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Attitude Estimation from Vector Measurements: Necessary and Sufficient Conditions and Convergent Observer Design ⋆

Bowen Yi *, **Lei Wang** †, **Ian R. Manchester** *

* *Australian Centre for Field Robotics, School of Aerospace, Mechanical and Mechatronic Engineering, The University of Sydney, NSW 2006, Australia*

† *State Key Laboratory of Industrial Control Technology, Zhejiang University, Hangzhou 310027, China*

E-mail: bowen.yi{ian.manchester}@sydney.edu.au; lei.wangzju@zju.edu.cn

Abstract: The paper addresses the problem of attitude estimation for rigid bodies using (possibly time-varying) vector measurements, for which we provide a *necessary and sufficient* condition of distinguishability. Such a condition is shown to be strictly weaker than those previously used for attitude observer design. Thereafter, we show that even for the single vector case the resulting condition is sufficient to design almost globally convergent attitude observers, and an explicit design is obtained. To overcome the weak excitation issue, the design makes full use of historical information. Simulation results illustrate the accurate estimation despite with noisy measurements.

Keywords: Nonlinear systems, observer design, observability, attitude estimation

1. INTRODUCTION

The attitude of a rigid body is its orientation with respect to an inertial reference frame. Attitude estimation is an essential element in a wide range of robotics and aerospace applications, in particular for control, navigation, and localization tasks. Many common sensor types, e.g. magnetometers, accelerometers, and monocular cameras, provide body-fixed measurements of quantities with a known inertial value, e.g. the earth's magnetic field and gravitational force, or the bearing to certain known landmarks. These are known as *complementary* measurements Trumpf et al. (2012). In some less common scenarios a set of known vectors in the body-fixed frame are measured in the inertial frame, e.g. measurements from two GPS receivers attached to the rigid body with a known base-line. These are known as *compatible* measurements Trumpf et al. (2012).

Estimation of attitude from multiple non-collinear vector measurements was formulated as a total least-squares problem over rotation matrices by Wahba (1965). Several efficient algorithms exist for its solution, including singular value decomposition methods, TRIAD, and QUEST Shuster and Oh (1981).

However, when estimating a time-varying attitude it often is beneficial to fuse the vector measurements with information from gyroscopes using a dynamical model. The resulting dynamic estimator commonly known as a filter or observer. These approaches can significantly reduce the impact of high-frequency measurement noise. Furthermore, in many applications there is only a single vector available for attitude estimation and in this case the attitude is not completely determined at a single moment. Applications for estimation from a single vector measurement include Sun sensors in eclipse periods Namvar and Safaei (2013), improving reliability with redundant measurements and simplifying designs Reis et al. (2021), as well as visual-inertial navigation with only two feature points visible in some periods.

Among filtering approaches, extended Kalman filter is the most widely-applied for attitude estimation. However the domain of attraction is intrinsically local since the filter is based on first-order linearization; see Crassidis et al. (2007) for a recent review. Alternatively, interest in nonlinear attitude observers was spurred by Salcudean's seminal work Salcudean (1991), and has achieved significant progress since then. There are many nonlinear attitude observers making direct use of vector measurements, e.g., with multiple measurements Mahony et al. (2008); Trumpf et al. (2012); Zlotnik and Forbes (2016) or single vector measurements Batista et al. (2012); Grip et al. (2011); Bahrami and Namvar (2017); Kinsey and Whitcomb (2007). The latter works impose a persistently non-constant condition on the single reference vector, or similar conditions in which the *uniformity* of excitation with respect to time plays an essential role to guarantee asymptotic convergence. In Trumpf et al. (2012), the authors provide a

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comprehensive treatment of observability of a rigid-body attitude kinematic model with vectorial outputs. However, as illustrated in (Trumpf et al., 2012, Remark 3.9), the condition is only sufficient but *not* necessary for distinguishability, a specific type of observability for nonlinear dynamical systems Besançon (2007); Bernard (2019). In this paper, we revisit the problem of observability analysis and propose two novel attitude observers. To be precise, the main contributions of the note are two-fold:

1) For the problem of attitude estimation from vector measurements, we provide the *necessary and sufficient* condition to distinguishability of the associated dynamical model, which is known as the necessity to reconstruct attitude over time in any deterministic estimators;
2) We show that the resulting distinguishability condition is also sufficient to design a continuous-time attitude observer. By focusing on single vector measurements, we provide a novel almost globally convergent attitude observer, which requires some significantly weaker conditions than existing methods.

