On necessary and sufficient conditions for H_∞ output feedback control of Markov jump linear systems

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***Abstract*— This note addresses the output feedback H_∞ control problem for continuous-time Markov jump linear systems. It is shown that the feasibility of a set of linear matrix inequalities is both sufficient and necessary for the existence of a solution. Under standard assumptions, we also give a Riccati-type sufficient and necessary condition for an H_∞-suboptimal controller to exist.**

I. Introduction

The H_∞ control problem for continuous-time Markov jump linear systems (MJLS) has attracted much attention in the recent years [4], [14], [3], [15], [2]. Developments regarding applications of H_∞ control to robust control and filtering can also been found in [18], [5], [6]. The problem was initiated by [4], where both the system state and the jump variable were assumed to be available to the controller. Reference [14] related this problem to a class of zero-sum differential games, where in both finite and infinite horizon cases, a complete set of solutions was derived for both state feedback and output feedback versions of the problem. Specifically, in the output feedback case, [14] gives a sufficient and a necessary conditions to guarantee that the dynamic game considered in that paper has the zero upper value. However, a gap exists between those conditions. Also, in the output feedback case, stabilizing properties of the resulting controller were not addressed. The reference [3] also studied the output feedback H_∞ control problem in terms of linear matrix inequalities (LMIs), while only a sufficient condition was provided for the existence of a solution.

In this note the output feedback H_∞ control problem for MJLS is addressed under the assumption that the jump variable is perfectly known to the controller, and the initial state of the plant is not subject to uncertainty. Our objective is to obtain tight conditions for the existence of a suboptimal stabilizing output feedback H_∞ controller which are both necessary and sufficient. A result extending the celebrated Strict Bounded Real Lemma [16] to the realm of jump parameter systems plays an instrumental role in the derivation of our result. This result has been established in [9] as a corollary from a more general statement. We present a direct and simpler proof of the Strict Bounded Real Lemma for continuous-time MJLS. Based on this lemma, it is proved that the feasibility of the set of LMIs introduced in the sufficiency result of [3] is also necessary for the corresponding suboptimal output feedback H_∞ control problem to have a solution. Under standard assumptions, we give a Riccati-type sufficient and necessary condition, which complements the sufficient condition derived in [14] in that it provides a stabilizing solution.

This note is organized as follows. In Section II the notation and preliminary results are presented. The strict bounded real lemma for continuous-time MJLS is given in Section III, as a counterpart to corresponding results for discrete-time MJLS [17] and deterministic systems [16]. Section IV contains our main results which address the sufficient and necessary condition for output feedback jump H_∞ controller synthesis, and the conclusion is given in Section V.

II. Preliminaries

Let $\mathcal{M}^{n,q}$ be the linear space made up of collections of s matrices, such that $\mathcal{M}^{n,q} = \{U = (U_1, \cdots, U_s) : U_i \in \mathbb{R}^{n\times q}\}$; also $\mathcal{M}^n = \mathcal{M}^{n,n}$. By $\mathcal{S}^n$ we denote the subset of $\mathcal{M}^n$ consisting of collections of symmetric matrices. For collections of matrices $U, V \in \mathcal{S}^n$, $U \geq V$ (resp. $U > V$) means that $U_i - V_i \geq 0$ (resp. $U_i - V_i > 0$), $i = 1, \cdots, s$. For instance, $U > 0$ means that $U_i > 0, i = 1, \cdots, s$. For the collections of matrices $X, Y \in \mathcal{M}^{n,q}$, $X+Y$ stands for $X+Y = (X_1+Y_1, \cdots, X_s+Y_s)$, and for $\alpha \in \mathbb{R}$, $\alpha X = (\alpha X_1, \cdots, \alpha X_s)$. Other mathematical operations on $\mathcal{M}^{n,q}$ are defined in a similar manner.

Consider the following MJLS described in a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ by

$$\mathcal{G}_0 : \begin{cases} \dot{x}(t) = A(\eta(t))x(t) + E(\eta(t))w(t), \\ z(t) = C_1(\eta(t))x(t), \qquad t \geq 0, \\ w \in L_2^m, \quad x(0) = x_0 \in R^n, \quad \eta(0) = \eta_0, \end{cases} \tag{1}$$

where $\mathcal{F}_t$ is the filteration generated by a continuous-time homogeneous stationary Markov chain $\{\eta(t), t \geq 0\}$ taking values in a finite set $\mathbb{S} = \{1, 2, \cdots, s\}$; the chain $\eta(t)$ is assumed to have a positive stationary initial distribution $\pi = [\pi_1, \cdots, \pi_k]$, i.e., $\pi_j > 0, \forall j \in \mathbb{S}$. Also, let $\Lambda = [\lambda_{ij}]_{s\times s}$ be the transition rate matrix of this process, with $\lambda_{ij} \geq 0, i \neq j$ and $\lambda_{ii} = -\sum_{j\neq i} \lambda_{ij} \leq 0, \forall i \in \mathbb{S}$, such that transition probabilities of the system mode variable $\eta(t)$ satisfy

$$\mathcal{P}\{\eta(t+\Delta) = j | \eta(t) = i\} = \begin{cases} \lambda_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \lambda_{ii}\Delta + o(\Delta) & \text{if } i = j. \end{cases}$$

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L_2^m denotes the Hilbert space of $\mathcal{F}_t$-measurable m-dimensional stochastic processes $v(t)$, such that

$$\|v\|_2^2 := \int_0^\infty \mathbf{E}\{|v(t)|^2\}dt < \infty;$$

Here, $\mathbf{E}$ denotes the expectation.

