgue's dominated convergence theorem, for $\forall \zeta \in R$ and $\forall t > 0$, there is

$$\begin{aligned}
&x_i^*(t+\zeta) - x_i^*(t)\\
&\quad = \lim_{j\to+\infty}\int_t^{t+\zeta} A_i(x_i(u+\mathcal{T}_{k_j}))\\
&\qquad \times \{-\mathcal{A}_{i+}(u)x_i(u+\mathcal{T}_{k_j}) - \mathcal{A}_{i-}(u)|x_i(u+\mathcal{T}_{k_j})|\\
&\qquad + \sum_{j=1}^n b_{ij}(u) f_j(x_j(u+\mathcal{T}_{k_j}))\\
&\qquad + \sum_{j=1}^n c_{ij}(u) g_j(x_j(u+\mathcal{T}_{k_j}-\tau_{ij})) + I_i(t)\\
&\qquad + \sum_{j=1}^n d_{ij}(u)\int_0^\infty h_j(x_j(u+\mathcal{T}_{k_j}-s))l_j(s)ds\}du\\
&\quad = \int_t^{t+\zeta} A_i(x_i^*(t))\{-\mathcal{A}_{i+}(u)x_i^*(t) - \mathcal{A}_{i-}(u)|x_i^*(t)|\\
&\qquad + \sum_{j=1}^n b_{ij}(u) f_j(x_j^*(t)) + \sum_{j=1}^n c_{ij}(u) g_j(x_j^*(t)) + I_i(t)\\
&\qquad + \sum_{j=1}^n d_{ij}(u)\int_0^\infty h_j(x_j^*(t))l_j(s)ds\}du.
\end{aligned}$$

For the arbitrary of ζ, $x^*(t)$ is a solution of MCGNNs (1). Based on Theorem 1, for $\forall \epsilon > 0$, when j is sufficient large, we have $|x_i(t+\mathcal{T}_{k_j}+\omega) - x_i(t+\mathcal{T}_{k_j})| \le \epsilon$. Suppose $j \to +\infty$, then $|x_i^*(t+\omega) - x_i^*(t)| \le \epsilon$ for $t \ge 0$. So far, we have proved that there is at least one almost periodic solution in each elements of Ω_1.

Finally, we will show that the almost periodic solutions in Ω_1 are Ψ-type stable.

Theorem 3: If the conditions in Theorem 1 hold, then the solutions of MCGNNs (1) are Ψ-type stable.

Proof: Denote the following Lyapunov function

$$\begin{aligned}
&\mathcal{V}(x(t),t)\\
&\quad = \sum_{i=1}^n \Bigg[k_i(\frac{\psi(t)}{\psi(t_0)})^\beta \Bigg|\int_{x_i^*(t)}^{x_i(t)} \frac{1}{A_i(s)}ds\Bigg|\\
&\qquad + \sum_{j=1}^n k_i\bar{\mathcal{C}}_{ij}\int_{t-\tau_{ij}}^t (\frac{\psi(s+\tau_{ij})}{\psi(t_0)})^\beta |g_j(x_j(s))\\
&\qquad - g_j(x_j^*(s))|ds + \sum_{j=1}^n k_i\bar{\mathcal{D}}_{ij}\int_0^\infty l_j(s)\int_{t-s}^t(\frac{\psi(u+s)}{\psi(t_0)})^\beta\\
&\qquad \times |h_j(x_j(t)) - h_j(x_j^*(u))|duds\Bigg].
\end{aligned} \tag{21}$$

For $e_i(t) = x_i(t) - x_i^*(t)$, as the corresponding proof in Theorem 1, we have

$$\mathcal{V}(x(t),t) \le (\frac{\psi(t)}{\psi(t_0)})^\beta \sum_{i=1}^n k_i \Xi_i |e_i(t)| < 0.$$

So

$$\frac{k_i}{\underline{A}_i}(\frac{\psi(t)}{\psi(t_0)})^\beta |e_i(t)| \le \mathcal{V}(x(t),t) \le \mathcal{V}(x(0),0),$$

which means that

$$|e_i(t)| \le \frac{\underline{A}_i}{k_i}(\frac{\psi(t_0)}{\psi(t)})^\beta \mathcal{V}(x(0),0).$$

Thus, we have

$$\frac{\ln|e_i(t)|}{\ln\psi(t)} \le -\beta\ln(\frac{\underline{A}_i}{k_i}\mathcal{V}(x(0),0)(\psi(t_0))^\beta).$$

From the Definition 3, the solutions are Ψ-type stable.

