SOME SUMMATION IDENTITIES USING GENERALIZED g-MATRICES

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1. INTRODUCTION

In a belated acknowledgment, Hoggatt [3] states:

The first use of the Q-matrix to generate the Fibonacci numbers appears in an abstract of a paper by Professor J. L. Brenner by the title "Lucas' Matrix." This abstract appeared in the March 1951 American Mathematical Monthly on pages 221 and 222. The basic exploitation of the Q-matrix appeared in 1960 in the San Jose State College Master's thesis of Charles H. King with the title "Some Further Properties of the Fibonacci Numbers." Further utilization of the Q-matrix appears in the Fibonacci Primer sequence parts I-V.

For a comprehensive history of the Q-matrix, see Gould [2]. Numerous analogs of the Q-matrix relating to third-order recurrences have been used. See, for instance, Waddill and Sacks [13], Shannon and Horadam [10], and Waddill [11]. Mahon [8] has made extensive use of matrices to study his third-order diagonal functions of the Pell polynomials. Recently, Waddill [12] considered a general Q-matrix. He defined and used the $k \times k$ matrix

$$ R = \begin{pmatrix} r_0 & r_1 & \cdots & r_{k-1} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} $$

in relation to a $k$-order linear recursive sequence $\{V_n\}$, where

$$ V_n = \sum_{i=0}^{k-1} r_i V_{n-i}, \quad n \geq k. $$

The matrix $R$ generalized the matrix $Q_k$ of Ivie [5].

In the notation of Horadam [4], write

$$ W_n = W_n(a, b; p, q) $$

so that

$$ W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = a, \quad W_1 = b, \quad n \geq 2. $$

With this notation, define

$$ \begin{cases} U_n = W_n(0, 1; p, q), \\ V_n = W_n(2, p, p, q), \end{cases} $$

Indeed, $\{U_n\}$ and $\{V_n\}$ are the fundamental and primordial sequences generated by (1.2). They have been studied extensively, particularly by Lucas [7]. Further information can be found in [1], [4], and [6].
The most commonly used matrix in relation to the recurrence relation (1.2) is

\[ M = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}. \]  

(1.4)

which, for \( p = -q = 1 \), reduces to the ordinary \( Q \)-matrix. In this paper we define a more general matrix \( M_{k, m} \) parametrized by \( k \) and \( m \) and reducing to \( M \) for \( k = m = 1 \). We use \( M_{k, m} \) to develop various summation identities involving terms from the sequences \( \{U_n\} \) and \( \{V_n\} \).

Our work is a generalization of the work of Mahon and Horadam [9] who used several pairs of \( 2 \times 2 \) matrices to generate summation identities involving terms from the Pell polynomial sequences

\[
\begin{cases}
P_n = W_n(0, 1; 2x, -1), \\
Q_n = W_n(2, 2x; 2x, -1).
\end{cases}
\]  

(1.5)

We generalize their work in two ways. First, we consider sequences generated by a more general recurrence relation. Second, our parametrization of the matrix \( M_{k, m} \) includes all the matrices considered by Mahon and Horadam as special cases.

2. THE MATRIX \( M_{k, m} \)

Before proceeding, we state some results which are used subsequently. None of these is new and each can be proved using Binet forms. If

\[ \Delta = p^2 - 4q, \]  

(2.1)

then

\[ U_{n+1} - qU_{n-1} = V_n, \]  

(2.2)

\[ V_{n+1} - qV_{n-1} = \Delta U_n, \]  

(2.3)

\[ V_{2k} - 2q^k = \Delta U_k^2, \]  

(2.4)

\[ U_{k+m} - q^mU_{k-m} = U_mV_k, \]  

(2.5)

\[ V_{k+m} - q^mV_{k-m} = \Delta U_kU_m, \]  

(2.6)

\[ U_{k+m}U_{k-m} - U_k^2 = -q^{k-m}U_m^2, \]  

(2.7)

\[ V_{k+m}V_{k-m} - V_k^2 = \Delta q^{k-m}U_m^2, \]  

