

A GENERALIZATION OF A RESULT OF D'OCAGNE

R. S. Melham and A. G. Shannon

University of Technology, Sydney 2007, Australia

(Submitted July 1993)

1. INTRODUCTION

In this paper we consider some aspects of sequences generated by the m^{th} order homogeneous linear recurrence relation

$$R_n = \sum_{i=1}^m a_i R_{n-i} \quad \text{for } m \geq 2, \quad (1.1)$$

where $a_m \neq 0$ and the underlying field is the complex numbers. To generate a sequence $\{R_n\}_{n=0}^{\infty}$, we specify initial values R_0, R_1, \dots, R_{m-1} . Indeed, this sequence can be extended to negative subscripts by using (1.1), and with this convention we simply write $\{R_n\}$.

For the case $m = 2$, we adopt the notation of Hordam [3] and write

$$W_n = W_n(\alpha, b; p, q), \quad (1.2)$$

meaning that

$$W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = \alpha, \quad W_1 = b. \quad (1.3)$$

If $(R_0, \dots, R_{m-2}, R_{m-1}) = (0, \dots, 0, 1)$, we write $\{R_n\} = \{U_n\}$. The sequence $\{U_n\}$ is called the fundamental sequence generated by (1.1). It is "fundamental" in the sense that, if $\{R_n\}$ is any sequence generated by (1.1), then there exist complex numbers b_0, \dots, b_{m-1} depending upon a_1, \dots, a_m and R_0, \dots, R_{m-1} such that

$$R_n = \sum_{i=0}^{m-1} b_i U_{n+i} \quad \text{for all integers } n. \quad (1.4)$$

In this regard, see Jarden [4], p. 114 or Dickson [1], p. 409, where this result is attributed to D'Ocagne. In §2 we generalize this idea.

For the Fibonacci and Lucas numbers, it can be proved that

$$L_n^2 + L_{n+1}^2 = 5(F_n^2 + F_{n+1}^2). \quad (1.5)$$

More generally, for the second-order fundamental and primordial sequences of Lucas [5] defined by

$$\begin{cases} U_n = W_n(0, 1; p, q), \\ V_n = W_n(2, p; p, q), \end{cases} \quad (1.6)$$

where $\Delta = p^2 - 4q \neq 0$, we have

$$-qV_n^2 + V_{n+1}^2 = \Delta(-qU_n^2 + U_{n+1}^2). \quad (1.7)$$

In §3 we demonstrate the existence of a result analogous to (1.7) for any two sequences generated by (1.1).

2. A GENERALIZATION OF D'OCAGNE'S RESULT

Let $\{R_n\}$ and $\{S_n\}$ be any two sequences generated by (1.1). Define the $(m+1) \times (m+1)$ determinant D_n , for all integers n , by

$$D_n = \begin{vmatrix} R_n & S_n & S_{n+1} & \cdots & S_{n+m-1} \\ R_{m-1} & S_{m-1} & S_m & \cdots & S_{2m-2} \\ R_{m-2} & S_{m-2} & S_{m-1} & \cdots & S_{2m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_0 & S_0 & S_1 & \cdots & S_{m-1} \end{vmatrix}.$$

Theorem 1: $D_n = 0$ for all integers n .

Proof: $D_0 = D_1 = \cdots = D_{m-1} = 0$ since, in each case, we have an $(m+1) \times (m+1)$ determinant with two identical rows. Now expanding D_n along the top row, we see that D_n is a linear combination of $R_n, S_n, \dots, S_{n+m-1}$. Therefore, since each of the sequences $\{R_n\}, \{S_n\}, \dots, \{S_{n+m-1}\}$ is generated by (1.1) then so is $\{D_n\}$. But $\{D_n\}$ has m successive terms that are zero and so all its terms are zero. This completes the proof. \square

We now come to the main result of this section.

Corollary 1: There exist constants c and c_{oj} , $0 \leq j \leq m-1$, such that

$$cR_n = \sum_{j=0}^{m-1} c_{oj} S_{n+j} \quad \text{for all integers } n. \quad (2.1)$$

Proof: Expand D_n along the top row. \square

Equation (2.1) generalizes D'Ocagne's result (1.4), where the b_i are normally specified without the use of determinants. If $\{S_n\} = \{U_n\}$, then c , which is the minor of R_n is unity and we obtain an equivalent form of D'Ocagne's result.

