LUCAS SEQUENCES AND FUNCTIONS OF A 4-BY-4 MATRIX

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1. INTRODUCTION

Define the sequences $\{U_n\}$ and $\{V_n\}$ for all integers n by

$$\begin{cases} U_n = pU_{n-1} - qU_{n-2}, & U_0 = 0,\ U_1 = 1, \\ V_n = pV_{n-1} - qV_{n-2}, & V_0 = 2,\ V_1 = p, \end{cases} \tag{1.1}$$

where p and q are real numbers with $q(p^2-4q) \neq 0$. These sequences were studied originally by Lucas [6], and have subsequently been the subject of much attention.

The Binet forms for U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n, \tag{1.2}$$

where

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2} \tag{1.3}$$

are the roots, assumed distinct, of $x^2 - px + q = 0$. We assume further that α / β is not an n^{th} root of unity for any n.

A well-known relationship between U_n and V_n is

$$V_n = U_{n+1} - qU_{n-1}, \tag{1.4}$$

which we use subsequently.

Recently, Melham [7] considered functions of a 3-by-3 matrix and obtained infinite sums involving squares of terms from the sequences (1.1). Here, using a similarly defined 4-by-4 matrix, we obtain new infinite sums involving cubes, and other terms of degree three, from the sequences (1.1). For example, closed expressions for

$$\sum_{n=0}^{\infty} \frac{U_n^3}{n!} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{U_n^2 U_{n+1}}{n!}$$

arise as special cases of results in Section 3 [see (3.4) and (3.5)]. Since the above mentioned paper of Melham contains a comprehensive list of references, we have chosen not to repeat them here.

Unfortunately, one of the matrices which we need to record does not fit comfortably on a standard page. We overcome this difficulty by simply listing elements in a table. Following convention, the (i, j) element is the element in the i^{th} row and j^{th} column.

2. THE MATRIX $A_{k,x}$

By lengthy but straightforward induction on n, it can be shown that the 4-by-4 matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & -q^3 \\ 0 & 0 & q^2 & 3pq^2 \\ 0 & -q & -2pq & -3p^2q \\ 1 & p & p^2 & p^3 \end{pmatrix} \tag{2.1}$$

is such that, for nonnegative integers n, A^n is as follows:

$$\begin{pmatrix} -q^3U_{n-1}^3 & -q^3U_{n-1}^2U_n & -q^3U_{n-1}U_n^2 & -q^3U_n^3 \\ 3q^2U_{n-1}^2U_n & q^2(2U_n^2U_{n-1}+U_{n+1}U_{n-1}^2) & q^2(U_n^3+2U_{n-1}U_nU_{n+1}) & 3q^2U_n^2U_{n+1} \\ -3qU_{n-1}U_n^2 & -q(U_n^3+2U_{n-1}U_nU_{n+1}) & -q(2U_n^2U_{n+1}+U_{n-1}U_{n+1}^2) & -3qU_nU_{n+1}^2 \\ U_n^3 & U_n^2U_{n+1} & U_nU_{n+1}^2 & U_{n+1}^3 \end{pmatrix}.$$

To complete the proof by induction, we make repeated use of the recurrence for $\{U_n\}$. For example, performing the inductive step for the (2, 2) position, we have

$$\begin{aligned} &-q^3(U_n^3+2U_{n-1}U_nU_{n+1})+3pq^2U_n^2U_{n+1} \\ &= q^2U_n[U_n(-qU_n)+2U_{n+1}(-qU_{n-1})+3pU_nU_{n+1}] \\ &= q^2U_n[U_n(U_{n+2}-pU_{n+1})+2U_{n+1}(U_{n+1}-pU_n)+3pU_nU_{n+1}] \\ &= q^2U_n[2U_{n+1}^2+U_nU_{n+2}] \\ &= q^2[2U_{n+1}^2U_n+U_{n+2}U_n^2], \text{ which is the required expression.} \end{aligned}$$

When $p=1$ and $q=-1$, the matrix A becomes

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

which is a 4-by-4 *Fibonacci matrix*. Other 4-by-4 Fibonacci matrices have been studied, for example, in [3] and [4].

