LUCAS SEQUENCES AND FUNCTIONS OF A 4-BY-4 MATRIX

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1. INTRODUCTION

Define the sequences \( \{U_n\} \) and \( \{V_n\} \) for all integers \( n \) by

\[
\begin{align*}
U_n &= pU_{n-1} - qU_{n-2}, \quad U_0 = 0, \quad U_1 = 1, \\
V_n &= pV_{n-1} - qV_{n-2}, \quad V_0 = 2, \quad V_1 = p,
\end{align*}
\]  

(1.1)

where \( p \) and \( q \) are real numbers with \( q(p^2 - 4q) \neq 0 \). These sequences were studied originally by Lucas [6], and have subsequently been the subject of much attention.

The Binet forms for \( U_n \) and \( V_n \) are

\[
\begin{align*}
U_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n, \\
\end{align*}
\]  

(1.2)

where

\[
\begin{align*}
\alpha &= \frac{p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2},
\end{align*}
\]  

(1.3)

are the roots, assumed distinct, of \( x^2 - px + q = 0 \). We assume further that \( \alpha / \beta \) is not an \( n^{th} \) root of unity for any \( n \).

A well-known relationship between \( U_n \) and \( V_n \) is

\[
V_n = U_{n+1} - qU_{n-1},
\]  

(1.4)

which we use subsequently.

Recently, Melham [7] considered functions of a 3-by-3 matrix and obtained infinite sums involving squares of terms from the sequences (1.1). Here, using a similarly defined 4-by-4 matrix, we obtain new infinite sums involving cubes, and other terms of degree three, from the sequences (1.1). For example, closed expressions for

\[
\sum_{n=0}^{\infty} \frac{U_n^3}{n!} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{U_n^2U_{n+1}}{n!}
\]

arise as special cases of results in Section 3 [see (3.4) and (3.5)]. Since the above mentioned paper of Melham contains a comprehensive list of references, we have chosen not to repeat them here.

Unfortunately, one of the matrices which we need to record does not fit comfortably on a standard page. We overcome this difficulty by simply listing elements in a table. Following convention, the \((i, j)\) element is the element in the \(i^{th}\) row and \(j^{th}\) column.
2. THE MATRIX $A_{k,x}$

By lengthy but straightforward induction on $n$, it can be shown that the 4-by-4 matrix

$$A = \begin{pmatrix}
0 & 0 & 0 & -q^3 \\
0 & 0 & q^2 & 3pq^2 \\
0 & -q & -2pq & -3p^2q \\
1 & p & p^2 & p^3
\end{pmatrix}$$

is such that, for nonnegative integers $n$, $A^n$ is as follows:

$$A^n = \begin{pmatrix}
-q^3U_{n-1} & -q^4U_{n-2} & -q^4U_{n-3} & -q^3U_{n-4} \\
3q^3U_{n-1}U_n & q^6(2U_{n-1}U_{n+1} + U_{n+1}U_{n+2}) & q^6(U_{n+1}^2 + 2U_{n+1}U_{n+2}) & 3q^3U_{n+1}U_{n+2} \\
-3qU_{n-1}U_n^2 & -q(U_n^2 + 2U_{n-1}U_{n+1}) & -q(2U_{n+1}^2U_{n+1} + U_{n+1}U_{n+2}) & -3qU_nU_{n+1}^2 \\
U_n^3 & U_n^2U_{n+1} & U_nU_{n+1}U_{n+2} & U_n^3
\end{pmatrix}$$

To complete the proof by induction, we make repeated use of the recurrence for $\{U_n\}$. For example, performing the inductive step for the (2, 2) position, we have

$$-q^3(U_n^3 + 2U_{n-1}U_{n+1}) + 3pq^2U_nU_{n+1} = q^2U_n(-qU_n + 2U_{n+1} - 3pU_{n+1})$$
$$= q^2U_n(U_n + 2U_{n+1} - 3pU_{n+1})$$
$$= q^2U_n[(U_n + 2U_{n+1} - 3pU_{n+1})$$
$$= q^2[2U_{n+1}U_n + U_{n+1}U_{n+2}]$$

which is the required expression.