3. A RESULT CONCERNING SUMS OF SQUARES

From (2.1) we have, for any integer i ,

$$cR_{n+i} = \sum_{j=0}^{m-1} c_{oj} S_{n+i+j}. \quad (3.1)$$

Using (1.1), the right side of (3.1) can be written in terms of $S_n, S_{n+1}, \dots, S_{n+m-1}$. That is, for any integer i there exist constants c_{ij} , $0 \leq j \leq m-1$, such that

$$cR_{n+i} = \sum_{j=0}^{m-1} c_{ij} S_{n+j}. \quad (3.2)$$

Write $\ell = \binom{m}{2}$. Then, for parameters d_0, d_1, \dots, d_ℓ we have, from (3.2),

$$c^2 \sum_{i=0}^{\ell} d_i R_{n+i}^2 = \sum_{j=0}^{m-1} S_{n+j}^2 \sum_{i=0}^{\ell} d_i c_{ij}^2 + 2 \sum_{0 \leq j < k \leq m-1} S_{n+j} S_{n+k} \sum_{i=0}^{\ell} d_i c_{ij} c_{ik}. \quad (3.3)$$

Consider the system of equations

$$\sum_{i=0}^{\ell} d_i c_{ij} c_{ik} = 0, \quad 0 \leq j < k \leq m-1, \tag{3.4}$$

in the unknowns $d_0, d_1, \dots, d_{\ell}$. Since (3.4) is a system of ℓ homogeneous linear equations in $\ell+1$ unknowns, there are an infinite number of solutions $(d_0, d_1, \dots, d_{\ell})$. Choose any nontrivial solution and put

$$e_i = c^2 d_i, \quad 0 \leq i \leq \ell, \\ f_j = \sum_{i=0}^{\ell} d_i c_{ij}^2, \quad 0 \leq j \leq m-1.$$

Making these substitutions in (3.3), we have succeeded in proving the following theorem.

Theorem 2: Let $\{R_n\}$ and $\{S_n\}$ be any two sequences generated by the recurrence (1.1). Then there exist constants $e_i, 0 \leq i \leq \ell = \binom{m}{2}$, and $f_i, 0 \leq i \leq m-1$, not all zero such that, for all integers n ,

$$\sum_{i=0}^{\ell} e_i R_{n+i}^2 = \sum_{i=0}^{m-1} f_i S_{n+i}^2. \tag{3.5}$$

Theorem 2 shows the existence of a result analogous to (1.7) for any two sequences generated by (1.1).

Example 1: Let $\{W_n\}$ and $\{S_n\}$ be any two sequences generated by the recurrence (1.3). Then, after some tedious algebra, we obtain the following determinantal identity:

$$\left| \begin{array}{cc|cc|cc} S_n^2 & S_{n+1}^2 & W_n^2 & W_{n+1}^2 & & \\ \left| \begin{array}{cc} W_2 & S_1 \\ W_1 & S_0 \end{array} \right| & \left| \begin{array}{cc} S_2 & W_1 \\ S_3 & W_2 \end{array} \right| & \left| \begin{array}{cc} S_2 & W_1 \\ S_3 & W_2 \end{array} \right| & q^2 \left| \begin{array}{cc} W_2 & S_1 \\ W_1 & S_0 \end{array} \right| & & \\ \left| \begin{array}{cc} S_1 & S_2 \\ S_2 & S_3 \end{array} \right| & & -q \left| \begin{array}{cc} W_1 & W_2 \\ W_2 & W_3 \end{array} \right| & & & \end{array} \right| = 0. \tag{3.6}$$

Example 2: For a fixed integer k , consider the sequences $\{F_{kn}\}$ and $\{L_{kn}\}$. They both satisfy the recurrence (1.3) with $p = L_k$ and $q = (-1)^k$. Substitution into (3.6) yields

$$5(F_{kn}^2 + (-1)^{k-1} F_{k(n+1)}^2) = L_{kn}^2 + (-1)^{k-1} L_{k(n+1)}^2. \tag{3.7}$$

Example 3: In (1.1), taking $m = 3$ and $a_1 = a_2 = a_3 = 1$, we have

$$R_n = R_{n-1} + R_{n-2} + R_{n-3}. \tag{3.8}$$

Feinberg [2] referred to sequences generated by (3.8) as Tribonacci sequences.

