ON k-GEODETIC GRAPHS AND GROUPS

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Abstract. We call a graph k-geodetic, for some $k \geq 1$, if it is connected and between any two vertices there are at most k geodesics. It is shown that any hyperbolic group with a k-geodetic Cayley graph is virtually-free. Furthermore, in such a group the centraliser of any infinite order element is an infinite cyclic group. These results were known previously only in the case that $k = 1$. A key tool used to develop the theorem is a new graph theoretic result concerning "ladder-like structures" in a k-geodetic graph.

1. Introduction

For any positive integer k, we will call a (possibly infinite) graph *k-geodetic* if the graph is connected and between any pair of vertices there are at most k geodesics. For example, a tree is 1-geodetic and the complete bipartite graph $K_{k,l}$ is $(\max\{k, l\})$-geodetic. While 1-geodetic graphs (known simply as *geodetic* graphs) [7, 4] and 2-geodetic graphs [10] have been studied, it seems that little work has been done on k-geodetic graphs. Our first result is a necessary condition for a graph to be k-geodetic. We introduce a technical notion of a *ladder-like structure* with parameters for height and width (see Definition 3.2).

Theorem A. Let m and k be positive integers. In any k-geodetic graph there is a universal bound on the height of ladder-like structures of width m.

A group G is called *k-geodetic* if it admits a finite inverse-closed generating set S such that the corresponding undirected Cayley graph $\mathrm{Cay}(G, S)$ is k-geodetic. It is clear that any finite group G is geodetic (with $S = G\backslash\{1_G\}$). The hyperbolic groups are a natural next class of groups to investigate. If G is hyperbolic, then geodesics fellow travel and we may use this property to construct ladder-like structures. We parlay this idea into our second result which demonstrates that the hyperbolic k-geodetic groups form a proper subclass of the virtually-free groups.

Theorem B. Let k be a positive integer. If G is a hyperbolic k-geodetic group, then G is virtually-free and in G the centraliser of any infinite order element is an infinite cyclic group.

We note that $\mathbb{Z} \times \mathbb{Z}_2$ fails the centraliser condition of Theorem B and so is an example of a virtually-free group that is not k-geodetic for any positive integer k.

In 1997, Shapiro [9] asked if the geodetic groups are exactly the *plain* groups. A group is *plain* if it is isomorphic to a free product of finitely many finite groups and finitely many copies of $\mathbb{Z}$. There is a natural choice of generating set of a plain group so that the Cayley graph is geodetic. Although Shapiro's question remains unanswered in general, some progress has been made in the special case of hyperbolic groups. Papasoglu [8, 1.4] showed that hyperbolic geodetic groups are in

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plain groups	$\subseteq$	hyperbolic geodetic groups	$\subseteq$	hyperbolic k-geodetic groups	$\subsetneq$	virtually free groups

FIGURE 1. Known containments.

fact virtually-free. Observing that hyperbolic geodetic groups admit presentation by finite Church-Rosser Thue systems, one may apply a result by Madlener and Otto[6] to conclude that in hyperbolic geodetic groups the centraliser of any infinite order element is infinite cyclic. Theorem B shows that hyperbolic k-geodetic groups satisfy the key constraints known to hold for hyperbolic geodetic groups.

Shapiro [9, p.6] proved that if G is virtually infinite cyclic and k-geodetic with respect to generating set S, then G is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$ and S is the standard generating set. Taken with the existing theory, Theorem B leaves us with the containments in Fig. 1 and is evidence in favour of the following conjecture.

Conjecture C. A hyperbolic group G is k-geodetic for some positive integer k if and only if G is geodetic. Furthermore, if G is infinite and $\mathrm{Cay}(G, S)$ is k-geodetic for some finite generating set S and some $k \geq 1$, then $\mathrm{Cay}(G, S)$ is geodetic.

We note the difference between finite and infinite groups in the above conjecture. For any finite group G, it is clear that $\mathrm{Cay}(G, G \setminus \{1_G\})$ is geodetic. For any positive integer k, the only infinite k-geodetic Cayley graphs we know are in fact geodetic.

Example 1.1. For any integer $k > 1$, we give an example of a group G and generating set S such that $\mathrm{Cay}(G, S)$ is k-geodetic but not $(k-1)$-geodetic as follows. We observe that the complete bipartite graph $K_{k,k}$ is k-geodetic and not $(k-1)$-geodetic. We now choose a group and generating set with Cayley graph $K_{k,k}$. Let G be the cyclic group of order $2k$, let a be an order $2k$ element in G and define $S := \{a^{2i+1} \mid 0 \leq i \leq k-1\}$. Then $\mathrm{Cay}(G, S)$ has k distinct geodesics of length 2 for each $a^{2i} \in G$ with $1 \leq i \leq k$ and a unique geodesic of length 1 for each $a^{2i+1} \in G$ with $0 \leq i \leq k-1$.

2. Preliminaries

Let $X = (V, E)$ be a locally-finite simple connected graph. For $a, b \in \mathbb{N}$ with $a \leq b$, define $[a, b]$ to be $\{a, a+1, \ldots, b\}$. A *path* in X is a map $\gamma : [0, n] \to V$ with $\{v_i, v_{i+1}\} \in E$ for each $0 \leq i \leq n-1$. The path γ has an *initial point*, *end point* and *length* given by $\gamma(0)$, $\gamma(n)$ and n respectively.

