

# SOME SUMMATION IDENTITIES USING GENERALIZED $Q$ -MATRICES

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## 1. INTRODUCTION

In a belated acknowledgment, Hoggatt [3] states:

The first use of the  $Q$ -matrix to generate the Fibonacci numbers appears in an abstract of a paper by Professor J. L. Brenner by the title "Lucas' Matrix." This abstract appeared in the March 1951 *American Mathematical Monthly* on pages 221 and 222. The basic exploitation of the  $Q$ -matrix appeared in 1960 in the San Jose State College Master's thesis of Charles H. King with the title "Some Further Properties of the Fibonacci Numbers." Further utilization of the  $Q$ -matrix appears in the *Fibonacci Primer* sequence parts I-V.

For a comprehensive history of the  $Q$ -matrix, see Gould [2]. Numerous analogs of the  $Q$ -matrix relating to third-order recurrences have been used. See, for instance, Waddill and Sacks [13], Shannon and Horadam [10], and Waddill [11]. Mahon [8] has made extensive use of matrices to study his third-order diagonal functions of the Pell polynomials. Recently, Waddill [12] considered a general  $Q$ -matrix. He defined and used the  $k \times k$  matrix

$$R = \begin{pmatrix} r_0 & r_1 & \cdots & r_{k-1} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

in relation to a  $k$ -order linear recursive sequence  $\{V_n\}$ , where

$$V_n = \sum_{i=0}^{k-1} r_i V_{n-1-i}, \quad n \geq k.$$

The matrix  $R$  generalized the matrix  $Q_r$  of Ivie [5].

In the notation of Horadam [4], write

$$W_n = W_n(a, b, p, q) \tag{1.1}$$

so that

$$W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = a, \quad W_1 = b, \quad n \geq 2. \tag{1.2}$$

With this notation, define

$$\begin{cases} U_n = W_n(0, 1, p, q), \\ V_n = W_n(2, p, p, q). \end{cases} \tag{1.3}$$

Indeed,  $\{U_n\}$  and  $\{V_n\}$  are the fundamental and primordial sequences generated by (1.2). They have been studied extensively, particularly by Lucas [7]. Further information can be found in [1], [4], and [6].

The most commonly used matrix in relation to the recurrence relation (1.2) is

$$M = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}, \quad (1.4)$$

which, for  $p = -q = 1$ , reduces to the ordinary  $Q$ -matrix. In this paper we define a more general matrix  $M_{k,m}$  parametrized by  $k$  and  $m$  and reducing to  $M$  for  $k = m = 1$ . We use  $M_{k,m}$  to develop various summation identities involving terms from the sequences  $\{U_n\}$  and  $\{V_n\}$ .

Our work is a generalization of the work of Mahon and Horadam [9] who used several pairs of  $2 \times 2$  matrices to generate summation identities involving terms from the Pell polynomial sequences

$$\begin{cases} P_n = W_n(0, 1; 2x, -1), \\ Q_n = W_n(2, 2x; 2x, -1). \end{cases} \quad (1.5)$$

We generalize their work in two ways. First, we consider sequences generated by a more general recurrence relation. Second, our parametrization of the matrix  $M_{k,m}$  includes all the matrices considered by Mahon and Horadam as special cases.

## 2. THE MATRIX $M_{k,m}$

Before proceeding, we state some results which are used subsequently. None of these is new and each can be proved using Binet forms. If

$$\Delta = p^2 - 4q, \quad (2.1)$$

then

$$U_{n+1} - qU_{n-1} = V_n, \quad (2.2)$$

$$V_{n+1} - qV_{n-1} = \Delta U_n, \quad (2.3)$$

$$V_{2k} - 2q^k = \Delta U_k^2, \quad (2.4)$$

$$U_{k+m} - q^m U_{k-m} = U_m V_k, \quad (2.5)$$

$$V_{k+m} - q^m V_{k-m} = \Delta U_k U_m \quad (2.6)$$

$$U_{k+m} U_{k-m} - U_k^2 = -q^{k-m} U_m^2, \quad (2.7)$$

$$V_{k+m} V_{k-m} - V_k^2 = \Delta q^{k-m} U_m^2, \quad (2.8)$$

$$U_{n+m} U_{n_1+m} - q^m U_n U_{n_1} = U_m U_{n+n_1+m}. \quad (2.9)$$

By induction it can be proved that, for the matrix  $M$  in (1.4),

$$M^n = \begin{pmatrix} U_{n+1} & -qU_n \\ U_n & -qU_{n-1} \end{pmatrix}, \quad (2.10)$$

where  $n$  is an integer.

