SOFR Term Structure Dynamics — Discontinuous Short Rates and Stochastic Volatility Forward Rates

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Abstract

As more and more jurisdictions transition from LIBOR–type interest rate benchmarks to new riskfree rate (RFR) benchmarks based on overnight rates, such as SOFR in the US, it is important to adapt interest rate term structure models to reflect this. In particular, overnight rates are largely driven by monetary policy and thus display dynamics that are (at least to first order) piecewise constant between central bank rate decisions, while forward rates continue to evolve in a more diffusive fashion. We construct a tractable multifactor, stochastic volatility term structure model which incorporates these features. Calibrating to prices for options on SOFR futures, we achieve a good fit to the market across available maturities and strikes in a single, consistent model. The model also provides novel insights into SOFR term rate behaviour (and implied volatilities) within the SOFR term rate accrual periods, as well as into empirical mean reversion dynamics.

1 Introduction

Gellert and Schlögl (2021) proposed a model inspired by the empirical behaviour of short rates, where the primary driver is the piecewise constant Fed Funds target rate changing on predictable dates[1], whereas the expectations of those changes embedded in the forward rates evolve in a continuous (i.e., diffusive) manner. This is achieved by defining volatilities for instantaneous forward rates within the Heath, Jarrow and Morton (1992) (HJM) framework in terms of indicator functions dependent on the number of meeting dates between the current and forward time. In the present paper, we extend this model based on further consideration of both empirical time series data as well as cross–sectional calibration to interest rate options. Examining the empirical dynamics of the factors driving the evolution of the term structure of interest rates, one finds evidence of excess kurtosis, which motivates a stochastic volatility extension of the model. In addition to stochastic volatility, in turns out that mean reversion is an important feature to include in the extended model. We also revisit the empirical behaviour of SOFR to find that based on most recent data, it now appears to be driven primarily by the Fed Funds target rate.

The model proposed in this paper endows each factor with its own stochastic volatility and mean reversion. One could label this a "Heston/Hull–White" dynamic. In literature, "Heston/Hull–White" usually refers to Heston (1993) stochastic volatility equity models with an interest rate driven by a Hull and White (1990) model, see for example Grzelak, Oosterlie and Weeran (2008). In the present paper, we instead consider a Markovian (in a small number of state variables), exponential affine interest term structure term structure model where the stochastic volatility follows a Heston–type dynamic, and which in the one–factor, deterministic volatility case collapses back to a Hull/White model. As in Gellert and Schlögl (2021), we use indicator functions in the instantaneous forward rate volatilities to obtain piecewise constant paths of the short rate, while maintaining diffusive dynamics of forward rates maturing beyond the next central bank meeting date.

Several papers consider various empirical aspects of SOFR. Skov and Skovmand (2021) use an arbitrage–free, diffusion–based model of interest rate term structures of the Nelson and Siegel (1987) type, and demonstrate it can, to a degree of accuracy, reflect SOFR futures prices, albeit without taking into account the piecewise constant nature of the underlying SOFR rate. Andersen and Bang (2020) address SOFR spikes in their proposed model also without considering the SOFR rate as being driven by the piecewise constant target rate. Heitfield and Park (2019) perform cross-sectional calibration to futures only and without considering stochastic dynamics of SOFR or forward rates. The model presented in the present paper is motivated by the empirical behaviour of SOFR and SOFR forward rates implied from futures, and we conduct a cross–sectional calibration to both futures and options.

The proposed model performs well in cross–sectional calibration due to having a sufficient amount of variables which control various aspects of model behaviour. This allows the model to be calibrated across different maturities, underlying futures accrual periods and option strikes. This flexibility in the context cross–sectional calibration is similar to

[1] Federal Open Market Committee (FOMC) meeting dates

prominent models deployed in practice: The SABR model, introduced in Hagan, Kumar, Lesniewski and Woodward (2002), can be calibrated to implied volatility convexity and skew across strikes, but generally requires a new calibration per option expiry (and swap/forward rate tenor). Short rate models, such as Hull and White (1990), are usually calibrated to only co-terminal swaptions chosen to match an underlying trade[2] and a singular strike per swaption. The lognormal LIBOR Market Model model[3] (LMM) is well suited for simultaneously calibrating to at the money swaptions across expiries and tenors. Most comparable in terms of ability to calibrate across expiry, underlying tenor and strike are stochastic volatility extensions to the LMM, see for example Piterbarg (2015) or Karlsson, Pilz and Schlögl (2017).

We also present an analysis of the model–implied behaviour of options in the accrual period. Interest in this behaviour is mostly driven by the practicalities of adapting existing LIBOR–based modelling to SOFR and therefore requires casting option behaviour in the accrual period to the behaviour of the dynamics of partially set forward term rates. In this context, we find that under simplifying assumptions our model is consistent with Lyashenko and Mercurio (2019). However, the model presented in this paper handles the case of partially set forwards more naturally than that paper, and also provides more granular insight into the decay characteristics of implied volatility within the accrual period.

Additionally, the proposed model has revealed a connection between forward rate empirical behaviour and short rate mean reversion. In the HJM framework, mean reversion is usually embedded a priori as a decay function of forward rate volatilities. We include both a decay function and a piecewise constant component[4] in the HJM volatility function. However, remarkably we find that the piecewise component derived directly from empirical data without any shape restrictions closely resembles the decay function associated with mean reversion. We discuss the implication from this result in detail in Section 4.

The rest of the paper is organised as follows. The motivation based on empirical (time series and cross–sectional) evidence is discussed in Section 2. The model is formulated in Section 3. The resulting term rate dynamics in the context of accrual period behaviour, mean reversion and option calibration are examined in Section 4. Cross–sectional calibration results are presented in Section 5. Section 6 concludes the paper.

2 Motivation

Three components can been seen to contribute to changes in SOFR:[5] spikes, a spread between the target policy rate and SOFR, and the policy rate itself. All three components historically contributed substantially to the observed daily variance of SOFR.

Spikes have been a well publicised feature of SOFR and reflect imbalances in the underlying overnight repo market. Spikes used to occur regularly at the end of the month

[2]for example call dates in a callable note

[3]See Brace, Gatarek and Musiela (1997), Miltersen, Sandmann and Sondermann (1997) and Jamshidian (1997).

[4]This is in order to achieve a piecewise constant forward rate structure corresponding to FOMC meeting dates.

[5]For a more detailed analysis, see Gellert and Schlögl (2021).

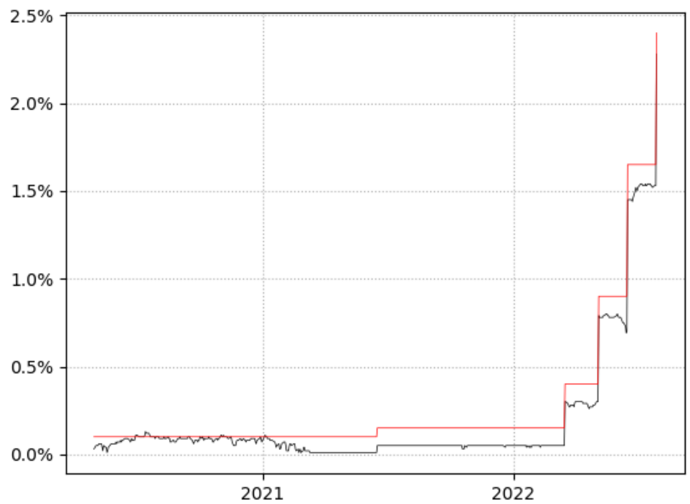

Figure 1: SOFR v FOMC target rate history May 2020 to August 2022

and occasionally on non-end–of–month dates. A very large spike in SOFR in September 2019 motivated the Federal Reserve to take action to effectively stabilise this rate. Since that time, as can be seen in Figure 1, spikes are no longer a feature of SOFR. The variance in SOFR is now dominated by changes in the policy target rate (around 99% of variance), with the remainder of the variance explained by a SOFR to target rate spread, see Figure 3. Consequently, the present paper focuses on modelling the policy target rate component.

Motivation for the construction of our model are the empirical dynamics of the forward rate states, extracted from the data assuming term structure which are piecewise flat between FOMC dates, without any assumptions regarding the driving stochastic dynamics. Applying principal component analysis to obtain orthogonal factors, the time series of these factors clearly fail tests for normality. The quantile/quantile (QQ) plots in Figure 2 compare the expected quantile values for a normal distribution (red line) against the empirical value (blue dots). The dominant three PCA factors shown exhibit clear leptokurtosis, with excess kurtosis of 63, 10 and 2, respectively. The stochastic volatility is a common and parsimonious modelling choice to reproduce this feature.

One of the consequences of linking the model to FOMC dates is that some of the empirical results have a direct economic interpretation. As explained in Gellert and Schlögl (2021), the first factor focuses the dynamics on policy rate changes at the upcoming FOMC meeting, while the higher–order factors tend to focus on FOMC meetings beyond the next one. Excess kurtosis is notably highest for the first factor, suggesting a possible economic link between high leptokurtosis and the next FOMC meeting. This is in line with evidence from interest rate options, which imply a higher stochastic volatility for shorter expiry options. Anecdotally, interest rate market participants tend to focus on the next FOMC

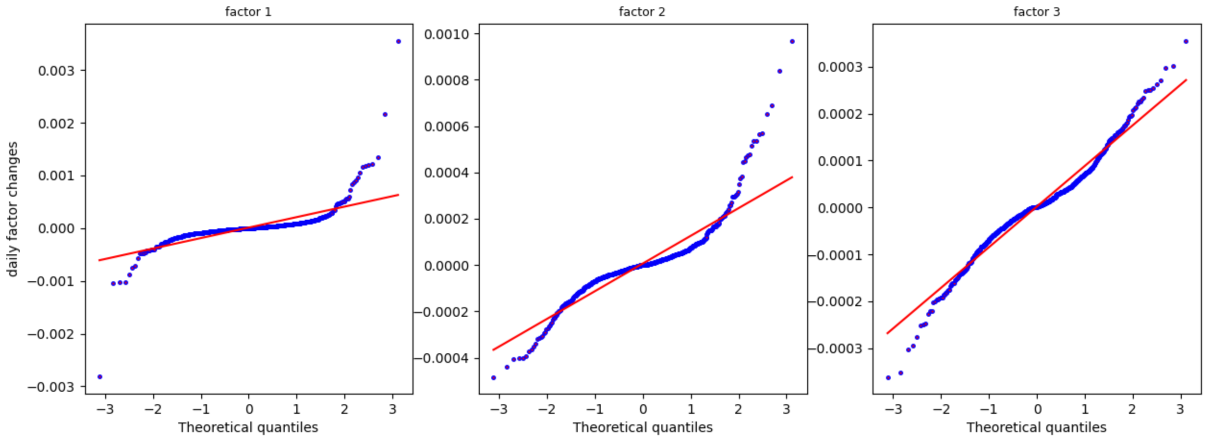

Figure 2: empirical factor states quantile quantile plots

meeting date and the Federal Reserve tends to focus on managing the expectations related to the next FOMC date. This tends to make the expectations related to the next FOMC date most susceptible to news and changing economic circumstances, which offers a possible explanation of the excess kurtosis term structure.

Another important aspect to consider is calibration to interest rate options. In general, calibration to interest rate options requires some degrees of freedom to calibrate the skewness and convexity of implied volatilities for a range of strikes. Interest rate options also tend to imply a term structure of volatility, skewness and convexity for a range of expiries and forward terms. Embedding stochastic volatility into each factor provides the ability to calibrate convexity and skewness[6] in addition to volatility level, with some control of the term structure of those features.

3 Model

3.1 Reconciling piecewise constant short rates with diffusive forward rates

The modelling aim is to reflect the observation that real–world policy targets for the overnight rate are constant between central bank decision dates, the timing of which is typically known beforehand. The literature refers to jumps with deterministic jump times as *stochastic discontinuities*, see for example Kim and Wright (2014), Keller-Ressel, Schmidt and Wardenga (2018), and Fontana, Grbac, Gümbel and Schmidt (2020). In addition, the model should produce forward rates which for maturities beyond the next central bank decision date exhibit a diffusive dynamic. As in Gellert and Schlögl (2021), this is achieved within the HJM framework by specifying instantaneous forward rate volatilities in terms of indicator functions based on the number of decision dates scheduled between the current

[6]Skewness is impacted by the correlation of stochastic volatility to forward rate changes.

Figure 3: SOFR breakdown in vertical order (i) target rates (ii) SOFR-Target Rate spread (iii) variance contribution

("calendar") time t and the forward rate maturity T. Here, we adapt this approach to the case of stochastic volatility. Starting point is the standard HJM result for forward rate dynamics with N factors under the spot risk–neutral measure:

$$f(t,T) = f(0,T) + \sum_{j=1}^{N} \int_{0}^{t} \sigma_j(u,T) \int_{u}^{T} \sigma_j(u,s) ds du + \sum_{j=1}^{N} \int_{0}^{t} \sigma_j(s,T) dW_j(s) \tag{1}$$

Define $\sigma_j(t,T)$ as a piecewise constant function between FOMC meeting dates:

$$\sigma_j(t,T) = \sigma_j \sum_{i=1}^{n} \gamma_{i,j} \mathbb{1}(i \leq \mathcal{A}_{t,T}) \tag{2}$$

where n is the total number of meetings dates and $\mathcal{A}_{t,T}$ reflects the number of meeting dates between t and T:

$$\mathcal{A}_{t,T} := \left|\{x_1, ..., x_m | t < x_i \leq T\}\right| \tag{3}$$

σ_j and $\gamma_{i,j}$ scale the volatility loading of each component of the driving (vector–valued) Brownian motion. σ_j allows control of the overall level of variance and is the key variable used in calibration to option prices. $\gamma_{i,j}$ scales the volatility based on the number of FOMC meeting dates between t and T. It can be empirically derived to reflect the covariance structure between forward rates.[7] Solving the stochastic integral yields:

$$
\begin{aligned}
f(t,T) - f(0,T) = &\sum_{j=1}^{n}\sum_{q=1}^{n}\sum_{i=1}^{n} \sigma_j^2 \gamma_{q,j}\gamma_{i,j} \int\limits_0^t \mathbb{1}(q \le \mathcal{A}_{u,T}) \int\limits_u^T \mathbb{1}(i \le \mathcal{A}_{u,s}) ds du \\
&+ \sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_j \gamma_{i,j} \mathbb{1}(i \le \mathcal{A}_{0,T}) W_j(t \wedge x_{\bar{\imath}(T)})
\end{aligned} \tag{4}
$$

where $\bar{\imath}(T) = \mathcal{A}_{0,T} - i + 1$. The solution reveals that the total variance is an increasing function of the number of meeting dates between 0 and T, up to the minimum of t and the last meeting date before T. This implies that the variance of the forward rate is zero if the forward date occurs prior to the next meeting date.

