Quantifying resource in catalytic resource theory

Anurag Anshu[*]   Min-Hsiu Hsieh[†]   Rahul Jain[‡]

August 8, 2017

Abstract

We consider a general resource theory that allows the use of free resource as a catalyst. We show that the amount of 'resource' contained in a given state, in the asymptotic scenario, is equal to the regularized *relative entropy of resource* of that state, which then yields a straightforward operational meaning to this quantity. Such an answer has been long sought for in any resource theory since the usefulness of a state in information-processing tasks is directly related to the amount of resource the state possesses in the beginning. While we need to place a few assumptions in our resource theoretical framework, it is still general enough and includes quantum resource theory of entanglement, coherence, asymmetry, non-uniformity, purity, contextuality, stabilizer computation and the classical resource theory of randomness extraction as special cases. Since our resource theoretic framework includes entanglement theory, our result also implies that the amount of noise one has to inject locally in order to erase all entanglement contained in an entangled state is equal to the regularized relative entropy of entanglement, resolving an open question posted in [*Groisman et al., Phys. Rev. A. 72: 032317, 2005*]. On the way to prove the main result, we also quantify the amount of resource contained in a state in the one-shot setting (where one only has a single copy of the state), in terms of the smooth max-relative entropy. Our one-shot result employs a recently developed technique of convex-split lemma.

1 Introduction

A question that is commonly asked in Physics is how a certain property of a physical system can be used to achieve useful tasks, and how to quantify the amount of such a property in a meaningful way. Various areas in Physics have developed methodology to answer these questions relevant to their own areas; however, many of the approaches are difficult, sometimes even impossible, to employ outside of their respective fields. Quantum resource theory then emerges when quantum information theory is found to provide a unified platform for characterizing the resource [1, 2] because in a nutshell, they can all be viewed as interconversion of different system states with system-dependent constraints. Since then, individual resource theory has been able to characterize targeted properties in some information-processing tasks using entropic quantities defined on system states.

---

[*]Center for Quantum Technologies, National University of Singapore, Singapore. `a0109169@u.nus.edu`

[†]Centre for Quantum Software and Information, University of Technology Sydney, NSW 2007, Australia. `Min-Hsiu.Hsieh@uts.edu.au`

[‡]Center for Quantum Technologies, National University of Singapore and MajuLab, UMI 3654, Singapore. `rahul@comp.nus.edu.sg`

The core of a resource theory is built upon two main system-dependent requirements for the resources and allowed operations; namely, (i) the existence of a set of states that are free and those not in the set are expensive; and (ii) the allowed operations are those that map the set of free states to itself. Various resource theories have been developed in the past decade. The most notable example is the resource theory of entanglement [3], where the set of free states corresponds to the collection of separable states and the allowed free operations are the local quantum operations and classical communication (LOCC). These two classes of states and operations have attracted stand-alone interests besides resource theory [4, 5]. Under this resource framework, one can then ask the amount of valuable resource (cf. quantum entanglement) possessed by an entangled state relative to the set of free states (cf. separable states). This question motivates the scenario of injecting noise locally to the system in order to destroy the quantum entanglement, i.e., the *randomness cost*. A complete characterization of this question has remained open; though, gapped upper and lower bounds have been provided in the asymptotic i.i.d. setting in Ref. [6]. It is worthwhile to mention that investigation of a variant of the above setting, where the local noise is used to destroy the total correlation in an entangled state, relates the minimal randomness cost to the quantum mutual information [6]. This seminal result gives the first operational meaning to this entropic quantity and advances significantly our understanding of entanglement theory. Being able to answer the optimal randomness cost to bring entangled states to separable states thus bears equivalent significance, if not more important, since the existence of entanglement is believed to make quantum systems superior to their classical counterparts and the amount of entanglement is generally linked to its information-processing power [7]. Likewise, this crucial question of the amount of valuable resource possessed by a state relative to its free resource is then adhesive to every resource theory, be it quantum coherence, thermodynamics, etc.

One can view the above erasing framework as a restricted model of state transformation, where the final state belongs to the free resource [8, 9]. In the general model of state transformation, it has been demonstrated that not every state transformation is possible, and could become possible if a catalyst is involved [10]. Even if the transformation process is possible in the beginning, adding catalyst would likely make the process much more efficient. Therefore, individual resource theories have included catalyst in their formalism [11, 12].

We thus consider a general resource theory framework that allows free resource being used as a catalyst. We demonstrate that this framework includes previous major resource theories of (i) entanglement [3, 6, 13, 14], (ii) coherence [15, 16, 14, 17], (iii) thermodynamics [18, 19, 20, 21, 22, 23], (iv) non-uniformity [11], (v) purity [24], (vi) randomness extractors [25, 26], (vii) contextuality [27], (viii) asymmetry [28] and (ix) stabilizer computation [29] as special instances. Treating resource theory in a general framework enjoys the major advantage that the resource conversion tasks done for one particular resource can already be adapted to other resources almost trivially. Hence, it has started to attract more attention [8, 9]. We establish that the entropic quantity, the *regularized relative entropy of resource* [8], characterizes the amount of useful resource in a given state relative to its free resource in the asymptotic i.i.d. setting. In other words, if one has infinitely many copies of the same state, the randomness cost (per copy of the state) for erasing the resource contained in the given state and bringing it to the closest free state is equal to this entropic quantity. This yields a crucial operational meaning to this entropic quantity. Since the resource theory of quantum entanglement is a special case of our framework, our result directly provides the missing answer[1] to the minimal randomness cost for erasing the amount of entanglement in an entangled state to a

---

[1]We remark that an independent work for this answer can be found in Ref. [30]

separable state, a question first posted in Ref. [6].

We also obtain matching upper and lower bounds on the randomness cost if one only has a single copy of a given state, i.e., the one-shot scenario. Our one-shot bounds are given in terms of smooth max-relative entropy; hence our result also provides a new operational meaning to this quantity in the resource theoretic framework. We emphasize that being able to obtain matching one-shot bounds in such a general resource theory framework is rare [8], and is due to the technical tool: the convex split lemma [31]. This again provides another excellent example that quantum information theory helps in the understanding of quantum physics.

The paper is organized as follows. Section 2 contains assumptions required for our catalytic resource framework. We then define our information-processing task of quantifying the amount of resource and present our main theorem. We also discuss several individual resource theories as special cases of our framework. Section 3 contains definition of entropic quantities and useful lemmas required in the proofs. The proofs of our main result are given in Sections 4–6. We conclude in Section 7.

2 Catalytic Resource Framework and Main Results

The question that we will answer in this general catalytic resource theory framework is how much resource is contained in a given resource state ρ (relative to its free resource). This question is crucial, as this amount directly relates to the power the state ρ possesses in achieving information-processing tasks.

We first present assumptions that we have to make in our resource theoretic framework. It is apparent that if no limit is set on the allowed operations and free resources, then it is almost impossible to obtain useful characterization, as also noted in Ref. [8]. Thus, we need a sufficient (yet small) number of assumptions to keep the theory interesting.

Our framework for both single-partite and multi-partite settings is discussed below, with the assumptions on free resources and allowed operations. Our assumptions capture the requirement we impose on an experimenter (or experimenters in the multi-partite setting) who wants to quantify a useful property of his/her physical system. For this, we start with the natural experimental framework, where the experimenter has decided upon the global Hilbert space and its tensor decomposition into several Hilbert spaces (for example, as 'physically separated' Hilbert spaces). Given this framework, we impose the conditions that are to be satisfied by the set of free resources and the allowed unitary operations. It is important to note that the conditions are only for a given framework; we do not restrict the *choice* of the global Hilbert space and its tensor decomposition by the experimenter.

For the notations and definitions used in this section, please refer to Section 3.

2.1 Single-partite case

As discussed above, an experimenter starts with a collection of quantum registers and performs operations on them. The joint state lives on a global Hilbert space $\mathcal{H}$. This Hilbert space possesses an ordered decomposition into r systems $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \mathcal{H}_r$. Given the Hilbert space and its ordered decomposition (the order of the decomposition may matter in a given physical scenario, hence we take it into account), we define the set of free states and allowed operations. The definition is analogous to the postulates given in Ref. [8].

Definition 1 (Single party resource theory). *Fix an integer $r \geq 1$. Given $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \mathcal{H}_r$ and registers $M_1, M_2, \ldots M_r$ where M_i corresponds to $\mathcal{H}_i$, the set of free resources and free operations are as follows.*

1. *For every non-empty ordered set $\boldsymbol{L} = \{s_1, s_2, \ldots s_{|\boldsymbol{L}|}\} \subseteq \{1, 2, \ldots r\}$ (where $\{s_1, s_2, \ldots s_{|\boldsymbol{L}|}\}$ are increasing sequence of integers), let $\mathcal{H}_{\boldsymbol{L}} = \mathcal{H}_{s_1} \otimes \mathcal{H}_{s_2} \otimes \ldots \mathcal{H}_{s_{|\boldsymbol{L}|}}$ and $\mathcal{F}_{\boldsymbol{L}} \subseteq \mathcal{D}(\mathcal{H}_{\boldsymbol{L}})$ satisfy the following properties. For brevity, we set $M_{\boldsymbol{L}} := M_{s_1} M_{s_2} \ldots M_{s_{|\boldsymbol{L}|}}$.*

   (a) *$\mathcal{F}_{\boldsymbol{L}}$ is a convex and closed set.*

   (b) *For any $\boldsymbol{L}' \subseteq \boldsymbol{L}$, $\mathcal{F}_{\boldsymbol{L}'} \otimes \mathcal{F}_{\boldsymbol{L} \setminus \boldsymbol{L}'} \subseteq \mathcal{F}_{\boldsymbol{L}}$.*
      - *This assumption states that if two quantum states are free resources, then their tensor product is a free resource as well.*

   (c) *For any $\sigma_{M_{\boldsymbol{L}}} \in \mathcal{F}_{\boldsymbol{L}}$ and $\boldsymbol{L}' \subseteq \boldsymbol{L}$, $\mathrm{Tr}_{M_{\boldsymbol{L}'}}(\sigma_{M_{\boldsymbol{L}}}) \in \mathcal{F}_{\boldsymbol{L} \setminus \boldsymbol{L}'}$.*
      - *This assumption states that if a quantum state on more than one registers is a free resource, then we obtain a free resource by partial trace over a subset of these registers.*