Whenever $\eta(t) = i \in \mathbb{S}$, we assume $A(\eta(t)) = A_i$; the matrices $A_i, i = 1, \cdots, s$, belong to a given collection of matrices $A = (A_1, \cdots, A_s) \in \mathcal{M}^n$. In a similar fashion, collections of matrices $E \in \mathcal{M}^{n,m}, C_1 \in \mathcal{M}^{p,n}$ are associated with the coefficients $E(\eta(t))$, $C_1(\eta(t))$, respectively.

Recall the definitions of the notions of mean square (MS) stability and MS stabilization.

Definition 1: [12], [7] The system $\mathcal{G}_0$ is said to be internally MS-stable, or equivalently, the pair (A, Λ) is said to be MS-stable, if the solution to equation (1) corresponding to $w \equiv 0$ has the property that $\lim_{t\to\infty} \mathbf{E}\{|x(t)|^2|\eta_0\} = 0$ for all $x_0 \in \mathbb{R}^n$ and $\eta_0 \in \mathbb{S}$.

In the above definition, $\mathbf{E}\{\cdot|\eta_0\}$ denotes the conditional expectation given $\eta(0) = \eta_0$. Clearly, MS-stability of the system $\mathcal{G}_0$ also implies that $\lim_{t\to\infty} \mathbf{E}\{|x(t)|^2\} = 0$ for all $x_0 \in \mathbb{R}^n$. Since $\pi_i > 0$, the two stability properties are equivalent.

Definition 2: [13], [7] The system $\mathcal{G}_0$ is said to be MS-stabilizable, or equivalently, the triplet (A, E, Λ) is said to be MS-stabilizable, if there exists $K \in \mathcal{M}^{m,n}$ such that the pair $(A + EK, \Lambda)$ is MS-stable.

The following propositions provide conditions to check MS-stability. Proposition 3 can be derived from [11], and similar results can also be found in [13], [3], [1]. Proposition 4 can be derived from the results of [7], [10].

Proposition 3: The following statements are equivalent:

(i). (A, Λ) is MS-stable.

(ii). The LMIs

$$A_i'P_i + P_iA_i + \sum_{j=1}^s \lambda_{ij}P_j < 0, \qquad i = 1, \cdots, s$$

are feasible for some $P > 0$ in $\mathcal{S}^n$.

(iii). For any given $T \in \mathcal{S}^n$, $T > 0$ (resp. $T \geq 0$), there exists a unique $P \in \mathcal{S}^n$ such that $P > 0$ (resp. $P \geq 0$) and

$$A_i'P_i + P_iA_i + \sum_{j=1}^s \lambda_{ij}P_j + T_i = 0, \qquad i = 1, \cdots, s. \quad (2)$$

Proposition 4: If (2) holds for some $P > 0$ and $T \geq 0$ in $\mathcal{S}^n$ and (A_i, T_i) is observable for all $i \in \mathbb{S}$, then the pair (A, Λ) is MS-stable.

The above result holds under a weaker assumption that (A, T, Λ) is W-detectable; see [7, Proposition 8]. However, we assume a stronger property that each pair (A_i, T_i) is observable, since this stronger assumption is consistent with other results on jump parameter systems used in this paper. Note that the observability of each matrix pair (A_i, T_i) implies the W-detectability of (A, T, Λ), see [7] for details. Another weaker version of Proposition 4 can be found in [10, Theorem 7.2] in the context of stochastically uniform observability.

Definition 5: [4] The H_∞-norm of an internally MS-stable system $\mathcal{G}_0$, $\|\mathcal{G}_0\|_\infty$, is the infimum of $\gamma > 0$, such that

$$\|z\|_2 < \gamma\|w\|_2$$

holds for all $w \in L_2^m$ with $w \not\equiv 0$ and $x_0 = 0$.

III. The Strict Bounded Real Lemma for Markov Jump Systems

In this section, we present a strict bounded real lemma for continuous-time MJLS. In [9], a more general version of this statement was derived for systems subjected both to multiplicative white noise disturbances and Markovian parameter jumps. Here we give a new and direct proof of the version relevant to systems of the form (1). We first prove the following preliminary lemma.