3.2 Introducing stochastic volatility

We introduce stochastic volatility into the model in a way that is inspired by what can be called a Heston/Hull–White (HHW) "quasi-Gaussian" model. This builds on the Gaussian Hull–White model by adding a Heston–type stochastic volatility component. Start with a one–factor quasi-Gaussian model (QG1) with the volatility function of instantaneous forward rates given by

$$
\sigma(t,T) = \chi(t)\phi(T) \tag{5}
$$

where $\chi(t)$ is generally stochastic. Under the spot risk–neutral measure we can write the dynamics of the instantaneous forward rates as

$$
df(t,T) = F(t,T)dt + \sigma(t,T)dW(t) \text{ where } F(t,T) = \sigma(t,T)\int\limits_t^T \sigma(t,u)du \tag{6}
$$

$$
\implies f(t,T) - f(0,T) = \int\limits_0^t F(s,T)ds + \phi(T)\int\limits_0^t \chi(s)dW(s) \tag{7}
$$

Then set $T = t$ and differentiate with respect to t to express the spot rate $r(t)$ in the form

$$
\begin{aligned}
r(t) - f(0,t) = x(t) &= \int\limits_0^t F(s,t)ds + \phi(t)\int\limits_0^t \chi(s)dW(s), \quad x(0) = 0 \\
dx(t) &= \frac{d}{dt}\left\{\int\limits_0^t F(s,t)ds\right\}dt + \phi'(t)\int\limits_0^t \chi(s)dW(s) + \phi(t)\chi(t)dW(t) \\
&= \frac{d}{dt}\left[\int\limits_0^t F(s,t)ds\right]dt + \frac{\phi'(t)}{\phi(t)}\left[x(t) - \int\limits_0^t F(s,t)ds\right]dt + \sigma(t,t)dW(t)
\end{aligned} \tag{8}
$$

[7]This is derived in Gellert and Schlögl (2021) using principal component analysis (PCA) and in fact corresponds to the PCA eigenvectors.

Define

$$\phi(T) = \exp\bigg(-\int_0^T \lambda(v)dv\bigg) \implies \frac{\phi'(t)}{\phi(t)} = -\lambda(t) \tag{9}$$

$$\chi(t) = \sigma(t)\exp\bigg(\int_0^t \lambda(v)dv\bigg) \implies \sigma(t,T) = \chi(t)\phi(T) = \sigma(t)\exp\bigg(-\int_t^T \lambda(v)dv\bigg) \tag{10}$$

therefore

$$F(t,T) = \sigma^2(t)\exp\bigg(-\int_t^T \lambda(v)dv\bigg)\int_t^T \exp\bigg(-\int_t^u \lambda(v)dv\bigg)du, \quad F(t,t) = 0 \tag{11}$$

Hence σ inherits the stochasticity of $\chi, \sigma(t,t) = \sigma(t)$ and the SDE changes to

$$dx(t) = \bigg\{\frac{d}{dt}\bigg[\int_0^t F(s,t)ds\bigg] + \lambda(t)\int_0^t F(s,t)ds\bigg\}dt - \lambda(t)x(t)dt + \sigma(t)dW(t) \tag{12}$$

in which the part of the drift term involving $F(t,T)$ simplifies to

$$\begin{aligned}\Phi(t) &= F(t,t) + \int_0^t \frac{\partial}{\partial T}F(s,T)\bigg|_{T=t} ds + \lambda(t)\int_0^t F(s,t)ds \\ &= \int_0^t \sigma^2(s)\exp\bigg(-2\int_s^t \lambda(v)dv\bigg)ds = \int_0^t \sigma^2(s,t)ds\end{aligned} \tag{13}$$

The volatility $\sigma(.)$ is made stochastic by incorporating a Heston process $v(.)$ in it:

$$\sigma(t) \to \sigma(t)\sqrt{v(t)} \tag{14}$$

which results in an affine system of stochastic differential equations, which can be expressed under the spot risk–neutral measure as

$$\begin{aligned}dx(t) &= [\Phi(t) - \lambda(t)x(t)]dt + \sigma(t)\sqrt{v(t)}dW(t), \quad x(0) = 0 \\ d\Phi(t) &= [\sigma^2(t)v(t) - 2\lambda(t)\Phi(t)]dt, \quad \Phi(0) = 0 \\ dv(t) &= \theta(t)(1 - v(t))dt + \alpha(t)\sqrt{v(t)}dU(t), \quad v(0) = 1\end{aligned} \tag{15}$$

with

$$\langle dW(.), dU(.)\rangle(t) = \rho dt \tag{16}$$

Bond price dynamics can be written as follows (see Appendix A for derivation):

$$B(t,T) = \exp\biggl(-\int_t^T f(t,u)du\biggr) = \frac{B(0,T)}{B(0,t)}\exp\biggl(-\Lambda(t,T)y(t) - \frac{1}{2}\Phi(t)\Lambda^2(t,T)\biggr) \tag{17}$$

where:

$$\Lambda(t,T) = \int_t^T \exp\biggl(-\int_t^u \lambda(v)dv\biggr)du \tag{18}$$

$$\Phi(t) = \int_0^t \sigma^2(s)\exp\biggl(-2\int_s^t \lambda(v)dv\biggr)ds \tag{19}$$

3.3 Piecewise constant short rates with diffusive forward rates under stochastic volatility

Merging the modelling of Sections 3.1 and 3.2, assume now that each factor evolves with its own, independent Heston–type stochastic volatility. That is, each factor in the model of Section 3.1 is extended in the same manner as the single factor in Section 3.2. This model thus inherits the piecewise constant short rates with diffusive forward rates, but with stochastic volatility dynamics.

This set–up provides ample flexibility to calibrate to the volatility term structure (since each factor impacts different aspects of the forward rate term structure), as well as option–implied volatility skew and smile across different expiries. The level of flexibility is regulated by the number of factors and degree of time–dependence of the model parameters.

Starting point is again the standard HJM result for forward rate dynamics with N factors under the spot risk–neutral measure:

$$f(t,T) = f(0,T) + \sum_{j=1}^{N}\int_0^t \sigma_j(u,T)\int_u^T \sigma_j(u,s)dsdu + \sum_{j=1}^{N}\int_0^t \sigma_j(s,T)dW_j(s) \tag{20}$$

We define the j-th component of the instantaneous forward rate volatility function as follows:

$$\sigma_j(t,T) = \sum_{i=1}^{n} \mathbb{I}_{\{i\leq \mathcal{A}(t,T)\}}\chi_j(t)\phi_j(T)\gamma_{i,j} \tag{21}$$

where

$$\phi_j(T) = \exp\biggl(-\int_0^T \lambda_j(s)ds\biggr) \tag{22}$$

and

$$\chi_j(t) = \sigma_j(t)\sqrt{v_j(t)}\exp\biggl(\int_0^t \lambda_j(s)ds\biggr) \tag{23}$$

$v(t)$ evolves with a Heston–type dynamic:

$$dv(t) = \theta(t)(1 - v(t))dt + \alpha(t)\sqrt{v(t)}dU(t), \quad v(0) = 1 \tag{24}$$

with

$$\langle dW_j(.), dU_j(.)\rangle(t) = \rho_j dt \tag{25}$$

and

$$\langle dW_i(.), dU_j(.)\rangle(t) = 0, \text{ for } i \neq j \tag{26}$$

The bond price dynamics for a single factor[8] can be written as (see Appendix A for derivation):

$$\begin{aligned}
B(t,T) &= \exp\Big(- \int_t^T f(t,u)du \Big) \qquad (27)\\
&= \frac{B(0,T)}{B(0,t)}\exp\Big(- \sum_{b=0}^{\eta(t)-2} \Lambda_{x_{\eta(T)-1}}(x_{b+1},T) y_{\eta(T)-1}(x_{b+1}) - \Lambda_{x_{\eta(T)-1}}(t,T) y_{\eta(T)-1}(t)\\
&- \sum_{k=\eta(t)}^{\eta(T)-2} \sum_{b=0}^{\eta(t)-2} \Lambda_{x_k}(x_{b+1}, x_{k+1}) y_k(x_{b+1}) - \sum_{k=\eta(t)}^{\eta(T)-2} \Lambda_{x_k}(t, x_{k+1}) y_k(t)\\
&- \frac{1}{2} \sum_{b=0}^{\eta(t)-2} \sum_{i=1}^{\eta(T)-1-b} \sum_{j=1}^{\eta(T)-1-b} \gamma_i \gamma_j \Phi(x_b, x_{b+1}) \{\Lambda^2(x_{b+1}, T) - \Lambda^2(x_{b+1}, x_{\eta(T)-1})\}\\
&- \frac{1}{2} \sum_{i=1}^{\eta(T)-\eta(t)} \sum_{j=1}^{\eta(T)-\eta(t)} \gamma_i \gamma_j \Phi(x_{\eta(t)-1}, t) \{\Lambda^2(t,T) - \Lambda^2(t, x_{\eta(T)-1})\}\\
&- \sum_{k=\eta(t)}^{\eta(T)-2} \frac{1}{2} \sum_{b=0}^{\eta(t)-2} \sum_{i=1}^{k-b} \sum_{j=1}^{k-b} \gamma_i \gamma_j \Phi(x_b, x_{b+1}) \{\Lambda^2(x_{b+1}, x_{k+1}) - \Lambda^2(x_{b+1}, x_k)\}\\
&- \sum_{k=\eta(t)}^{\eta(T)-2} \frac{1}{2} \sum_{i=1}^{(k-\eta(t)+1)} \sum_{j=1}^{(a-\eta(t)+1)} \gamma_i \gamma_j \Phi(x_{\eta(t)-1}, t) \{\Lambda^2(t, x_{k+1}) - \Lambda^2(t, x_k)\}\Big) \qquad (28)
\end{aligned}$$

3.4 SOFR term rates

The transition of the key (US dollar) interest rate index from LIBOR to SOFR (with similar transitions for many other curencies) imposes on the market a change from benchmark rates set for a longer term (usually three months) to rates with an effective term of one business day. Transitioning to daily frequency for derivative instruments would not be desirable for many reasons, including burdening the system with a large increase in transaction volumes

[8]Since the model specification results in driving factors which are mutually independent, the generalisation of this expression to the multifactor case is straightforward, though notationally tedious.

to settle daily flows. Instead, the market is adopting an approach where instruments are still defined with longer term rates, but those term rates are now calculated using either a compounding or averaging of SOFR over the term. This is what is typically called "term SOFR."[9]

A LIBOR term would be defined by the start date T_i and an end date T_k of the period over which it applies. A SOFR term for the corresponding dates is defined as a set of discrete dates $\{T_i, , , T_k\}$ on which SOFR is observed. The most common definition of term SOFR is based on compounding over the term (usually 3m):

$$S(T_i, T_k) = \tau_{i,k}\left[\prod_{j=i}^{k}(1 + s(T_j)\delta_j) - 1\right] \tag{29}$$

where $\tau_{i,k}$ is the year fraction of the term length and $s(t)$ is the SOFR observed set for T_j. δ_j is the year fraction for the period between T_j and T_{j+1}, in order to account for days on which SOFR is not observed (weekends and holidays). For the empirical results presented in this paper, we make the assumption that the daily SOFR rate is approximated by the continuous short rate $r(t)$.

3.5 Pricing Futures

Define a 3M SOFR futures contract $F(T_i, T_k)$ with accrual period starting at T_i and ending at T_k, with payoff measurable at T_k:

$$F(T_i, T_k) = 100\Big(1 - S(T_i, T_k)\Big) \tag{30}$$

where $\delta_{i,k}$ is the year fraction between T_i and T_k. Using the generic futures pricing theorem, the time t futures price $F(t, T_i, T_k)$ is given by the expected value at t under spot risk–neutral measure, i.e.

$$F(t, T_i, T_k) = E_\beta\Big[F(T_i, T_k)|\mathcal{F}_t\Big] \tag{31}$$

3.6 Pricing Options on Futures

Options on 3M SOFR futures exist for a variety of strikes and expiries. They are specified with American–style exercise, but we use them to approximate European–style implied volatilities, as is common in practice. We do not address the impact of the American exercise in this paper, instead we use these options to demonstrate the ability of the model to calibrate to a variety of strikes and expiries. The value of a call option at time t,

[9] If one takes into account the "multicurve" phenomenon observed in interest rate markets, these "SOFR term rates" are more akin to rates implied by overnight index swaps (OIS) than actual term rates such as LIBOR, see Alfeus, Grasselli and Schlögl (2020) and Backwell, Macrina, Schlögl and Skovmand (2019). However, here we only consider instruments referencing SOFR, so this distinction is not needed in the present paper.

expiring at $T_e < T_i$ with strike K, on the futures contract, can be expressed as the expected discounted payoff under the spot risk–neutral measure:

$$C(t,T_e,F(T_i,T_k),K) = E_\beta\left[\frac{1}{\beta(T_e)}(F(T_i,T_k)-K)^+|\mathcal{F}_t\right] \tag{32}$$

3.7 Simulating the model

As an initial proof of concept, particularly the ability of the model to calibrate to options on SOFR futures, we price options by Monte Carlo simulation.[10] For the stochastic integral component in Equation (20), we have:

$$\int_0^t \sigma_j(s,T)dW_j(s) = \sum_{i=1}^n \mathbb{I}_{\{i\leq\mathcal{A}(t,T)\}}\gamma_{i,j}\phi_j(T)\int_0^t \sigma_j(s)\sqrt{v_j(s)}\exp\left(\int_0^s \lambda_j(q)dq\right)dW_j(s) \tag{33}$$

Setting constant parameters $\sigma_j(s) = \sigma_j$ and $\lambda_j(q) = \lambda_j$:

$$\int_0^t \sigma_j(s)\sqrt{v_j(s)}\exp\left(\int_0^s \lambda_j(q)dq\right)dW_j(s) = \sigma_j\int_0^t \sqrt{v_j(s)}e^{s\lambda_j}dW_j(s) \tag{34}$$

The stochastic component is approximated as follows:

$$\int_0^t \sqrt{v_j(s)}e^{s\lambda_j}dW_j(s) \approx \frac{1}{N}\sum_{p=1}^N g(t) \tag{35}$$

where

$$g_j(t) = \int_0^t \sqrt{v_j(s)}e^{s\lambda_j}dW_j(s) \tag{36}$$

calculated with Euler discretisation:

$$\Delta g_j(s) = \sqrt{v_j(s)}e^{s\lambda_j}\Delta W_j(s) \tag{37}$$

where $\Delta W_j(s) \sim N(0,\sqrt{\Delta t})$, $v_j(s) = v_j(s-\Delta t) + \Delta v_j(s)$ and:

$$\Delta v_j(s) = \theta(1-v_j(s-\Delta t))\Delta t + \alpha\sqrt{v_j(s-\Delta t)}\Delta U_j(s), v_j(0) = 1 \tag{38}$$

where $\Delta U_j(s) \sim N(0,\sqrt{\Delta t})$ and $\langle\Delta W_j(.),\Delta U_j(.)\rangle(s) = \rho_j\Delta t$

[10]The HHW stochastic volatility dynamics assumed in our model would permit the derivation of semi-analytical option pricing formulae using Fourier transform techniques, but we leave such derivations to future work.

4 Term rate dynamics

4.1 Accrual period

A prevalent approach in the LIBOR to SOFR transition, as reflected in literature (see Lyashenko and Mercurio (2019)), is the adaptation of existing LIBOR–based modelling to SOFR. A highly practical problem stemming from this approach is the behaviour of options in the accrual period of the SOFR term rate, i.e. for term forwards $S(T_i, T_k)$ at time $T_i < t \leq T_k$. This occurs when the expiry of the option is set past the beginning of the accrual period.

Examples of impacted options are in–arrears SOFR caps and exchange–traded options on 1M SOFR futures.[11] Options on averaging and compounding SOFR term rates can be simply thought of as average rate options on the short rate. However, existing LIBOR–based pricing models require an artificially induced decay of the "SOFR term rate" volatility within its accrual period. To this effect, Lyashenko and Mercurio (2019) suggest implied volatility as a linearly decaying function of the accrual time.