The characteristic equation of A is

$$\lambda^4 - p(p^2-2q)\lambda^3 + q(p^2-2q)(p^2-q)\lambda^2 - pq^3(p^2-2q)\lambda + q^6 = 0.$$

Since $p=\alpha+\beta$ and $q=\alpha\beta$, it is readily verified that α^3, $\alpha^2\beta$, $\alpha\beta^2$, and β^3 are the eigenvalues λ_j $(j=1,2,3,4)$ of A. These eigenvalues are nonzero and distinct because of our assumptions in Section 1.

Associated with A, we define the matrix $A_{k,x}$ by

$$A_{k,x} = xA^k, \tag{2.2}$$

where x is an arbitrary real number and k is a nonnegative integer. From the definition of an eigenvalue, it follows immediately that $x\alpha^{3k}$, $x\alpha^{2k}\beta^k$, $x\alpha^k\beta^{2k}$, and $x\beta^{3k}$ are the eigenvalues of $A_{k,x}$. Again, they are nonzero and distinct.

3. THE MAIN RESULTS

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series whose domain of convergence includes the eigenvalues of $A_{k,x}$. Then we have, from (2.2),

$$f(A_{k,x})=\sum_{n=0}^{\infty}a_nA_{k,x}^n=\sum_{n=0}^{\infty}a_nx^nA^{kn}. \tag{3.1}$$

The final sum in (3.1) can be expressed as a 4-by-4 matrix whose entries we record in the following table.

(i, j)	(i, j) element of $f(A_{k,x})$
(1, 1)	$-q^3\sum_{n=0}^{\infty}a_nx^nU_{kn-1}^3$
(1, 2)	$-q^3\sum_{n=0}^{\infty}a_nx^nU_{kn-1}^2U_{kn}$
(1, 3)	$-q^3\sum_{n=0}^{\infty}a_nx^nU_{kn-1}U_{kn}^2$
(1, 4)	$-q^3\sum_{n=0}^{\infty}a_nx^nU_{kn}^3$
(2, 1)	$3q^2\sum_{n=0}^{\infty}a_nx^nU_{kn-1}^2U_{kn}$
(2, 2)	$q^2\sum_{n=0}^{\infty}a_nx^n(2U_{kn}^2U_{kn-1}+U_{kn+1}U_{kn-1}^2)$
(2, 3)	$q^2\sum_{n=0}^{\infty}a_nx^n(U_{kn}^3+2U_{kn-1}U_{kn}U_{kn+1})$
(2, 4)	$3q^2\sum_{n=0}^{\infty}a_nx^nU_{kn}^2U_{kn+1}$
(3, 1)	$-3q\sum_{n=0}^{\infty}a_nx^nU_{kn-1}U_{kn}^2$
(3, 2)	$-q\sum_{n=0}^{\infty}a_nx^n(U_{kn}^3+2U_{kn-1}U_{kn}U_{kn+1})$
(3, 3)	$-q\sum_{n=0}^{\infty}a_nx^n(2U_{kn}^2U_{kn+1}+U_{kn-1}U_{kn+1}^2)$
(3, 4)	$-3q\sum_{n=0}^{\infty}a_nx^nU_{kn}U_{kn+1}^2$
(4, 1)	$\sum_{n=0}^{\infty}a_nx^nU_{kn}^3$
(4, 2)	$\sum_{n=0}^{\infty}a_nx^nU_{kn}^2U_{kn+1}$
(4, 3)	$\sum_{n=0}^{\infty}a_nx^nU_{kn}U_{kn+1}^2$
(4, 4)	$\sum_{n=0}^{\infty}a_nx^nU_{kn+1}^3$