Remark 4: Different from EPs, the multistability conclusion of almost periodic solutions can not be obtained directly. So we need Theorem 1-3 to analyze the multiple Ψ-type stability in system (1).

Remark 5: If we choose $f_j(\cdot) = g_j(\cdot) = h_j(\cdot)$, $\psi(t) = e^t$, then we can get the relevant multistability conclusions in [12].

C. Estimation of the attraction basins

Next, we will estimate the attraction basin. From the definition of $\mathcal{F}_{i+}(t,\alpha)$ and $\mathcal{F}_{i-}(t,\alpha)$, it is obvious to get the conclusion that $\mathcal{F}_{i+}(t,+\infty) = -\infty$ and $\mathcal{F}_{i-}(t,-\infty) = +\infty$. So for the time being, the following formula can be established

$$\mathcal{F}_{i+}(t,u_1) = -\iota_i < 0, \qquad \mathcal{F}_{i-}(t,u_2) = \iota_i > 0,$$

where $\iota_i > 0$ is a positive constant, $u_1 \ge p_i^{(K)}$, $u_2 \le q_i^{(0)}$ and $t > 0$. According to Lemma 1, there is $\mathcal{F}_{i-}(t,q_i^{(k-1)}) < \mathcal{F}_{i+}(t,q_i^{(k-1)}) < 0$. Combine with $\mathcal{F}_{i-}(t,p_i^{(k)}) > 0$ and $\mathcal{F}_{i-}(t,\cdot)$ is continuous, so there is at least one $u_2^* \in (q_i^{(k-1)}, p_i^{(k)})$ satisfying $\mathcal{F}_{i-}(t,u_2^*) = 0$. Suppose there are multiple u_2^* and $U_{i,k}^*$ is the max one of them, which means

$$U_{i,k}^* = \max\{u_2^*|\mathcal{F}_{i-}(t,u_2^*) = 0, u_2^* \in (q_i^{(k-1)}, p_i^{(k)}), t > 0\}.$$

So $\mathcal{F}_{i-}(t,U_{i,k}^*) = 0$ and $\mathcal{F}_{i-}(t,u_2) > 0$ where $u_2 \in (U_{i,k}^*, p_i^{(k)}]$. Also, define

$$H_{i,k}^* = \min\{h_1^*|\mathcal{F}_{i+}(t,h_1^*) = 0, h_1^* \in (q_i^{(k-1)}, p_i^{(k)}), t > 0\}.$$

Similarly, $\mathcal{F}_{i-}(t,H_{i,k}^*) = 0$ and $\mathcal{F}_{i-}(t,h_1) < 0$ where $h_1 \in [q_i^{(k-1)}, H_{i,k}^*)$, $k = 1,2,\cdots,K$. So the lower bound of $\mathcal{F}_{i-}(t,\cdot)$ and the upper bound of $\mathcal{F}_{i+}(t,\cdot)$ are shown as following respectively

$$\begin{aligned}
L_i &= \inf\{\mathcal{F}_{i-}(t,x)|x \in (-\infty, p_i^{(0)}]\},\\
U_i &= \sup\{\mathcal{F}_{i+}(t,x)|x \in [q_i^{(K)}, +\infty)\}.
\end{aligned}$$