Lemma 6: If (A, Λ) is MS-stable, then there exist $\Delta > 0 \in \mathcal{S}^n$ such that

$$A_i'\Delta_i + \Delta_iA_i + \Delta_iE_iE_i'\Delta_i + \sum_{j=1}^s \lambda_{ij}\Delta_j < 0$$

for all $i \in \mathbb{S}$.

Proof: Since (A, Λ) is MS-stable, by Proposition 3 there exist $\tilde{\Delta}_i > 0, i = 1, \cdots, s$, such that

$$A_i'\tilde{\Delta}_i + \tilde{\Delta}_iA_i + \sum_{j=1}^s \lambda_{ij}\tilde{\Delta}_j + Q_i = 0,$$

where $Q_i > 0, i = 1, \cdots, s$, are given matrices. We can find a sufficiently small $\varepsilon > 0$ such that

$$\varepsilon\tilde{\Delta}_iE_iE_i'\tilde{\Delta}_i < Q_i, \quad i = 1, \cdots, s.$$

Letting $\Delta_i = \varepsilon\tilde{\Delta}_i$ completes the proof. ■

Theorem 7 (Strict Bounded Real Lemma): The following statements are equivalent.

(i). $\mathcal{G}_0$ is internally MS-stable and $\|\mathcal{G}_0\|_\infty < \gamma$.

(ii). The coupled algebraic Riccati equations

$$\begin{aligned} A_i'P_i + P_iA_i &+ \sum_{j=1}^s \lambda_{ij}P_j \\ &+ \gamma^{-2}P_iE_iE_i'P_i + C_{1i}'C_{1i} = 0, \quad (3) \\ &i = 1, \cdots, s \end{aligned}$$

have a stabilizing solution $P \geq 0$ in $\mathcal{S}^n$, i.e., $(A + \gamma^{-2}EE'P, \Lambda)$ is MS-stable.

(iii). The coupled algebraic Riccati inequalities

$$\begin{aligned} A_i'\tilde{P}_i + \tilde{P}_iA_i &+ \sum_{j=1}^s \lambda_{ij}\tilde{P}_j \\ &+ \gamma^{-2}\tilde{P}_iE_iE_i'\tilde{P}_i + C_{1i}'C_{1i} < 0, \quad (4) \\ &i = 1, \cdots, s \end{aligned}$$

have a positive solution $\tilde{P} > 0$ in $\mathcal{S}^n$.

Furthermore, if either of these statements holds then $\tilde{P} > P$.

Proof: $(i) \Rightarrow (ii)$: This statement is a special case of [14, Theorem 3.2] without the assumption of observability of $(A_i, C_{1i}'C_{1i})$ [1] for each $i \in \mathbb{S}$.

[1]This observability assumption is only needed in [14] to show that P is positive definite. Since in (ii) we claim a weaker condition $P \geq 0$, the observability is not needed in our case.

$(ii) \Rightarrow (iii)$: Since $P \geq 0$ is the stabilizing solution of (3), by Lemma 6, there exist $\Delta_i > 0, i = 1, \cdots, s$, such that

$$(A_i + \gamma^{-2}E_iE_i'P_i)'\Delta_i + \Delta_i(A_i + \gamma^{-2}E_iE_i'P_i) + \gamma^{-2}\Delta_iE_iE_i'\Delta_i + \sum_{j=1}^{s}\lambda_{ij}\Delta_j < 0 \tag{5}$$

for all $i \in \mathbb{S}$. After adding (5) and (3), we have

$$A_i'(P_i+\Delta_i) + (P_i+\Delta_i)A_i + \sum_{j=1}^{s}\lambda_{ij}(P_j+\Delta_j) + \gamma^{-2}(P_i+\Delta_i)E_iE_i'(P_i+\Delta_i) + C_{1i}'C_{1i} < 0.$$

Therefore, $\tilde{P}_i = P_i + \Delta_i > 0, i = 1, \cdots, s$, satisfy (4).

$(iii) \Rightarrow (i)$: (A, Λ) is MS-stable by Proposition 3. We conclude from (4) that there exist $\tilde{P}_i > 0$, $i = 1, \cdots, s$, such that

$$A_i'\tilde{P}_i + \tilde{P}_iA_i + \sum_{j=1}^{s}\lambda_{ij}\tilde{P}_j + \gamma^{-2}\tilde{P}_iE_iE_i'\tilde{P}_i + C_{1i}'C_{1i} + \tilde{Q}_i = 0, \quad i = 1, \cdots, s,$$

where $\tilde{Q}_i > 0$. Define

$$\tilde{z}(t) = \begin{bmatrix} z(t) \\ \tilde{Q}^{\frac{1}{2}}(\eta(t))x(t) \end{bmatrix},$$

where $\tilde{Q}(\eta(t)) = \tilde{Q}_i$, when $\eta(t) = i$, and $z(t)$, $x(t)$ are from $\mathcal{G}_0$. Since $(A_i, C_{1i}'C_{1i} + \tilde{Q}_i)$ is observable for each $i \in \mathbb{S}$, hence it follows from [4, Theorem 3.2] that for all $w \in L_2^m$ with $w \not\equiv 0$ and $x_0 = 0$, $\gamma\|w\|_2 > \|\tilde{z}\|_2 > \|z\|_2$, thus $\|\mathcal{G}_0\|_\infty < \gamma$.

