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Study of Intuitionistic Fuzzy Super Matrices and Its Application in Decision Making

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Abstract Recent developments in fuzzy theory have been of great use in providing a framework for the understanding of situations involving decision-making. However, these tools have limitations, such as the fact that multi-attribute decision-making problems cannot be described in a single matrix. Fuzzy and intuitionistic fuzzy matrices are important tools for these types of problems since they can help to solve them. We presented a new super matrix theory in the intuitionistic fuzzy environment in order to overcome these restrictions. This theory is able to readily cope with problems that include numerous attributes while addressing belongingness and non-belonging criteria. Hence, it introduces a fresh perspective into our thinking, which in turn enables us to generalize our findings and arrive at more sound conclusions. For the purpose of theoretical development, we define a variety of different kinds of intuitionistic fuzzy super matrices and present a number of essential algebraic operations in order to make it more applicable to situations that take place in the real world. One multi-criteria decision-making problem based on super matrix theory is discussed here for the sake of validating and illustrating the applicability of the established findings. In addition to this, we suggest a general multi-criteria decision-making algorithm that makes use of intuitionistic fuzzy super matrix theory. This algorithm is more dynamic than both intuitionistic fuzzy matrix and fuzzy super matrix theories, and it can be applied to the resolution of a wide range of issues. The validation of the proposed theory is done by taking a real-world example to show its importance.

Keywords Fuzzy Matrices, Intuitionistic Fuzzy Matrices, Intuitionistic Fuzzy Super Matrices, Multi-criteria Decision Making

1. Introduction

Matrix theory is an important tool in mathematics as it is utilized to express numerous forms of relations between the objects and their attributes in several fields of research including science, engineering, medical, and many others. Generally, this relationship has not existed in the form of a binary relation that is entries of these matrices are not 0 or 1, but its membership values lie between 0 and 1. In fact, in almost all real-world problems, we receive imprecise and vague data and it is very important to study them as it provides useful information. For the study of such data efficiently and effectively, we require a certain technique that handles uncertainties and vagueness. Several theories have evolved over time to deal with the various types of uncertainties, imprecision, and vagueness that occur in real-world problems, including probability theory, Zadeh's fuzzy set theory [1], Atanassov Intuitionistic fuzzy set theory [2], Rough set theory introduced by Pawlak [3], neutrosophic set theory as an extension of intuitionistic fuzzy set given by Smarandache [4] and many others, which are defined and their applications are discussed (see for instance [5-14]). These theories were developed in order to address their inherent limitations. For example, fuzzy set, which is an extension of the conventional notion of the set, deals with vagueness and provides the degree of belongingness or membership value of an object within an interval [0,1]. We can say, this is a broad interpretation of real-world problems, based on degrees of truth rather than the typical Boolean logic which is true/false or 1/0. Initially, fuzzy matrix algebra [15-16] was proposed to solve fuzzy relations. But fuzzy set theory provides the information of the membership value (degree of belongingness of any data in a matrix) only and it cannot provide a valid answer for non-membership value (degree of non-belongingness of any data in a matrix). Because of these limitations in dealing with uncertainties by fuzzy sets, Atanassov in 1986 introduced the theory of intuitionistic fuzzy sets as a generalization of fuzzy sets that contain both membership and non-membership values. New theories using intuitionistic fuzzy sets have been developed along with their operations [17-20]. Thomason [21], for the first time, described fuzzy matrices, whose elements are drawn from the unit interval [0,1], discussing the convergence of powers of a fuzzy matrix. Pal et al. [13] investigated and created the idea of intuitionistic fuzzy matrices (IFMs) as an extension of fuzzy matrices. At present, many theories exist that give novel theoretical structures for intuitionistic fuzzy matrices (IFMs) and their operations under pre-defined constraints (see [22-27]). These ideas are quite helpful in dealing with decision-making problems. Some of the applications of these ideas may be found in [19,26,28-37], which span from scientific problems to real-life decision-making scenarios with multiple criteria. Intuitionistic fuzzy matrices (IFMs) have been proposed to describe intuitionistic fuzzy relations on finite universes with more or less imprecise relationships between components. A membership and non-membership value that reflects positive and negative characteristics of the presented data make up an intuitionistic fuzzy matrix. A more general class of matrix is the super matrix, in which entries are itself matrices containing elements that can be scalars or other matrices. Using this idea of super matrix Kalra and Khan [14] provided fuzzy super matrix theory in 2010.

The concept of fuzzy super matrix theory motivates us to extend the fuzzy super matrix theory in an intuitionistic environment. Thus, we have provided some novel generalized intuitionistic fuzzy super matrix concepts and their application in the multi-criteria decision-making problem to justify the usefulness of the theory. The following is the flow of the paper: preliminaries related to fuzzy matrices are presented in section 2, and intuitionistic fuzzy super matrix theory with fundamental algebraic operations is established in section 3. Finally, the use of intuitionistic fuzzy super matrices in a decision-making problem is demonstrated in section 4. Lastly, we have concluded our work by discussing its usefulness.

2. Preliminaries

In this section, we present a list of several basic definitions related to fuzzy set, fuzzy matrix, intuitionistic fuzzy matrix and super matrix that are well known in the literature and helpful in establishing our results.

Definition 2.1. Fuzzy Sets [1]

If U is a collection of objects denoted generically by x, and $\mu_A(x)$ is the membership value in the interval [0,1], then a fuzzy set A in U is a set of ordered pairs defined as:

$$A = \{(x, \mu_A(x)) | \forall x \in U\}.$$

Definition: 2.2. Support [1]

The support of a fuzzy set A is the set of all points in the universe of discourse U such that the membership function of A is non-zero.

Definition: 2.3. Core [1]

The core of a fuzzy set A is the crisp set of all points in the universe of discourse U such that the membership function of A is 1.

$$core \quad A = \{\mu_A(x) = 1, \qquad \forall x \in U\}.$$

Definition 2.4. Convex Fuzzy Set [1]

Let $x_1, x_2 \in R$, a fuzzy set A is convex, if $\forall \lambda \in [0,1]$, we have

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \qquad \mu_A(x_2)).$$

Definition 2.5. Height of a Fuzzy Set [1]

The height of a fuzzy set A is defined as:

$$h(A) = \max(\mu_A(x), \qquad x \in U).$$

Definition 2.6. Normal Fuzzy Set [1]

The fuzzy set is said to be normal if

$$h(A) = 1.$$

Definition 2.7. Fuzzy Number [1]

A fuzzy number is an extension of a regular number in the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible value has its own weight between 0 and 1. This weight is called the membership function. A fuzzy number is both convex and normal.

Definition 2.8. Intuitionistic Fuzzy Set [2]

An intuitionistic fuzzy set A over universal set U is represented by

$$A = \{(x, \mu_A(x), v_A(x)) | x \in U\},$$

where $\mu_A(x)$ represents the membership value of x in A, and $v_A(x)$ represents the non-membership value of x in A. Here, $\mu_A, v_A: U \to [0, 1]$ with the condition $0 \leq \mu_A(x) + v_A(x) \leq 1$. For each intuitionistic fuzzy subset A in U, $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$ is called hesitancy degree of x in A.

Definition 2.9. Fuzzy Matrix [12]

A matrix A is called as a fuzzy matrix, when $A = [a_{ij}]_{m\times n}$, where, $a_{ij} \in [0, 1]$ and $1 \leq i \leq m$, $1 \leq j \leq n$.

Definition 2.10. Intuitionistic Fuzzy Matrix [13]

Let $A = [a_{ij}]_{m\times n}$ be a matrix of order $m \times n$. If all its elements are intuitionistic fuzzy values, then A is called an intuitionistic fuzzy matrix.

Definition 2.11. Fuzzy Super Matrix [14]

Let us consider a fuzzy matrix

$$A = \begin{bmatrix} X_{11} & X_{12} \dots & X_{1n} \\ X_{21} & X_{22} \dots & X_{2n} \\ \vdots & \vdots & \vdots \\ X_{m1} & X_{m2} \dots & X_{mn} \end{bmatrix},$$

where, X_{ij} ($1 \leq i \leq m$ and $1 \leq j \leq n$) are fuzzy submatrices of A; with entries in the closed interval $[0, 1]$, such that number of rows in fuzzy submatrices X_{i1}, $X_{i2}, \dots, X_{in}$ for each $i = 1, 2, \dots m$ are equal and similarly number of columns in fuzzy submatrices X_{1j}, $X_{2j}, \dots, X_{mj}$ for each $j = 1, 2, \dots n$ are equal, then A is a general fuzzy super matrix.