   (d) *It holds that $\frac{\mathbb{I}_{M_r}}{|M_r|} \in \mathcal{F}_{\{r\}}$, where $|M_r|$ is the dimension of register M_r.*
      - *Under this assumption, we require the maximally mixed quantum state to be a free resource on the last register.*

2. *The complete set of free resources is $\mathcal{F} := \cup_{\boldsymbol{L}} \mathcal{F}_{\boldsymbol{L}}$.*

3. *For a fixed $\boldsymbol{L}$, the free operations $\mathcal{U}_{\boldsymbol{L}}$ are defined as the set of all unitaries $U : \mathcal{H}_{\boldsymbol{L}} \to \mathcal{H}_{\boldsymbol{L}}$ that satisfy*

   (a) *$\sigma \in \mathcal{F}_{\boldsymbol{L}} \implies U \sigma U^{\dagger} \in \mathcal{F}_{\boldsymbol{L}}$.*

   (b) *$U \in \mathcal{U}_{\boldsymbol{L}}$ if and only if $U^{\dagger} \in \mathcal{U}_{\boldsymbol{L}}$.*

4. *The set of all free operations are $\mathcal{U} := \cup_{\boldsymbol{L}} \mathcal{U}_{\boldsymbol{L}}$.*

We remark that these assumptions we required are natural and mild [8]. Above, we also observe that the set $\mathcal{U}_{\boldsymbol{L}}$ (for every $\boldsymbol{L}$) forms a group.

Now we are in a position to formally define our task, that we call an $(\varepsilon, \log |J|)$-transformation of ρ_M to $\mathcal{F}$.

Task 2. *Let $\varepsilon > 0$, $r \geq 1$ be an integer and fix a Hilbert space $\mathcal{H}$ with decomposition $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_r$, which is chosen by the experimenter. Let the register M correspond to $\mathcal{H}_M = \mathcal{H}_1$, register J correspond to $\mathcal{H}_J = \mathcal{H}_r$ and register E correspond to $\mathcal{H}_E = \mathcal{H}_2 \otimes \ldots \mathcal{H}_{r-1}$. An experimenter holds a quantum state ρ_M. Using a state $\mu_{EJ} \in \mathcal{F}$, she applies a unitary $U \in \mathcal{U}$ to obtain a joint quantum state Θ_{MEJ}:*

$$\Theta_{MEJ} = U(\rho_M \otimes \mu_{EJ})U^{\dagger}.$$

It is required that there exists a $\sigma_{ME} \in \mathcal{F}$ such that $\mathrm{Pur}(\Theta_{ME}, \sigma_{ME}) \leq \varepsilon$, where the chosen distance measure is defined in Equation (1). The number of discarded qubits is $\log |J|$.

In above task, no restriction is made on the transformation of the free resource state μ_{EJ}. But in many cases, it is desirable that this quantum state be returned close to its original form and hence act as a catalyst. For example, the framework of catalytic decoupling [32] studies such a scenario. Our achievability result shall belong to such class of transformations. Hence we have the following definition.

Definition 3. *Task 2 is said to be a $(\varepsilon, \log |J|)$-catalytic transformation of ρ_M if $\mu_{EJ} = \mu_E \otimes \mu_J$ and $\sigma_{ME} = \sigma_M \otimes \mu_E$ for some $\sigma_M \in \mathcal{F}$.*

Task 2 is motivated by the work [6], which considered the problem of transforming a bipartite quantum state $\rho_{AB}^{\otimes n}$ into a product state $\rho_{A^n} \otimes \rho_{B^n}$ with the aid of shared randomness and local unitaries. It was shown that the number of bits of randomness required (in other words, the randomness cost) is $\approx n \cdot \mathrm{I}(A : B)_\rho$. Task 2 is along the lines of the framework considered in above work, but with an additional freedom of allowing the use of additional free resources that can aid in the transformation of the desired quantum state. Moreover, when the register J is classical (in a more precise sense to be discussed below), we can interpret $\log |J|$ to be the randomness cost of the protocol. In Definition 3, we have provided further restriction that the free resource (expect for the randomness used) be returned with small error.

We provide a near optimal characterization of the randomness cost of Task 2 in Theorem 5 below. Our achievability protocol requires a further assumption related to the last register (and its connection to rest of the registers), as this register serves as the source of 'classical randomness'. This assumption is as follows.

Assumption 4. *Invoke the notation in Definition 1. Then there exists a canonical basis $\mathcal{B} = \{|1\rangle, |2\rangle, \ldots |\ell_r\rangle\}$ on $\mathcal{H}_r$ (where ℓ_r is the dimension of $\mathcal{H}_r$) such that*

$$\mathcal{F} = conv\left\{ \frac{1}{\ell_r} \sum_j U_j \sigma U_j^\dagger \otimes |j\rangle\langle j|_{M_r} : \sigma \in \mathcal{F}_{\{1,2,\ldots r-1\}}, U_j \in \mathcal{U}_{\{1,2,\ldots r-1\}} \right\},$$

and the experimeter only performs an operation from the set

$$\left\{ \sum_{j=1}^{\ell_r} U_j \otimes |j\rangle\langle j|_{M_r} : U_j \in \mathcal{U}_{\{1,2,\ldots r-1\}} \right\}.$$

Moreover, suppose it holds that $\mathcal{F}_{\{j\}} = \mathcal{F}_{\{1\}}$ for all $j < r$. Then $\sigma_{M_1 M_2 \ldots M_r} \in \mathcal{F}$ implies $\sigma_{M_j M_1 \ldots M_{j-1} M_{j+1} \ldots M_r} \in \mathcal{F}$ for all $j < r$.

Under this assumption, all the states $\mu_{EJ} \in \mathcal{F}$ in Task 2 are classical-quantum states, with register J being classical and μ_J being diagonal in the canonical basis. Thus, the assumption allows us to interpret $\log |J|$ as the randomness cost of the protocol. We have the following theorem.

Theorem 5. *Fix $\epsilon, \delta > 0$, and a quantum state ρ_M.*

- *Achievability: Suppose Assumption 4 holds. Then there exists an $(\varepsilon + \delta, k + 2\log \frac{1}{\delta})$-catalytic transformation of ρ_M to $\mathcal{F}$, where $k := \min_{\sigma_M \in \mathcal{F}} \mathrm{D}_{\max}^\varepsilon(\rho_M \| \sigma_M)$.*

5

- *Converse: For every $(\varepsilon, \log |J|)$- transformation of ρ_M to $\mathcal{F}$, it holds that*

$$\log |J| \geq \min_{\sigma_M \in \mathcal{F}} \mathrm{D}^\varepsilon_{\max}(\rho_M \| \sigma_M),$$

  if Assumption 4 is true. Otherwise,

$$\log |J| \geq \frac{1}{2} \min_{\sigma_M \in \mathcal{F}} \mathrm{D}^\varepsilon_{\max}(\rho_M \| \sigma_M).$$

The proof of this theorem is given in Section 4. Now we consider the asymptotic i.i.d. properties of our results.

Definition 6. *Suppose Assumption 4 holds. We say that the asymptotic randomness rate of catalytic transformation of ρ_M is R, if for every $\varepsilon > 0$, there exists an $n_0(\varepsilon)$ such that for all $n \geq n_0(\varepsilon)$, there exists a (ε, nR)-catalytic transformation of $\rho_M^{\otimes n}$ to $\mathcal{F}$. Define the following relative entropy of resource:*

$$E(\rho) = \inf_{\sigma_M \in \mathcal{F}} D(\rho_M \| \sigma_M)$$

and the regularized relative entropy of resource:

$$E^\infty(\rho) = \lim_{n \to \infty} \frac{1}{n} E(\rho_M^{\otimes n}).$$

Let $\mathcal{F}_n$ denote the set of free resources for the register $M^n := M \times M \times \ldots M$. Define the constant

$$C(\mathcal{F}) := \lim_{n \to \infty} \frac{1}{n} \inf_{\tau \in \mathcal{F}_n} \| \log \tau \|_\infty.$$

Using Theorem 5, we obtain the following.

Theorem 7. *Suppose the resource theory $\mathcal{F}$ is such that $C(\mathcal{F}) < \infty$. Then for a quantum state ρ_M, the asymptotic randomness rate of catalytic transformation of ρ is given by $E^\infty(\rho)$.*

It completely characterizes the per copy randomness requirement in the asymptotic i.i.d. setting. The proof is given in Section 6. Moreover, we show that the number of qubits of catalytic free resource grows only polynomially in n (Theorem 28).

2.2 Multi-partite case

Our formalism also extends to the multi-partite case. We consider the case of t parties and introduce some minor modifications to Definition 1 and Assumption 4. Let $\mathcal{H}$ be the global Hilbert space in the multiparty setting. We start with a decomposition of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \ldots \otimes \mathcal{H}_r$, where $\mathcal{H}_i$ is a Hilbert space shared by all the t parties.

Definition 8 (Multi-partite resource theory). *Fix an integer $r \geq 1$. Given*

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \ldots \otimes \mathcal{H}_r$$

and registers $M_1^1 M_1^2 \ldots M_1^t$, $M_2^1 M_2^2 \ldots M_2^t$, $\ldots$, $M_r^1 M_r^2 \ldots M_r^t$, where $M_i^1 M_i^2 \ldots M_i^t$ corresponds to $\mathcal{H}_i$ and M_i^j is held by j-th party, the set of free resources and free operations are as follows.

1. For every non empty ordered set

$$L = \{s_1, s_2, \ldots s_{|L|}\} \subseteq \{1, 2, \ldots r\}$$

let $\mathcal{F}_L \subseteq \mathcal{D}(\mathcal{H}_{s_1} \otimes \mathcal{H}_{s_2} \otimes \ldots \otimes \mathcal{H}_{s_L})$ satisfy the following properties. For brevity, we let M_L be the register corresponding to set L.

