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# Turán and Ramsey Problems for Alternating Multilinear Maps

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**Abstract:** Guided by the connections between hypergraphs and exterior algebras, we study Turán and Ramsey type problems for alternating multilinear maps. This study lies at the intersection of combinatorics, group theory, and algebraic geometry, and has origins in the works of Lovász (*Proc. Sixth British Combinatorial Conf.*, 1977), Buhler, Gupta, and Harris (*J. Algebra*, 1987), and Feldman and Propp (*Adv. Math.*, 1992).

Our main result is a Ramsey theorem for alternating bilinear maps. Given $s,t \in \mathbb{N}$, $s,t \geq 2$, and an alternating bilinear map $\phi : V \times V \to U$ with $\dim(V) \geq s \cdot t^4$, we show that there exists either a dimension-$s$ subspace $W \leq V$ such that $\dim(\mathrm{span}(\phi(W,W))) = 0$, or a dimension-$t$ subspace $W \leq V$ such that $\dim(\mathrm{span}(\phi(W,W))) = \binom{t}{2}$. This result has natural group-theoretic (for finite $p$-groups) and geometric (for Grassmannians) implications, and leads to new Ramsey-type questions for varieties of groups and Grassmannians.

**Key words and phrases:** Ramsey and Turán problems, extremal combinatorics, alternating multilinear maps, exterior algebras, $p$-groups, Grassmannians

## 1 Introduction

The main result of this paper is a Ramsey theorem for alternating bilinear maps, or equivalently, for linear spaces of alternating bilinear forms.

To state the result we need some definitions. Let $\mathbb{F}$ be a field. Let $U$ be an $n$-dimensional vector space over $\mathbb{F}$. We use $W \leq U$ to denote that $W$ is a subspace of $U$. Recall that a bilinear form $f : U \times U \to \mathbb{F}$ is *alternating*, if for any $u \in U$, $f(u,u) = 0$. For $W \leq U$, the *restriction* of $f$ to $W$ is denoted as $f|_W : W \times W \to \mathbb{F}$. Let $\Lambda(U)$ be the linear space of alternating bilinear forms on $U$. For $\mathcal{A} \leq \Lambda(U)$ and $W \leq U$, $\mathcal{A}|_W := \{f|_W \mid f \in \mathcal{A}\} \leq \Lambda(W)$.

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**Theorem 1.1.** *Let $\mathbb{F}$ be a field, $s,t \in \mathbb{N}$, $s,t \geq 2$, and $n \geq s \cdot t^4$. Let $U$ be an $n$-dimensional vector space over $\mathbb{F}$. Then for any $\mathcal{A} \leq \Lambda(U)$, either there exists $V \leq U$, $\dim(V) = s$, such that $\mathcal{A}|_V$ is the zero space, or there exists $W \leq U$, $\dim(W) = t$, such that $\mathcal{A}|_W = \Lambda(W)$.*

We will prove Theorem 1.1 in Section 3, after presenting some preliminaries in Section 2. Theorem 1.1 has natural interpretations in group theory and geometry, and it can be naturally understood as a linear Ramsey theorem in the context of Turán and Ramsey problems for alternating multilinear maps. These will be explained in Section 4 and 5.

A closely related result is Weaver's "quantum" Ramsey theorem [Wea17]. That result concerns the so-called operator systems from quantum information, which are actually linear spaces of matrices over $\mathbb{C}$ satisfying certain conditions. Cliques and anticliques can be defined for such matrix spaces, which are in close analogy with the totally-isotropic spaces and the complete spaces studied here. In [Wea17], it was shown that when $n \geq 8s^{11}$, any operator system has either an $s$-clique or an $s$-anticlique.

The initial strategy for our proof (in particular Steps 1 and 2, see Section 3) follows closely some ideas in Weaver's proof in [Wea17] and Sims' work on enumerating $p$-groups [Sim65, Sec. 2]. However, new ideas are indeed required, as in the alternating matrix space setting, there are no diagonal matrices which are crucial for Weaver's proof in the operator system setting. Furthermore, our result works over any field.

## 2 Preliminaries

### 2.1 Some notation

For $n \in \mathbb{N}$, $[n] := \{1, \dots, n\}$. The set of size-$\ell$ subsets of $[n]$ is denoted as $\binom{[n]}{\ell}$.

### 2.2 Vector spaces

For a field $\mathbb{F}$, $\mathbb{F}^n$ is the linear space consisting of length-$n$ column vectors over $\mathbb{F}$. We use $e_i$ to denote the $i$th standard basis vector in $\mathbb{F}^n$. The dual space of $\mathbb{F}^n$ is denoted as $(\mathbb{F}^n)^*$, and the dual vector of $v \in \mathbb{F}^n$ is denoted as $v^*$. For $S \subseteq \mathbb{F}^n$, $\mathrm{span}(S)$ denotes the subspace spanned by vectors in $S$. For $v \in \mathbb{F}^n$ and $i \in [n]$, $v(i)$ denotes the $i$th component of $v$. For $S \subseteq \mathbb{F}^n$, $S^\perp := \{v \in \mathbb{F}^n \mid \forall u \in S, v^t u = 0\}$.

### 2.3 Matrices

We use $\mathrm{M}(\ell \times n, \mathbb{F})$ to denote the linear space of $\ell \times n$ matrices over $\mathbb{F}$, and set $\mathrm{M}(n,\mathbb{F}) := \mathrm{M}(n \times n, \mathbb{F})$. For $A \in \mathrm{M}(\ell \times n, \mathbb{F})$, $A(i,j)$ is the $(i,j)$th entry of $A$. We shall use $\mathbf{0}$ to denote all-zero vectors or matrices of appropriate sizes. A matrix $A \in \mathrm{M}(n,\mathbb{F})$ is *alternating*, if for any $v \in \mathbb{F}^n$, $v^t A v = 0$. For $(i,j) \in [n] \times [n]$, $E_{i,j} \in \mathrm{M}(n,\mathbb{F})$ is the $n \times n$ matrix with the $(i,j)$th entry being 1, and the rest entries being 0. For $\{i,j\} \in \binom{[n]}{2}$, $i < j$, $A_{i,j} \in \mathrm{M}(n,\mathbb{F})$ is the $n \times n$ alternating matrix with the $(i,j)$th entry being 1, the $(j,i)$th entry being $-1$, and the rest entries being 0. The general linear group of degree $n$ over $\mathbb{F}$ is denoted by $\mathrm{GL}(n,\mathbb{F})$.

### 2.4 Alternating matrix spaces

Let $\Lambda(n,\mathbb{F})$ be the linear space of $n\times n$ alternating matrices over $\mathbb{F}$. Subspaces of $\Lambda(n,\mathbb{F})$ are called alternating matrix spaces.

Two alternating matrix spaces $\mathcal{A},\mathcal{B}\leq\Lambda(n,\mathbb{F})$ are *isometric*, if there exists $T\in\mathrm{GL}(n,\mathbb{F})$, such that $\mathcal{A}=T^{\mathrm{t}}\mathcal{B}T:=\{T^{\mathrm{t}}BT\mid B\in\mathcal{B}\}$.

Let $\mathcal{A}\leq\Lambda(n,\mathbb{F})$. Suppose $U\leq\mathbb{F}^n$ is of dimension $d$, and let $T\in\mathrm{M}(n\times d,\mathbb{F})$ be a matrix whose column vectors span $U$. The restriction of $\mathcal{A}$ to $U$ via $T$ is $\mathcal{A}|_{U,T}:=\{T^{\mathrm{t}}AT\mid A\in\mathcal{A}\}\leq\Lambda(d,\mathbb{F})$. Given another $T'\in\mathrm{M}(n\times d,\mathbb{F})$ whose columns also span $U$, $\mathcal{A}|_{U,T}$ and $\mathcal{A}|_{U,T'}$ are isometric. Therefore, we may write $\mathcal{A}|_U$ as the restriction of $\mathcal{A}$ to $U$ via some $T\in\mathrm{M}(n\times d,\mathbb{F})$ whose columns span $U$.

We say that $U$ is a *totally-isotropic space*[1] for $\mathcal{A}$, if $\dim(\mathcal{A}|_U)=0$. That is, for any $u,u'\in U$ and $A\in\mathcal{A}$, we have $u^{\mathrm{t}}Au'=0$. We say that $U$ is a *complete space* for $\mathcal{A}$, if $\dim(\mathcal{A}|_U)=\binom{d}{2}$.

Given $v\in\mathbb{F}^n$, the *degree* of $v$ in $\mathcal{A}$, $\deg_{\mathcal{A}}(v)$, is the dimension of $\{Av\mid A\in\mathcal{A}\}\leq\mathbb{F}^n$. When $\mathcal{A}$ is clear from the context, we may simply write $\deg(v)$ instead of $\deg_{\mathcal{A}}(v)$. The *minimum degree* of $\mathcal{A}$, denoted as $\delta(\mathcal{A})$, is the minimum over the degrees over non-zero $v$ in $\mathcal{A}$.

Given $S\subseteq\mathbb{F}^n$, the *radical space* of $S$ in $\mathcal{A}$ is $\mathrm{rad}_{\mathcal{A}}(S):=\{u\in\mathbb{F}^n\mid\forall A\in\mathcal{A},v\in S,u^{\mathrm{t}}Av=0\}$. We may simply write $\mathrm{rad}_{\mathcal{A}}(\{v\})$ as $\mathrm{rad}_{\mathcal{A}}(v)$. When $\mathcal{A}$ is clear from the context, we may simply write $\mathrm{rad}(S)$ instead of $\mathrm{rad}_{\mathcal{A}}(S)$. We also define the *radical space* of $\mathcal{A}\leq\Lambda(n,\mathbb{F})$ as $\mathrm{rad}(\mathcal{A}):=\mathrm{rad}_{\mathcal{A}}(\mathbb{F}^n)=\{v\in\mathbb{F}^n\mid\forall A\in\mathcal{A},Av=0\}$.