In order to complete the proof, we now show that $\tilde{P} > P$ provided that statements (i)-(iii) hold. By subtracting (4) from (3), we have

$$(A_i + \gamma^{-2}E_iE_i'P_i)'\tilde{\Delta}_i + \tilde{\Delta}_i(A_i + \gamma^{-2}E_iE_i'P_i) + \gamma^{-2}\tilde{\Delta}_iE_iE_i'\tilde{\Delta}_i + \sum_{j=1}^{s}\lambda_{ij}\tilde{\Delta}_j < 0, \quad i = 1, \cdots, s,$$

where $\tilde{\Delta}_i = \tilde{P}_i - P_i, i \in \mathbb{S}$. Since statement (ii) ascertains that $(A + \gamma^{-2}EE'P, \Lambda)$ is MS-stable, by Proposition 3, $\tilde{P}_i > P_i, i \in \mathbb{S}$. ■

IV. The Main Results

Consider the following linear control system,

$$\mathcal{G}: \begin{cases} \dot{x}(t) = A(\eta(t))x(t) + B(\eta(t))u(t) + E(\eta(t))w(t), \\ z(t) = C_1(\eta(t))x(t) + D_1(\eta(t))u(t), \\ y(t) = C_2(\eta(t))x(t) + D_2(\eta(t))w(t), \qquad t \geq 0 \\ w \in L_2^m,\ x(0) = x_0 \in R^n,\ \eta(0) = \eta_0, \end{cases} \tag{6}$$

where $A \in \mathcal{M}^n, E \in \mathcal{M}^{n,m}, C_1 \in \mathcal{M}^{p,n}$ are as defined before, and $B \in \mathcal{M}^{n,r}, D_1 \in \mathcal{M}^{p,r}, C_2 \in \mathcal{M}^{q,n}, D_2 \in \mathcal{M}^{q,m}$ are defined in a similar fashion. The H_∞ control problem considered in this paper is to find an internally MS-stabilizing controller (i.e., such that the closed-loop system in which $w \equiv 0$ is MS-stable) of the following form

$$\mathcal{G}_c: \begin{cases} \dot{x_c}(t) = A_c(\eta(t))x(t) + B_c(\eta(t))y(t), \\ u(t) = K_c(\eta(t))x_c(t), \end{cases} \tag{7}$$

such that $\|\mathcal{G}_{cl}\|_\infty < \gamma$; here $\mathcal{G}_{cl} = (\tilde{A}, \tilde{E}, \tilde{C})$ denotes the closed loop system mapping w to z, and

$$\tilde{A}_i = \begin{bmatrix} A_i & B_iK_{ci} \\ B_{ci}C_{2i} & A_{ci} \end{bmatrix}, \quad \tilde{E}_i = \begin{bmatrix} E_i \\ B_{ci}D_{2i} \end{bmatrix},$$
$$\tilde{C}_i = \begin{bmatrix} C_{1i} & D_{1i}K_{ci} \end{bmatrix}, \qquad i = 1, \cdots, s.$$

Theorem 8: The following statements are equivalent.

(i). There exists an output feedback controller $\mathcal{G}_c$ of the form (7) such that the corresponding closed-loop system $\mathcal{G}_{cl}$ is internally MS-stable and $\|\mathcal{G}_{cl}\|_\infty < \gamma$.

(ii). The following LMIs with variables $X \in \mathcal{S}^n$, $Y \in \mathcal{S}^n$, $F \in \mathcal{M}^{r,n}$, $L \in \mathcal{M}^{n,q}$ are feasible:

$$\begin{bmatrix} V_i(X,L) & X_iE_i + L_iD_{2i} \\ E_i'X_i + D_{2i}'L_i' & -\gamma^2 I \end{bmatrix} < 0, \tag{8a}$$

$$\begin{bmatrix} W_i(Y,F) & (C_{1i}Y_i + D_{1i}F_i)' & R_i(Y) \\ C_{1i}Y_i + D_{1i}F_i & -I & 0 \\ R_i'(Y) & 0 & S_i(Y) \end{bmatrix} < 0, \tag{8b}$$

$$\begin{bmatrix} Y_i & I \\ I & X_i \end{bmatrix} > 0, \tag{8c}$$

$$i = 1, \cdots, s,$$

where

$$V_i(X,L) = A_i'X_i + X_iA_i + L_iC_{2i} + C_{2i}'L_i' + C_{1i}'C_{1i} + \sum_{j=1}^{s}\lambda_{ij}X_j,$$
$$W_i(Y,F) = A_iY_i + Y_iA_i' + B_iF_i + F_i'B_i' + \lambda_{ii}Y_i + \gamma^{-2}E_iE_i',$$
$$R_i(Y) = [\sqrt{\lambda_{i1}}Y_i \cdots \sqrt{\lambda_{i(i-1)}}Y_i, \sqrt{\lambda_{i(i+1)}}Y_i \cdots \sqrt{\lambda_{is}}Y_i],$$
$$S_i(Y) = -\mathrm{diag}(Y_1 \cdots Y_{i-1}, Y_{i+1} \cdots Y_s).$$