(2.8)

\[ U_{n+m}U_{n+m} - q^nU_nU_{n+m} = U_mU_{n+m+n+m}. \]  

(2.9)

By induction it can be proved that, for the matrix \( M \) in (1.4),

\[ M^n = \begin{pmatrix} U_{n+1} & -qU_n \\ U_n & -qU_{n-1} \end{pmatrix}. \]  

(2.10)

where \( n \) is an integer.
We now give a generalization of the matrix $M$. Associated with the recurrence (1.2) and with \{${U_n}$\} as in (1.3), define

$$M_{k,m} = \begin{pmatrix} U_{k+m} & -q^m U_k \\ U_k & -q^m U_{k-m} \end{pmatrix},$$

(2.11)

where $k$ and $m$ are integers. By induction and making use of (2.9), it can be shown that, for all integral $n,$

$$M^n_{k,m} = U_{nk+m} \begin{pmatrix} U_{nk} & -q^m U_{nk} \\ U_{nk} & -q^m U_{nk-m} \end{pmatrix}.$$  

(2.12)

When $k = m = 1,$ we see that $M_{k,m}$ reduces to $M$ and $M^n_{k,m}$ reduces to $M^n.$

3. SUMMATION IDENTITIES

We now use the matrix $M_{k,m}$ to produce summation identities involving terms from \{${U_n}$\} and \{${V_n}$\}. Using (2.5) and (2.7), we find that the characteristic equation of $M_{k,m}$ is

$$\lambda^2 - U_{m,k} \lambda + q^k U_m^2 = 0$$

(3.1)

and, by the Cayley-Hamilton theorem,

$$M^2_{k,m} = U_{m,k} M_{k,m} + q^k U_m^2 I = 0,$$  

(3.2)

where $I$ is the $2 \times 2$ unit matrix. From (3.2), we have

$$(U_{m,k} M_{k,m} - q^k U_m^2 I)^n M^j_{k,m} = M^{2n+j}_{k,m},$$

(3.3)

and expanding yields

$$\sum_{i=0}^n \binom{n}{i} (-1)^{n-i} q^{k(n-i)} U_{m,k} M^{2n+i}_{k,m} M^j_{k,m} = M^{2n+j}_{k,m}.$$  

(3.4)

Using (2.12) to equate upper left entries gives

$$\sum_{i=0}^n \binom{n}{i} (-1)^{n-i} q^{k(n-i)} U_{m,k} M^{2n+i}_{k,m} M^j_{k,m} = U^{(2n+j)k+m}_{(2n+j)k+m}.$$  

(3.5)

Again from (3.2),

$$(M^2_{k,m} + q^k U_m^2 I)^n = U_{m,k}^n M^n_{k,m},$$  

(3.6)

and expanding we have

$$\sum_{i=0}^n \binom{n}{i} q^{k(n-i)} U_{m,k}^{2(n-i)} M^{2i}_{k,m} M^n_{k,m} = U_{m,k}^n V^n_{k,m}.$$  

(3.7)