In contrast, the model proposed in this paper handles the case of partially set forwards naturally and also provides an alternative insight into the decay characteristics of implied volatility within the accrual period. In such a factor model of SOFR forward rates the entire term structure is available at any forward simulation point without additional simulation cost. The appropriate dynamics are embedded in the partially set term forwards by evolving each SOFR forward rate up to its observation/accrual time. As shown in Figure 4, setting a constant volatility level σ and removing the indicator functions in the HJM volatility function results in a linearly decaying implied volatility. This is consistent with the *ad hoc* assumption in Lyashenko and Mercurio (2019), but here it results directly from model behaviour.

However, a different behaviour of implied volatility appears when forward rate volatility is driven by FOMC meetings and therefore there is zero volatility in the period between the end of the accrual period and the last FOMC meeting within the accrual period. As shown in Figure 4, this results in an accelerating decay in implied volatility to zero at the final meeting date prior to the end of the accrual period. Note that this analysis does ignore volatility of the SOFR to policy target rate spread, but this is a second order effect. Spread volatility contributes relatively little to the variance of SOFR, suggesting the implication on the accrual period dynamics is likely to be accurate — a prediction of the model which can be verified once options on 1M SOFR futures become sufficiently liquid in the market.

[11] Options on 3M futures expire prior to the accrual period, hence are not impacted by behaviour during the accrual period.

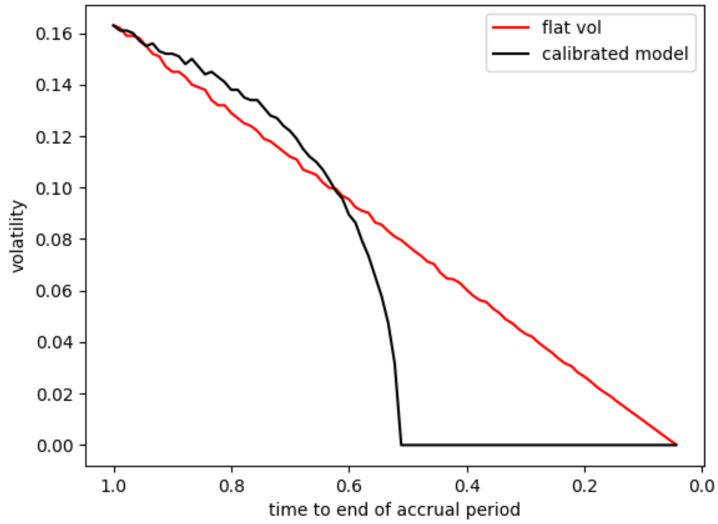

Figure 4: Accrual period term volatility comparison

4.2 Mean reversion

Mean reversion is embedded in the model in the definition of $\sigma_j(t,T)$ which can be rewritten as follows:

$$\sigma_j(t,T) = \sigma_j(t)\sqrt{v_j(t)}\exp\bigg(-\int_t^T \lambda_j(s)ds\bigg)\sum_{i=1}^{n}\mathbb{I}_{\{i\le\mathcal{A}(t,T)\}}\gamma_{i,j} \tag{39}$$

On the one hand, mean reversion is reflected in the term $\exp\bigg(-\int_t^T \lambda_j(s)ds\bigg)$, resulting in the volatility of instantaneous forward rates decaying as a function of time to maturity $(T-t)$. On the other hand, the $\gamma_{i,j}$ vector scales the volatility function based on the number of FOMC meetings between t and T and as such has an inherent dependence on $T-t$. Therefore, for a given $\lambda_j(s)$ function, it is possible to define $\gamma_{i,j}$ such that:

$$\sum_{i=1}^{n}\mathbb{I}_{\{i\le\mathcal{A}(t,T)\}}\gamma_{i,j} \approx \exp\bigg(-\int_t^T \lambda_j(s)ds\bigg) \tag{40}$$

That is, it is possible to set $\lambda_j = 0$ and mimic mean reverting dynamics with the appropriate choice of $\gamma_{i,j}$. In Gellert and Schlögl (2021), $\gamma_{i,j}$ is derived from PCA of forward states implied from SOFR and Fed Funds futures. The states are derived assuming piecewise flat structure between FOMC dates without any assumptions regarding the driving dynamics, in turn allowing for empirical assessment of the state dynamics. The first factor, i.e for $j=1$, explains a large proportion (around 80%) of the forward state variance. It has a clear economic interpretation of focusing forward rate dynamics on the changing expectations

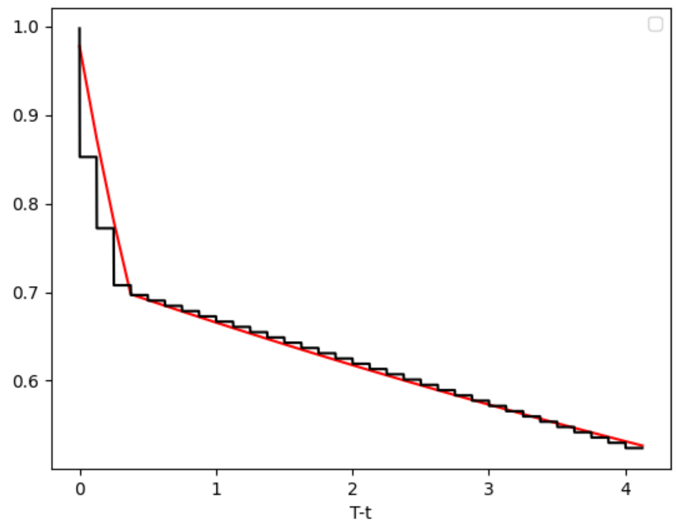

Figure 5: Comparison of $\exp\left(-\int_t^T \lambda_j(s)ds\right)$(red) and $\sum_{i=1}^n \mathbb{I}_{\{i\le\mathcal{A}(t,T)\}}\gamma_{i,j}$(black)

related to the change in policy rate at the FOMC date immediately following t. This in itself is an intuitively agreeable insight: forward rate dynamics are largely driven by changing expectations of the next move in the policy rate. However, it is critical to mimicking mean reverting behaviour in this way that $\gamma_{i,j}$ has the opposite sign between $\gamma_{1,1}$ and $\gamma_{i,1}$ for $i > 1$. Inspection of the empirically derived $\gamma_{i,j}$ vector for $j = 1$, reveals that it is now possible to choose $\lambda_j(s)$ such that:

$$\exp\left(-\int_t^T \lambda_j(s)ds\right) \approx \sum_{i=1}^n \mathbb{I}_{\{i\le\mathcal{A}(t,T)\}}\gamma_{i,j} \tag{41}$$

by setting:

$$\lambda_j(s) = \begin{cases} 0.9, & s-t<0.5 \\ 0.08, & \text{otherwise} \end{cases} \tag{42}$$

which results in the comparison shown in Figure 5, demonstrating it how it is possible to obtain the behaviour implied by the (PCA–derived) γ from an appropriate choice of $\lambda_j(s)$. It is clear that most of the difference stems from the continuous and piecewise definitions, but both approaches are very similar in terms of embedding mean reversion dynamics. Thus the estimation of the proposed model creates an implicit connection between forward rate dynamics driven by the next policy rate change and mean reverting behaviour of the short rate.

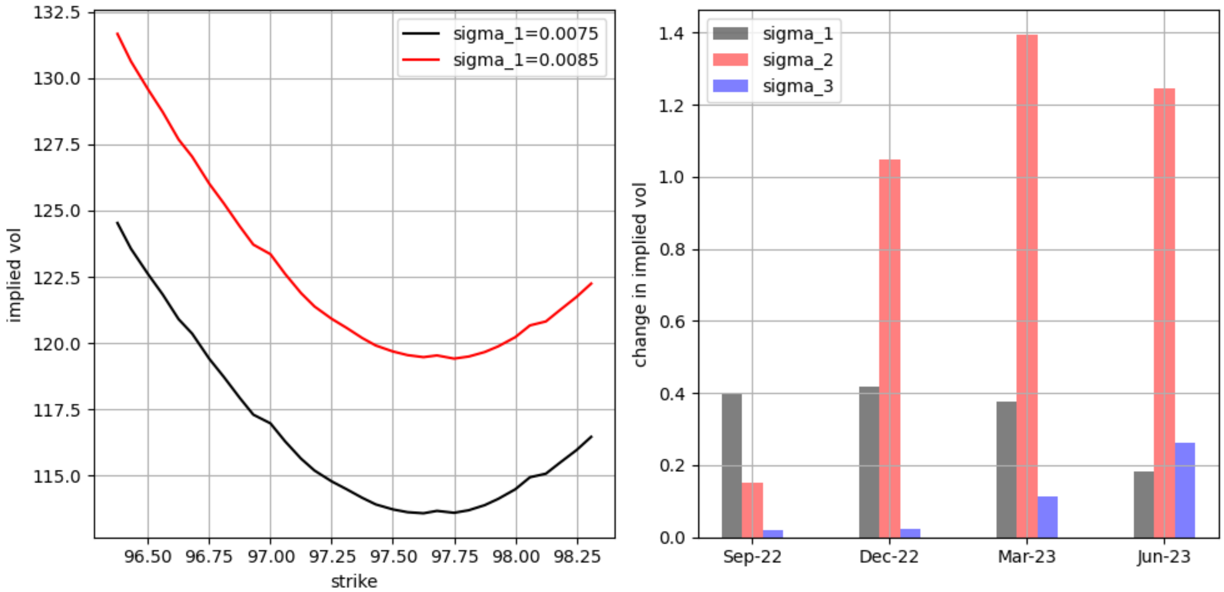

Figure 6: (LHS) Implied volatility for different values of σ. (RHS) ATM implied volatility sensitivity across contracts to changes in σ.

4.3 Factor Sensitivities

Calibration to interest rate options requires the model to fit the first four moments of terminal distributions across different expiries and tenors. The moments of a terminal distribution at a specific expiry are usually characterised in terms of implied volatilities across different strikes. In this representation the first moment corresponds to a horizontal shift in the implied volatilities (across strikes), second moment a vertical shift (across all implied volatilities), third moment a gradient shift and the fourth moment a shift in convexity. Using this characterisation we demonstrate the flexibility of the model proposed in this paper to control the moments of the distribution as well as their term structures across different expiries.

Using a model calibration on the 10-June-2022 to the first four quarterly SOFR options, including all available strikes, we study the sensitivity of implied volatilities to the model parameters. Starting with σ, see Figure 6, it is apparent that changing this variable results in a parallel shift in the implied volatilities, thereby controlling the second moment. The table in Fig.6, shows how different factors impact different expiries with factor 1 focused on short expiries, while factors 2 and 3 increasingly focus on the longer expiries, providing calibration flexibility across the term structure.

The α parameter determines the level of stochastic volatility in the models, usually associated with the fourth moment. As shown in Figure 7, changing the α parameter results in a change in convexity as well as the level of volatilities. Control of just convexity, without changing ATM volatilities, is possible by combining offsetting changes in the σ parameter. The table in Figure 7 shows different impact on convexity from different factors across

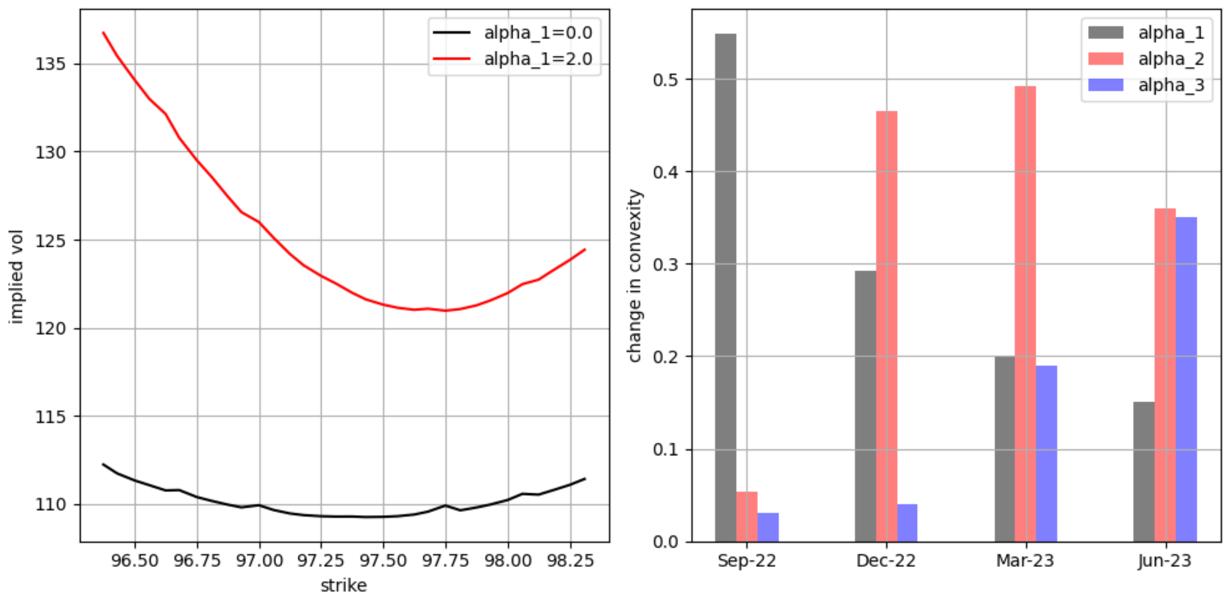

Figure 7: (LHS) Implied volatility for different values of α. (RHS) Implied volatility convexity sensitivity across contracts to changes in α.

expiries, enabling the model to calibrate to different stochastic volatility term structures.

The ρ parameter determines the correlation between stochastic volatility and the forward rates. As can be seen in Figure 8, changing the ρ parameter results in a gradient change in implied volatilities, corresponding to a change in the third moment. Similarly to the other parameters, the impact on implied volatility skewness varies for different factors across expiries, allowing the model be calibrated to different correlation term structures.

λ and θ are two variables associated with mean reversion. The λ parameter controls mean reversion of the forward rates while the θ parameter controls the mean reversion of the stochastic volatility. From an implied volatility perspective, as shown in Figure 9, the mean reversion parameters work in reverse to their corresponding volatility parameters. The λ parameters offset the impact from σ and result in a parallel change in implied volatility with the opposite sign to the change in the parameter. The θ parameter reverses the α parameter and therefore results in both a level and convexity change in the implied volatilities.

Thus the model proposed has the flexibility to attempt simultaneous calibration to option–implied volatilities across both strikes and expiries. Additional flexibility for calibration comes from the ability to define the variables as functions of time. In the next section, we demonstrate the model's ability to calibrate to market data on options on SOFR futures.