When $p = 1$ and $q = -1$, the matrix $A$ becomes

$$A = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 3 \\
0 & 1 & 2 & 3 \\
1 & 1 & 1 & 1
\end{pmatrix}$$

which is a 4-by-4 Fibonacci matrix. Other 4-by-4 Fibonacci matrices have been studied, for example, in [3] and [4].

The characteristic equation of $A$ is

$$\lambda^4 - p(p^2 - 2q)\lambda^3 + q(p^2 - 2q)(p^2 - q)\lambda^2 - pq^3(p^2 - 2q)\lambda + q^6 = 0.$$ 

Since $p = \alpha + \beta$ and $q = \alpha\beta$, it is readily verified that $\alpha^3$, $\alpha^2\beta$, $\alpha\beta^2$, and $\beta^3$ are the eigenvalues $\lambda_j$ ($j = 1, 2, 3, 4$) of $A$. These eigenvalues are nonzero and distinct because of our assumptions in Section 1.

Associated with $A$, we define the matrix $A_{k,x}$ by

$$A_{k,x} = xA^k,$$  

(2.2)
where \( x \) is an arbitrary real number and \( k \) is a nonnegative integer. From the definition of an eigenvalue, it follows immediately that \( x\alpha^{2k}, x\alpha^{2k}\beta^k, x\alpha^k\beta^{2k}, \) and \( x\beta^{3k} \) are the eigenvalues of \( A_{k,x} \). Again, they are nonzero and distinct.

3. THE MAIN RESULTS

Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be a power series whose domain of convergence includes the eigenvalues of \( A_{k,x} \). Then we have, from (2.2),

\[
f(A_{k,x}) = \sum_{n=0}^{\infty} a_n A_{k,x}^n = \sum_{n=0}^{\infty} a_n x^n A^{kn}.
\]

The final sum in (3.1) can be expressed as a 4-by-4 matrix whose entries we record in the following table.

<table>
<thead>
<tr>
<th>(i, j)</th>
<th>(i, j) element of ( f(A_{k,x}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td>(-q^3 \sum_{n=0}^{\infty} a_n x^n U_{k-1}^3)</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>(-q^3 \sum_{n=0}^{\infty} a_n x^n U_{k-1}^3 U_{kn})</td>
</tr>
<tr>
<td>(1, 3)</td>
<td>(-q^3 \sum_{n=0}^{\infty} a_n x^n U_{kn}^2 U_{k-1})</td>
</tr>
<tr>
<td>(1, 4)</td>
<td>(-q^3 \sum_{n=0}^{\infty} a_n x^n U_{kn}^2)</td>
</tr>
<tr>
<td>(2, 1)</td>
<td>(3q^2 \sum_{n=0}^{\infty} a_n x^n U_{k-1}^2 U_{kn})</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>(q^3 \sum_{n=0}^{\infty} a_n x^n (2U_{kn}^2 U_{k-1} + U_{k-1} U_{k+1}^2))</td>
</tr>
<tr>
<td>(2, 3)</td>
<td>(q^3 \sum_{n=0}^{\infty} a_n x^n (U_{kn}^3 + 2U_{k-1} U_{k} U_{k+1}))</td>
</tr>
<tr>
<td>(2, 4)</td>
<td>(3q^2 \sum_{n=0}^{\infty} a_n x^n U_{k-1}^2 U_{k+1})</td>
</tr>
<tr>
<td>(3, 1)</td>
<td>(-3q^3 \sum_{n=0}^{\infty} a_n x^n U_{kn} U_{k-1}^2)</td>
</tr>
<tr>
<td>(3, 2)</td>
<td>(-q^3 \sum_{n=0}^{\infty} a_n x^n (U_{kn}^3 + 2U_{k-1} U_{k} U_{k+1}))</td>
</tr>
<tr>
<td>(3, 3)</td>
<td>(-q^3 \sum_{n=0}^{\infty} a_n x^n (2U_{kn}^2 U_{k+1} + U_{k+1} U_{k+1}^2))</td>
</tr>
<tr>
<td>(3, 4)</td>
<td>(-3q^3 \sum_{n=0}^{\infty} a_n x^n U_{kn} U_{k+1}^2)</td>
</tr>
<tr>
<td>(4, 1)</td>
<td>(3q^2 \sum_{n=0}^{\infty} a_n x^n U_{kn}^3)</td>
</tr>
<tr>
<td>(4, 2)</td>
<td>(\sum_{n=0}^{\infty} a_n x^n U_{kn} U_{k+1}^2)</td>
</tr>
<tr>
<td>(4, 3)</td>
<td>(\sum_{n=0}^{\infty} a_n x^n U_{kn} U_{k+1}^2)</td>
</tr>
<tr>
<td>(4, 4)</td>
<td>(\sum_{n=0}^{\infty} a_n x^n U_{kn} U_{k+1}^2)</td>
</tr>
</tbody>
</table>
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On the other hand, from the theory of functions of matrices ([2] and [5]), it is known that

