On eigenvalue bounds for the finite-state birth-death process intensity matrix

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Abstract. The paper sets forth a novel eigenvalue interlacing property across the finite-state birth-death process intensity matrix and two clearly identified submatrices as an extension of Cauchy's interlace theorem for Hermitian matrix eigenvalues. A supplemental proof involving an examination of probabilities acquired from specific movements across states and a derivation of a form for the eigenpolynomial of the matrix through convolution and Laplace transform is then presented towards uncovering a similar characteristic for the general Markov chain transition rate matrix. Consequently, the proposition generates bounds for each eigenvalue of the original matrix, easing numerical computation. To conclude, the applicability of the property to some real square matrices upon transformation is explored.

1. Introduction

Stochastic processes are indispensable in mathematical modelling [1]. A simple but analytically troublesome class of stochastic models is the birth-death process, which is a special case of continuous-time Markov chain that has only two transition types – "births," which increase the state by 1, and "deaths," which decrease the state by 1 [2]. It is understood on this account that in a birth-death process, transitions from state i can only go to either state $i+1$ or $i-1$. The state of the process may be thought of as the size of some population [3]. When the state increases by 1, one says that a birth occurs; when the state decreases by 1, one says that a death occurs. Plainly, the rate at which births occur at state i, denoted by λ_i, is the birth rate. This is not to be confused with λ, by which eigenvalues are represented. On the other hand, the rate at which deaths occur at state i, denoted by μ_i, is the death rate.

Essential to the process above are the intensity matrices, also called transition rate matrices or generator matrices. These describe the rate at which a continuous-time Markov chain transitions or moves within states and determine the process [4]. Inferring from this, the manipulation of matrices is commonplace in putting the theory of birth-death processes to use. A problem, however, arises as the process becomes greater in scale, that is to say, its states increase in number. Computations using the transition rate matrix may prove inefficient with matrices of large orders as the cost increases proportional to the matrix dimensions.

The inquiry comes to grips with this trouble through operating with eigenvalues. As eigenvalues are intrinsic to a matrix, there are valuable attributes pertaining to the processes that are stored in the eigenvalues. For example, probing into the steady-state probabilities of some Markov chain comprises finding the eigenvalues of either the transition probability matrix or the transition rate matrix. Knowledge of the eigenvalues is also useful in the solution of the Kolmogorov equations.

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the well-known characteristic equation:

$$\det(A - \lambda I) = 0.$$

Expansion of this determinant yields a polynomial of degree n, which in itself is another difficulty. Note that the roots of the general polynomial of degree n can be solved for algebraically if and only if $n \leq 4$ [5]. In response, various techniques in determining bounds for eigenvalues do exist in literature and such are often sufficient to solve problems involving eigenvalue calculation.

The more prominent studies that use matrix elements for bounding include the Gerschgorin circle theorem [6] and the ovals of Cassini [7]. Moreover, a relation between eigenvalues and powers of the matrix and an upper bound for the modulus of eigenvalues worked out by A. Householder [8] served as foundation to a more precise bound by E. Lorch [9]. Several eigenvalue bounding theorems through related matrices [10] were also advanced by H. Wittmeyer as early as 1936. The aforementioned are simply a few of the many.

Undeniably, one needs to approximate eigenvalues in situations that solving for them is costly just the same. The proposition introduced in the text addresses this very complication. Here, a strategy for eigenvalue bounding that necessitates the extraction of two principal submatrices from the intensity matrix of a birth-death process is investigated. More precisely, it is to be shown that the sequence of eigenvalues from the main generator matrix interlace with the pooled eigenvalues of the specified submatrices. In connection with the proposition, the distinctiveness is rooted in its ability to simultaneously produce unique bounds for every eigenvalue and in the possibility of a pseudo-recursive computation.

The succeeding sections are organized as follows. The second section illustrates the proposed property involving interlacing eigenvalues employing a simple 7×7 example. The main contribution – the eigenvalue interlacing property proposition yielding eigenvalue bounds – is then stated and proven twofold in the third section. The fourth section discusses the key outcome through an application, looking into the similarity of a general matrix to a block tridiagonal transition rate matrix via some transformation. To close, a synthesis, including some recommendations for future study, is contained in the fifth chapter.