On the other hand, from the theory of functions of matrices ([2] and [5]), it is known that

$$f(A_{k,x}) = c_0 I + c_1 A_{k,x} + c_2 A_{k,x}^2 + c_3 A_{k,x}^3, \tag{3.2}$$

where I is the 4-by-4 identity matrix, and where c_0, c_1, c_2, and c_3 can be obtained by solving the system

$$\begin{cases} c_0 + c_1 x\alpha^{3k} + c_2 x^2\alpha^{6k} + c_3 x^3\alpha^{9k} = f(x\lambda_1^k) = f(x\alpha^{3k}), \\ c_0 + c_1 x\alpha^{2k}\beta^k + c_2 x^2\alpha^{4k}\beta^{2k} + c_3 x^3\alpha^{6k}\beta^{3k} = f(x\lambda_2^k) = f(x\alpha^{2k}\beta^k), \\ c_0 + c_1 x\alpha^k\beta^{2k} + c_2 x^2\alpha^{2k}\beta^{4k} + c_3 x^3\alpha^{3k}\beta^{6k} = f(x\lambda_3^k) = f(x\alpha^k\beta^{2k}), \\ c_0 + c_1 x\beta^{3k} + c_2 x^2\beta^{6k} + c_3 x^3\beta^{9k} = f(x\lambda_4^k) = f(x\beta^{3k}). \end{cases}$$

With the use of Cramer's rule, and making use of the Binet form for U_n, we obtain, after much tedious algebra,

$$\begin{aligned} c_0 = {} & \frac{-f(x\alpha^{3k})\beta^{6k}}{U_k U_{2k} U_{3k}(\alpha-\beta)^3} + \frac{f(x\alpha^{2k}\beta^k)\alpha^k\beta^{3k}}{U_k^2 U_{2k}(\alpha-\beta)^3} \\ & - \frac{f(x\alpha^k\beta^{2k})\alpha^{3k}\beta^k}{U_k^2 U_{2k}(\alpha-\beta)^3} + \frac{f(x\beta^{3k})\alpha^{6k}}{U_k U_{2k} U_{3k}(\alpha-\beta)^3}, \end{aligned}$$

$$\begin{aligned} c_1 = {} & \frac{f(x\alpha^{3k})\beta^{3k}(\alpha^{2k}+\beta^{2k}+\alpha^k\beta^k)}{x\alpha^{2k}U_k U_{2k} U_{3k}(\alpha-\beta)^3} - \frac{f(x\alpha^{2k}\beta^k)(\alpha^{3k}+\beta^{3k}+\alpha^{2k}\beta^k)}{x\alpha^{2k}U_k^2 U_{2k}(\alpha-\beta)^3} \\ & + \frac{f(x\alpha^k\beta^{2k})(\alpha^{3k}+\beta^{3k}+\alpha^k\beta^{2k})}{x\beta^{2k}U_k^2 U_{2k}(\alpha-\beta)^3} - \frac{f(x\beta^{3k})\alpha^{3k}(\alpha^{2k}+\beta^{2k}+\alpha^k\beta^k)}{x\beta^{2k}U_k U_{2k} U_{3k}(\alpha-\beta)^3}, \end{aligned}$$

$$\begin{aligned} c_2 = {} & \frac{-f(x\alpha^{3k})\beta^k(\alpha^{2k}+\beta^{2k}+\alpha^k\beta^k)}{x^2\alpha^{3k}U_k U_{2k} U_{3k}(\alpha-\beta)^3} + \frac{f(x\alpha^{2k}\beta^k)(\alpha^{3k}+\beta^{3k}+\alpha^k\beta^{2k})}{x^2\alpha^{3k}\beta^{2k}U_k^2 U_{2k}(\alpha-\beta)^3} \\ & - \frac{f(x\alpha^k\beta^{2k})(\alpha^{3k}+\beta^{3k}+\alpha^{2k}\beta^k)}{x^2\alpha^{2k}\beta^{3k}U_k^2 U_{2k}(\alpha-\beta)^3} + \frac{f(x\beta^{3k})\alpha^k(\alpha^{2k}+\beta^{2k}+\alpha^k\beta^k)}{x^2\beta^{3k}U_k U_{2k} U_{3k}(\alpha-\beta)^3}, \end{aligned}$$

$$\begin{aligned} c_3 = {} & \frac{f(x\alpha^{3k})}{x^3\alpha^{3k}U_k U_{2k} U_{3k}(\alpha-\beta)^3} - \frac{f(x\alpha^{2k}\beta^k)}{x^3\alpha^{3k}\beta^{2k}U_k^2 U_{2k}(\alpha-\beta)^3} \\ & + \frac{f(x\alpha^k\beta^{2k})}{x^3\alpha^{2k}\beta^{3k}U_k^2 U_{2k}(\alpha-\beta)^3} - \frac{f(x\beta^{3k})}{x^3\beta^{3k}U_k U_{2k} U_{3k}(\alpha-\beta)^3}. \end{aligned}$$