3. Intuitionistic Fuzzy Super Matrix Theory

This section introduces the new notion of the Intuitionistic fuzzy super matrix, as well as the fundamental operations that can be performed on it.

3.1. Novel Intuitionistic Fuzzy Super Matrices

Definition 3.1.1. Intuitionistic Fuzzy Super Matrix (IFSUPM)

A super matrix $A_{p\times q} = [X_{m\times n}]$ defined over a fuzzy set IFSUPM $_{m\times n}$ is known as an Intuitionistic fuzzy super matrix if all the entries of this matrix are the Intuitionistic fuzzy submatrices of the form $[X_{ij}] \in IFSUPM_{m\times n}$ for i = 1, 2,… , m and j = 1, 2,… , n. Here $a_{ij} \in (\mu_j(c_i), \nu_j(c_i))$, where $\mu_j(c_i)$ is called membership of c_i and $\nu_j(c_i)$ is called non-membership of c_i in the given intuitionistic fuzzy super matrix. Here c_i belongs to the Universal set U.

Here every submatrix is an Intuitionistic Fuzzy Matrix, and each element of the submatrix is an intuitionistic fuzzy number of the form (α, β), where α represents the membership grade and β represents the non-membership grade in the intuitionistic fuzzy set.

Example 3.1.1.

Consider an Intuitionistic fuzzy super matrix $A_{2\times 3}$ having 2 rows and 3 columns of submatrices given as follows:

$$A_{2\times 3} = \begin{bmatrix} (A_{11})_{2\times 2} & (A_{12})_{2\times 1} & (A_{13})_{2\times 3} \\ (A_{21})_{1\times 2} & (A_{22})_{1\times 1} & (A_{23})_{1\times 3} \end{bmatrix}$$

As an example, one can define entries for the IFSUPM as follows:

$$A_{2\times 3} = \left[\begin{array}{cc:c:ccc} (0.2, 0.5) & (0.6, 0.3) & (0.7, 0.1) & (0.1, 0.4) & (0.4, 0.5) & (0.6, 0.1) \\ (0.1, 0.8) & (0.4, 0.5) & (0.3, 0.5) & (0.5, 0.3) & (0.5, 0.1) & (0.7, 0.2) \\ \hdashline (0.3, 0.6) & (0.2, 0.4) & (0.6, 0.3) & (0.8, 0.1) & (0.9, 0.1) & (0.3, 0.4) \end{array}\right]$$

As A is a super matrix, the submatrices of the same row have the same row order and submatrices of the same column have the same column order. Here the dotted line represents the partitioning of the super matrix.

Definition 3.1.2. Intuitionistic Fuzzy Super Universal Matrix

An Intuitionistic fuzzy super matrix of order $m \times n$, denoted by U is called an Intuitionistic fuzzy super universal matrix if all its elements are $(1, 0)$.

Definition 3.1.3. Intuitionistic Fuzzy Super Row Matrix

Let us consider $A_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, and $a_{ij} \in (\mu_j(c_i), \nu_j(c_i))$ to be an Intuitionistic fuzzy super matrix.

Here, the Intuitionistic fuzzy super row matrix contains rows of order 1 $(p = 1)$ and the number of columns can be more than 1 $(q \geq 1)$. Their submatrix also contains all the rows of order 1.

An example of an Intuitionistic fuzzy super row matrix is given as follows:

$$A_{1\times 2} = [\ [(0.2, 0.4)\ (0.9, 0.1)]_{1\times 2} \ \vdots\ [(0.1, 0.5)\ (0.4, 0.3)\ (0.8, 0.1)]_{1\times 3}].$$

Definition 3.1.4. Intuitionistic Fuzzy Super Column Matrix

Let us consider $A_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, and $a_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$ to be an Intuitionistic fuzzy super matrix.

Then, an Intuitionistic fuzzy super column matrix has the column of order $1(q = 1)$, and its submatrices also have column of order 1. The number of rows can be more than 1 $(p \geq 1)$.

An example of Intuitionistic fuzzy super column matrix is given as follows:

$$A_{2\times 1} = \begin{bmatrix} \begin{bmatrix} (0.3, 0.5) \\ (0.6, 0.2) \\ (0.8, 0.1) \end{bmatrix}_{3\times 1} \\ \cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix} (0.1, 0.6) \\ (0.2, 0.7) \\ (0.5, 0.5) \\ (0.6, 0.3) \\ (0.4, 0.4) \end{bmatrix}_{5\times 1} \end{bmatrix}$$

Definition 3.1.5. Intuitionistic Fuzzy Super Rectangular Matrix

Let us consider $A_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, and $a_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$ to be an Intuitionistic fuzzy super matrix.

If $p \neq q$, then A is said to be an Intuitionistic fuzzy super rectangular matrix.

The following matrix is an example of Intuitionistic fuzzy super rectangular matrix:

$$A_{2\times 1} = \begin{bmatrix} \begin{bmatrix} (0.1, 0.6) \\ (0.2, 0.3) \end{bmatrix}_{2\times 1} \vdots \begin{bmatrix} (0.1, 0.7) \\ (0.4, 0.6) \end{bmatrix}_{2\times 1} \vdots \begin{bmatrix} (0.7, 0.1) \\ (0.3, 0.6) \end{bmatrix}_{2\times 1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ [(0.3, 0.1)]_{1\times 1} \vdots [(0.1, 0.9)]_{1\times 1} \vdots [(0.4, 0.4)]_{1\times 1} \end{bmatrix}_{2\times 1}$$

Definition 3.1.6. Intuitionistic Fuzzy Super Square Matrix

Let us consider $A_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, and $a_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$ to be an Intuitionistic fuzzy super matrix, if $p = q$, then A is said to be an Intuitionistic fuzzy super square matrix.

For example:

$$A_{2\times 2} = \begin{bmatrix} [(0.4, 0.2)]_{1\times 1} \vdots [(0.2, 0.6) \quad (0.8, 0.1)]_{1\times 2} \\ \begin{bmatrix} (0.3, 0.4) \\ (0.2, 0.1) \end{bmatrix}_{2\times 1} \vdots \begin{bmatrix} (0.6, 0.1) & (0.3, 0.6) \\ (0.2, 0.5) & (0.7, 0.1) \end{bmatrix}_{2\times 2} \end{bmatrix}$$

Remark 3.1.

The submatrices can be of any order. They can be square as well as rectangular matrices.

Definition 3.1.7. Intuitionistic Fuzzy Super Null Matrix

An Intuitionistic fuzzy super null matrix has every entry of every submatrix as (0, 1), denoted by Φ. As an example, it can be written as:

$$A_{2\times 2} = \begin{bmatrix} \begin{bmatrix} (0,1) \\ (0,1) \end{bmatrix}_{2\times 1} \vdots \begin{bmatrix} (0,1) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) \end{bmatrix}_{2\times 3} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix} (0,1) \\ (0,1) \\ (0,1) \end{bmatrix}_{3\times 1} \vdots \begin{bmatrix} (0,1) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) \end{bmatrix}_{3\times 3} \end{bmatrix}$$

Definition 3.1.8. Intuitionistic Fuzzy Super Upper Triangular Matrix

Let us consider $A_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, and $a_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$ to be an Intuitionistic fuzzy super matrix with $p = q$. An Intuitionistic fuzzy super upper triangular matrix is an Intuitionistic fuzzy super square matrix in which all submatrices below the leading diagonal have all entries as (0, 1). The remaining submatrices on the diagonal as well as above the leading diagonal can have any entry.