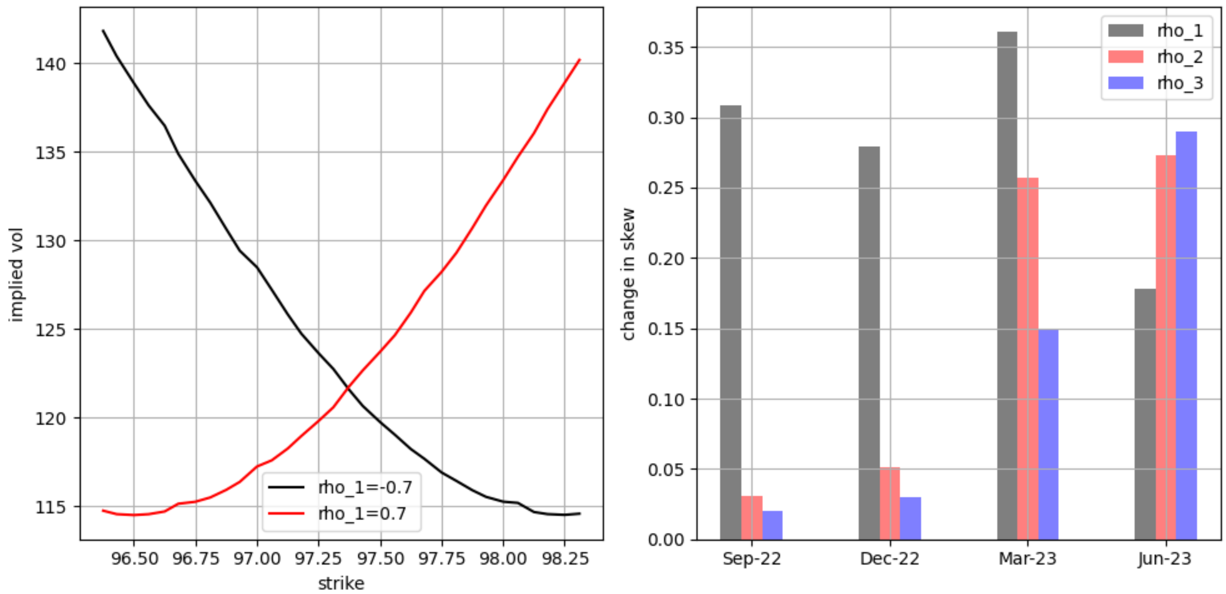

Figure 8: (LHS) Implied volatility for different values of ρ. (RHS) Implied volatility skew sensitivity across contracts to changes in ρ.

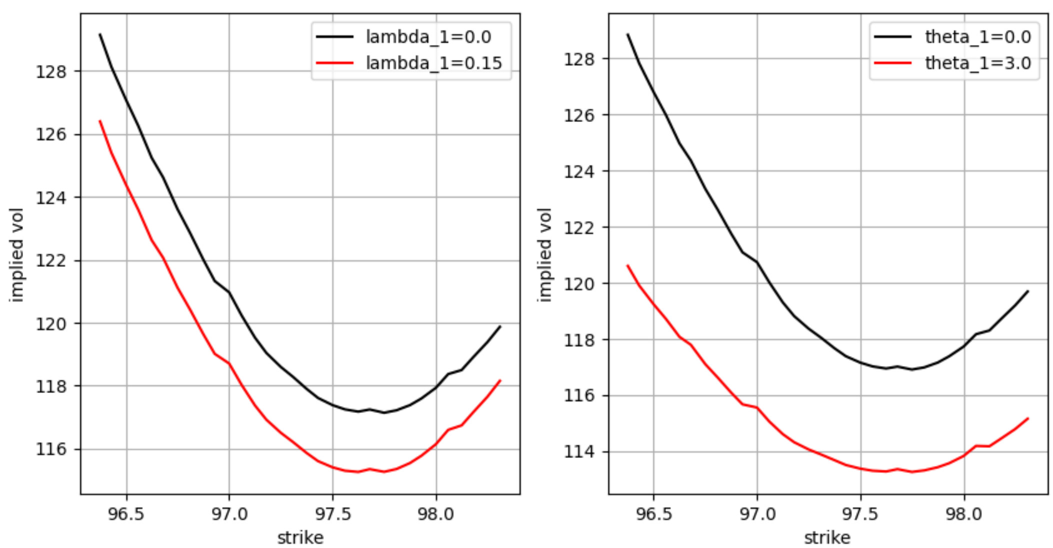

Figure 9: Varying lambda and theta.

5 Calibration to options

The ability to calibrate to cross sectional option data is an important feature of interest rate models. Although it violates the model assumptions, it is standard practice in industry to

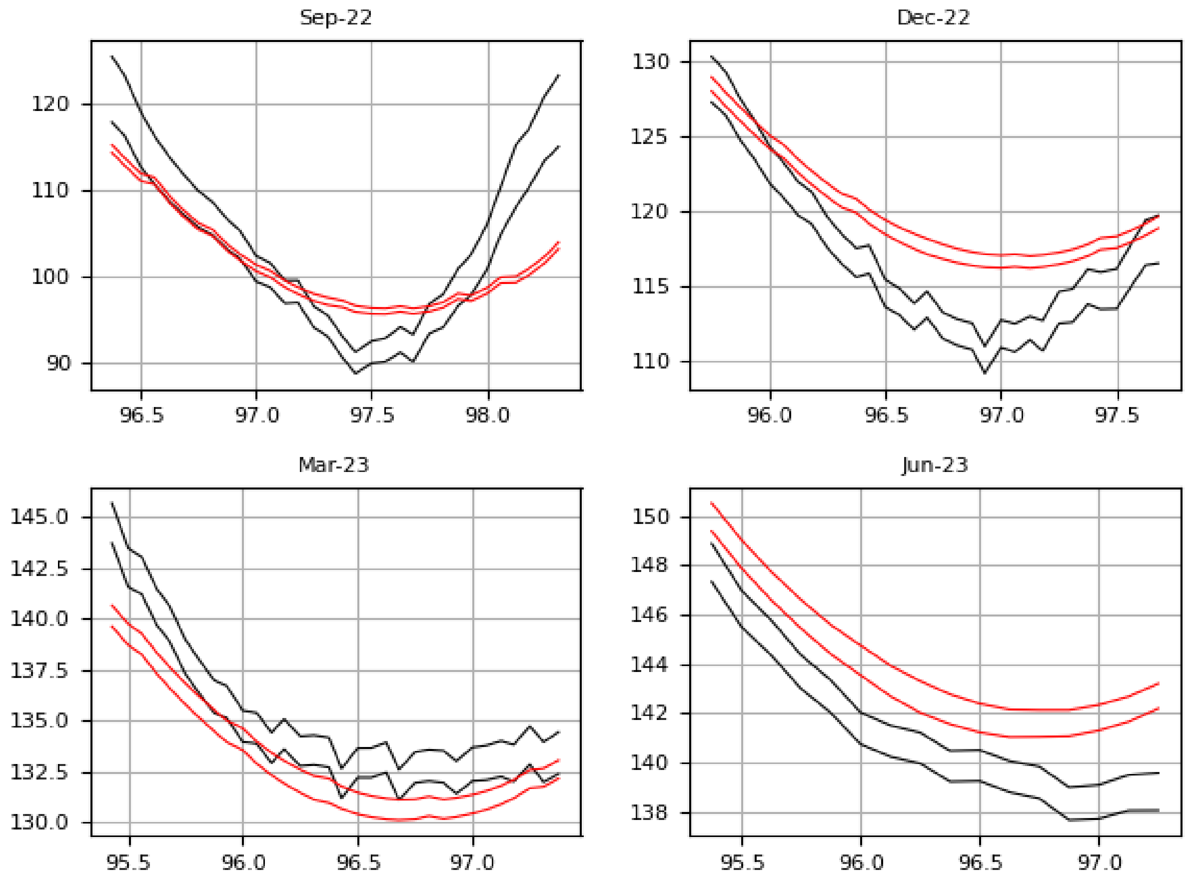

Figure 10: Calibration with constant parameters.

recalibrate interest rate models on a daily basis to vanilla instruments such as swaptions and caps. Calibration using options on futures is less common, but options on SOFR futures have been one of the first SOFR–related option instruments to trade since the inception of the new benchmark. These are also the only SOFR–related options traded directly on an exchange, meaning that the price information is widely available for research purposes.

At the time of writing, most of the market liquidity in options on SOFR futures is concentrated on the front four options on 3M SOFR futures. Arguably, shorter expiry interest rate options are the most difficult to fit, due to steep and highly variable term structures in implied volatilities and implied kurtosis, as is evident in the data set used for this section. This makes calibration to these options a good proof–of–concept test to assess the model's calibration capability.

For the calibration we take the γ parameters from the empirical estimation performed in Gellert and Schlögl (2021). The remaining parameters σ, α, λ, θ and ρ are calibrated to option prices. In the calibration, σ controls the general level of volatility, and the mean reversion parameter λ gives some control of volatility levels across expiries. α controls the

parameters	σ_1	σ_2	σ_3	λ_1	λ_2	λ_3	α_1	α_2	α_3	ρ_1	ρ_2	ρ_3	θ_1	θ_2	θ_3
constant	0.0081	0.006	0.0041	0.01	0	0.16	1.57	0.82	0	-0.2	0	0	0	0	0
time dependent	0.00663	0.00587	0.00447	0.02	0.004	0.35	[3.142, 1.35, 3.2, 0.86]	[0.76, 0.66, 0.6, 0.22]	[3.0, 0.6, 0.5, 4.1]	-0.14	-0.025	-0.83	0.1	0	11.0

Table 1: parameter table

level of kurtosis, and the stochastic volatility mean reversion parameter α gives some control of kurtosis across expiries. The correlation parameter ρ controls the implied volatility skew.

Each of the model calibration parameters can be defined as a function of time. Combined with the ability to choose the number of HJM factors, this provides significant flexibility in the model for calibration. We begin by performing the calibration with the parameters constant across time before adding time dependent parameters. The results are presented as a comparison of normal (i.e., "Bachelier") volatilities implied from the bid/offer prices taken from settlement price information on the 10-June-2022, with the 5% confidence interval for the calibrated model price based on simulation results.

The calibration results shown in Figure 10 show that the model can be fitted to general volatility levels, skew and convexity, though insufficiently to match market–implied volatilities. The main features of market–implied volatilities is the sharply declining convexity as a function of expiry. Another feature is the term structure in skew slightly declining as a function of expiry. As can be seen in Table 1, with constant parameters the calibration focuses on α_0, which is the stochastic volatility parameter associated with the first factor. This understates the convexity on the first expiry and overstates for the longest expiry, thus effectively freezing factor 2 and 3 stochastic volatility (α_1, α_2) at zero. These results suggest the introduction of time dependent stochastic volatility parameters.

We define stochastic volatility as a function of t, with parameters piecewise constant between the option expiry dates. As shown in Figure 11, this change provides enough flexibility across different expiries to results in a large improvement in model fit. With the added time dependence, the first stochastic volatility parameter α_0 has increased for short expiries and decreased for longer expiries, as opposed to the results obtained with constant parameters.

This example demonstrates the flexibility of the model. Based on the calibration with constant parameters, we were able to make an informed choice with respect to which parameters could be made time dependent to benefit the calibration. We only had to change three of the fifteen available parameters to achieve a much better calibration results, albeit for a limited set of calibration instruments. We believe this same approach could be repeated for a larger set of more traditional calibration instruments, making other parameters time dependent or increasing the number the factors if required.

6 Conclusion

The model proposed in this paper is an outcome of a data–driven approach focused on the new SOFR benchmark. Primarily, it accounts for the (to first order) piecewise constant nature of SOFR by using a stochastic volatility model in which the short rate $r(t)$ picks up the underlying driving randomness (modelled as a diffusion) only on central bank meeting (FOMC) dates, resulting in "stochastic discontinuities" (i.e., jumps at known times). These

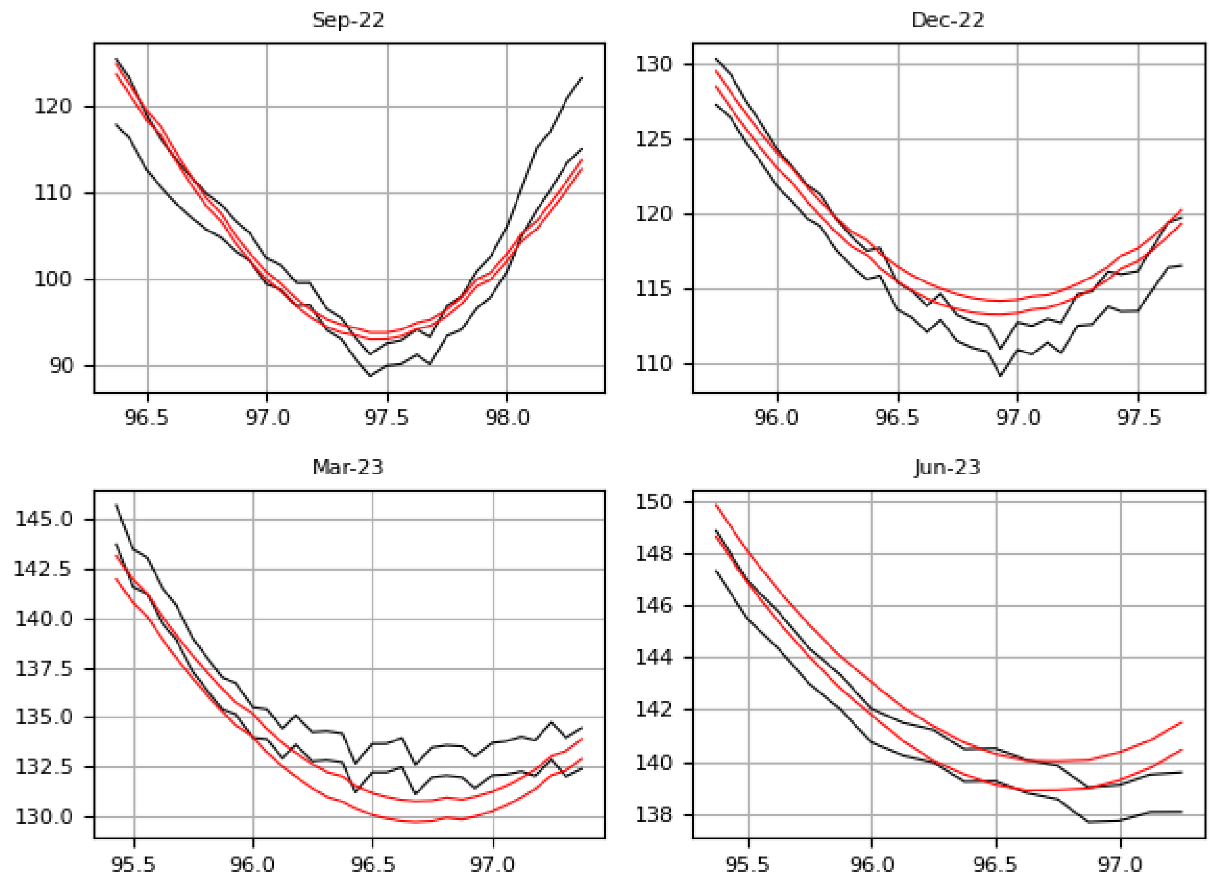

Figure 11: Calibration with time dependent parameters.

empirically inspired features also allow the model to be calibrated to interest rates options across different expiries, forward times and strikes.

Thus the model is directly connected to FOMC meeting dates, the regular economic event most important for US interest rate options markets. Arguably, other significant economic events and data funnel into FOMC policy rate decisions, as well as into market expectations of these decisions reflected in derivative pricing. The calibration of this stochastic discontinuity feature to a history of SOFR futures prices has revealed a connection between interest rate mean reversion and FOMC policy rate expectations. It turns out that the piecewise constant volatility component mimics instantaneous forward rate volatility decaying function in time to maturity and therefore has a similar effect as traditional mean reversion in the model. This reveals a direct connection between the evolution of FOMC policy rate expectations reflected in SOFR futures prices and traditional mean reverting stochastic dynamics.

The primary driver of SOFR futures prices are changes in expectations related to the next FOMC meeting, which in turn tends to be negatively correlated with changes in expectations for subsequent meetings, creating variance decay as a function of time to maturity. Historically, the Federal Reserve in managing economic cycles acts to mean revert interest rates. The market expects the Federal Reserve to continue to act this way and using our modelling set–up this expectation is actually detectable in the evolution of SOFR futures prices.