There is a metric $d_X : V \times V \to \mathbb{N}$ such that $d_X(u, v)$ is the length of a minimal length path between u and v. We call such a path a *geodesic*. We say that X is *k-geodetic* if for any pair of vertices the number of distinct geodesics between them is less than or equal to k. In the special case that $k = 1$, we say that X is *geodetic*. For our arguments that follow, we will require precise notions relating to fellow travelling.

Definition 2.1. Let $\gamma_i \colon [0, n_i] \to V$ for $i \in \{1, 2\}$ be paths in V and $n = \max\{n_1, n_2\}$. Then the paths are said to *m-fellow travel* if

$$d_X(\gamma_1(t), \gamma_2(t)) \in [0, m],$$

for all $t \in [0, n]$. Note that if $n_i < n$, we define $\gamma_i(t) = \gamma_i(n_i)$ for all $t \in [n_i + 1, n]$.

Definition 2.2. Let $\gamma_1, \gamma_2 \colon [0, n] \to V$ be paths of length n. For a given $m > 0$, we say γ_1 and γ_2 are:

(i) *m-apart at* $i \in [0, n]$ if $d_X(\gamma_1(i), \gamma_2(i)) = m$;
(ii) *m-close at* $i \in [0, n]$ if $d_X(\gamma_1(i), \gamma_2(i)) \in [1, m]$;
(iii) *asynchronously disjoint* if for all distinct $i, j \in [0, n]$ we have $\gamma_1(i) \neq \gamma_2(j)$;
(iv) *co-travelling* if $\gamma_1(i) = \gamma_2(j)$ and $\gamma_1(i+1) = \gamma_2(j+1)$ for some $i, j \in [0, n-1]$, and *synchronously co-travelling* if $i = j$.

Furthermore, we define $a_m(\gamma_1, \gamma_2) := |\{i \in [0, n] \mid d_X(\gamma_1(i), \gamma_2(i)) = m\}|$ and $c_m(\gamma_1, \gamma_2) := |\{i \in [0, n] \mid d_X(\gamma_1(i), \gamma_2(i)) \in [1, m]\}|$; so $a_m(\gamma_1, \gamma_2)$ records the number of times that γ_1 and γ_2 are m-apart, while $c_m(\gamma_1, \gamma_1)$ records the number of times they are m-close.

Definition 2.3. A *geodesic triangle* in X is the union of three geodesic paths $\alpha \colon [0, n_\alpha] \to V$, $\beta \colon [0, n_\beta] \to V$ and $\gamma \colon [0, n_\gamma] \to V$, such that $\alpha(n_\alpha) = \beta(0)$, $\beta(n_\beta) = \gamma(0)$ and $\gamma(n_\gamma) = \alpha(0)$. The geodesic triangle is *non-degenerate* if $\alpha(a), \beta(b), \gamma(c)$ are pairwise distinct for all $a \in [1, n_\alpha]$, $b \in [1, n_\beta]$ and $c \in [1, n_\gamma]$; otherwise it is *degenerate*.

Definition 2.4. A *geodesic bigon* in X is the union of two geodesic paths $\alpha \colon [0, n] \to V$ and $\beta \colon [0, n] \to V$ such that $\alpha(0) = \beta(0)$ and $\beta(n) = \gamma(n)$. The geodesic bigon is *non-degenerate* if $\alpha(i) \neq \beta(i)$ for all $i \in [1, n-1]$; otherwise it is *degenerate*.

Let G be a group and $S \subseteq G \setminus \{1_G\}$ a finite inverse-closed generating set. The undirected Cayley graph of G with respect to S, denoted $\mathrm{Cay}(G, S)$ is the graph with vertex set G and edge set $\{\{g, h\} \in G \times G \mid g^{-1}h \in S\}$. Since S generates G, $\mathrm{Cay}(G, S)$ is connected. Since S is finite, $\mathrm{Cay}(G, S)$ is locally-finite. Since $1_G \notin S$ and $S \subset G$, $\mathrm{Cay}(G, S)$ is simple. We call S an *alphabet* and denote the set of finite words over the alphabet S by S^*. We write $|u|$ for the length of the word $u \in S^*$; the unique word of length 0 is called the empty word and denoted λ. Let $S^+ := S^* \setminus \{\lambda\}$. For any $w = w_1 w_2 \dots w_n \in S^*$, a word of the form $w_i w_{i+1} \dots w_j$ with $1 \leq i \leq j \leq n$ is called a *factor* of w. A word $w \in S^+$ is called *primitive* if there is no word $u \in S^*$ such that $w = u^m$ for some $m > 1$. If a word w is not primitive, then we call the minimal length word u such that $w = u^m$ for some $m > 1$ the *primitive root* for w. For any $g \in G$, we write $|g|_{G,S}$ for the length of a shortest word $w \in S^*$ such that w spells g. For every $u \in G$, there is a bijective correspondence between paths in $\mathrm{Cay}(G, S)$ with initial vertex u and words in S^*; minimal length words spelling a group element g correspond to geodesic paths in $\mathrm{Cay}(G, S)$ from u to ug. We write $u = v$ if $u, v \in S^*$ are identical as words. We use use the symbol $\equiv$ to denote that the left hand side and right hand side evaluate to the same element in G. For any $g \in G$ and $r > 0$, we write $B_r(g)$ for the set $\{h \in G \mid d_X(g, h) < r\}$. The centraliser of an element $g \in G$ is defined to be $C_G(g) := \{h \in G \mid gh \equiv hg\}$.