Using (2.12) to equate upper left entries gives

$$\sum_{i=0}^n \binom{n}{i} q^{k(n-i)} U_{2i} M^{2i}_{k,m} = V^n_{k,m}.$$  

(3.8)
Once again, from (3.2),
\[(M_{2k,m} - q^k U_m I)^2 = U_m (V_{2k} - 2q^k) M_{2k,m} = \Delta U_m U^{2n}_{k} \Delta U_m U^{2n}_{k},\]  
(3.9)
and expanding, after taking \(n^\text{th}\) powers, we have
\[\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i q^i (2n-i) U_m^{2n-i} M_{2k,m} = \Delta U_m U^{2n}_{k} \Delta U_m U^{2n}_{k}.\]  
(3.10)
Equating upper left entries yields
\[\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i q^i (2n-i) U_{2ik+m} = \Delta U_m U^{2n}_{k} U_{2nk+m}.\]  
(3.11)
From (3.9),
\[(M_{2k,m} - q^k U_m I)^{2n+1} = \Delta U_m U^{2n}_{k} (M_{2k,m} - q^k U_m M_{2k,m}).\]  
(3.12)
Equating upper left entries yields, after simplifying,
\[\sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^i q^i (2n+1-i) U_{2ik+m} = \Delta U_m U^{2n}_{k} (U_{2(n+1)k+m} - q^k U_{2nk+m}),\]  
(3.13)
and using (2.5) to simplify the right side gives
\[\sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^i q^i (2n+1-i) U_{2ik+m} = \Delta U_m U^{2n+1}_{k} U_{(2n+1)k+m}.\]  
(3.14)
This should be compared to (3.11).
Manipulating the characteristic equation (3.1), we have \((2\lambda - U_m V_k)^2 = \Delta U_m^2 V_k^2\), so that
\[(2M_{k,m} - U_m V_k I)^2 = \Delta U_m^2 V_k^2 I.\]  
(3.15)
Expanding gives
\[\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i 2^i U_m^{2n-i} V_k^i M_{k,m} = \Delta U_m^2 V_k^2 U_m.\]  
(3.16)
Equating upper left entries and also lower left entries yields, respectively,
\[\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i 2^i V_k^{2n-i} U_{ik+m} = \Delta U_k^2 U_m I,\]  
(3.17)
\[\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i 2^i V_k^{2n-i} U_{ik} = 0.\]  
(3.18)
We note that (3.17) reduces to (3.18) when \(m = 0\).
Multiplying both sides of (3.15) by \((2M_{k,m} - U_m V_k I)\) and expanding gives
\[\sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^i 2^i U_m^{2n+1-i} V_k^{2n+1-i} M_{k,m} = \Delta U_m^2 U_k^2 (2M_{k,m} - U_m V_k I).\]  
(3.19)
Equating upper left entries yields
\[
\sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^{n-i} 2^{i+1} V_k^{2n+1-i} U_k = \Delta^n V_k^{2n+1} U_m, \tag{3.20}
\]
which should be compared to (3.17).

Now, using (3.5), we have
\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} q^{k(n-i)} U_k^{i} (U_{(i+j)k+m+1} - q U_{(i+j)k+m}) = U_{(2n+j)k+m+1} - q U_{(2n+j)k+m-1},
\]
and (2.2) shows that this simplifies to
\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} q^{k(n-i)} U_k^{i} (U_{(i+j)k+m}) = V_{(2n+j)k+m}. \tag{3.21}
\]

Making use of (2.2) and (2.3) and working in the same manner with identities (3.8), (3.11), (3.14), (3.17), and (3.20) yields, respectively,
\[
\sum_{i=0}^{n} \binom{n}{i} q^{k(n-i)} V_{2ik+m} = V_k^{n} V_{nk+m}, \tag{3.22}
\]
\[
\sum_{i=0}^{2n} \binom{2n}{i} (-1)^{n-i} q^{k(2n-i)} V_{2ik+m} = \Delta^n V_k^{2n} V_{2nk+m}, \tag{3.23}
\]
\[
\sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^{n-i} q^{k(2n+1-i)} V_{2ik+m} = \Delta^n V_k^{2n+1} U_{(2n+1)k+m}, \tag{3.24}
\]
\[
\sum_{i=0}^{2n} \binom{2n}{i} (-1)^{n-i} 2^{i} V_k^{2n-i} V_{2ik+m} = \Delta^n V_k^{2n} V_m, \tag{3.25}
\]
\[
\sum_{i=0}^{2n} \binom{2n}{i} (-1)^{n-i} 2^{i} V_k^{2n-i} V_{2ik+m} = \Delta^n V_k^{2n} U_m. \tag{3.26}
\]

In what follows, we make use of the following result:
\[
M_{k,m}^n M_{k,m}^n = U_{n+k}^{n+k} \left( \begin{array}{cc}
U_{nk+nk} & -q^n U_{nk+nk} \\
0 & -q^n U_{nk+nk-nm}
\end{array} \right). \tag{3.27}
\]

This is proved by multiplying the matrices on the left and using (2.9).