For convenience, the following notations are introduced

$$\begin{aligned}
&\bar{U}_{i0}^* \triangleq -\infty, \quad \bar{H}_{i,k}^* \triangleq +\infty,\\
&\bar{U}_{i,k}^* \triangleq \sup_{t\ge0} U_{i,k}^*, \quad \bar{H}_{i,k}^* \triangleq \inf_{t\ge0} H_{i,k}^*,
\end{aligned}$$

where $k = 1,2,\cdots,K$. It is obvious that $\bar{U}_{i,k}^* < p_i^{(k)}$ and $\bar{H}_{i,k}^* > q_i^{(k)}$ for all k. Suppose

$$\begin{aligned}
\bar{\Omega}_1 = \Bigg\{&\bar{\Gamma}'_\gamma = \prod_{i=1}^n\Bigg(\bigcup_{k=0}^K(\bar{U}_{i,k}^*, \bar{H}_{i,k}^*)^{\gamma_{k+1}^{(i)}}\Bigg)\Bigg|\\
&\gamma = (\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(n)}),\\
&\gamma^{(i)} = (\gamma_1^{(i)}, \cdots, \gamma_{K+1}^{(i)}) \in \Delta(K+1)\Bigg\}.
\end{aligned}$$

Similar with Ω_1, $\bar{\Omega}_1$ has $(K+1)^n$ elements, and there are $\Gamma'_\gamma \subseteq \bar{\Gamma}'_\gamma$ established. Then we will show that the elements in $\bar{\Omega}_1$ are also the attraction basins.

Theorem 4: The elements in $\bar{\Omega}_1$ are the enlarged attraction basins compared with Ω_1 of MCGNNs (1).

Proof: Similar to the proof of the positive invariance, we firstly show that $\bar{H}^*_{i-1,k} \leq \bar{U}^*_{i,k}$ holds for $k = 1, 2, \cdots, K$ and $i = 1, 2, \cdots, n$. From the definition of $H^*_{i-1,k}$, it is obvious that $\mathcal{F}_{i+}(t, H^*_{i-1,k}) = 0$ and $\mathcal{F}_{i-}(t, H^*_{i-1,k}) < \mathcal{F}_{i+}(t, H^*_{i-1,k}) = 0$. So $H^*_{i-1,k} \leq U^*_{i,k}$. Thus we have $\bar{H}^*_{i-1,k} \leq \bar{U}^*_{i,k}$. For arbitrary initial condition $\phi(t)$, if $\phi_i(t_0) \in (\bar{U}^*_{i,0}, p_i^{(0)}]$, then there is $\dot{x}_i(t) \geq \mathcal{F}_{i-}(t, x_i(t)) \geq L_i \geq \iota_i > 0$ for $t > 0$, thus we have $x_i(t) \geq L_i t + \phi_i(t_0)$. So there exist $T \geq t_0$ such that $x_i(T) \in (p_i^{(0)}, q_i^{(0)})$.

Similarly, it is easy to prove that there is some time T satisfying $x_i(t)$ will never escape from $(p_i^{(k)}, q_i^{(k)})$ after time T for $k = 1, 2, \cdots, K$.

Hence $\bar{\Omega}_1$ is the enlarged attraction basin of MCGNNs (1). Here we finish our proof. At this point, we have completed the discussion on Ψ-type multistability of almost periodic solutions for MCGNNs.

IV. Illustrative Examples

Two simulation examples are shown to verify the obtained results in this section.

Example 1: Consider a 2-neurons MCGNNs (1) with the following coefficients

$$A_1(x_1(t)) = 1 + \frac{2}{\sin(x_1(t)) + 2},$$
$$A_2(x_2(t)) = 2 - \frac{1}{\cos(x_2(t)) + 2},$$
$$\mathcal{A}_{1+}(t) = 10 - 0.5|\cos(t)|, \quad \mathcal{A}_{2+}(t) = 8 + 0.4|\sin(t)|,$$
$$\mathcal{A}_{1-}(t) = 0.5\sin(t), \quad \mathcal{A}_{2-}(t) = 0.2\cos(t).$$