We refer the reader to [1] for basic definitions and results regarding hyperbolic geodesic metric spaces. A locally-finite simple connected graph X is a geodesic metric space. Let T be a geodesic triangle in X with vertices T_1, T_2 and T_3 and sides γ_1, γ_2 and γ_3. Take the points p_i on each γ_i to be those that have

$$d_X(T_1, p_2) = d_X(T_1, p_3), d_X(T_2, p_1) = d_X(T_2, p_3), d_X(T_3, p_1) = d_X(T_3, p_2).$$

For a real number $\delta > 0$, we say T is *δ-thin* if for each $i \in \{1, 2, 3\}$ and distinct $j, k \in \{1, 2, 3\} \setminus \{i\}$ the sub-paths of γ_j and γ_k from T_i to p_j and p_k respectively

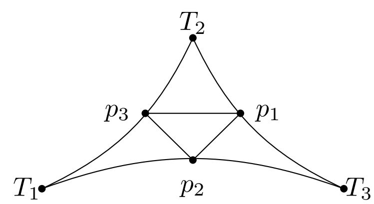

FIGURE 2. Hyperbolic space has δ-thin geodesic triangles

δ-fellow travel. We say X is *hyperbolic* if there exists $\delta > 0$ such that all geodesic triangles are δ-thin. We say a group G is a *hyperbolic group* if $\mathrm{Cay}(G, S)$ is a hyperbolic for some (and hence any) finite generating set S. We have the well-known fellow traveller property in hyperbolic groups [3, Lemma 2.3.2 and Thm. 3.4.5]:

Proposition 2.5. *Let G be a hyperbolic group and $X = Cay(G, S)$ for some finite generating set S. Then for any $c \geq 0$ there exists an $m_c > 0$ such that any two geodesics $\gamma_i : [0, n_i] \to X$ with $i \in \{1, 2\}$, $\gamma_1(0) = \gamma_2(0)$ and $d_X(\gamma_1(n_1), \gamma_2(n_2)) \leq c$ will m_c-fellow travel.*

3. Ladder-like structures are bounded

We will show that in a k-geodetic graph, there is a bound on the number of times a pair of asynchronously disjoint geodesics may be m-apart and m-close.

Lemma 3.1. *Let X be a k-geodetic graph and let u, v be vertices in X. If there exist distinct paths $\alpha_0, \ldots, \alpha_k \colon [0, n] \to X$ with initial point u and terminal point v, then there exists a path β from u to v of length $n-1$ or $n-2$.*

Proof. Since there are $k+1$ paths of length n, none of them can be geodesics. Consider the sequence of paths $\alpha_0|_{[0,i]}$ for $i \in [0, n]$. Let

$$i_0 := \min\{i \in [0, n] \mid \alpha_0|_{[0,i]} \text{ is not a geodesic}\}.$$

Define β_0 to be a geodesic from u to $\alpha_0(i_0)$. Then β_0 has length j for some $j \in [i_0 - 2, i_0 - 1]$, since a shorter path contradicts the minimality of i_0. Define a path β by

$$\beta(i) := \begin{cases} \beta_0(i) & \text{for } i \in [0, j], \\ \alpha_0(i + i_0 - j) & \text{for } i \in [j+1, n - i_0 + j]. \end{cases}$$

Then β is a path from u to v with length $n-1$ or $n-2$. □

Definition 3.2. Let m and r be positive integers. A *ladder-like structure* of *width* m and *height* r is a pair of asynchronously disjoint geodesics γ_x and γ_y with $a_m(\gamma_x, \gamma_y) = r$.

Proposition 3.3. *Let m and k be positive integers. There exists a constant $A(m, k)$ such that no ladder-like structure of width m has a height exceeding $A(m, k)$ in any k-geodetic graph.*

Proof. Let k and m be positive integers. Define $r := k \prod_{i=2}^{2m+1}(ik+1)$ and $A(m, k) := mr$. Let X be a k-geodetic graph. For contradiction, suppose there exist two asynchronously disjoint geodesics γ_x and γ_y in X that form a ladder-like structure of width m and height $A(m,k)+1$. For each $i \in [0,r]$, define the points x_i on γ_x and y_i on γ_y to be $(im+1)$-th occurrence of γ_x and γ_y being m-apart, ignoring all other occurrences that γ_x and γ_y are m-apart. Hence, there exists a diagram for γ_x and γ_y where each $d_i \geq m$ as depicted in Fig. 3. The top row from x_0 to x_r is a depiction of γ_x, the bottom row from y_0 to y_r is a depiction of γ_y and $d_i \geq m$ for each $i \in [1,r]$. For each $j \in [0,r]$, the path from x_j to y_j is a geodesic γ_j of length m. The vertices in $\{x_0,\ldots,x_r\} \cup \{y_0,\ldots,y_r\}$ are pairwise disjoint: because γ_x is a geodesic, $x_i = x_j$ if and only if $i = j$; because γ_y is a geodesic, $y_i = y_j$ if and only of $i = j$; because γ_x and γ_y are asynchronously disjoint, $x_i \neq y_j$ for any i,j such that $i \neq j$; because the ladder has width m, $d(x_i,y_i) = m > 0$ for any i. Furthermore, since $d_i \geq m$ for each $i \in [1,r]$ we have that x_{i+1} does not lie on γ_i for any i. For clarity in the arguments to follow, we schematically depict this part of the graph as shown in Fig. 4.