In the context of cross–sectional calibration, stochastic volatility allows the model to fit skewness and convexity across strikes, a prominent feature in interest rate options prices. We demonstrated this on options on SOFR futures. The model also could be adapted to other calibration instruments, such as caps and swaptions.

Our research approach has been to allow empirical data to inform our modelling choices. The aim of the approach was to identify and create a model which reflects key empirical features of SOFR. The resulting model accommodates cross–sectional calibration arguably better than leading industry models, in addition to providing genuine economic insights related to the evolution of interest rates.

Appendix A

A.1 Single dimensional case

Define the following:

$$\sigma(t,T) = \chi(t)\phi(T) \tag{43}$$

$$\phi(T) = \exp\biggl(- \int_0^T \lambda(v)dv\biggr) \tag{44}$$

$$\chi(t) = \sigma(t)\exp\biggl(\int_0^t \lambda(v)dv\biggr) \tag{45}$$

$$\Lambda(t,T) = \int_t^T \exp\biggl(- \int_t^u \lambda(v)dv\biggr)du \tag{46}$$

$$\Phi(t) = \int_0^t \sigma^2(s)\exp\biggl(- 2\int_s^t \lambda(v)dv\biggr)ds \tag{47}$$

HJM result:

$$
\begin{aligned}
f(t,T) &= f(0,T) + \int_0^t \sigma(s,T)\int_s^T \sigma(s,u)duds + \int_0^t \sigma(s,T)dW(s) \\
&= f(0,T) + \int_0^t \chi(s)\phi(T)\int_s^T \chi(s)\phi(u)duds + \int_0^t \chi(s)\phi(T)dW(s) \\
&= f(0,T) + \int_0^t \sigma(s)\exp\bigg(\int_0^s \lambda(v)dv\bigg)\exp\bigg(-\int_0^T \lambda(v)dv\bigg) \\
&\times \int_s^T \sigma(s)\exp\bigg(\int_0^s \lambda(v)dv\bigg)\exp\bigg(-\int_0^u \lambda(v)dv\bigg)duds \\
&+ \int_0^t \sigma(s)\exp\bigg(\int_0^s \lambda(v)dv\bigg)\exp\bigg(-\int_0^T \lambda(v)dv\bigg)dW(s) \\
&= f(0,T) + \int_0^t \sigma^2(s)\exp\bigg(-\int_s^T \lambda(v)dv\bigg)\int_s^T \exp\bigg(-\int_s^u \lambda(v)dv\bigg)duds \\
&+ \int_0^t \sigma(s)\exp\bigg(-\int_s^T \lambda(v)dv\bigg)dW(s) \\
&= f(0,T) + \int_0^t \sigma^2(s)\exp\bigg(-\int_s^T \lambda(v)dv\bigg)\Lambda(s,T)ds + \int_0^t \sigma(s)\exp\bigg(-\int_s^T \lambda(v)dv\bigg)dW(s)
\end{aligned}
$$

let $y(t) = \int_0^t \sigma^2(s)\exp\Big(-\int_s^t \lambda(v)dv\Big)\Lambda(s,t)ds + \exp\Big(\int_t^T \lambda(v)dv\Big)\int_0^t \sigma(s)\exp\Big(-\int_s^T \lambda(v)dv\Big)dW(s)$

substitute $\int_0^t \sigma(s)\exp\Big(-\int_s^T \lambda(v)dv\Big)dW(s) = \frac{y(t)-\int_0^t \sigma^2(s)\exp\Big(-\int_s^t \lambda(v)dv\Big)\Lambda(s,t)ds}{\exp\Big(\int_t^T \lambda(v)dv\Big)}$

$$
\begin{aligned}
f(t,T) &= f(0,T) + \int_0^t \sigma^2(s)\exp\Big(-\int_s^T \lambda(v)dv\Big)\Lambda(s,T)ds + \int_0^t \sigma(s)\exp\Big(-\int_s^T \lambda(v)dv\Big)dW(s) \\
&= f(0,T) + \exp\Big(-\int_t^T \lambda(v)dv\Big)y(t) + \int_0^t \sigma^2(s)\exp\Big(-\int_s^T \lambda(v)dv\Big)\Lambda(s,T)ds \\
&- \exp\Big(-\int_t^T \lambda(v)dv\Big)\int_0^t \sigma^2(s)\exp\Big(-\int_s^t \lambda(v)dv\Big)\Lambda(s,t)ds \\
&= f(0,T) + \exp\Big(-\int_t^T \lambda(v)dv\Big)y(t) + \exp\Big(-\int_t^T \lambda(v)dv\Big) \\
&\times \int_0^t \sigma^2(s)\Big\{\exp\Big(\int_t^T \lambda(v)dv\Big)\exp\Big(-\int_s^T \lambda(v)dv\Big)\Lambda(s,T) - \exp\Big(-\int_s^t \lambda(v)dv\Big)\Lambda(s,t)\Big\}ds \\
&= f(0,T) + \exp\Big(-\int_t^T \lambda(v)dv\Big)y(t) + \exp\Big(-\int_t^T \lambda(v)dv\Big) \\
&\times \int_0^t \sigma^2(s)\exp\Big(-\int_s^t \lambda(v)dv\Big)\Big\{\Lambda(s,T) - \Lambda(s,t)\Big\}ds \\
&= f(0,T) + \exp\Big(-\int_t^T \lambda(v)dv\Big)y(t) \\
&+ \Lambda(t,T)\exp\Big(-\int_t^T \lambda(v)dv\Big)\int_0^t \sigma^2(s)\exp\Big(-2\int_s^t \lambda(v)dv\Big)ds \\
&= f(0,T) + \exp\Big(-\int_t^T \lambda(v)dv\Big)y(t) + \Phi(t)\Lambda(t,T)\exp\Big(-\int_t^T \lambda(v)dv\Big)
\end{aligned}
$$

therefore:

$$
\begin{aligned}
\int_t^T f(t,u)du &= \int_t^T \Big(f(0,u) + \exp\Big(-\int_t^u \lambda(v)dv\Big)y(t) + \Phi_j(t)\Lambda(t,u)\exp\Big(-\int_t^u \lambda(v)dv\Big)\Big)du \\
&= \int_t^T f(0,u)du + y(t)\int_t^T \exp\Big(-\int_t^u \lambda(v)dv\Big)du + \Phi(t)\int_t^T \Lambda(t,u)\exp\Big(-\int_t^u \lambda(v)dv\Big)du \\
&= \int_t^T f(0,u)du + \Lambda(t,T)y(t) + \Phi(t)\int_t^T \Lambda(t,u)d\Lambda(t,u) \\
&= \int_t^T f(0,u)du + \Lambda(t,T)y(t) + \frac{1}{2}\Phi(t)\Lambda^2(t,T)
\end{aligned}
$$

Therefore, the bond price is given by

$$
B(t,T) = \exp\Big(-\int_t^T f(t,u)du\Big) = \frac{B(0,T)}{B(0,t)}\exp\Big(-\Lambda(t,T)y(t) - \frac{1}{2}\Phi(t)\Lambda^2(t,T)\Big) \qquad (48)
$$

A.2 Single dimensional case with piecewise continuous short rate

Define the following:

$$\sigma(t,T) = \sum_{i=1}^{n} \mathbb{I}_{\{i \leq \mathcal{A}(t,T)\}} \chi(t)\phi(T)\gamma_i \tag{49}$$

$$\phi(T) = \exp\bigg(- \int_0^T \lambda(v)dv \bigg) \tag{50}$$

$$\chi(t) = \sigma(t)\exp\bigg(\int_0^t \lambda(v)dv \bigg) \tag{51}$$

$$\Lambda(t,T) = \int_t^T \exp\bigg(- \int_t^u \lambda(v)dv \bigg) du \tag{52}$$

$$\Lambda_a(t,T) = \int_a^T \exp\bigg(- \int_t^u \lambda(v)dv \bigg) du \tag{53}$$

$$\Phi(t) = \int_0^t \sigma^2(s)\exp\bigg(- 2\int_s^t \lambda(v)dv \bigg) ds \tag{54}$$

A.2.1 trivial case $t < T < x_1$

HJM result:

$$f(t,T) = f(0,T) + \int_0^t \sigma(s,T) \int_t^T \sigma(s,u)duds + \int_0^t \sigma(s,T)dW(s) = f(0,T) \tag{55}$$

$$\int_t^T f(t,u)du = \int_t^T f(0,u)du \tag{56}$$

Therefore, the bond price is given by

$$B(t,T) = \exp\bigg(- \int_t^T f(t,u)du \bigg) = \frac{B(0,T)}{B(0,t)} \tag{57}$$

A.2.2 basic case $t < x_1 < T < x_2$

$$\int_t^T f(t,u)du = \int_t^{x_1} f(t,u)du + \int_{x_1}^T f(t,u)du \tag{58}$$
$$= \int_t^{x_1} f(0,u)du + \int_{x_1}^T f(t,u)du \tag{59}$$

To solve $\int_{x_1}^T f(t,u)du$, restrict $T \in [x1, x2]$ and $t < x_1$, HJM result:

$$\begin{aligned}
f(t,T) &= f(0,T) + \int_0^t \sigma(s,T)\int_s^T \sigma(s,u)duds + \int_0^t \sigma(s,T)dW(s) \\
&= f(0,T) + \int_0^t \sum_{i=1}^n \mathbb{I}_{\{i\leq \mathcal{A}(s,T)\}}\chi(s)\phi(T)\gamma_i \int_s^T \sum_{j=1}^n \mathbb{I}_{\{j\leq \mathcal{A}(s,u)\}}\chi(s)\phi(u)\gamma_j duds \\
&+ \int_0^t \sum_{i=1}^n \mathbb{I}_{\{i\leq \mathcal{A}(s,T)\}}\chi(s)\phi(T)\gamma_i dW(s) \\
&= f(0,T) + \int_0^t \chi(s)\phi(T)\gamma_1 \int_s^T \mathbb{I}_{\{s<x_1\}}\mathbb{I}_{\{u>x_1\}}\chi(s)\phi(u)\gamma_1 duds + \int_0^t \chi(s)\phi(T)\gamma_1 dW(s) \\
&= f(0,T) + \gamma_1^2 \int_0^t \chi(s)\phi(T)\mathbb{I}_{\{s<x_1\}} \int_s^T \mathbb{I}_{\{u>x_1\}}\chi(s)\phi(u)duds + \gamma_1 \int_0^t \chi(s)\phi(T)dW(s) \\
&= f(0,T) + \gamma_1^2 \int_0^t \chi(s)\phi(T) \int_s^T \mathbb{I}_{\{u>x_1\}}\chi(s)\phi(u)duds + \gamma_1 \int_0^t \chi(s)\phi(T)dW(s) \\
&= f(0,T) + \gamma_1^2 \int_0^t \chi(s)\phi(T) \int_{x_1}^T \chi(s)\phi(u)duds + \gamma_1 \int_0^t \chi(s)\phi(T)dW(s) \\
&= f(0,T) + \gamma_1^2 \int_0^t \sigma(s)\exp\Big(\int_0^s \lambda(v)dv\Big)\exp\Big(-\int_0^T \lambda(v)dv\Big) \\
&\times \int_{x_1}^T \sigma(s)\exp\Big(\int_0^s \lambda(v)dv\Big)\exp\Big(-\int_0^u \lambda(v)dv\Big)duds \\
&+ \gamma_1 \int_0^t \sigma(s)\exp\Big(\int_0^s \lambda(v)dv\Big)\exp\Big(-\int_0^T \lambda(v)dv\Big)dW(s) \\
&= f(0,T) + \gamma_1^2 \int_0^t \sigma^2(s)\exp\Big(-\int_s^T \lambda(v)dv\Big)\int_{x_1}^T \exp\Big(-\int_s^u \lambda(v)dv\Big)duds \\
&+ \gamma_1 \int_0^t \sigma(s)\exp\Big(-\int_s^T \lambda(v)dv\Big)dW(s) \\
&= f(0,T) + \gamma_1^2 \int_0^t \sigma^2(s)\exp\Big(-\int_s^T \lambda(v)dv\Big)\Lambda_{x_1}(s,T)ds + \gamma_1 \int_0^t \sigma(s)\exp\Big(-\int_s^T \lambda(v)dv\Big)dW(s)
\end{aligned}$$

let

$$y(t) = \gamma_1^2 \int_0^t \sigma^2(s)\exp\Big(-\int_s^t \lambda(v)dv\Big)\Lambda_{x_1}(s,t)ds + \exp\Big(\int_t^T \lambda(v)dv\Big)\gamma_1 \int_0^t \sigma(s)\exp\Big(-\int_s^T \lambda(v)dv\Big)dW(s) \tag{60}$$

substitute

$$\gamma_1 \int_0^t \sigma(s) \exp\Big(- \int_s^T \lambda(v) dv \Big) dW(s) = \frac{y(t) - \gamma_1^2 \int_0^t \sigma^2(s) \exp\Big(- \int_s^t \lambda(v) dv \Big) \Lambda_{x_1}(s,t) ds}{\exp\Big(\int_t^T \lambda(v) dv \Big)}$$

$$\begin{aligned}
f(t,T) &= f(0,T) + \gamma_1^2 \int_0^t \sigma^2(s) \exp\Big(- \int_s^T \lambda(v) dv \Big) \Lambda_{x_1}(s,T) ds + \gamma_1 \int_0^t \sigma(s) \exp\Big(- \int_s^T \lambda(v) dv \Big) dW(s) \\
&= f(0,T) + \exp\Big(- \int_t^T \lambda(v) dv \Big) y(t) + \gamma_1^2 \int_0^t \sigma^2(s) \exp\Big(- \int_s^T \lambda(v) dv \Big) \Lambda_{x_1}(s,T) ds \\
&- \gamma_1^2 \exp\Big(- \int_t^T \lambda(v) dv \Big) \int_0^t \sigma^2(s) \exp\Big(- \int_s^t \lambda(v) dv \Big) \Lambda_{x_1}(s,t) ds \\
&= f(0,T) + \exp\Big(- \int_t^T \lambda(v) dv \Big) y(t) + \gamma_1^2 \exp\Big(- \int_t^T \lambda(v) dv \Big) \\
&\times \int_0^t \sigma^2(s) \Big\{ \exp\Big(\int_t^T \lambda(v) dv \Big) \exp\Big(- \int_s^T \lambda(v) dv \Big) \Lambda_{x_1}(s,T) - \exp\Big(- \int_s^t \lambda(v) dv \Big) \Lambda_{x_1}(s,t) \Big\} ds \\
&= f(0,T) + \exp\Big(- \int_t^T \lambda(v) dv \Big) y(t) + \gamma_1^2 \exp\Big(- \int_t^T \lambda(v) dv \Big) \\
&\times \int_0^t \sigma^2(s) \exp\Big(- \int_s^t \lambda(v) dv \Big) \Big\{ \Lambda_{x_1}(s,T) - \Lambda_{x_1}(s,t) \Big\} ds \\
&= f(0,T) + \exp\Big(- \int_t^T \lambda(v) dv \Big) y(t) + \gamma_1^2 \Lambda(t,T) \exp\Big(- \int_t^T \lambda(v) dv \Big) \int_0^t \sigma^2(s) \exp\Big(- 2 \int_s^t \lambda(v) dv \Big) ds \\
&= f(0,T) + \exp\Big(- \int_t^T \lambda(v) dv \Big) y(t) + \gamma_1^2 \Phi(t) \Lambda(t,T) \exp\Big(- \int_t^T \lambda(v) dv \Big)
\end{aligned}$$

therefore:

$$\begin{aligned}
\int_{x_1}^T f(t,u) du &= \int_{x_1}^T \Big(f(0,u) + \exp\Big(- \int_t^u \lambda(v) dv \Big) y(t) + \gamma_1^2 \Phi(t) \Lambda(t,u) \exp\Big(- \int_t^u \lambda(v) dv \Big) \Big) du \\
&= \int_{x_1}^T f(0,u) du + y(t) \int_{x_1}^T \exp\Big(- \int_t^u \lambda(v) dv \Big) du + \gamma_1^2 \Phi(t) \int_{x_1}^T \Lambda(t,u) \exp\Big(- \int_t^u \lambda(v) dv \Big) du \\
&= \int_{x_1}^T f(0,u) du + \Lambda_{x_1}(t,T) y(t) + \gamma_1^2 \Phi(t) \int_{x_1}^T \Lambda(t,u) d\Lambda(t,u) \\
&= \int_{x_1}^T f(0,u) du + \Lambda_{x_1}(t,T) y(t) + \frac{1}{2} \gamma_1^2 \Phi(t) \{ \Lambda^2(t,T) - \Lambda^2(t,x_1) \}
\end{aligned}$$

therefore:

$$\int_t^{x_1} f(t,u) du + \int_{x_1}^T f(t,u) du = \int_t^T f(0,u) du + \Lambda_{x_1}(t,T) y(t) + \frac{1}{2} \gamma_1^2 \Phi(t) \{ \Lambda^2(t,T) - \Lambda^2(t,x_1) \}$$