   (a) $\mathcal{F}_L$ is a convex and closed set.
   (b) For any $L' \subseteq L$, $\mathcal{F}_{L'} \otimes \mathcal{F}_{L \setminus L'} \subseteq \mathcal{F}_L$.
   (c) For any $\sigma_{M_L} \in \mathcal{F}_L$ and $L' \subseteq L$, $\mathrm{Tr}_{M_{L'}}(\sigma_{M_L}) \in \mathcal{F}_{L \setminus L'}$.

2. The complete set of free resources is $\mathcal{F} := \cup_L \mathcal{F}_L$.

3. For a fixed L, the free operations $\mathcal{U}_L$ are defined as the set of all unitaries U that satisfy

   (a) $\sigma \in \mathcal{F}_L \implies U\sigma U^\dagger \in \mathcal{F}_L$.
   (b) $U \in \mathcal{U}_L$ if and only if $U^\dagger \in \mathcal{U}_L$.

4. The set of all free operations are $\mathcal{U} := \cup_L \mathcal{U}_L$.

In an analogous fashion to Task 2, we define the multi-partite task.

Task 9 (An $(\varepsilon, \log|J|, t)$-transformation of $\rho_{M^1 M^2 \ldots M^t}$ to $\mathcal{F}$). *Let $\varepsilon > 0$, $r \geq 1$ be an integer and fix a Hilbert space $\mathcal{H}$ with decomposition $\mathcal{H}_1 \otimes \mathcal{H}_2 \ldots \otimes \mathcal{H}_r$, which is chosen by the experimenters. Let the register $M^1 M^2 \ldots M^t$ correspond to $\mathcal{H}_{M^1 M^2 \ldots M^t} = \mathcal{H}_1$, register $J_1 J_2 \ldots J_t$ correspond to $\mathcal{H}_{J_1 J_2 \ldots J_t} = \mathcal{H}_r$ (with register J_i held by i-th party) and register E correspond to $\mathcal{H}_E = \mathcal{H}_2 \otimes \mathcal{H}_3 \ldots \otimes \mathcal{H}_{r-1}$. The experimenters share a quantum state $\rho_{M^1 M^2 \ldots M^t}$. Using a state $\mu_{E J_1 J_2 \ldots J_t} \in \mathcal{F}$, they apply the unitary $U \in \mathcal{U}$ to obtain a joint quantum state $\Theta_{M^1 M^2 \ldots M^t E J_1 J_2 \ldots J_t}$:*

$$\Theta_{M^1 M^2 \ldots M^t E J_1 J_2 \ldots J_t} = U(\rho_{M^1 M^2 \ldots M^t} \otimes \mu_{E J_1 J_2 \ldots J_t})U^\dagger.$$

It is required that there exists a $\sigma_{M^1 M^2 \ldots M^t E} \in \mathcal{F}$ such that $\mathrm{Pur}(\Theta_{M^1 M^2 \ldots M^t E}, \sigma_{M^1 M^2 \ldots M^t E}) \leq \varepsilon$, where the chosen distant measure is defined in Equation (1).

We can similarly define a catalytic transformation in this framework.

Definition 10. *Task 9 is said to be a $(\varepsilon, \log|J|, t)$-catalytic transformation of $\rho_{M^1 M^2 \ldots M^t}$ if $\mu_{E J_1 J_2 \ldots J_t} = \mu_E \otimes \mu_{J_1 J_2 \ldots J_t}$ and $\sigma_{M^1 M^2 \ldots M^t E} = \sigma_{M^1 M^2 \ldots M^t} \otimes \mu_E$ for some $\sigma_{M^1 M^2 \ldots M^t} \in \mathcal{F}$.*

We will also need the following assumption for achievability protocol.

Assumption 11. *Invoke the notation from Definition 8. There exists a canonical basis $\mathcal{B}_j = \{|1\rangle, |2\rangle, \ldots |\ell_r\rangle\}$ on $\mathcal{H}_r^j$ (where the dimension ℓ_r of $\mathcal{H}_r^j$ is independent of j) such that*

$$\mathcal{F} = \mathrm{conv}\left\{\frac{1}{\ell_r} \sum_{k=1}^{\ell_r} U_k \sigma U_k^\dagger \otimes |k\rangle\langle k|_{M_r^1} \otimes |k\rangle\langle k|_{M_r^2} \otimes \ldots |k\rangle\langle k|_{M_r^t} : U_k \in \mathcal{U}_{\{1,2,\ldots r-1\}}\right\}$$

and experimenter only performs an operation from the set

$$\left\{\sum_k U_k \otimes |k\rangle\langle k|_{M_r^1} \otimes |k\rangle\langle k|_{M_r^2} \otimes \ldots |k\rangle\langle k|_{M_r^t} : U_k \in \mathcal{U}_{\{1,2,\ldots r-1\}}\right\}.$$

7

For brevity, define

$$I_{\ell_r,t} := \frac{1}{\ell_r} \sum_{k=1}^{\ell_r} |k\rangle\langle k|_{M_r^1} \otimes |k\rangle\langle k|_{M_r^2} \otimes \ldots |k\rangle\langle k|_{M_r^t}.$$

Moreover, suppose it holds that $\mathcal{F}_{\{j\}} = \mathcal{F}_{\{1\}}$ for all $j < r$. Then

$$\sigma_{M_1^1 M_1^2 \ldots M_1^t, M_2^1 M_2^2 \ldots M_2^t, \ldots M_r^1 M_r^2 \ldots M_r^t} \in \mathcal{F}$$

implies

$$\sigma_{M_j^1 M_j^2 \ldots M_j^t, M_2^1 M_2^2 \ldots M_2^t, \ldots M_{j-1}^1 M_{j-1}^2 \ldots M_{j-1}^t, M_1^1 M_1^2 \ldots M_1^t, M_{j+1}^1 M_{j+1}^2 \ldots M_{j+1}^t, \ldots M_r^1 M_r^2 \ldots M_r^t} \in \mathcal{F}$$

for all $j < r$.

We are now in a position to prove the following theorem.

Theorem 12. *Fix $\epsilon, \delta > 0$, and a quantum state $\rho_{M^1 M^2 \ldots M^t}$.*

- *Achievability: Suppose Assumption 11 holds. There exists an $(\varepsilon + \delta, k + 2\log\frac{1}{\delta}, t)$-catalytic transformation of $\rho_{M^1 M^2 \ldots M^t}$ to $\mathcal{F}$, where $k := \min_{\sigma_{M^1 M^2 \ldots M^t} \in \mathcal{F}} D_{\max}^\varepsilon(\rho_{M^1 M^2 \ldots M^t} \| \sigma_{M^1 M^2 \ldots M^t})$.*

- *Converse: For every $(\varepsilon, \log|J|, t)$-transformation of $\rho_{M^1 M^2 \ldots M^t}$ to $\mathcal{F}$, it holds that*

$$\log|J| \geq \min_{\sigma_{M^1 M^2 \ldots M^t} \in \mathcal{F}} D_{\max}^\varepsilon(\rho_{M^1 M^2 \ldots M^t} \| \sigma_{M^1 M^2 \ldots M^t}),$$

  if Assumption 11 is true.

The proof of this theorem follows similar to the proof of Theorem 5 and is given in Corollaries 26 and 27 in Section 5.

2.3 Known resource theories

To conclude this section, we show that our general resource framework includes at least the following individual resource theory.

1. Resource theory of entanglement [3, 6, 13, 14]: In this resource theory, the set of free states, $\mathcal{F}$, is the collection of separable states. The perfectly correlated state $I_{\ell,2}$ is a separable quantum state for all $\ell \geq 1$, and hence is a free resource. The formalism also extends to multi-partite entanglement, where the free resources are convex combination of product quantum states.

2. Resource theory of coherence [15, 16, 14, 17]: In this resource theory, the set of free states, $\mathcal{F}$, is the collection of diagonal states in a pre-determined basis. This is captured by Definition 1, where we choose a basis on each Hilbert space $\mathcal{H}_i$ and take the free resources $\mathcal{F}_{\{i\}}$ to be the set of diagonal quantum states. The properties such as being closed under partial trace and tensor product are easily seen to be satisfied. Moreover, maximally mixed state is diagonal in any basis, and hence belongs to the free resources.

3. Resource theory of asymmetry [28]: In this resource theory, the set of free resources are the states that are invariant under some group transformation. To construct $\mathcal{F}_{\{\ell\}}$ for some $\ell \geq 2$, one fixes a group G with a unitary representation $\{U_g\}_{g \in G}$ in dimension ℓ. The set $\mathcal{F}_{\{\ell\}}$ is the collection of states σ that satisfy $U_g \sigma U_g^\dagger = \sigma$. The resource theory can then be constructed on tensor product of such Hilbert spaces (see Section 2.B in [28]). To verify the partial trace condition (Item 1(c)), let $\sigma_{M_1 M_2}$ be such that $(U_g \otimes V_{g'}) \sigma_{M_1 M_2} (U_g^\dagger \otimes V_{g'}^\dagger) = \sigma_{M_1 M_2}$. Then

$$
\begin{aligned}
\sigma_{M_1} &= \mathrm{Tr}_{M_2}(\sigma_{M_1 M_2}) = \mathrm{Tr}_{M_2}\left(U_g \otimes V_{g'})\sigma_{M_1 M_2}(U_g^\dagger \otimes V_{g'}^\dagger)\right) \\
&= \mathrm{Tr}_{M_2}\left(U_g \otimes V_{g'}^\dagger V_{g'})\sigma_{M_1 M_2}(U_g^\dagger \otimes \mathrm{I})\right) = \mathrm{Tr}_{M_2}\left(U_g \otimes \mathrm{I})\sigma_{M_1 M_2}(U_g^\dagger \otimes \mathrm{I})\right) \\
&= U_g \sigma_{M_1} U_g^\dagger.
\end{aligned}
$$

Symmetry under permutation of registers of identical dimension can also be verified in similar way, as $\sigma_{M_1 M_2}$ must be invariant under $U_g \otimes U_{g'}$ for all g, g'.

4. Resource theory of nonuniformality [11] and purity [24]: In this resource theory, the only free state is the completely mixed state. This is easily captured by Definition 1. This is equivalent to the formalism of randomness extractors [25, 26].

