Pattern-Reconfigurable Sparse Linear Array Synthesis Under Minimum Element Spacing Control by Alternating Sequential Quadratic Programming

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***Abstract*—A new method called alternating sequential quadratic programming (ASQP) is proposed to synthesize pattern-reconfigurable sparse linear arrays with minimum element spacing control. The method can find the common element positions and multiple sets of excitations for generating multiple reconfigurable patterns which accurately meet their given upper and lower pattern bounds. In addition, by introducing auxiliary weighting coefficients and collective excitation coefficient vectors and choosing them as optimization variables alternately, the proposed method can appropriately incorporate the minimum element spacing constraint into the pattern synthesis. Synthesized results show that the proposed method can give satisfactory reconfigurable pattern performance but save much more elements compared with some existing methods.**

***Index Terms*—Pattern synthesis, pattern-reconfigurable array, alternating sequential quadratic programming, sparse array, minimum spacing constraint.**

I. Introduction

RECONFIGURABLE antenna arrays capable of generating multiple different patterns by only altering element excitations for different application occasions have been in great demand of many radar and wireless communication systems [1]–[4]. In the past decades, a few methods have been developed to synthesize pattern-reconfigurable arrays [5]–[8]. However, these synthesis methods are usually limited to dealing with uniformly spaced linear arrays, and consequently they sometimes require a large number of elements to simultaneously meet multiple different pattern requirements.

In recent years, some advanced techniques exploiting nonuniform element spacings have been presented to effectively reduce the number of elements in the synthesis of pattern-reconfigurable antenna arrays [9]–[18]. These techniques mainly include the extended matrix pencil methods (MPM) [9], [10], the joint sparse recovery technique [11], the extended reweighted L1-norm minimization (RL1NM) [12], the multiple measurement vectors FOCal underdetermined system solver (M-FOCUSS) [13], the extended alternating convex optimization (ACO) [14], and some others [15]–[18]. For multiple patterns with different desired shapes, they can successfully find the common element positions and multiple sets of excitation distributions. However, most of them need to preset multiple reference pattern shapes including amplitude and phase distributions. As we know, artificially setting multiple reference patterns is not easy, and one has to resort some other synthesis techniques to determine the multiple reference patterns in advance. In addition, matching the radiated multiple patterns to the reference ones in both amplitude and phase distributions is difficult, and an inappropriate setting of reference patterns will reduce the available solution space. Besides, some of techniques such as in [11]–[13] cannot control the minimum element spacing of the synthesized pattern-reconfigurable array, and consequently the obtained minimum element spacings may be very small. This may cause the synthesized array geometry impractical.

As is known, synthesizing an antenna array with multiple shaped patterns by using as fewer elements as possible is typically a constrained nonlinear optimization problem. The sequential quadratic programming (SQP) technique has been recognized as a convenient and effective tool for solving nonlinear problems [19], and it has been successfully applied to synthesize excitations for either focused or shaped power patterns [20]–[22]. In the SQP-based synthesis techniques, pattern shape control by setting lower and upper bounds can be easily processed. However, the original SQP in [20]–[22] only consider single pattern synthesis for a given array geometry, and it cannot be applied to optimize the common element positions with multiple sets of excitations for multiple different pattern requirements. In this work, we present an novel alternating sequential quadratic programming (ASQP) method in which collective excitation coefficient vectors and auxiliary weighting coefficients are introduced and then chosen as optimization variables alternately so that the common element positions along with multiple sets of excitations can be optimized by iteratively performing two alternating optimization problems and the minimum element spacing constraint can be appropriately processed in this procedure. In addition, a refining strategy by updating active element patterns (AEPs) is adopted such that the mutual coupling can be also included in the synthesis. Three examples of synthesizing different

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pattern-reconfigurable sparse linear arrays are given to validate the effectiveness and advantages of the proposed method.