The example of Intuitionistic fuzzy super upper triangular matrix is as follows:

$$A_{3\times 3} = \begin{bmatrix} [(0.4, 0.5) \quad (0.2, 0.6)]_{1\times 2} \vdots [(0.1, 0.9)]_{1\times 1} \vdots [(0.8, 0.1) \quad (0.2, 0.7)]_{1\times 2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix} (0,1) & (0,1) \\ (0,1) & (0,1) \end{bmatrix}_{2\times 2} \vdots \begin{bmatrix} (0.3, 0.2) \\ (0.1, 0.2) \end{bmatrix}_{2\times 1} \vdots \begin{bmatrix} (0.6, 0.1) & (0.3, 0.6) \\ (0.2, 0.5) & (0.7, 0.1) \end{bmatrix}_{2\times 2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix} (0,1) & (0,1) \\ (0,1) & (0,1) \end{bmatrix}_{2\times 2} \vdots \begin{bmatrix} (0,1) \\ (0,1) \end{bmatrix}_{2\times 1} \vdots \begin{bmatrix} (0.2, 0.5) & (0.6, 0.3) \\ (0.1, 0.8) & (0.4, 0.5) \end{bmatrix}_{2\times 2} \end{bmatrix}.$$

Definition 3.1.9. Intuitionistic Fuzzy Super Lower Triangular Matrix

Let us consider $A_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, and $a_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$ to be an Intuitionistic fuzzy super matrix, with $p = q$. An intuitionistic fuzzy super lower triangular matrix is an Intuitionistic fuzzy super square matrix which has all the submatrices above the leading diagonal with all entries as (0, 1). The remaining submatrices on the leading diagonal as well as below the leading diagonal can have any entry.

$$A_{3\times 3} = \begin{bmatrix} [(0.2, 0.7)]_{1\times 1} & \vdots\ [(0,1)]_{1\times 1} & \vdots\ [(0,1)\ \ (0,1)]_{1\times 2} \\ \cdots & \cdots & \cdots \\ \begin{bmatrix}(0.6, 0.1)\\(0.8, 0.1)\end{bmatrix}_{2\times 1} & \vdots \begin{bmatrix}(0.3, 0.4)\\(0.1, 0.6)\end{bmatrix}_{2\times 1} & \vdots \begin{bmatrix}(0,1) & (0,1)\\(0,1) & (0,1)\end{bmatrix}_{2\times 2} \\ \cdots & \cdots & \cdots \\ [(0.5, 0.4)]_{1\times 1} & \vdots\ [(0.4, 0.5)]_{1\times 1} & \vdots\ [(0.4, 0.3)\ \ (0.1, 0.7)]_{1\times 2} \end{bmatrix}$$

Remark 3.2

The submatrices on the diagonal as well as above or below the diagonal cannot have all entries as (0, 1), otherwise it is an intuitionistic fuzzy super null matrix.

Definition 3.1.10. Intuitionistic Fuzzy Super Diagonal Matrix

Let us consider $A_{p\times q} = [X_{m\times n}]$, where, $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, and $a_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$ to be an intuitionistic fuzzy super matrix, with $p = q$. An Intuitionistic Fuzzy Super Diagonal Matrix is an intuitionistic fuzzy super square matrix where all the submatrices which are not on diagonal are intuitionistic null matrices, and none of the diagonal submatrix is an intuitionistic null matrix. The example of intuitionistic fuzzy super diagonal matrix is given below:

$$A_{3\times 3} = \begin{bmatrix} [(0.2, 0.3)]_{1\times 1} & \vdots\ [(0,1)\ \ (0,1)]_{1\times 2} & \vdots\ [(0,1)]_{1\times 1} \\ \cdots & \cdots & \cdots \\ [(0,1)]_{1\times 1} & \vdots\ [(0.9,0.1)(0.8,0.1)]_{1\times 2} & \vdots\ [(0,1)]_{1\times 1} \\ \cdots & \cdots & \cdots \\ \begin{bmatrix}(0,1)\\(0,1)\end{bmatrix}_{2\times 1} & \vdots \begin{bmatrix}(0,1)(0,1)\\(0,1)(0,1)\end{bmatrix}_{2\times 2} & \vdots \begin{bmatrix}(0.3, 0.2)\\(0.1, 0.4)\end{bmatrix}_{2\times 1} \end{bmatrix}$$

Definition 3.1.11. Intuitionistic Fuzzy Super scalar Matrix

Let us consider $A_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, and $a_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$ to be an Intuitionistic fuzzy super matrix, with $p = q$.

An Intuitionistic fuzzy super scalar matrix is an Intuitionistic fuzzy super identity matrix which has been multiplied with a scalar k where $0 \le k \le 1$.

Here is the example in which we have considered $k = 0.4$:

$$A_{3\times 3} = \begin{bmatrix} \begin{bmatrix}(0.4, 0.0) & (0,1)\\(0,1) & (0.4, 0)\end{bmatrix}_{2\times 2} & \vdots \begin{bmatrix}(0,1) & (0,1) & (0,1)\\(0,1) & (0,1) & (0,1)\end{bmatrix}_{2\times 3} & \vdots \begin{bmatrix}(0,1) & (0,1)\\(0,1) & (0,1)\end{bmatrix}_{2\times 2} \\ \cdots & \cdots & \cdots \\ \begin{bmatrix}(0,1) & (0,1)\\(0,1) & (0,1)\\(0,1) & (0,1)\end{bmatrix}_{3\times 2} & \vdots \begin{bmatrix}(0.4, 0.0) & (0,1) & (0,1)\\(0,1) & (0.4, 0.0) & (0,1)\\(0,1) & (0,1) & (0.4, 0.0)\end{bmatrix}_{3\times 3} & \vdots \begin{bmatrix}(0,1) & (0,1)\\(0,1) & (0,1)\\(0,1) & (0,1)\end{bmatrix}_{3\times 2} \\ \cdots & \cdots & \cdots \\ \begin{bmatrix}(0,1) & (0,1)\\(0,1) & (0,1)\\(0,1) & (0,1)\end{bmatrix}_{3\times 2} & \vdots \begin{bmatrix}(0,1) & (0,1) & (0,1)\\(0,1) & (0,1) & (0,1)\end{bmatrix}_{2\times 3} & \vdots \begin{bmatrix}(0.4, 0.0) & (0,1)\\(0,1) & (0.4, 0)\end{bmatrix}_{2\times 2} \end{bmatrix}$$

3.2. Algebra of Intuitionistic Fuzzy Super Matrices

3.2.1. Addition of Intuitionistic Fuzzy Super Matrices:

Let us consider two intuitionistic fuzzy super matrices $A_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, $a_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$ and $B_{p\times q} = [Y_{m\times n}]$, where $Y_{m\times n} = [b_{ij}] \in IFSUPM_{m\times n}$, $b_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$, then the addition is only defined if both A and B have the same order and their corresponding submatrices also have the same order. The addition of the submatrices is already defined by Guu et al. [16] as every submatrix is an intuitionistic fuzzy matrix.

Example 3.2.1

Let us consider two Intuitionistic fuzzy super matrices A and B having 2 rows and 2 columns given as below:

$$A_{2\times 2} = \begin{bmatrix} [(0.4, 0.3)] & \vdots\ [(0.2, 0.6) \quad (0.7, 0.1)] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix}(0.1, 0.6)\\(0.3, 0.5)\end{bmatrix} & \vdots \begin{bmatrix}(0.1, 0.7) & (0.2, 0.5)\\(0.9, 0.1) & (0.4, 0.5)\end{bmatrix} \end{bmatrix}$$

and

$$B_{2\times 2} = \begin{bmatrix} [(0.2, 0.6)] & \vdots\ [(0.5, 0.4) \quad (0.3, 0.6)] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix}(0.6, 0.1)\\(0.7, 0.2)\end{bmatrix} & \vdots \begin{bmatrix}(0.1, 0.6) & (0.6, 0.2)\\(0.4, 0.5) & (0.3, 0.7)\end{bmatrix} \end{bmatrix}.$$

Here the addition is possible because order of A and B are the same, and also the order of corresponding submatrices are the same. That is, for the above two matrices A and B, their entries A_{11} and B_{11} have order 1×1; A_{12} and B_{12} have 1×2; A_{21} and B_{21} have 2×1; and A_{22} and B_{22} have order 2×2 intuitionistic fuzzy matrices.