\[ f(A_{k,x}) = c_0 I + c_1 A_{k,x} + c_2 A^2_{k,x} + c_3 A^3_{k,x}, \]

where \( I \) is the 4-by-4 identity matrix, and where \( c_0, c_1, c_2, \) and \( c_3 \) can be obtained by solving the system

\[
\begin{align*}
0 + c_0 x \alpha^k + c_2 x^2 \alpha^{6k} + c_3 x^3 \alpha^{9k} &= f(x \lambda^k), \\
0 + c_0 x \alpha^k \beta^k + c_2 x^2 \alpha^{4k} \beta^{2k} + c_3 x^3 \alpha^{6k} \beta^{3k} &= f(x \lambda \beta^k), \\
0 + c_0 x \alpha^k \beta^{2k} + c_2 x^2 \alpha^{2k} \beta^{4k} + c_3 x^3 \alpha^{3k} \beta^{6k} &= f(x \lambda^2 \beta^k), \\
0 + c_0 x \alpha^k \beta^{3k} + c_2 x^2 \alpha^{3k} \beta^{3k} + c_3 x^3 \alpha^{3k} \beta^{9k} &= f(x \lambda^2 \beta^k).
\end{align*}
\]

With the use of Cramer's rule, and making use of the Binet form for \( U_n \), we obtain, after much tedious algebra,

\[
c_0 = \frac{-f(x \lambda^3 \beta^k)}{U_1 U_2 U_3 U_4 (\alpha - \beta)} + \frac{f(x \lambda^2 \beta^k)(\alpha^3 \beta^k + \beta^3 \alpha^k + 2\beta^k \alpha^k)}{U_1^2 U_2 U_3 (\alpha - \beta)^3} \]
\[
- \frac{f(x \lambda^2 \beta^2 k)(\alpha^3 \beta^k + \beta^3 \alpha^k + 2\beta^k \alpha^k)}{U_1 U_2^2 U_3 (\alpha - \beta)^3},
\]
\[
c_1 = \frac{f(x \lambda^2 \beta^3 k)(\alpha^2 \beta^k + \beta^2 \alpha^k + 2\beta^k \alpha^k)}{x \alpha^2 \beta^k U_1 U_2 U_3 (\alpha - \beta)^3} - \frac{f(x \lambda^3 \beta^k)(\alpha^3 \beta^k + \beta^3 \alpha^k + 2\beta^k \alpha^k)}{x \alpha^2 \beta^k U_1 U_2 U_3 (\alpha - \beta)^3} \]
\[
+ \frac{f(x \lambda^2 \beta^2 k)(\alpha^3 \beta^k + \beta^3 \alpha^k + 2\beta^k \alpha^k)}{x \alpha^2 \beta^k U_1 U_2 U_3 (\alpha - \beta)^3},
\]
\[
c_2 = \frac{f(x \lambda^2 \beta^3 k)(\alpha^2 \beta^k + \beta^2 \alpha^k + 2\beta^k \alpha^k)}{x \alpha^2 \beta^k U_1 U_2 U_3 (\alpha - \beta)^3} - \frac{f(x \lambda^3 \beta^k)(\alpha^3 \beta^k + \beta^3 \alpha^k + 2\beta^k \alpha^k)}{x \alpha^2 \beta^k U_1 U_2 U_3 (\alpha - \beta)^3} \]
\[
+ \frac{f(x \lambda^2 \beta^2 k)(\alpha^3 \beta^k + \beta^3 \alpha^k + 2\beta^k \alpha^k)}{x \alpha^2 \beta^k U_1 U_2 U_3 (\alpha - \beta)^3},
\]
\[
c_3 = \frac{f(x \lambda^2 \beta^3 k)(\alpha^2 \beta^k + \beta^2 \alpha^k + 2\beta^k \alpha^k)}{x \alpha^2 \beta^k U_1 U_2 U_3 (\alpha - \beta)^3} - \frac{f(x \lambda^3 \beta^k)(\alpha^3 \beta^k + \beta^3 \alpha^k + 2\beta^k \alpha^k)}{x \alpha^2 \beta^k U_1 U_2 U_3 (\alpha - \beta)^3} \]
\[
+ \frac{f(x \lambda^2 \beta^2 k)(\alpha^3 \beta^k + \beta^3 \alpha^k + 2\beta^k \alpha^k)}{x \alpha^2 \beta^k U_1 U_2 U_3 (\alpha - \beta)^3}.
\]