2. Illustration

A motivation to the alternative bounding procedure that hires the eigenvalue interlacing property concerning two principal submatrices is presented. The transition rate matrix B for a seven-state process is considered as example:

$$B = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & -5 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & -9 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 & -11 & 6 \\ 0 & 0 & 0 & 0 & 0 & 6 & -6 \end{bmatrix}.$$

One first solves for the eigenvalues, expanding the determinant of $B - \lambda I$, where I is the identity matrix of order seven, as customary. Roots of the resulting polynomial, omitted for brevity, are then computed numerically. It is known that $\lambda \approx -17.65, -11.23, -6.92, -3.88, -1.80, -0.53, 0$. Each of the eigenvalues of B are written as b_i, $0 \leq i \leq 6$, in increasing fashion, *i.e.*,

$$\{-17.65, -11.23, -6.92, -3.88, -1.80, -0.53, 0\} = \{b_0, b_1, b_2, b_3, b_4, b_5, b_6\}.$$

Now, the upper left and lower right 3×3 principal submatrices of B,

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -5 \end{bmatrix} \text{ and } C = \begin{bmatrix} -9 & 5 & 0 \\ 5 & -11 & 6 \\ 0 & 6 & -6 \end{bmatrix},$$

are put to inspection. In a similar manner, the eigenvalues of A and C are found. Submatrix A has eigenvalues $\lambda \approx -6.29, -2.29, -0.42$. For submatrix C, $\lambda \approx -17.24, -7.88, -0.88$. Lastly, the two eigenvalue triples from A and C are joined to form an increasing sequence:

$$\{-17.24, -7.88, -6.29, -2.29, -0.88, -0.42\} = \{d_0, d_1, d_2, d_3, d_4, d_5\}.$$

It may be observed that

$$-17.65 \le -17.24 \le -11.23 \le -7.88 \le \cdots \le -0.88 \le -0.53 \le -0.42 \le 0$$

or, as represented earlier by variables,

$$b_0 \le d_0 \le b_1 \le d_1 \le b_2 \le d_2 \le b_3 \le d_3 \le b_4 \le d_4 \le b_5 \le d_5 \le b_6.$$

To reiterate, the set of eigenvalues from the original matrix B interlace with the set of eigenvalues coming from principal submatrices A and C.

3. Discussion

In setting about going into the section, one formally defines what interlacing means.

Definition 3.1. Consider two sequences of real numbers $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ and $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_m$ with $m < n$. The second sequence is said to interlace the first whenever

$$\lambda_i \ge \mu_i \ge \lambda_{n-m+i}$$

for $i = 1, 2, \ldots, m$. If $m = n - 1$, the interlacing inequalities become

$$\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \cdots \ge \mu_m \ge \lambda_n,$$

which clarifies the name.

For convenience, the term interlacing is reserved for this particular case and generalized interlacing shall be used otherwise [11].

The central proposition – the interlacing property proposition on the eigenvalues of birth-death process generator matrices – is then stated and proved. The standard representation for the intensity matrix of a finite-state birth-death process is to be noted.

Remark 3.2. The transition rate matrix Q of a birth-death process with $n+1$ states has the form

$$Q = \begin{bmatrix} -a_0 & \lambda_0 & 0 & 0 & \cdots & & 0 \\ \mu_1 & -a_1 & \lambda_1 & 0 & \cdots & & 0 \\ 0 & \mu_2 & -a_2 & \lambda_2 & \cdots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & & \mu_{n-2} & -a_{n-2} & \lambda_{n-2} & 0 \\ 0 & \cdots & & 0 & \mu_{n-1} & -a_{n-1} & \lambda_{n-1} \\ 0 & \cdots & & 0 & 0 & \mu_n & -a_n \end{bmatrix},$$

where $a_0 = \lambda_0 > 0$, $a_i = \mu_i + \lambda_i$, $\mu_i > 0$ and $\lambda_i > 0$ for $1 \le i \le n-1$, and $0 < \mu_n = a_n$ [4].