Moreover, if (8) is feasible, the corresponding controller $\mathcal{G}_c$ is given by

$$K_{ci} = F_iY_i^{-1},$$
$$B_{ci} = (Y_i^{-1} - X_i)^{-1}L_i,$$
$$A_{ci} = (Y_i^{-1} - X_i)^{-1}M_iY_i^{-1}, \tag{9}$$

where

$$M_i = -A_i' - X_iA_iY_i - X_iB_iF_i - L_iC_{2i}Y_i - C_{1i}'(C_{1i}Y_i + D_{1i}F_i) - \gamma^{-2}(X_iE_i + L_iD_{2i})E_i' - \sum_{j=1}^{s}\lambda_{ij}Y_j^{-1}Y_i.$$

Proof: From [3, Theorem 4.2], claim (ii) is equivalent to the fact that there exists an output feedback controller $\mathcal{G}_c$ of the form (7) such that the coupled Riccati inequalities

$$\tilde{A}_i'P_i + P_i\tilde{A}_i + \sum_{j=1}^{s}\lambda_{ij}P_j + \gamma^{-2}P_i\tilde{E}_i\tilde{E}_i'P_i + \tilde{C}_i'\tilde{C}_i < 0, \quad i = 1, \cdots, s$$

have a positive solution $P > 0$ in $\mathcal{S}^n$. The latter fact is equivalent to claim (i) by Theorem 7. ■

Theorem 8 gives a sufficient and necessary condition in the LMI form for the H_∞ output feedback control problem for MJLS (6), (7) to have a suboptimal solution. The sufficiency part of this theorem was established in [3]. In what follows we will give the Riccati-type sufficient and necessary condition under standard assumptions, as a counterpart of the well known deterministic result [8]. To proceed, the following assumptions of reference [14] for the system (6) are made, which are usual in the Riccati approach.

Assumption 9: For all $i \in \mathbb{S}$, $N_i := D_{2i}D_{2i}' > 0$, $E_iD_{2i}' = 0$.

Assumption 10: For all $i \in \mathbb{S}$, $R_i := D_{1i}'D_{1i} > 0$, $C_{1i}'D_{1i} = 0$.

Assumption 11: (A, B, Λ) is MS-stabilizable.

Assumption 12: The pair $(A_i, C_{1i}'C_{1i})$ is observable for each $i \in \mathbb{S}$.

Introduce the following set of coupled generalized algebraic Riccati equations (GAREs), generalized algebraic Riccati inequalities (GARIs) and the coupling conditons:

$$A_i'Z_i + Z_iA_i - Z_i(B_iR_i^{-1}B_i' - \gamma^{-2}E_iE_i')Z_i + C_{1i}'C_{1i} + \sum_{j=1}^{s} \lambda_{ij}Z_j = 0, \qquad (10)$$

$$A_i'Z_i + Z_iA_i - Z_i(B_iR_i^{-1}B_i' - \gamma^{-2}E_iE_i')Z_i + C_{1i}'C_{1i} + \sum_{j=1}^{s} \lambda_{ij}Z_j < 0, \qquad (11)$$

$$A_i'\Theta_i + \Theta_iA_i + \Theta_iE_iE_i'\Theta_i - C_{2i}'N_i^{-1}C_{2i} + \gamma^{-2}C_{1i}'C_{1i} + \sum_{j=1}^{s} \lambda_{ij}\Theta_j < 0, \qquad (12)$$

$$\gamma^2\Theta_i - Z_i > 0 \qquad (13)$$

$$i = 1, \cdots, s.$$

Theorem 13: Under Assumptions 9-10, there exists an output feedback controller $\mathcal{G}_c$ (7) such that the closed-loop system $\mathcal{G}_{cl}$ is internally MS-stable and $\|\mathcal{G}_{cl}\|_\infty < \gamma$ if and only if the coupled GARIs (11) admit a positive definite solution $Z > 0$ in $\mathcal{S}^n$, and the coupled GARIs (12) admit a positive definite solution $\Theta > 0$ in $\mathcal{S}^n$, such that (13) holds for all $i \in \mathbb{S}$. Moreover, if these conditions hold, the corresponding controller $\mathcal{G}_c$ is given by

$$\begin{aligned} K_{ci} &= -R_i^{-1}B_i'Z_i, \\ B_{ci} &= \gamma^2(\gamma^2\Theta_i - Z_i)^{-1}C_{2i}'N_i^{-1}, \\ A_{ci} &= A_i - (B_iR_i^{-1}B_i' - \gamma^{-2}E_iE_i')Z_i - B_{ci}C_{2i} \\ &\quad + (\gamma^2\Theta_i - Z_i)^{-1}R_Z(i), \end{aligned}$$

where $R_Z(i)$ denotes the matrix on the left-hand side of (11).