The symmetry in these expressions emerges if we compare the coefficients of \( f(x \alpha^3 \beta^k) \) and \( f(x \beta^3 \alpha^k) \) and the coefficients of \( f(x \alpha^2 \beta^k) \) and \( f(x \alpha \beta^2 \alpha^k) \).

Now, if we consider (3.1) and (3.2) and the expressions for the entries of \( A^n \), and equate entries in the (4, 1) position, we obtain

\[
\sum_{n=0}^{\infty} a_n x^n U_{4n} = c_0 U_1^3 + c_2 x U_2^3 + c_3 x^3 U_3^3.
\]

Finally, with the values of \( c_1, c_2, \) and \( c_3 \) obtained above, we obtain, with much needed help from the software package "Mathematica":

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\[ \sum_{n=0}^{\infty} a_n x^n U_{kn}^3 = \frac{f(x\alpha^{3k}) - 3f(x\alpha^{2k}\beta^k) + 3f(x\alpha^k\beta^{2k}) - f(x\beta^{3k})}{(\alpha - \beta)^3}. \]  
\[ \text{(3.4)} \]

In precisely the same manner, we equate appropriate entries in (3.1) and (3.2) to obtain

\[ \sum_{n=0}^{\infty} a_n x^n U_{kn}^2 U_{kn+1} \]
\[ = \frac{\alpha f(x\alpha^{3k}) - (2\alpha + \beta) f(x\alpha^{2k}\beta^k) + (\alpha + 2\beta) f(x\alpha^k\beta^{2k}) - \beta f(x\beta^{3k})}{(\alpha - \beta)^3}. \]
\[ \text{(3.5)} \]

\[ \sum_{n=0}^{\infty} a_n x^n U_{kn}^3 U_{kn+1} \]
\[ = \frac{\alpha^2 f(x\alpha^{3k}) - (\alpha^2 + 2\alpha\beta) f(x\alpha^{2k}\beta^k) + (\beta^2 + 2\alpha\beta) f(x\alpha^k\beta^{2k}) - \beta^2 f(x\beta^{3k})}{(\alpha - \beta)^3}. \]
\[ \text{(3.6)} \]

\[ \sum_{n=0}^{\infty} a_n x^n U_{kn+1} \]
\[ = \frac{\alpha^3 f(x\alpha^{3k}) - 3\alpha^2 \beta f(x\alpha^{2k}\beta^k) + 3\alpha\beta^2 f(x\alpha^k\beta^{2k}) - \beta^3 f(x\beta^{3k})}{(\alpha - \beta)^3}. \]
\[ \text{(3.7)} \]

\[ \sum_{n=0}^{\infty} a_n x^n U_{kn-1} U_{kn}^2 \]
\[ = \frac{\beta f(x\alpha^{3k}) - (\alpha + 2\beta) f(x\alpha^{2k}\beta^k) + (2\alpha + \beta) f(x\alpha^k\beta^{2k}) - \alpha f(x\beta^{3k})}{\alpha \beta (\alpha - \beta)^3}. \]
\[ \text{(3.8)} \]