### 2.5 Alternating multilinear maps

An $\ell$-linear map $\phi:(\mathbb{F}^n)^{\times\ell}\to\mathbb{F}^m$ is *alternating*, if for any $(v_1,\dots,v_\ell)\in(\mathbb{F}^n)^{\times\ell}$ where $v_i=v_j$ for some $i\neq j$, $\phi(v_1,\dots,v_\ell)=0$. Here, $(\mathbb{F}^n)^{\times\ell}$ denotes the $\ell$-fold Cartesian product of $\mathbb{F}^n$. Two alternating $\ell$-linear maps $\phi,\psi:(\mathbb{F}^n)^{\times\ell}\to\mathbb{F}^m$ are *isomorphic*, if there exists $(S,T)\in\mathrm{GL}(n,\mathbb{F})\times\mathrm{GL}(m,\mathbb{F})$, such that for any $v_1,\dots,v_\ell\in\mathbb{F}^n$, $\phi(S(v_1),\dots,S(v_\ell))=T(\phi(v_1,\dots,v_\ell))$.

### 2.6 3-way arrays

Matrices are 2-way arrays, i.e. an array with two indices. We shall also need the notion of *3-way arrays*, namely arrays with three indices. We use $\mathrm{M}(\ell\times m\times n,\mathbb{F})$ to denote the linear space of 3-way arrays with the index set being $[\ell]\times[m]\times[n]$. Let $\mathtt{A}\in\mathrm{M}(\ell\times m\times n,\mathbb{F})$ be a 3-way array. Following [KB09, GQ21], we define the following. The *frontal slices* of $\mathtt{A}$ are $A_1,\dots,A_n\in\mathrm{M}(\ell\times m)$, where $A_k(i,j)=\mathtt{A}(i,j,k)$. The *tube fibres* of $\mathtt{A}$ are $v_{i,j}\in\mathbb{F}^n$, $i\in[\ell]$, $j\in[m]$, where $v_{i,j}(k)=\mathtt{A}(i,j,k)$.

3-way arrays are also referred to as 3-tensors in some literature. We adopt 3-way arrays, because 3-tensors are usually considered to be 3-way arrays together with the natural action of $\mathrm{GL}(\ell,\mathbb{F})\times\mathrm{GL}(m,\mathbb{F})\times\mathrm{GL}(n,\mathbb{F})$. In this paper, as the reader will see soon, 3-way arrays are used to record the structure constant of alternating bilinear maps. Therefore, the group action of relevance in this context is by $\mathrm{GL}(n,\mathbb{F})\times\mathrm{GL}(m,\mathbb{F})$ where $\mathrm{GL}(n,\mathbb{F})$ acts covariantly on the first two indices.

[1] Totally-isotropic spaces are also called totally singular [Atk73], isotropic [BGH87], and $\ell$-singular [DS10] (for alternating $\ell$-linear maps) in the literature. We adopt the terminologies of totally-isotropic and anisotropic (see Section 5.8), which are from the study of bilinear forms [MH73], and have been used in e.g. [BMW17].

### 2.7 Relations between 3-way arrays, bilinear maps, and matrix spaces

It is not hard to see that alternating bilinear maps, alternating matrix spaces, and 3-way arrays are closely related. We spell out some details here.

An alternating bilinear map $\phi : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^m$ can be represented as a tuple of alternating matrices $\mathbf{A} = (A_1, \ldots, A_m) \in \Lambda(n, \mathbb{F})^m$, such that for any $u, v \in \mathbb{F}^n$, $\phi(u, v) = (u^t A_1 v, \ldots, u^t A_m v)^t$.

From an alternating matrix tuple $\mathbf{A} \in \Lambda(n, \mathbb{F})^m$, we can construct a 3-way array $\mathtt{A} \in \mathrm{M}(n \times n \times m, \mathbb{F})$ whose frontal slices are $A_i$'s. We can also construct an alternating matrix space $\mathcal{A} = \mathrm{span}\{A_1, \ldots, A_m\} \leq \Lambda(n, \mathbb{F})$.

Let $\mathcal{A} \leq \Lambda(n, \mathbb{F})$ be an $m$-dimensional alternating matrix space. Let $\mathbf{A} = (A_1, \ldots, A_m) \in \Lambda(n, \mathbb{F})$ be an ordered linear basis of $\mathcal{A}$. Then $\mathbf{A}$ gives rise to an alternating bilinear map $\phi$ and a 3-way array $\mathtt{A}$ as above. Note that different ordered bases of $\mathcal{A}$ yield different but isomorphic alternating bilinear maps.

## 3 Proof of Theorem 1.1

### 3.1 Restatement of Theorem 1.1

By fixing a basis for $U$ and identifying alternating bilinear forms with alternating matrices, Theorem 1.1 can be restated as follows.

**Theorem 1.1, restated.** Let $\mathbb{F}$ be a field, $s, t \in \mathbb{N}$, $s, t \geq 2$, and $n \geq s \cdot t^4$. For any alternating matrix space $\mathcal{A} \leq \Lambda(n, \mathbb{F})$, either there exists an $s$-dimensional totally-isotropic space of $\mathcal{A}$, or there exists a $t$-dimensional complete space of $\mathcal{A}$.

### 3.2 Proof outline

To start with, by restricting to any subspace of dimension $s \cdot t^4$, we can assume that $\mathcal{A} \leq \Lambda(n, \mathbb{F})$ where $n = s \cdot t^4$.

The proof consists of four steps. Before going into the details, let us first outline the objective for each step.

- *Step 1* This step either constructs a $s$-dimensional totally-isotropic space of $\mathcal{A}$, or computes a $n'$-dimensional $P \leq \mathbb{F}^n$, such that the minimum degree $\delta(\mathcal{A}|_P) \geq t^4$.

  Let $\mathcal{B} = \mathcal{A}|_P \leq \Lambda(n', \mathbb{F})$. The goal of the next three steps is to construct a dimension-$(t+1)$ complete space for $\mathcal{B}$.

- *Step 2* Let $t' = t^2$. This step constructs a $Q_2 \in \mathrm{M}(n' \times (t'+1), \mathbb{F})$, of rank-$(t'+1)$, such that $\mathcal{C} = Q_2^t \mathcal{B} Q_2 \leq \Lambda(t'+1, \mathbb{F})$ (the restriction of $\mathcal{B}$ to the subspace of $\mathbb{F}^{n'}$ spanned by the columns of $Q_2$) contains matrices $C_1, \ldots, C_{t'} \in \mathcal{C}$, where $C_i = \begin{bmatrix} \tilde{C}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $\tilde{C}_i$ has size $(i+1) \times (i+1)$, and $C_i(i, i+1) = 1$.

- *Step 3* Let $r = t + \binom{t}{2}$, and recall that $t' = t^2$. This step constructs $Q_3 \in \mathrm{GL}(t'+1,\mathbb{F})$, such that $\mathcal{D} = Q_3^{\mathrm{t}} \mathcal{C} Q_3 \leq \Lambda(t'+1,\mathbb{F})$ contains $D_1,\ldots,D_r \in \Lambda(t'+1,\mathbb{F})$ satisfying (1) for $i \in [t]$, $D_i = \begin{bmatrix} \tilde{D}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ where $\tilde{D}_i \in \Lambda(i+1,\mathbb{F})$ and $D_i(i,i+1) = 1$, and (2) for $i \in [\binom{t}{2}]$, $D_{t+i} = \begin{bmatrix} \tilde{D}_{t+i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 1 & \mathbf{0} \\ \mathbf{0} & -1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$, where $\tilde{D}_{t+i}$ is of size $t+1+2(i-1)$.
- *Step 4* Based on $D_1,\ldots,D_r$ from Step 3, this step constructs $Q_4 \in \mathrm{GL}(t'+1,\mathbb{F})$, such that $W = \mathrm{span}\{e_1,\ldots,e_{t+1}\}$ is a complete space for $Q_4^{\mathrm{t}} \mathcal{D} Q_4$. It is clear that the complete space $W$ of $Q_4^{\mathrm{t}} \mathcal{D} Q_4$ translates to a complete space for $\mathcal{A}|_P$ through $Q_2$, $Q_3$, and $Q_4$, giving us the desired complete space for $\mathcal{A}$.

Each step relies on a lemma which could be of independent interest. In the following, we explain these steps in detail.

### 3.3 Step 1

The first step relies on the following lemma. Recall that the minimum degree $\delta(\cdot)$ is defined in Section 2.4.

**Lemma 3.1.** *Let $s,d \in \mathbb{N}$, $s,d \geq 2$, and $n = s \cdot d$. For any $\mathcal{A} \leq \Lambda(n,\mathbb{F})$, either there exists a totally-isotropic space of dimension $s$ of $\mathcal{A}$, or there exists $P \leq \mathbb{F}^n$, such that $\dim(P) \geq 2d$, and $\delta(\mathcal{A}|_P) \geq d$.*

*Proof.* Consider the following procedure in at most $s$ rounds. The basic idea is that in each round, if there exists a non-zero vector $v$ of degree $< d$, we restrict the current alternating matrix space to $\mathrm{rad}(v)$.