II. Formulation and Algorithm

A. Problem Description

Consider a pattern-reconfigurable linear array with N antenna elements that are located along x-axis with an uniform spacing of d. Assume that this array can radiate M patterns with different shapes by varying its excitation distributions. The mth ($m = 1, 2, \ldots, M$) power pattern of this array can be written as

$$\left|F^{(m)}(\theta)\right|^2 = \left|\mathbf{a}^T(\theta)\mathbf{w}^{(m)}\right|^2 \tag{1}$$

where $\mathbf{w}^{(m)} = \left[w_1^{(m)}, w_2^{(m)}, \cdots, w_N^{(m)}\right]^T$, and

$$\mathbf{a}(\theta) = \left[a_1(\theta)e^{j\beta d\sin\theta}, a_2(\theta)e^{j\beta 2d\sin\theta}, \ldots, a_N(\theta)e^{j\beta Nd\sin\theta}\right]^T \tag{2}$$

In the above, $j = \sqrt{-1}$, $\beta = 2\pi/\lambda$ is the wavenumber in the free space, $a_n(\theta)$ is the phase-adjusted pattern of the nth element and $w_n^{(m)}$ means the nth element excitation for the mth pattern. To constrain all the M patterns within their required shapes, we can put the following pattern constraints:

$$\text{Const.}\begin{cases} L^{(m)}(\theta) \leq \left|F^{(m)}(\theta)\right|^2 \leq U^{(m)}(\theta),\ \forall\theta \in \Omega_{ML}^{(m)} \\ \left|F^{(m)}(\theta)\right|^2 \leq P_{SL}^{(m)}(\theta),\ \forall\theta \in \Omega_{SL}^{(m)} \\ (m = 1, 2, \cdots, M) \end{cases} \tag{3}$$

where $L^{(m)}(\theta)$ and $U^{(m)}(\theta)$ represent the lower and upper bound functions over the mainlobe region $\Omega_{ML}^{(m)}$, and $P_{SL}^{(m)}(\theta)$ means an upper bound over the sidelobe region $\Omega_{SL}^{(m)}$. By choosing appropriate $L^{(m)}(\theta)$, $U^{(m)}(\theta)$ and $P_{SL}^{(m)}(\theta)$, we can constrain each pattern mainlobe shape and sidelobe distribution without using reference amplitude and phase patterns.

The problem of synthesizing a pattern-reconfigurable sparse array can be formulated as determining the common element positions and multiple sets of excitation distributions for the M patterns satisfying the Const. (3). It can be further transformed as picking up the best element positions from the N densely spaced elements (i.e., $d \ll \lambda$) while meeting the multiple different pattern requirements. However, one element can be discarded only if all the excitations for this element corresponding to multiple patterns are zeros. To tackle this issue, we introduce a collective excitation coefficient vector (CECV) for the nth element which is given by

$$\tilde{\mathbf{w}}_n = \left[w_n^{(1)}, w_n^{(2)}, \ldots, w_n^{(M)}\right]^T \tag{4}$$

Thus, the energy of the CECV for the nth ($n = 1, 2, \ldots, N$) element can be defined by $\|\tilde{\mathbf{w}}_n\|_2^2$. Naturally, when $\|\tilde{\mathbf{w}}_n\|_2 = 0$, the nth element can be discarded.

However, when selecting the element positions from the densely placed potential elements, another issue is that the minimum element spacing must be accurately controlled. Besides, the dynamic range ratio (DRR) of excitation amplitudes should be also controlled to simplify antenna array design. Hence, by considering all the multiple pattern constraints, the minimum element spacing and DRR control, the concerned problem can be formulated as

$$\begin{aligned} &\min\ \left\| \{\|\tilde{\mathbf{w}}_1\|_2, \|\tilde{\mathbf{w}}_2\|_2, \cdots, \|\tilde{\mathbf{w}}_N\|_2\} \right\|_0 \\ &\text{s.t.}\begin{cases} \text{Const. (4)} \\ d_{\text{min}} \geq Qd \\ \dfrac{\max_s\{w_s^{(m)}\}}{\min_s\{w_s^{(m)}\}} \leq \rho, w_s^{(m)} \text{ are selected excitations} \end{cases} \end{aligned} \tag{5}$$

where $\{w_1^{(m)}, w_2^{(m)}, \cdots, w_S^{(m)}\} \subset \{w_1^{(m)}, w_2^{(m)}, \cdots, w_N^{(m)}\}$ are the excitations of the selected elements for the mth patterns, $d_{\text{min}} \geq Qd$ denotes the minimum element spacing constraint and Q is an integer number.