\[ \sum_{n=0}^{\infty} a_n x^n (U_{kn}^3 + 2U_{kn-1} U_{kn} U_{kn+1}) \]
\[ = \frac{3\alpha \beta (f(x\alpha^{3k}) - f(x\beta^{3k})) - (\alpha + 2\beta)(2\alpha + \beta)(f(x\alpha^{2k}\beta^k) - f(x\alpha^k\beta^{2k}))}{\alpha \beta (\alpha - \beta)^3}. \]
\[ \text{(3.9)} \]

\[ \sum_{n=0}^{\infty} a_n x^n (2U_{kn+1}^2 U_{kn} U_{kn+1}^2) \]
\[ = \frac{3\alpha^2 \beta f(x\alpha^{3k}) - \alpha (\alpha + 2\beta)^2 f(x\alpha^{2k}\beta^k) + \beta (2\alpha + \beta)^2 f(x\alpha^k\beta^{2k}) - 3\alpha^2 \beta^2 f(x\beta^{3k})}{\alpha \beta (\alpha - \beta)^3}. \]
\[ \text{(3.10)} \]

\[ \sum_{n=0}^{\infty} a_n x^n U_{kn-1} U_{kn} \]
\[ = \frac{\beta^2 f(x\alpha^{3k}) - \beta (2\alpha + \beta) f(x\alpha^{2k}\beta^k) + \alpha (\alpha + 2\beta) f(x\alpha^k\beta^{2k}) - \alpha^2 f(x\beta^{3k})}{\alpha^2 \beta^3 (\alpha - \beta)^3}. \]
\[ \text{(3.11)} \]

\[ \sum_{n=0}^{\infty} a_n x^n (2U_{kn}^2 U_{kn-1} + U_{kn-1} U_{kn+1}^2) \]
\[ = \frac{3\alpha \beta^2 f(x\alpha^{3k}) - \beta (2\alpha + \beta)^2 f(x\alpha^{2k}\beta^k) + \alpha (\alpha + 2\beta)^2 f(x\alpha^k\beta^{2k}) - 3\alpha^2 \beta f(x\beta^{3k})}{\alpha^2 \beta^3 (\alpha - \beta)^3}. \]
\[ \text{(3.12)} \]
\[
\sum_{n=0}^{\infty} a_n x^n U_{kn}^3
= \beta^3 f(x^3) - 3\alpha \beta^2 f(x^2 \beta) + 3\alpha^2 \beta f(x \alpha \beta^2) - \alpha^3 f(x \beta^3)
\]
\[
\frac{\alpha^3 \beta^3 (\alpha - \beta)^3}{\alpha^3 (\alpha - \beta)^3}
\]

From (3.4) and (3.9), we obtain
\[
\sum_{n=0}^{\infty} a_n x^n U_{kn+1} U_{kn}^2
= \alpha \beta (f(x^3) - f(x \beta^3)) - \alpha^2 + \alpha \beta + \beta^2 (f(x^2 \beta^2) - f(x \alpha \beta^2))
\]
\[
\alpha \beta (\alpha - \beta)^3
\]

Similarly, (3.5) and (3.10) and then (3.8) and (3.12) yield, respectively,
\[
\sum_{n=0}^{\infty} a_n x^n U_{kn} U_{kn+1}^2
= \alpha^2 \beta f(x^3) - \alpha^2 + 2\beta^2 f(x^2 \beta^2) + \beta (2\alpha^2 + \beta^2) (f(x^2 \beta^2) - f(x \alpha \beta^2)) - \alpha^2 \beta f(x \beta^3)
\]
\[
\alpha^2 \beta (\alpha - \beta)^3
\]

Finally, from (1.2), we have
\[
V_{kn}^3 = U_{kn+1}^3 - 3q U_{kn+1} U_{kn-1} + 3q^2 U_{kn+1} U_{kn-1} - q^3 U_{kn-1}
\]
This, together with (3.7), (3.13), (3.15), and (3.16), yields
\[
\sum_{n=0}^{\infty} a_n x^n V_{kn}^3
= f(x^3) + 3f(x^2 \beta^2) + 3f(x \alpha \beta^2) + f(x \beta^3)
\]
(3.17)
after some tedious manipulation involving the use of the equality \(\alpha \beta = q\).