To illustrate the addition of two intuitionistic fuzzy super matrices, the addition of first row-first column- element is done below:

$$A_{11} + B_{11} = [(0.4, 0.3)] + [(0.2, 0.6)]$$
$$= [max(0.4, 0.2), min(0.3, 0.6)]$$
$$= [(0.4, 0.3)]$$

Similarly, we operate for other submatrices. Then the addition of two Intuitionistic fuzzy submatrices is given as,

$$A + B = \begin{bmatrix} [(0.4, 0.3) & \vdots\ [(0.5, 0.4) \quad (0.7, 0.1)]] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix}(0.6, 0.2)\\(0.7, 0.2)\end{bmatrix} & \vdots \begin{bmatrix}(0.1, 0.6) & (0.6, 0.2)\\(0.9, 0.1) & (0.4, 0.5)\end{bmatrix} \end{bmatrix}$$

Remark 3.3

For A which is an Intuitionistic fuzzy matrix, $A + A = A$ due to the max-min rule.

Similarly subtraction of Intuitionistic fuzzy super matrices is defined as:

3.2.2. Subtraction of Intuitionistic Fuzzy Super Matrix

For two Intuitionistic fuzzy super matrices $A_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, $a_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$ and $B_{p\times q} = [Y_{m\times n}]$, where $Y_{m\times n} = [b_{ij}] \in IFSUPM_{m\times n}$, $b_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$, the subtraction is defined if both A and B have the same order and their corresponding submatrices also have the same order.

Example 3.2.2

Considering same Intuitionistic fuzzy super matrices A and B, i.e.,

$$A_{2\times 2} = \begin{bmatrix} [(0.4, 0.3)] & \vdots\ [(0.2, 0.6) \quad (0.7, 0.1)] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix}(0.1, 0.6)\\(0.3, 0.5)\end{bmatrix} & \vdots \begin{bmatrix}(0.1, 0.7) & (0.2, 0.5)\\(0.9, 0.1) & (0.4, 0.5)\end{bmatrix} \end{bmatrix}$$

and

$$B_{2\times 2} = \begin{bmatrix} [(0.2, 0.6)] & \vdots\ [(0.5, 0.4) \quad (0.3, 0.6)] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix}(0.6, 0.1)\\(0.7, 0.2)\end{bmatrix} & \vdots \begin{bmatrix}(0.1, 0.6) & (0.6, 0.2)\\(0.4, 0.5) & (0.3, 0.7)\end{bmatrix} \end{bmatrix}.$$

Now the subtraction is possible because the order of A and B are the same, and the order of corresponding submatrices is also the same. In case of subtraction we follow the below procedure for each entry:

$$A_{11} - B_{11} = [(0.4, 0.3)] - [(0.2, 0.6)]$$
$$= [min(0.4, 0.2), max(0.3, 0.6)]$$
$$= [(0.2, 0.6)]$$

After calculating each entry the subtraction of two Intuitionistic fuzzy submatrices is given as,

$$A - B = \begin{bmatrix} [(0.2, 0.6) & \vdots\ [(0.2, 0.6) \quad (0.3, 0.6)]] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix}(0.1, 0.6)\\(0.3, 0.5)\end{bmatrix} & \vdots \begin{bmatrix}(0.1, 0.7) & (0.2, 0.5)\\(0.4, 0.5) & (0.3, 0.7)\end{bmatrix} \end{bmatrix}$$

To define multiplication of Intuitionistic fuzzy super matrices, we need to revisit the concept of multiplication of Intuitionistic fuzzy matrices as they are the entries in the form of submatrices in the novel concept of Intuitionistic fuzzy super matrices. It was discussed by Saw et al. [17] given as follows:

3.2.3. Multiplication of Intuitionistic Fuzzy Matrices (IFM) [17]

If $A_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$ and $B_{n\times p} = [b_{jk}] \in IFSUPM_{n\times p}$, then the multiplication of A and B is defined as $A * B = [C_{ik}]_{m\times p} = \left(min\ \left(\mu_{A_j},\ \mu_{B_j}\right),\ max\ \left(\nu_{A_j},\ \nu_{B_j}\right)\right), \forall i, j.$

An example to illustrate the multiplication of two

Intuitionistic fuzzy matrices is given below:

Example 3.2.3

Consider

$$A = \begin{bmatrix} (0.8, 0.1) & (0.4, 0.5) \\ (0.7, 0.3) & (0.4, 0.6) \end{bmatrix}_{2\times 2}$$

and

$$B = \begin{bmatrix} (0.6, 0.3) & (0.8, 0.2) \\ (0.7, 0.3) & (0.5, 0.5) \end{bmatrix}_{2\times 2}$$

as two Intuitionistic fuzzy matrices, then their product is given as follows:

$$A * B = \begin{bmatrix} (0.6, 0.3) & (0.8, 0.2) \\ (0.6, 0.3) & (0.7, 0.3) \end{bmatrix}_{2\times 2}$$

Remark 3.4

The multiplication of two Intuitionistic fuzzy matrices A and B is not commutative in general, i.e. $A * B \neq B * A$.

Before we begin with multiplication, let us point out basic features of super matrices. In a super matrix, every submatrix of a given row has the same row order and every submatrix of a given column has the same column order. Because of this, it becomes possible to define multiplication of two Intuitionistic Fuzzy Super Matrices. We define multiplication of Intuitionistic Fuzzy Super Matrices in the following way:

3.2.4. Multiplication of Intuitionistic Fuzzy Super Matrices

Let A be an Intuitionistic fuzzy super matrix with order 3×4, then A has the following form:

$$A_{3\times 4} = \begin{bmatrix} (A_{11})_{x\times a} & (A_{12})_{x\times b} & (A_{13})_{x\times c} & (A_{14})_{x\times d} \\ (A_{21})_{y\times a} & (A_{22})_{y\times b} & (A_{23})_{y\times c} & (A_{24})_{y\times d} \\ (A_{31})_{z\times a} & (A_{32})_{z\times b} & (A_{33})_{z\times c} & (A_{34})_{z\times d} \end{bmatrix}.$$

Let B be another intuitionistic fuzzy super matrix. Now for A and B to be conformable for multiplication, order of column of matrix A should be equal to the order of row of matrix B. Thus, for above matrix A, row order of the matrix B should be equal to 4. Thus we define a matrix B of order 4×3 given as follows:

$$B_{4\times 3} = \begin{bmatrix} (B_{11})_{a\times p} & (B_{12})_{a\times q} & (B_{13})_{a\times r} \\ (B_{21})_{b\times p} & (B_{22})_{b\times q} & (B_{23})_{b\times r} \\ (B_{31})_{c\times p} & (B_{32})_{c\times q} & (B_{33})_{c\times r} \\ (B_{41})_{d\times p} & (B_{42})_{d\times q} & (B_{43})_{d\times r} \end{bmatrix}$$

Now the matrices A and B are conformable to each other. Here, the product $A*B$ will have order 3×3, where we notice that addition is possible as every multiplied submatrix of that element has the same order. Thus, we get the submatrices of the following form:

$$AB_{3\times 3} = \begin{bmatrix} (S_{11})_{x\times p} & (S_{12})_{x\times q} & (S_{13})_{x\times r} \\ (S_{21})_{y\times p} & (S_{22})_{y\times q} & (S_{23})_{y\times r} \\ (S_{31})_{z\times p} & (S_{32})_{z\times q} & (S_{33})_{z\times r} \end{bmatrix}$$

where,

$$(S_{11})_{x\times p} = (A_{11})_{x\times a} \quad (B_{11})_{a\times p} + (A_{12})_{x\times b} \quad (B_{21})_{b\times p} + (A_{13})_{x\times c} \quad (B_{31})_{c\times p} + (A_{14})_{x\times d} \quad (B_{41})_{d\times p}$$

$$(S_{12})_{x\times q} = (A_{11})_{x\times a} \quad (B_{12})_{a\times q} + (A_{12})_{x\times b} \quad (B_{22})_{b\times q} + (A_{13})_{x\times c} \quad (B_{32})_{c\times q} + (A_{14})_{x\times d} \quad (B_{42})_{d\times q}$$

$$(S_{13})_{x\times r} = (A_{11})_{x\times a} \quad (B_{13})_{a\times r} + (A_{12})_{x\times b} \quad (B_{32})_{b\times r} + (A_{13})_{x\times c} \quad (B_{33})_{c\times r} + (A_{14})_{x\times d} \quad (B_{43})_{d\times r}$$