We now give a generalization of the matrix  $M$ . Associated with the recurrence (1.2) and with  $\{U_n\}$  as in (1.3), define

$$M_{k,m} = \begin{pmatrix} U_{k+m} & -q^m U_k \\ U_k & -q^m U_{k-m} \end{pmatrix} \quad (2.11)$$

where  $k$  and  $m$  are integers. By induction and making use of (2.9), it can be shown that, for all integral  $n$ ,

$$M_{k,m}^n = U_m^{n-1} \begin{pmatrix} U_{nk+m} & -q^m U_{nk} \\ U_{nk} & -q^m U_{nk-m} \end{pmatrix}. \quad (2.12)$$

When  $k = m = 1$ , we see that  $M_{k,m}$  reduces to  $M$  and  $M_{k,m}^n$  reduces to  $M^n$ .

### 3. SUMMATION IDENTITIES

We now use the matrix  $M_{k,m}$  to produce summation identities involving terms from  $\{U_n\}$  and  $\{V_n\}$ . Using (2.5) and (2.7), we find that the characteristic equation of  $M_{k,m}$  is

$$\lambda^2 - U_m V_k \lambda + q^k U_m^2 = 0 \quad (3.1)$$

and, by the Cayley-Hamilton theorem,

$$M_{k,m}^2 - U_m V_k M_{k,m} + q^k U_m^2 I = 0, \quad (3.2)$$

where  $I$  is the  $2 \times 2$  unit matrix. From (3.2), we have

$$(U_m V_k M_{k,m} - q^k U_m^2 I)^n M_{k,m}^j = M_{k,m}^{2n+j}, \quad (3.3)$$

and expanding yields

$$\sum_{i=0}^n \binom{n}{i} (-1)^{n-i} q^{k(n-i)} U_m^{2n-i} V_k^i M_{k,m}^{i+j} = M_{k,m}^{2n+j}. \quad (3.4)$$

Using (2.12) to equate upper left entries gives

$$\sum_{i=0}^n \binom{n}{i} (-1)^{n-i} q^{k(n-i)} V_k^i U_{(i+j)k+m} = U_{(2n+j)k+m}. \quad (3.5)$$

Again from (3.2),

$$(M_{k,m}^2 + q^k U_m^2 I)^n = U_m^n V_k^n M_{k,m}^n, \quad (3.6)$$

and expanding we have

$$\sum_{i=0}^n \binom{n}{i} q^{k(n-i)} U_m^{2(n-i)} M_{k,m}^{2i} = U_m^n V_k^n M_{k,m}^n. \quad (3.7)$$

Using (2.12) to equate upper left entries gives

$$\sum_{i=0}^n \binom{n}{i} q^{k(n-i)} U_{2ik+m} = V_k^n U_{nk+m}. \quad (3.8)$$

Once again, from (3.2),

$$(M_{2k,m} - q^k U_m I)^2 = U_m (V_{2k} - 2q^k) M_{2k,m} = \Delta U_m U_k^2 M_{2k,m}, \quad (3.9)$$

and expanding, after taking  $n^{\text{th}}$  powers, we have

$$\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i q^{k(2n-i)} U_m^{2n-i} M_{2k,m}^i = \Delta^n U_m^n U_k^{2n} M_{2k,m}^n. \quad (3.10)$$

Equating upper left entries yields

$$\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i q^{k(2n-i)} U_{2ik+m} = \Delta^n U_k^{2n} U_{2nk+m}. \quad (3.11)$$

From (3.9),

$$(M_{2k,m} - q^k U_m I)^{2n+1} = \Delta^n U_m^n U_k^{2n} (M_{2k,m}^{n+1} - q^k U_m M_{2k,m}^n). \quad (3.12)$$

Equating upper left entries yields, after simplifying,

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^{i+1} q^{k(2n+1-i)} U_{2ik+m} = \Delta^n U_k^{2n} (U_{2(n+1)k+m} - q^k U_{2nk+m}), \quad (3.13)$$

and using (2.5) to simplify the right side gives

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^{i+1} q^{k(2n+1-i)} U_{2ik+m} = \Delta^n U_k^{2n+1} V_{(2n+1)k+m}. \quad (3.14)$$

This should be compared to (3.11).