B. The Proposed Alternating Sequential Quadratic Programming (ASQP) Synthesis Method

The problem in (5) is involved with a constrained ℓ_0-norm minimization which is a computationally expensive NP-hard optimization problem. As is known, the reweighted ℓ_1-norm optimization techniques are alternative ways to effectively produce sparse solutions, but the non-convex lower bound constraints for the mainlobe power shape control and the minimum element spacing constraint are not easy to handle. Here an ASQP method is presented to synthesize multiple pattern-reconfigurable sparse linear array with the minimum element spacing and DRR control. In the proposed ASQP, a set of weighting coefficients denoted by $\{g_1, g_2, \ldots, g_N\}$ are introduced. These coefficients are not fixed, instead they together with multiple sets of element excitation vectors are alternately chosen as the optimization variables. Besides, an auxiliary optimization variable $v^{(m)}$ is introduced to control the DRR, which can be formulated as $v^{(m)} \leq w_s^{(m)} \leq v^{(m)}\rho$ where $\{w_1^{(m)}, \cdots, w_S^{(m)}\}$ are the excitations corresponding to the element positions selected at the previous step. Mathematically, the ASQP method performs the following alternating optimization problems iteratively

$$\begin{cases} \min\limits_{\left\{w_n^{(m)}; v^{(m)} \middle| \substack{n=1,\cdots,N \\ m=1,\cdots,M}\right\}} \sum\limits_{n=1}^{N} g_n \|\tilde{\mathbf{w}}_n\|_2, \quad \text{for given } g_n\text{s} \\ \text{s.t.}\begin{cases} \mathbf{0} \leq \mathbf{I}^{(m)} \leq \mathbf{1}, -\boldsymbol{\pi} \leq \boldsymbol{\varphi}^{(m)} \leq \boldsymbol{\pi} \\ v^{(m)} \leq \mathbf{A}_s^T\mathbf{I}^{(m)} \leq v^{(m)}\rho \\ L^{(m)}(\theta) \leq \left|F^{(m)}(\theta)\right|^2 \leq U^{(m)}(\theta),\ \forall\theta \in \Omega_{ML}^{(m)} \\ \left|F^{(m)}(\theta)\right|^2 \leq P_{SL}^{(m)}, \forall\theta \in \Omega_{SL}^{(m)} \end{cases} & \text{(6a)} \\ \min\limits_{\{g_n \in R; | n=1,2,\cdots,N\}} \sum\limits_{n=1}^{N} g_n \|\tilde{\mathbf{w}}_n\|_2, \quad \text{for given } \tilde{\mathbf{w}}_n\text{s} \\ \text{s.t.}\begin{cases} 0 \leq g_n \leq 1 \\ \sum\limits_{n=i}^{i+Q-1} g_n \geq Q-1 \ (\text{for } i = 1, \cdots, N-Q+1) \end{cases} & \text{(6b)} \end{cases}$$

where $\mathbf{I}^{(m)}$, $\boldsymbol{\varphi}^{(m)}$ denote the excitation amplitudes and phases of $\mathbf{w}^{(m)}$, and the vector $\mathbf{A}_s \in \Re^{N\times 1}$ is utilized to control the DRR of the sth selected excitation amplitude at the previous step. The main principle of controlling DRR is as follows: 1) find the indexes for '0' in $\{g_1, g_2, \ldots, g_N\}$ obtained at the previous step and store these indexes in a vector $\mathbf{p}$; 2) then set $\mathbf{A}_s(\mathbf{p}(s)) = 1$ for all $s = 1, \cdots, \text{length}(\mathbf{p})$, and all other elements of $\mathbf{A}_s$ are set as zeros. In Problem (6a), the CECV $\tilde{\mathbf{w}}_n$s are optimization variables and g_n $(n = 1, \cdots, N)$ are the weighting coefficients, while they are reversed in Problem (6b). Solving Problem (6a) by using the SQP can give a solution with multiple sets of element excitations which satisfy all the pattern bounds and DRR constraint, except that the minimum element spacing will depend on the distribution of g_ns. Then, for given $\tilde{\mathbf{w}}_n$s obtained in Problem (6a), solving Problem (6b) will give a solution which has at the least $(Q-1)$ elements with ‘1’ and at the most one element with ‘0’ for each Q-length segment of g_ns. Thus, in the next iteration, the CECV $\tilde{\mathbf{w}}_n$s corresponding to $g_n = 1$ will be severely penalized to zeros while the CECVs corresponding to $g_n = 0$ will be retained. By iteratively solving problems (6a) and (6b), we can obtain a sparse solution which meets all the pattern constraints, the minimum element spacing and the DRR constraint.