$$(S_{21})_{y\times p} = (A_{21})_{y\times a} \quad (B_{11})_{a\times p} + (A_{22})_{y\times b} \quad (B_{21})_{b\times p} + (A_{23})_{y\times c} \quad (B_{31})_{c\times p} + (A_{24})_{y\times d} \quad (B_{41})_{d\times p}$$

$$(S_{22})_{y\times q} = (A_{21})_{y\times a} \quad (B_{12})_{a\times q} + (A_{22})_{y\times b} \quad (B_{22})_{b\times q} + (A_{23})_{y\times c} \quad (B_{32})_{c\times q} + (A_{24})_{y\times d} \quad (B_{42})_{d\times q}$$

$$(S_{23})_{y\times r} = (A_{21})_{y\times a} \quad (B_{13})_{a\times r} + (A_{22})_{y\times b} \quad (B_{32})_{b\times r} + (A_{23})_{y\times c} \quad (B_{33})_{c\times r} + (A_{24})_{y\times d} \quad (B_{43})_{d\times r}$$

$$(S_{31})_{z\times p} = (A_{31})_{z\times a} \quad (B_{11})_{a\times p} + (A_{32})_{z\times b} \quad (B_{21})_{b\times p} + (A_{33})_{z\times c} \quad (B_{31})_{c\times p} + (A_{34})_{z\times d} \quad (B_{41})_{d\times p}$$

$$(S_{32})_{z\times q} = (A_{31})_{z\times a} \quad (B_{12})_{a\times q} + (A_{32})_{z\times b} \quad (B_{22})_{b\times q} + (A_{33})_{z\times c} \quad (B_{32})_{c\times q} + (A_{34})_{z\times d} \quad (B_{42})_{d\times q}$$

$$(S_{33})_{z\times r} = (A_{31})_{z\times a} \quad (B_{13})_{a\times r} + (A_{32})_{z\times b} \quad (B_{32})_{b\times r} + (A_{33})_{z\times c} \quad (B_{33})_{c\times r} + (A_{34})_{z\times d} \quad (B_{43})_{d\times r}$$

Remark: 3.5.

(i). In the above matrices a, b, c, d, x, y, z, p, q, r are used to represent the row and column order of the submatrices.

(ii). The matrix multiplication of two Intuitionistic fuzzy super matrices is not commutative in general, i.e. $AB \neq BA$.

Now we illustrate the multiplication of two Intuitionistic fuzzy super matrices.

Example 3.2.4

Let $A_{2\times 2}$ and $B_{2\times 2}$ be two Intuitionistic fuzzy super matrices given as below:

$$A_{2\times 2}=\begin{bmatrix}[(0.1,\quad 0.4)]_{1\times 1} \vdots [(0.6,0.1)\quad (0.2,0.3)]_{1\times 2}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots..\\ \begin{bmatrix}(0.2,0.2)\\(0.3,0.5)\end{bmatrix}_{2\times 1} \vdots \begin{bmatrix}(0.2,0.4) & (0.7,0.1)\\(0.1,0.6) & (0.5,0.2)\end{bmatrix}_{2\times 2}\end{bmatrix}$$

and

$$B_{2\times 2}=\begin{bmatrix}[(0.2,\quad 0.3)]_{1\times 1} \vdots [(0.4,0.2)\quad (0.5,0.5)]_{1\times 2}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots..\\ \begin{bmatrix}(0.4,0.2)\\(0.6,0.3)\end{bmatrix}_{2\times 1} \vdots \begin{bmatrix}(0.4,0.6) & (0.5,0.5)\\(0.1,0.4) & (0.4,0.1)\end{bmatrix}_{2\times 2}\end{bmatrix}$$

then,

$$AB_{2\times 2}=\begin{bmatrix}[(0.1,0.4)]+[(0.4,0.3)] \vdots [(0.1,0.4)\ \ (0.1,0.5)]+[(0.4,0.4)\ \ (0.5,0.3)]\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.\\ \begin{bmatrix}(0.2,0.3)\\(0.2,0.5)\end{bmatrix}+\begin{bmatrix}(0.6,0.3)\\(0.5,0.3)\end{bmatrix} \vdots \begin{bmatrix}(0.2,0.2) & (0.2,0.5)\\(0.3,0.5) & (0.3,0.5)\end{bmatrix}+\begin{bmatrix}(0.2,0.4) & (0.4,0.5)\\(0.1,0.4) & (0.4,0.2)\end{bmatrix}\end{bmatrix}$$

which reduces to

$$AB_{2\times 2}=\begin{bmatrix}[(0.4,0.3)] \vdots [(0.4,0.4)\quad (0.5,0.3)]\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots..\\ \begin{bmatrix}(0.6,0.3)\\(0.5,0.3)\end{bmatrix} \vdots \begin{bmatrix}(0.2,0.2) & (0.4,0.5)\\(0.3,0.4) & (0.3,0.2)\end{bmatrix}\end{bmatrix}$$

3.2.5. Equality of Intuitionistic Fuzzy Super Matrices

Two Intuitionistic Fuzzy Super Matrices are equal to each other if their orders as well as order of their submatrices are the same, and also they both have the same Intuitionistic fuzzy numbers in the same arrangement. Keeping these things in mind, we propose that for

$A_{p\times q}=[X_{m\times n}]$, where $X_{m\times n}=[a_{ij}]\in \quad IFSUPM_{m\times n}$, and $a_{ij}\in\left(\mu_j(c_i),\ \nu_j(c_i)\right)$,

$B_{p\times q}=[Y_{m\times n}]$, where $Y_{m\times n}=[b_{ij}]\in \quad IFSUPM_{m\times n}$, and $b_{ij}\in\left(\mu_j(c_i),\ \nu_j(c_i)\right)$,

$C_{p\times q}=[Z_{m\times n}]$, where $Z_{m\times n}=[c_{ij}]\in \quad IFSUPM_{m\times n}$, and $c_{ij}\in\left(\mu_j(c_i),\ \nu_j(c_i)\right)$,

we have following results related to equality:

1) If $A=B$, then $B=A$.
2) If $A=B$ and $B=C$, then $A=C$.

Remark 3.6

For $A=B$, $A-B$ is not 0 due to subtraction rules, instead it turns out that for $A=B$, $A-B \ =A$ or $A-B=B$.

Proposition 3.2.1

For, $A_{p\times q}=[X_{m\times n}]$, where $X_{m\times n}=[a_{ij}]\in IFSUPM_{m\times n}$, and $a_{ij}\in\left(\mu_j(c_i),\ \nu_j(c_i)\right)$,
$B_{p\times q}=[Y_{m\times n}]$, where $Y_{m\times n}=[b_{ij}]\in \quad IFSUPM_{m\times n}$, and $b_{ij}\in\left(\mu_j(c_i),\ \nu_j(c_i)\right)$,
$C_{p\times q}=[Z_{m\times n}]$, where $Z_{m\times n}=[c_{ij}]\in \quad IFSUPM_{m\times n}$, and $c_{ij}\in\left(\mu_j(c_i),\ \nu_j(c_i)\right)$, the following laws hold:

1) Commutative law: A+B = B+A
2) Associative law: A+(B+C) = (A+B) + C = A+B+C
3) Additive identity: $A+0=A$ (where 0 is a super matrix with same order as A with corresponding submatrices of same order with every element as $(0,1]$.

3.2.6. Scalar Multiplication of Intuitionistic Fuzzy Super Matrices

Scalar multiplication is defined for $k\in(0,1]$, which when multiplied with an Intuitionistic fuzzy super matrix A gives kA. Here, k is multiplied with each Intuitionistic fuzzy number (α,β) of every submatrix.

Example 3.2.6

Let $A_{2\times2}$ be an Intuitionistic fuzzy super matrix given as below:

$$A_{2\times2} = \begin{bmatrix} [(0.4, 0.3)] \vdots [(0.2, 0.6) \quad (0.7, 0.1)] \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.. \\ \begin{bmatrix}(0.1, 0.6)\\(0.3, 0.5)\end{bmatrix} \vdots \begin{bmatrix}(0.1, 0.7) & (0.2, 0.5)\\(0.9, 0.1) & (0.4, 0.5)\end{bmatrix} \end{bmatrix}.$$

When $k = 0.5$, then scalar multiplication is given as:

$$kA = \begin{bmatrix} [(0.2, 0.15)] \vdots [(0.1, 0.3) \quad (0.35, 0.05)] \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.. \\ \begin{bmatrix}(0.05, 0.3)\\(0.15, 0.25)\end{bmatrix} \vdots \begin{bmatrix}(0.05, 0.35) & (0.1, 0.25)\\(0.45, 0.05) & (0.2, 0.25)\end{bmatrix} \end{bmatrix}$$

Proposition 3.2.2

The following proposition holds true in the case of an Intuitionistic fuzzy super matrix: $k(gA) = (kg)A$ (kg should be in the interval $(0,1]$.