More specifically, in the first round, if there are no non-zero $v_1 \in \mathbb{F}^n$ such that $\deg_{\mathcal{A}}(v_1) < d$, then $\mathbb{F}^n$ satisfies what we need for $P$. Otherwise, there exists a non-zero $v_1 \in \mathbb{F}^n$ such that $\deg_{\mathcal{A}}(v_1) < d$. Let $S_1 = \mathrm{span}\{v_1\}$, $T_1 = \mathrm{rad}_{\mathcal{A}}(S_1)$, and $\mathcal{A}_1 = \mathcal{A}|_{T_1}$. By the alternating property, $v_1 \in T_1$, so $S_1 \leq T_1$. Make $R_1$ a complement subspace of $S_1$ in $T_1$. By $\deg_{\mathcal{A}}(v_1) < d$, we have $\dim(T_1) \geq (s-1)d+1$ and $\dim(R_1) \geq (s-1)d$. Also note that $S_1 \leq \mathrm{rad}(\mathcal{A}_1)$. We then continue to the next round.

Then before the $i$th round, $i = 2,\ldots,s-1$, we have obtained from the $(i-1)$th round the following data.

- $S_{i-1} = \mathrm{span}\{v_1,\ldots,v_{i-1}\} \leq \mathbb{F}^n$, $\dim(S_{i-1}) = i-1$, and $S_{i-1}$ is a totally-isotropic space of $\mathcal{A}$.
- $T_{i-1} \leq \mathbb{F}^n$, $\dim(T_{i-1}) \geq (s-(i-1))d + (i-1)$, and $S_{i-1} \leq T_{i-1}$.
- $R_{i-1}$, a complement subspace of $S_{i-1}$ in $T_{i-1}$. Note that $\dim(R_{i-1}) \geq (s-(i-1))d \geq (s-(s-1-1))d = 2d$.
- $\mathcal{A}_{i-1} = \mathcal{A}|_{T_{i-1}}$, and $S_{i-1} \leq \mathrm{rad}(\mathcal{A}_{i-1})$.

We then try to find a non-zero $v_i \in R_{i-1}$ such that $\deg_{\mathcal{A}_{i-1}}(v_i) < d$. If no such $v_i$ exist, then $R_{i-1}$ satisfies what we need for $P$. Otherwise, let $S_i = \mathrm{span}\{v_1,\ldots,v_i\}$, $T_i = \mathrm{rad}_{\mathcal{A}_{i-1}}(S_i) = \mathrm{rad}_{\mathcal{A}_{i-1}}(v_i)$, and $\mathcal{A}_i = \mathcal{A}|_{T_i}$. Set $R_i$ to be a complement subspace of $S_i$ in $T_i$. As $\deg_{\mathcal{A}_{i-1}}(v_i) < d$, $\dim(T_i) \geq (s-i)d + i$, and $\dim(R_i) \geq (s-i)d$. Clearly $S_i \leq \mathrm{rad}(\mathcal{A}_i)$, so in particular, $S_i$ is a totally-isotropic space for $\mathcal{A}$. We then continue to the $(i+1)$th round.

Now suppose we just enter the $s$th round. At this point, we have $\dim(T_{s-1}) \geq d+(s-1)$, and $\dim(R_{s-1}) \geq d$. Take any non-zero $v_s \in R_{s-1}$, and set $S_s = \mathrm{span}\{v_1,\dots,v_s\}$. Since $S_{s-1} \leq \mathrm{rad}(\mathcal{A}_{s-1})$, $S_s$ is a totally-isotropic space of dimension $s$. □

Back to our original setting, recall that $n = s \cdot t^4$. Let $d = t^4$, so $n = s \cdot d$. Applying Lemma 3.1 gives us either a totally-isotropic space of dimension $s$, or a subspace $P$ such that $n' = \dim(P) \geq 2d$ and $\delta(\mathcal{A}|_P) \geq d$. In the former case, we can conclude the proof of Theorem 1.1. In the latter case, we will construct a dimension-$(t+1)$ totally-isotropic space for $\mathcal{B}$ in the next three steps.

### 3.4 Step 2

The second step relies on the following lemma.

**Lemma 3.2.** *Let $d, t' \in \mathbb{N}$, $d = t'^2$. Suppose $\mathcal{B} \leq \Lambda(n', \mathbb{F})$ satisfies that $n' \geq 2d$ and $\delta(\mathcal{B}) \geq d$. Then there exists $Q \in \mathrm{M}(n' \times (t'+1), \mathbb{F})$, of rank-$(t'+1)$, such that $\mathcal{C} = Q^{\mathrm{t}}\mathcal{B}Q \leq \Lambda(t'+1,\mathbb{F})$ contains matrices $C_1,\dots,C_{t'} \in \mathcal{C}$, where $C_i = \begin{bmatrix} \tilde{C}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $\tilde{C}_i$ has size $(i+1)\times(i+1)$, and $C_i(i,i+1) = 1$.*

*Proof.* Consider the following procedure in at most $t'$ rounds.

In the first round, take any non-zero $w_1 \in \mathbb{F}^{n'}$, and find $B_1 \in \mathcal{B}$, such that $B_1 w_1 \neq \mathbf{0}$. Then there exists $w_2 \in \mathbb{F}^{n'}$ such that $w_2^{\mathrm{t}} B_1 w_1 \neq 0$. Such $B_1$ exists, as $\dim(\mathrm{rad}_{\mathcal{B}}(w_1)) \leq n'-d$. Set $T_2 = (\mathrm{span}\{B_1w_1, B_1w_2\})^{\perp}$, and $W_2 = \mathrm{span}\{w_1,w_2\}$. By $w_2^{\mathrm{t}}B_1w_1 \neq 0$, $T_2 \cap W_2 = 0$. By the alternating property, $\dim(W_2) = 2$.

Then before the $i$th round, $i = 2,\dots,t'$, we have obtained $w_1,\dots,w_i \in \mathbb{F}^{n'}$, $B_1,\dots,B_{i-1} \in \mathcal{B}$, such that $\forall j \in [i-1]$, (1) $w_j^{\mathrm{t}} B_j w_{j+1} \neq 0$, and (2) $\forall 1 \leq k \leq i$, $\forall j+1 < \ell \leq i$, $w_k^{\mathrm{t}} B_j w_\ell = 0$ . We also have $T_i = (\mathrm{span}\{B_j w_k \mid j \in [i-1], k \in [i]\})^{\perp}$, and $W_i = \mathrm{span}\{w_1,\dots,w_i\}$, such that $T_i \cap W_i = 0$.

We claim that there exist $w_{i+1} \in T_i$ and $B_i \in \mathcal{B}$, such that $w_{i+1}^{\mathrm{t}} B_i w_i \neq 0$. To see this, note that $\dim(T_i) \geq n' - (i-1)\cdot i$ and $\dim(\mathrm{rad}_{\mathcal{B}}(w_i)) \leq n'-d$. So as long as $d > (i-1)\cdot i$, we can take any $w_{i+1} \in T_i \setminus \mathrm{rad}_{\mathcal{B}}(w_i)$, for which there exists $B_i \in \mathcal{B}$ such that $w_{i+1}^{\mathrm{t}} B_i w_i \neq 0$.

Let $T_{i+1} = (\mathrm{span}\{B_j w_k \mid j \in [i], k \in [i+1]\})^{\perp}$, and $W_{i+1} = \mathrm{span}\{w_1,\dots,w_{i+1}\}$. We claim that $T_{i+1} \cap W_{i+1} = 0$. If not, suppose $w = \alpha_1 w_1 + \cdots + \alpha_i w_i + \alpha_{i+1} w_{i+1} \in T_{i+1}$. Let $j$ be the smallest integer such that $\alpha_j \neq 0$. If $j \leq i$, then $w^{\mathrm{t}} B_j w_{j+1}$ is non-zero. If $j = i+1$, then $w^{\mathrm{t}} B_i w_i$ is non-zero. In either case, this is a contradiction to the assumption that $w \in T_{i+1}$. We also note that, by examining $w_{i+1}^{\mathrm{t}} B_i w_i$, we have $B_i \notin \mathrm{span}\{B_1,\dots,B_{i-1}\}$.

We perform the above operations, and after the $t'$-th round we get the desired $w_1,\dots,w_{t'+1}$ and $B_1,\dots,B_{t'} \in \Lambda(n',\mathbb{F})$. This requires $d > (t'-1)t'$, which is fine as we have set $d = t'^2$.

Let $Q = \begin{bmatrix} w_1 & \dots & w_{t'+1} \end{bmatrix} \in \mathrm{M}(n' \times (t'+1), \mathbb{F})$, and let $\mathcal{C} = Q^{\mathrm{t}}\mathcal{B}Q \leq \Lambda(t'+1,\mathbb{F})$. We claim that $\mathcal{C}$ satisfies the requirements of this lemma. Recall that for any $i \in [t']$, $B_i$ satisfies that $w_i^{\mathrm{t}} B_i w_{i+1} \neq 0$. Furthermore, for any $i+1 < \ell \leq t+1$ and $1 \leq k \leq t+1$, $w_k^{\mathrm{t}} B_i w_\ell = 0$. Set $w_i^{\mathrm{t}} B_i w_{i+1} = \alpha_i$. Let $C_i = Q^{\mathrm{t}}(\frac{1}{\alpha_i} B_i) Q \in \mathcal{C}$. Then $C_i(i,i+1) = \frac{1}{\alpha_i} w_i^{\mathrm{t}} B_i w_{i+1} = 1$, and for any $i+1 < \ell \leq t+1$ and $1 \leq k \leq t+1$,

$C_i(k,\ell) = \frac{1}{\alpha_i} w_k^{\mathrm{t}} B_i w_\ell = 0$. Intuitively, this just means that $C_i$ is of the form $\begin{bmatrix} \tilde{C}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where $\tilde{C}_i$ is of size $(i+1)\times(i+1)$, and $C_i(i,i+1) = 1$. □

From Step 1, we have $\mathcal{B} \leq \Lambda(n',\mathbb{F})$ with $n' \geq 2d$ and $\delta(\mathcal{B}) \geq d$. Recall that $d = t^4$, and set $t' = t^2$. Applying Lemma 3.2 to $\mathcal{B}$, $d$, $t'$ produces $Q_2 \in \mathrm{M}(n' \times (t'+1),\mathbb{F})$ which achieves the objective of this step.