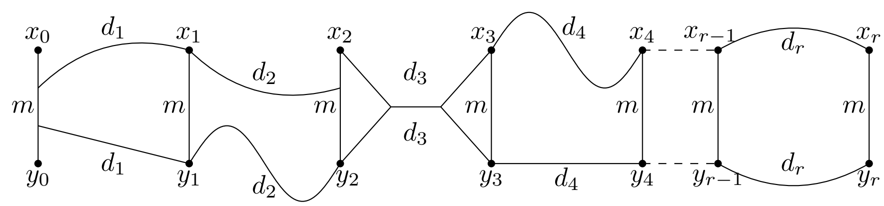

FIGURE 3. An example ladder-like structure of width m and height h.

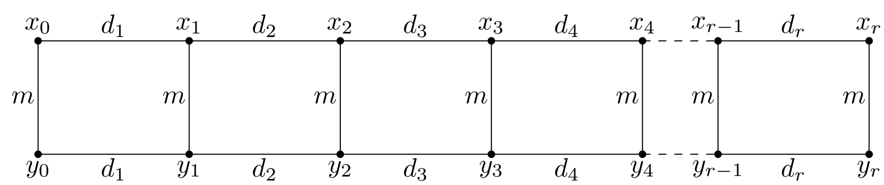

FIGURE 4. A schematic ladder-like structure of width m and height h

In this paragraph we demonstrate the existence of a 'short' path from x_0 to y_k. For each $j \in [0,k]$ we define a path α_j from x_0 to y_k as follows: α_j travels along γ_x from x_0 to x_j, then travels along γ_j to y_j, and finally travels along γ_y to y_k. Although the paths $\alpha_0,\ldots,\alpha_j$ are not necessarily internally disjoint, they are distinguished by which of the points $\{x_0,\ldots,x_r\} \cup \{y_0,\ldots,y_r\}$ they visit. Hence, we have exhibited $k+1$ distinct paths from x_0 to y_k of length $m + \sum_{i=1}^{k} d_i$. By Lemma 3.1, there is a path β from x_0 to y_k such that

$$|\beta| = m - 1 + \sum_{i=1}^{k} d_i \text{ or } |\beta| = m - 2 + \sum_{i=1}^{k} d_i.$$

Since γ_x and γ_k are geodesics, any path from x_0 to y_k that passes through x_k has length at least $m + \sum_{i=1}^{k} d_i$; hence β does not pass through x_k.

For each $j \in [1,2k]$, we repeat the above argument for paths from x_{jk} to $y_{(j+1)k}$. We deduce that for each $j \in [0,2k]$, there is a path from x_{jk} to $y_{(j+1)k}$ that does not pass through $x_{(j+1)k}$ and has length

$$m-1+\sum_{i=jk+1}^{(j+1)k} d_i \text{ or } m-2+\sum_{i=jk+1}^{(j+1)k} d_i.$$

For each $j \in [0, 2k]$, extend these paths so that their initial vertex is x_0, travelling along γ_x to x_j, and their terminal vertex is $y_{k(2k+1)}$, travelling along γ_y from $y_{(j+1)k}$. By the pigeonhole principle, at least $k+1$ of the extended paths have the same length. Since X is k-geodetic, by Lemma 3.1 there is a path from x_0 to $y_{k(2k+1)}$ of length

$$m-p+\sum_{i=1}^{k(2k+1)} d_i$$

with $p \in [2, 4]$. Continuing these arguments we see that there is a path of length

$$m-p+\sum_{i=1}^{f(c)} d_i$$

from x_0 to $y_{f(c)}$ for some $p \in [c, 2c]$, where $f(c) = k\prod_{i=2}^{c}(ik+1)$. By our assumption we can take $c = 2m+1$, which gives $p > 2m$. This implies the existence of a path from x_0 to x_r going via y_r that is shorter than travelling along the geodesic γ_x. We have a contradiction showing the ladder-like structure with width m cannot have height exceeding $A(m, k)$. □

This completes the proof of Theorem A.

Corollary 3.4. *Let m and k be positive integers. There exists a constant $C(m, k)$ such that asynchronously disjoint geodesics cannot be m-close more than $C(m, k)$ times in any k-geodetic graph.*

Proof. The result follows directly from Proposition 3.3 and the pigeonhole principle; giving a constant $C(m, k) < mA(m, k) + 1$ bounding how many times asynchronously disjoint geodesics can be m-close. □

4. Hyperbolic k-geodetic groups are virtually-free

We will now focus on hyperbolic groups with k-geodetic Cayley graphs, with the key result being that they are virtually-free.