Proof: Note that by Schur complement and Assumption 9, (8a) is equivalent to

$$A_i'\hat{\Theta}_i + \hat{\Theta}_iA_i + \hat{\Theta}_iE_iE_i'\hat{\Theta}_i - C_{2i}'N_i^{-1}C_{2i} + \gamma^{-2}C_{1i}'C_{1i} + \sum_{j=1}^{s} \lambda_{ij}\hat{\Theta}_j + (C_{2i} + \gamma^{-2}N_iL_i')'N_i^{-1}(C_{2i} + \gamma^{-2}N_iL_i') < 0; \qquad (14)$$

here we define $\hat{\Theta}_i = \gamma^{-2}X_i$. Similarly, defining $\hat{Z}_i = Y_i^{-1}$, under Assumption 10, (8b) is equivalent to

$$A_i'\hat{Z}_i + \hat{Z}_iA_i - \hat{Z}_i(B_iR_i^{-1}B_i' - \gamma^{-2}E_iE_i')\hat{Z}_i + C_{1i}'C_{1i} + \sum_{j=1}^{s} \lambda_{ij}\hat{Z}_j + \hat{Z}_i(R_iF_i + B_i')'R_i^{-1}(R_iF_i + B_i')\hat{Z}_i < 0, \qquad (15)$$

Also, (8c) becomes

$$\gamma^2\hat{\Theta}_i - \hat{Z}_i > 0, \qquad (16)$$

If (11), (12), (13) admit some solutions $\Theta_i > 0$, $Z_i > 0$, then $\hat{\Theta}_i = \Theta_i$, $\hat{Z}_i = Z_i$, $L_i = -\gamma^2C_{2i}'N_i^{-1}$, $F_i = -R_i^{-1}B_i'$ satisfy (14), (15), (16). By Theorem 8, the sufficiency is verified. The controller is obtained by substituting these parameters into (9).

The necessity part follows directly from Theorem 8 and (14), (15), (16). This completes the proof. ■

In the special case of state feedback control, where the controller has full access to the state of the system, we have $C_2(\eta(t)) = I, D_2(\eta(t)) = 0$ in (6). In this case, we only need condition (8b) of Theorem 8, and conditions (8a) and (8c) do not arise. Then the following statements hold in this special case.

Corollary 14: There exists a state feedback controller $\mathcal{G}_{sc}$ such that the closed-loop system $\mathcal{G}_{cl}$ is internally MS-stable and $\|\mathcal{G}_{cl}\|_\infty < \gamma$ if and only if LMIs (8b) with variables $Y \in \mathcal{S}^n$, $F \in \mathcal{M}^{r,n}$ are feasible.

Corollary 15: Under Assumption 10, there exists a state feedback controller $\mathcal{G}_{sc}$ such that the closed-loop system $\mathcal{G}_{cl}$ is internally MS-stable and $\|\mathcal{G}_{cl}\|_\infty < \gamma$ if and only if the coupled GARIs (11) admit a positive definite solution $Z > 0$ in $\mathcal{S}^n$.

Corollary 16: Under Assumptions 10-12, the following statements are equivalent.

(i). There exists a state feedback controller $\mathcal{G}_{sc}$ such that the closed-loop system $\mathcal{G}_{cl}$ is internally MS-stable and $\|\mathcal{G}_{cl}\|_\infty < \gamma$.

(ii). The coupled GAREs (10) admit a positive definite solution $Z > 0$ in $\mathcal{S}^n$.

(iii). The coupled GAREs (10) admit a minimal positive definite solution $Z^{(*)} > 0$ in $\mathcal{S}^n$.

(iv). The coupled GARIs (11) admit a positive definite solution $\tilde{Z} > 0$ in $\mathcal{S}^n$.

Moreover, if either of these statements holds, the following facts hold true.

- The pair $(A - BR^{-1}B'\Omega, \Lambda)$ is MS-stable, where $\Omega = Z$, $Z^{(*)}$, or $\tilde{Z}$, respectively.
- The pair $(A - (BR^{-1}B' - \gamma^{-2}EE')Z^{(*)}, \Lambda)$ is MS-stable.
- $\tilde{Z} > Z^{(*)}$, and there exists a non-increasing sequence $\{\tilde{Z}^{(n)}\}$ of strict positive definite matrices, where $\tilde{Z}^{(n)} \in \mathcal{S}^n$ satisfies (11), converging to $Z^{(*)}$.

