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Bonferroni Weighted Logarithmic Averaging Distance Operator Applied to Investment Selection Decision Making

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Citation: Alfaro-Garcia, V.G.; Blanco-Mesa, F.; León-Castro, E.; Merigo, J.M. Bonferroni Weighted Logarithmic Averaging Distance Operator Applied to Investment Selection Decision Making. *Mathematics* **2022**, *10*, 2100. https://doi.org/10.3390/math10122100

Academic Editors: José Álvarez-García, María de la Cruz del Río-Rama and Mar Arenas-Parra

Received: 18 May 2022
Accepted: 14 June 2022
Published: 16 June 2022

Abstract: Distance measures in ordered weighted averaging (OWA) operators allow the modelling of complex decision making problems where a set of ideal values or characteristics are required to be met. The objective of this paper is to introduce extended distance measures and logarithmic OWA-based decision making operators especially designed for the analysis of financial investment options. Based on the immediate weights, Bonferroni means and logarithmic averaging operators, in this paper we introduce the immediate weights logarithmic distance (IWLD), the immediate weights ordered weighted logarithmic averaging distance (IWOWLAD), the hybrid weighted logarithmic distance (HWLD), the Bonferroni ordered weighted logarithmic averaging distance (B-OWLAD) operator, the Bonferroni immediate weights ordered weighted logarithmic averaging distance (B-IWOWLAD) operator and the Bonferroni hybrid weighted logarithmic distance (HWLD). A financial decision making illustrative example is proposed, and the main benefits of the characteristic design of the introduced operators is shown, which include the analysis of the interrelation between the modelled arguments required from the decision makers and the stakeholders, and the comparison to an ideal set of characteristics that the possible companies in the example must portray. Moreover, some families, particular cases and brief examples of the proposed operators, are studied and presented. Finally, among the main advantages are the modeling of diverse perspectives, attitudinal characteristics and complex scenarios, through the interrelation and comparison between the elements with an ideal set of characteristics given by the decision makers and a set of options.

Keywords: logarithmic averaging operators; distance measures; immediate weights; Bonferroni means; OWA operators

MSC: 03B52; 90B50; 47S40

1. Introduction

The rapid advancement of fuzzy methods for analyzing and solving different analysis problems highlights applications in decision making processes in highly changing and complex environments [1–3]. Among the methods, it can highlight the characteristic objects method (COMET), which allows one to determine the preferences of each alternative considering the distance between the objects to be evaluated [4–6]. In addition, the TOPSIS method provides solutions to linear cases, since it allows one to rank the alternatives by considering two reference points of the positive ideal solution and the negative ideal solution, simultaneously [7,8]. Following the same idea, an extension called the DARIA-TOPSIS method has been proposed, which provides aggregated efficiency results of the performance of the evaluated alternatives, taking into account the dynamics of the changes over the time interval under investigation [9]. Likewise, the Stable Preference Ordering Towards Ideal Solution (SPOTIS) method allows preference ordering, established from the scoring matrix of the MCDM problem, considering the comparisons with respect to the chosen ideal solution by transforming the original incomplete problem into a well-defined one by specifying the minimum and maximum bounds of each criterion involved in the problem [10]. Finally, within these methods for decision making, aggregation operators are considered, which offer us the obtention of a single representative value of a set of elements [11,12]. Thus, these new approaches are of great help to parameterize the criteria of the decision maker and give greater relevance to the meaning of the information than in its own measurement [13]. Among the existing aggregation operators in the literature [14], the ordered weighted average (OWA) operator [15] stands out. It has been widely accepted by the scientific community and multiple extensions and applications have been developed [16].

The newly introduced operator aggregates information combining a weighting vector and mechanism that reorders the arguments depending on diverse criteria and the attitudinal nature of the decision maker. Several extensions of aggregation operators have been proposed since the original presentation of the ordered weighted average OWA operator, many extensions have also been proposed, including induced operators [17–19], heavy and prioritized [20–22] distances [23], linguistic operators [24], moving averages [25,26], Bonferroni means [27] and logarithmic averaging operators [28,29].

Given that there is a wide range of operators and extensions, this study focuses on the proposals for extensions and applications with Bonferroni means [30,31] and the OWA operator [15] related to decision making in business management [32]. The characteristic design of the Bonferroni mean allows compensation for potential errors when dealing with multiple comparisons. The compensation is constructed with the included r and q elements, these correct possible errors when addressing multiple comparisons, thus adjusting the analyzed data set [33]. Within these proposals, distance measures applied by personnel selection and entrepreneurship are highlighted, which allow one to have a threshold in the comparison process, the distance and the weighted order reflect the importance of the argument and its ordered position; they consider the degree of importance of the information in the ordering [34]. In addition, induced variables are applied on sale forecasting and enterprise risk management, which allow one to reorder the information by using order-inducing variables to obtain the maximum and minimum operators and deal with heterogeneity and uncertainty by information asymmetry [35]. Furthermore, central tendency measures such as variance and covariance are applied on the enterprise risk management strategy and the R&D investment problem [33], which allow, on the one hand, to adjust the variance and the standard deviation depending on the behavior, attitude and intuition of the decision makers, offering the alternative of a soft variance. On the other hand, one can generate a wider representation of the possible scenarios when it is under- or over-estimating the covariance and value of the set of joint variations for which the best result is close to 1 or -1. Additionally, probability is applied to agricultural commodities price [36], which can be used to create scenarios in uncertainty using a V^i vector that intercorrelates the arguments, and a weighted and probability vector that represents the attitude and expectancy of the decision maker towards a problem.