## 3.5 Step 3

The objective of this step is fulfilled by the following lemma.

**Lemma 3.3.** *Let $t' = t^2$, and $r = t + \binom{t}{2}$. Suppose $\mathcal{C} \leq \Lambda(t'+1,\mathbb{F})$ contains matrices $C_1,\ldots,C_{t'} \in \mathcal{C}$, where $C_i = \begin{bmatrix} \tilde{C}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $\tilde{C}_i$ has size $(i+1)\times(i+1)$, and $C_i(i,i+1) = 1$. Then there exists $Q \in \mathrm{GL}(t'+1,\mathbb{F})$, such that $\mathcal{D} = Q^{\mathrm{t}}\mathcal{C}Q \leq \Lambda(t'+1,\mathbb{F})$ contains $D_1,\ldots,D_r \in \Lambda(t'+1,\mathbb{F})$ satisfying (1) for $i \in [t]$, $D_i = \begin{bmatrix} \tilde{D}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ where $\tilde{D}_i \in \Lambda(i+1,\mathbb{F})$ and $D_i(i,i+1) = 1$, and (2) for $i \in [\binom{t}{2}]$, $D_{t+i}$ is of the form*

$$\begin{bmatrix} \tilde{D}_{t+i} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & 0 & 1 & 0 & \dots & 0 \\ \mathbf{0} & -1 & 0 & 0 & \dots & 0 \\ \mathbf{0} & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{1}$$

*where $\tilde{D}_{t+i} \in \Lambda(t+1+2(i-1),\mathbb{F})$.*

That is, in the matrix $D_{t+i}$, the only nonzero entries of the $(t+2i)$th and $(t+2i+1)$th rows and columns are at the $(t+2i,t+2i+1)$ and $(t+2i+1,t+2i)$ positions. The reason for imposing this condition will be clear later from Observation 3.6.

*Proof of Lemma 3.3.* Observe that the $(t+2i,t+2i+1)$ and $(t+2i+1,t+2i)$ entries of $C_{t+2i}$ are 1 and $-1$, respectively. So we can put $C_{t+2i}$ in the desired form, by multiplying appropriate elementary matrices (i.e. $I+\alpha\cdot E_{t+2i,j}$ for $j < t+2i$ and appropriate $\alpha \in \mathbb{F}$) on the left and their transposes on the right, to set other entries on the $(t+2i+1)$th row and column to be 0. Some care is required to ensure that during this process, other $C_{t+2j}$'s, if they are already in this form, are not affected.

Therefore, we apply appropriate matrices to $C_{t+2i}$, for $i$ in a *decreasing* order, namely starting from $i = \binom{t}{2}$ and then going to $i = 1$. To see that this does not affect those $C_{t+2j}$'s which were already in this form, let us examine $C_{t+2i}$, $C_{t+2i+1}$, and $C_{t+2i+2}$, when we put $C_{t+2i}$ into the form as in (1). Note that

$C_{t+2i+2}$ has been put into the desired form as in (1). That is,

$$C_{t+2i} = \begin{bmatrix} \hat{C}_{t+2i} & * & * & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ * & 0 & 1 & 0 & 0 & \dots & 0 \\ * & -1 & 0 & 0 & 0 & \dots & 0 \\ \mathbf{0} & 0 & 0 & 0 & 0 & \dots & 0 \\ \mathbf{0} & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

$$C_{t+2i+1} = \begin{bmatrix} \hat{C}_{t+2i+1} & * & * & * & \mathbf{0} & \dots & \mathbf{0} \\ * & 0 & * & * & 0 & \dots & 0 \\ * & * & 0 & 1 & 0 & \dots & 0 \\ * & * & -1 & 0 & 0 & \dots & 0 \\ \mathbf{0} & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

$$C_{t+2i+2} = \begin{bmatrix} \hat{C}_{t+2i+2} & * & * & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ * & 0 & * & 0 & 0 & \dots & 0 \\ * & * & 0 & 0 & 0 & \dots & 0 \\ \mathbf{0} & 0 & 0 & 0 & 1 & \dots & 0 \\ \mathbf{0} & 0 & 0 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

where $\hat{C}_{t+2i}$, $\hat{C}_{t+2i+1}$, and $\hat{C}_{t+2i+2}$ are in $\Lambda(s+2i-1,\mathbb{F})$. From the above, when we use $(t+2i,t+2i+1)$ and $(t+2i+1,t+2i)$ entries to set other entries on the $(t+2i)$th and $(t+2i+1)$th rows and columns in $C_{t+2i}$ to be zero, such operations do not affect the $(t+2i+2)$th and $(t+2i+3)$th rows and columns of $C_{t+2i+2}$, nor the $(t+2j)$th and $(t+2j+1)$th rows and columns of $C_{t+2j}$ for $j \geq i+1$ in general. Furthermore, such operations do not change $C_{t+2j}$ for $j \leq i-1$ at all. It follows that after these operations, all $C_{t+2i}$, $i \in [\binom{t}{2}]$, are in the form of (1) as desired.

Suppose $(C_1,\dots,C_{t'})$ are changed to $(C'_1,\dots,C'_{t'})$ after these operations, which implicitly define $Q \in \mathrm{GL}(t'+1,\mathbb{F})$. We then do the following. For $i \in [t]$, let $D_i = C'_i$. For $i \in [\binom{t}{2}]$, let $D_{t+i} = C'_{t+2i}$. We then obtain $r = t + \binom{t}{2} = \binom{t+1}{2}$ matrices $D_1,\dots,D_r \in \Lambda(t'+1,\mathbb{F})$ with $t' = t^2$ which satisfy the properties as required by this lemma, concluding the proof. □

## 3.6 Step 4

To start with, we need an observation on complete spaces as follows. Given an $m$-dimensional $\mathcal{A} \leq \Lambda(n,\mathbb{F})$, let $\mathbf{A} = (A_1,\dots,A_m) \in \Lambda(n,\mathbb{F})^m$ be an ordered basis of $\mathcal{A}$. From $\mathbf{A}$, we construct a 3-way array $\mathtt{A} \in \mathrm{M}(n \times n \times m,\mathbb{F})$ whose frontal slices are $A_i$'s. Let $f_{i,j} \in \mathbb{F}^m$, $i,j \in [n]$, be the tube fibres of $\mathtt{A}$. Let $W = \mathrm{span}\{e_1,\dots,e_t\} \leq \mathbb{F}^n$, where $e_i$'s are standard basis vectors. We then note the following characterisation of $W$ to be a complete space of $\mathcal{A}$.

**Observation 3.4.** *Let $\mathcal{A}$, $f_{i,j}$, and $W$ be as above. Then $W$ is a complete space of $\mathcal{A}$, if and only if, $f_{i,j}$, $1 \le i < j \le t$, are linearly independent.*

The following lemma completes Step 4.

**Lemma 3.5.** *Let $t' = t^2$, and $r = t + \binom{t}{2}$. Suppose $\mathcal{D} \le \Lambda(t'+1,\mathbb{F})$ contains $D_1,\dots,D_r \in \Lambda(t'+1,\mathbb{F})$, such that (1) for $i \in [t]$, $D_i = \begin{bmatrix} \tilde{D}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ where $\tilde{D}_i \in \Lambda(i+1,\mathbb{F})$ and $D_i(i,i+1) = 1$, and (2) for $i \in [\binom{t}{2}]$, $D_{t+i}$ is of the form $i \in [\binom{t}{2}]$, $D_{t+i} = \begin{bmatrix} \tilde{D}_{t+i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 1 & \mathbf{0} \\ \mathbf{0} & -1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$, where $\tilde{D}_{t+i} \in \Lambda(t+1+2(i-1),\mathbb{F})$. Then there exists $Q \in \mathrm{GL}(t'+1,\mathbb{F})$, such that $W = \mathrm{span}\{e_1,\dots,e_{t+1}\}$ is a complete space for $Q^{\mathrm{t}}\mathcal{D}Q$.*

*Proof.* Construct a 3-way array $\mathtt{D}$ of size $(t'+1)\times(t'+1)\times r$, where the $i$th frontal slice is $D_i$. Let $f_{i,j} \in \mathbb{F}^r$ be the $(i,j)$th tube fibre of $\mathtt{D}$. Note that the tube fibres $f_{1,2}, f_{2,3}, \dots, f_{t,t+1}$ and $f_{t+2,t+3}, f_{t+4,t+5}, \dots, f_{t',t'+1}$ are linearly independent.

Let us arrange the tube fibres $f_{i,j}$, $1 \le i \le t-1$, $i+2 \le j \le t$, in the reverse lexicographic order, and relabel them accordingly as $\tilde{f}_k$ for $k \in [\binom{t}{2}]$. That is, $\tilde{f}_1 = f_{1,3}$, $\tilde{f}_2 = f_{1,4}$, $\tilde{f}_3 = f_{2,4}$, $\tilde{f}_4 = f_{1,5}$, $\tilde{f}_5 = f_{2,5}$, and so on.

Our goal is to apply appropriate elementary matrices (on the left and their transposes on the right) to $\mathtt{D}$ to make the tube fibres at positions $(k,\ell)$, $1 \le k < \ell \le t+1$, linearly independent. If this could be achieved, Observation 3.4 ensures that $W = \mathrm{span}\{e_1,\dots,e_{t+1}\}$ is a complete space of the resulting alternating matrix space.

