A construction of higher-rank lattice rules

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Abstract

Lattice rules are quasi-Monte Carlo methods for numerical multiple integration that are based on the selection of an s-dimensional integration lattice. The abscissa set is the intersection of the integration lattice with the unit hypercube. It is well-known that the abscissa set of a lattice rule can be generated by a number of fixed rational vectors. In general, different sets of generators produce different integration lattices and rules, and a given rule has many different generator sets. The rank of the rule is the minimum number of generators required to span the abscissa set.

A lattice rule is usually specified by a generator set, and the quality of the rule varies with the choice of generator set. This paper describes a new method for the construction of generator sets for higher-rank rules that is based on techniques arising from the theory of simultaneous Diophantine approximation. The method extends techniques currently applied in the rank 1 case.

1 Introduction

An s-dimensional lattice rule is a quasi-Monte Carlo rule which can be written in the form

$$Q(f) = \frac{1}{N} \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_m=0}^{n_m-1} f\left(\left\{\sum_{i=1}^{m} j_i \mathbf{g}_i / n_i\right\}\right), \tag{1}$$

where each quadrature point occurs only once in the sum. In (1), $\mathbf{g}_1, \ldots, \mathbf{g}_m \in \mathbb{Z}^s$ and the braces indicate that addition is modulo $\mathbb{Z}^s$, and $N = \prod_{i=1}^{m} n_i$, where $1 \leq m \leq s$, $n_1 > 1$ and $n_{i+1} \mid n_i$, for $i = 1, \cdots, m-1$. The value of m is called the *rank* of the rule and the vectors $\mathbf{g}_i/n_i$ constitute a *generator set* for the rule. The quadrature points of a rule, together with the integer lattice in $\mathbb{Z}^s$, generate the *integration lattice* corresponding to the rule. The *dual lattice*, denoted by $L^\perp$, is defined by

$$L^\perp = \{\mathbf{h} \in \mathbb{Z}^s : \mathbf{x} \cdot \mathbf{h} \in \mathbb{Z} \text{ for all } \mathbf{x} \in L\}. \tag{2}$$

A commonly used figure of merit for lattice rules is

$$P_\alpha = \sum_{\mathbf{h} \in L^\perp - \{\mathbf{0}\}} \frac{1}{r(\mathbf{h})^\alpha},$$

where $r(\mathbf{h}) = \prod_{i=1}^{s} \max(1, |h_i|)$ and α is an even positive integer. For surveys of the theory of lattice rules the reader is referred to [Niederreiter, 1992] and [Sloan and Joe, 1994].

A rank 1 rule is said to be *simple* if it has a generator $\mathbf{g}/N$ such that $\gcd(g_i, N) = 1$ for some $i \in \{1, \ldots, s\}$. In [Langtry, 1996] a vector continued fraction algorithm was used to construct sequences of rank 1 simple lattice rules, and in [Langtry, 1998] this approach was adapted to the construction of certain maximal rank rules. In each case a single generator is determined by each iteration of the continued fraction algorithm—in the case of maximal rank rules a full generator set for the rule is then determined by a process of scaling and copying the corresponding rank 1 rule. The results of these papers agree with suggestions by earlier authors [Disney and Sloan, 1992] that certain maximal rank rules formed by copying rank 1 rules may be at least competitive with rank 1 rules of similar orders. Sloan and Joe [Sloan and Joe, 1994, Ch. 7] make a similar observation with respect to certain intermediate rank rules used to construct embedded sequences of rules. In this paper we investigate the feasibility of constructing intermediate rank rules directly. In particular, we seek to extend the continued fraction algorithm in a way which allows the construction of generator sets with more than one generator. The rank of such a rule may then be found by the techniques described in [Langtry, 1995,Lyness and Keast, 1995].

2 Background

In general terms, in order for a rule generated by rational vectors $\mathbf{g}_j/n_j$ ($j = 1, \ldots, m$) to possess a good figure of merit (in particular, a small value of P_α) it is necessary that the smallest values of $r(\mathbf{h})$ for $\mathbf{h} \in L^\perp$ be large. Now it follows from (2) that $\mathbf{h} \in L^\perp$ if and only if for $j = 1, \ldots, m$ we have $\mathbf{h} \cdot \mathbf{g}_j/n_j = -c_{0,j}$ for some $c_{0,j} \in \mathbb{Z}$. That is, $\Gamma_{\mathbf{c}_j}(1, \mathbf{g}_j/n_j) = 0$, where $\mathbf{c}_j = (c_{0,j}, \mathbf{h})$ and $\Gamma_{\mathbf{c}_j}$ is the linear form

$$\Gamma_{\mathbf{c}_j}(x_0, \mathbf{x}) = c_{0,j} x_0 + \mathbf{h} \cdot \mathbf{x}.$$

That is, we would like to choose the generators so that the only values of $\mathbf{h}$ for which $\Gamma_{\mathbf{c}_j}(1, \mathbf{g}_j/n_j) = 0$, for each $j = 1, \ldots, m$, are large in magnitude. Since each $\Gamma_{\mathbf{c}_j}$ is continuous on $\mathbb{R}^{s+1}$, for fixed $\mathbf{x}_j \in \mathbb{R}^s$ with $\mathbf{x}_j$ close to $\mathbf{g}_j/n_j$, this is equivalent to requiring that (allowing $\mathbf{c}_j$ to vary) the smallest values

of $\Gamma_{\mathbf{c}_j}(1, \mathbf{x}_j)$ be large relative to the size of $\mathbf{h}$. A less stringent requirement is that they be large relative to the size of $\mathbf{c}_j$, for each $j = 1, \ldots, m$.