The constructive tool we adopt in observer design is the parameter estimation-based observer (PEBO), which was recently proposed in Euclidean space Ortega et al. (2015, 2021), and extended to matrix Lie groups by the authors in Yi et al. (2021a,b). Its basic idea is to translate system states observation into the estimation of certain constant quantities. The interested reader may refer to Yi et al. (2018) for the geometric interpretation to PEBOs. In contrast to the case with at least two non-collinear vectors in Yi et al. (2021a,b), in this paper we consider a more challenging scenario with only a single vector measurement available under a weak excitation condition. We are unaware of any previous results capable to deal with such a case. In the proposed observer design, after translating the problem into on-line parameter identification, we propose a mechanism to integrate both the historical and current information to achieve uniform convergence.

Notation. $I_n \in \mathbb{R}^{n\times n}$ represents the identity matrix of dimension n, and $0_n \in \mathbb{R}^n$ and $0_{n\times m} \in \mathbb{R}^{n\times m}$ denote the zero column vector of dimension n and the zero matrix of dimension $n \times m$, respectively. We use $\mathbb{N}$ to represent the set of all natural integers, and $\mathbb{N}_+$ for the set of positive integers. We also define the skew-symmetric matrix

$$\mathcal{J} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Given a square matrix $A \in \mathbb{R}^{n\times n}$ and a vector $x \in \mathbb{R}^n$, the Frobenius norm is defined as $\|A\| = \sqrt{\mathrm{tr}(A^\top A)}$, and $|x|$ represents the standard Euclidean norm. The n-sphere is defined as $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, and we use $SO(3)$ to represent the special orthogonal group, and $\mathfrak{so}(3)$ is the associated Lie algebra as the set of skew-symmetric matrices satisfying $SO(3) = \{R \in \mathbb{R}^{3\times 3} | R^\top R = I_3, \ \det(R) = 1\}$. Given a variable $R \in SO(3)$, we use $|R|_I$ to represent the normalized distance to I_3 on $SO(3)$ with $|R|_I^2 := \frac{1}{4}\mathrm{tr}(I_3 - R)$. The operator skew$(\cdot)$ is defined as $\mathrm{skew}(A) := \frac{1}{2}(A - A^\top)$ for a square matrix A. Given $a \in \mathbb{R}^3$, we define the operator $(\cdot)_\times$ as

$$a_\times := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \in \mathfrak{so}(3),$$

and its inverse operator is defined as $\mathrm{vex}(a_\times) = a$.

2. PROBLEM FORMULATION

The aim of this note is to study the observability and observer design of the rotation matrix representing the coordinates of the body-fixed frame $\{B\}$ with respect to the coordinates of the inertial frame $\{I\}$, which lives in the group $SO(3)$. Its dynamics is given by

$$\dot{R} = R\omega_\times, \quad R(0) = R_0 \tag{1}$$

with the rotational velocity $\omega \in \mathbb{R}^3$ measured in the body-fixed coordinate. Assume there is a vector $g \in \mathbb{S}^2$, known in the inertial frame, being measured in the body-fixed frame, and the output is

$$y_{\mathtt{B}} = R^\top g \tag{2}$$

with $y_{\mathtt{B}} \in \mathbb{S}^2$, which is known as complementary measurement. We also consider the compatible measurement $y_{\mathtt{I}}$, i.e., a known vector $b \in \mathbb{S}^2$ in the body-fixed frame is measured in the inertial frame

$$y_{\mathtt{I}} = Rb \tag{3}$$

with $y_{\mathtt{I}} \in \mathbb{S}^2$. It is referred to (Trumpf et al., 2012, Sec. II) for more details about the names “complementary” and “compatible”.