Manipulating the characteristic equation (3.1), we have  $(2\lambda - U_m V_k)^2 = \Delta U_m^2 V_k^2$ , so that

$$(2M_{k,m} - U_m V_k I)^{2n} = \Delta^n U_m^{2n} U_k^{2n} I. \quad (3.15)$$

Expanding gives

$$\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i 2^i U_m^{2n-i} V_k^{2n-i} M_{k,m}^i = \Delta^n U_m^{2n} U_k^{2n} I. \quad (3.16)$$

Equating upper left entries and also lower left entries yields, respectively,

$$\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i 2^i V_k^{2n-i} U_{ik+m} = \Delta^n U_k^{2n} U_m, \quad (3.17)$$

$$\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i 2^i V_k^{2n-i} U_{ik} = 0. \quad (3.18)$$

We note that (3.17) reduces to (3.18) when  $m = 0$ .

Multiplying both sides of (3.15) by  $(2M_{k,m} - U_m V_k I)$  and expanding gives

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^{i+1} 2^i U_m^{2n+1-i} V_k^{2n+1-i} M_{k,m}^i = \Delta^n U_m^{2n} U_k^{2n} (2M_{k,m} - U_m V_k I). \quad (3.19)$$

Equating upper left entries yields

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^{i+1} 2^i V_k^{2n+1-i} U_{ik+m} = \Delta^n U_k^{2n+1} V_m, \quad (3.20)$$

which should be compared to (3.17).

Now, using (3.5), we have

$$\sum_{i=0}^n \binom{n}{i} (-1)^{n-i} q^{k(n-i)} V_k^i (U_{(i+j)k+m+1} - q U_{(i+j)k+m-1}) = U_{(2n+j)k+m+1} - q U_{(2n+j)k+m-1},$$

and (2.2) shows that this simplifies to

$$\sum_{i=0}^n \binom{n}{i} (-1)^{n-i} q^{k(n-i)} V_k^i V_{(i+j)k+m} = V_{(2n+j)k+m}. \quad (3.21)$$

Making use of (2.2) and (2.3) and working in the same manner with identities (3.8), (3.11), (3.14), (3.17), and (3.20) yields, respectively,

$$\sum_{i=0}^n \binom{n}{i} q^{k(n-i)} V_{2ik+m} = V_k^n V_{nk+m}, \quad (3.22)$$

$$\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i q^{k(2n-i)} V_{2ik+m} = \Delta^n U_k^{2n} V_{2nk+m}, \quad (3.23)$$

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^{i+1} q^{k(2n+1-i)} V_{2ik+m} = \Delta^{n+1} U_k^{2n+1} U_{(2n+1)k+m}, \quad (3.24)$$

$$\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i 2^i V_k^{2n-i} V_{ik+m} = \Delta^n U_k^{2n} V_m, \quad (3.25)$$

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^{i+1} 2^i V_k^{2n+1-i} V_{ik+m} = \Delta^{n+1} U_k^{2n+1} U_m. \quad (3.26)$$

In what follows, we make use of the following result:

$$M_{k,m}^n M_{k_1,m}^{n_1} = U_m^{n+n_1-1} \begin{pmatrix} U_{nk+n_1k_1+m} & -q^m U_{nk+n_1k_1} \\ U_{nk+n_1k_1} & -q^m U_{nk+n_1k_1-m} \end{pmatrix}. \quad (3.27)$$

This is proved by multiplying the matrices on the left and using (2.9).

Consider now the special case of (3.2), where  $k = m$ . Then, using (2.5),

$$M_{k,k}^2 = U_{2k} M_{k,k} - q^k U_k^2 I. \quad (3.28)$$

Using (3.28) and (2.9), we can show by induction that, for  $n \geq 2$ ,

$$M_{k,k}^n = U_k^{n-2} (U_{nk} M_{k,k} - q^k U_k U_{(n-1)k} I). \quad (3.29)$$

The binomial theorem applied to (3.29) gives

$$U_k^{(n-2)s} \sum_{i=0}^s \binom{s}{i} (-1)^{s-i} q^{k(s-i)} U_k^{s-i} U_{(n-1)k}^{s-i} U_{nk}^i M_{k,k}^{i+j} = M_{k,k}^{ns+j} \quad (3.30)$$

Equating lower left entries of the relevant matrices then yields

$$\sum_{i=0}^s \binom{s}{i} (-1)^{s-i} q^{k(s-i)} U_{(n-1)k}^{s-i} U_{nk}^i U_{(i+j)k} = U_k^s U_{(ns+j)k} \quad (3.31)$$

Multiplying both sides of (3.30) by  $M_{k_1,k}$  and using (3.27) to equate lower left entries gives

$$\sum_{i=0}^s \binom{s}{i} (-1)^{s-i} q^{k(s-i)} U_{(n-1)k}^{s-i} U_{nk}^i U_{(i+j)k+k_1} = U_k^s U_{(ns+j)k+k_1}, \quad (3.32)$$

which generalizes (3.31).