Proof: The implication $(i) \Rightarrow (iii)$ is proved in [14, Theorem 3.2]. The implication $(iii) \Rightarrow (ii)$ is obvious. The implication $(ii) \Rightarrow (i)$ is proved in [4, Theorem 3.2]. The fact that claims (i) and (iv) are equivalent follows from Corollary 15.

Suppose that either of claims $(i)-(iv)$ holds, then all of them hold as we have shown. The MS-stability of $(A-BR^{-1}B'\Omega,\Lambda)$, where $\Omega=Z, Z^{(*)}$, or $\tilde{Z}$, respectively, is proved in [4, Theorem 3.2]. The MS-stability of $(A-(BR^{-1}B'-\gamma^{-2}EE')Z^{(*)},\Lambda)$ is proved in [14, Theorem 3.2]. The fact that $\tilde{Z}>Z^{(*)}$ and the existence of the non-increasing convergent sequence $\{\tilde{Z}^{(n)}\}$ follow from the proof of [14, Corollary 3.1]. ■

We are now in a position to present the main results of this paper.

Theorem 17: Under Assumption 9-12, the following statements are equivalent.

(i). There exists an output feedback controller $\mathcal{G}_c$ (7) such that the closed-loop system $\mathcal{G}_{cl}$ is internally MS-stable and $\|\mathcal{G}_{cl}\|_\infty<\gamma$.

(ii). GAREs (10) admit a positive definite solution $Z>0$ in $\mathcal{S}^n$, and GARIs (12) admit a positive definite solution $\Theta>0$ in $\mathcal{S}^n$, such that (13) holds for all $i\in\mathbb{S}$.

(iii). GAREs (10) admit a minimal positive definite solution $Z>0$ in $\mathcal{S}^n$, and GARIs (12) admit a positive definite solution $\Theta>0$ in $\mathcal{S}^n$, such that (13) holds for all $i\in\mathbb{S}$.

Moreover, if either of these statements holds, the following controller solves the H_∞ control problem under consideration:

$$\begin{aligned}\dot{x}_c(t)=&[A(\eta(t))-(B(\eta(t))R^{-1}(\eta(t))B'(\eta(t))\\&-\gamma^{-2}E(\eta(t))E'(\eta(t)))Z(\eta(t))]x_c(t)\\&+\gamma^2[\gamma^2\Theta(\eta(t))-Z(\eta(t))]^{-1}C_2'(\eta(t))N^{-1}(\eta(t))\\&\cdot[y(t)-C_2(\eta(t))x_c(t)],\\u(t)=&-R^{-1}(\eta(t))B'(\eta(t))Z(\eta(t))x_c(t),\end{aligned}\tag{17}$$

where $R(\eta(t)),N(\eta(t))$, $Z(\eta(t)),\Theta(\eta(t))$ are defined on $[0,+\infty)$ as:

$$\begin{aligned}&R(\eta(t)):=R_i,\quad N(\eta(t)):=N_i,\\&Z(\eta(t)):=Z_i,\quad \Theta(\eta(t)):=\Theta_i,\quad \text{if } \eta(t)=i.\end{aligned}$$

Proof: $(i)\Rightarrow(iii)$: From Theorem 13, (11), (12), (13) admit some solutions $\Theta>0, Z>0$ in $\mathcal{S}^n$. This further implies that the GAREs (10) admit a minimal positive definite solution $Z^{(*)}>0$ in $\mathcal{S}^n$ by Corollary 16. Then $\gamma^2\Theta>Z>Z^{(*)}$. Therefore $\Theta,Z^{(*)}$ verify (10), (12), (13).

The implication $(iii)\Rightarrow(ii)$ is obvious.

$(ii)\Rightarrow(i)$: Suppose that (10), (12), (13) admit some solutions $Z_i>0$, $\Theta_i>0$ in $\mathcal{S}^n$. Consider the controller (17) and write the closed loop system as

$$\begin{aligned}\dot{\tilde{x}}(t)=&\begin{bmatrix}A(\eta(t)) & B(\eta(t))K_c(\eta(t))\\ B_c(\eta(t))C_2(\eta(t)) & A_c(\eta(t))\end{bmatrix}\tilde{x}(t)\\&+\begin{bmatrix}E(\eta(t))\\ B_c(\eta(t))D_2(\eta(t))\end{bmatrix}w(t)\\ \triangleq& A_{cl}(\eta(t))\tilde{x}(t)+B_{cl}(\eta(t))w(t),\\ z(t)=&\begin{bmatrix}C_1(\eta(t)) & D_1(\eta(t))K_c(\eta(t))\end{bmatrix}\tilde{x}(t),\end{aligned}\tag{18}$$

where $\tilde{x}=(x',x_c')'$ and

$$\begin{aligned}&\Psi(\eta(t))=\gamma^2\Theta(\eta(t))-Z(\eta(t)),\\&K_c(\eta(t))=-R^{-1}(\eta(t))B'(\eta(t))Z(\eta(t)),\\&B_c(\eta(t))=\gamma^2\Psi^{-1}(\eta(t))C_2'(\eta(t))N^{-1}(\eta(t)),\\&A_c(\eta(t))=A(\eta(t))-[B(\eta(t))R^{-1}(\eta(t))B'(\eta(t))\\&\qquad-\gamma^{-2}E(\eta(t))E'(\eta(t))]Z(\eta(t))-B_c(\eta(t))C_2(\eta(t)).\end{aligned}$$