Therefore, the bond price is given by

$$B(t,T)=\exp\bigg(-\int_t^T f(t,u)du\bigg)=\frac{B(0,T)}{B(0,t)}\exp\bigg(-\Lambda_{x_1}(t,T)y(t)-\frac{1}{2}\gamma_1^2\Phi(t)\{\Lambda^2(t,T)-\Lambda^2(t,x_1)\}\bigg) \tag{61}$$

A.2.3 more general case $t<x_1<T$

In general we will need $\int_{x_a}^T f(t,u)du$ where $T<x_{a+1}$, therefore restrict $T\in[x_a,x_{a+1}]$ and $t<x_1$, HJM result:

$$\begin{aligned}
f(t,T)&=f(0,T)+\int_0^t\sigma(s,T)\int_s^T\sigma(s,u)duds+\int_0^t\sigma(s,T)dW(s)\\
&=f(0,T)+\int_0^t\sum_{i=1}^n\mathbb{I}_{\{i\le\mathcal{A}(s,T)\}}\chi(s)\phi(T)\gamma_i\int_s^T\sum_{j=1}^n\mathbb{I}_{\{j\le\mathcal{A}(s,u)\}}\chi(s)\phi(u)\gamma_j duds\\
&+\int_0^t\sum_{i=1}^n\mathbb{I}_{\{i\le\mathcal{A}(s,T)\}}\chi(s)\phi(T)\gamma_i dW(s)\\
&=f(0,T)+\int_0^t\chi(s)\phi(T)\sum_{i=1}^a\gamma_i\int_s^T\sum_{j=1}^n\mathbb{I}_{\{j\le\mathcal{A}(s,u)\}}\chi(s)\phi(u)\gamma_j duds+\int_0^t\chi(s)\phi(T)\sum_{i=1}^a\gamma_i dW(s)
\end{aligned}$$

Now

$$\int_s^T\sum_{j=1}^n\mathbb{I}_{\{j\le\mathcal{A}(s,u)\}}\chi(s)\phi(u)\gamma_j du=\chi(s)\sum_{j=1}^n\gamma_j\int_s^T\mathbb{I}_{\{j\le\mathcal{A}(s,u)\}}\phi(u)du \tag{62}$$

$$\begin{aligned}
\int_s^T\mathbb{I}_{\{j\le\mathcal{A}(s,u)\}}\phi(u)du&=\int_s^{x_1}\mathbb{I}_{\{j\le\mathcal{A}(s,u)\}}\phi(u)du+\sum_{k=1}^{a-1}\int_{x_k}^{x_{k+1}}\mathbb{I}_{\{j\le\mathcal{A}(s,u)\}}\phi(u)du+\int_{x_a}^T\mathbb{I}_{\{j\le\mathcal{A}(s,u)\}}\phi(u)du\\
&=\sum_{k=1}^{a-1}\int_{x_k}^{x_{k+1}}\mathbb{I}_{\{j\le\mathcal{A}(s,u)\}}\phi(u)du+\int_{x_a}^T\mathbb{I}_{\{j\le\mathcal{A}(s,u)\}}\phi(u)du
\end{aligned}$$

Therefore:

$$\begin{aligned}
\sum_{j=1}^n\gamma_j\int_s^T\mathbb{I}_{\{j\le\mathcal{A}(s,u)\}}\phi(u)du&=\sum_{j=1}^n\gamma_j\sum_{k=1}^{a-1}\int_{x_k}^{x_{k+1}}\mathbb{I}_{\{j\le\mathcal{A}(s,u)\}}\phi(u)du+\sum_{j=1}^n\gamma_j\int_{x_a}^T\mathbb{I}_{\{j\le\mathcal{A}(s,u)\}}\phi(u)du\\
&=\sum_{j=1}^{a-1}\gamma_j\int_{x_j}^{x_a}\phi(u)du+\sum_{j=1}^a\gamma_j\int_{x_a}^T\phi(u)du\\
&=\sum_{j=1}^{a-1}\gamma_j\int_{x_j}^T\phi(u)du+\gamma_a\int_{x_a}^T\phi(u)du\\
&=\sum_{j=1}^a\gamma_j\int_{x_j}^T\phi(u)du
\end{aligned}$$

Therefore:

$$
\begin{aligned}
f(t,T) &= f(0,T) + \int_0^t \chi(s)\phi(T)\sum_{i=1}^{a}\gamma_i\int_s^T\sum_{j=1}^{n}\mathbb{I}_{\{j\le\mathcal{A}(s,u)\}}\chi(s)\phi(u)\gamma_j duds + \int_0^t\chi(s)\phi(T)\sum_{i=1}^{a}\gamma_i dW(s)\\
&= f(0,T) + \int_0^t \chi(s)\phi(T)\sum_{i=1}^{a}\gamma_i\chi(s)\sum_{j=1}^{a}\gamma_j\int_{x_j}^T\phi(u)duds + \int_0^t\chi(s)\phi(T)\sum_{i=1}^{a}\gamma_i dW(s)\\
&= f(0,T) + \sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\int_0^t\chi(s)\phi(T)\int_{x_j}^T\chi(s)\phi(u)duds + \sum_{i=1}^{a}\gamma_i\int_0^t\chi(s)\phi(T)dW(s)\\
&= f(0,T) + \sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\int_0^t\sigma(s)\exp\biggl(\int_0^s\lambda(v)dv\biggr)\exp\biggl(-\int_0^T\lambda(v)dv\biggr)\\
&\times\int_{x_j}^T\sigma(s)\exp\biggl(\int_0^s\lambda(v)dv\biggr)\exp\biggl(-\int_0^u\lambda(v)dv\biggr)duds\\
&+\sum_{i=1}^{a}\gamma_i\int_0^t\sigma(s)\exp\biggl(\int_0^s\lambda(v)dv\biggr)\exp\biggl(-\int_0^T\lambda(v)dv\biggr)dW(s)\\
&= f(0,T) + \sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\int_0^t\sigma^2(s)\exp\biggl(-\int_s^T\lambda(v)dv\biggr)\int_{x_j}^T\exp\biggl(-\int_s^u\lambda(v)dv\biggr)duds\\
&+\sum_{i=1}^{a}\gamma_i\int_0^t\sigma(s)\exp\biggl(-\int_s^T\lambda(v)dv\biggr)dW(s)\\
&= f(0,T) + \sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\int_0^t\sigma^2(s)\exp\biggl(-\int_s^T\lambda(v)dv\biggr)\Lambda_{x_j}(s,T)ds\\
&+\sum_{i=1}^{a}\gamma_i\int_0^t\sigma(s)\exp\biggl(-\int_s^T\lambda(v)dv\biggr)dW(s)
\end{aligned}
$$

Let

$$
\begin{aligned}
y_a(t) &= \sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\int_0^t\sigma^2(s)\exp\biggl(-\int_s^t\lambda(v)dv\biggr)\Lambda_{x_j}(s,t)ds\\
&\quad+\exp\biggl(\int_t^T\lambda(v)dv\biggr)\sum_{i=1}^{a}\gamma_i\int_0^t\sigma(s)\exp\biggl(-\int_s^T\lambda(v)dv\biggr)dW(s)
\end{aligned}
$$

Substitute

$$
\sum_{i=1}^{a}\gamma_i\int_0^t\sigma(s)\exp\biggl(-\int_s^T\lambda(v)dv\biggr)dW(s) = \frac{y_a(t)-\sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\int_0^t\sigma^2(s)\exp\Bigl(-\int_s^t\lambda(v)dv\Bigr)\Lambda_{x_j}(s,t)ds}{\exp\Bigl(\int_t^T\lambda(v)dv\Bigr)}
$$

$$
\begin{aligned}
f(t,T) = f(0,T) &+ \sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\int_0^t \sigma^2(s)\exp\Big(-\int_s^T\lambda(v)dv\Big)\Lambda_{x_j}(s,T)ds \\
&+ \sum_{i=1}^{a}\gamma_i\int_0^t\sigma(s)\exp\Big(-\int_s^T\lambda(v)dv\Big)dW(s) \\
&= f(0,T) + \exp\Big(-\int_t^T\lambda(v)dv\Big)y_a(t) + \sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\int_0^t\sigma^2(s)\exp\Big(-\int_s^T\lambda(v)dv\Big)\Lambda_{x_j}(s,T)ds \\
&- \sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\exp\Big(-\int_t^T\lambda(v)dv\Big)\int_0^t\sigma^2(s)\exp\Big(-\int_s^t\lambda(v)dv\Big)\Lambda_{x_j}(s,t)ds \\
&= f(0,T) + \exp\Big(-\int_t^T\lambda(v)dv\Big)y(t) + \sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\exp\Big(-\int_t^T\lambda(v)dv\Big) \\
&\times\int_0^t\sigma^2(s)\Big\{\exp\Big(\int_t^T\lambda(v)dv\Big)\exp\Big(-\int_s^T\lambda(v)dv\Big)\Lambda_{x_j}(s,T) - \exp\Big(-\int_s^t\lambda(v)dv\Big)\Lambda_{x_j}(s,t)\Big\}ds \\
&= f(0,T) + \exp\Big(-\int_t^T\lambda(v)dv\Big)y_a(t) + \sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\exp\Big(-\int_t^T\lambda(v)dv\Big) \\
&\times\int_0^t\sigma^2(s)\exp\Big(-\int_s^t\lambda(v)dv\Big)\Big\{\Lambda_{x_j}(s,T)-\Lambda_{x_j}(s,t)\Big\}ds \\
&= f(0,T) + \exp\Big(-\int_t^T\lambda(v)dv\Big)y_a(t) \\
&+ \sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\Lambda(t,T)\exp\Big(-\int_t^T\lambda(v)dv\Big)\int_0^t\sigma^2(s)\exp\Big(-2\int_s^t\lambda(v)dv\Big)ds \\
&= f(0,T) + \exp\Big(-\int_t^T\lambda(v)dv\Big)y_a(t) + \sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\Phi(t)\Lambda(t,T)\exp\Big(-\int_t^T\lambda(v)dv\Big)
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\int_{x_a}^T f(t,u)du &= \int_{x_a}^T\Big(f(0,u) + \exp\Big(-\int_t^u\lambda(v)dv\Big)y_a(t) + \sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\Phi(t)\Lambda(t,u)\exp\Big(-\int_t^u\lambda(v)dv\Big)\Big)du \\
&= \int_{x_a}^T f(0,u)du + y_a(t)\int_{x_a}^T\exp\Big(-\int_t^u\lambda(v)dv\Big)du \\
&+ \sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\Phi(t)\int_{x_a}^T\Lambda(t,u)\exp\Big(-\int_t^u\lambda(v)dv\Big)du \\
&= \int_{x_a}^T f(0,u)du + \Lambda_{x_a}(t,T)y_a(t) + \sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\Phi(t)\int_{x_a}^T\Lambda(t,u)d\Lambda(t,u) \\
&= \int_{x_a}^T f(0,u)du + \Lambda_{x_a}(t,T)y_a(t) + \frac{1}{2}\sum_{i=1}^{a}\sum_{j=1}^{a}\gamma_i\gamma_j\Phi(t)\{\Lambda^2(t,T)-\Lambda^2(t,x_a)\}
\end{aligned}
$$

Define $\eta(t) = \min\{k | x_k > t\}$, now:

$$\int_t^T f(t,u)du = \int_t^{x_{\eta(t)}} f(t,u)du + \int_{x_{\eta(T)-1}}^T f(t,u)du + \sum_{k=\eta(t)}^{\eta(T)-2} \int_{x_k}^{x_{k+1}} f(t,u)du$$
$$= \int_t^T f(0,u)du + \Lambda_{x_{\eta(T)-1}}(t,T) y_{x_{\eta(T)-1}}(t)$$
$$+ \frac{1}{2} \sum_{i=1}^{x_{\eta(T)-1}} \sum_{j=1}^{x_{\eta(T)-1}} \gamma_i \gamma_j \Phi(t) \{\Lambda^2(t,T) - \Lambda^2(t, x_{\eta(T)-1})\}$$
$$+ \sum_{k=\eta(t)}^{\eta(T)-2} \left(\Lambda_{x_k}(t, x_{k+1}) y_k(t) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} \gamma_i \gamma_j \Phi(t) \{\Lambda^2(t, x_{k+1}) - \Lambda^2(t, x_k)\} \right)$$

Therefore, the bond price is given by

$$B(t,T) = \exp\left(- \int_t^T f(t,u)du \right)$$
$$= \frac{B(0,T)}{B(0,t)} \exp\left(\Lambda_{x_{\eta(T)-1}}(t,T) y_{x_{\eta(T)-1}}(t) + \frac{1}{2} \sum_{i=1}^{x_{\eta(T)-1}} \sum_{j=1}^{x_{\eta(T)-1}} \gamma_i \gamma_j \Phi(t) \{\Lambda^2(t,T) - \Lambda^2(t, x_{\eta(T)-1})\} \right.$$
$$\left. + \sum_{k=\eta(t)}^{\eta(T)-2} \left(\Lambda_{x_k}(t, x_{k+1}) y_k(t) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} \gamma_i \gamma_j \Phi(t) \{\Lambda^2(t, x_{k+1}) - \Lambda^2(t, x_k)\} \right) \right)$$

A.2.4 general case $t < T$

In general we will need $\int_{x_a}^T f(t,u)du$ where $T < x_{a+1}$, therefore restrict $T \in [x_a, x_{a+1}]$ and $t < T$. Define $\eta(t) = \min\{b | x_b \geq t\}$, now HJM result:

$$f(t,T) = f(0,T) + \int_0^t \sigma(s,T) \int_s^T \sigma(s,u) duds + \int_0^t \sigma(s,T) dW(s)$$
$$= f(0,T) + \int_{x_{\eta(t)-1}}^t \sigma(s,T) \int_s^T \sigma(s,u) duds + \sum_{b=0}^{\eta(t)-2} \int_{x_b}^{x_{b+1}} \sigma(s,T) \int_s^T \sigma(s,u) duds$$
$$+ \int_{x_{\eta(t)-1}}^t \sigma(s,T) dW(s) + \sum_{b=0}^{\eta(t)-2} \int_{x_b}^{x_{b+1}} \sigma(s,T) dW(s) \tag{63}$$