Before closing this section, let us recall some definitions used throughout the paper.

Definition 1. (*Distinguishability*, Bernard (2019)) Consider an open set $\mathcal{X} \subset \mathbb{R}^n$ and a complete nonlinear system

$$\begin{aligned} \dot{x} &= f(x,t) \\ y &= h(x,t) \end{aligned} \tag{4}$$

with state $x \in \mathbb{R}^n$ and output $y \in \mathbb{R}^m$. The system (4) is distinguishable on $\mathcal{X}$ if for all $(x_a, x_b) \in \mathcal{X} \times \mathcal{X}$,

$$h\big(X(t;t_0,x_a),t\big) = h\big(X(t;t_0,x_b),t\big), \ \forall t \ge t_0 \implies x_a = x_b,$$

in which $X(t;t_0,x_a)$ represents the solution at time t of (4) through x_0 at time t_0. In this paper, we focus on the particular case $t_0 = 0$.

Definition 2. (*Persistent and interval excitation*, Ortega et al. (2020)) Given a bounded signal $\phi : \mathbb{R}_+ \to \mathbb{R}^n$, it is persistently excited (PE) if

$$\int_t^{t+T} \phi(s)\phi^\top(s)ds \ge \delta I_n, \ \forall t \ge 0$$

for some $T > 0, \delta > 0$; or intervally excited (IE), if there exists $T \ge 0$ such that

$$\int_0^{T} \phi(s)\phi^\top(s)ds \ge \delta I_n$$

for some $\delta > 0$.

3. NECESSARY AND SUFFICIENT CONDITIONS TO OBSERVABILITY

First, we consider the observability for the case with multiple measurements

$$\begin{aligned} y_{\mathtt{B},i} &= R^\top g_i, \qquad i \in \ell_1 := \{1,\ldots,n_1\} \\ y_{\mathtt{I},j} &= Rb_j, \qquad j \in \ell_2 := \{1,\ldots,n_2\} \end{aligned} \tag{5}$$

with $n_1, n_2 \in \mathbb{N}$.[1] It is clear that the single measurement is corresponding to the case $n_1 + n_2 = 1$, for which we will construct an asymptotically convergent observer in the next section.

In the following proposition, we uncover a necessary and sufficient condition to the distinguishability for attitude estimation.

Proposition 3. The time-varying system (1) with the output (5), and $n := n_1 + n_2 \geq 1$, is distinguishable if and only if there exist two moments $t_1, t_2 \geq 0$ such that

$$\sum_{i,l\in\ell_1,j,k\in\ell_2} \Big|g_i(t_1)\times g_l(t_2)\Big| + \Big|g_i(t_1)_\times R_0\Phi(0,t_2)b_j(t_2)\Big| + \Big|b_j(t_1)_\times \Phi(t_1,t_2)b_k(t_2)\Big| > 0, \tag{6}$$

in which $\Phi(s,t)$ is the state transition matrix of the time-varying system matrix $\omega_\times(t)$ from s to t.

Proof. The state transition matrix $\Phi(s,t)$ of the linear time-varying (LTV) system

$$\dot{x} = \omega_\times x$$

with $x \in \mathbb{R}^3$ defined as

$$\begin{aligned} \frac{\partial}{\partial t}\Phi(t,s) &= \omega_\times(t)\Phi(t,s) \\ \Phi(s,s) &= I_3. \end{aligned}$$

It is equivalent to define

$$\Phi(s,t) = Q(s)^{-1}Q(t),$$

in which $Q \in SO(3)$ is generated by the dynamics

$$\dot{Q} = Q\omega_\times, \quad Q(0) = I_3 \tag{7}$$

with $Q \in SO(3)$. From

$$\frac{d}{dt}(RQ^\top) = \dot{R}Q^\top - RQ^\top\dot{Q}Q^\top = 0,$$

we have for all $t, s \geq 0$

$$\begin{aligned} R(t)Q(t)^\top = R(0)Q(0)^\top \iff R(t) &= R_0Q(t) \\ \iff R(t) &= R(s)\Phi(s,t), \end{aligned}$$

with $R_0 := R(0)$.