Proposition 3.3. Let Q be the transition rate matrix for the finite-state birth-death process. List the eigenvalues of Q as $\mathcal{Q} = \{q_0, q_1, \ldots, q_n\}$, in increasing order.

Consider the principal submatrices upon dividing Q relative to discretionary row and column number q with $0 \le q \le n$:

$$P = \begin{bmatrix} -a_0 & \lambda_0 & 0 & 0 & \cdots & & 0 \\ \mu_1 & -a_1 & \lambda_1 & 0 & \cdots & & 0 \\ 0 & \mu_2 & -a_2 & \lambda_2 & \cdots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & & \mu_{q-3} & -a_{q-3} & \lambda_{q-3} & 0 \\ 0 & \cdots & & 0 & \mu_{q-2} & -a_{q-2} & \lambda_{q-2} \\ 0 & \cdots & & 0 & 0 & \mu_{q-1} & -a_{q-1} \end{bmatrix}$$

and

$$R = \begin{bmatrix} -a_{q+1} & \lambda_{q+1} & 0 & 0 & \cdots & & 0 \\ \mu_{q+2} & -a_{q+2} & \lambda_{q+2} & 0 & \cdots & & 0 \\ 0 & \mu_{q+3} & -a_{q+3} & \lambda_{q+3} & \cdots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & & \mu_{n-2} & -a_{n-2} & \lambda_{n-2} & 0 \\ 0 & \cdots & & 0 & \mu_{n-1} & -a_{n-1} & \lambda_{n-1} \\ 0 & \cdots & & 0 & 0 & \mu_n & -a_n \end{bmatrix}.$$

Arrange the eigenvalues of P as an increasing sequence $\mathcal{P} = \{p_0, p_1, \ldots, p_{q-1}\}$. Similarly, write the eigenvalues of R as $\mathcal{R} = \{r_{q+1}, r_{q+2}, \ldots, r_n\}$, also in increasing order.

As result, the sequence of eigenvalues $\mathcal{Q} \setminus (\mathcal{P} \cap \mathcal{R})$ from generator matrix Q interlace with the sequence of eigenvalues $\mathcal{P} \cup \mathcal{R}$ from principal submatrices P and R. Furthermore, should $\mathcal{P} \cap \mathcal{R}$ be non-empty, every element of the intersection proves to be an eigenvalue of Q.

Remark 3.4. Each of the sequences $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ contains nonpositive and distinct elements [6].

For the first method of proof, one takes advantage of the similarity of a transition rate matrix Q to a Hermitian matrix. The proposition is proven as an extension of Cauchy's interlace theorem [12].

Proof. Let $\sigma(Q)$ denote the set of eigenvalues of Q including multiplicities. The matrix Q is similar to a real, symmetric, and tridiagonal S via a diagonal

$$D = \begin{bmatrix} d_0 & 0 & \cdots & 0 \\ 0 & d_1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix},$$

where $d_0 = 1$ and $d_{i+1} = d_i \cdot \sqrt{\frac{\mu_i}{\lambda_{i-1}}}$ for $0 \le i \le n-1$.

On the other hand, the matrix $P \oplus R$ is the $n \times n$ leading principal submatrix of MQM^T, where

$$M = I_q \oplus \begin{bmatrix} 0 & I_{n-q} \\ 1 & 0 \end{bmatrix}$$

is a permutation matrix, and is similar to the $n \times n$ leading principal submatrix of the real and symmetric MSM^T.

By Cauchy's interlace theorem for eigenvalues of Hermitian matrices [12], the pooled eigenvalues of P and R interlace with those of Q.

To conclude the proof, if $\lambda \in \sigma(P) \cap \sigma(R)$, then the cofactor expansion of $\det(Q - \lambda I)$ along row $q+1$ is 0, showing that $\sigma(P) \cap \sigma(R) \subset \sigma(Q)$.   □

As mentioned, the first proof gives the desired result due to the matrix being block tridiagonal. should one wish to extend the proposition to a general Markov chain transition rate matrix, nonetheless, another approach is needed. A probabilistic second proof is provided.