The symmetry in these expressions emerges if we compare the coefficients of $f(x\alpha^{3k})$ and $f(x\beta^{3k})$ and the coefficients of $f(x\alpha^{2k}\beta^k)$ and $f(x\alpha^k\beta^{2k})$.

Now, if we consider (3.1) and (3.2) and the expressions for the entries of A^n, and equate entries in the (4, 1) position, we obtain

$$\sum_{n=0}^{\infty} a_n x^n U_{kn}^3 = c_1 x U_k^3 + c_2 x^2 U_{2k}^3 + c_3 x^3 U_{3k}^3. \tag{3.3}$$

Finally, with the values of c_1, c_2, and c_3 obtained above, we obtain, with much needed help from the software package "Mathematica":

$$\sum_{n=0}^{\infty} a_n x^n U_{kn}^3 = \frac{f(x\alpha^{3k}) - 3f(x\alpha^{2k}\beta^k) + 3f(x\alpha^k\beta^{2k}) - f(x\beta^{3k})}{(\alpha-\beta)^3}. \tag{3.4}$$

In precisely the same manner, we equate appropriate entries in (3.1) and (3.2) to obtain

$$\sum_{n=0}^{\infty} a_n x^n U_{kn}^2 U_{kn+1} = \frac{\alpha f(x\alpha^{3k}) - (2\alpha+\beta) f(x\alpha^{2k}\beta^k) + (\alpha+2\beta) f(x\alpha^k\beta^{2k}) - \beta f(x\beta^{3k})}{(\alpha-\beta)^3}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn} U_{kn+1}^2 = \frac{\alpha^2 f(x\alpha^{3k}) - (\alpha^2+2\alpha\beta) f(x\alpha^{2k}\beta^k) + (\beta^2+2\alpha\beta) f(x\alpha^k\beta^{2k}) - \beta^2 f(x\beta^{3k})}{(\alpha-\beta)^3}, \tag{3.6}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn+1}^3 = \frac{\alpha^3 f(x\alpha^{3k}) - 3\alpha^2\beta f(x\alpha^{2k}\beta^k) + 3\alpha\beta^2 f(x\alpha^k\beta^{2k}) - \beta^3 f(x\beta^{3k})}{(\alpha-\beta)^3}, \tag{3.7}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1} U_{kn}^2 = \frac{\beta f(x\alpha^{3k}) - (\alpha+2\beta) f(x\alpha^{2k}\beta^k) + (2\alpha+\beta) f(x\alpha^k\beta^{2k}) - \alpha f(x\beta^{3k})}{\alpha\beta(\alpha-\beta)^3}, \tag{3.8}$$

$$\sum_{n=0}^{\infty} a_n x^n (U_{kn}^3 + 2U_{kn-1}U_{kn}U_{kn+1}) = \frac{3\alpha\beta(f(x\alpha^{3k}) - f(x\beta^{3k})) - (\alpha+2\beta)(2\alpha+\beta)(f(x\alpha^{2k}\beta^k) - f(x\alpha^k\beta^{2k}))}{\alpha\beta(\alpha-\beta)^3}, \tag{3.9}$$

$$\sum_{n=0}^{\infty} a_n x^n (2U_{kn}^2 U_{kn+1} + U_{kn-1}U_{kn+1}^2) = \frac{3\alpha^2\beta f(x\alpha^{3k}) - \alpha(\alpha+2\beta)^2 f(x\alpha^{2k}\beta^k) + \beta(2\alpha+\beta)^2 f(x\alpha^k\beta^{2k}) - 3\alpha\beta^2 f(x\beta^{3k})}{\alpha\beta(\alpha-\beta)^3}, \tag{3.10}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1}^2 U_{kn} = \frac{\beta^2 f(x\alpha^{3k}) - \beta(2\alpha+\beta) f(x\alpha^{2k}\beta^k) + \alpha(\alpha+2\beta) f(x\alpha^k\beta^{2k}) - \alpha^2 f(x\beta^{3k})}{\alpha^2\beta^2(\alpha-\beta)^3}, \tag{3.11}$$