Remark 3.7

The usual identities of matrix multiplication which are mentioned below do not exist for Intuitionistic fuzzy super matrices:

(i). $k(A + B) = kA + kB$
(ii). $(k + g)A = kA + gA$ $(0 <= k + g <= 1)$
(iii). $k(AB) = (kA)B = A(kB)$

3.2.7. Intuitionistic Fuzzy Super Identity Matrix

An Intuitionistic Fuzzy Super Identity Matrix is an Intuitionistic fuzzy super square matrix, where all the submatrices on the diagonal are Intuitionistic fuzzy identity matrices, which means that all the Intuitionistic fuzzy numbers on the diagonals are (1, 0) and remaining are (0, 1).

Thus, all the submatrices on the diagonal should be square matrices as well. And the remaining submatrices which are not on the main diagonal should be intuitionistic fuzzy null matrices, which is illustrated below:

$$I_{3\times3} = \begin{bmatrix} \begin{bmatrix}(1,0) & (0,1)\\(0,1) & (1,0)\end{bmatrix}_{2\times2} \vdots \begin{bmatrix}(0,1) & (0,1) & (0,1)\\(0,1) & (0,1) & (0,1)\end{bmatrix}_{2\times3} \vdots \begin{bmatrix}(0,1) & (0,1)\\(0,1) & (0,1)\end{bmatrix}_{2\times2} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \\ \begin{bmatrix}(0,1) & (0,1)\\(0,1) & (0,1)\\(0,1) & (0,1)\end{bmatrix}_{3\times2} \vdots \begin{bmatrix}(1,0) & (0,1) & (0,1)\\(0,1) & (1,0) & (0,1)\\(0,1) & (0,1) & (1,0)\end{bmatrix}_{3\times3} \vdots \begin{bmatrix}(0,1) & (0,1)\\(0,1) & (0,1)\\(0,1) & (0,1)\end{bmatrix}_{3\times2} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \\ \begin{bmatrix}(0,1) & (0,1)\\(0,1) & (0,1)\end{bmatrix}_{2\times2} \vdots \begin{bmatrix}(0,1) & (0,1) & (0,1)\\(0,1) & (0,1) & (0,1)\end{bmatrix}_{2\times3} \vdots \begin{bmatrix}(1,0) & (0,1)\\(0,1) & (1,0)\end{bmatrix}_{2\times2} \end{bmatrix}$$

Proposition 3.2.3

Let $A_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, and $a_{ij} \in \left(\mu_j(c_i), \nu_j(c_i)\right)$ be an Intuitionistic fuzzy super matrix, then $AI = IA$, where AI and IA multiplication is conformable.

Remark 3.8

We observe that several rules of distributive property which hold true for normal matrices are invalid here, such as:

1. Associative Property: $A(BC) = (AB)C = ABC$
2. Left Distributive Property: $A(B + C) = AB + AC$
3. Right Distributive Property: $(A + B)C = AC + BC$

3.2.8. Transpose of Intuitionistic Fuzzy Super Matrix

We define transpose of an Intuitionistic fuzzy super matrix by switching the row column indices, hence flipping the matrix over its diagonal. Here, we also take the transpose of the submatrices.

The transpose of an Intuitionistic fuzzy super matrix is shown with an example given below.

Example 3.2.8

Let $A_{2\times2}$ be an Intuitionistic fuzzy super matrix

$$A_{2\times2} = \begin{bmatrix} [(0.4, 0.3)]_{1\times1} \vdots [(0.2, 0.6) \quad (0.7, 0.1)]_{1\times2} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.. \\ \begin{bmatrix}(0.1, 0.6)\\(0.3, 0.5)\end{bmatrix}_{2\times1} \vdots \begin{bmatrix}(0.1, 0.7) & (0.2, 0.5)\\(0.9, 0.1) & (0.4, 0.5)\end{bmatrix}_{2\times2} \end{bmatrix}$$

then the transpose of the above matrix is given as,

$$A^T = \begin{bmatrix} [(0.4, 0.3)]_{1\times1} \vdots [(0.1, 0.6) \quad (0.3, 0.5)]_{1\times2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix}(0.2, 0.6)\\(0.7, 0.1)\end{bmatrix}_{2\times1} \vdots \begin{bmatrix}(0.1, 0.7) & (0.9, 0.1)\\(0.2, 0.5) & (0.4, 0.5)\end{bmatrix}_{2\times2} \end{bmatrix}$$

Proposition 3.2.4

Let $A_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, and $a_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$ and $B_{p\times q} = [Y_{m\times n}]$, where $Y_{m\times n} = [b_{ij}] \in IFSUPM_{m\times n}$, and $b_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$, then some of the properties of transpose which holds true are given as:

(i) $(A+B)' = A' + B'$
(ii) $(kA)' = kA'$
(iii) $(A')' = A$

Remark 3.9

The following property does not exist for transpose: $(AB)' = \quad B' \quad A'$.

3.2.9. Intuitionistic Fuzzy Super Symmetric Matrix

Let $A_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, and $a_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$, is an Intuitionistic fuzzy super matrix. If $A = A^T$, then A is called an Intuitionistic fuzzy super symmetric matrix.

Example 3.2.9

Consider a matrix $A_{2\times2}$, which is an Intuitionistic fuzzy super matrix, given as below,

$$A_{2\times2} = \begin{bmatrix} [(0.1, 0.6)] \vdots [(0.4, 0.1) \quad (0.3, 0.5)] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix}(0.4, 0.1)\\(0.3, 0.5)\end{bmatrix} \vdots \begin{bmatrix}(0.1, 0.2) & (0.4, 0.3)\\(0.4, 0.3) & (0.3, 0.6)\end{bmatrix} \end{bmatrix}$$

then transpose of the above matrix is,

$$A^T = \begin{bmatrix} [(0.1, 0.6)] \vdots [(0.4, 0.1) \quad (0.3, 0.5)] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix}(0.4, 0.1)\\(0.3, 0.5)\end{bmatrix} \vdots \begin{bmatrix}(0.1, 0.2) & (0.4, 0.3)\\(0.4, 0.3) & (0.3, 0.6)\end{bmatrix} \end{bmatrix}$$

Here, we can see that $A = A^T$. Thus, the given matrix is a Symmetric Intuitionistic fuzzy super matrix.

3.2.10. Complement of an Intuitionistic Fuzzy Super Matrix

Following the definition of the Intuitionistic fuzzy matrix complement [36], the complement of Intuitionistic fuzzy super matrix $A_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, and $a_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$ can be defined as an Intuitionistic fuzzy super matrix $A^C_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, and $a_{ij} \in \left(\nu_j(c_i), \quad \mu_j(c_i)\right)$.

Proposition 3.2.5.

Let $A_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, and $a_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$ then,

(i). $(A^C)^C = A$
(ii). $\Phi^C = U$
(iii). $(A+U)^C = \Phi$
(iv). $(A+B)^C = (B+A)^C$

Example 3.2.10

Consider a matrix $A_{2\times2}$, which is an Intuitionistic fuzzy super matrix and is given below as:

$$A = \begin{bmatrix} [(0.1, 0.6)] \vdots [(0.4, 0.1) \quad (0.3, 0.5)] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix}(0.4, 0.1)\\(0.3, 0.5)\end{bmatrix} \vdots \begin{bmatrix}(0.1, 0.2) & (0.4, 0.3)\\(0.4, 0.3) & (0.3, 0.6)\end{bmatrix} \end{bmatrix}$$

Thus, the complement of the above matrix is given as,

$$A^C = \begin{bmatrix} [(0.6, 0.1)] \vdots [(0.1, 0.4) \quad (0.5, 0.3)] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix}(0.1, 0.4)\\(0.5, 0.3)\end{bmatrix} \vdots \begin{bmatrix}(0.2, 0.1) & (0.3, 0.4)\\(0.3, 0.4) & (0.6, 0.3)\end{bmatrix} \end{bmatrix}$$

4. Application of Intuitionistic Fuzzy Super Matrix in Multi-criterion Decision Making

In this section, we have discussed the application of Intuitionistic fuzzy super matrix in a decision-making problem. We have provided some basic definitions such as value super matrix, score super matrix and total score of super matrix, using which interpretation and solution of multi criterion decision making problem is discussed.