Again from (3.29), after transposing terms and raising to a power  $s$ , we obtain

$$\sum_{i=0}^s \binom{s}{i} q^{k(s-i)} U_k^{k(s-i)} U_{(n-1)k}^{s-i} U_{nk}^{s-i} M_{k,k}^{ni} = U_k^{(n-2)s} U_{nk}^s M_{k,k}^s, \quad (3.33)$$

which yields

$$\sum_{i=0}^s \binom{s}{i} q^{k(s-i)} U_k^i U_{(n-1)k}^{s-i} U_{nk} = U_{nk}^s U_{sk} \quad (3.34)$$

Multiplying both sides of (3.33) by  $M_{k_1,k}$  and using (3.27) to equate lower left entries gives

$$\sum_{i=0}^s \binom{s}{i} q^{k(s-i)} U_k^i U_{(n-1)k}^{s-i} U_{nk+k_1} = U_{nk}^s U_{sk+k_1}, \quad (3.35)$$

which generalizes (3.34).

Continuing in this manner after yet again transposing terms in (3.29) and raising to a power  $s$ , we obtain

$$\sum_{i=0}^s \binom{s}{i} (-1)^i U_k^{(n-2)(s-i)} U_{nk}^{s-i} M_{k,k}^{(n-1)i+s} = q^{ks} U_k^{(n-1)s} U_{(n-1)k}^s I. \quad (3.36)$$

Equating upper left entries and lower left entries yields, respectively,

$$\sum_{i=0}^s \binom{s}{i} (-1)^i U_k^i U_{nk}^{s-i} U_{((n-1)i+s+1)k} = q^{ks} U_{(n-1)k}^s U_k, \quad (3.37)$$

$$\sum_{i=0}^s \binom{s}{i} (-1)^i U_k^i U_{nk}^{s-i} U_{((n-1)i+s)k} = 0. \quad (3.38)$$

Multiplying (3.36) by  $M_{k_1,k}$  and equating lower left entries yields

$$\sum_{i=0}^s \binom{s}{i} (-1)^i U_k^i U_{nk}^{s-i} U_{((n-1)i+s)k+k_1} = q^{ks} U_{(n-1)k}^s U_{k_1}. \quad (3.39)$$

We note that, when  $k_1 = k$ , (3.39) reduces to (3.37) and when  $k_1 = 0$ , (3.39) reduces to (3.38).

Now, manipulating (3.32), (3.35), and (3.39) in the same way that (3.5) was manipulated to yield (3.21), we obtain, respectively,

$$\sum_{i=0}^s \binom{s}{i} (-1)^{s-i} q^{k(s-i)} U_{(n-1)k}^{s-i} U_{nk}^i V_{(i+j)k+k_1} = U_k^s V_{(ns+j)k+k_1}, \quad (3.40)$$

$$\sum_{i=0}^s \binom{s}{i} q^{k(s-i)} U_k^i U_{(n-1)k}^{s-i} V_{nik+k_1} = U_{nk}^s V_{sk+k_1}, \quad (3.41)$$

$$\sum_{i=0}^s \binom{s}{i} (-1)^i U_k^i U_{nk}^{s-i} V_{((n-1)i+s)k+k_1} = q^{ks} U_{(n-1)k}^s V_{k_1}. \quad (3.42)$$

#### 4. THE MATRIX $X_k$

We have found a matrix having the property of generating terms from  $\{U_n\}$  and  $\{V_n\}$  simultaneously. It is a generalization of the matrix  $W$  introduced by Mahon and Horadam [9]. Define

$$X_k = \begin{pmatrix} V_k & U_k \\ \Delta U_k & V_k \end{pmatrix}, \quad k \text{ an integer.} \quad (4.1)$$

Then by induction we have, for integral  $n$ ,

$$X_k^n = 2^{n-1} \begin{pmatrix} V_{nk} & U_{nk} \\ \Delta U_{nk} & V_{nk} \end{pmatrix}. \quad (4.2)$$