Motivated by these studies above, this study presents a new extension to logarithmic averaging operators, which have been studied by [28] with the generalized ordered weighted logarithm averaging (GOWLA) operator and by [29] with ordered weighted logarithmic averaging distance (OWLAD). The aim of this paper is to introduce an extension of the OWA operator that combines Bonferroni means and the logarithmic aggregation operator called the Bonferroni ordered weighted logarithmic average (B-OWLA) operator. Likewise, some other extensions using distance measures, such as Bonferroni hybrid weighted logarithmic distance (B-HWLD), Bonferroni ordered weighted logarithmic average distance (B-OWLAD) and Bonferroni immediate weights OWA logarithmic distance

(B-IWOWLAD), are proposed. The novelty of the paper is the aggregation of multiple criteria capturing the interrelation between arguments and comparisons between an ideal and a real possibility using logarithmic distances. A mathematical example focused on the investing decision making problem is proposed, and different rankings are generated depending on the analyzed operator. The main observed benefits are the modelling of diverse perspectives, attitudinal characteristics and different scenarios, including the interrelation between the elements and the comparison to an ideal set of characteristics retrieved from the decision makers' requirements and the performance of the possible set of options.

The paper is structured as follows: Section 2 presents the preliminaries and foundations of the OWA operator, and the ordered weighted logarithmic average (OWLA) operator. Section 3 introduces the B-OWLA operator and its main characteristics and properties. Section 4 presents a mathematical application of investment selection decision making using the proposed operators. Finally, the conclusions of the paper are presented in Section 5.

2. Preliminaries

This section presents the foundations of this study. We examine some basic definitions of distance measures and OWA operators, Bonferroni operators, immediate weights, logarithmic averaging operators and some of their extensions.

2.1. Distance Measures

The Hamming distance [37] calculates the differences between two elements, sets or strings. For sets A and B, the weighted Hamming distance is defined as follows.

Definition 1. *A weighted Hamming distance of dimension n results from a mapping* $d_{WH}: [0,\ 1]^n \times [0,\ 1]^n \rightarrow [0,\ 1]$ *that has an associated weighing vector W of dimension n with the sum of the weights being 1 and* $w_j \in [0,1]$*, such that:*

$$d_{WH}(\langle x_1, y_1 \rangle, \ldots, \langle x_n, y_n \rangle) = \sum_{j=1}^{n} w_j |x_i - y_i|, \quad (1)$$

where x_i *and* y_i *are the ith arguments of the sets X and Y.*

2.2. OWA Operators and Other Extensions

In [15], the authors proposed the ordered weighted average operator (OWA), which allows one to aggregate information using a weighting vector and reordering mechanism according to different criteria and the attitudinal character of the decision maker.

Definition 2. *An OWA operator of n dimension results from a mapping* $OWA: R^n \rightarrow R$ *associated to a weighting vector W of n dimension, such that* $\sum_{j=1}^{n} w_j = 1$ *and* $w_j \in [0,1]$ *according to:*

$$OWA(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j, \quad (2)$$

where b_j *is the jth major element of the* a_i *collection.*

In [38], the authors proposed the OWAD operator.

Definition 3. *An OWAD operator of dimension n is a mapping* $OWAD: [0,\ 1]^n \times [0,\ 1]^n \rightarrow [0,\ 1]$ *with a characteristic weighting vector W, such that* $\sum_{j=1}^{n} w_j = 1$ *and* $w_j \in [0,1]$*, according to:*

$$OWAD(\langle x_1, y_1 \rangle, \ldots, \langle x_n, y_n \rangle) = \sum_{j=1}^{n} w_j D_j, \quad (3)$$

where D_j represents the jth largest of the $|x_i - y_i|$.

Definition 4. *An OWLAD operator of dimension n is a mapping $OWLAD : \Omega^n \times \Omega^n \to \Omega$ that has an associated weighting vector W, with $\sum_i w_i = 1$ and $w_i \in [0,1]$, such as:*

$$OWLAD(x_1, y_1, x_2, y_2, \ldots, x_n, y_n,) = \exp\left\{\sum_{j=1}^{n} w_i \ln D_j\right\}, \tag{4}$$

where D_j represents the jth largest of $|x_i - y_i|$ over all i and $|x_i - y_i|$ is the argument variable, which is represented in the form of individual distances.

The Bonferroni mean was proposed by [30] and it allows one to use multiple aggregation criteria that use the product of the means of two elements a_i and a_j, thus capturing the interrelation among the arguments. This procedure is performed to implement satisfaction criteria [39]. Rearranging the terms following [27], we can formulate the Bonferroni mean as:

$$B(a_1, a_2, \ldots, a_n) = \left(\sum_{k=1}^{n} a_k^r \left(\frac{1}{1-n} \sum_{j=1; j\neq k}^{n} a_j^q\right)\right)^{\frac{1}{r+q}}, \tag{5}$$

where r and q are parameters such that $r, q \geq 0$ and the arguments $a \geq 0$.