$$\sum_{n=0}^{\infty} a_n x^n (2U_{kn}^2 U_{kn-1} + U_{kn+1}U_{kn-1}^2) = \frac{3\alpha\beta^2 f(x\alpha^{3k}) - \beta(2\alpha+\beta)^2 f(x\alpha^{2k}\beta^k) + \alpha(\alpha+2\beta)^2 f(x\alpha^k\beta^{2k}) - 3\alpha^2\beta f(x\beta^{3k})}{\alpha^2\beta^2(\alpha-\beta)^3}, \tag{3.12}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1}^3 = \frac{\beta^3 f(x\alpha^{3k}) - 3\alpha\beta^2 f(x\alpha^{2k}\beta^k) + 3\alpha^2\beta f(x\alpha^k\beta^{2k}) - \alpha^3 f(x\beta^{3k})}{\alpha^3\beta^3(\alpha-\beta)^3}. \tag{3.13}$$

From (3.4) and (3.9), we obtain

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1} U_{kn} U_{kn+1} = \frac{\alpha\beta(f(x\alpha^{3k}) - f(x\beta^{3k})) - (\alpha^2+\alpha\beta+\beta^2)(f(x\alpha^{2k}\beta^k) - f(x\alpha^k\beta^{2k}))}{\alpha\beta(\alpha-\beta)^3}. \tag{3.14}$$

Similarly, (3.5) and (3.10) and then (3.8) and (3.12) yield, respectively,

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1} U_{kn+1}^2 = \frac{\alpha^2\beta f(x\alpha^{3k}) - \alpha(\alpha^2+2\beta^2) f(x\alpha^{2k}\beta^k) + \beta(2\alpha^2+\beta^2) f(x\alpha^k\beta^{2k}) - \alpha\beta^2 f(x\beta^{3k})}{\alpha\beta(\alpha-\beta)^3}, \tag{3.15}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn+1} U_{kn-1}^2 = \frac{\alpha\beta^2 f(x\alpha^{3k}) - \beta(2\alpha^2+\beta^2) f(x\alpha^{2k}\beta^k) + \alpha(\alpha^2+2\beta^2) f(x\alpha^k\beta^{2k}) - \alpha^2\beta f(x\beta^{3k})}{\alpha^2\beta^2(\alpha-\beta)^3}. \tag{3.16}$$

Finally, from (1.2), we have $V_{kn}^3 = U_{kn+1}^3 - 3qU_{kn+1}^2 U_{kn-1} + 3q^2 U_{kn+1} U_{kn-1}^2 - q^3 U_{kn-1}^3$. This, together with (3.7), (3.13), (3.15), and (3.16), yields

$$\sum_{n=0}^{\infty} a_n x^n V_{kn}^3 = f(x\alpha^{3k}) + 3f(x\alpha^{2k}\beta^k) + 3f(x\alpha^k\beta^{2k}) + f(x\beta^{3k}) \tag{3.17}$$

after some tedious manipulation involving the use of the equality $\alpha\beta = q$.

4. APPLICATIONS

We now specialize (3.4) and (3.17) to the Chebyshev polynomials to obtain some attractive sums involving third powers of the sine and cosine functions.