Proof. Let $\{Q(t),\ t \geq 0\}$ be a birth-death process on state space $\mathcal{S} = \{0,1,\ldots,n\}$. The intensity matrix for this process is Q from the proposition. Consider the transition probability below.

$$x_{q,q}(t) = \mathrm{P}(\{Q(t) = q \mid Q(0) = q\}) \tag{3.1}$$

The following probabilities are then introduced.

$$y_{q,q}(t) = \mathrm{P}(\{Q(s) = q \ \forall \ s \in [0,t] \mid Q(0) = q\}) \tag{3.2}$$

$$y_{q,q+1}(t)dt = \mathrm{P}(\{Q(s) = q \ \forall \ s \in [0,t] \wedge Q(u) = q+1 \ \exists \ u \in (t,t+dt) \mid Q(0) = q\}) \tag{3.3}$$

$$y_{q,q-1}(t)dt = \mathrm{P}(\{Q(s) = q \ \forall \ s \in [0,t] \wedge Q(u) = q-1 \ \exists \ u \in (t,t+dt) \mid Q(0) = q\}) \tag{3.4}$$

$$y_{q+1}(t)dt = \mathrm{P}(\{Q(s) \geq q+1 \ \forall \ s \in [0,t] \wedge Q(u) = q \ \exists \ u \in (t,t+dt) \mid Q(0) = q+1\}) \tag{3.5}$$

$$y_{q-1}(t)dt = \mathrm{P}(\{Q(s) \leq q-1 \ \forall \ s \in [0,t] \wedge Q(u) = q \ \exists \ u \in (t,t+dt) \mid Q(0) = q-1\}) \tag{3.6}$$

From probabilities (3.2) to (3.6), (3.1) may be rewritten as

$$x_{q,q}(t) = y_{q,q}(t) + y_{q,q+1}(t) * y_{q+1}(t) * x_{q,q}(t) + y_{q,q-1}(t) * y_{q-1}(t) * x_{q,q}(t),$$

where $f(t) * g(t)$ represents the convolution of $f(t)$ and $g(t)$. This indicates that the leading probability may actually be broken into three cases comprising the other probabilities, with the first one being the situation in which the process remains in the same state for the whole time period.

It is intuitive, then, that the next step would be to take the Laplace transform of both sides of the acquired equation:

$$\mathcal{L}[x_{q,q}(t)] = \mathcal{L}[y_{q,q}(t)] + \mathcal{L}[y_{q,q+1}(t) * y_{q+1}(t) * x_{q,q}(t)] + \mathcal{L}[y_{q,q-1}(t) * y_{q-1}(t) * x_{q,q}(t)]$$

$$\Leftrightarrow x_{q,q}^{\mathcal{L}}(s) = \frac{y_{q,q}^{\mathcal{L}}(s)}{1 - y_{q,q+1}^{\mathcal{L}}(s) \cdot y_{q+1}^{\mathcal{L}}(s) - y_{q,q-1}^{\mathcal{L}}(s) \cdot y_{q-1}^{\mathcal{L}}(s)},$$

where $h^{\mathcal{L}}(s)$ denotes the Laplace transform of $h(t)$. Performing the transform is tantamount to transitioning from a time perspective to a frequency perspective – from t to s. The subsequent assertions abide by the manuscript by J. Keilson [13], a focal source for parts of the proof.

Moreover,

$$y_{q,q}^{\mathcal{L}}(s) = \frac{1}{s + a_q},$$

$$y_{q,q+1}^{\mathcal{L}}(s) = \frac{\lambda_q}{s + a_q}, \text{ and}$$

$$y_{q,q-1}^{\mathcal{L}}(s) = \frac{\mu_q}{s + a_q}$$

are known to be true [13]. One may then revise:

$$x_{q,q}^{\mathcal{L}}(s) = \frac{1}{s + a_q - \lambda_q \cdot y_{q+1}^{\mathcal{L}}(s) - \mu_q \cdot y_{q-1}^{\mathcal{L}}(s)}.$$

The succeeding step is to represent $y_{q+1}(t)$ and $y_{q-1}(t)$ in terms of the eigenvalues of principal submatrices R and P, respectively.