5. Resource theory of Thermodynamics [18, 19, 20, 21, 22, 23]: In this resource theory, the free states are Gibbs quantum states, that are states of the form $\rho_\beta(H) = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$, for an arbitrary Hamiltonian $H > 0$. To capture it in Definition 1, we assign a Gibbs state as the only free resources in the set $\mathcal{F}_{\{i\}}$ (for all i). Thus, for a system with r registers, the free resource is $\otimes_{i=1}^r \rho_\beta(H_i)$. The condition that $U \otimes_{i=1}^r \rho_\beta(H_i) U^\dagger = \otimes_{i=1}^r \rho_\beta(H_i)$ is equivalent to the condition $[U, \sum_i H_i] = 0$. Moreover, this also implies $[U^\dagger, \sum_i H_i] = 0$. Finally, the maximally mixed state is a Gibbs quantum state with Hamiltonian $H = 0$. This establishes consistency with Definition 1.

Theorem 7 applies as long as $C(\mathcal{F}) < \infty$. For a Gibbs state $\rho_\beta(H)$, it holds that

$$
\| \log \rho_\beta(H) \|_\infty \leq \|\beta H\|_\infty + \log \mathrm{Tr}(e^{-\beta H}) \leq \|\beta H\|_\infty + \log d,
$$

where d is the support size of the Gibbs state. Thus, Theorem 7 applies for bounded $\|\beta H\|_\infty$.

6. Resource theory of contextuality [27]: In this resource theory, the free resources are the set of conditional probability distributions that can be described 'classically' (a notion that is made more precise in [27]). All the conditional probability distributions in this theory satisfy a consistency condition (which can be viewed as an analogue of the non-signaling condition in the case of non-locality). It can be observed that the set of free resources satisfies the conditions in Definition 1, being convex and closed under permutations and tensor product. Uniform probability distribution (or the maximally mixed state) also belongs to the free resource.

7. Resource theory of stabilizer computation [29]: In this resource theory, the free resources are the convex hull of the set of all pure states that can be generated by the action of a Clifford unitary on a standard state (for example, $|0\rangle$). Thus, maximally mixed state belongs to this set. This set is convex and closed. The free resources can be extended to the tensor product of registers along the lines as discussed in [28].

9

3 Preliminaries

Let $\mathcal{H}_A$ be the Hilbert space associated to a register A. Let $|A|$ denote the dimension of $\mathcal{H}_A$. Let $\mathcal{D}(\mathcal{H}_A)$ be the set of all normalized quantum states acting on $\mathcal{H}_A$. For two subsets $\mathcal{F}_A \subseteq \mathcal{D}(\mathcal{H}_A), \mathcal{F}_B \subseteq \mathcal{D}(\mathcal{H}_B)$, let $\mathcal{F}_A \otimes \mathcal{F}_B := \{\rho_A \otimes \sigma_B : \rho_A \in \mathcal{F}_A, \sigma_B \in \mathcal{F}_B\}$. Let $U : \mathcal{H}_A \rightarrow \mathcal{H}_A$ be a unitary operator acting on $\mathcal{H}_A$.

The definitions below have been adapted from the references [33, 34, 35, 36, 37, 38, 39] For $\rho_A, \sigma_A \in \mathcal{D}(\mathcal{H}_A)$ such that $\operatorname{supp}(\rho_A) \subset \operatorname{supp}(\sigma_A)$, its relative entropy is given by

$$\mathrm{D}(\rho_A \| \sigma_A) := \operatorname{Tr}(\rho_A \log \rho_A) - \operatorname{Tr}(\rho_A \log \sigma_A),$$

and is equal to infinity otherwise. Similarly, the relative entropy variance of $\rho_A, \sigma_A \in \mathcal{D}(\mathcal{H}_A)$ is defined as

$$\mathrm{V}(\rho \| \sigma) := \operatorname{Tr}(\rho(\log \rho - \log \sigma)^2) - (\mathrm{D}(\rho \| \sigma))^2,$$

when $\operatorname{supp}(\rho_A) \subset \operatorname{supp}(\sigma_A)$.

The max-relative entropy of $\rho_A, \sigma_A \in \mathcal{D}(\mathcal{H}_A)$ such that $\operatorname{supp}(\rho_A) \subset \operatorname{supp}(\sigma_A)$ is

$$\mathrm{D}_{\max}(\rho_A \| \sigma_A) := \inf\{\lambda \in \mathbb{R} : 2^\lambda \sigma_A \geq \rho_A\}.$$

The smooth max-relative entropy $\mathrm{D}_{\max}^\varepsilon(\rho_A \| \sigma_A)$ is

$$\mathrm{D}_{\max}^\varepsilon(\rho_A \| \sigma_A) := \sup_{\rho'_A \in \mathcal{B}^\varepsilon(\rho_A)} \mathrm{D}_{\max}\left(\rho'_A \| \sigma_A\right),$$

where the ε-ball, $\mathcal{B}^\varepsilon(\rho_A) := \{\rho'_A | \operatorname{Pur}(\rho_A, \rho'_A) \leq \varepsilon\}$, is defined via the purified distance

$$\operatorname{Pur}(\rho_A, \sigma_A) := \sqrt{1 - \mathrm{F}^2(\rho_A, \sigma_A)}. \tag{1}$$

In the above, $\mathrm{F}(\rho_A, \sigma_A) := \|\sqrt{\rho_A}\sqrt{\sigma_A}\|_1$ is the fidelity between two states.

We will use the following facts.

Fact 13 (Triangle inequality for purified distance, [39, 34]). *For quantum states ρ_A, σ_A, τ_A,*

$$\operatorname{Pur}(\rho_A, \sigma_A) \leq \operatorname{Pur}(\rho_A, \tau_A) + \operatorname{Pur}(\tau_A, \sigma_A).$$

Fact 14 (Uhlmann's Theorem, [40]). *Let $\rho_A, \sigma_A \in \mathcal{D}(\mathcal{H}_A)$. Let $|\rho\rangle\langle\rho|_{AB} \in \mathcal{D}(\mathcal{H}_{AB})$ be a purification of ρ_A. There exists a purification $|\theta\rangle\langle\theta|_{AB}$ of θ_A such that,*

$$\mathrm{F}(|\theta\rangle\langle\theta|_{AB}, |\rho\rangle\langle\rho|_{AB}) = \mathrm{F}(\rho_A, \sigma_A).$$

Fact 15. *Let $\rho_A, \sigma_A \in \mathcal{D}(\mathcal{H}_A)$ and $\rho_{AB} \in \mathcal{D}(\mathcal{H}_{AB})$ be an extension of ρ_A. Then there exists an extension σ_{AB} of σ_A such that*

$$\mathrm{F}(\rho_{AB}, \sigma_{AB}) = \mathrm{F}(\rho_A, \sigma_A).$$

Proof. Introduce a register C and consider a purification $|\rho\rangle\langle\rho|_{ABC}$ of ρ_{AB}. There exists a purification $|\sigma\rangle\langle\sigma|_{ABC}$ of σ_A such that $\mathrm{F}(|\rho\rangle\langle\rho|_{ABC}, |\sigma\rangle\langle\sigma|_{ABC}) = \mathrm{F}(\rho_A, \sigma_A)$, due to Uhlmann's Theorem (Fact 14). Thus, σ_{AB} is the desired extension due to the relation

$$\mathrm{F}(\rho_A, \sigma_A) \geq \mathrm{F}(\rho_{AB}, \sigma_{AB}) \geq \mathrm{F}(|\rho\rangle\langle\rho|_{ABC}, |\sigma\rangle\langle\sigma|_{ABC}) = \mathrm{F}(\rho_A, \sigma_A).$$

$\square$

Fact 16. *Let $\rho_A, \sigma_A \in \mathcal{D}(\mathcal{H}_A)$ and $\rho_{AB} \in \mathcal{D}(\mathcal{H}_{AB})$ be a classical-quantum extension of ρ_A, that is $\rho_{AB} = \sum_j p_j \rho_A^j \otimes |j\rangle\langle j|_B$, where $\rho_A = \sum_j p_j \rho_A^j$. Then there exists a classical-quantum extension σ_{AB} of σ_A such that*

$$\mathrm{F}(\rho_{AB}, \sigma_{AB}) = \mathrm{F}(\rho_A, \sigma_A),$$

and $\mathrm{supp}(\sigma_B) \subseteq \mathrm{supp}(\rho_B)$.

Proof. From Fact 15, there exists an extension σ'_{AB} of σ_A such that $\mathrm{F}(\rho_{AB}, \sigma'_{AB}) = \mathrm{F}(\rho_A, \sigma_A)$. Now, consider the map $\Lambda : B \to B$ defined as $\Lambda(\omega_B) = \sum_j |j\rangle\langle j|\omega_B|j\rangle\langle j|$. Observe that $\mathrm{I}_A \otimes \Lambda_B(\rho_{AB}) = \rho_{AB}$. Define $\sigma''_{AB} := \mathrm{I}_A \otimes \Lambda_B(\sigma'_{AB})$. It holds that σ''_{AB} is a classical-quantum extension of σ_A. Moreover,

$$\mathrm{F}(\rho_A, \sigma_A) \geq \mathrm{F}(\rho_{AB}, \sigma''_{AB}) \geq \mathrm{F}(\rho_{AB}, \sigma'_{AB}) = \mathrm{F}(\rho_A, \sigma_A).$$

Let Π_B be the projector onto the support of ρ_B. Define the quantum state $\sigma_{AB} := \frac{\Pi_B \sigma''_{AB} \Pi_B}{\mathrm{Tr}(\sigma''_B \Pi_B)}$. Observe that σ_{AB} is also a classical-quantum extension of σ_A and $\mathrm{supp}(\sigma_B) \in \mathrm{supp}(\rho_B)$. Then

$$
\begin{aligned}
\mathrm{F}(\rho_{AB}, \sigma_{AB}) &= \mathrm{Tr}\left(\sqrt{\sqrt{\rho_{AB}}\sigma_{AB}\sqrt{\rho_{AB}}}\right) \\
&= \frac{1}{\sqrt{\mathrm{Tr}(\sigma''_B \Pi_B)}}\mathrm{Tr}\left(\sqrt{\sqrt{\rho_{AB}}\Pi_B\sigma''_{AB}\Pi_B\sqrt{\rho_{AB}}}\right) \\
&= \frac{1}{\sqrt{\mathrm{Tr}(\sigma''_B \Pi_B)}}\mathrm{Tr}\left(\sqrt{\sqrt{\rho_{AB}}\sigma''_{AB}\sqrt{\rho_{AB}}}\right) \\
&= \frac{1}{\sqrt{\mathrm{Tr}(\sigma''_B \Pi_B)}}\mathrm{F}(\rho_{AB}, \sigma''_{AB}) \\
&\geq \mathrm{F}(\rho_{AB}, \sigma''_{AB}).
\end{aligned}
$$