To do that, consider the following operations in $\binom{t}{2}$ rounds. After the $i$th round, we wish to maintain that

$$f_{1,2}, f_{2,3}, \dots, f_{t,t+1}, \tilde{f}_1, \dots, \tilde{f}_i, f_{t+2(i+1),t+2i+3}, f_{t+2(i+2),t+2i+5}, \dots, f_{t',t'+1}$$

are linearly independent. Note that $t' = t + 2\cdot\binom{t}{2}$. If this could be achieved, after $\binom{t}{2}$ rounds,

$$f_{1,2}, f_{2,3}, \dots, f_{t,t+1}, \tilde{f}_1, \dots, \tilde{f}_{\binom{t}{2}}$$

would be linearly independent.

Recall that before the first round starts, we have that the tube fibres $f_{1,2}$, $f_{2,3}$, $\dots$, $f_{t,t+1}$ and $f_{t+2,t+3}, f_{t+4,t+5}, \dots, f_{t',t'+1}$ are linearly independent.

Suppose now we have completed the $i$th round. Let us explain the operations in the $(i+1)$th round. We first check if

$$f_{1,2}, f_{2,3}, \dots, f_{t,t+1}, \tilde{f}_1, \dots, \tilde{f}_i, \tilde{f}_{i+1}, f_{s+2(i+2),s+2i+5}, f_{s+2(i+3),s+2i+7}, \dots, f_{t',t'+1},$$

are linearly independent.

If so, we proceed to the next round.

If not, we have that $\tilde{f}_{i+1}$ is in the linear span of

$$f_{1,2}, f_{2,3}, \dots, f_{t,t+1}, \tilde{f}_1, \dots, \tilde{f}_i, f_{t+2(i+2),t+2i+5}, f_{t+2(i+3),t+2i+7}, \dots, f_{t',t'+1}.$$

Now we wish to add $f_{t+2(i+1),t+2i+3}$ to $\tilde{f}_{i+1}$. This is because, since after the $i$th round we had that

$$f_{1,2}, f_{2,3}, \ldots, f_{t,t+1}, \tilde{f}_1, \ldots, \tilde{f}_i, f_{t+2(i+1),t+2i+3}, f_{t+2(i+2),t+2i+5}, \ldots, f_{t',t'+1}, \tag{2}$$

are linearly independent, we have that

$$f_{1,2}, f_{2,3}, \ldots, f_{t,t+1}, \tilde{f}_1, \ldots, \tilde{f}_i, \tilde{f}_{i+1} + f_{t+2(i+1),t+2i+3}, f_{t+2(i+2),t+2i+5}, \ldots, f_{t',t'+1}, \tag{3}$$

are also linearly independent.

But we cannot add $f_{t+2(i+1),t+2i+3}$ to $\tilde{f}_{i+1}$ directly. Indeed, the legitimate operations are left multiplying elementary matrices (namely $I+E_{i,j}$) and right multiplying their transposes. These correspond to performing row and column operations on $\mathtt{D}$ viewed as a matrix $(f_{i,j})_{i,j\in[t'+1]}$ whose entries are vectors. We will make use of this perspective in the following.

Suppose $\tilde{f}_{i+1}$ corresponds to $f_{j,k}$ for some $1 \leq j < k \leq t+1$. In order to add $f_{t+2(i+1),t+2i+3}$ to $\tilde{f}_{i+1}$, we can first add the $(t+2(i+1))$th row to the $j$th row, and to maintain the alternating property, add the $t+2(i+1)$th column to the $j$th column. Then we add the $(t+(2i+3))$th column to the $k$th column, and to maintain the alternating property, add the $(t+(2i+3))$th row to the $k$th row. This does add $f_{t+2(i+1),t+2i+3}$ to $\tilde{f}_{i+1}$.

However, it is possible that during the above procedure, some of the vectors in

$$\{f_{1,2}, f_{2,3}, \ldots, f_{t,t+1}, \tilde{f}_1, \ldots, \tilde{f}_i\}$$

get altered as well. (It is easy to see that $f_{t+2(i+2),t+2i+5}, \ldots, f_{t',t'+1}$ are not changed.) For example, when adding the $(t+2(i+1))$th row to the $j$th row, $f_{j,j+1}$ and those $\tilde{f}_{i'}$ corresponding to $f_{j,j'}$ could be added by certain vectors as well. Therefore, instead of getting those vectors in (3), we get

$$\begin{aligned} &f_{1,2}+g_{1,2}, f_{2,3}+g_{2,3}, \ldots, f_{t,t+1}+g_{t,t+1}, \tilde{f}_1+\tilde{g}_1, \ldots, \tilde{f}_i+\tilde{g}_i, \\ &\qquad \tilde{f}_{i+1}+f_{t+2(i+1),t+2i+3}+\tilde{g}_{i+1}, f_{t+2(i+2),t+2i+5}, \ldots, f_{t',t'+1}, \end{aligned} \tag{4}$$

where $g_{j,j+1}$ and $\tilde{g}_k \in \mathbb{F}^r$.

Therefore, we need to show that those vectors in (4) are linearly independent. The following observation is crucial for this.

**Observation 3.6.** *We have that $g_{j,j+1}$ and $\tilde{g}_k$ are in* $\mathrm{span}\{f_{t+2(i+2),t+2i+5}, \ldots, f_{t',t'+1}\}$.

*Proof.* Note that $g_{j,j+1}$ and $\tilde{g}_k$ come from those fibre tubes $f_{p,t+2(i+1)}$ and $f_{q,t+2i+3}$, where $1 \leq p,q \leq t+1$. As can be seen from (1), these fibre tubes are in the linear span of $f_{t+2(i+2),t+2i+5}, \ldots, f_{t',t'+1}$, because the only non-zero entries on the $(t+2i+2)$th and $(t+2i+3)$th columns of $D_{t+i+1}$ are in the $(t+2i+3, t+2i+2)$ and $(t+2i+2, t+2i+3)$th positions. □

Given this observation, the linear independence of vectors in (4) follows from the linear independence of vectors in (2). Indeed, by a change of basis, we can assume that the vectors in (2) form a set of standard basis vectors in the order they are listed. So putting those vectors in (2) as column vectors in a matrix form simply gives

$$\begin{bmatrix} I_{t+i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{\binom{t+1}{2}-(t+i+1)} \end{bmatrix},$$

where $I_k$ denotes the $k \times k$ identity matrix. Now putting the vectors in (4) as column vectors in a matrix gives

$$\begin{bmatrix} I_{t+i} & * & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} \\ * & * & I_{\binom{t+1}{2}-(t+i+1)} \end{bmatrix},$$

where $*$ means that the entries could be arbitrary. This matrix is clearly of full-rank, proving the linear independence of vectors in (4). Note that the entries in the lower-left submatrix comes from $g_{i,i+1}$ and $\tilde{g}_j$, and the entries in the $(t+i+1)$-th column comes from $f_{t+2(i+1),t+(2i+3)} + \tilde{g}_{i+1}$. This also explains the necessity of Step 3, as otherwise the upper-left $(t+i+1) \times (t+i+1)$ submatrix would be of the form $\begin{bmatrix} I_{t+i} & * \\ * & 1 \end{bmatrix}$, which could be not full-rank.

Now that we achieved what we wanted in the $(i+1)$th round, also note that the above operations do not affect the $2j$ and $2j+1$ rows and columns of $D_{t+j}$ for $j > i+1$. This means that we can have the same set up to perform the above operations (with increased row and column indices) in the next round. This concludes the proof of Lemma 3.5. □

This concludes the proof of Theorem 1.1. □

# 4 Applications of Theorem 1.1

To explain the applications of Theorem 1.1 in group theory and geometry, we recast Theorem 1.1 in terms of alternating bilinear maps, which follows easily by the relationship between alternating matrix spaces and alternating bilinear maps explained in Section 2.7.

Let $\phi : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^m$ be an alternating bilinear map. Let $W \leq \mathbb{F}^n$ be a subspace. We say that $W$ is a *totally-isotropic* space for $\phi$, if $\phi(W,W) = 0$. We say that $W$ is a *complete space* for $\phi$, if $\dim(\mathrm{span}(\phi(W,W))) = \binom{\dim(W)}{2}$.

**Theorem 4.1** (Theorem 1.1 for alternating bilinear maps). *Let $s,t \in \mathbb{N}$, $s,t \geq 2$. Let $\phi : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^m$ be an alternating bilinear map over $\mathbb{F}$, where $n \geq s \cdot t^4$. Then $\phi$ has either a dimension-$s$ totally-isotropic space, or a dimension-$t$ complete space.*

## 4.1 Implications and new questions in group theory

Let $p$ be an odd prime. Let $\mathfrak{B}_{p,2}$ be the class of finite $p$-groups of class 2 and exponent $p$. That is, a finite group $G$ is in $\mathfrak{B}_{p,2}$, if the commutator subgroup $[G,G]$ is contained in the centre $\mathrm{Z}(G)$, and every $g \in G$ satisfies that $g^p = \mathrm{id}$. We also define $\mathfrak{B}_{p,2,n} \subseteq \mathfrak{B}_{p,2}$, such that $G \in \mathfrak{B}_{p,2}$ is in $\mathfrak{B}_{p,2,n}$ if and only if a minimal generating set of $G$ is of size $n$, or equivalently, if $G/[G,G] \cong \mathbb{Z}_p^n$.