Definition 5. *Let W be an OWA weighting vector of dimension $n-1$ with components $w_i \in [0,1]$ when $\sum_i w_i = 1$. Following, we define this procedure as $OWA_W(V^i) = \left(\sum_{j=1}^{n-1} w_i a_{\pi_k(j)}\right)$, where $a_{\pi_k(j)}$ is the largest element in the $n-1$ tuple $V^i = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$. The Bonferroni OWA [27] is a mean-type aggregation operator and is defined following:*

$$B - OWA(a_1, \ldots, a_n) = \left(\frac{1}{n}\sum_{i}^{n} a_i^r OWA_w\left(V^i\right)\right)^{\frac{1}{r+q}}, \tag{6}$$

where $\left(V^i\right)$ is the vector of all a_j except a_i.

Following the studies in Bonferroni means, several extensions are proposed as has occurred with distance measures; in that sense, one of them is B-OWAD [40].

Definition 6. *A B-OWAD distance for sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ is defined by:*

$$B\text{-}OWAD(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) = \left(\frac{1}{n}\sum_{i}^{n} D_i^r OWAD_{w_i}\left(V^i\right)\right)^{\frac{1}{r+q}}, \tag{7}$$

where $OWAD_{w_i}(V^i) = \left(\frac{1}{n-1}\sum_{\substack{j=1 \\ j\neq i}}^{n} D_j^q\right)$ with $\left(V^i\right)$ is the vector for all $|x_j - y_j|$ with exception to $|x_i - y_i|$ also, ω_i is a $n-1$ vector W_i associated to α_i, which components w_{ij} are the OWA weights. Likewise, D_i is the kth smallest of the individual distance $|x_i - y_i|$.

3. Bonferroni Ordered Weighted Logarithmic Average (B-OWLA) Operator and Distance Measurement

The objective of the present paper to propose an extension to the OWA operator defined as the Bonferroni ordered weighted logarithmic average (B-OWLA) operator and distance measure. Initially, the proposals are presented considering the immediate weighted OWA logarithmic distance and then in combination with the Bonferroni mean.

Based on this definition and using OWLAD [29], a new extension is formulated, called the IWOWLAD operator.

Definition 7. *An IWOWLAD operator of dimension n is a mapping* $IWOWLAD: \Omega^n \times \Omega^n \rightarrow \Omega$ *that has an associated weighted vector W of dimension n* $w_j \in [0,1]$ *and* $\sum_{j=1}^{n} w_j = 1$*, such that:*

$$IWOWLAD(\langle x_1, y_1 \rangle, \ldots, \langle x_n, y_n \rangle) = exp\left\{ \sum_{j=1}^{n} \hat{v}_j \left(lnb_j \right) \right\}, \quad (8)$$

where b_j *is the jth largest of* $|x_i - y_i|$*, each* $|x_i - y_i|$ *has an associated WA* v_i*,* v_j *is the associated WA of* b_j*, and* $\hat{v}_j = \left(w_j v_j / \sum_{j=1}^{n} w_j v_j \right)$*. From* $ln|x_i - y_i| \geq 0$*, it follows that* $exp\ (ln|x_i - y_i|) \geq exp(0)$*. Thus,* $ln|x_i - y_i| \geq 1$*.*

Example 1. *Let* $X = (9,\ 7,\ 11)$ *and* $Y = (17,\ 18,\ 13)$ *be two sets of arguments.* $v_j = (0.12,\ 0.09,\ 0.13)$ *is the weighting vector associated with WA and* w_i *is the weighting vector of the argument* $ln|x_i - y_i|$ *associated with* α_i *whose component is* v_{ij}*. Here we shall let* $\alpha_1 = 0.26$*,* $\alpha_2 = 0.31$ *and* $\alpha_3 = 0.43$*. In addition:* $V^1 = |11–13|$ $V^2 = |9–17|$ *and* $V^3 = |7–18|$*. In addition:* $\sum_{j=1}^{n} w_i v_j = (0.26 \times 0.12) + (0.31 \times 0.09) + (0.43 \times 0.13) = 0.115$*. Using this we get:*

$$IWOWLAD = \exp\left(\frac{0.12 \times 0.26}{0.115} \times \ln|11–13| + \frac{0.09 \times 0.31}{0.115} \times \ln|9–17| + \frac{0.13 \times 0.43}{0.115} \times \ln|7–18| \right)$$

$$IWOWLAD = 7.855$$

Definition 8. *The HWLD operator of dimension n is a mapping* $HWLD: \Omega^n \times \Omega^n \rightarrow \Omega$ *with an associated weighting vector V, such that* $\sum_{j=1}^{n} v_j = 1$*,* $v_j \geq 0$ $v_j \in [0,1]$ *and a W characteristic weighting vector of the argument* $|x_i - y_i|$*, with* $\sum_{j=1}^{n} w_j = 0$*,* $w_j \geq 0$ $w_j \in [0,1]$ *and m is a balancing coefficient, defined as:*