Therefore, in general we need to solve $\int_{x_b}^t \sigma(s,T)\int_s^T \sigma(s,u)duds$ and $\int_{x_b}^t \sigma(s,T)dW(s)$ where $t \in [x_b, x_{b+1}]$. Now:

$$\begin{aligned}
\int_{x_b}^t \sigma(s,T)dW(s) &= \int_{x_b}^t \sum_{i=1}^n \mathbb{I}_{\{i\leq \mathcal{A}(s,T)\}}\chi(s)\phi(T)\gamma_i dW(s) \\
&= \int_{x_b}^t \chi(s)\phi(T)\sum_{i=1}^{a-b}\gamma_i dW(s) \\
&= \sum_{i=1}^{a-b}\gamma_i \int_{x_b}^t \sigma(s)\exp\Big(\int_0^s \lambda(v)dv\Big)\exp\Big(-\int_0^T \lambda(v)dv\Big)dW(s) \\
&= \sum_{i=1}^{a-b}\gamma_i \int_{x_b}^t \sigma(s)\exp\Big(-\int_s^T \lambda(v)dv\Big)dW(s)
\end{aligned}$$

Also:

$$\begin{aligned}
\int_{x_b}^t \sigma(s,T)\int_s^T \sigma(s,u)duds &= \int_{x_b}^t \sum_{i=1}^n \mathbb{I}_{\{i\leq \mathcal{A}(s,T)\}}\chi(s)\phi(T)\gamma_i \int_s^T \sum_{j=1}^n \mathbb{I}_{\{j\leq \mathcal{A}(s,u)\}}\chi(s)\phi(u)\gamma_j duds \\
&= \int_{x_b}^t \chi(s)\phi(T)\sum_{i=1}^{a-b}\gamma_i \int_s^T \sum_{j=1}^n \mathbb{I}_{\{j\leq \mathcal{A}(s,u)\}}\chi(s)\phi(u)\gamma_j duds \\
&= \int_{x_b}^t \chi(s)\phi(T)\sum_{i=1}^{a-b}\gamma_i\chi(s)\sum_{j=1}^n \gamma_j \int_s^T \mathbb{I}_{\{j\leq \mathcal{A}(s,u)\}}\phi(u)duds
\end{aligned}$$

Now, implicitly with $s \in [x_b, t]$:

$$\begin{aligned}
\int_s^T \mathbb{I}_{\{j\leq \mathcal{A}(s,u)\}}\phi(u)du &= \int_s^{x_1} \mathbb{I}_{\{j\leq \mathcal{A}(s,u)\}}\phi(u)du + \sum_{k=1}^{a-1}\int_{x_k}^{x_{k+1}} \mathbb{I}_{\{j\leq \mathcal{A}(s,u)\}}\phi(u)du + \int_{x_a}^T \mathbb{I}_{\{j\leq \mathcal{A}(s,u)\}}\phi(u)du \\
&= \sum_{k=b+1}^{a-1}\int_{x_k}^{x_{k+1}} \mathbb{I}_{\{j\leq \mathcal{A}(s,u)\}}\phi(u)du + \int_{x_a}^T \mathbb{I}_{\{j\leq \mathcal{A}(s,u)\}}\phi(u)du
\end{aligned}$$

Therefore:

$$\begin{aligned}
\sum_{j=1}^n \gamma_j \int_s^T \mathbb{I}_{\{j\leq \mathcal{A}(s,u)\}}\phi(u)du &= \sum_{j=1}^n \gamma_j \sum_{k=b+1}^{a-1}\int_{x_k}^{x_{k+1}} \mathbb{I}_{\{j\leq \mathcal{A}(s,u)\}}\phi(u)du + \sum_{j=1}^n \gamma_j \int_{x_a}^T \mathbb{I}_{\{j\leq \mathcal{A}(s,u)\}}\phi(u)du \\
&= \sum_{j=1}^{a-b-1}\gamma_j \int_{x_{b+j}}^{x_a}\phi(u)du + \sum_{j=1}^{a-b}\gamma_j \int_{x_a}^T \phi(u)du \\
&= \sum_{j=1}^{a-b-1}\gamma_j \int_{x_{b+j}}^{T}\phi(u)du + \gamma_{a-b}\int_{x_a}^T \phi(u)du \\
&= \sum_{j=1}^{a-b}\gamma_j \int_{x_{b+j}}^{T}\phi(u)du
\end{aligned}$$

Therefore:

$$\int_{x_b}^{t}\sigma(s,T)\int_{s}^{T}\sigma(s,u)duds = \int_{x_b}^{t}\chi(s)\phi(T)\sum_{i=1}^{a-b}\gamma_i\chi(s)\sum_{j=1}^{n}\gamma_j\int_{s}^{T}\mathbb{I}_{\{j\leq\mathcal{A}(s,u)\}}\phi(u)duds$$

$$= \int_{x_b}^{t}\chi(s)\phi(T)\sum_{i=1}^{a-b}\gamma_i\sum_{j=1}^{a-b}\gamma_j\int_{x_{b+j}}^{T}\chi(s)\phi(u)duds$$

$$= \sum_{i=1}^{a-b}\sum_{j=1}^{a-b}\gamma_i\gamma_j\int_{x_b}^{t}\chi(s)\phi(T)\int_{x_{b+j}}^{T}\chi(s)\phi(u)duds$$

$$= \sum_{i=1}^{a-b}\sum_{j=1}^{a-b}\gamma_i\gamma_j\int_{x_b}^{t}\sigma(s)\exp\bigg(\int_{0}^{s}\lambda(v)dv\bigg)\exp\bigg(-\int_{0}^{T}\lambda(v)dv\bigg)$$

$$\times\int_{x_{b+j}}^{T}\sigma(s)\exp\bigg(\int_{0}^{s}\lambda(v)dv\bigg)\exp\bigg(-\int_{0}^{u}\lambda(v)dv\bigg)duds$$

$$= \sum_{i=1}^{a-b}\sum_{j=1}^{a-b}\gamma_i\gamma_j\int_{x_b}^{t}\sigma^2(s)\exp\bigg(-\int_{s}^{T}\lambda(v)dv\bigg)\int_{x_{b+j}}^{T}\exp\bigg(-\int_{s}^{u}\lambda(v)dv\bigg)duds$$

$$= \sum_{i=1}^{a-b}\sum_{j=1}^{a-b}\gamma_i\gamma_j\int_{x_b}^{t}\sigma^2(s)\exp\bigg(-\int_{s}^{T}\lambda(v)dv\bigg)\Lambda_{x_{b+j}}(s,T)ds$$

Rewrite (63):

$$f(t,T) = f(0,T) + \int_{0}^{t}\sigma(s,T)\int_{s}^{T}\sigma(s,u)duds + \int_{0}^{t}\sigma(s,T)dW(s)$$

$$= f(0,T) + \sum_{b=0}^{\eta(t)-2}\bigg\{\int_{x_b}^{x_{b+1}}\sigma(s,T)\int_{s}^{T}\sigma(s,u)duds + \int_{x_b}^{x_{b+1}}\sigma(s,T)dW(s)\bigg\}$$

$$+ \int_{x_{\eta(t)-1}}^{t}\sigma(s,T)\int_{s}^{T}\sigma(s,u)duds + \int_{x_{\eta(t)-1}}^{t}\sigma(s,T)dW(s)$$

$$= f(0,T) + \sum_{b=0}^{\eta(t)-2}\bigg\{\sum_{i=1}^{a-b}\sum_{j=1}^{a-b}\gamma_i\gamma_j\int_{x_b}^{x_{b+1}}\sigma^2(s)\exp\bigg(-\int_{s}^{T}\lambda(v)dv\bigg)\Lambda_{x_{b+j}}(s,T)ds$$

$$+ \sum_{i=1}^{a-b}\gamma_i\int_{x_b}^{x_{b+1}}\sigma(s)\exp\bigg(-\int_{s}^{T}\lambda(v)dv\bigg)dW(s)\bigg\}$$

$$+ \sum_{i=1}^{(a-\eta(t)+1)}\sum_{j=1}^{(a-\eta(t)+1)}\gamma_i\gamma_j\int_{x_{\eta(t)-1}}^{t}\sigma^2(s)\exp\bigg(-\int_{s}^{T}\lambda(v)dv\bigg)\Lambda_{x_{(\eta(t)-1+j)}}(s,T)ds$$

$$+ \sum_{i=1}^{a-\eta(t)+1}\gamma_i\int_{x_{\eta(t)-1}}^{t}\sigma(s)\exp\bigg(-\int_{s}^{T}\lambda(v)dv\bigg)dW(s)$$

Let

$$y_a(t) = \sum_{i=1}^{(a-\eta(t)+1)} \sum_{j=1}^{(a-\eta(t)+1)} \gamma_i \gamma_j \int_{x_{\eta(t)-1}}^{t} \sigma^2(s) \exp\bigg(- \int_s^t \lambda(v) dv \bigg) \Lambda_{x_{(\eta(t)-1+j)}}(s,t) ds$$
$$+ \exp\bigg(\int_t^T \lambda(v) dv \bigg) \sum_{i=1}^{a-\eta(t)+1} \gamma_i \int_{x_{\eta(t)-1}}^{t} \sigma(s) \exp\bigg(- \int_s^T \lambda(v) dv \bigg) dW(s)$$

Substitute

$$\sum_{i=1}^{a-\eta(t)+1} \gamma_i \int_{x_{\eta(t)-1}}^{t} \sigma(s) \exp\bigg(- \int_s^T \lambda(v) dv \bigg) dW(s)$$
$$= \frac{y_a(t) - \sum_{i=1}^{(a-\eta(t)+1)} \sum_{j=1}^{(a-\eta(t)+1)} \gamma_i \gamma_j \int_{x_{\eta(t)-1}}^{t} \sigma^2(s) \exp\bigg(- \int_s^t \lambda(v) dv \bigg) \Lambda_{x_{(\eta(t)-1+j)}}(s,t) ds}{\exp\bigg(\int_t^T \lambda(v) dv \bigg)}$$

Define

$$\Phi(u,t) = \int_u^t \sigma^2(s) \exp\bigg(-2 \int_s^t \lambda(v) dv \bigg) ds \tag{64}$$

Therefore

$$\sum_{i=1}^{a-b}\sum_{j=1}^{a-b}\gamma_i\gamma_j\int_{x_b}^{x_{b+1}}\sigma^2(s)\exp\bigg(-\int_s^T\lambda(v)dv\bigg)\Lambda_{x_{b+j}}(s,T)ds+\sum_{i=1}^{a-b}\gamma_i\int_{x_b}^{x_{b+1}}\sigma(s)\exp\bigg(-\int_s^T\lambda(v)dv\bigg)dW(s)$$

$$=\exp\bigg(-\int_{x_{b+1}}^T\lambda(v)dv\bigg)y_a(x_{b+1})+\sum_{i=1}^{a-b}\sum_{j=1}^{a-b}\gamma_i\gamma_j\int_{x_b}^{x_{b+1}}\sigma^2(s)\exp\bigg(-\int_s^T\lambda(v)dv\bigg)\Lambda_{x_{b+j}}(s,T)ds$$

$$-\sum_{i=1}^{a-b}\sum_{j=1}^{a-b}\gamma_i\gamma_j\exp\bigg(-\int_{x_{b+1}}^T\lambda(v)dv\bigg)\int_{x_b}^{x_{b+1}}\sigma^2(s)\exp\bigg(-\int_s^{x_{b+1}}\lambda(v)dv\bigg)\Lambda_{x_{b+j}}(s,x_{b+1})ds$$

$$=\exp\bigg(-\int_{x_{b+1}}^T\lambda(v)dv\bigg)y_a(x_{b+1})+\sum_{i=1}^{a-b}\sum_{j=1}^{a-b}\bigg\{\gamma_i\gamma_j\exp\bigg(-\int_{x_{b+1}}^T\lambda(v)dv\bigg)$$

$$\times\bigg[\int_{x_b}^{x_{b+1}}\sigma^2(s)\bigg\{\exp\bigg(\int_{x_{b+1}}^T\lambda(v)dv\bigg)\exp\bigg(-\int_s^T\lambda(v)dv\bigg)\Lambda_{x_{b+j}}(s,T)$$

$$-\exp\bigg(-\int_s^{x_{b+1}}\lambda(v)dv\bigg)\Lambda_{x_{b+j}}(s,x_{b+1})\bigg\}ds\bigg]\bigg\}$$

$$=\exp\bigg(-\int_{x_{b+1}}^T\lambda(v)dv\bigg)y_a(x_{b+1})+\sum_{i=1}^{a-b}\sum_{j=1}^{a-b}\gamma_i\gamma_j\Lambda(x_{b+1},T)\exp\bigg(-\int_{x_{b+1}}^T\lambda(v)dv\bigg)$$

$$\times\int_{x_b}^{x_{b+1}}\sigma^2(s)\exp\bigg(-2\int_s^{x_{b+1}}\lambda(v)dv\bigg)ds$$

$$=\exp\bigg(-\int_{x_{b+1}}^T\lambda(v)dv\bigg)y_a(x_{b+1})+\sum_{i=1}^{a-b}\sum_{j=1}^{a-b}\gamma_i\gamma_j\Phi(x_b,x_{b+1})\Lambda(x_{b+1},T)\exp\bigg(-\int_{x_{b+1}}^T\lambda(v)dv\bigg)$$

and

$$\sum_{i=1}^{(a-\eta(t)+1)}\sum_{j=1}^{(a-\eta(t)+1)}\gamma_i\gamma_j\int_{x_{\eta(t)-1}}^{t}\sigma^2(s)\exp\Big(-\int_s^T\lambda(v)dv\Big)\Lambda_{x_{(\eta(t)-1+j)}}(s,T)ds$$

$$+\sum_{i=1}^{a-\eta(t)+1}\gamma_i\int_{x_{\eta(t)+1}}^{t}\sigma(s)\exp\Big(-\int_s^T\lambda(v)dv\Big)dW(s)$$

$$=\exp\Big(-\int_t^T\lambda(v)dv\Big)y_a(t)+\sum_{i=1}^{(a-\eta(t)+1)}\sum_{j=1}^{(a-\eta(t)+1)}\gamma_i\gamma_j\int_{x_{\eta(t)-1}}^{t}\sigma^2(s)\exp\Big(-\int_s^T\lambda(v)dv\Big)\Lambda_{x_{(\eta(t)-1+j)}}(s,T)ds$$

$$-\sum_{i=1}^{(a-\eta(t)+1)}\sum_{j=1}^{(a-\eta(t)+1)}\gamma_i\gamma_j\exp\Big(-\int_t^T\lambda(v)dv\Big)\int_{x_{\eta(t)-1}}^{t}\sigma^2(s)\exp\Big(-\int_s^t\lambda(v)dv\Big)\Lambda_{x_{(\eta(t)-1+j)}}(s,t)ds$$

$$=\exp\Big(-\int_t^T\lambda(v)dv\Big)y_a(t)+\sum_{i=1}^{(a-\eta(t)+1)}\sum_{j=1}^{(a-\eta(t)+1)}\Big\{\gamma_i\gamma_j\exp\Big(-\int_t^T\lambda(v)dv\Big)$$

$$\times\Big[\int_{x_{\eta(t)-1}}^{t}\sigma^2(s)\Big\{\exp\Big(\int_t^T\lambda(v)dv\Big)\exp\Big(-\int_s^T\lambda(v)dv\Big)\Lambda_{x_{(\eta(t)-1+j)}}(s,T)$$

$$-\exp\Big(-\int_s^t\lambda(v)dv\Big)\Lambda_{x_{(\eta(t)-1+j)}}(s,t)\Big\}ds\Big]\Big\}$$

$$=\exp\Big(-\int_t^T\lambda(v)dv\Big)y_a(t)+\sum_{i=1}^{(a-\eta(t)+1)}\sum_{j=1}^{(a-\eta(t)+1)}\gamma_i\gamma_j\Lambda(t,T)\exp\Big(-\int_t^T\lambda(v)dv\Big)$$

$$\times\int_{x_{\eta(t)-1}}^{t}\sigma^2(s)\exp\Big(-2\int_s^t\lambda(v)dv\Big)ds$$

$$=\exp\Big(-\int_t^T\lambda(v)dv\Big)y_a(t)+\sum_{i=1}^{(a-\eta(t)+1)}\sum_{j=1}^{(a-\eta(t)+1)}\gamma_i\gamma_j\Phi(x_{\eta(t)-1},t)\Lambda(t,T)\exp\Big(-\int_t^T\lambda(v)dv\Big)$$