There are two important group families in $\mathfrak{B}_{p,2}$. First, elementary abelian $p$-groups, $\mathbb{Z}_p^s$, are in $\mathfrak{B}_{p,2}$. Second, for any $t \in \mathbb{N}$, there is $F_{p,2,t}$, the *relatively free* $p$-groups of class 2 and exponent $p$ with $t$ generators, defined as the quotient of the free group in $t$ generators by the subgroup generated by all words of the form $x^p$ and $[[x,y],z]$. Note that $F_{p,2,t}$ can be viewed as a universal group in $\mathfrak{B}_{p,2,t}$, in that any group in $\mathfrak{B}_{p,2,t}$ is isomorphic to the quotient of $F_{p,2,t}$ by a subgroup of $[F_{p,2,t}, F_{p,2,t}]$.

Baer's correspondence [Bae38] connects $\mathfrak{B}_{p,2}$ with alternating bilinear maps over $\mathbb{F}_p$. Indeed, this correspondence leads to an isomorphism between the categories of groups in $\mathfrak{B}_{p,2}$ and of alternating bilinear maps over $\mathbb{F}_p$ (cf. [Wil09, Sec. 3]). It is then not surprising to see correspondences between structures of $\mathfrak{B}_{p,2}$ and of alternating bilinear maps over $\mathbb{F}_p$. Examples include abelian subgroups vs totally-isotropic spaces [Alp65], central decompositions vs orthogonal decompositions [Wil09, LQ20], hyperbolic pairs vs totally-isotropic decompositions [BMW17, BCG$^+$21].

Theorem 4.1 has a natural interpretation in the context of $p$-groups of class 2 and exponent $p$ as follows.

**Corollary 4.2.** *Let $G \in \mathfrak{B}_{p,2,n}$, where $n \geq s \cdot t^4$. Then $G$ has either an abelian subgroup $S \leq G$ such that $S[G,G]/[G,G] \cong \mathbb{Z}_p^s$, or a subgroup isomorphic to $F_{p,2,t}$.*

*Proof.* Let $G \in \mathfrak{B}_{p,2,n}$. Then $G/[G,G] \cong \mathbb{Z}_p^n$ and suppose $[G,G] \cong \mathbb{Z}_p^m$. The commutator map $[,]$ induces an alternating bilinear map $\phi : G/[G,G] \times G/[G,G] \to [G,G]$.

Given a subgroup $H \leq G/[G,G]$ such that $H \cong \mathbb{Z}_p^s$, let $S_H$ be a subgroup of $G$ of the smallest order satisfying $S_H[G,G]/[G,G] = H$. Then it is known, at least since [Alp65], that $H$ is a totally-isotropic space of $\phi$ if and only if $S_H$ is abelian. It is also straightforward to verify that $H$ is a complete space if and only if $S_H$ is isomorphic to $F_{p,2,s}$. From these, the corollary follows immediately from Theorem 4.1. □

Readers familiar with varieties of groups [Neu67] may recognise that abelian groups and relatively free groups are the two opposite structures in a variety of groups. In this sense, Corollary 4.2 may be viewed as a group-theoretic version of the classical Ramsey theorem for graphs [Ram30].

Corollary 4.2 also leads to the following family of questions. Recall that a variety of groups, $\mathfrak{C}$, is the class of all groups satisfying a set of laws. Examples include abelian groups, nilpotent groups of class $c$, and solvable groups of class $c$. Let $\mathfrak{C}_t$ be the subclass of $\mathfrak{C}$, consisting of groups that can be generated by $t$ elements. The relatively free group of rank $t$ in $\mathfrak{C}_t$, $F_{\mathfrak{C},t}$, as the free group on $t$ generators modulo the laws defining $\mathfrak{C}$. For a group $G$ and $S \subseteq G$, let $\Phi(G)$ be the Frattini subgroup of $G$. Then the following Ramsey problem for $\mathfrak{C}$ can be formulated.

**Question 4.3** (Ramsey problem for a variety of groups $\mathfrak{C}$)**.** Let $G \in \mathfrak{C}_n$ and $s,t \in \mathbb{N}$. Is it true that, if $n > f_{\mathfrak{C}}(s,t)$ for some function $f_{\mathfrak{C}} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, then either there exists an abelian subgroup $S \leq G$ such that $S\Phi(G)/\Phi(G)$ is of rank $s$, or $G$ has a subgroup isomorphic to $F_{\mathfrak{C},t}$.

There are several deep Ramsey-type results for nilpotent groups [Lei98, BL03, JJR17], mostly following the lines of the van der Waerden theorem [vdW27] and the Hales-Jewett theorem [HJ63]. Question 4.3 asks to show the existence of large enough subgroups of certain types in a group from a variety of groups [Neu67], which is in a closer analogy with the graph Ramsey theory.

## 4.2 Implications and questions for Grassmannians

Theorem 4.1 can also be interpreted in the context of hyperplane sections of Grassmannians. To introduce this implication we need some further terminologies.

Let $\phi : (\mathbb{F}^n)^{\times \ell} \to \mathbb{F}^m$ be an alternating $\ell$-linear map. We say that $V \leq \mathbb{F}^n$ is a *totally-isotropic space* of $\phi$, if for any $v_1, \dots, v_\ell \in V$, $\phi(v_1, \dots, v_\ell) = 0$. We say that $W \leq \mathbb{F}^n$ of dimension $\geq \ell$ is an *anisotropic space* of $\phi$, if for any linearly independent $w_1, \dots, w_\ell \in W$, $\phi(w_1, \dots, w_\ell) \neq 0$.

As pointed out by Feldman and Propp [FP92, Sec. 6], an alternating $\ell$-linear map $\phi : (\mathbb{F}^n)^\ell \to \mathbb{F}^m$ defines an $m$-fold hyperplane section $H$ on the Grassmannian $\mathrm{Gr}(n,\ell)$, the variety of $\ell$-dimensional subspaces of $\mathbb{F}^n$. For $W \leq \mathbb{F}^n$, $\mathrm{Gr}(W,\ell)$ is a subvariety of $\mathrm{Gr}(n,\ell)$. Feldman and Propp noted that $W$ is a totally-isotropic space if and only if $\mathrm{Gr}(W,\ell)$ is contained in $H$. It is also easy to see that $W$ is anisotropic if and only if $H$ intersects $\mathrm{Gr}(W,\ell)$ trivially.

Let $\ell = 2$, and note that a complete space is anisotropic. Due to the geometric interpretations of anisotropic spaces and totally-isotropic spaces explained above, Theorem 4.1 implies the following.

**Corollary 4.4.** *When $n \geq s \cdot t^4$, any $m$-fold hyperplane section of $\mathrm{Gr}(n,2)$ either contains $\mathrm{Gr}(W,2)$ for some $s$-dimensional $W \leq \mathbb{F}^n$, or intersects $\mathrm{Gr}(W,2)$ trivially for some $t$-dimensional $W \leq \mathbb{F}^n$.*

Note that Corollary 4.4 deals with $m$-fold hyperplane sections of $\mathrm{Gr}(n,2)$. This leads to the question as whether a similar statement for $\mathrm{Gr}(n,\ell)$, $\ell > 2$, holds.

# 5 Discussions: Theorem 1.1 as a linear Ramsey theorem

In this section we discuss on an analogy between graphs and alternating bilinear maps, and more generally, between hypergraphs and alternating multilinear maps, in the context of extremal combinatorics.

Through this analogy, Theorem 1.1 can be naturally understood as a linear algebraic Ramsey theorem. A previous "linear Ramsey theorem" of Feldman and Propp [FP92] could be more properly understood as a "linear Turán theorem." An intriguing open problem, which can be viewed as a linear Turán's problem for hypergraphs, is proposed in Section 5.7.

We also present two small results. First, we present a result that complements Theorem 1.1 (Proposition 5.2). Second, we generalise a correspondence between independent sets and totally-isotropic spaces in [BCG+21] from graphs to hypergraphs (Proposition 5.1).

## 5.1 Graphs and alternating matrix spaces

Following ideas traced back to Tutte [Tut47] and Lovász [Lov79], we construct an alternating matrix space from a graph.

Let $G = ([n], E)$ be a simple, undirected graph. Suppose $E = \{\{i_1, j_1\}, \ldots, \{i_m, j_m\}\} \subseteq \binom{[n]}{2}$, where for $k \in [m]$, $i_k < j_k$ and for $1 \leq k < k' \leq m$, $(i_k, j_k) < (i_{k'}, j_{k'})$ in the lexicographic order. For $k \in [m]$, let $A_k$ be the alternating matrix $A_{i_k, j_k}$ (defined in Section 2) over $\mathbb{F}$, and set $\mathcal{A}_G := \mathrm{span}\{A_1, \ldots, A_m\} \leq \Lambda(n, \mathbb{F})$.

The basic observation of Tutte and Lovász is that $G$ has a perfect matching if and only if $\mathcal{A}_G$ contains a full-rank matrix. This classical example is the precursor of several recent discoveries relating properties of $G$, including independent sets and vertex colourings [BCG+21], vertex and edge connectivities [LQ20], and isomorphism and homomorphism notions [HQ21], with properties of $\mathcal{A}_G$. Graph-theoretic questions and techniques have also been translated to study alternating matrix spaces or alternating bilinear maps in [LQ17, BCG+21, Qia21] with applications to group theory and quantum information.

## 5.2 Hypergraphs and alternating multilinear maps (and exterior algebras)

Naturally extending the construction of alternating matrix spaces from graphs, the following classical construction of alternating $\ell$-linear maps from $\ell$-uniform hypergraphs is also traced back to Lovász

[Lov77].

Given $\{i_1,\dots,i_\ell\} \in \binom{[n]}{\ell}$ where $i_1 < \cdots < i_\ell$, there is the alternating $\ell$-linear form $e_{i_1}^* \wedge \cdots \wedge e_{i_\ell}^*$. From an $\ell$-uniform hypergraph $H = ([n],E)$ where $E \subseteq \binom{[n]}{\ell}$ and $|E| = m$, we can construct an alternating $\ell$-linear map $\phi_H : (\mathbb{F}^n)^{\times \ell} \to \mathbb{F}^m$ by first applying the above alternating $\ell$-linear form construction to every hyperedge in $E$, and then ordering these forms by the lexicographic ordering of $\binom{[n]}{\ell}$.