Definition 4.1. Value Super Matrix $(V(A))$

Let the matrix $A_{p\times q} = [X_{m\times n}]$, where $X_{m\times n} = [a_{ij}] \in IFSUPM_{m\times n}$, and $a_{ij} \in \left(\mu_j(c_i),\ \nu_j(c_i)\right)$, be an Intuitionistic fuzzy super matrix. The value super matrix is formed when non-membership value is subtracted from membership value of each Intuitionistic fuzzy number in every submatrix of the Intuitionistic fuzzy super matrix.

Definition 4.2. Score Super Matrix $(S(A))$

Let V(A) and V(B) be the value super matrices of two IFSUPM A and B of same order, then the score super matrix $S_{[A,B]}$ is formed by subtracting each value of $V(B)$ from $V(A)$, defined as:

$$S_{[A,B]} = V(A) - V(B)$$

Definition 4.3. Total Score of Super Matrix $(T(A))$

The total score of super matrix can be defined as the sum of all values in the submatrix of a score super matrix.

4.1. Algorithm

A multi-criterion decision making problem can be successfully evaluated using aforementioned method, we can generalize the method for n entries using the following algorithm:

1. Create two or more finite number of Intuitionistic Fuzzy Super Matrices of same order and with corresponding sub matrices of same order, for example $A_{p\times q}$, $B_{p\times q}$, $C_{p\times q}$, etc.
2. Obtain complement of each Intuitionistic Fuzzy Super Matrix considered, for example A^c, B^c, C^c, etc.
3. Using operation of addition for Intuitionistic Fuzzy Super Matrices, we obtain the addition of matrices and its complement, i.e., $(A+B+C+\cdots)$ and $(A^c+B^c+C^c+\cdots)$
4. Calculate value super matrix for both the resultant IFSUPM obtained in last step, such as $V(A+B+C+\cdots)$ and $V(A^c+B^c+C^c+\cdots)$.
5. Compute the Score matrix using value super matrices calculated above. For example, $S_{[A+B+C\ldots],[A^c+B^c+C^c\ldots]}$
6. Add all the values in each submatrix of the score matrix to obtain Total score matrix T.
7. The values found indicate the total score considered for each entity for which a sub matrix was assigned.
8. The singular numeric values found in step number 7 for each entity can be used to rank all entities and perform comparative analysis.

4.2. A Brief Introduction to Intuitionistic Scoring

Before continuing with application of IFSUPM, a narrative demonstration is provided below, which illustrates the scoring system in IFSUPM. The intuitionistic fuzzy values used are represented as (μ,ν), where $0\le\mu\le 1$, $0\le\nu\le 1$, and $\mu+\nu+\pi=1$, where π is the hesitancy. Here μ denotes the membership of our subject to the set, while ν denotes the non-membership. For example, if candidate A gets a value of (1,0) in the criteria of "corporate finance" then it denotes that the judge has deemed their skill to be outstanding with regards to the criteria and believes that they completely fulfill it. Similarly, if candidate B gets a value of (0,1) in the criteria of "peer interaction" then that means that the judge believes that their peer interaction ability is very poor and they do not satisfy the requirements laid out for that criteria. Similarly, if candidate C gets a value of (0.4,0.3) in "corporate finance", which would mean they partially fulfill the criteria, with their skills being 'average', and a 0.3 score of non-membership showing that their performance left something to be desired. However, there is also the presence of a hesitancy of 0.3 (1 – 0.4 – 0.3), with the hesitancy value denoting how unsure the judge is in determining the membership and non-membership of the candidate. In essence, the values can be seen as 0-1(worst to best fit) for $'\mu'$ or membership , 0-1(best to worst fit) for $'\nu'$ or non-membership, and 0-1(no hesitation to complete hesitation in judgement) for 'π' or hesitancy.

4.3. Illustration of the Algorithm: A Case Study

Two professors would like to evaluate and rank four students based on their performance in two subjects. Suppose Corporate Finance and Accounting are two such subjects. They are being evaluated on two basis, critical thinking and peer interaction. In this multi-criterion decision making problem, the entities are the four students, the parameters are the two subjects and the criterions are the two basis.

Let the four students be $S=\{S_1,S_2,S_3,S_4\}$ and the parameters are $P=\{P_1,P_2\}$ for the subjects Corporate Finance and Accounting respectively.Let the criterions be $C=\{C_1,C_2\}$ for critical thinking and peer interaction respectively, then P X C gives us the ordered pair (parameter, criteria) to be scored for each entity (student) by all judges (Professors).

Intuitionistic Fuzzy Set is created over P X C indicating the belongingness and non-belongingness of each student in the form of ordered pairs (α,β). The professors provides a score to each student with an intuitionistic fuzzy number in the form of ordered pair.

Let A and B be two IFSUPM, each graded by one professor. The corresponding submatrices of A and B are for each student and the intuitionistic fuzzy number in each submatrix is an ordered pair of P X C.

$$A=\begin{bmatrix} \begin{array}{c} \begin{array}{cc} & \begin{array}{cc} P_1 & P_2 \end{array} \\ \begin{array}{c} C_1 \\ C_2 \end{array} & \begin{bmatrix} (0.3,0.5) & (0.1,0.2) \\ (0.4,0.2) & (0.6,0.2) \end{bmatrix} \end{array} \\ S_1 \end{array} & \begin{array}{c} \begin{array}{cc} & \begin{array}{cc} P_1 & P_2 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \end{array} & \begin{bmatrix} (0.8,0.1) & (0.7,0.3) \\ (0.1,0.3) & (0.3,0.2) \end{bmatrix} \end{array} \\ S_2 \end{array} \\ \begin{array}{c} \begin{array}{cc} & \begin{array}{cc} P_1 & P_2 \end{array} \\ \begin{array}{c} C_1 \\ C_2 \end{array} & \begin{bmatrix} (0.5,0.1) & (0.6,0.2) \\ (0.3,0.5) & (0.7,0.1) \end{bmatrix} \end{array} \\ S_3 \end{array} & \begin{array}{c} \begin{array}{cc} & \begin{array}{cc} P_1 & P_2 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \end{array} & \begin{bmatrix} (0.4,0.3) & (0.3,0.2) \\ (0.1,0.8) & (0.9,0.1) \end{bmatrix} \end{array} \\ S_4 \end{array} \end{bmatrix}$$

and

$$B=\begin{bmatrix} \begin{array}{c} \begin{array}{cc} & \begin{array}{cc} P_1 & P_2 \end{array} \\ \begin{array}{c} C_1 \\ C_2 \end{array} & \begin{bmatrix} (0.2,0.6) & (0.3,0.4) \\ (0.5,0.4) & (0.3,0.5) \end{bmatrix} \end{array} \\ S_1 \end{array} & \begin{array}{c} \begin{array}{cc} & \begin{array}{cc} P_1 & P_2 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \end{array} & \begin{bmatrix} (0.7,0.2) & (0.1,0.3) \\ (0.5,0.5) & (0.1,0.3) \end{bmatrix} \end{array} \\ S_2 \end{array} \\ \begin{array}{c} \begin{array}{cc} & \begin{array}{cc} P_1 & P \end{array} \\ \begin{array}{c} C_1 \\ C_2 \end{array} & \begin{bmatrix} (0.9,0.1) & (0.4,0.3) \\ (0.7,0.2) & (0.3,0.6) \end{bmatrix} \end{array} \\ S_3 \end{array} & \begin{array}{c} \begin{array}{cc} & \begin{array}{cc} P_1 & P_2 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \end{array} & \begin{bmatrix} (0.5,0.4) & (0.9,0.1) \\ (0.3,0.4) & (0.5,0.4) \end{bmatrix} \end{array} \\ S_4 \end{array} \end{bmatrix}$$