We will use a characterisation of virtually-free groups as seen in [5]. Let $e > 0$, then a language L over an alphabet S is *e-locally excluding* over S if there exists a finite set F of words of length at most e such that any word not in L has a factor in F. Then G is virtually-free if and only if there exists a finite inverse-closed generating set S such that the language of geodesics is e-locally excluding over S for some $e > 0$.

Proposition 4.1. *Let k be a positive integer. Any hyperbolic k-geodetic group G is virtually-free.*

Proof. Let G be a hyperbolic group which admits a finite inverse-closed generating set S such that $X = \mathrm{Cay}(G, S)$ is k-geodetic. By Proposition 2.5, there exists $m > 0$ such that any two geodesics $\gamma_i : [0, n_i] \to X$ with $i \in \{1, 2\}$, $\gamma_1(0) = \gamma_2(0)$ and $d_X(\gamma_1(n_1), \gamma_2(n_2)) \leq 1$, will m-fellow travel. We claim that the language of all geodesic words for G with respect to S is an $(C(\lceil m \rceil, k) + 1)$-locally excluding language, where $C(\lceil m\rceil, k)$ is the bound given in Corollary 3.4. Define the finite set

$$F := \{w \in S^* \mid |w| \leq C(\lceil m\rceil, k) + 1 \text{ and } w \text{ not a geodesic}\}.$$

Suppose $w \in S^*$ is not a geodesic. Then there exists $u, v \in S^*$ and $x \in S$ such that $w = uxv$ and u is a geodesic but ux is not. If the last letter of u is x^{-1}, then the factor $x^{-1}x \in F$. Now assume that the last letter of u is not x^{-1}, so that the terminal vertex of ux does not lie on the path u. Let w' be a geodesic representative of ux. Clearly, $|w'|$ is either $|u|$ or $|u| - 1$. For compatibility with Definition 2.2, we let w'' equal w' if $|w'| = |u|$ and $w'x^{-1}$ if $|w'| = |u| - 1$. Then w'' and u are asynchronously disjoint and m-fellow travel. Furthermore, since the terminal vertex of w' does not lie on the path of u, there exists words u_1 and u_2 such that $u = u_1u_2$, $|u_2| > 0$ and the words u and w'' do not co-travel after $|u_1|$ steps. Since u and w'' are m-fellow travelling, they must be $\lceil m\rceil$-close after $|u_1|$ steps. By Corollary 3.4, u and w'' are $\lceil m\rceil$-close at most $C(\lceil m\rceil, k)$ times, so $|u_2| \leq C(\lceil m\rceil, k)$. Therefore, the factor of w given by u_2x is not a geodesic and it appears in F. Thus the language of geodesics of G is $(C(\lceil m\rceil, k) + 1)$-locally excluding over S. □

We also have the following fact regarding non-degenerate triangles and bigons that are useful in later arguments:

Lemma 4.2. *Let k be a positive integer and G a hyperbolic group with inverse-closed generating set S such that Cay(G, S) is k-geodetic. Then the non-degenerate geodesic triangles and bigons in Cay(G, S) have bounded side-length.*

Proof. Let k be a positive integer and suppose that G is k-geodetic. Since G is a hyperbolic group, there exists a $\delta > 0$ such that geodesic triangles in Cay(G, S) are δ-thin. Suppose we have a non-degenerate geodesic triangle in Cay(G, S) with at least one side of length greater than $2C(\delta, k)$, where $C(\delta, k)$ is found in the proof of Corollary 3.4. Then we have asynchronously disjoint geodesics that are δ-close and more than $C(\delta, k)$ times. This contradicts Corollary 3.4. This also shows that non-degenerate geodesic bigons have bounded side-length since any non-degenerate geodesic bigon forms a non-degenerate geodesic triangle. □

5. Centralisers of infinite order elements

In this section we investigate centralisers of infinite order elements in groups with k-geodetic Cayley graph. This will lead to a proof of the second part of Theorem B, restricting which virtually-free groups can be k-geodetic. Our result and proof is motivated by Madlener-Otto's [6] analogous result for groups presented by finite Church-Rosser Thue systems.

We recall a classical combinatorial result for words over any alphabet. The result is due to Lyndon and Schützenberger and can be found in [2, Thm. 6.5].

Lemma 5.1. *Let $x, y, z \in S^*$ be words over an alphabet S.*

(a) *If $x \neq \lambda$ and $zx = yz$, then there are $s, t \in S^*$ and $q \in \mathbb{N}$ such that $x = st, y = ts$ and $z = (ts)^q t$.*
(b) *If $xy = yx$, then both x and y are powers of the same word.*

Lemma 5.2. *Let k be a positive integer, let G be a group with a finite inverse-closed generating set S such that Cay(G, S) is k-geodetic. If $u \in S^+$ is a primitive word such that u^r is a geodesic for all $r \geq 1$ and u evaluates to $g \in G$, then $C_G(g) = \langle g\rangle$.*

Proof. For the sake of contradiction, suppose that $C_G(g) \neq \langle g \rangle$. Then there exists $h \in C_G(g)$ such that $h \notin \langle g \rangle$. Let $v \in S^*$ be a geodesic word evaluating to $h \in G$. Let α be the ray in $\mathrm{Cay}(G, S)$ from the vertex 1_G with label u^∞ and let β be the ray from the vertex h with label u^∞. Then α is the top path, and β the bottom path, in a structure shown schematically in Fig. 5.