$$HWLD(x, y) = \exp\left\{ \sum_{j=1}^{n} v_j \Delta \ln\left(x_{\sigma(j)}, y_{\sigma(j)} \right) \right\}^{\frac{1}{\lambda}},\ \lambda > 0, \quad (9)$$

where $\Delta\left(x_{\sigma(j)}, y_{\sigma(j)} \right)$ *is the jth largest of weighted arguments* $\Delta\ \ln\left(x_j, y_j \right)$*. Here* $\Delta\ \ln\left(x_j, y_j \right) = m w_i \ln|x_i - y_i|^{\lambda}, i = 1,\ 2,\ \ldots n$*. From* $\ln|x_i - y_i|^{\lambda} \geq 0$*, it follows that* $\exp\ (\ln|x_i - y_i|^{\lambda}) \geq \exp(0)$*. Thus,* $|x_i - y_i|^{\lambda} \geq 1$*.*

Example 2. *Let* $X = (9,\ 7,\ 11)$ *and* $Y = (17,\ 18,\ 13)$ *be two sets of arguments.* $v_i = (0.12,\ 0.09,\ 0.13)$ *is the weighting vector associated with HWLD and* w_i *is the weighting vector of the argument* $|x_i - y_i|$ *associated with* α_i*, whose component is* v_{ij}*; this value is specified by a value* α_i*. Here we shall let* $\alpha_1 = 0.26$*,* $\alpha_2 = 0.31$ *and* $\alpha_3 = 0.43$*. We take* $\lambda = 1$*. In addition:* $V^1 = |11–13|$ $V^2 = |9–17|$ *and* $V^3 = |7–18|$*. Using this we get:*

$$\begin{aligned} HWLD_{v_1}\left(V^1\right) = \ & \exp((3 \times 0.09 \times 0.26 \times \ln|11–13|) + (3 \times 0.12 \times 0.31 \times \ln|9–17|\,) \\ & +(3 \times 0.13 \times 0.43 \times \ln|7–18|))^{1/1} \end{aligned}$$

$$HWLD = \exp(0.0648 + 0.1740 + 0.4021) = 1.898$$

To this point, the proposed extensions combine the features of the immediate weighted OWA, logarithmic and distance measures. For HWLD, it considers the simplification of large values and the importance of the argument and the weight on the ordered position of each value rather than the importance of each value itself. For IWOWLAD, the ordered weighted mean considers the degree of importance of the simplified information. Both consider the comparison between an ideal scenario and the actual conditions.

Definition 9. *The Bonferroni OWLA operator of dimension n is a mapping B-OWLA* : $\Omega^n \times \Omega^n \to \Omega$ *with a weighting vector W associated, such that* $\sum_i w_i = 1$ *and* $w_i \in [0,1]$, *following:*

$$B\text{-}OWLA(a_1, \ldots, a_n) = \exp\left(\frac{1}{n}\sum_{j=1}^{n} \ln(a_i^r)\left(OWLA_w\left(V^i\right)\right)\right)^{\frac{1}{r+q}}, \quad (10)$$

where $\left(OWLA_W\left(V^i\right)\right) = \left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} \ln(a_j^q)\right)$ *with* $\left(V^i\right)$ *being the vector of all* a_j *with exception to* a_i *and w is a* $n-1$ *vector* W_i *associated with* α_i *which component* w_{ij} *is the OWA weight. Here W is an OWA weighing vector of dimension* $n-1$ *which components* $w_i \in [0,1]$ *when* $\sum_i w_i = 1$. *Here, we can define* $OWA_W\left(V^i\right) = \left(\sum_{j=1}^{n-1} w_i \ln(a_{\pi_k(j)})\right)$, *where* $a_{\pi_k(j)}$ *is the largest element in the tuple* $w_i = \frac{1}{n-1}$ *for all i and* V^i.

Example 3. *Let* $X = (7,\ 9,\ 11)$ *be the set of arguments.* w_i *is the weighting vector of the argument* x_i *associated with* α_i *with component* v_{ij}. *Here we shall let* $\alpha_1 = 0.26$, $\alpha_2 = 0.31$ *and* $\alpha_3 = 0.43$. *We take* $r = q = 0.5$. *Using this we get:*

$$OWLA_1 = (0.26 \times (\ln(9) + \ln(11))) = 1.1947$$

$$OWLA_2 = (0.31 \times (\ln(7) + \ln(11))) = 1.3465$$

$$OWLA_3 = (0.43 \times (\ln(7) + \ln(9))) = 1.7815$$

$$B\text{-}OWLA = exp\left(\frac{1}{3} \times (\ln(7) \times 1.4244) + (\ln(9) \times 1.1293) + (\ln(11) \times 1.7815)\right)^{\frac{1}{0.5+0.5}}$$

$$B\text{-}OWLA = 24.172$$

Following Bonferroni distance measures definitions [41], the Bonferroni ordered weighted logarithmic average distance (B-OWLAD) operator are proposed:

Definition 10. *A B-OWLAD distance for the sets* $X = \{x_1, \ldots, x_n\}$ *and* $Y = \{y_1, \ldots, y_n\}$ *is given by a dimension n, which is a mapping B-OWLAD* : $\Omega^n \times \Omega^n \to \Omega$ *with an associated weighting vector W of dimension n* $w_j \in [0,1]$ *and* $\sum_{j=1}^{n} w_j = 1$, *such that:*

$$B\text{-}OWLAD(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) = \exp\left(\frac{1}{n}\sum_i \ln(D_i^r) OWLAD_{w_i}\left(V^i\right)\right)^{\frac{1}{r+q}} \quad (11)$$

where $OWLAD_{w_i}\left(V^i\right) = \left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} \ln\left(\mathrm{D}_j^q\right)\right)$ *with* $\left(V^i\right)$ *being the vector of all* $|x_j - y_j|$ *with exception to* $|x_i - y_i|$ *and* w_i *an* $n-1$ *vector* W_i *i* α_i *which components* w_{ij} *i* D_i *is the kth i* $|x_i - y_i|$. *Furthermore, B-OWLAD has the following properties: (1)* $B\text{-}OWLAD^{r,q}(0,\ 0, \ldots, 0) = 0$; *(2)* $B\text{-}OWLAD^{r,q}(a,\ a, \ldots, a) = d$, *if* $d_k = d$, *for all k; (3)* $B\text{-}OWLAD^{r,q}(a_1,\ a_2, \ldots, a_n) \geq B\text{-}OWLAD^{r,q}(d,\ d, \ldots, d)$, *i.e.,* $BON\text{-}OWLAD^{r,q}$ *is monotic, if* $a_k \geq d_k$, *for all k; (4)* $\max_k\{a_k\} \leq B\text{-}OWLAD^{r,q}(a_1,\ a_2, \ldots, a_n) \leq \min\{a_k\}$. *In addition, if* $q = 0$ *, then* $B\text{-}OWLAD^{r,0}(a_1,\ a_2, \ldots, a_n) = \left(\frac{1}{n}\sum_{k=1}^{n} D_k^r\right)^{1/r}$. *If* $r = 2$ *and* $q = 0$, *then B-OWLAD reduces to the square mean distance:* $B\text{-}OWLAD^{r,0}(a_1,\ a_2, \ldots, a_n) = \left(\frac{1}{n}\sum_{k=1}^{n} D_k^2\right)^{1/2}$. *If* $r = 1$ *and* $q = 0$, *then B-OWLAD is reduced to the average distance:* $B\text{-}OWLAD^{r,0}(a_1,\ a_2, \ldots, a_n) = \frac{1}{n}\sum_{k=1}^{n} D_k$. *If* $r \to +\infty$ *and* $q = 0$, *then B-OWLAD reduces to the max operator:* $\lim_{r\to+\infty} B\text{-}OWLAD^{r,0}(a_1,\ a_2, \ldots, a_n) = \max\{D_k\}$. *If* $r \to 0$ *and* $q = 0$, *then B-OWLAD is reduced to the geometric mean distance:* $\lim_{r\to 0} B\text{-}OWLAD^{r,0}(a_1,\ a_2, \ldots, a_n) =$

$\left(\prod_{k=1}^{n} D_k\right)^{1/n}$. *If* $r = q = 1$, *then B-OWLAD reduces to the following expression:* $B\text{-}OWLAD^{1,1}(a_1,\ a_2,\ldots,a_n) = \left(\frac{1}{n(n-1)}\right)\sum_{\substack{k,j=1\\k\neq j}}^{n} D_k D_j$.

Definition 11. *A B-IWOWLAD operator distance for the sets* $X = \{x_1,\ldots,x_n\}$ *and* $Y = \{y_1 \ldots, y_n\}$ *of dimension n, which is a mapping* $B\text{-}IWOWLAD: \Omega^n \times \Omega^n \to \Omega$ *with an associated weighted vector W of dimension n* $w_j \in [0,1]$ *and* $\sum_{j=1}^{n} w_j = 1$, *following:*

$$B\text{-}IWOWLAD(\langle x_1,y_1\rangle,\ldots,\langle x_n,y_n\rangle) = \exp\left(\frac{1}{n}\sum_{i=1}^{n} ln(D_i^r) IWOWLAD_{w_i}\left(V^i\right)\right)^{\frac{1}{r+q}}, \quad (12)$$

where $IWOWLAD_{w_i}(V^i) = \left(\frac{1}{1-n}\sum_{\substack{j=1\\j\neq i}}^{n}(v_j/\sum_{j=1}^{n} v_j)\ln(D_j^q)\right)$ *with* (V^i) *as the vector of all* $|x_j - y_j|$ *with exception to* $|x_i - y_i|$ *and* w_i *being an* $n-1$ *vector* W_i *associated to* α_i *which components* w_{ij} *are a weighting vector* v_i *associated with the WLA and the OWLA weights. Likewise,* D_i *is the kth smallest of the individual distance* $|x_i - y_i|$. *From* $\ln|x_i - y_i| \geq 0$, *it follows that* $\exp(\ln|\mathrm{x_i} - \mathrm{y_i}|) \geq \exp(0)$. *Please note that,* $\ln|x_i - y_i| \geq 1$. *Here, if* $w_j = 1/n$ *for all j, we get the B-IWLD and if* $v_j = 1/n$ *for all j, we get the B-OWLAD operator. We compute the Bonferroni immediate weight logarithmic (B-IWL) operator if one of the sets is empty.*