For $(R_0, R_1, R_2) = (0, 0, 1)$ write $\{R_n\} = \{U_n\}$.

For $(R_0, R_1, R_2) = (3, 1, 3)$ write $\{R_n\} = \{V_n\}$.

Then $\{V_n\}$ bears the same relation to $\{U_n\}$ as does the Lucas sequence to the Fibonacci sequence (see [6], p. 300).

Now assuming a relationship between $\{U_n\}$ and $\{V_n\}$ of the form (3.5) and solving for the coefficients e_i and f_i yields

$$34V_n^2 - 30V_{n+1}^2 + V_{n+2}^2 + 9V_{n+3}^2 = -154U_n^2 + 176U_{n+1}^2 + 726U_{n+2}^2. \quad (3.9)$$

Alternatively, we have

$$46U_n^2 - 50U_{n+1}^2 - 114U_{n+2}^2 + 54U_{n+3}^2 = -7V_n^2 + 12V_{n+1}^2 - V_{n+2}^2. \quad (3.10)$$

4. OPEN QUESTION

Is there a result analogous to (3.5) for higher powers?

REFERENCES

1. L. E. Dickson. *History of the Theory of Numbers*. Vol. 1. New York: Chelsea, 1966.
2. M. Feinberg. "Fibonacci-Tribonacci." *The Fibonacci Quarterly* **1.3** (1963):71-74.
3. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* **3.3** (1965):161-76.
4. D. Jarden. *Recurring Sequences*. Jerusalem: Riveon Lematematika, 1966.
5. E. Lucas. *Théorie des Nombres*. Paris: Blanchard, 1961.
6. M. E. Waddill. "Using Matrix Techniques To Establish Properties of a Generalized Tribonacci Sequence." In *Applications of Fibonacci Numbers*, Vol. 4. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. The Netherlands: Kluwer, 1991.

AMS Classification Numbers: 11B37, 11B39



The Fibonacci Quarterly

Official Publication of The Fibonacci Association



[Journal Home](#) | [Editorial Board](#) | [List of Issues](#)
[How to Subscribe](#) | [General Index](#) | [Fibonacci Association](#)

Volume 33

Number 2

May 1995

CONTENTS

[Cover Page](#)

- G. F. C. de Bruyn
Formulas $a + a^2 2^p + a^3 3^p + \dots + a^n n^p$ 98
[Full text](#)
- Wai-fong Chuan
Generating Fibonacci Words 104
[Full text](#)
- Wai-fong Chuan
Extraction Property of the Golden Sequence 113
[Full text](#)
- A. Rotkiewicz and K. Ziemak
On Even Pseudoprimes 123
[Full text](#)
- R. S. Melham and A. G. Shannon
Generalizations of Some Simple Congruences 126
[Full text](#)
- P. R. Subramanian
Nonzero Zeros of the Hermite Polynomials are Irrational 131
[Full text](#)
- R. S. Melham and A. G. Shannon
A Generalization of a Result of D'Ocagne 135
[Full text](#)
- Helmut Prodinger
Geometric Distributions and Forbidden Subwords 139
[Full text](#)
- Ray Melham and Derek Jennings
On the General Linear Recurrence Relation 142
[Full text](#)
- N. G. Gamkrelidze
On a Probabilistic Property of the Fibonacci Sequence 147

[Full text](#)

Arnold Knopfmacher and M. E. Mays
Pierce Expansions of Ratios and Fibonacci and Lucas Numbers and Polynomials 153

[Full text](#)

Peter Hope
Exponential Growth of Random Fibonacci Sequences 164

[Full text](#)

L. C. Hsu
A Difference-Operational Approach to the Möbius Inversion Formulas 169

[Full text](#)

Jun Wang
On the k^{th} Derivative Sequences of Fibonacci and Lucas Polynomials 174

[Full text](#)

Krystyna Bialek
A Note on Choudhry's Results 179

[Full text](#)

Edited by Stanley Rabinowitz
Elementary Problems and Solutions 181

[Full text](#)

Edited by Raymond E. Whitney
Advanced Problems and Solutions 187

[Full text](#)

[Back Cover](#)

Copyright © 2010 [The Fibonacci Association](#). All rights reserved.