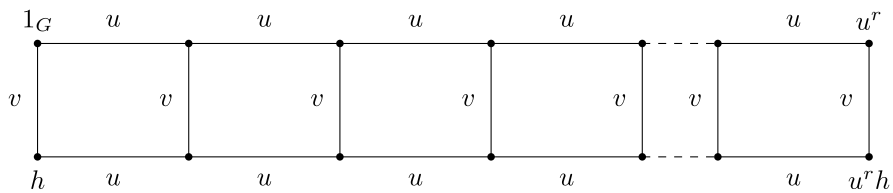

FIGURE 5. A schematic of h commuting with powers of u

In this paragraph we show that α, β must co-travel but not synchronously, that is, must join after some prefixes $\alpha' \neq \beta'$ of u^∞ as depicted in Fig. 6. Since α and β are labelled by the same word but start at distinct vertices in a Cayley graph, they cannot synchronously co-travel. Furthermore, by Proposition 3.3 they cannot be asynchronously disjoint for arbitrarily large r, so we know they must join asynchronously. We then have the diagram depicted in Fig. 6, where α' and β' are prefixes of some powers of u.

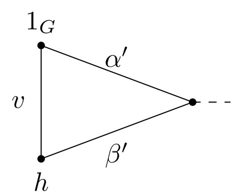

FIGURE 6. A depiction of the asynchronous joining

Let $u = u_1 \dots u_{|u|}$, so $\alpha' = u^{m_1} u_1 \dots u_i$ and $\beta' = u^{m_2} u_1 \dots u_j$ for some $i, j \in [0, |u|]$. If $i = j$, then $v = u^{m_2 - m_1}$ which is not possible because $h \notin \langle g \rangle$; so we may assume that $i \neq j$. Now continue moving along α and β in Fig. 6 starting with $u_{i+1} \dots u_{|u|}$ and $u_{j+1} \dots u_{|u|}$ then powers of u. If α and β bifurcate, then by applying Proposition 3.3 starting from the bifurcation point, α and β will only remain disjoint for a bounded number of steps. Furthermore, α and β cannot bifurcate and meet again more than $\log_2 k$ times. Hence, α and β co-travel forever after some point. First assume $i > j$. Now consider Figure 7:

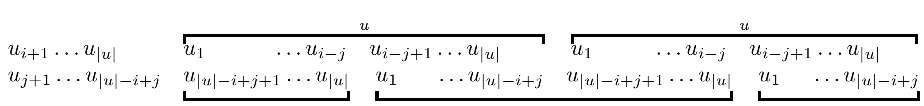

FIGURE 7. Equating α and β as they co-travel forever

By equating words, we deduce that $u^2 = xuy$ where $x = u_1 \dots u_{|i-j|}$ and $y = u_1 \dots u_{|u|-i+j}$. Hence $u = xu' = u''y$ for some words u' and u''. Since $|x| + |y| = |u|$, we must have $|u'| = |y|, |u''| = |x|$ so $u = xy$. Hence $xyxy = u^2 = xuy = xxyy$, so $xy = yx$. If $i < j$, a similar argument shows that gives $x'y'x'y' = u^2 = x'x'y'y'$ for some x' and y'. By part (b) of Lemma 5.1, we find that u is not primitive. □

We will now consider the language of geodesic words for all powers of an infinite order element in a hyperbolic group with k-geodetic Cayley graph.

Proposition 5.3. *Let k be a positive integer, let G be a group with a finite inverse-closed generating set S such that $Cay(G, S)$ is k-geodetic, and let $g \in G$ be an element of infinite order. For all $n \geq 0$, let L_n be the set of geodesic words for g^n with respect to S. If G is hyperbolic then $\bigcup_{n\geq 0} L_n$ is a regular language.*

Proof. For a fixed $r > 0$, there exists a $p \in \mathbb{N}$ such that $g^n \in \{x \in G \mid |x|_{G,S} \geq r\}$ for all $n \geq p$. This is because there is a maximal number of times that powers of g can visit $B_r(1_G)$. Diagrammatically we represent the words in L_n as a shaded region from 1_G to g^n, as seen in Fig. 8.

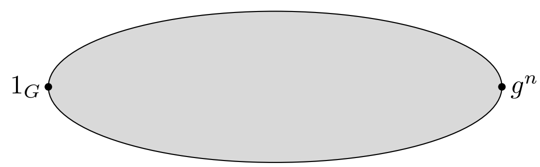

FIGURE 8. Diagrammatic representation of L_n

Claim 1. For a fixed $M_1 > 0$, there exists an n_0 such that

$$L_n = \{a_{(i)}bc_{(j)} \mid i \in [1, k_1], j \in [1, k_2], k_1k_2 \leq k, |b| \geq M_1\},$$

for all $n \geq n_0$, where each $a_{(i)}$ is a geodesic representative for some $g_a \in G$ and each $c_{(j)}$ is a geodesic representatives of some $g_c \in G$.