Example 4. *Let* $X = (9,\ 7,\ 11)$ *and* $Y = (17,\ 18,\ 13)$ *be sets of arguments.* $v_j = (0.12,\ 0.09,\ 0.13)$ *is the characteristic weighting vector with WA and* w_i *is the vector of the argument* $ln|x_i - y_i|$ *with* α_i *which components are* v_{ij}. *Here we define* $\alpha_1 = 0.26$, $\alpha_2 = 0.31$ *and* $\alpha_3 = 0.43$. *We take* $r = q = 0.5$. *In addition:* $V^1 = |9–17|$ *and* $|7–18|$, $V^2 = |11–13|$ *and* $|7–18|$ *and* $V^3 = |11–13|$ *and* $|9–17|$. *In addition,* $\sum_{j=1}^{n} w_i v_j = (0.26 \times 0.12) + (0.31 \times 0.09) + (0.43 \times 0.13) = 0.115$. *Using this we get:*

$$IWOWLAD_{v_1}\left(V^1\right) = \frac{0.09 \times 0.26}{0.115} \times \ln|7–17| + \frac{0.13 \times 0.26}{0.115} \times \ln|7–18| = 1.1278$$

$$IWOWLAD_{v_2}\left(V^2\right) = \frac{0.12 \times 0.31}{0.115} \times \ln|11–13| + \frac{0.13 \times 0.31}{0.115} \times \ln|7–18| = 1.0645$$

$$IWOWLAD_{v_3}\left(V^3\right) = \frac{0.12 \times 0.43}{0.115} \times \ln|11–13| + \frac{0.09 \times 0.43}{0.115} \times \ln|9–17| = 1.0107$$

$$B\text{-}IWOWLAD = \exp\left(\left(\frac{1}{3} \times ((\ln|11–13| \times 1.1278) + (\ln|7–18| \times 1.0645) + (\ln|9–17| \times 1.0107))\right)^1\right) = 6.088$$

Definition 12. *A Bonferroni hybrid weighted logarithmic distance for the sets* $X = \{x_1, x_2,\ldots,x_n\}$ *and* $Y = \{y_1, y_2,\ldots,y_n\}$ *is given by dimension n, which is a mapping* $B\text{-}HWLD: \Omega^n \times \Omega^n \to \Omega$ *that has an associated weighting vector V, with* $\sum_{j=1}^{n} v_j = 1$, $v_j \geq 0$ $v_j \in [0,\ 1]$ *letting a weighting vector W of* $|x_i - y_i|$, *with* $\sum_{j=1}^{n} w_j = 0$, $w_j \geq 0$ $w_j \in [0,\ 1]$ *and m as a balancing coefficient, such as:*

$$B\text{-}HWLD(\langle x_1,y_1\rangle,\ldots,\langle x_n,y_n\rangle) = \left(\left(\frac{1}{n}\sum_{i}^{n}\ln(D_i^r) HWLD_{v_i}\left(V^i\right)\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{r+q}}, \quad (13)$$

where $HWLD_{w_i}(V^i) = \left(\frac{1}{1-n}\sum_{\substack{j=1\\j\neq i}}^{n} v_i n \ln\left(D_j^q\right)^{\lambda}\right)$ *with* (V^i) *as the vector of all* $|x_j - y_j|$ *i* $|x_i - y_i|$, w_i *as an* $n-1$ *vector* V_i *associated with* α_i *which components of the argument* $|x_i - y_i|$ *are weights, a weighting vector* v_i *associated with the HWD and n is a balancing coefficient. Also,*

D_i is the kth smallest of the individual distance $|x_i - y_i|$. From $\ln|x_i - y_i|^{\lambda} \geq 0$, it follows that $\exp\ (\ln|x_i - y_i|^{\lambda}) \geq \exp(0)$. *Thus,* $|x_i - y_i|^{\lambda} \geq 1$.

In addition, the characteristics given for HWLD and IWOWLAD, when combined with the B-OWA operator, allow one (a) to make an ordination according to the degree of or-ness and and-ness that reflect the decision maker's attitude and (b) to make multiple comparisons by compensating for the possible error in making the comparisons.

To understand each of the proposals, Table 1 and Figure 1 are presented.

Table 1. Extension comparison.

Operators	Extensions	Values
HWD EXT	HWLD B-HWLD	1.898 1.865
B-OWA EXT	B-OWLA B-OWLAD	24.172 6.594
IWOWA EXT	IWOWLA IWOWLAD B-IWOWLA B-IWOWLAD	17.438 7.855 22.477 6.088

Source: Own elaboration.