Lovász's construction was originally stated in terms of subspaces of exterior algebras. Since then, the use of exterior algebras has lead to the elegant extensions of Bollobás's Two Families Theorem [Bol65] by Frankl [Fra82], Kalai [Kal84] and Alon [Alo85], as well as Kalai's algebraic shifting method [Kal02]. The recent work by Scott and Wilmer [SW21] systematically extends several basic results from the extremal combinatorics of hypergraphs to subspaces of exterior algebras.

Subspaces of exterior algebras, linear spaces of alternating multilinear forms, and alternating multilinear maps are of course closely related. Indeed, our main result can be formulated in terms of exterior algebras, just as Feldman and Propp did for their main result in [FP92, Corollary 2].

### 5.3 Independent sets and totally-isotropic spaces

Recall that for an $\ell$-uniform hypergraph $H = ([n],E)$, $S \subseteq [n]$ is an *independent set* in $H$, if $S$ does not contain any hyperedge from $E$. The *independence number* of $H$, $\alpha(H)$, is the maximum size over all independent sets in $H$.

Let $\phi : (\mathbb{F}^n)^{\times \ell} \to \mathbb{F}^m$ be an alternating $\ell$-linear map. Recall that totally-isotropic spaces of $\phi$ are defined in Section 4.2. The *totally-isotropic number* of $\phi$, $\alpha(\phi)$, is the maximum dimension over all totally-isotropic spaces of $\phi$.

In [BCG$^+$21], it is shown that when $\ell = 2$, i.e. for a graph $G$, $\alpha(G) = \alpha(\phi_G)$. We generalise that result to any $\ell$ in the following proposition, which justifies viewing totally-isotropic spaces as a linear algebraic analogue of independent sets.

**Proposition 5.1.** *Let $H$ be an $\ell$-uniform hypergraph, and let $\phi_H$ be the alternating $\ell$-linear map constructed from $H$ as in Section 5.2. Then we have $\alpha(H) = \alpha(\phi_H)$.*

*Proof.* Let $H = ([n],E)$, $E \subseteq \binom{[n]}{\ell}$. Let $\phi_H : (\mathbb{F}^n)^{\times \ell} \to \mathbb{F}^m$ be the alternating $\ell$-linear map constructed from $H$ via the construction in Section 5.2.

Suppose $S \subseteq [n]$ is an independent set of $H$. Let $V = \mathrm{span}\{e_i \mid i \in S\} \leq \mathbb{F}^n$. It is easy to verify that $V$ is a totally-isotropic space of $\phi_H$. It follows that $\alpha(H) \geq \alpha(\phi_H)$.

Suppose $V \leq \mathbb{F}^n$ is a totally-isotropic space of $\phi_H$ of dimension $d$. Let $B \in \mathrm{M}(n \times d, \mathbb{F})$ be a matrix whose columns span $V$. For $i \in [n]$, let $w_i \in \mathbb{F}^d$ such that $w_i^t$ is the $i$th row of $B$. As $\mathrm{rk}(B) = d$, there exists $i_1,\dots,i_d$, $1 \leq i_1 < \cdots < i_d \leq n$, such that $w_{i_j}$, $j \in [d]$, are linearly independent.

We claim that $\{i_1,\dots,i_d\} \in [n]$ is an independent set of $H$. If not, by relabelling the vertices, we can assume that $\{i_1,\dots,i_\ell\} \in E$. As $V$ is a totally-isotropic space, we have $e_{i_1}^* \wedge \cdots \wedge e_{i_\ell}^*$, when restricted to $V$, is the zero map. It follows that $w_{i_1} \wedge \cdots \wedge w_{i_\ell} = 0$, contradicting that $w_{i_1},\dots,w_{i_\ell}$ are linearly independent. The claim is proved.

We then derive that $\alpha(\phi_H) \geq \alpha(H)$, concluding the proof. □

### 5.4 Three numbers related to Turán's theorem

For an $\ell$-uniform hypergraph $H = ([n], E)$, the size of $H$ is denoted as $\text{size}(H) = |E|$. We define three closely related numbers regarding $n$, $\text{size}(H)$, $\ell$, and the independence number $\alpha(H)$. The first number is just the celebrated Turán number for hypergraphs [Tur61, Sid95]. The other two numbers originate from [BGH87] and [FP92], respectively, in the context of alternating multilinear maps.

Let $n, m, \ell, a \in \mathbb{N}$. The *Turán number* is

$$\mathrm{T}(n, a, \ell) = \min\{\text{size}(H) \mid H \text{ is } \ell\text{-uniform, } n\text{-vertex hypergraph, and } \alpha(H) \le a\}.$$

The *Feldman-Propp number* is

$$\mathrm{FP}(a, m, \ell) = \min\{n \in \mathbb{N} \mid \forall \ell\text{-uniform, } n\text{-vertex, } m\text{-edge hypergraph } H, \alpha(H) > a\}.$$

We also define the number

$$\alpha(n, m, \ell) = \min\{\alpha(H) \mid H \text{ is } \ell\text{-uniform, } n\text{-vertex, } m\text{-edge hypergraph}\}.$$

It is easy to see the relations of these three numbers, $\alpha(n,m,\ell)$, $\mathrm{T}(n,a,\ell)$, and $\mathrm{FP}(a,m,\ell)$. On the one hand, $\mathrm{T}(n,a,\ell) \le m$ implies the existence of some $n$-vertex, $m$-edge, $\ell$-uniform hypergraph $H$ with $\alpha(H) \le a$, which in turn implies that $\alpha(n,m,\ell) \le a$ and $\mathrm{FP}(a,m,\ell) > n$. On the other hand, $\mathrm{T}(n,a,\ell) > m$ implies that for any $n$-vertex, $m$-edge, $\ell$-uniform hypergraph $H$, $\alpha(H) > a$, which in turn implies that $\alpha(n,m,\ell) > a$ and $\mathrm{FP}(a,m,\ell) \le n$.

In the alternating multilinear map setting, by replacing independence numbers with totally-isotropic numbers defined in Section 5.3, we can define $\alpha_{\mathbb{F}}(n,m,\ell)$, $\mathrm{T}_{\mathbb{F}}(n,a,\ell)$, and $\mathrm{FP}_{\mathbb{F}}(a,m,\ell)$ for those alternating multilinear maps $\phi : (\mathbb{F}^n)^{\times \ell} \to \mathbb{F}^m$ with $\alpha(\phi) = a$. Note that an immediate consequence of Proposition 5.1 is that $\alpha_{\mathbb{F}}(n,m,\ell) \le \alpha(n,m,\ell)$.

### 5.5 Turán meets Buhler, Gupta, and Harris

The celebrated Turán's theorem [Tur41] is a cornerstone of extremal graph theory [Bol04]. Formulated in terms of independent sets, Turán's theorem gives that

$$\alpha(n,m,2) \ge \left\lceil \frac{n^2}{2m+n} \right\rceil, \tag{5}$$

where the equality can be achieved.

In the alternating multilinear map setting, the quantity $\alpha_{\mathbb{F}}(n,m,2)$ has been studied by Buhler, Gupta, and Harris [BGH87]. The main result of [BGH87] states that for any $m > 1$, we have

$$\alpha_{\mathbb{F}}(n,m,2) \le \left\lfloor \frac{m+2n}{m+2} \right\rfloor, \tag{6}$$

where the equality is attainable over any[2] algebraically closed[3] field. The inequality was also obtained earlier by Ol'shanskii [Ol'78] over finite fields. This allows us to show the following result that complements Theorem 1.1.

[2]While in [BGH87] the main result was stated for fields of characteristic $\neq 2$, the proof works for any characteristic.

[3]For fields that are not algebraically closed, the equality may not be achieved. See [BGH87, Sec. 3] and [GQ06].

**Proposition 5.2.** *There exists an alternating bilinear map $\phi : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^m$, $n = \Theta(s \cdot t^2)$, such that $\phi$ has neither a dimension-$s$ totally-isotropic space, nor a dimension-$t$ complete space.*

*Proof.* Let $m = \lfloor \binom{t-1}{2} \rfloor$. Let $n = \lfloor \frac{(m+2)(s-2)}{2} \rfloor + 1$, which implies that $\frac{m+2n}{m+2} \leq s-1$. Note that $n = \Theta(s \cdot m) = \Theta(s \cdot t^2)$.

By Ol'shanskii ( [Ol'78, Lemma 2]) and Buhler, Gupta, and Harris ( [BGH87, Main Theorem]), $\alpha_{\mathbb{F}}(n,m,2) \leq \lfloor \frac{m+2n}{m+2} \rfloor$. It follows that there exists $\phi : (\mathbb{F}^n)^{\times \ell} \to \mathbb{F}^m$ with $\alpha(\phi) \leq s-1$. Furthermore, by $m = \lfloor \binom{t-1}{2} \rfloor$, any complete spaces of $\phi$ is of dimension $\leq t-1$. The result then follows. □

Closing the gap between $\Omega(s \cdot t^2)$ from Proposition 5.2, and $s \cdot t^4$ from Theorem 1.1, is an interesting open problem.

**Remark 5.3.** Comparing Equations 5 and 6, we see that $\alpha(n,m,2)$ and $\alpha_{\mathbb{F}}(n,m,2)$ behave quite differently. For example, by Equation 5, every graph with $n$ vertices and $2n$ edges has an independent set of size at least $n/5$. On the other hand, by Equation 6, there exists an alternating bilinear map $\phi : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^{2n}$ with no totally-isotropic spaces of dimension $\geq 2$.