The connection weight matrices are

$$B(t) = \begin{pmatrix} 15 + 0.5\cos(t) & 0.2\sin(t) \\ 0.2\sin(t) & 15 + \sin(t) \end{pmatrix},$$
$$C(t) = \begin{pmatrix} 0.5\sin(t) & 0.1\cos(t) \\ 0.1\cos(t) & 0.5\sin(t) \end{pmatrix},$$
$$D(t) = \begin{pmatrix} 0.2\cos(t) & 0.1\sin(t) \\ 0.1\cos(t) & 0.2\sin(t) \end{pmatrix}.$$

The external inputs are

$$I_1(t) = 6(\sin(\pi t) + \sin(t)), \quad I_2(t) = 4(\cos(\pi t) + \cos(t)).$$

Moreover, the activation functions are as following

$$f(u) = ue^{-\frac{3(u^2-1)}{2}},$$
$$g(u) = \frac{1 - e^{-10u}}{1 + e^{-10u}} \frac{1 - 0.5e^{15(|u|-10)}}{1 + e^{15(|u|-10)}},$$
$$h(u) = \begin{cases} -1.1207(u+2), u \in [-2, -0.6), \\ 2.615u, \quad u \in [-0.6, 0.6], \\ -1.1207(u-2), u \in (0.6, 2], \\ 0, \quad u \in (-\infty, -2) \cup (2, +\infty). \end{cases}$$

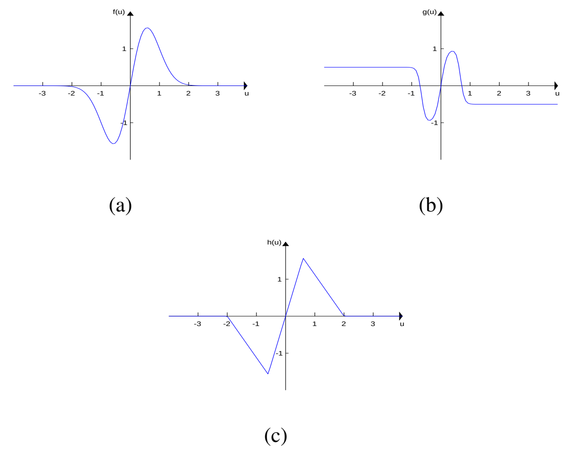

Fig. 1. Three different kinds of activation functions in Example 1. (a). The Crespi activation function; (b). The Morita activation function; (c). The piecewise linear activation function.

In order to facilitate understanding, we show the function images of the three activation functions here.

Here we choose $K = 1$. According to Assumption 4, the whole state space is divided into $(2K+1)^2 = 9$ parts. The specific state space partition is shown in Table 1. From (4), (5), we have

$$\mathcal{F}_{i+}(t, -3) < 0, \quad \mathcal{F}_{i-}(t, -0.58) > 0,$$
$$\mathcal{F}_{i+}(t, 0.58) < 0, \quad \mathcal{F}_{i-}(t, 3) > 0.$$

Table 1. State space partition.

p_1^0	q_1^0	p_1^1	q_1^1
-3	-0.58	0.58	3
p_2^0	q_2^0	p_2^1	q_2^1
-3	-0.58	0.58	3

Then we choose $\psi(t) = e^t$, from Theorem 1-3, it has $(K+1)^n = 4$ Ψ-type stable almost periodic solutions. 100 random initial conditions are selected for computer simulation. The trajectories are drawn in Fig. 2-4 respectively. In Fig. 2-3, it can be clearly observed that the state of neuron $x_1(t)$ gradually converges to the vicinity of points 1.138 and -1.136. Moreover, from partial enlarged figures (a), (b) within [0.18, 0.184] in Fig. 2, the solutions of $x_1(t)$ are almost periodic. In Fig. 3, the state of neuron $x_2(t)$ gradually converges to the vicinity of points 1.185 and -1.183. Also, from partial enlarged figures (a), (b) within [0.212, 0.218] in Fig. 3, the solutions of $x_2(t)$ are almost periodic. In Fig. 4, we can observe that the 100 initial values converge to 4 almost periodic solutions in the entire state space, which is in line with our theoretical results.