Let $\{T_n(t)\}_{n=0}^{\infty}$ and $\{S_n(t)\}_{n=0}^{\infty}$ denote the Chebyshev polynomials of the first and second kinds, respectively. Then

$$\left.\begin{aligned} S_n(t) &= \frac{\sin n\theta}{\sin\theta} \\ T_n(t) &= \cos n\theta \end{aligned}\right\}, \quad t = \cos\theta,\ n \ge 0.$$

Indeed, $\{S_n(t)\}_{n=0}^{\infty}$ and $\{2T_n(t)\}_{n=0}^{\infty}$ are the sequences $\{U_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$, respectively, generated by (1.1), where $p = 2\cos\theta$ and $q = 1$. Thus,

$$\alpha = \cos\theta + i\sin\theta = e^{i\theta} \quad \text{and} \quad \beta = \cos\theta - i\sin\theta = e^{-i\theta},$$

which are obtained from (1.3). Further information about Chebyshev polynomials can be found, for example, in [1].

We use the following well-known power series, each of which has the complex plane as its domain of convergence:

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \tag{4.1}$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \tag{4.2}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \tag{4.3}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}. \tag{4.4}$$

Now, in (3.4), taking $U_n = \sin n\theta / \sin\theta$ and replacing f by the functions in (4.1)-(4.4), we obtain , after replacing all occurrences of $k\theta$ by ϕ,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \sin^3(2n+1)\phi}{(2n+1)!} = \frac{3\cos(x\cos\phi)\sinh(x\sin\phi) - \cos(x\cos 3\phi)\sinh(x\sin 3\phi)}{4}, \tag{4.5}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \sin^3 2n\phi}{(2n)!} = \frac{-3\sin(x\cos\phi)\sinh(x\sin\phi) + \sin(x\cos 3\phi)\sinh(x\sin 3\phi)}{4}, \tag{4.6}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1} \sin^3(2n+1)\phi}{(2n+1)!} = \frac{3\cosh(x\cos\phi)\sin(x\sin\phi) - \cosh(x\cos 3\phi)\sin(x\sin 3\phi)}{4}, \tag{4.7}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n} \sin^3 2n\phi}{(2n)!} = \frac{3\sinh(x\cos\phi)\sin(x\sin\phi) - \sinh(x\cos 3\phi)\sin(x\sin 3\phi)}{4}. \tag{4.8}$$

Similarly, in (3.17), taking $V_n = 2\cos n\theta$ and replacing f by the functions in (4.1)-(4.4), we obtain, respectively,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \cos^3(2n+1)\phi}{(2n+1)!} = \frac{3\sin(x\cos\phi)\cosh(x\sin\phi) + \sin(x\cos 3\phi)\cosh(x\sin 3\phi)}{4}, \tag{4.9}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \cos^3 2n\phi}{(2n)!} = \frac{3\cos(x\cos\phi)\cosh(x\sin\phi) + \cos(x\cos 3\phi)\cosh(x\sin 3\phi)}{4}, \tag{4.10}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1} \cos^3(2n+1)\phi}{(2n+1)!} = \frac{3\sinh(x\cos\phi)\cos(x\sin\phi) + \sinh(x\cos 3\phi)\cos(x\sin 3\phi)}{4}, \tag{4.11}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n} \cos^3 2n\phi}{(2n)!} = \frac{3\cosh(x\cos\phi)\cos(x\sin\phi) + \cosh(x\cos 3\phi)\cos(x\sin 3\phi)}{4}. \tag{4.12}$$

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REFERENCES

1. M. Abramowitz & I. A. Stegun. *Handbook of Mathematical Functions*. New York: Dover, 1972.
2. R. Bellman. *Introduction to Matrix Analysis*. New York: McGraw-Hill, 1970.
3. O. Brugia & P. Filipponi. "Functions of the Kronecker Square of the Matrix Q." In *Applications of Fibonacci Numbers* **2**:69-76. Ed. A. N. Philippou et al. Dordrecht: Kluwer, 1988.
4. P. Filipponi. "A Family of 4-by-4 Fibonacci Matrices." *The Fibonacci Quarterly* **35.4** (1997):300-08.
5. F. R. Gantmacher. *The Theory of Matrices*. New York: Chelsea, 1960.
6. E. Lucas. "Théorie des Fonctions Numériques Simplement Periodiques." *Amer. J. Math.* **1** (1878):184-240, 289-321.
7. R. S. Melham. "Lucas Sequences and Functions of a 3-by-3 Matrix." *The Fibonacci Quarterly* **37.2** (1999):111-16.

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