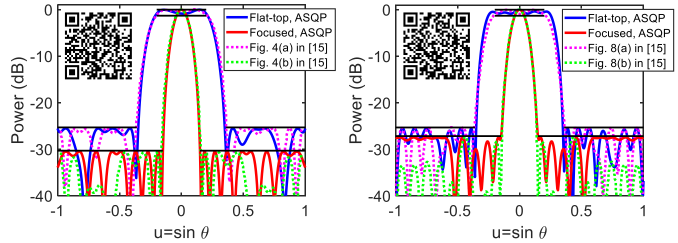
Inspired by [24]–[27], the element mutual coupling can be also incorporated by using a refining strategy. At the initial step, an isolated element pattern (IEP) can be obtained by full-wave simulation of a single antenna element, and the array manifold vector $\mathbf{a}(\theta)$ can be constructed by using the phase-shifted relationship of $\mathrm{a}_n(\theta) \approx \mathrm{a}_i(\theta)e^{j\beta(n-i)d\sin\theta}$. Then, the proposed ASQP is performed to find the common element positions and multiple sets of element excitations. At the kth $(k \geq 1)$ step, full-wave simulation of the antenna array with the positions obtained at the previous step is performed, and then, the new simulated AEPs are used to approximate and update all the element patterns of $a_n(\theta)$ at their nearby potential positions. This refining strategy can be iteratively performed until the obtained element positions do not change.

III. Numerical Results

A. Synthesizing a Sparse Array with Focused/Shaped Patterns

In the first example, we consider synthesizing a sparse linear array radiating a focused pattern and a flat-top pattern which were synthesized in [15] by using 20 $\lambda/2$-spaced elements. In [15], when the DRR constraint is not considered, the synthesized flat-top pattern has a ripple of less than ± 0.43 dB in $|\sin\theta| \leq 0.2$, and maximum sidelobe level (SLL) is below -25.5 dB in $|\sin\theta| \geq 0.35$. The maximum SLL of the synthesized focused pattern is below $-30.5\,$dB in $|\sin\theta| \geq 0.15$. When the constraint of $DRR = 4.51$ is considered in [15], the maximum SLL of the focused beam is increased to -27.45 dB and the other performance of the synthesized dual patterns are the same as those without the DRR constraint.

Now, the proposed ASQP method is applied to synthesize the same dual patterns without/with the constraint of $DRR = 4.51$. The initial linear array is set as 381 potential elements with a dense spacing of $\lambda/40$. The minimum element spacing is set as $d_{\min} \geq 0.5\lambda$. That is, setting $Q = 20$ in Problem (6b). The synthesized dual pattern results by the ASQP are shown in Fig. 1(a) and (b), and the obtained element positions and excitations are contained in the QR codes. It can be seen that both of the dual patterns synthesized without and with the DRR constraint by the proposed ASQP precisely meet the required pattern upper and lower bounds. The synthesized arrays without/with the DRR constraint require only 16 and 18 elements, respectively. This means that 20% and 10% elements are saved compared with the result of [15]. Moreover, both of the obtained minimum element spacings for the cases of without/with the DRR constraint meet the required constraint of $d_{\min} \geq 0.5\lambda$ as shown in QR codes.



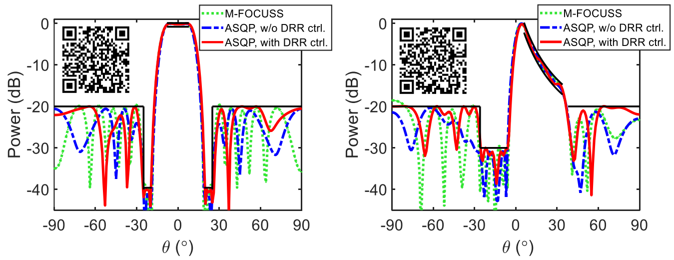
(a) (b)

Fig. 1. The synthesized dual patterns by the proposed ASQP and the results in [15]. (a) and (b) show the results obtained without/with the DRR constraint.