Now we collect all the measured outputs in the vector

$$\bar{y} = \mathrm{col}(y_{\mathtt{B},1}, \ldots, y_{\mathtt{B},n_1}, y_{\mathtt{I},1}, \ldots, y_{\mathtt{I},n_2}).$$

With a slight abuse of notation, we denote the output signal $\bar{y}$ from the initial condition $R_0 \in SO(3)$ as $\bar{y}(t; R_0)$. In terms of Definition 1, the system is distinguishable from $t = 0$ if and only if

$$\bar{y}(t; R_a) \not\equiv \bar{y}(t; R_b) \quad \Longrightarrow \quad R_a \neq R_b \tag{8}$$

for any $R_a, R_b \in SO(3)$. Clearly, the above condition (8) is equivalent to the *identifiability* of the constant matrix $R_0 \in SO(3)$ from the nonlinear regression equation

$$\bar{y} = h(R_0, t) \tag{9}$$

with the mapping

$$h(R_0,t) := \begin{bmatrix} Q^\top(t)R_0^\top g_1(t) \\ \vdots \\ Q^\top(t)R_0^\top g_{n_1}(t) \\ R_0Q(t)b_1(t) \\ \vdots \\ R_0Q(t)b_{n_2}(t) \end{bmatrix}.$$

The regressor equation (9) can be rewritten as

$$Y(t) = R_0^\top\phi(t), \quad R_0 \in SO(3) \tag{10}$$

with $Y \in \mathbb{R}^{3\times n}$ and $\phi \in \mathbb{R}^{3\times n}$ given by

$$\begin{aligned} Y &:= Q\,[y_{\mathtt{B},1}, \ldots, y_{\mathtt{B},n_1}, b_1, \ldots, b_{n_2}] \\ \phi &:= [g_1, \ldots, g_{n_1}, y_{\mathtt{I},1}, \ldots, y_{\mathtt{I},n_2}]. \end{aligned}$$

Note that $Q(t)$ is an available signal for attitude estimation. Hence, the identifiability of the constant matrix R_0 on $SO(3)$ from the nonlinear regression model (9) is equivalent to the solvability of the Wahba problem for the regression model (10) over time Wahba (1965) – invoking that (10) holds for all $t \geq 0$. That is the existence of moments $t_1, t_2 \geq 0$ such that

$$\phi_i(t_1) \times \phi_j(t_2) \neq 0 \tag{11}$$

for some $i, j \in \{1, \ldots, n\}$, with ϕ_i representing the i-th column vector of ϕ.

The last step of the proof is to show that the condition (11) is equivalent to (6). There are totally three possible cases when (11) holds true: 1) $i, j \in \{1, \ldots, n_1\}$, 2) $i, j \in \{n_1 + 1, \ldots, n\}$, and 3) $i \in \{1, \ldots, n_1\}, j \in \{n_1 + 1, \ldots, n\}$.[2] For the first case, the condition (11) is equivalent to

$$\sum_{i,l\in\ell_1} \Big|g_i(t_1)\times g_l(t_2)\Big| > 0. \tag{12}$$

The second case is equivalent to for some $j, k \in \ell_2$

$$\begin{aligned} & y_{\mathtt{I},j}(t_1) \times y_{\mathtt{I},k}(t_2) \neq 0 \\ \iff & [R(t_1)b_j(t_1)]_\times R(t_2)b_k(t_2) \neq 0 \\ \iff & R(t_1)[b_j(t_1)]_\times R(t_1)^\top R(t_2)b_k(t_2) \neq 0 \\ \iff & [b_j(t_1)]_\times R(t_1)^\top R(t_2)b_k(t_2) \neq 0 \\ \iff & [b_j(t_1)]_\times \Phi(t_1,t_2)b_k(t_2) \neq 0 \end{aligned} \tag{13}$$

where in the second implication we use the identity $(Rb)_\times = Rb_\times R^\top$, the full-rankness of $R(t_1)$ in the third implication, and in the last

$$\begin{aligned} R(t_1)^\top R(t_2) &= Q(t_1)^\top R_0^\top R_0 Q(t_2) \\ &= \Phi(t_1,t_2). \end{aligned}$$