This completes the proof. $\qquad\square$

Fact 17. *Let Θ_{AB} be a bipartite quantum state. Let Π_B be the projector onto the support of Θ_B. Then*

$$\Theta_{AB} \preceq |B|\Theta_A \otimes \Pi_B.$$

Fact 18. *Let Θ_{AB} be a classical quantum state with B as the quantum part. Let Π_B be the projector onto the support of Θ_B. Then*

$$\Theta_{AB} \preceq \Theta_A \otimes \Pi_B.$$

Fact 19 ([35, 36]). *Let $\varepsilon \in (0,1)$ and n be an integer. Let $\rho^{\otimes n}, \sigma^{\otimes n}$ be quantum states. Define $\Phi(x) = \int_{-\infty}^x \frac{e^{-t^2/2}}{\sqrt{2\pi}}dt$. It holds that*

$$\mathrm{D}_{\max}^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = n\mathrm{D}(\rho\|\sigma) + \sqrt{n\mathrm{V}(\rho\|\sigma)}\Phi^{-1}(\varepsilon) + O(\log n),$$

Fact 20. *For the function $\Phi(x) = \int_{-\infty}^x \frac{e^{-t^2/2}}{\sqrt{2\pi}}dt$ and $\varepsilon \leq \frac{1}{2}$, it holds that $|\Phi^{-1}(\varepsilon)| \leq 2\sqrt{\log\frac{1}{2\varepsilon}}$.*

Proof. We have

$$\Phi(-x) = \int_{-\infty}^{-x} \frac{e^{-t^2/2}}{\sqrt{2\pi}}dt = \int_0^\infty \frac{e^{-(-x-t)^2/2}}{\sqrt{2\pi}}dt \leq e^{-x^2/2}\int_0^\infty \frac{e^{-(-t)^2/2}}{\sqrt{2\pi}}dt = \frac{1}{2}e^{-x^2/2}.$$

Thus, $\Phi^{-1}(\varepsilon) \geq -2\sqrt{\log\frac{1}{2\varepsilon}}$, which completes the proof. $\qquad\square$

Fact 21. *Let ρ_1 be a quantum state and $\{\mathcal{E}_2, \mathcal{E}_3, \ldots\}$ be a collection of quantum maps. Define a series of quantum states $\{\rho_2, \rho_3, \ldots\}$ recursively as $\rho_i = \mathcal{E}_i(\rho_{i-1})$. It holds that*

$$\mathrm{Pur}(\rho_i, \rho_1) \le (i-1) \max_i \cdot \{\mathrm{Pur}(\mathcal{E}_i(\rho_1), \rho_1)\}.$$

Proof. Consider

$$\mathrm{Pur}(\rho_i, \rho_1) = \mathrm{Pur}(\mathcal{E}_i(\rho_{i-1}), \rho_1) \le \mathrm{Pur}(\mathcal{E}_i(\rho_{i-1}), \mathcal{E}_i(\rho_1)) + \mathrm{Pur}(\mathcal{E}_i(\rho_1), \rho_1) \le \mathrm{Pur}(\rho_{i-1}, \rho_1) + \mathrm{Pur}(\mathcal{E}_i(\rho_1), \rho_1).$$

This completes the proof. $\qquad\square$

Lemma 22 (Convex-split lemma [31]). *Let $\varepsilon > 0$, ρ_M and σ_M be quantum states, n be an integer and $k = \mathrm{D}_{\max}^{\varepsilon}(\rho_M \| \sigma_M)$. Consider the following quantum state*

$$\tau_{M_1 M_2 \ldots M_n} = \frac{1}{n} \sum_{j=1}^{n} \rho_{M_j} \otimes \sigma_{M_1} \otimes \sigma_{M_2} \ldots \otimes \sigma_{M_{j-1}} \otimes \sigma_{M_{j+1}} \ldots \otimes \sigma_{M_n}$$

where $\forall j \in [n] : \rho_{M_j} = \rho_M$ and $\sigma_{M_j} = \sigma_M$. Then,

$$\mathrm{Pur}(\tau_{M_1 M_2 \ldots M_n}, \sigma_{M_1} \otimes \sigma_{M_2} \ldots \otimes \sigma_{M_n}) \le \varepsilon + \sqrt{\frac{2^k}{n}}.$$

4 Proof of Theorem 5

4.1 Achievability

Theorem 23 (Achievability). *Fix $\varepsilon, \delta > 0$ and ρ_M. Suppose Assumption 4 holds. There exists an $(\varepsilon + \delta, k + 2\log\frac{1}{\delta})$-catalytic transformation of ρ_M to $\mathcal{F}$, where $k := \min_{\sigma_M \in \mathcal{F}} \mathrm{D}_{\max}^{\varepsilon}(\rho_M \| \sigma_M)$.*

Proof. Let $n := \frac{2^k}{\delta^2}$ and let $\sigma_M \in \mathcal{F}$ be the quantum state achieving the minimum in the definition of k. Let J be a random variable taking values uniformly in $\{1, 2, \ldots n\}$. Introduce registers $M_1, M_2, \ldots M_n \equiv M$. Let $\mu_{M_1 M_2 \ldots M_n} := \sigma_M^{\otimes n}$ be the quantum state in $\mathcal{F}$ which an experimenter uses as a catalyst.

The protocol is as follows. Experimenter introduces the quantum state $\mu_{M_1 M_2 \ldots M_n} \otimes \frac{1_J}{n}$, which belongs to $\mathcal{F}$ due to Definition 1 (Item 1(b)). Controlled on the value j in J, the experimenter swaps register M with M_j. This operation belongs to $\mathcal{U}$ from Assumption 4. Let the resulting global quantum state be $\tau_{M M_1 M_2 \ldots M_n J}$. We have $\tau_{M M_1 M_2 \ldots M_n} = \sigma_M \otimes \tau_{M_1 M_2 M_2 \ldots M_n}$ and

$$\tau_{M_1 M_2 \ldots M_n} = \frac{1}{n} \sum_{j=1}^{n} \rho_{M_j} \otimes \sigma_{M_1} \otimes \sigma_{M_2} \ldots \otimes \sigma_{M_{j-1}} \otimes \sigma_{M_{j+1}} \ldots \otimes \sigma_{M_n}.$$

Using convex-split lemma (Lemma 22), this allows us to conclude that

$$\mathrm{Pur}(\tau_{M_1 M_2 \ldots M_n}, \sigma_M^{\otimes n}) \le \varepsilon + \delta.$$

This completes the proof. $\qquad\square$

Remark 1: The number of qubits used as a catalyst in above task is $\log|M| \cdot \frac{2^k}{\delta^2}$.

4.2 Converse Proof

In this section, we prove a converse bound for resource transformation, showing near optimality of Theorem 23. We will separately consider the cases where Assumption 4 holds and where it does not.

Theorem 24. *Fix an $\varepsilon > 0$. For every $(\varepsilon, \log|J|)$- transformation of ρ_M to $\mathcal{F}$, it holds that*

$$\log|J| \geq \min_{\sigma_M \in \mathcal{F}} \mathrm{D}^{\varepsilon}_{\max}(\rho_M \| \sigma_M),$$

if Assumption 4 is true.

Proof. Consider any protocol that starts with quantum states $\rho_M \otimes \mu_{EJ}$ and applies the unitary $U := \sum_j U_j \otimes |j\rangle\langle j|_J$ (where $U_j : \mathcal{H}_E \otimes \mathcal{H}_M \to \mathcal{H}_E \otimes \mathcal{H}_M$) to obtain $\Theta_{MEJ} = U(\rho_M \otimes \mu_{EJ})U^{\dagger}$. By Assumption 4, μ_{EJ} is a classical-quantum state with J being the classical register. There exist $\sigma_{ME} \in \mathcal{F}$ such that $\mathrm{Pur}(\Theta_{ME}, \sigma_{ME}) \leq \varepsilon$.

From Fact 16, there exists a classical-quantum extension Θ'_{MEJ} of σ_{ME} (that is, $\Theta'_{ME} = \sigma_{ME}$ and J is the classical register in Θ'_{MEJ}) such that

$$\mathrm{Pur}(\Theta_{MEJ}, \Theta'_{MEJ}) = \mathrm{Pur}(\Theta_{ME}, \sigma_{ME}) \leq \varepsilon. \tag{2}$$

Since Θ'_{MEJ} is a classical-quantum state, it holds by Fact 18 that

$$\Theta'_{MEJ} \preceq \Theta'_{ME} \otimes \mathrm{I}_J = |J| \cdot \sigma_{ME} \otimes \frac{\mathrm{I}_J}{|J|} \implies \mathrm{D}_{\max}\left(\Theta'_{MEJ} \| \sigma_{ME} \otimes \frac{\mathrm{I}_J}{|J|}\right) \leq \log|J|.$$

From this, we conclude that

$$\mathrm{D}_{\max}\left(\mathrm{Tr}_{EJ}\left(U^{\dagger}\Theta'_{MEJ}U\right) \| \mathrm{Tr}_{EJ}\left(U^{\dagger}\left(\sigma_{ME} \otimes \frac{\mathrm{I}_J}{|J|}\right)U\right)\right) \leq \log|J|. \tag{3}$$

From Equation (2),

$$\mathrm{Pur}\left(\mathrm{Tr}_{EJ}\left(U^{\dagger}\Theta'_{MEJ}U\right), \rho_M\right) \leq \mathrm{Pur}(U^{\dagger}\Theta'_{MEJ}U, \rho_M \otimes \mu_{EJ}) = \mathrm{Pur}(\Theta'_{MEJ}, \Theta_{MEJ}) \leq \varepsilon. \tag{4}$$

From Definition 1 (Items $1(a), 3$),

$$\mathrm{Tr}_J\left(U^{\dagger}\left(\sigma_{ME} \otimes \frac{\mathrm{I}_J}{|J|}\right)U\right) = \frac{1}{|J|}\sum_j U_j^{\dagger}(\sigma_{ME})U_j \in \mathcal{F}.$$

Thus, again from Definition 1 (Item $1(c)$),

$$\mathrm{Tr}_{EJ}\left(U\left(\sigma_M \otimes \mu'_E \otimes \frac{\mathrm{I}_J}{|J|}\right)U^{\dagger}\right) \in \mathcal{F}. \tag{5}$$

Combining Equations (3), (4) and (5), we obtain

$$\min_{\sigma'_M \in \mathcal{F}} \mathrm{D}^{\varepsilon}_{\max}(\rho_M \| \sigma'_M) \leq \log|J|.$$

This completes the proof.