Noting that  $X_1^{m+n} = X_1^m \cdot X_1^n$  produces the well-known identities

$$2V_{m+n} = V_m V_n + \Delta U_m U_n, \quad (4.3)$$

$$2U_{m+n} = V_m U_n + U_m V_n. \quad (4.4)$$

The characteristic equation for  $X_k$  is

$$\lambda^2 - 2V_k \lambda + 4q^k = 0 \quad (4.5)$$

and so, by the Cayley-Hamilton theorem

$$X_k^2 - 2V_k X_k + 4q^k I = 0. \quad (4.6)$$

Using (4.3) and (4.4), we see that

$$X_k^n X_{k_1} = 2^n \begin{pmatrix} V_{nk+k_1} & U_{nk+k_1} \\ \Delta U_{nk+k_1} & V_{nk+k_1} \end{pmatrix}. \quad (4.7)$$

Considering the case  $k = 1$ , we can show by induction, with the aid of (4.6), that

$$X_1^n = 2^{n-1} (U_n X_1 - 2q U_{n-1} I), \quad n \geq 2, \quad (4.8)$$

which is analogous to (3.29).

It is interesting to note that the methods applied to  $M_{k,m}$  when applied to  $X_k$  produce most of the summation identities that we have obtained so far. The exceptions are the identities that arose by using (3.29). The analogous procedure for  $X_k$  is to use (4.8), but the identities that arise are less general. For example, (4.8) produces

$$\sum_{i=0}^s \binom{s}{i} (-1)^{s-i} q^{s-i} U_{n-1}^{s-i} U_n^i U_{i+j+k_1} = U_{ns+j+k_1}, \quad (4.9)$$

which is a special case of (3.32).

### 5. THE MATRIX $N_{k,m}$

We have found yet another matrix defined in a similar manner to  $M_{k,m}$  whose powers also generate terms of the sequences  $\{U_n\}$  and  $\{V_n\}$ . Define

$$N_{k,m} = \begin{pmatrix} V_{k+m} & -q^m V_k \\ V_k & -q^m V_{k-m} \end{pmatrix}. \quad (5.1)$$

Then for all integral  $n$ ,

$$N_{k,m}^{2n} = U_m^{2n-1} \Delta^n \begin{pmatrix} U_{2nk+m} & -q^m U_{2nk} \\ U_{2nk} & -q^m U_{2nk-m} \end{pmatrix}, \quad (5.2)$$

$$N_{k,m}^{2n-1} = U_m^{2n-2} \Delta^{n-1} \begin{pmatrix} V_{(2n-1)k+m} & -q^m V_{(2n-1)k} \\ V_{(2n-1)k} & -q^m V_{(2n-1)k-m} \end{pmatrix}. \quad (5.3)$$

The characteristic equation of  $N_{k,m}$  is

$$\lambda^2 - \Delta U_k U_m \lambda - \Delta q^k U_m^2 = 0, \quad (5.4)$$

and so

$$N_{k,m}^2 - \Delta U_k U_m N_{k,m} - \Delta q^k U_m^2 I = 0. \quad (5.5)$$

Using the previous techniques and due to the manner in which powers of  $N_{k,m}$  are defined, we have found some interesting summation identities. We note, however, that some of the methods applied to  $M_{k,m}$  do not apply to  $N_{k,m}$ . For example, we could find no succinct counterpart to (3.29). We state only the essential details and omit summation identities that we have obtained previously.

Manipulating (5.5), we can write

$$\Delta U_m (U_k N_{k,m} + q^k U_m I) = N_{k,m}^2 \quad (5.6)$$

and

$$(2N_{k,m} - \Delta U_k U_m I)^2 = \Delta U_m^2 V_k^2 I. \quad (5.7)$$

From (5.6) and (5.7), we have

$$\Delta^n U_m^n (U_k N_{k,m} + q^k U_m I)^n = N_{k,m}^{2n}, \quad (5.8)$$

$$(2N_{k,m} - \Delta U_k U_m I)^{2n} = \Delta^n U_m^{2n} V_k^{2n} I, \quad (5.9)$$

$$(2N_{k,m} - \Delta U_k U_m I)^{2n+1} = \Delta^n U_m^{2n} V_k^{2n} (2N_{k,m} - \Delta U_k U_m I). \quad (5.10)$$