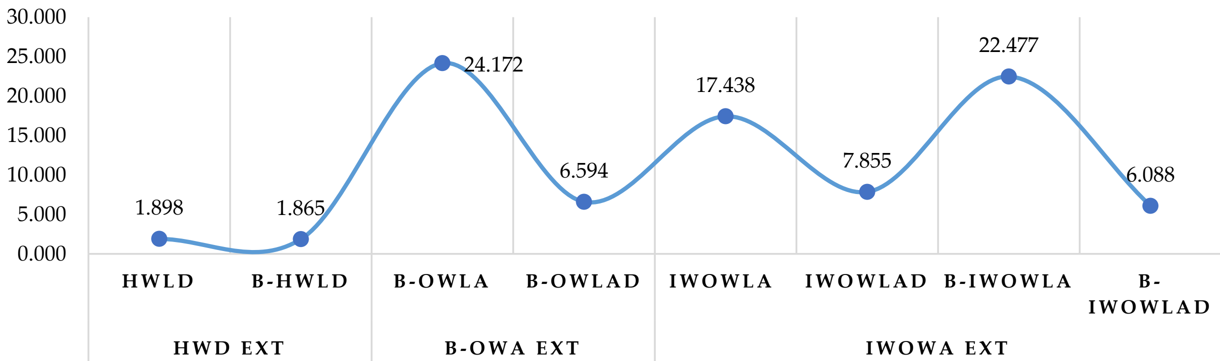

Figure 1. Extension comparison. Source: own elaboration.

Looking at Figure 1, it is worth noting that one of the main advantages of these operators is the possibility of modeling complex problems in decision making, considering (a) the aggregation of multiple criteria capturing the interrelation between arguments and (b) the possibility of making comparisons between an ideal and a real possibility using logarithmic distances. Thus, this allows one to have an ordering mechanism that allows one to make complex calculations in decision processes with high degrees of uncertainty.

4. B-OWLAD Operator in Investing Decision Making

The introduced operators, moreover, the B-OWLAD operator, allows the modelling of highly complex problems in a wide-ranging set of areas such as engineering, medicine, energy and environmental sciences; nonetheless, some of the most direct applications can be foreseen in business, management and accounting, especially in investment decision making processes. Here the direct requirements from the decision makers and stakeholders make it necessary to include perceptions, beliefs and risk aversion, thus attitudinal characteristics to the modelling. Moreover, the set of ideal proposals retrieved from the performance of selected options make the OWA operators and, more specifically, the Bonferroni-based OWA logarithmic distance operators, a suitable option for the development of an illustrative example. In this section we propose a set of initial conditions that must be met by a series of options, and the final ranking yields the most viable option in this highly complex scenario. The next five steps describe the general approach to B-OWLAD modelling.

Step 1. The requirements that the company where you want to invest must be determined (Table 2). This information was obtained from a Mexican investor from the state of Sinaloa with more than 15 years of experience, so the requirements will drastically change depending on the investor, state or country and composition of the market that is being analyzed. In addition, it is important to note that this is conducted for medium to large companies, so if the process is to be applied to small enterprises, the variables will change accordingly to the realities of the companies.

Table 2. Company requirements.

Variables	Goal
Minimum income	1,000,000.00 USD
Minimum years in the market	10.00 years
Minimum number of employees	50.00
Minimum net profit	100,000.00 USD
Maximum debt level	65.00

Source: own elaboration.

Step 2. With the requirements established, the investor must consider only enterprises that meet all requirements, because with the formulation, one considers absolute numbers for the use of logarithmic numbers and if some variables have a negative distance value the results can be misunderstood because minus three is the same as plus three (considering absolute numbers). Considering this, there are three different companies to invest in (Table 3). The information of the companies was obtained from a database of 20 companies that were available to invest in based on the information provided by the investors.

Table 3. Companies to invest.

Variables	Company A	Company B	Company C
Income	1,878,327.00	1,633,852.00	1,796,604.00
Years in the market	16.00	19.00	17.00
Number of employees	84.00	87.00	75.00
Net profit	222,187.00	236,471.00	195,871.00
Debt level	44.00	54.00	49.00

Source: own elaboration.

Step 3. The first step when all the information is available is to obtain the distance between the companies' values and the requirements. For example, the income distance is calculated by the company value less the requirement, and likewise for years in the market, number of employees and net profit. In the case of debt level, the formulation is maximum debt level less company value (Table 4). In addition, the logarithmic distances are calculated with the natural logarithmic of the distance (Table 5).

Table 4. Distances between the company and the requirements.

Variables	Company A	Company B	Company C
Income distance	878,327.00	633,852.00	796,604.00
Years in the market distance	6.00	9.00	7.00
Number of employees distance	34.00	37.00	25.00
Net profit distance	122,187.00	136,471.00	95,871.00
Debt level distance	21.00	11.00	16.00

Source: own elaboration.

Table 5. Logarithmic distances.

Variables	Company A	Company B	Company C
Income distance	13.69	13.36	13.59
Years in the market distance	1.79	2.20	1.95
Number of employees distance	3.53	3.61	3.22
Net profit distance	11.71	11.82	11.47
Debt level distance	3.04	2.40	2.77

Source: own elaboration.

Step 4. With the information provided in Table 5, the logarithmic average distance (LAD), B-OWLAD and B-IWOWLAD operator are used (See Table 6).

Table 6. Results using different aggregation operators.

Operator	Company A	Company B	Company C
LAD	4.598×10^{14}	3.169×10^{14}	2.138×10^{14}
B-OWLAD	1.593×10^{13}	1.615×10^{13}	4.832×10^{12}
B-IWOWLAD	8.666×10^{10}	1.395×10^{11}	3.896×10^{10}

Source: own elaboration.