Note that the last line of the condition (13) can be compactly written as

$$\sum_{j,k\in\ell_2} \Big|b_j(t_1)_\times \Phi(t_1,t_2)b_k(t_2)\Big| > 0. \tag{14}$$

Similarly, we get that for the third case the condition (11) is equivalent to

$$\sum_{i\in\ell_1,\ j\in\ell_2} \Big|g_i(t_1)_\times R_0\Phi(0,t_2)b_j(t_2)\Big| > 0. \tag{15}$$

Combining these three cases, it is sufficient to obtain (6). On the other hand, since each term in (6) is non-negative, if the condition (6) holds, at least one of the above cases should be satisfied. We complete the proof. □

For the case with *only* complementary or compatible measurements ($n_1 \cdot n_2 = 0$), then the distinguishability condition becomes

$$\sum_{i,l\in\ell_1,j,k\in\ell_2} \Big|g_i(t_1)\times g_l(t_2)\Big| + \Big|b_j(t_1)_\times \Phi(t_1,t_2)b_k(t_2)\Big| > 0.$$

If there are two types of measurements, the identifiability depends on the initial attitude R_0, and this implies that some region in $SO(3)$ may be not distinguishable for a

[1] If $n_i = 0$ $(i = 1, 2)$, then the set ℓ_i is defined as the empty set $\emptyset$.

[2] We do not distinguish the order of i and j.

given specific trajectory. However, the following corollary shows that such a region has zero Lebesgue measure in the group $SO(3)$. Note that the condition below does not rely on the initial attitude.

Corollary 4. If the condition (6) is replaced by the initial attitude-independent term

$$\sum_{i,l\in\ell_1,j,k\in\ell_2} \Big|g_i(t_1)\times g_l(t_2)\Big| + \Big|g_i(t_1)_\times \Phi(0,t_2)b_j(t_2)\Big| + \Big|b_j(t_1)_\times \Phi(t_1,t_2)b_k(t_2)\Big| > 0, \tag{16}$$

the distinguishability is guaranteed almost surely.[3]

Proof. It is omitted for space limitation. □

Remark 5. In Trumpf et al. (2012) the authors propose the following *sufficient* (but not necessary, shown in (Trumpf et al., 2012, Remark 3.9)) condition to distinguishability of the given system.

$$\lambda_2\left(\sum_{i\in\ell_1}\int_0^T g_i(s)g_i^\top(s)ds\right) + \left\|\int_0^T \sum_{j\in\ell_2}\left(\omega_\times b_j(s)+\frac{d}{ds}b_j(s)\right)ds\right\| > 0, \tag{17}$$

for some $T>0$, with $\lambda_2(\cdot)$ representing the second largest eigenvalue of a square matrix. Note that in the above condition it is necessary to impose (piece-wise) smoothness of the signals b_j. In the following corollary, we show that the above condition is sufficient to the proposed necessary and sufficient condition (6).

Corollary 6. Consider the time-varying system (1) with the output (5), and $n := n_1 + n_2 \geq 1$. If (17) holds, then the condition in Proposition 3 is also verified.

Proof. It is omitted for space limitation. □

4. ATTITUDE OBSERVER FOR A SINGLE VECTOR MEASUREMENT

In this section, we show that the distinguishability condition – identified in Proposition 3 – is sufficient to design a continuous-time observer with almost globally asymptotically convergent estimate to the unknown attitude. Since the scenario with only a single vector measurement is more challenging than the multiple vector case, we focus on the former in this section. The main results can be extended to the case with multiple vector measurements in a straightforward manner.