Consider now the special case of (3.2), where \( k = m \). Then, using (2.5),
\[
M_{k,k}^2 = U_{2k}^{2k} M_{k,k} - q^k U_{2k}^2 I. \tag{3.28}
\]

Using (3.28) and (2.9), we can show by induction that, for \( n \geq 2 \),
\[
M_{k,k}^n = U_{n-k}^{n-k} \left( U_{nk} M_{k,k} - q^k U_{n-k} I \right). \tag{3.29}
\]
The binomial theorem applied to (3.29) gives

$$U_k^{(n-2)s} \sum_{i=0}^{s} \binom{s}{i} (-1)^{s-i} q^{k(s-i)} U_k^{i-1} U_{(n-1)k}^{i-1} U_{nk}^i M_{k,k,k}^i = M_{k,k,k}^{n+i}$$

(3.30)

Equating lower left entries of the relevant matrices then yields

$$\sum_{i=0}^{s} \binom{s}{i} (-1)^{s-i} q^{k(s-i)} U_k^{i-1} U_{(n-1)k}^{i-1} U_{nk}^i U_{(i+j)k} = U_k^i U_{(n+j)k}. $$

(3.31)

Multiplying both sides of (3.30) by $M_{k,k}$ and using (3.27) to equate lower left entries gives

$$\sum_{i=0}^{s} \binom{s}{i} (-1)^{s-i} q^{k(s-i)} U_k^{i-1} U_{(n-1)k}^{i-1} U_{nk}^i U_{(i+j)k+k_i} = U_k^i U_{(n+j)k+k_i}, $$

(3.32)

which generalizes (3.31).

Again from (3.29), after transposing terms and raising to a power $s$, we obtain

$$\sum_{i=0}^{s} \binom{s}{i} q^{k(s-i)} U_k^{(n-1)(s-i)} U_{(n-1)k}^{i-1} M_{k,k}^i = U_k^{(n-2)s} U_{nk}^s M_{k,k,k}^s, $$

(3.33)

which yields

$$\sum_{i=0}^{s} \binom{s}{i} q^{k(s-i)} U_k^{i-1} U_{(n-1)k}^{i-1} U_{nk}^i U_{sk} = U_k^i U_{sk}.$$  

(3.34)

Multiplying both sides of (3.33) by $M_{k,k}$ and using (3.27) to equate lower left entries gives

$$\sum_{i=0}^{s} \binom{s}{i} q^{k(s-i)} U_k^{i-1} U_{(n-1)k}^{i-1} U_{nk}^i U_{sk+k_i} = U_k^i U_{sk+k_i}, $$

(3.35)

which generalizes (3.34).

Continuing in this manner after yet again transposing terms in (3.29) and raising to a power $s$, we obtain

$$\sum_{i=0}^{s} \binom{s}{i} (-1)^i U_k^{(n-2)(s-i)} U_{nk}^{i-1} M_{k,k}^{(n-1)+i} = q^{k^2} U_k^{(n-1)s} U_{(n-1)k}^i I. $$

(3.36)

Equating upper left entries and lower left entries yields, respectively,

$$\sum_{i=0}^{s} \binom{s}{i} (-1)^i U_k^i U_{nk}^{i-1} U_{(n-1)+i+k} = q^{k^2} U_{(n-1)k}^i U_k $$

(3.37)

$$\sum_{i=0}^{s} \binom{s}{i} (-1)^i U_k^i U_{nk}^{i-1} U_{(n-1)+i} = 0. $$

(3.38)

Multiplying (3.36) by $M_{k,k}$ and equating lower left entries yields

$$\sum_{i=0}^{s} \binom{s}{i} (-1)^i U_k^i U_{nk}^{i-1} U_{(n-1)+i+k_i} = q^{k^2} U_k^{(n-1)k} U_{k_i}. $$

(3.39)
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We note that, when $k_1 = k$, (3.39) reduces to (3.37) and when $k_1 = 0$, (3.39) reduces to (3.38).