Example 2: In order to show that our conclusions are still correct for the degraded MCGNNs without almost periodic solutions, we will show the degraded MCGNNs model with multiple stable EPs. The specific coefficients are shown as

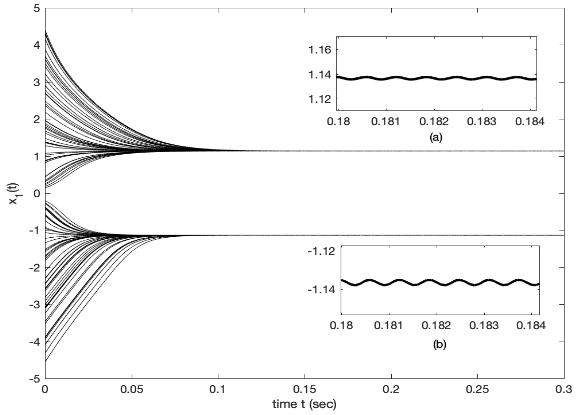

Fig. 2. Two almost periodic solution trajectories of neuron state $x_1(t)$.

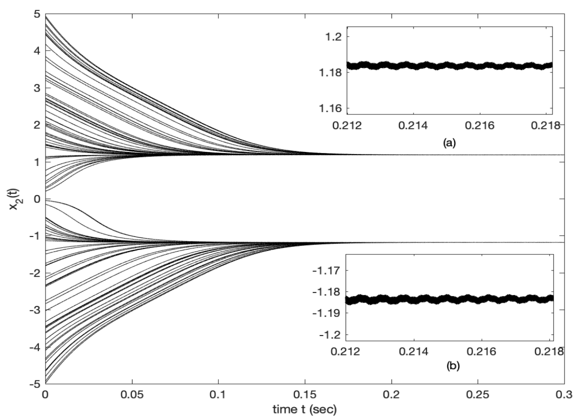

Fig. 3. Two almost periodic solution trajectories of neuron state $x_2(t)$.

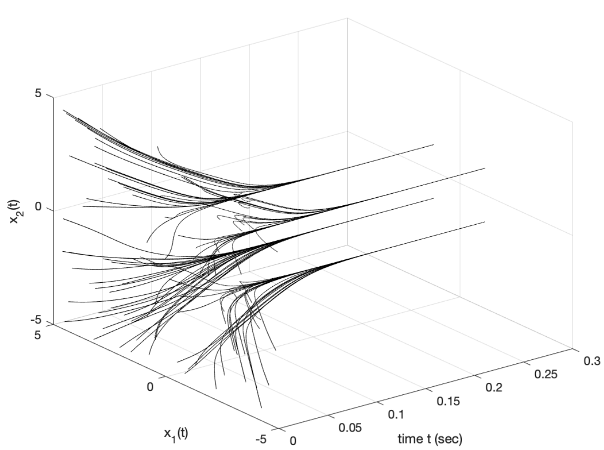

Fig. 4. Four stable almost periodic solution trajectories of neuron states $(x_1(t), x_2(t))$ with time t.

following

$$A_1(x_1(t)) = 1, \quad A_2(x_2(t)) = 1,$$
$$\mathcal{A}_{1+}(t) = 1.5, \quad \mathcal{A}_{2+}(t) = 1.5,$$
$$\mathcal{A}_{1-}(t) = 0.1, \quad \mathcal{A}_{2-}(t) = 0.1.$$