4.1 Attitude Observer Using Integral Correction Term

Let us consider the observer design with a single complementary measurement (2). In the first observer design, we construct a dynamic extension – following the PEBO methodology Ortega et al. (2015) – in order to reformulate attitude estimation as an on-line consistent parameter identification problem. By adding an elaborated construction of "integral"-type correction term, we are able to achieve asymptotic stability of the observer.

[3] We refer to the initial attitude set which makes the system lose distinguishability having zero Lebesgue measure in the entire state space.

Proposition 7. For the system (1) with the complementary output (2), we assume that all signals are piece-wisely continuous and the reference satisfies the distinguishability condition, i.e.,

$$\exists t_1,t_2>0,\quad \big|g(t_1)\times g(t_2)\big|>0, \tag{18}$$

with a *known* bound $T>0$ on the distinguishability interval.[4] The attitude observer

$$\dot Q = Q\omega_\times \tag{19}$$

with $Q(0)\in SO(3)$ and

$$\dot{\hat Q}_c = \eta_\times \hat Q_c,\quad \hat R = \hat Q_c^\top Q \tag{20}$$

with

$$\begin{aligned}\eta &= \gamma_{\mathtt P}(\hat Q_c g)\times(Qy_{\mathtt B}) + \gamma_{\mathtt I}\xi\\ \xi &= 2\mathrm{vex}\big(\mathrm{skew}(A\hat Q_c^\top)\big)\\ \dot A &= \begin{cases} Qy_{\mathtt B}g^\top, & t\in[0,T)\\ 0_{3\times3}, & t\geq T\end{cases}\end{aligned}$$

with the gains $\gamma_{\mathtt P},\gamma_{\mathtt I}>0$ and $A(0)=0_{3\times3}$, guarantees $\hat R(t)\in SO(3)$ for all $t\geq0$ and the convergence

$$\lim_{t\to\infty}\|\hat R(t)-R(t)\|=0 \tag{21}$$

almost globally.

Proof. Due to the space limitation, we only provide the sketch of the proof.

By defining a variable

$$E(R,Q)=QR^\top,$$

we have

$$\dot E = \dot Q R^\top - QR^\top \dot R R^\top = 0.$$

Therefore, there exists a constant matrix $Q_c\in SO(3)$ such that

$$Q(t)R^\top(t)=Q_c,\quad \forall t\in[0,+\infty). \tag{22}$$

Note that $Q(t)$ is an available signal by construction, and Q_c is unknown. Invoking (22) and the full-rankness of Q, the estimation of R is equivalent to the one of Q_c.

Based on the above idea, we construct the following auxiliary system

$$\Sigma_c:\begin{cases}\dot Q_c = Q_c(\omega_c)_\times\\ y_c = Q_c b_c,\end{cases} \tag{23}$$

in which $Q_c\in SO(3)$ is constant thus $\omega_c=0_3$, the output

$$y_c(t) \;:=\; Q(t)y_{\mathtt B}(t),$$

and the "body-fixed coordinate" reference

$$b_c := g.$$

It is clear that the system Σ_c is exactly in the same form as the kinematic model with a compatible measurement (1) and (3). Consider the candidate Lyapunov function $V(\tilde Q_c)=3-\mathrm{tr}(\tilde Q_c)$, which has its minimal value zero at $\tilde Q_c = I_3$. It yields

$$\begin{aligned}\dot V &= -\,\mathrm{tr}(\eta_\times\tilde Q_c)\\ &\leq -\,\lambda_{\mathtt{min}}(\Gamma)\|\mathrm{skew}(\tilde Q_c)\|^2\end{aligned}$$

after some complicated but straightforward calculations, in which we have defined the variable Γ as

$$\Gamma = \Gamma_{\mathtt P} + \Gamma_{\mathtt I}$$

with

$$\Gamma_{\mathtt P}(t) := \gamma_{\mathtt P}(I-y_c(t)y_c^\top(t))$$

[4] Namely, there exists a known $T>0$ such that $0\leq t_1<t_2\leq T$.