Now, manipulating (3.32), (3.35), and (3.39) in the same way that (3.5) was manipulated to yield (3.21), we obtain, respectively,

$$
\sum_{i=0}^{s}(s\choose i)(-1)^{s-i}q^{k(s-i)}U_{k}^{(s-i)j}U_{n+1}^{i}V_{(s+i)k+k_i} = U_{k}^{(s+1)j+k_i},
$$

(3.40)

$$
\sum_{i=0}^{s}(s\choose i)q^{k(s-i)}U_{k}^{(s-i)j}U_{n+1}^{i}V_{n+k+k_i} = U_{n}^{s}V_{sk+k_i},
$$

(3.41)

$$
\sum_{i=0}^{s}(s\choose i)(-1)^{s-i}U_{n}^{i}U_{n+1}^{j}V_{(n+1)i+k+k_i} = q^{k(s+i)}U_{(n+1)i+k}V_{k_i},
$$

(3.42)

4. THE MATRIX $X_k$

We have found a matrix having the property of generating terms from $\{U_n\}$ and $\{V_n\}$ simultaneously. It is a generalization of the matrix $W$ introduced by Mahon and Horadam [9]. Define

$$
X_k = \begin{pmatrix}
V_k & U_k \\
\Delta U_k & V_k
\end{pmatrix},
$$

(4.1)

Then by induction we have, for integral $n$,

$$
X_k^n = 2^{n-1}\begin{pmatrix}
V_{nk} & U_{nk} \\
\Delta U_{nk} & V_{nk}
\end{pmatrix}.
$$

(4.2)

Noting that $X_k^{m+n} = X_k^m \cdot X_k^n$ produces the well-known identities

$$
2V_{m+n} = V_{n}V_{m} + \Delta U_{n}U_{n},
$$

(4.3)

$$
2U_{m+n} = V_{n}U_{m} + U_{n}V_{n}.
$$

(4.4)

The characteristic equation for $X_k$ is

$$
\lambda^2 - 2V_k \lambda + 4q^k = 0
$$

(4.5)

and so, by the Cayley-Hamilton theorem

$$
X_k^2 - 2V_k X_k + 4q^k I = 0.
$$

(4.6)

Using (4.3) and (4.4), we see that

$$
X_k^n X_{k_1} = 2^n\begin{pmatrix}
V_{nk+k_1} & U_{nk+k_1} \\
\Delta U_{nk+k_1} & V_{nk+k_1}
\end{pmatrix}.
$$

(4.7)

Considering the case $k = 1$, we can show by induction, with the aid of (4.6), that

$$
X_1^n = 2^{n-1}(U_n X_1 - 2q U_{n-1}I), \quad n \geq 2,
$$

(4.8)

which is analogous to (3.29).
It is interesting to note that the methods applied to $M_{k,m}$ when applied to $X_k$ produce most of the summation identities that we have obtained so far. The exceptions are the identities that arose by using (3.29). The analogous procedure for $X_k$ is to use (4.8), but the identities that arise are less general. For example, (4.8) produces

$$
\sum_{i=0}^{k} \binom{s}{i} (-1)^{s-i} q^{s-i} U_n^{i-k} U_{n+j+k_i} = U_{n+j+k_i},
$$

(4.9)

which is a special case of (3.32).

5. THE MATRIX $N_{k,m}$

We have found yet another matrix defined in a similar manner to $M_{k,m}$ whose powers also generate terms of the sequences $\{U_n\}$ and $\{V_n\}$. Define

$$
N_{k,m} = \begin{pmatrix}
V_{k+m} & -q^m V_k \\
V_k & -q^m V_{k-m}
\end{pmatrix}
$$

(5.1)

Then for all integral $n$,

$$
N_{k,m}^{2n} = U_m^{2n-1} \Delta^n \begin{pmatrix}
U_{2nk+m} & -q^m U_{2nk} \\
U_{2nk} & -q^m U_{2nk-m}
\end{pmatrix},
$$

(5.2)

$$
N_{k,m}^{2n-1} = U_m^{2n-2} \Delta^{n-1} \begin{pmatrix}
V_{(2n-1)k+m} & -q^m V_{(2n-1)k} \\
V_{(2n-1)k} & -q^m V_{(2n-1)k-m}
\end{pmatrix}
$$