The results of [Ol'78, BGH87] were obtained in the context of abelian subgroups of finite $p$-groups, following the works of Burnside [Bur13] and Alperin [Alp65]. Abelian subgroups of general finite groups were studied by Erdős and Straus [ES76] and Pyber [Pyb97].

The techniques of [Ol'78, BGH87] are worth noting. The upper bound for $\alpha_{\mathbb{F}}(n,m,2)$ is through a probabilistic argument in the linear algebra or geometry settings. As Pyber indicated [Pyb97], this is one of the first applications of the random method in group theory. The lower bound for $\alpha_{\mathbb{F}}(n,m,2)$ in [BGH87] relies on methods from the intersection theory in algebraic geometry [HT84].

### 5.6 Turán meets Feldman and Propp

After proving his theorem in [Tur41], Turán proposed the corresponding problem for hypergraphs [Tur61], which asks to determine $\mathrm{T}(n,a,\ell)$ for any $\ell$. This problem greatly stimulates the development of extremal combinatorics; one prominent example is Razborov's invention of flag algebras [Raz07]. More results and developments can be found in surveys by Sidorenko [Sid95] and Keevash [Kee11].

The corresponding problem for alternating multilinear maps was studied by Feldman and Propp [FP92], with applications to geometry and quantum mechanics. Their main result is a lower bound of $\alpha_{\mathbb{F}}(n,m,\ell)$. This lower bound is more easily described in the form of an upper bound of the Feldman-Propp number $\mathrm{FP}_{\mathbb{F}}(a,m,\ell)$, which is a recursive function and can grow as fast as Ackermann's function.

Interestingly, Feldman and Propp termed their result as a "linear Ramsey theorem," and compared it with the classical Ramsey theorem in [FP92, Sec. 3]. By the relations among the three numbers explained in Section 5.4, this is a misnomer, and it is really a linear Turán theorem. In particular, Feldman and Propp's argument to prove the upper bound for $\mathrm{FP}_{\mathbb{F}}(a,m,\ell)$ carries over to $\mathrm{FP}(a,m,\ell)$ in a straightforward way.

### 5.7 Turán's problem for alternating multilinear maps

Let $\mathbb{F}$ be an algebraically closed field. From the experience in settling $\alpha_{\mathbb{F}}(n,m,2)$ in [BGH87], the lower bound derived from intersection theory matches the upper bound derived from a probabilistic argument.

For $\alpha_{\mathbb{F}}(n,m,\ell)$, the same probabilistic argument, observed already in [FP92], gives an upper bound which is the smallest integer $a$ satisfying $n < \frac{m}{a} \cdot \binom{a}{\ell} + a$. The lower bound obtained from [FP92] is far from this upper bound. It seems to us that there are certain substantial difficulties to generalise the intersection-theoretic calculations in [HT84] which support the lower bound for $\alpha(n,m,2)$ in [BGH87] to provide better lower bound for $\alpha(n,m,\ell)$. Therefore, we believe that improving the lower bound of $\alpha_{\mathbb{F}}(n,m,\ell)$ for $\ell > 2$ is a fascinating open problem.

### 5.8 Two opposite structures for totally-isotropic spaces

Let $\phi : (\mathbb{F}^n)^{\times \ell} \to \mathbb{F}^m$ be an alternating $\ell$-linear form. For any $W \le \mathbb{F}^n$, the restriction of $\phi$ to $W$ naturally gives $\phi|_W : W^{\times \ell} \to \mathbb{F}^m$. Then $W$ is a totally-isotropic space if and only if $\phi|_W$ is the zero map, i.e. $\dim(\mathrm{span}(\phi(W,\dots,W))) = 0$. The opposite situation then is naturally when $\phi|_W$ is the full map, i.e. $\dim(\mathrm{span}(\phi(W,\dots,W))) = \binom{\dim(W)}{\ell}$. So we define $W \le \mathbb{F}^n$ to be a *complete space* of $\phi$, if $\dim(W) \ge \ell$ and $\dim(\mathrm{span}(\phi(W,\dots,W))) = \binom{\dim(W)}{\ell}$.

That $W$ being a totally-isotropic space for $\phi$ can also be formulated as: for any $w_1,\dots,w_\ell \in W$, $\phi(w_1,\dots,w_\ell) = 0$. From this viewpoint, the opposite structure of totally-isotropic spaces would be anisotropic spaces defined in Section 4.2.

While it is clear that a complete space is anisotropic, the converse is not necessarily true. This is because there exists a non-full alternating $\ell$-linear map $\phi$ such that for any linearly independent $(v_1,\dots,v_\ell)$, $\phi(v_1,\dots,v_\ell) \neq 0$. When $\ell = 2$ this is already seen in Remark 5.3, and for general $\ell$ this follows easily from the upper bound of $\alpha_{\mathbb{F}}(n,m,\ell)$ indicated in Section 5.7. This distinction between complete spaces and anisotropic spaces does not arise for those alternating multilinear maps constructed from hypergraphs, as a non-complete hypergraph certainly misses a hyperedge.

Besides being natural structures opposite to totally-isotropic spaces, complete spaces and anisotropic spaces have known connections to group theory and geometry, respectively. These connections have been explained in Section 4 to deduce Corollaries 4.2 and 4.4 from Theorem 1.1.

We note that a result relating complete spaces (or anisotropic spaces) and cliques, in the spirit of Proposition 5.1 relating totally-isotropic spaces and independent sets, is not possible in general. This can be seen from the algorithmic viewpoint, say when $\ell = 2$. It is well-known computing the maximum clique size is NP-hard on graphs. Computing the maximum complete space dimension can be achieved in randomised polynomial time, when the field size is large enough, by the Schwartz-Zippel lemma [Sch80, Zip79]. In [BCG+21], it is shown that the problem of deciding whether an alternating bilinear map is anisotropic subsumes the problem of deciding quadratic residuosity modulo squarefree composite numbers, a difficult number-theoretic problem.

### 5.9 Ramsey numbers for alternating multilinear maps

Based on which of complete spaces and anisotropic spaces play the role of cliques, two Ramsey numbers can be defined for alternating multilinear maps.

**Definition 5.4.** For a field $\mathbb{F}$ and $s,t,\ell \in \mathbb{N}$, $s,t \ge \ell \ge 2$, the *Ramsey number for complete spaces*, $\mathrm{R}^{\mathrm{c}}_{\mathbb{F}}(s,t,\ell)$, is the minimum number $n \in \mathbb{N}$ such that any alternating $\ell$-linear map $\phi : (\mathbb{F}^n)^{\times \ell} \to \mathbb{F}^m$ has either a totally-isotropic space of dimension $s$, or a complete space of dimension $t$.

The *Ramsey number for anisotropic spaces*, $\mathrm{R}^{\mathrm{a}}_{\mathbb{F}}(s,t,\ell)$, is defined in the same way, except that we replace complete spaces with anisotropic spaces in the above.

It is clear that $\mathrm{R}^{\mathrm{a}}_{\mathbb{F}}(s,t,\ell) \leq \mathrm{R}^{\mathrm{c}}_{\mathbb{F}}(s,t,\ell)$.

As with hypergraph Ramsey numbers, the first question is whether $\mathrm{R}^{\mathrm{c}}_{\mathbb{F}}(s,t,\ell)$ and $\mathrm{R}^{\mathrm{a}}_{\mathbb{F}}(s,t,\ell)$ are finitely upper bounded, and if so, provide an explicit bound as tight as possible. Improving bounds for hypergraph Ramsey numbers is a major research topic in Ramsey theory, with classical works by Ramsey [Ram30] and Erdős and Rado [ER52], and the recent breakthrough by Conlon, Fox, and Sudakov [CFS10]. However, directly applying those methods for the hypergraph Ramsey theorem seems not to work. One reason is that the "correlation" in the linear algebraic world prohibits the divide and conquer paradigm. For example, one way to prove the $\ell$-uniform hypergraph Ramsey theorem is to use a recursive relation for Ramsey numbers as

$$\mathrm{R}(s,t,\ell) \leq \mathrm{R}(\mathrm{R}(s-1,t,\ell),\mathrm{R}(s,t-1,\ell),\ell-1)+1.$$

Such a relation does not carry over to $\mathrm{R}^{\mathrm{a}}_{\mathbb{F}}$ and $\mathrm{R}^{\mathrm{c}}_{\mathbb{F}}$, at least not directly. The problem lies in the $\mathrm{R}(s,t-1,\ell)$ and then $t-1$ branch, as we cannot "merge" the two conditions into one to obtain the desired dimension-$t$ anisotropic or complete space.

To the best of our knowledge, it is not even known that $\mathrm{R}^{\mathrm{c}}$ and $\mathrm{R}^{\mathrm{a}}$ are finitely bounded for general $\ell$. We therefore propose the following conjecture.

**Conjecture 5.5.** The Ramsey numbers for complete spaces and anisotropic spaces, $\mathrm{R}^{\mathrm{c}}_{\mathbb{F}}(s,t,\ell)$ and $\mathrm{R}^{\mathrm{c}}_{\mathbb{F}}(s,t,\ell)$, are upper bounded by explicit functions in $s$, $t$, and $\ell$.

Our main result just proves Conjecture 5.5 in the case of $\ell = 2$. It is also somewhat surprising, as it gives a *polynomial upper bound* for $\mathrm{R}^{\mathrm{c}}$, and therefore $\mathrm{R}^{\mathrm{a}}$, in the case of $\ell = 2$. As the reader already saw in Section 3, its proof strategy is very different from those for proving the graph Ramsey theorem.

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