and

$$\Gamma_{\mathtt{I}}(t) \;:=\; \begin{cases} \gamma_{\mathtt{I}} \displaystyle\int_0^t \Big(I - y_c(s) y_c^\top(s)\Big) ds, & t \in [0,T] \\ \Gamma_{\mathtt{I}}(T), & t > T. \end{cases}$$

From the condition (18), it is sufficient to show the positive definiteness of Γ, yielding the local exponential stability of $\tilde{Q}_c = I_3$. It is indeed almost globally asymptotically stable with more technically involved analysis, which is omitted here. Invoking the algebraic relation $R = Q_c^\top Q$, we may conclude the results. □

Remark 8. In the above attitude observer design, the error term η contains two parts

$$\eta \;=\; \underbrace{\gamma_{\mathtt{P}}(\hat{Q}_c g) \times (Qy)}_{\text{current}} \;+\; \underbrace{\gamma_{\mathtt{I}}\xi}_{\text{historical}} \;,$$

which may be viewed as an observer design using a “proportional + integral”-type error term. The first term only utilizes the current information, making it behave as an on-line design. The second “integral” term enables to achieve asymptotic convergence of the estimation error under the extremely weak identifiability condition (18). The gain parameters $\gamma_{\mathtt{P}}, \gamma_{\mathtt{I}}$ can be used as the weights on how we trust the current and historical data.

Remark 9. The bound $T > 0$ is used in the dynamics of the variable A in order to be able to guarantee its boundedness. Indeed, the bound T is not necessarily known apriori, since the distinguishability condition (18) is an easily-checkable condition on measured quantities. The proposed scheme may be modified as an *adaptive* design in which such a condition is checked online continuously, and the dynamics of A simply changes until the condition holds. It is also natural to replace the condition (18) by $|g(t_1) \times g(t_2)| > \delta$ for some $\delta > 0$, to deal with sensor noise.

Remark 10. As shown above, the “integral” term only accumulates information in the interval $[0, T]$, which, however, does not have the sort of “fading memory” property on past measurements. As long as the excitation condition holds, which is easily monitored on-line, the observer performance can be improved considering the moving interval $[t - T, t]$ rather than $[0, T]$ in Proposition 7. Another possible route to deal with weak excitation is to extend the approach in Wang et al. (2021) to manifolds.

Remark 11. For the case with a single compatible measurement (3), we may still get the auxiliary model (23) by designing the dynamic extension $\dot{Q} = Q\omega_\times$, but with the new definitions of $y_c := y_{\mathtt{I}}$ and $g_c := Qb$. Then, the above two designs are capable to solve the problem with slight modifications accordingly.

5. SIMULATIONS

Some simulations have been carried out with realistic considerations to evaluate the performance of the proposed observers. We consider a single time-varying inertial vector

$$g(t) = \begin{cases} e_1, & t \in [0, 5)\text{s} \\ e_3, & t \geq 5\text{s}, \end{cases} \tag{24}$$

in which e_i represents the i-th standard Euclidean basis in $\mathbb{R}^3$. Clearly, it satisfies the sufficient excitation condition (18), but not for the persistently non-constant reference vector assumption in many works Batista et al. (2012). The attitude of the rigid-body starts from the initial condition $R(0) = \text{diag}(-1, -1, 1)$ under the rotational velocity $\omega = [0.23 \;\; -0.5 \;\; 0.15]^\top$. We take the measurement noise into consideration for both the angular velocity readings and the vector measurements.

We evaluate the performance of the scheme in Proposition 7. The observer is initialized from $Q(0) = I_3, \hat{Q}_c(0) = I_3$, with the selection of gains $\gamma_{\mathtt{P}} = 3$ and $\gamma_{\mathtt{I}} = 1$. It corresponds to the initial yaw, pitch and roll estimates all being 0°. The results of simulations are shown in Fig. 1 in the form of Euler angles, and also see the norm of the estimation error $|\tilde{R}|_I$ in Fig. 2, which is drawn in a logarithmic scale for the y-axis. During $[0, 5]$ s, the error $\tilde{R}$ is converging to some non-zero constant under a constant vector measurement. This is because a single vector output makes two of three Euler angles partially observable Martin and Sarras (2018). After 5s the model satisfies the distinguishability, and then all Euler angles converge to their true values. Note that the proposed scheme is robust *vis-à-vis* measurement noise.