(5.3)

The characteristic equation of $N_{k,m}$ is

$$
\lambda^2 - \Delta U_k U_m \lambda - \Delta q^k U_m^2 = 0,
$$

(5.4)

and so

$$
N_{k,m}^2 - \Delta U_k U_m N_{k,m} - \Delta q^k U_m^2 I = 0.
$$

(5.5)

Using the previous techniques and due to the manner in which powers of $N_{k,m}$ are defined, we have found some interesting summation identities. We note, however, that some of the methods applied to $M_{k,m}$ do not apply to $N_{k,m}$. For example, we could find no succinct counterpart to (3.29). We state only the essential details and omit summation identities that we have obtained previously.

Manipulating (5.5), we can write

$$
\Delta U_m (U_k N_{k,m} + q^k U_m I) = N_{k,m}^2
$$

(5.6)

and

$$
(2N_{k,m} - \Delta U_k U_m I)^2 = \Delta U_m^2 q^2 I.
$$

(5.7)

From (5.6) and (5.7), we have

$$
\Delta U_m^2 (U_k N_{k,m} + q^k U_m I)^n = N_{k,m}^2
$$

(5.8)
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\[
(2N_{k,m} - AU_kU_mI)^{2n} = \Delta^n U_k^{2n} V_k^{2n} I,
\]

(5.9)

\[
(2N_{k,m} - AU_kU_mI)^{2n+1} = \Delta^n U_k^{2n+1} V_k^{2n+1} (2N_{k,m} - AU_kU_mI).
\]

(5.10)

Now expanding each of (5.8)-(5.10) and equating upper left entries of the relevant matrices leads, respectively, to

\[
\sum_{i=0}^{n} \binom{n}{i} q^{k(n-i)} \Delta^k U_k^{i} V_{ik+m} + \sum_{i=1, \text{even}}^{n} \binom{n}{i} q^{k(n-i)} \Delta^k U_k^{i} V_{ik+m} = U_{2nk+m},
\]

(5.11)

\[
\sum_{i=0}^{2n} \binom{2n}{i} 2^{i} \Delta^k U_k^{2n-i} V_{ik+m} - \sum_{i=1, \text{odd}}^{2n} \binom{2n}{i} 2^{i} \Delta^k U_k^{2n-i} V_{ik+m} = V_{2nU_m},
\]

(5.12)

\[
\sum_{i=1, \text{odd}}^{2n+1} \binom{2n+1}{i} 2^{i} \Delta^k U_k^{2n+1-i} V_{ik+m} - \sum_{i=1, \text{even}}^{2n} \binom{2n}{i} 2^{i} \Delta^k U_k^{2n+1-i} V_{ik+m} = V_{2n+1} U_m.
\]

(5.13)

Finally, making use of (2.2) and (2.3) and applying to (5.11)-(5.13) the same technique used to obtain (3.21), we have

\[
\sum_{i=0}^{n} \binom{n}{i} q^{k(n-i)} \Delta^k U_k^{i} V_{ik+m} + \sum_{i=1, \text{even}}^{n} \binom{n}{i} q^{k(n-i)} \Delta^k U_k^{i} V_{ik+m} = V_{2nk+m},
\]

(5.14)

\[
\sum_{i=0}^{2n} \binom{2n}{i} 2^{i} \Delta^k U_k^{2n-i} V_{ik+m} - \sum_{i=1, \text{odd}}^{2n} \binom{2n}{i} 2^{i} \Delta^k U_k^{2n-i} V_{ik+m} = V_{2nU_m},
\]

(5.15)

\[
\sum_{i=1, \text{odd}}^{2n+1} \binom{2n+1}{i} 2^{i} \Delta^k U_k^{2n+1-i} V_{ik+m} - \sum_{i=1, \text{even}}^{2n} \binom{2n}{i} 2^{i} \Delta^k U_k^{2n+1-i} V_{ik+m} = \Delta V_{2n+1} U_m.
\]

(5.16)

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