The probability $y_{q+1}(t)$, monotonic, may be written as the sum

$$-\sum_{i=q+1}^{n} c_i \cdot r_i \cdot e^{r_i t},$$

where $c_i > 0$ for $i = q+1, \ldots, n$ and $\sum_{i=q+1}^{n} c_i \leq 1$ [11]. If $a_i = \lambda_i + \mu_i$ for all $i = q+1, \ldots, n-1$ and $a_n = \mu_n$, then $\sum_{i=q+1}^{n} c_i = 1$. If $a_i > \lambda_i + \mu_i$ for some $i \in \{q+1, q+2, \ldots, n-1\}$ or $a_n > \mu_n$, then $\sum_{i=q+1}^{n} c_i < 1$.

Similarly, $y_{q-1}(t)$ may be written as

$$-\sum_{i=0}^{q-1} c_i \cdot p_i \cdot e^{p_i t},$$

where $c_i > 0$ for $i = 0, \ldots, q-1$ and $\sum_{i=0}^{q-1} c_i \leq 1$ [13]. If $a_i = \lambda_i + \mu_i$ for all $i = 1, \ldots, q-1$ and $a_0 = \mu_0$, then $\sum_{i=0}^{q-1} c_i = 1$. If $a_i > \lambda_i + \mu_i$ for some $i \in \{1, 2, \ldots, q-1\}$ or $a_0 > \mu_0$, then $\sum_{i=0}^{q-1} c_i < 1$.

Taking the Laplace transforms of the representations above,

$$y_{q+1}^{\mathcal{L}}(s) = -\sum_{i=q+1}^{n} c_i \cdot r_i \cdot \frac{1}{(s - r_i)} \text{ and } y_{q-1}^{\mathcal{L}}(s) = -\sum_{i=0}^{q-1} c_i \cdot p_i \cdot \frac{1}{(s - p_i)}$$

are the outcomes [13].

Now, one substitutes and gets

$$x_{q,q}^{\mathcal{L}}(s) = \delta(s) \cdot \frac{1}{\epsilon(s)},$$

where

$$\delta(s) = \prod_{i=0}^{q-1} (s - p_i) \cdot \prod_{i=q+1}^{n} (s - r_i),$$

$$\epsilon(s) = (s + a_q) \cdot \delta(s) + \mu_q \cdot \sum_{i=0}^{q-1} c_i \cdot p_i \cdot \delta_i(s) + \lambda_q \cdot \sum_{i=q+1}^{n} c_i \cdot r_i \cdot \delta_i(s),$$

$$\delta_i(s) = \delta(s) \cdot \frac{1}{s - p_i} \text{ for } i = 0, 1, \ldots, q-1, \text{ and}$$

$$\delta_i(s) = \delta(s) \cdot \frac{1}{s - r_i} \text{ for } i = q+1, q+2, \ldots, n.$$

By nature of $y_{q-1}^{\mathcal{L}}(s)$ and $y_{q+1}^{\mathcal{L}}(s)$, it is clear that $\epsilon(s)$ is the eigenpolynomial of intensity matrix Q [13]. Cases are presented.

If $p_i = r_j$ for some $i \in \{0, 1, \ldots, q-1\}$ and $j \in \{q+1, q+2, \ldots, n\}$, $\epsilon(p_i) = \epsilon(r_j) = 0$. This means that $p_i = r_j$ is an eigenvalue of Q. Should there be equal eigenvalues, they are designated as $q_1', q_2', \ldots, q_k'$. Let the remaining eigenvalues from both P and R be $b_1, b_2, \ldots, b_{n-k}$, in increasing order. The polynomial $\epsilon(s)$ may then be recast as

$$\epsilon(s) = (s - q_1') \cdot (s - q_2') \cdots (s - q_{k-1}') \cdot (s - q_k') \cdot \epsilon_E(s).$$

Since all eigenvalues from P and R are negative and $c_i > 0$ for all i, $(-1)^i \cdot \epsilon_E(b_i) > 0$ for $i = 1, 2, \ldots, n-k$. This means that the roots of $\epsilon_E(s)$ – the eigenvalues of Q except for $q_1', q_2', \ldots, q_k'$ – interlace with $b_1, b_2, \ldots, b_{n-k}$.