Therefore:

$$f(t,T) = f(0,T) + \sum_{b=0}^{\eta(t)-2} \bigg\{ \sum_{i=1}^{a-b}\sum_{j=1}^{a-b} \gamma_i\gamma_j \int_{x_b}^{x_{b+1}} \sigma^2(s)\exp\bigg(- \int_s^T \lambda(v)dv\bigg)\Lambda_{x_{b+j}}(s,T)ds$$

$$+ \sum_{i=1}^{a-b} \gamma_i \int_{x_b}^{x_{b+1}} \sigma(s)\exp\bigg(- \int_s^T \lambda(v)dv\bigg)dW(s)\bigg\}$$

$$+ \sum_{i=1}^{(a-\eta(t)+1)} \sum_{j=1}^{(a-\eta(t)+1)} \gamma_i\gamma_j \int_{x_{\eta(t)-1}}^{t} \sigma^2(s)\exp\bigg(- \int_s^T \lambda(v)dv\bigg)\Lambda_{x_{(\eta(t)-1+j)}}(s,T)ds$$

$$+ \sum_{i=1}^{a-\eta(t)+1} \gamma_i \int_{x_{\eta(t)-1}}^{t} \sigma(s)\exp\bigg(- \int_s^T \lambda(v)dv\bigg)dW(s)$$

$$= f(0,T) + \sum_{b=0}^{\eta(t)-2} \bigg\{ \exp\bigg(- \int_{x_{b+1}}^T \lambda(v)dv\bigg) y_a(x_{b+1})$$

$$+ \sum_{i=1}^{a-b}\sum_{j=1}^{a-b} \gamma_i\gamma_j \Phi(x_b, x_{b+1})\Lambda(x_{b+1},T)\exp\bigg(- \int_{x_{b+1}}^T \lambda(v)dv\bigg)\bigg\}$$

$$+ \exp\bigg(- \int_t^T \lambda(v)dv\bigg) y_a(t) + \sum_{i=1}^{(a-\eta(t)+1)} \sum_{j=1}^{(a-\eta(t)+1)} \gamma_i\gamma_j \Phi(x_{\eta(t)-1}, t)\Lambda(t,T)\exp\bigg(- \int_t^T \lambda(v)dv\bigg)$$

$$= f(0,T) + \sum_{b=0}^{\eta(t)-2} \exp\bigg(- \int_{x_{b+1}}^T \lambda(v)dv\bigg) y_a(x_{b+1}) + \exp\bigg(- \int_t^T \lambda(v)dv\bigg) y_a(t)$$

$$+ \sum_{b=0}^{\eta(t)-2} \sum_{i=1}^{a-b}\sum_{j=1}^{a-b} \gamma_i\gamma_j \Phi(x_b, x_{b+1})\Lambda(x_{b+1},T)\exp\bigg(- \int_{x_{b+1}}^T \lambda(v)dv\bigg)$$

$$+ \sum_{i=1}^{(a-\eta(t)+1)} \sum_{j=1}^{(a-\eta(t)+1)} \gamma_i\gamma_j \Phi(x_{\eta(t)-1}, t)\Lambda(t,T)\exp\bigg(- \int_t^T \lambda(v)dv\bigg)$$

Therefore:

$$\int_{x_a}^T f(t,u)du = \int_{x_a}^T \Bigg(f(0,u) + \sum_{b=0}^{\eta(t)-2} \exp\Big(- \int_{x_{b+1}}^u \lambda(v)dv\Big) y_a(x_{b+1}) + \exp\Big(- \int_t^u \lambda(v)dv\Big) y_a(t)$$

$$+ \sum_{b=0}^{\eta(t)-2} \sum_{i=1}^{a-b} \sum_{j=1}^{a-b} \gamma_i\gamma_j \Phi(x_b, x_{b+1})\Lambda(x_{b+1},u)\exp\Big(- \int_{x_{b+1}}^u \lambda(v)dv\Big)$$

$$+ \sum_{i=1}^{(a-\eta(t)+1)} \sum_{j=1}^{(a-\eta(t)+1)} \gamma_i\gamma_j \Phi(x_{\eta(t)-1}, t)\Lambda(t,u)\exp\Big(- \int_t^u \lambda(v)dv\Big)\Bigg) du$$

$$= \int_{x_a}^T f(0,u)du + \sum_{b=0}^{\eta(t)-2} y_a(x_{b+1}) \int_{x_a}^T \exp\Big(- \int_{x_{b+1}}^u \lambda(v)dv\Big) du + y_a(t) \int_{x_a}^T \exp\Big(- \int_t^u \lambda(v)dv\Big) du$$

$$+ \sum_{b=0}^{\eta(t)-2} \sum_{i=1}^{a-b} \sum_{j=1}^{a-b} \gamma_i\gamma_j \Phi(x_b, x_{b+1}) \int_{x_a}^T \Lambda(x_{b+1},u)\exp\Big(- \int_{x_{b+1}}^u \lambda(v)dv\Big) du$$

$$+ \sum_{i=1}^{(a-\eta(t)+1)} \sum_{j=1}^{(a-\eta(t)+1)} \gamma_i\gamma_j \Phi(x_{\eta(t)-1}, t) \int_{x_a}^T \Lambda(t,u)\exp\Big(- \int_t^u \lambda(v)dv\Big) du$$

$$= \int_{x_a}^T f(0,u)du + \sum_{b=0}^{\eta(t)-2} \Lambda_{x_a}(x_{b+1},T) y_a(x_{b+1}) + \Lambda_{x_a}(t,T) y_a(t)$$

$$+ \sum_{b=0}^{\eta(t)-2} \sum_{i=1}^{a-b} \sum_{j=1}^{a-b} \gamma_i\gamma_j \Phi(x_b, x_{b+1}) \int_{x_a}^T \Lambda(x_{b+1},u) d\Lambda(x_{b+1},u)$$

$$+ \sum_{i=1}^{(a-\eta(t)+1)} \sum_{j=1}^{(a-\eta(t)+1)} \gamma_i\gamma_j \Phi(x_{\eta(t)-1}, t) \int_{x_a}^T \Lambda(t,u) d\Lambda(t,u)$$

$$= \int_{x_a}^T f(0,u)du + \sum_{b=0}^{\eta(t)-2} \Lambda_{x_a}(x_{b+1},T) y_a(x_{b+1}) + \Lambda_{x_a}(t,T) y_a(t)$$

$$+ \frac{1}{2} \sum_{b=0}^{\eta(t)-2} \sum_{i=1}^{a-b} \sum_{j=1}^{a-b} \gamma_i\gamma_j \Phi(x_b, x_{b+1}) \{\Lambda^2(x_{b+1},T) - \Lambda^2(x_{b+1},x_a)\}$$

$$+ \frac{1}{2} \sum_{i=1}^{(a-\eta(t)+1)} \sum_{j=1}^{(a-\eta(t)+1)} \gamma_i\gamma_j \Phi(x_{\eta(t)-1}, t) \{\Lambda^2(t,T) - \Lambda^2(t,x_a)\}$$

Now

$$\int_t^T f(t,u)du = \int_t^{x_{\eta(t)}} f(t,u)du + \int_{x_{\eta(T)-1}}^T f(t,u)du + \sum_{k=\eta(t)}^{\eta(T)-2} \int_{x_k}^{x_{k+1}} f(t,u)du$$

$$= \int_t^T f(0,u)du + \sum_{b=0}^{\eta(t)-2} \Lambda_{x_{\eta(T)-1}}(x_{b+1},T) y_{\eta(T)-1}(x_{b+1}) + \Lambda_{x_{\eta(T)-1}}(t,T) y_{\eta(T)-1}(t)$$

$$+ \frac{1}{2} \sum_{b=0}^{\eta(t)-2} \sum_{i=1}^{\eta(T)-1-b} \sum_{j=1}^{\eta(T)-1-b} \gamma_i \gamma_j \Phi(x_b, x_{b+1}) \{\Lambda^2(x_{b+1},T) - \Lambda^2(x_{b+1}, x_{\eta(T)-1})\}$$

$$+ \frac{1}{2} \sum_{i=1}^{\eta(T)-\eta(t)} \sum_{j=1}^{\eta(T)-\eta(t)} \gamma_i \gamma_j \Phi(x_{\eta(t)-1}, t) \{\Lambda^2(t,T) - \Lambda^2(t, x_{\eta(T)-1})\}$$

$$+ \sum_{k=\eta(t)}^{\eta(T)-2} \Bigg[\sum_{b=0}^{\eta(t)-2} \Lambda_{x_k}(x_{b+1}, x_{k+1}) y_k(x_{b+1}) + \Lambda_{x_k}(t, x_{k+1}) y_k(t)$$

$$+ \frac{1}{2} \sum_{b=0}^{\eta(t)-2} \sum_{i=1}^{k-b} \sum_{j=1}^{k-b} \gamma_i \gamma_j \Phi(x_b, x_{b+1}) \{\Lambda^2(x_{b+1}, x_{k+1}) - \Lambda^2(x_{b+1}, x_k)\}$$

$$+ \frac{1}{2} \sum_{i=1}^{(k-\eta(t)+1)} \sum_{j=1}^{(a-\eta(t)+1)} \gamma_i \gamma_j \Phi(x_{\eta(t)-1}, t) \{\Lambda^2(t, x_{k+1}) - \Lambda^2(t, x_k)\} \Bigg]$$

$$= \int_t^T f(0,u)du$$

$$+ \sum_{b=0}^{\eta(t)-2} \Lambda_{x_{\eta(T)-1}}(x_{b+1},T) y_{\eta(T)-1}(x_{b+1})$$

$$+ \Lambda_{x_{\eta(T)-1}}(t,T) y_{\eta(T)-1}(t)$$

$$+ \sum_{k=\eta(t)}^{\eta(T)-2} \sum_{b=0}^{\eta(t)-2} \Lambda_{x_k}(x_{b+1}, x_{k+1}) y_k(x_{b+1})$$

$$+ \sum_{k=\eta(t)}^{\eta(T)-2} \Lambda_{x_k}(t, x_{k+1}) y_k(t)$$

$$+ \frac{1}{2} \sum_{b=0}^{\eta(t)-2} \sum_{i=1}^{\eta(T)-1-b} \sum_{j=1}^{\eta(T)-1-b} \gamma_i \gamma_j \Phi(x_b, x_{b+1}) \{\Lambda^2(x_{b+1},T) - \Lambda^2(x_{b+1}, x_{\eta(T)-1})\}$$

$$+ \frac{1}{2} \sum_{i=1}^{\eta(T)-\eta(t)} \sum_{j=1}^{\eta(T)-\eta(t)} \gamma_i \gamma_j \Phi(x_{\eta(t)-1}, t) \{\Lambda^2(t,T) - \Lambda^2(t, x_{\eta(T)-1})\}$$

$$+ \sum_{k=\eta(t)}^{\eta(T)-2} \frac{1}{2} \sum_{b=0}^{\eta(t)-2} \sum_{i=1}^{k-b} \sum_{j=1}^{k-b} \gamma_i \gamma_j \Phi(x_b, x_{b+1}) \{\Lambda^2(x_{b+1}, x_{k+1}) - \Lambda^2(x_{b+1}, x_k)\}$$

$$+ \sum_{k=\eta(t)}^{\eta(T)-2} \frac{1}{2} \sum_{i=1}^{(k-\eta(t)+1)} \sum_{j=1}^{(a-\eta(t)+1)} \gamma_i \gamma_j \Phi(x_{\eta(t)-1}, t) \{\Lambda^2(t, x_{k+1}) - \Lambda^2(t, x_k)\}$$

Therefore, the bond price is given by

$$
\begin{aligned}
B(t,T) &= \exp\Big(- \int_t^T f(t,u)du \Big) \\
&= \frac{B(0,T)}{B(0,t)} \exp\Big(- \sum_{b=0}^{\eta(t)-2} \Lambda_{x_{\eta(T)-1}}(x_{b+1},T) y_{\eta(T)-1}(x_{b+1}) \\
&- \Lambda_{x_{\eta(T)-1}}(t,T) y_{\eta(T)-1}(t) \\
&- \sum_{k=\eta(t)}^{\eta(T)-2} \sum_{b=0}^{\eta(t)-2} \Lambda_{x_k}(x_{b+1},x_{k+1}) y_k(x_{b+1}) \\
&- \sum_{k=\eta(t)}^{\eta(T)-2} \Lambda_{x_k}(t,x_{k+1}) y_k(t) \\
&- \frac{1}{2} \sum_{b=0}^{\eta(t)-2} \sum_{i=1}^{\eta(T)-1-b} \sum_{j=1}^{\eta(T)-1-b} \gamma_i\gamma_j \Phi(x_b,x_{b+1}) \{\Lambda^2(x_{b+1},T) - \Lambda^2(x_{b+1},x_{\eta(T)-1})\} \\
&- \frac{1}{2} \sum_{i=1}^{\eta(T)-\eta(t)} \sum_{j=1}^{\eta(T)-\eta(t)} \gamma_i\gamma_j \Phi(x_{\eta(t)-1},t) \{\Lambda^2(t,T) - \Lambda^2(t,x_{\eta(T)-1})\} \\
&- \sum_{k=\eta(t)}^{\eta(T)-2} \frac{1}{2} \sum_{b=0}^{\eta(t)-2} \sum_{i=1}^{k-b} \sum_{j=1}^{k-b} \gamma_i\gamma_j \Phi(x_b,x_{b+1}) \{\Lambda^2(x_{b+1},x_{k+1}) - \Lambda^2(x_{b+1},x_k)\} \\
&- \sum_{k=\eta(t)}^{\eta(T)-2} \frac{1}{2} \sum_{i=1}^{(k-\eta(t)+1)} \sum_{j=1}^{(a-\eta(t)+1)} \gamma_i\gamma_j \Phi(x_{\eta(t)-1},t) \{\Lambda^2(t,x_{k+1}) - \Lambda^2(t,x_k)\} \Big)
\end{aligned}
$$

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