4. APPLICATIONS

We now specialize (3.4) and (3.17) to the Chebyshev polynomials to obtain some attractive sums involving third powers of the sine and cosine functions.

Let \(\{T_n(t)\}_{n=0}^{\infty}\) and \(\{S_n(t)\}_{n=0}^{\infty}\) denote the Chebyshev polynomials of the first and second kinds, respectively. Then
\[
S_n(t) = \left\{ \begin{array}{ll}
\sin n\theta & n \geq 0,
\end{array} \right.
T_n(t) = \cos n\theta,
\]

Indeed, \(\{S_n(t)\}_{n=0}^{\infty}\) and \(\{2T_n(t)\}_{n=0}^{\infty}\) are the sequences \(\{U_n\}_{n=0}^{\infty}\) and \(\{V_n\}_{n=0}^{\infty}\), respectively, generated by (1.1), where \(p = 2 \cos \theta\) and \(q = 1\). Thus,
\[
\alpha = \cos \theta + i \sin \theta = e^{i\theta} \quad \text{and} \quad \beta = \cos \theta - i \sin \theta = e^{-i\theta},
\]

which are obtained from (1.3). Further information about Chebyshev polynomials can be found, for example, in [1].

We use the following well-known power series, each of which has the complex plane as its domain of convergence:

\[
\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!},
\]

(4.1)

\[
\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!},
\]

(4.2)

\[
\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!},
\]

(4.3)

\[
cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.
\]

(4.4)

Now, in (3.4), taking \(U_n = \sin n\theta / \sin \theta\) and replacing \(f\) by the functions in (4.1)-(4.4), we obtain, after replacing all occurrences of \(k\theta\) by \(\phi\),

\[
\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \sin^3(2n+1)\phi}{(2n+1)!} = \frac{3 \cos(x \cos \phi) \sinh(x \sin \phi) - \cos(x \cos 3\phi) \sin(x \sin 3\phi)}{4},
\]

(4.5)

\[
\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \sin^3 2n\phi}{(2n)!} = -3 \sin(x \cos \phi) \sin(x \sin \phi) + \sin(x \cos 3\phi) \sin(x \sin 3\phi),
\]

(4.6)

\[
\sum_{n=0}^{\infty} \frac{x^{2n+1} \sin^3(2n+1)\phi}{(2n+1)!} = \frac{3 \cosh(x \cos \phi) \sin(x \sin \phi) - \cosh(x \cos 3\phi) \sin(x \sin 3\phi)}{4},
\]

(4.7)

\[
\sum_{n=0}^{\infty} \frac{x^{2n} \sin^3 2n\phi}{(2n)!} = \frac{3 \sinh(x \cos \phi) \sin(x \sin \phi) - \sinh(x \cos 3\phi) \sin(x \sin 3\phi)}{4}.
\]

(4.8)

Similarly, in (3.17), taking \(V_n = 2 \cos n\theta\) and replacing \(f\) by the functions in (4.1)-(4.4), we obtain, respectively,

\[
\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \cos^3(2n+1)\phi}{(2n+1)!} = \frac{3 \sin(x \cos \phi) \cosh(x \sin \phi) + \sin(x \cos 3\phi) \cosh(x \sin 3\phi)}{4},
\]

(4.9)

\[
\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \cos^3 2n\phi}{(2n)!} = \frac{3 \cos(x \cos \phi) \cosh(x \sin \phi) + \cos(x \cos 3\phi) \cosh(x \sin 3\phi)}{4},
\]

(4.10)

\[
\sum_{n=0}^{\infty} \frac{x^{2n+1} \cos^3(2n+1)\phi}{(2n+1)!} = \frac{3 \sinh(x \cos \phi) \cosh(x \sin \phi) + \sinh(x \cos 3\phi) \cosh(x \sin 3\phi)}{4},
\]

(4.11)

\[
\sum_{n=0}^{\infty} \frac{x^{2n} \cos^3 2n\phi}{(2n)!} = \frac{3 \cosh(x \cos \phi) \cos(x \sin \phi) + \cosh(x \cos 3\phi) \cos(x \sin 3\phi)}{4}.
\]

(4.12)

Finally, we mention that much of the tedious algebra in this paper was accomplished with the help of "Mathematica".
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