If there are no equal eigenvalues from P and R, then one has $b_1, b_2, \ldots, b_n$, arranged as well. Analogous to the previous case, $(-1)^i \cdot \epsilon_E(b_i) > 0$ for $i = 1, 2, \ldots, n$. This means that the roots of $\epsilon(s)$ – the eigenvalues of Q – interlace with $b_1, b_2, \ldots, b_n$. $\qquad\square$

4. Application

While the paper gives emphasis on transition rate matrices for birth-death processes, the proposition may apply to some real square matrices. The section tackles a small-scale example of a general 5×5 matrix being transformed into a generator matrix of the same order [14]. Take, for instance, the matrix

$$A_1 = \begin{bmatrix} \frac{2\sqrt{2}-35}{8} & \frac{2\sqrt{2}+1}{4} & -\frac{7}{4} & \frac{-4\sqrt{2}-11}{8} & \frac{-9\sqrt{2}+2}{8} \\ \frac{2\sqrt{2}+1}{4} & \frac{2\sqrt{2}-5}{2} & -\frac{1}{2} & \frac{-4\sqrt{2}+1}{4} & \frac{3\sqrt{2}+2}{4} \\ -\frac{7}{4} & -\frac{1}{2} & -4 & -\frac{3}{4} & -\frac{7\sqrt{2}}{4} \\ -\frac{11}{8} & \frac{1}{4} & -\frac{3}{4} & \frac{-6\sqrt{2}-19}{8} & \frac{-\sqrt{2}+6}{8} \\ \frac{-9\sqrt{2}-2}{8} & \frac{3\sqrt{2}-2}{4} & -\frac{7\sqrt{2}}{4} & \frac{-\sqrt{2}+10}{8} & \frac{-2\sqrt{2}-11}{4} \end{bmatrix}.$$

The matrix clearly does not satisfy the properties of an intensity matrix; nevertheless, its reduction to a block tridiagonal matrix is possible through the Householder matrix [15]

$$Q_1 = \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} & 0 & -\frac{1}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{2} & 0 & \frac{3}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{4} & \frac{1}{2} \end{bmatrix}$$

generated from the unit vector

$$v_1 = \begin{bmatrix} \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{4} \\ \frac{1}{2} \end{bmatrix}.$$

With $A_2 = Q_1 A_1 Q_1$, one is left with

$$A_2 = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 \\ 2 & -5 & 3 & 0 & 0 \\ 0 & 3 & -4 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 3 & -3 \end{bmatrix},$$

a generator matrix to which the earlier proposition applies. Both A_1 and A_2 share the same set of eigenvalues. Bounds for the eigenvalues of the former and accordingly, bounds for the eigenvalues of the latter may therefore be determined.

5. Conclusion

The research culminates in the proposition and proof of the uncovered eigenvalue interlacing property across the finite-state birth-death process intensity matrix and two carefully selected principal submatrices. Furthermore, a fruitful application of the result is covered.

A suggestion for future work is identifying the pattern in which the pooled eigenvalues are to interlace. As of this writing, there has been limited remarks on the matter. Although it is known that the merged set of eigenvalues is to interlace with the set of eigenvalues of the transition rate matrix, the arrangement of eigenvalues in the inequality is yet to be foretold.

An attempt at a different principal submatrix selection may also be worthwhile. In the study, one has omitted a row and a column; what may be done is to choose two overlapping principal submatrices. For a 9×9 matrix, for example, the possibility of an interlacing property proposition proceeding from two 5×5 principal submatrices may be investigated.

To end, the validity of the property in other Markov chain intensity matrices is also hoped to be discovered. After all, the supplementary proof was pursued mainly for this purpose. There may be variations of the proposition for other processes.

Acknowledgments

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