B. Synthesizing a Sparse Array with Dual Shaped Patterns

In the second example, we consider synthesizing a sparse linear array with two reconfigurable shaped patterns including a flat-top beam and a cosec-squared beam, both with multi-region SLL requirements. Assume that the flat-top pattern has a ripple of less than $\pm 0.1\,$dB in the mainlobe region $\theta \in [-18°, 18°]$, and its SLL bound is set below $-40\,$dB for $\theta \in [-25°, -19°] \cup [19°, 25°]$ and below $-20\,$dB for the other region. The desired cosec-squared pattern has a ripple of less than $\pm 0.6\,$dB in the mainlobe region $\theta \in [-5°, 37°]$, and its SLL bound is set below $-30\,$dB for the region $\theta \in [-26°, -6°]$ and below $-20\,$dB for the other region. The dual patterns with the above requirements were synthesized in [13] by using the M-FOCUSS with 16 nonuniformly spaced elements. In [13], reference dual patterns were needed which were generated using the technique in [23]. The DRR cannot be controlled, and it was 14.28 in [13].

Now, we apply the proposed ASQP method to synthesize the same dual patterns with as fewer elements as possible. In this example, the initial linear array is assumed to have 191 potential elements with a spacing of $\lambda/20$. The minimum interspacing is set as $d_{\min} \geq 0.5\lambda$ $(Q = 10)$. By applying the ASQP, a sparse array with only 12 elements is obtained. The obtained dual pattern results by the ASQP are shown in Fig. 2(a) and (b), and for comparison, the results synthesized by the M-FOCUSS in [13] are re-plotted in these figures. As can be seen, the two methods obtain similar mainlobe shapes for both flat-top and cosec-squared patterns, and all of the obtained mainlobe shapes meet the required upper and lower bounds. In the sidelobe region, the dual patterns obtained by the proposed ASQP satisfy all the required multiple SLL bounds, while the patterns by the M-FOCUSS are slightly beyond the required bounds in the low SLL region. Besides, we add the constraint of $DRR \leq 8$ to synthesize the same dual-pattern case. The obtained dual patterns by the ASQP with the DRR constraint are also shown in Fig. 2(a) and (b), and they also meet all the pattern bounds. In this situation, 14 elements are required with the DRR constraint. All the obtained common element positions and excitations by the



(a) (b)

Fig. 2. The synthesized reconfigurable dual patterns by the proposed ASQP and the M-FOCUSS in [13]. (a) Flat-top pattern, and (b) cosec-squared pattern.

proposed ASQP without/with the DRR constraint are stored in QR codes in Fig. 2. It is shown that the obtained minimum spacings by the ASQP without/with the DRR constraint are 0.65λ and 0.5λ, both meeting the requirement of $d_{\min} \geq 0.5\lambda$. Element savings without/with the DRR constraint are 25% and 12.5% compared with the 16-element array obtained in [13].

C. Synthesizing a Sparse Array with Triple Patterns

In the last example, we consider synthesizing a sparse array with more complicated triple patterns including a focused, a flat-top and a cosec-squared pattern. Assume that the focused beam has a first-null beamwidth of 16°, the flat-top pattern has a ripple of ± 0.5 dB in the mainlobe region of $\theta \in [-23^\circ, 23^\circ]$, and the cosec-squared pattern has a ripple of ± 0.5 dB in the mainlobe region of $\theta \in [-8^\circ, 42^\circ]$. The SLL bounds for all the three patterns are set as -20 dB in their sidelobe regions. The three reconfigurable patterns were synthesized by several methods including the M-FOCUSS in [13], the joint sparse recovery (JSR) technique in [11], the extended RL1NM in [12] and the extended ACO in [14]. The M-FOCUSS in [13] used reference patterns that were synthesized by the modified Woodward-Lawson method in [5]. The latter three methods used the upper bounds to control the sidelobe distributions of the three patterns, but they still adopted the reference functions to generate the reconfigurable three mainlobe shapes. The required number of elements as well as the obtained SLLs and the minimum element spacings for the four methods are given in Table I. As can be seen, the JSR technique requires 18 elements and the other three methods need 16 elements. The obtained minimum element spacings by the JSR and the extended RL1NM are much less than 0.5λ due to lack of the minimum element spacing control.