Also, the connection weight matrices are $B(t) = \begin{bmatrix} 15 & 0.2 \\ 0.2 & 15 \end{bmatrix}$, $C(t) = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}$, $D(t) = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}$, and the external inputs are $I_1(t) = 6, I_2(t) = 4$. The activation functions are as follows

$$f(x) = g(x) = h(x) = xe^{-\frac{3(x^2-1)}{2}}.$$

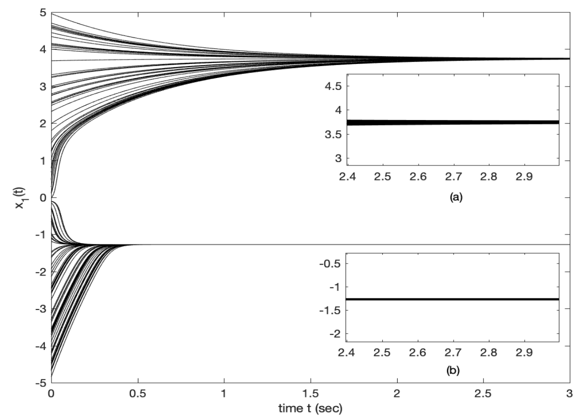

Fig. 5. Two stable EPs trajectories of neuron state $x_1(t)$.

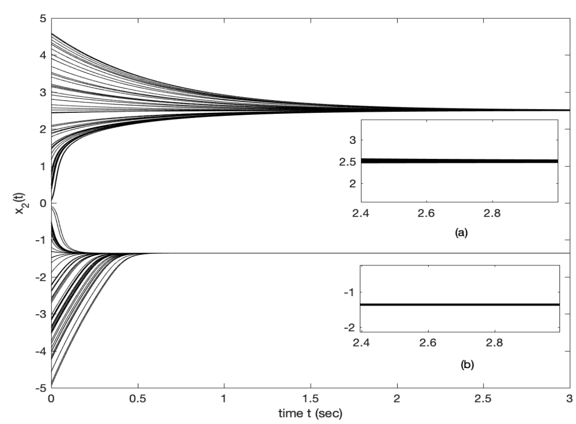

Fig. 6. Two stable EPs trajectories of neuron state $x_2(t)$.

Similarly, we choose $K = 1$. Example 2 also satisfies the theorem conditions, so there should be $(K+1)^n = 2^2 = 4$ EPs. Through computer simulation, we also verified the theoretical results. The state trajectories are shown in Fig. 5-7. It is easy to observe that there are 2 stable EPs in Fig. 5,6 respectively. To be more precise, in Fig. 5, the states of neuron $x_1(t)$ converges to the points 3.738 and -1.273, in Fig. 6, the states of neuron $x_2(t)$ converges to the points 2.533 and -1.36. Also, it is clear that there are 4 stable EPs in Fig.7, and this verifies that our conclusions are still correct when considering the degraded systems.

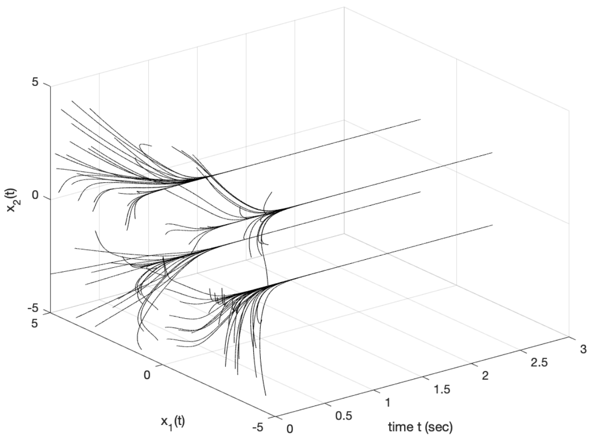

Fig. 7. Four stable EPs trajectories of neuron states $(x_1(t), x_2(t))$.

V. Conclusion

In this paper, some novel criteria about Ψ-type multistability of MCGNNs have been proved, which improves the related existing results. The multistability conclusions are interest issues in view of associative memory and image processing. There are still some questions needed to be study. One of the important practical application issues is whether the Ψ-type multistability results about almost periodic solutions obtained in this paper could be extended to the stochastic MCGNNs. And how robust are the neural networks under the condition of Ψ-type stable? We will investigate these topics in future.

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