$\square$

13

Now we consider the converse bound without Assumption 4.

Theorem 25. *Fix an $\varepsilon > 0$. For every $(\varepsilon, \log|J|)$- transformation of ρ_M to $\mathcal{F}$, it holds that*

$$\log|J| \geq \frac{1}{2} \min_{\sigma_M \in \mathcal{F}} \mathrm{D}_{\max}^{\varepsilon}(\rho_M \| \sigma_M).$$

Proof. Consider any protocol that starts with quantum states $\rho_M \otimes \mu_{EJ}$ and applies the unitary $U \in \mathcal{U}$ to obtain $\Theta_{MEJ} = U(\rho_M \otimes \mu_{EJ})U^{\dagger}$. There exist $\sigma_{ME} \in \mathcal{F}$ such that $\mathrm{Pur}(\Theta_{ME}, \sigma_{ME}) \leq \varepsilon$.

From Fact 15, there exists an extension Θ'_{MEJ} of σ_{ME} (that is, $\Theta'_{ME} = \sigma_{ME}$) such that

$$\mathrm{Pur}(\Theta_{MEJ}, \Theta'_{MEJ}) = \mathrm{Pur}(\Theta_{ME}, \sigma_{ME}) \leq \varepsilon. \tag{6}$$

It holds by Fact 17 that

$$\Theta'_{MEJ} \preceq |J|\Theta'_{ME} \otimes \mathrm{I}_J = |J|^2 \cdot \sigma_{ME} \otimes \frac{\mathrm{I}_J}{|J|} \implies \mathrm{D}_{\max}\left(\Theta'_{MEJ} \| \sigma_{ME} \otimes \frac{\mathrm{I}_J}{|J|}\right) \leq 2\log|J|.$$

From this, we conclude that

$$\mathrm{D}_{\max}\left(\mathrm{Tr}_{EJ}\left(U^{\dagger}\Theta'_{MEJ}U\right) \| \mathrm{Tr}_{EJ}\left(U^{\dagger}\left(\sigma_{ME} \otimes \frac{\mathrm{I}_J}{|J|}\right)U\right)\right) \leq 2\log|J|. \tag{7}$$

From Equation (6),

$$\mathrm{Pur}\left(\mathrm{Tr}_{EJ}\left(U^{\dagger}\Theta'_{MEJ}U\right), \rho_M\right) \leq \mathrm{Pur}(U^{\dagger}\Theta'_{MEJ}U, \rho_M \otimes \mu_{EJ}) = \mathrm{Pur}(\Theta'_{MEJ}, \Theta_{MEJ}) \leq \varepsilon. \tag{8}$$

From Definition 1 (Items $1(b), 1(d), 3$),

$$\mathrm{Tr}_J\left(U^{\dagger}\left(\sigma_{ME} \otimes \frac{\mathrm{I}_J}{|J|}\right)U\right) \in \mathcal{F}.$$

Thus, again from Definition 1 (Item $1(c)$),

$$\mathrm{Tr}_{EJ}\left(U\left(\sigma_M \otimes \mu'_E \otimes \frac{\mathrm{I}_J}{|J|}\right)U^{\dagger}\right) \in \mathcal{F}. \tag{9}$$

Combining Equations (7), (8) and (9), we obtain

$$\min_{\sigma'_M \in \mathcal{F}} \mathrm{D}_{\max}^{\varepsilon}(\rho_M \| \sigma'_M) \leq 2\log|J|.$$

This completes the proof.

$\square$

5 Proof of Theorem 12.

Achievability

The proof of achievability follows along lines similar to proof of Theorem 23. We state it as a corollary.

Corollary 26. *Fix $\varepsilon, \delta > 0$ and $\rho_{M^1 M^2 \dots M^t}$. Suppose Assumption 11 holds. Then there exists an $(\varepsilon + \delta, k + 2\log\frac{1}{\delta}, t)$-catalytic transformation of $\rho_{M^1 M^2 \dots M^t}$ to $\mathcal{F}$, where*

$$k := \min_{\sigma_{M^1 M^2 \dots M^t} \in \mathcal{F}} \mathrm{D}^{\varepsilon}_{\max}(\rho_{M^1 M^2 \dots M^t} \| \sigma_{M^1 M^2 \dots M^t}).$$

Proof. Let $n := \frac{2^k}{\delta^2}$ and $\sigma_{M^1 M^2 \dots M^t} \in \mathcal{F}$ be the quantum state achieving the minimum in the definition of k. The experimenters possess the state $\mathrm{I}_{n,t}$ in registers $J_1 J_2 \dots J_t$, such that $|J_i| = n$ for all i. Introduce registers $M_1^i, M_2^i, \dots M_n^i \equiv M^i$. They use the quantum state

$$\mu_{M_1^1 M_1^2 \dots M_1^t, M_2^1 M_2^2 \dots M_2^t, \dots M_n^1 M_n^2 \dots M_n^t} := \sigma^{\otimes n}_{M^1 M^2 \dots M^t}$$

as a catalyst. The protocol closely follows the protocol given in the proof of Theorem 23, where the party i swaps the registers M^i and M_j^i controlled on the value j in register J_i. It can be verified from Definition 8 and Assumption 11 that all these operations belong to $\mathcal{F}$ and $\mathcal{U}$. The proof of completeness follows similarly using convex split lemma (Lemma 22). $\square$

Converse

Using argument similar to that of Theorem 24, we have the following corollary.

Corollary 27. *Fix an $\varepsilon > 0$. For every $(\varepsilon, \log|J|, t)$- transformation of $\rho_{M^1 M^2 \dots M^t}$ to $\mathcal{F}$, it holds that*

$$\log|J| \geq \min_{\sigma_{M^1 M^2 \dots M^t} \in \mathcal{F}} \mathrm{D}^{\varepsilon}_{\max}(\rho_{M^1 M^2 \dots M^t} \| \sigma_{M^1 M^2 \dots M^t}),$$

if Assumption 11 is true.

Proof. Consider any protocol that starts with quantum states $\rho_{M^1 M^2 \dots M^t} \otimes \mu_{E J_1 J_2 \dots J_t}$ and applies the unitary $U := \sum_k U_k \otimes |k\rangle\langle k|_{J_1} \otimes |k\rangle\langle k|_{J_2} \otimes \dots |k\rangle\langle k|_{J_t}$ (where $U_k : \mathcal{H}_{M^1 M^2 \dots M^t} \otimes \mathcal{H}_E \to \mathcal{H}_{M^1 M^2 \dots M^t} \otimes \mathcal{H}_E$) to obtain $\Theta_{M^1 M^2 \dots M^t E J_1 J_2 \dots J_t} = U(\rho_{M^1 M^2 \dots M^t} \otimes \mu_{E J_1 J_2 \dots J_t}) U^\dagger$. By Assumption 11, $\mu_{E J_1 J_2 \dots J_t}$ is a classical-quantum state with registers $J_1 J_2 \dots J_t$ being classical and $\mu_{J_1 J_2 \dots J_t}$ belonging to $\mathrm{supp}(\mathrm{I}_{J,t})$. Furthermore, there exists a quantum state $\sigma_{M^1 M^2 \dots M^t E} \in \mathcal{F}$ such that $\mathrm{Pur}(\Theta_{M^1 M^2 \dots M^t E}, \sigma_{M^1 M^2 \dots M^t E}) \leq \varepsilon$.

From Fact 16, there exists a classical-quantum extension $\Theta'_{M^1 M^2 \dots M^t E J_1 J_2 \dots J_t}$ of $\sigma_{M^1 M^2 \dots M^t E}$ (that is, $\Theta'_{M^1 M^2 \dots M^t E} = \sigma_{M^1 M^2 \dots M^t E}$ and $J_1 J_2 \dots J_t$ is the classical register in $\Theta'_{M^1 M^2 \dots M^t E J_1 J_2 \dots J_t}$) such that

$$\mathrm{Pur}(\Theta_{M^1 M^2 \dots M^t E J_1 J_2 \dots J_t}, \Theta'_{M^1 M^2 \dots M^t E J_1 J_2 \dots J_t}) = \mathrm{Pur}(\Theta_{M^1 M^2 \dots M^t E}, \sigma_{M^1 M^2 \dots M^t E}) \leq \varepsilon. \quad (10)$$

Moreover, $\mathrm{supp}(\Theta'_{J_1 J_2 \dots J_t}) \subseteq \mathrm{supp}(\mathrm{I}_{J,t})$. Thus, by Fact 18

$$\Theta'_{M^1 M^2 \dots M^t E J_1 J_2 \dots J_t} \preceq |J| \Theta'_{M^1 M^2 \dots M^t E} \otimes \mathrm{I}_{|J|,t}$$

which implies

$$\mathrm{D}_{\max}\left(\Theta'_{M^1 M^2 \dots M^t E J_1 J_2 \dots J_t} \| \sigma_{M^1 M^2 \dots M^t E} \otimes \mathrm{I}_{|J|,t}\right) \leq \log|J|.$$

From this, we conclude that

$$\mathrm{D}_{\max}\left(\mathrm{Tr}_{E J_1 J_2 \dots J_t}\left(U^\dagger \Theta'_{M^1 M^2 \dots M^t E J_1 J_2 \dots J_t} U\right) \| \mathrm{Tr}_{E J_1 J_2 \dots J_t}\left(U^\dagger\left(\sigma_{M^1 M^2 \dots M^t E} \otimes \mathrm{I}_{|J|,t}\right) U\right)\right) \leq \log|J|. \quad (11)$$