Proof of Claim 1. By Lemma 4.2, there is a bound B_1 on the length of non-degenerate geodesic bigons. Furthermore, $B_2 = \log_2(k)$ is the maximal number of times that geodesics for the same group element can furcate then rejoin forming non-degenerate geodesic bigons. Then for any $n \geq 0$, $B = B_1B_2$ is the maximum number of total steps that the geodesic words of g^n are not synchronously co-travelling. There is an $n_0 \geq 0$ such that $|g^n|_{G,S} > B + (M_1 - 1)(B_2 + 1) + 1$ for all $n \geq n_0$. Hence, the geodesics of g^n all co-travel for at least $(M_1 - 1)(B_2 + 1) + 1$ steps, and there are at most B_2+1 disjoint segments that are separated by a shaded region of non-unique geodesic segments. By the pigeonhole principle at least one of these disjoint segments has length M_1. Let the word for such a segment be denoted by b. The set of geodesic words from 1_G to where the segment b begins are denoted $a_{(i)}$, where $i \in [1, k_1]$ for some $k_1 \leq k$, and the geodesic words from where b ends are denoted $c_{(j)}$, where $j \in [1, k_2]$ for some $k_2 \leq k$. Note that $k_1k_2 \leq k$, since otherwise we would have $|L_n| > k$, contradicting Cay(G, S) being k-geodetic. ■

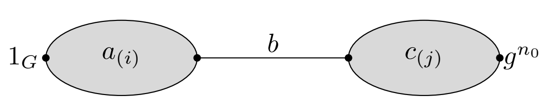

FIGURE 9. A unique factor b in all elements of L_n with $n \geq n_0$

Claim 2. For a fixed $M_2 > 0$, there exists an n_* so that $|g^{n_*+1}|_{G,S} > |g^{n_*}|_{G,S}$ and

$$L_{n_*} = \{\alpha_{(i)}\beta\gamma_{(j)} \mid i \in [1,k_1], j \in [1,k_2], |\alpha_{(i)}|, |\gamma_{(j)}| \geq M_2, k_1k_2 \leq k\},$$

where each $\alpha_{(i)}$ is a geodesic representative for some $g_\alpha \in G$ and each $\gamma_{(j)}$ is a geodesic representative for some $g_\gamma \in G$.

Proof of Claim 2. Take $M_1 = 2M_2$ from Claim 1. Then shift the prefix of b of length M_2 into the left shaded region and shift the suffix of b of length M_2 into the right shaded region. Then for each $n \geq n_0$, we have

$$L_n = \{\alpha_{(i)}\beta\gamma_{(j)} \mid i \in [1,k_1], j \in [1,k_2], |\alpha_{(i)}|, |\gamma_{(j)}| \geq M_2, k_1k_2 \leq k\},$$

where each $\alpha_{(i)}$ is a geodesic representative for some $g_\alpha \in G$ and each $\gamma_{(j)}$ is a geodesic representative for some $g_\gamma \in G$. By the opening statement in the proof of this proposition we can choose $n_* \geq n_0$ to be such that $|g^{n_*+1}|_{G,S} > |g^{n_*}|_{G,S}$. ∎

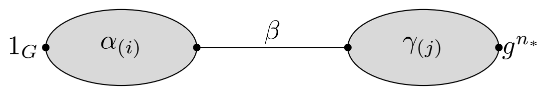

FIGURE 10. Schematic of L_{n_*}

We wish to ensure that the length of geodesics in both shaded regions in Figure 10 is at least the maximal side-length of a non-degenerate geodesic triangle (Lemma 4.2), which we denote by Δ. Hence, in Claim 2 choose n_* to correspond to some $M_2 \geq \Delta$. Now consider the geodesics from 1_G to g^{n_*+1} depicted in Fig. 11.

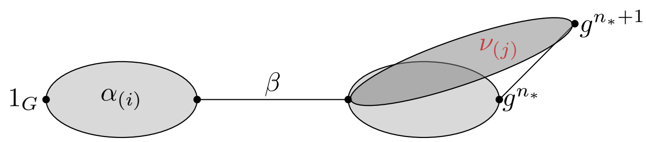

FIGURE 11. Geodesics from 1_G to g^{n_*+1}

Since $|\gamma_{(i)}| \geq \Delta$, the geodesics in L_{n_*+1} share a prefix up to the end of the word β to an element of L_{n_*}. Hence, $L_{n_*+1} = \{\alpha_{(i)}\beta\nu_{(j)} \mid i \in [1,k_1], j \in [1,k_3]\}$ for some positive integer k_3 with $k_1k_3 \leq k$. Instead, let us now consider the geodesics from g^{-1} to g^{n_*} depicted in Fig. 12:

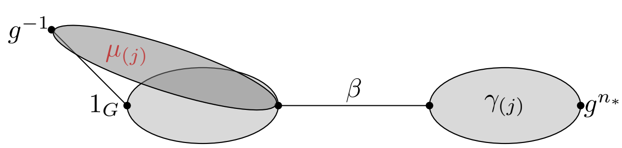

FIGURE 12. Geodesics from g^{-1} to g^{n_*}

Since $|\alpha_{(i)}| \geq \Delta$, the geodesics in L_{n_*+1} share a suffix to an element of L_{n_*} up to the start of the word β. Hence, $L_{n_*+1} = \{\mu_{(i)}\beta\gamma_{(j)} \mid i \in [1,k_4], j \in [1,k_2]\}$ for some positive integer k_4 with $k_4k_2 \leq k$. Therefore, for each $i \in [1,k_1]$ and $j \in [1,k_3]$ there is an $l \in [1,k_2]$ and $m \in [1,k_4]$ such that $\alpha_{(i)}\beta\nu_{(j)} = \mu_{(m)}\beta\gamma_{(l)}$. Since $|g_{n_*+1}|_{G,S} > |g_{n_*}|_{G,S}$, we know that $\alpha_{(i)}$ is a prefix of $\mu_{(m)}$ and $\gamma_{(i)}$ is a suffix of $\nu_{(m)}$. Then we have $\beta x = y\beta$ where $\nu_{(l)} = \alpha_{(i)}y$ and $\mu_{(m)} = x\gamma_{(j)}$. Invoking part (a) of Lemma 5.1 we have $t, s \in S^*$ such that $x = st, y = ts$ and $\beta = (ts)^q t$ for some $q \in \mathbb{N}$. So

$$L_{n_*} = \{\alpha_{(i)}(ts)^q t\gamma_{(j)} \mid i \in [1, k_1], j \in [1, k_2]\}$$

and

$$L_{n_*+1} = \{\alpha_{(i)}(ts)^{q+1} t\gamma_{(j)} \mid i \in [1, k_1], j \in [1, k_2]\}.$$

Since the prefixes $\alpha_{(i)}$ and suffixes $\gamma_{(j)}$ are preserved we can inductively deduce that

$$L_{n_*+c} = \{\alpha_{(i)}(ts)^{q+c} t\gamma_{(j)} \mid i \in [1, k_1], j \in [1, k_2]\}.$$

Hence, we conclude

$$\bigcup_{n\geq 0} L_n = (\bigcup_{n\geq 0}^{n_*-1} L_n) \cup \{\alpha_{(i)}(ts)^{q+c} t\gamma_{(j)} \mid i \in [1, k_1], j \in [1, k_2], c \geq 0\},$$

which is regular. □

We are now ready to prove the second part of Theorem B. The proof follows from the proofs of [6, Thm. 2.3 & Corollary 2.4], but we include it with our own notation for completeness.

Proposition 5.4. *Let k be a positive integer. The centraliser of any infinite order element is infinite cyclic in a hyperbolic k-geodetic group.*

Proof. Let G be a hyperbolic k-geodetic group, and let S be a finite generating set such that $\mathrm{Cay}(G, S)$ is k-geodetic. Let $g \in G$ be an infinite order element.

By Proposition 5.3, if L_n is the set of geodesic words for g^n, then $\bigcup_{n\geq 0} L_n$ is regular. By the pumping lemma for regular languages there is a subset of $\bigcup_{n\geq 0} L_n$ given by $\{xw^i z \mid i \geq 0\}$ such that $|w| \neq 0$. Let y be the primitive root of w, so $w = y^m$ for some $m \geq 0$. Since w^i is a geodesic for all $i \geq 0$, all powers of y are geodesics. For any $i \geq 0$ there exists an index j_i such that $xy^{mi}z$ is a geodesic representative of g^{j_i}. Since there are at most k representatives for a given g^{j_i}, we can choose an $n \geq 0$ such that $j_n < j_{n+1}$.

Since $xw^{n+1}z \equiv g^{j_{n+1}}$ and $xw^n z \equiv g^{j_n}$ we find that $g^{j_{n+1}-j_n} \equiv z^{-1}wz$. Let $h \in C_G(g)$, so

$$y^m(zhz^{-1}) \equiv zz^{-1}wzhz^{-1} \equiv zg^{j_{n+1}-j_n}hz^{-1} \equiv zhg^{j_{n+1}-j_n}z^{-1} \equiv (zhz^{-1})y^m,$$

and by Lemma 5.2 $zhz^{-1} \in \langle y \rangle$. We have shown that any $h \in C_G(g)$ is contained in $\langle z^{-1}yz \rangle \cong \mathbb{Z}$, so $C_G(g) \leq \mathbb{Z}$. The result follows since the only non-trivial subgroups of an infinite cyclic group are infinite cyclic. □

Propositions 4.1 and 5.4 together yield Theorem B. Furthermore, Proposition 5.4 immediately yields the following result.

Corollary 5.5. *Let k be a positive integer and let G be a group with a finite inverse-closed generating set S such that $\mathrm{Cay}(G, S)$ is k-geodetic. If G is hyperbolic, and $g, h \in G$ are commuting non-trivial elements, then:*

(a) *If g has finite order, then gh and h have finite order.*
(b) *If g has infinite order, then h has infinite order and either g and h are inverses or gh has infinite order.*

Remark 5.6. In general, the centraliser of an infinite order element of a virtually-free group is virtually-cyclic (see [1, III. Γ. Cor. 3.10]), so Proposition 5.4 excludes many virtually-free groups from being k-geodetic groups.

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