Now expanding each of (5.8)-(5.10) and equating upper left entries of the relevant matrices leads, respectively, to

$$\sum_{\substack{i=0 \\ i \text{ even}}}^n \binom{n}{i} q^{k(n-i)} \Delta^{\frac{i}{2}} U_k^i U_{ik+m} + \sum_{\substack{i=1 \\ i \text{ odd}}}^n \binom{n}{i} q^{k(n-i)} \Delta^{\frac{i-1}{2}} U_k^i V_{ik+m} = U_{2nk+m}, \quad (5.11)$$

$$\sum_{\substack{i=0 \\ i \text{ even}}}^{2n} \binom{2n}{i} 2^i \Delta^{\frac{2n-i}{2}} U_k^{2n-i} U_{ik+m} - \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-1} \binom{2n}{i} 2^i \Delta^{\frac{2n-1-i}{2}} U_k^{2n-i} V_{ik+m} = V_k^{2n} U_m, \quad (5.12)$$

$$\sum_{\substack{i=1 \\ i \text{ odd}}}^{2n+1} \binom{2n+1}{i} 2^i \Delta^{\frac{2n+1-i}{2}} U_k^{2n+1-i} V_{ik+m} - \sum_{\substack{i=0 \\ i \text{ even}}}^{2n} \binom{2n+1}{i} 2^i \Delta^{\frac{2n+2-i}{2}} U_k^{2n+1-i} U_{ik+m} = V_k^{2n+1} V_m. \quad (5.13)$$

Finally, making use of (2.2) and (2.3) and applying to (5.11)-(5.13) the same technique used to obtain (3.21), we have

$$\sum_{\substack{i=0 \\ i \text{ even}}}^n \binom{n}{i} q^{k(n-i)} \Delta^{\frac{i}{2}} U_k^i V_{ik+m} + \sum_{\substack{i=1 \\ i \text{ odd}}}^n \binom{n}{i} q^{k(n-i)} \Delta^{\frac{i+1}{2}} U_k^i U_{ik+m} = V_{2nk+m}, \quad (5.14)$$

$$\sum_{\substack{i=0 \\ i \text{ even}}}^{2n} \binom{2n}{i} 2^i \Delta^{\frac{2n-i}{2}} U_k^{2n-i} V_{ik+m} - \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-1} \binom{2n}{i} 2^i \Delta^{\frac{2n+1-i}{2}} U_k^{2n-i} U_{ik+m} = V_k^{2n} V_m, \quad (5.15)$$

$$\sum_{\substack{i=1 \\ i \text{ odd}}}^{2n+1} \binom{2n+1}{i} 2^i \Delta^{\frac{2n+3-i}{2}} U_k^{2n+1-i} U_{ik+m} - \sum_{\substack{i=0 \\ i \text{ even}}}^{2n} \binom{2n+1}{i} 2^i \Delta^{\frac{2n+2-i}{2}} U_k^{2n+1-i} V_{ik+m} = \Delta V_k^{2n+1} U_m. \quad (5.16)$$

## REFERENCES

1. G. E. Bergum & V. E. Hoggatt, Jr. "Sums and Products for Recurring Sequences." *The Fibonacci Quarterly* **13.2** (1975):115-20.
2. H. W. Gould. "A History of the Fibonacci  $Q$ -Matrix and a Higher-Dimensional Problem." *The Fibonacci Quarterly* **19.3** (1981):250-57.
3. V. E. Hoggatt, Jr. "Belated Acknowledgment." *The Fibonacci Quarterly* **6.3** (1968):85.
4. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* **3.3** (1965):161-76.
5. J. Ivie. "A General  $Q$ -Matrix." *The Fibonacci Quarterly* **10.3** (1972):255-61, 264.
6. D. Jarden. *Recurring Sequences*. Jerusalem: Riveon Lematematika, 1966.
7. E. Lucas. *Théorie des Nombres*. Paris: Albert Blanchard, 1961.
8. Br. J. M. Mahon. "Diagonal Functions of Pell Polynomials." Ph.D. dissertation presented to the University of New England, 1987.

9. Br. J. M. Mahon & A. F. Horadam. "Pell Polynomial Matrices." *The Fibonacci Quarterly* **25.1** (1987):21-28.
10. A. G. Shannon & A. F. Horadam. "Some Properties of Third-Order Recurrence Relations." *The Fibonacci Quarterly* **10.2** (1972):135-45.
11. M. E. Waddill. "Using matrix Techniques To Establish Properties of a Generalized Tribonacci Sequence." In *Applications of Fibonacci Numbers 4*:299-308. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1991.
12. M. E. Waddill. "Using Matrix Techniques To Establish Properties of  $k$ -Order Linear Recursive Sequences." In *Applications of Fibonacci Numbers 5*:601-15. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1993.
13. M. E. Waddill & L. Sacks. "Another Generalized Fibonacci Sequence." *The Fibonacci Quarterly* **5.3** (1967):209-22.

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