Introducing

$$\Sigma_i=\begin{bmatrix}\gamma^2\Theta_i & -\Psi_i\\ -\Psi_i & \Psi_i\end{bmatrix},\qquad i\in\mathbb{S},$$

it can be shown that the matrices $\Sigma_i, i\in\mathbb{S}$ satisfy the following GAREs:

$$\begin{aligned}&\Sigma_iA_{cl}(i)+A_{cl}'(i)\Sigma_i+\sum_{j=1}^{s}\lambda_{ij}\Sigma_j\\&\qquad+\gamma^{-2}\Sigma_iB_{cl}(i)B_{cl}'(i)\Sigma_i+\Gamma_i=0,\end{aligned}\tag{19}$$

where

$$\Gamma_i=\begin{bmatrix}C_{1i}'C_{1i} & 0\\ 0 & Z_iB_iR_i^{-1}B_i'Z_i\end{bmatrix}+\begin{bmatrix}-R_\Theta(i) & R_\Theta(i)\\ R_\Theta(i) & -R_\Theta(i)\end{bmatrix},\tag{20}$$

and $R_\Theta(i)$ is the matrix on left-hand side of (12). Now we will show that

$$(A_{cl}(i),\Gamma_i)\tag{21}$$

is observable for each $i\in\mathbb{S}$.

Suppose there exist $k\in\mathbb{S}$, a complex scalar α and a vector $\tilde{y}=(y',y_c')'\in\mathbb{R}^{2n}$ such that

$$(A_{cl}(k)-\alpha I)\tilde{y}=0,\tag{22}$$

$$\Gamma_i\tilde{y}=0.\tag{23}$$

Since the two entries of Γ_i in (20) are nonnegative definite, we have

$$\begin{bmatrix}C_{1k}'C_{1k} & 0\\ 0 & Z_kB_kR_k^{-1}B_k'Z_k\end{bmatrix}\tilde{y}=0,\tag{24}$$

$$\begin{bmatrix}-R_\Theta(k) & R_\Theta(k)\\ R_\Theta(k) & -R_\Theta(k)\end{bmatrix}\tilde{y}=0.\tag{25}$$

Thus, from (24),

$$C_{1k}'C_{1k}\ y=0,\tag{26}$$

$$B_k'Z_k\ y_c=0.\tag{27}$$

Substituting (27) into (22), we obtain

$$(A_k-\alpha I)y=0.\tag{28}$$

Combining (28), (26) with Assumption 12, we must have $y=0$. Then $R_\Theta(k)y_c=0$ follows from (25), which leads to $y_c=0$ by the fact that $R_\Theta(k)<0$. This proves the observability of the pair in (21) for all $i\in\mathbb{S}$.

Now back to equation (19). Since each Σ_i is positive definite and the observability of the pair (21) for all $i\in\mathbb{S}$ indicates that the pair

$$\left(A_{cl}(i),\gamma^{-2}\Sigma_iB_{cl}(i)B_{cl}'(i)\Sigma_i+\Gamma_i\right)$$

is observable for all $i \in \mathbb{S}$, then (A_{cl}, Λ) is MS-stable by Proposition 4.

Finally, since (A_{cl}, Λ) is MS-stable and the pair in (21) is observable for all $i \in \mathbb{S}$, hence it follows from [4, Theorem 3.2] that $\|\mathcal{G}_{cl}\|_\infty < \gamma$; see also proof of Theorem 7, part $[(iii) \Rightarrow (i)]$. ∎

V. Conclusion

In this paper, we have explored the infinite horizon output feedback H_∞ control problem for continuous-time MJLS. As noted in the introduction, there is a gap between the existing sufficiency and necessity results on this control problem presented in [14]. Specifically, the necessity result presented in [14] is formulated in terms of a set of coupled backward generalized Riccati differential equations (GRDEs). Finding a bounded solution of such a set of differential equation on an infinite time interval is a much more complicated task than testing the feasibility of (12). Instead, for the case of perfectly known initial condition, our result in Theorem 17 utilizes GARIs (12) and is both sufficient and necessary for the existence of an H_∞ suboptimal controller. That is, our result is tighter, and it also provides an easier way to check whether a solution to the problem exists.

Also note that the sufficient condition presented in [14, Theorem 4.3] only ensures that the upper value of the zero-sum game is zero. Stabilizing properties of the minimax control strategy were not addressed. By using strict inequalities in (12) rather than non-strict inequalities in [14], we have proved that the suboptimal controller (17) is actually stabilizing, which completes the H_∞ theory of continuous MJPS.

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