The complement of the Intuitionistic fuzzy super matrices A and B are calculated as,

$$A^C=\begin{bmatrix} \begin{bmatrix} (0.5,0.3) & (0.2,0.1) \\ (0.2,0.4) & (0.2,0.6) \end{bmatrix} & \begin{bmatrix} (0.1,0.8) & (0.3,0.7) \\ (0.3,0.1) & (0.2,0.3) \end{bmatrix} \\ \begin{bmatrix} (0.4,0.5) & (0.4,0.6) \\ (0.5,0.3) & (0.1,0.7) \end{bmatrix} & \begin{bmatrix} (0.5,0.4) & (0.2,0.3) \\ (0.8,0.1) & (0.1,0.9) \end{bmatrix} \end{bmatrix}$$

and

$$B^C=\begin{bmatrix} \begin{bmatrix} (0.6,0.2) & (0.4,0.3) \\ (0.4,0.5) & (0.5,0.3) \end{bmatrix} & \begin{bmatrix} (0.2,0.7) & (0.3,0.1) \\ (0.3,0.5) & (0.3,0.1) \end{bmatrix} \\ \begin{bmatrix} (0.1,0.9) & (0.3,0.4) \\ (0.2,0.7) & (0.6,0.3) \end{bmatrix} & \begin{bmatrix} (0.4,0.5) & (0.1,0.9) \\ (0.4,0.3) & (0.4,0.5) \end{bmatrix} \end{bmatrix}$$

The addition of two IFSUPM is computed as,

$$A+B=\begin{bmatrix}\begin{bmatrix}(0.3,0.5) & (0.3,0.2)\\(0.5,0.2) & (0.6,0.2)\end{bmatrix} & \begin{bmatrix}(0.8,0.1) & (0.7,0.3)\\(0.5,0.3) & (0.3,0.2)\end{bmatrix}\\\begin{bmatrix}(0.9,0.1) & (0.6,0.2)\\(0.7,0.2) & (0.7,0.1)\end{bmatrix} & \begin{bmatrix}(0.5,0.4) & (0.9,0.1)\\(0.3,0.4) & (0.9,0.1)\end{bmatrix}\end{bmatrix}$$

and the sum of its complement is

$$A^C+B^C=\begin{bmatrix}\begin{bmatrix}(0.6,0.2) & (0.4,0.1)\\(0.4,0.4) & (0.5,0.3)\end{bmatrix} & \begin{bmatrix}(0.2,0.7) & (0.3,0.1)\\(0.3,0.1) & (0.3,0.1)\end{bmatrix}\\\begin{bmatrix}(0.4,0.5) & (0.3,0.4)\\(0.5,0.3) & (0.6,0.3)\end{bmatrix} & \begin{bmatrix}(0.5,0.4) & (0.2,0.3)\\(0.8,0.1) & (0.4,0.5)\end{bmatrix}\end{bmatrix}$$

The value of the Intuitionistic fuzzy super matrix is determined as,

$$V(A+B)=\begin{bmatrix}\begin{bmatrix}-0.2 & 0.1\\0.3 & 0.4\end{bmatrix} & \begin{bmatrix}0.7 & 0.4\\0.2 & 0.1\end{bmatrix}\\\begin{bmatrix}0.8 & 0.4\\0.5 & 0.6\end{bmatrix} & \begin{bmatrix}0.1 & 0.8\\-0.1 & 0.8\end{bmatrix}\end{bmatrix}$$

and

$$V(A^C+B^C)=\begin{bmatrix}\begin{bmatrix}0.4 & 0.3\\0.0 & 0.2\end{bmatrix} & \begin{bmatrix}-0.5 & 0.2\\0.2 & 0.2\end{bmatrix}\\\begin{bmatrix}-0.1 & -0.1\\0.2 & 0.3\end{bmatrix} & \begin{bmatrix}0.1 & -0.1\\0.7 & -0.1\end{bmatrix}\end{bmatrix}$$

The score value of the Intuitionistic fuzzy super matrix is obtained as follows:

$$S_{((A+B),(A^C+B^C))}=\begin{bmatrix}\begin{matrix} & P_1 & P_2\\C_1 & -0.6 & -0.2\\C_2 & 0.3 & 0.2\end{matrix} & \begin{matrix}P_1 & P_2\\1.2 & 0.2\\0.0 & -0.1\end{matrix}\\S_1 & S_2\\\begin{matrix} & P_1 & P_2\\C_1 & 0,9 & 0.5\\C_2 & 0.3 & 0.3\end{matrix} & \begin{matrix} & P_1 & P_2\\P_1 & 0.0 & 0.9\\P_2 & -0.8 & 0.9\end{matrix}\\S_3 & S_4\end{bmatrix}$$

The individual criterion is based on the sum of the value score calculated as:

$$=\begin{bmatrix}[-0.3 \quad 0.0] & [1.2 \quad 0.1]\\ [1.2 \quad 0.8] & [-0.8 \quad 1.8]\end{bmatrix}$$

$$\text{Total Score }=\begin{bmatrix}-0.3 & 1.3\\S_1 & S_2\\2.0 & 1.0\\S_3 & S_4\end{bmatrix}.$$

The total score of all four students S_1, S_2, S_3, S_4 are calculated above and their values are -0.3, 1.3, 2.0 and 1.0, respectively. It was found that order of Rank are $S_3 > S_2 > S_4 > S_1$.

Here we observe that student S_3 is the best amongst all. The criterions we took, i.e critical thinking and peer interaction cannot be expressed by a percentage like we do for marks, score and rating but here our judges assign a membership grade and a non-membership grade.

In a similar fashion, this algorithm can be applied to a multi-criterion decision making problem with any number of entities (in this case, students), any number of parameters (in this case, subjects), any number of criterions (in this case, the basis of evaluation) and can be judged by any number of evaluators (in this case, Professor). The only condition is that the entity is evaluated for all criterions of each parameter.

The score obtained at the end is a condensed representation of a candidate's value in each ordered pair of P X C (parameters X criterions). We require scores to rank, compare and evaluate candidates on factors which are qualitative. Using a numeric value, we can make inferences on the used sample set.

4.4. Novelty of the Method: Comparison of IFM and IFSUPM

To understand the novelty of the aforementioned method, we attempt to model the same illustration as before using an Intuitionistic Fuzzy Matrix. We can create the matrices in a similar manner:

	[Corporate Finance	Accounting
S_1	(0.4, 0.5)	(0.6, 0.4)
S_2	(0.6, 0.3)	(0.5, 0.5)
S_3	(0.7, 0.2)	(0.8, 0.1)
S_4	(0.5, 0.5)	(0.9, 0.1)]

	[Peer Interaction	Critical Thinking
S_1	(0.6, 0.3)	(0.4, 0.6)
S_2	(0.4, 0.2)	(0.2, 0.1)
S_3	(0.8, 0.1)	(0.5, 0.4)
S_4	(0.9, 0.1)	(0.7, 0.2)]

Here we observe that instead of combining data in one IFSUPM, we have two IFM. Moreover, we cannot evaluate the intersection of parameters and their criterions. As seen above, students are scored on 'Corporate Finance' and 'Critical thinking' separately, whereas with IFSUPM method we are able to score a student based on their "critical thinking ability in the subject of corporate finance".

This explains how super matrices help us to model multi-criterion decision making problems. Using the IFSUPM approach over IFM, we add a new dimension to our set as we are allowed to scrutinize every ordered pair of P X C where P is a set of parameters and C is a set of criterions. Combining two matrices into one super matrix also saves computational cost.

5. Conclusions

In this paper, we have introduced the concept of Intuitionistic Fuzzy Super Matrices (IFSUPM) and illustrated an application of the IFSUPM. We have developed their algebra – addition, subtraction, multiplication and their properties like transpose, scalar multiplication and other general properties. The proposed method has an advantage over other decision-making methods like Intuitionistic fuzzy matrices and fuzzy super matrices as shown in the research paper in the form of an example as it allows us to make multi-criteria decisions. In particular, the IFSUPM method performs better than other decision-making methods based on the fuzzy set as in IFSUPM, we have used intuitionistic fuzzy set which includes both belongingness and non-belongingness values. This opens a new dimension of thinking, hence allowing us to generalize and to make decisions in a more robust manner. The illustration is taken for few parameters, but it can be generalized to finite number of parameters.

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