From Equation (10),

$$\mathrm{Pur}\left(\mathrm{Tr}_{EJ_1J_2...J_t}\left(U^\dagger\Theta'_{M^1M^2...M^tEJ_1J_2...J_t}U\right),\rho_{M^1M^2...M^t}\right)$$

$$\leq \mathrm{Pur}(U^\dagger\Theta'_{M^1M^2...M^tEJ_1J_2...J_t}U,\rho_{M^1M^2...M^t}\otimes\mu_{EJ_1J_2...J_t})$$

$$= \mathrm{Pur}(\Theta'_{M^1M^2...M^tEJ_1J_2...J_t},\Theta_{M^1M^2...M^tEJ_1J_2...J_t})\leq\varepsilon. \qquad (12)$$

From Definition 8 (Items $1(a),3$),

$$\mathrm{Tr}_{J_1J_2...J_t}\left(U^\dagger\left(\sigma_{M^1M^2...M^tE}\otimes\mathrm{I}_{|J|,t}\right)U\right) = \frac{1}{|J|}\sum_k U_k^\dagger(\sigma_{M^1M^2...M^tE})U_k\in\mathcal{F}.$$

Thus, again from Definition 8 (Item $1(c)$),

$$\mathrm{Tr}_{EJ_1J_2...J_t}\left(U\left(\sigma_{M^1M^2...M^t}\otimes\mu'_E\otimes\mathrm{I}_{|J|,t}\right)U^\dagger\right)\in\mathcal{F}. \qquad (13)$$

Combining Equations (11), (12) and (13), we obtain

$$\min_{\sigma'_{M^1M^2...M^t}\in\mathcal{F}}\mathrm{D}^\varepsilon_{\max}(\rho_{M^1M^2...M^t}\|\sigma'_{M^1M^2...M^t})\leq\log|J|.$$

This completes the proof. $\qquad\qquad\square$

6 Proof of Theorem 7

First we show that the number of qubits of catalysts scales only polynomially in the number of copies of the quantum state to be transformed.

Theorem 28. *Suppose Assumption 4 holds. Let $\varepsilon,\gamma > 0$ be such that $\gamma^2\leq\varepsilon$ and m be a sufficiently large integer. For every ρ_M, there exists a $(\varepsilon,m(R+\gamma))$-catalytic transformation of $\rho_M^{\otimes m}$ to $\mathcal{F}$, such that the number of qubits of catalyst used upper bounded by*

$$|M|\cdot\frac{2\log m\cdot V}{\gamma^2}\cdot m^{\frac{2\cdot V\cdot(R+\gamma)}{\gamma^2}},$$

where $R:=\min_{\sigma_M\in\mathcal{F}}\mathrm{D}(\rho_M\|\sigma_M)$ and $V:=\mathrm{V}(\rho_M\|\sigma_M^*)$, where σ_M^* achieves the minimum in the definition of R.

Proof. Fix a quantum state $\sigma_M\in\mathcal{F}$ and let

$$q:=\frac{2\log m\cdot\mathrm{V}(\rho_M\|\sigma_M)}{\gamma^2},k:=\mathrm{D}^{\frac{\varepsilon q}{2m}}_{\max}(\rho_M^{\otimes q}\|\sigma_M^{\otimes q})+2\log\frac{2m}{\varepsilon q}.$$

Let J be a random variable taking values uniformly in $\{1,2,\ldots2^k\}$. Introduce registers $M'_1,M'_2,\ldots M'_{2^k}$ such that each M'_i is equivalent to q copies of M. Experimenter introduces the quantum state

$$\sigma_M^{\otimes q\cdot2^k}:=(\sigma^{\otimes q})_{M'_1}\otimes\ldots(\sigma^{\otimes q})_{M'_{2^k}}.$$

16

She generates $\frac{m}{q}$ copies $J_1, J_2, \ldots J_{\frac{m}{q}}$ of the random variable J. Let $\mathcal{P}$ be the $(\frac{\varepsilon q}{m}, k)$-catalytic transformation protocol for $\rho_M^{\otimes q}$, as guaranteed by Theorem 23. The desired protocol $\mathcal{P}'$ is as follows, where the quantum state $\rho_M^{\otimes m}$ is divided into $\frac{m}{q}$ blocks as

$$\rho_M^{\otimes m} = (\rho^{\otimes q})_{M_1} \otimes (\rho^{\otimes q})_{M_2} \otimes \ldots (\rho^{\otimes q})_{M_{\frac{m}{q}}}.$$

Here, M_i is equivalent to q copies of M for all $i \in \{1, 2, \ldots \frac{m}{q}\}$.

1. Set $i = 1$.

2. Experimenter introduces the random variable J_i and runs the protocol $\mathcal{P}$ for the i-th block with quantum state $(\rho^{\otimes q})_{M_i}$.

3. She throws away the random variable J_i.

4. Set $i \leftarrow i + 1$. Go to step 2.

Let Θ be the quantum state on registers $M_1, M_2, \ldots M_{\frac{m}{q}}, M'_1, M'_2, \ldots M'_{2^k}$ after the protocol ends. For iteration i, let $\mathcal{P}'_i$ be the overall quantum map applied on these registers after step 3 finishes. Observe that $\mathcal{P}'_i$ acts only on registers $M_i, M'_1, M'_2, \ldots M'_{2^k}$. From Theorem 23, we have that
$$\mathrm{Pur}\left(\mathcal{P}'_i\left((\rho^{\otimes q})_{M_i} \otimes \sigma_M^{\otimes q \cdot 2^k}\right), (\sigma^{\otimes q})_{M_i} \otimes \sigma_M^{\otimes q \cdot 2^k}\right) \leq \frac{\varepsilon q}{m}.$$
Thus, from Fact 21,

$$\mathrm{Pur}(\Theta, (\sigma^{\otimes q})_{M_1} \otimes \ldots (\sigma^{\otimes q})_{M_{\frac{m}{q}}} \otimes (\sigma^{\otimes q})_{M'_1} \otimes \ldots (\sigma^{\otimes q})_{M'_{2^k}}) \leq (\frac{m}{q} - 1)\frac{\varepsilon q}{m} \leq \varepsilon.$$

Thus, the protocol achieves an $(\varepsilon, \frac{mk}{q})$-catalytic transformation of $\rho_M^{\otimes m}$. The theorem now follows from the following upper bound on k.

$$\begin{aligned}
k &= \mathrm{D}_{\max}^{\frac{\varepsilon q}{2m}}(\rho_M^{\otimes q} \| \sigma_M^{\otimes q}) + 2 \log \frac{2m}{\varepsilon q} \\
&\leq q \cdot \mathrm{D}(\rho_M \| \sigma_M) + \sqrt{q \cdot \mathrm{V}(\rho_M \| \sigma_M)}\Phi^{-1}\left(\frac{\varepsilon q}{2m}\right) + O(\log \frac{2m}{\varepsilon q}) \\
&\leq q \cdot \mathrm{D}(\rho_M \| \sigma_M) + \sqrt{q \log \frac{2m}{\varepsilon q} \cdot \mathrm{V}(\rho_M \| \sigma_M)} + O(\log \frac{m}{\varepsilon q}) \\
&= q \left(\mathrm{D}(\rho_M \| \sigma_M) + \sqrt{\frac{1}{q} \log \frac{2m}{\varepsilon q} \cdot \mathrm{V}(\rho_M \| \sigma_M)} + O(\frac{\log \frac{m}{\varepsilon q}}{q})\right) \\
&\leq q \left(\mathrm{D}(\rho_M \| \sigma_M) + \gamma\right),
\end{aligned}$$

where we have used Facts 19 and 20, the choice of q and the relation $\gamma^2 \leq \varepsilon$.

$\square$

Now we provide optimal asymptotic i.i.d. analysis of Task 2. For this we require the following result, essentially proved in [41], and restated in our setting.

Lemma 29 ([41]). *Let $\rho, \rho' \in \mathcal{D}(\mathcal{H}_M)$ be quantum states on register M with $\|\rho - \rho'\|_1 := \varepsilon \leq \frac{1}{3}$. Let $\mathcal{F} \subseteq \mathcal{D}(\mathcal{H}_M)$ be a convex set. Then it holds that*

$$|\inf_{\sigma \in \mathcal{F}} \mathrm{D}(\rho\|\sigma) - \inf_{\sigma' \in \mathcal{F}} \mathrm{D}(\rho'\|\sigma')| \leq \varepsilon \left(\log M + \inf_{\tau \in \mathcal{F}} \|\log \tau\|_\infty \right) + \varepsilon \log \frac{1}{\varepsilon} + 4\varepsilon.$$

Using above lemma, we have the following theorem.