Now, we apply the proposed ASQP to synthesize the same three reconfigurable patterns. The initial array is still assumed to have 191 elements with a uniform spacing of $\lambda/20$, and the minimum interspacing requirement is set as $d_{\min} \geq 0.5\lambda$ ($Q = 10$). The synthesized three patterns by our proposed method are shown in Fig. 4. The synthesized patterns exactly satisfy the prescribed bound constraints both in the mainlobe and sidelobe regions. Moreover, the required element number for generating the three different patterns by the proposed method is only 13 which is much less than the ones required by the previous four methods, and the obtained SLL is more accurate than those in [11]–[13]. In this example, the obtained minimum element spacing by the proposed ASQP is 0.55λ which still meets the requirement of $d_{\min} \geq 0.5\lambda$.

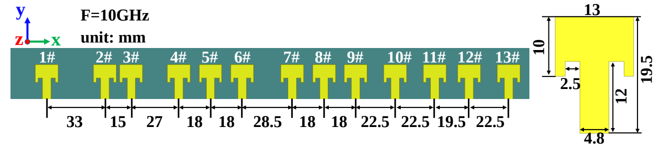
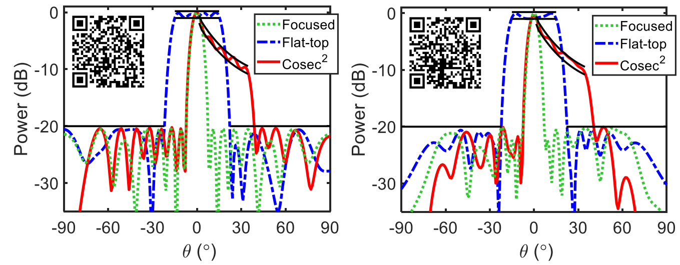


Fig. 3. Geometry of the obtained 13-element triple-pattern microstrip antenna array (a substrate with $\epsilon_r = 2.2$ and a thickness of 1.57 mm is used).



(a) (b)

Fig. 4. The obtained triple-pattern results by the ASQP for (a) the ideal array factor and (b) the microstrip array considering mutual coupling.

TABLE I
THE SYNTHESIZED TRIPLE-PATTERN RECONFIGURABLE SPARSE ARRAYS BY THE PROPOSED ASQP AND OTHER FOUR METHODS IN [11]–[14]

Synthesis method	J. Sparse Recovery	Ext. RL1NM	M-FOCUSS	Ext. ACO	Proposed ASQP
N	18	16	16	16	13
SLL (dB)	-19.12	-19.82	-19.53	-20.04	-20.07
$d_{\min}$ (λ)	0.21	0.35	0.53	0.50	0.55

Next, we consider synthesizing a sparse microstrip antenna array with the same triple pattern requirements. First, a microstrip antenna element working at 10 GHz is designed, and its geometry is shown in Fig. 3. Thus, the IEP of this single element can be obtained by full-wave simulation, and an initial array manifold vector $\mathbf{a}(\theta)$ can be constructed. Then, the simulated AEPs are updated iteratively by full-wave simulation of the antenna array with the positions synthesized by the ASQP at the previous step. In this example, the refining step is performed only twice, and the final optimized array has 13 nonuniformly spaced microstrip antenna elements as shown in Fig. 3. Fig. 4(b) shows the obtained full-wave simulated triple patterns. Since the mutual coupling for the real antenna structure is considered in the synthesis procedure, the obtained triple patterns by full-wave simulation still meet all the pattern bounds exactly. The obtained gains of the focused, flat-top, and cosec-squared patterns are 17.9 dBi, 11.42 dBi and 14.65 dBi.

IV. Conclusions

We have presented a novel alternating sequential programming (ASQP) method for synthesizing pattern-reconfigurable sparse linear arrays. Some complicated shaped pattern constraints, the minimum element spacing and DRR constraint can be incorporated into the proposed method. A refining strategy can be adopted to include the mutual coupling. Three examples for synthesizing different pattern-reconfigurable sparse arrays have been provided. Compared with many existing synthesis techniques, the proposed method does not need to preset any reference patterns, and thus it can exploit more degrees of freedom to further reduce the required element number while remaining satisfactory multiple pattern performance.

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