Step 5. In this step, an analysis of the results is conducted. In this case, it is possible to visualize three different rankings based on the aggregation operator that was used. It will be company A, B and C for the LAD operator, B, A and C for the B-OWLAD operator and B, A and C for the B-IWOWLAD. For selection of the ranking, it is best to consider the operator that aggregates more information; this will be the B-IWOWLAD, which also gives the same ranking as the B-OWLAD, but a difference is generated with the LAD operator. The difference is obtained because the relative importance of the different variables is not the same in the decision making process, but it is in the case of LAD. In addition, the use of Bonferroni means, which can capture the interrelationship of the variables, instead of just regular means such as the case of the LAD operator, provide the decision maker with a better analysis of the information. Finally, the difference between the results of company A and B are minimal, but considering that we can invest in just one company, this small difference is important and, because of that, analyzing the information in a more complex way is required.

Finally, these new aggregation operators can be used for different problems where two sets of information must be analyzed (because they are designed to be applied using distances), where the information that needs to be analyzed has an important difference in values (which is why the logarithmic is used) and the interrelationship of the arguments is important (the main attribute of Bonferroni means). Some additional examples that can be provided are decision making problems in topics related to human resource selection, products development, analysis of routes of distribution, supplier or client selection and any other related areas where the decision meets the above requirements.

5. Conclusions

The objective of this paper was to present extended distance measures and logarithmic OWA-based decision making tools applied to financial investment processes. Distance measures in OWA operators have proven to be effective in cases where a goal, optimal situation or an ideal scenario is compared to a series of options. This is especially interesting in financial decision making processes, where a set of ideal conditions must be met, e.g., to fulfill the expectations of the stakeholders.

Aiming to provide a wide-ranging set of decision making tools, this paper introduces Bonferroni means [30], immediate weights [42] and logarithmic OWA [43]-based operators. Specifically, the paper introduces the immediate weights logarithmic distance (IWLD), the immediate weights ordered weighted logarithmic averaging distance (IWOWLAD), the hybrid weighted logarithmic distance (HWLD), the Bonferroni ordered weighted logarithmic averaging distance (B-OWLAD) operator, the Bonferroni immediate weights ordered weighted logarithmic averaging distance (B-IWOWLAD) operator and the Bonferroni hybrid weighted logarithmic distance (HWLD). Some particular cases, families and generalizations, such as the immediate weighted logarithmic (IWL) operator, the Bonferroni immediate weights logarithmic (B-IWL) operator, the Bonferroni Minkowski logarithmic distance (B-WLD) operator, the Bonferroni weighted logarithmic distance (BWHLD) operator, the Bonferroni geometric logarithmic distance (B-BGLD) operator, the Bonferroni weighted Euclidean logarithmic distance (BWELD) operator and the Bonferroni hybrid logarithmic distance (B-HLD), were also studied.

Some of the benefits of introducing immediate weights is the fusion of information into a single formulation, including the weighted average, hence a degree of importance, and the OWA operator; thus, a degree of or-ness or optimism [42]. On the other hand, Bonferroni means allow a multi-comparison of input arguments and their interrelationship [30]. Additionally, the logarithmic averaging operators based on an optimal deviation model allow the analysis of complex inputs and smooth the estimations [43]. These characteristics allow the generation of robust models capable of handling highly complex inputs and decision makers' requirements.

To illustrate the characteristics of the introduced operators, a financial decision making illustrative example is proposed. Here, a series of ideal investment characteristics are introduced and diverse companies are compared. In general, diverse rankings are generated depending on the selected operator. Nonetheless, the benefits of using these operators are observed as the modelling allows the aggregation of multiple criteria, including the interrelation between the arguments and the analysis of an ideal set of characteristics from the stakeholders and its comparison to the performance of the selected companies. However, the main limitation of the proposal is that only positive integers can be treated, limiting applications with other types of numbers such as decimals or negative numbers. This implies that possible applications can be given with data that seek the max-max of the information in a positive way.

With the limitations outlined above, new approaches to guide future research can focus on modeling other complex and uncertain phenomena that can be represented with fuzzy numbers [44,45] (to be able to use crisp numbers), linguistic variables [46,47] (to be able to use endecadary scales), Pythagorean membership [48,49] (Pythagorean principles) and interval numbers [50,51], and induced [43,52] and heavy aggregations [53,54] (taking into account the attitude of the decision maker), where these methods have the potential to provide solutions for the treatment of highly complex scenarios.

Author Contributions: Methodology V.G.A.-G.; Formal Analysis, F.B.-M.; manuscript draft preparation—review and editing E.L.-C. and J.M.M. All authors have read and agreed to the published version of the manuscript.

Funding: ANID InES Ciencia Abierta INCA210005.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This work was supported by Universidad Pedagogica y Tecnologica de Colombia (Grant number SGI 3323), and the research was supported by Red Sistemas Inteligentes y Expertos Modelos Computacionales Iberoamericanos (SIEMCI), project number 522RT0130 in Programa Iberoamericano de Ciencia y Tecnología para el Desarrollo (CYTED).

Conflicts of Interest: The authors declare no conflict of interest.

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