Theorem 30. *Suppose Assumption 4 holds and $C(\mathcal{F}) < \infty$. Then for every ρ_M, the asymptotic randomness rate of catalytic transformation of ρ_M is equal to*

$$\lim_{n \to \infty} \frac{1}{n} \min_{\sigma \in \mathcal{F}} \mathrm{D}(\rho_M^{\otimes n}\|\sigma).$$

Proof. The upper bound follows by applying Theorem 28 to the quantum state $\rho_M^{\otimes n}$. For the lower bound, we appeal to the converse in Theorem 24 to conclude that for the (ε, nR)-catalytic transformation of $\rho_M^{\otimes n}$ to $\mathcal{F}$, we require

$$R \geq \frac{1}{n} \min_{\sigma \in \mathcal{F}} \mathrm{D}_{\max}^\varepsilon(\rho_M^{\otimes n}\|\sigma) \geq \frac{1}{n} \min_{\sigma \in \mathcal{F}} \mathrm{D}^\varepsilon(\rho_M^{\otimes n}\|\sigma).$$

Now, observe that

$$\min_{\sigma \in \mathcal{F}} \mathrm{D}^\varepsilon(\rho_M^{\otimes n}\|\sigma) = \min_{\sigma \in \mathcal{F}} \min_{\rho' \in \mathcal{B}^\varepsilon(\rho_M^{\otimes n})} \mathrm{D}(\rho'\|\sigma) = \min_{\rho' \in \mathcal{B}^\varepsilon(\rho_M^{\otimes n})} \min_{\sigma \in \mathcal{F}} \mathrm{D}(\rho'\|\sigma).$$

Let $\rho' \in \mathcal{B}^\varepsilon(\rho_M^{\otimes n})$ be the quantum state that achieves the minimum above. From Lemma 29, it holds that

$$\frac{1}{n}|\mathrm{D}(\rho'\|\sigma) - \mathrm{D}(\rho^{\otimes n}\|\sigma)| \leq \varepsilon C(\mathcal{F}) + \varepsilon \log M + \frac{5}{n}\varepsilon \log \frac{1}{\varepsilon}.$$

Thus,

$$R \geq \frac{1}{n} \min_{\sigma \in \mathcal{F}} \mathrm{D}(\rho_M^{\otimes n}\|\sigma) - \varepsilon(\log |M| + C(\mathcal{F}))) - \frac{5}{n}\varepsilon \log \frac{1}{\varepsilon}.$$

Letting $n \to \infty$ and $\varepsilon \to 0$, we conclude the proof. □

7   Conclusion

In this work we quantity the amount of 'resource' contained in a given state, in terms of regularized relative entropy of resource, in a general resource theoretic framework that allows to use free resource as a catalyst. This then yields a new operational interpretation to this entropic quantity, which has a nice geometric interpretation. Our general catalytic resource framework, while having some restriction on the free resource and free operations, is still general enough to include several well-developed resource theories in the literature, including resource theory of entanglement, resource theory of coherence, resource theory of non-uniformity, classical resource theory of randomness extraction, etc.

Our main result follows by first providing a one-shot bound that characterizes the amount of randomness required in order to erase the resource contained in one copy of a given state.

The one-shot bound is given by smooth max-relative entropy. This also gives the smooth max-relative entropy a new operational meaning in the resource theoretic framework. To the best of our knowledge, this is the first one-shot result that works for a general resource theoretic framework. This result is obtained via the convex-split lemma.

Answering the amount of useful resource possessed by a given state is just a first step toward a general resource theory. Understanding what other information-processing tasks could be done in this general framework is thus a meaningful and fruitful open question.

Acknowledgement

References

[1] I. Devetak, A. W. Harrow, and A. J. Winter, "A resource framework for quantum shannon theory," *IEEE Transactions on Information Theory*, vol. 54, pp. 4587–4618, Oct 2008.

[2] N. Datta and M.-H. Hsieh, "The apex of the family tree of protocols: optimal rates and resource inequalities," *New Journal of Physics*, vol. 13, no. 9, p. 093042, 2011.

[3] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, "Quantum entanglement," *Rev. Mod. Phys.*, vol. 81, pp. 865–942, Jun 2009.

[4] E. Chitambar and M.-H. Hsieh, "Asymptotic state discrimination and a strict hierarchy in distinguishability norms," *Journal of Mathematical Physics*, vol. 55, no. 11, p. 112204, 2014.

[5] E. Chitambar, D. Leung, L. Mančinska, M. Ozols, and A. Winter, "Everything you always wanted to know about locc (but were afraid to ask)," *Communications in Mathematical Physics*, vol. 328, no. 1, pp. 303–326, 2014.

[6] B. Groisman, S. Popescu, and A. Winter, "Quantum, classical, and total amount of correlations in a quantum state," *Phys. Rev. A*, vol. 72, p. 032317, Sep 2005.

[7] P. W. Shor, "Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer," *SIAM Review*, vol. 41, no. 2, pp. 303–332, 1999.

[8] F. G. S. L. Brandão and G. Gour, "Reversible framework for quantum resource theories," *Phys. Rev. Lett.*, vol. 115, p. 070503, Aug 2015.

[9] G. Gour, "Quantum resource theories in the single-shot regime," *Phys. Rev. A*, vol. 95, p. 062314, Jun 2017.

[10] Y. Feng, R. Duan, and M. Ying, "Catalyst-assisted probabilistic entanglement transformation," *IEEE Transactions on Information Theory*, vol. 51, pp. 1090–1101, March 2005.

[11] G. Gour, M. P. Müller, V. Narasimhachar, R. W. Spekkens, and N. Y. Halpern, "The resource theory of informational nonequilibrium in thermodynamics," *Physics Reports*, vol. 583, pp. 1 – 58, 2015. The resource theory of informational nonequilibrium in thermodynamics.

[12] F. Brandao, M. Horodecki, N. Ng, J. Oppenheim, and S. Wehner, "The second laws of quantum thermodynamics," *Proceedings of the National Academy of Sciences*, vol. 112, no. 11, pp. 3275–3279, 2015.

[13] F. G. S. L. Brandao and M. B. Plenio, "Entanglement theory and the second law of thermodynamics," *Nature Physics*, vol. 4, pp. 873–877, 2008.

[14] E. Chitambar and M.-H. Hsieh, "Relating the resource theories of entanglement and quantum coherence," *Phys. Rev. Lett.*, vol. 117, p. 020402, Jul 2016.

[15] T. Baumgratz, M. Cramer, and M. B. Plenio, "Quantifying coherence," *Phys. Rev. Lett.*, vol. 113, p. 140401, Sep 2014.

[16] A. Winter and D. Yang, "Operational resource theory of coherence," *Phys. Rev. Lett.*, vol. 116, p. 120404, Mar 2016.

[17] M. B. P. Alexander Streltsov, Gerardo Adesso, "Quantum coherence as a resource." arXiv:1609.02439, 2016.

[18] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, "Resource theory of quantum states out of thermal equilibrium," *Phys. Rev. Lett.*, vol. 111, p. 250404, Dec 2013.

[19] F. Brandão, M. Horodecki, N. Ng, J. Oppenheim, and S. Wehner, "The second laws of quantum thermodynamics," *Proceedings of the National Academy of Sciences*, vol. 112, no. 11, pp. 3275–3279, 2015.

[20] M. Horodecki and J. Oppenheim, "Fundamental limitations for quantum and nano thermodynamics," *Nature Communications*, vol. 4, no. 2059, 2013.

[21] P. Faist, F. Dupuis, J. Oppenheim, and R. Renner, "Fundamental limitations for quantum and nano thermodynamics," *Nature Communications*, vol. 6, no. 7669, 2015.

[22] G. Gour, M. P. Müller, V. Narasimhachar, R. W. Spekkens, and N. Y. Halpern, "The resource theory of informational nonequilibrium in thermodynamics," *Physics Reports*, vol. 583, pp. 1 – 58, 2015.

[23] V. Narasimhachar and G. Gour, "Low-temperature thermodynamics with quantum coherence," *Nature Communications*, vol. 6, no. 7689, 2015.

[24] M. Horodecki, P. Horodecki, and J. Oppenheim, "Reversible transformations from pure to mixed states and the unique measure of information," *Phys. Rev. A*, vol. 67, p. 062104, Jun 2003.

[25] L. Trevisan, "Extractors and pseudorandom generators," *J. ACM*, vol. 48, pp. 860–879, July 2001.

[26] L. Trevisan and S. Vadhan, "Extracting randomness from samplable distributions," in *Proceedings of the 41st Annual Symposium on Foundations of Computer Science*, FOCS '00, (Washington, DC, USA), pp. 32–, IEEE Computer Society, 2000.

[27] K. Horodecki, A. Grudka, P. Joshi, W. Kłobus, and J. Łodyga, "Axiomatic approach to contextuality and nonlocality," *Phys. Rev. A*, vol. 92, p. 032104, Sep 2015.

[28] E. Wakakuwa, "Symmetrizing cost of quantum states," *Phys. Rev. A*, vol. 95, p. 032328, Mar 2017.

[29] V. Veitch, S. A. H. Mousavian, D. Gottesman, and J. Emerson, "The resource theory of stabilizer quantum computation," *New Journal of Physics*, vol. 16, no. 1, p. 013009, 2014.

[30] M. Berta and C. Majenz, "Disentanglement cost of quantum states." https://arxiv.org/abs/1708.00360, 2017.

[31] A. Anshu, V. Devabathini, and R. Jain, "Near optimal bounds on quantum communication complexity of single-shot quantum state redistribution." http://arxiv.org/abs/1410.3031, 2014.

[32] C. Majenz, M. Berta, F. Dupuis, R. Renner, and M. Christandl, "Catalytic decoupling of quantum information," *Phys. Rev. Lett.*, vol. 118, p. 080503, Feb 2017.

[33] H. Umegaki, "Conditional expectation in an operator algebra. iv. entropy and information," *Kodai Math. Sem. Rep.*, vol. 14, no. 2, pp. 59–85, 1962.

[34] M. Tomamichel, "A framework for non-asymptotic quantum information theory." PhD Thesis, ETH Zurich, http://arXiv,org/abs/1203.2142, 2012.

[35] M. Tomamichel and M. Hayashi, "A hierarchy of information quantities for finite block length analysis of quantum tasks," *IEEE Transactions on Information Theory*, vol. 59, pp. 7693–7710, Nov 2013.

[36] K. Li, "Second-order asymptotics for quantum hypothesis testing," *Ann. Statist.*, vol. 42, pp. 171–189, 02 2014.

[37] N. Datta, "Min- and max- relative entropies and a new entanglement monotone," *IEEE Transactions on Information Theory*, vol. 55, pp. 2816–2826, 2009.

[38] R. Jain, J. Radhakrishnan, and P. Sen, "Privacy and interaction in quantum communication complexity and a theorem about the relative entropy of quantum states," in *The 43rd Annual IEEE Symposium on Foundations of Computer Science, 2002. Proceedings.*, pp. 429–438, 2002.

[39] A. Gilchrist, N. K. Langford, and M. A. Nielsen, "Distance measures to compare real and ideal quantum processes," *Phys. Rev. A*, vol. 71, p. 062310, Jun 2005.

[40] A. Uhlmann, "The "transition probability" in the state space of a *-algebra," *Rep. Math. Phys.*, vol. 9, pp. 273–279, 1976.

[41] M. J. Donald and M. Horodecki, "Continuity of relative entropy of entanglement," *Physics Letters A*, vol. 264, no. 4, pp. 257 – 260, 1999.