At the end, let us compare the proposed schemes to the complementary attitude observer in Trumpf et al. (2012), whose convergence is guaranteed by a persistent excitation condition. Clearly, this is not satisfied by the inertial reference vector in (24). We consider the same initial guess $\hat{R}(0) = I_3$, and the observer gain is selected as 1. We show the simulation results for both the noisy and noise-free cases in Fig. 3, and as expected, the estimate $\hat{R}$ fails to converge to its true values.

6. CONCLUDING REMARKS

In this paper, we studied the observability and observer design for the attitude estimation problem with vectorial measurements. By translating the observation problem into one of on-line parameter identification, we provided the *necessary and sufficient* condition to the distinguishability for the dynamical model on $SO(3)$, which is complementary to the existing necessary conditions in the literature. As shown later, though the resulting distinguishability condition is quite weak, we are still able to use it to derive a continuous-time attitude observer with almost global asymptotic stability guaranteed for the single vector case. Finally, simulation results were presented with satisfactory performance in the presence of measurement noise.

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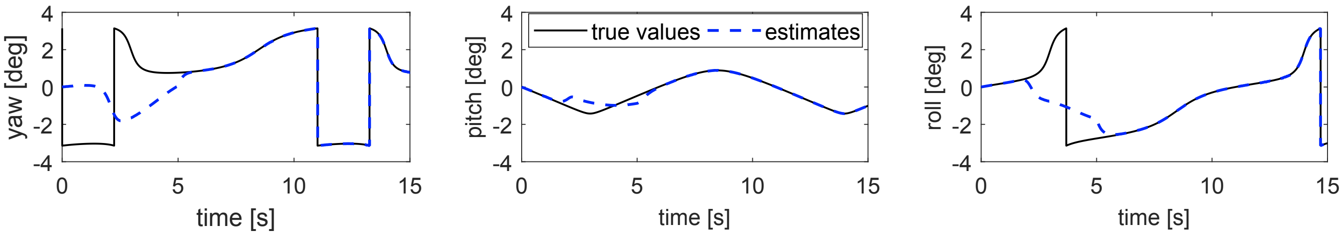

Fig. 1. Performance of the attitude observer in Proposition 7 with Euler angles

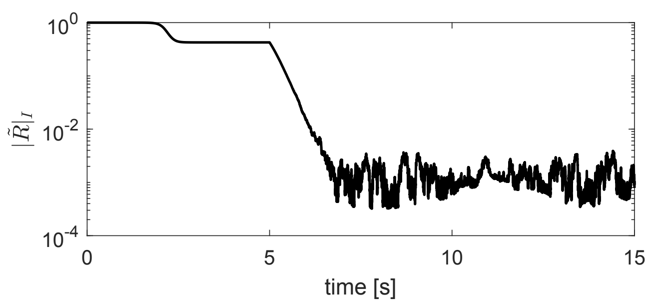

Fig. 2. Norm of estimation error $|\tilde{R}|_I$

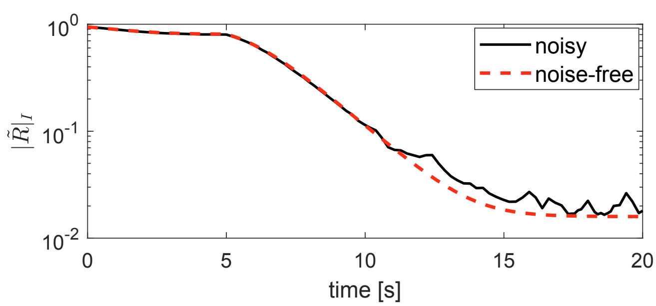

Fig. 3. Performance of the complementary attitude observer in Trumpf et al. (2012)

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