This problem is closely related to a problem arising in the theory of Diophantine approximation which we may express as follows. Let $\mathbf{1} = (1, \ldots, 1)^T$ and, for a given $t_1 \times t_2$ matrix X, denote by $(\mathbf{1}, X^T)$ the matrix with first column $\mathbf{1}$ and remaining columns those of X^T. Let $\Gamma_{\mathbf{c}}(\mathbf{1}, X) = \left(\mathbf{1}, X^T\right)\mathbf{c}$, where $\mathbf{c} \in \mathbb{Z}^{t_1+1}$. Find those $\mathbf{c}$ for which $\Gamma_{\mathbf{c}}(\mathbf{1}, X)$ best approximates $\mathbf{0}$, relative to the size of $\mathbf{c}$—that is, find $\mathbf{c}$ and $\gamma = (\gamma_1, \ldots, \gamma_{t_2})^T = \Gamma_{\mathbf{c}_j}(\mathbf{1}, X)$ such that γ is as small as possible (for the given X) when $\mathbf{c}$ is small. For the purpose of constructing generator sets for lattice rules, we seek to identify matrices X such that these small values of γ are relatively poor (that is, large), or equivalently, such that to obtain small values of γ we must have $\mathbf{c}$ relatively large. It then follows from a classical transference theorem [Cassels, 1957, Ch. V, Theorem II] that the matrix X is poorly approximated by rational matrices (of lower denominator, if X is itself rational).

In the case where $t_1 = t_2 = 1$, these problems reduce to finding numbers x that are badly approximated by rationals (where $x \notin \mathbb{Q}$), or by rationals of lower denominator (where $x \in \mathbb{Q}$), relative to the size of the denominator. These are precisely those numbers whose continued fractions have small partial quotients. The classical two-dimensional Fibonacci rules (see for example [Sloan and Joe, 1994, §4.3]) provide an example of their use in the construction of lattice rules. The ordinary continued fraction algorithm in one dimension has many applications, one of which is the construction of rational numbers which are good approximations to the number, say x, represented by the continued fraction. This is a rational number in the case of a terminating continued fraction, and an irrational in the non-terminating case.

In the case where $t_1 > 1$ and $t_2 = 1$, vectors $\mathbf{x}$ that are badly approximable in the sense of the preceding paragraph are characterised by having best rational approximations whose denominators are not widely separated [Lagarias, 1982]. Further, it is known [Langtry, 1996, Corollary 4.2] that if $\mathbf{x} \in \mathbb{R}$ is badly approximable then so are its best rational approximations, except possibly a finite number of them. In [Langtry, 1996] these observations were exploited in the construction of sequences of rank 1 rules with good figures of merit from the sequence of best rational approximation vectors yielded by the Szekeres continued fraction algorithm [Szekeres, 1970]. In this paper we generalise this algorithm to the case of $t_2 > 1$ and obtain sequences of lattice rules of rank $m \geq 1$.

3 Multidimensional continued fractions

In [Szekeres, 1970], Szekeres describes a general framework for extending the ordinary continued fraction algorithm to vectors in higher dimensions, giving a particular example. Later authors [Brentjes, 1981a,Brentjes, 1981b,Lagarias, 1994] have elaborated this approach. Szekeres' approach is based on a geometric interpretation of the one-dimensional algorithm in which Farey dissections are made of an initial interval with one end-point fixed, characterised as a one-dimensional simplex with a fixed vertex, yielding a sequence of simplexes containing the number x to be approximated. The vertices of the simplexes determine the intermediary approximations and convergents of the ordinary continued fraction algorithm. Successive dissections continue from one direction (that is, with one vertex fixed) until an approximation error relative to the reciprocal of the denominator (that is, a value of $\Gamma_{\mathbf{c}}(1, x)$, for suitable $\mathbf{c}$) is found which is larger than that corresponding to the previous best approximation. The preceding approximation is a convergent of the continued fraction and is a best rational approximation to x. Since it is known that successive convergents lie on opposite sides of x, succeeding dissections occur from the opposite direction.

The higher-dimensional analogue, described in [Szekeres, 1970], of the ordinary continued fraction algorithm is designed to yield good approximations to a vector $\mathbf{x} \in \mathbb{R}^t$, and, correspondingly, small values $\Gamma_{\mathbf{c}}(1, \mathbf{x}) = c_0 + \sum_{i=1}^{t} c_i x_i$ at $(1, \mathbf{x})$ of the linear form $\Gamma_{\mathbf{c}}$. An initial simplex, containing $\mathbf{x}$, with vertices in $\mathbb{R}^t$ is identified. Corresponding to this simplex is a $(t+1) \times (t+1)$ matrix $M^{(0)}$ in which are stored the numerators and denominators of the nearest $t+1$ integer vectors to $\mathbf{x}$. These are stored as rational vectors: the denominator and numerator of each vector is stored as a column of the matrix—the denominator is stored in the initial row (with row index 0) and the numerator in the remaining rows of the column. The initial simplex is then iteratively reduced by applying generalised Farey dissections to its edges. Each dissection yields a new smaller simplex containing $\mathbf{x}$ and having t vertices in common with the preceding simplex. It has one new vertex, which yields a new rational approximation to $\mathbf{x}$. Corresponding to the n'th simplex is a matrix $M^{(n)}$ containing $t+1$ rational approximations to $\mathbf{x}$, which correspond to the intermediary approximations and convergents of the one-dimensional algorithm.

At each iteration two things must be determined: (i) whether the current approximation is (in some appropriate sense) a 'best' approximation, and (ii) which edge of the simplex should be dissected in the next iteration. Examples are known [Lagarias, 1980] for which the Szekeres algorithm does not find all best rational approximations to a given $\mathbf{x}$. The best approximations found by the algorithm are called *principal approximations*. The initial principal approximation is that represented by the initial column (with column index 0) of

$M^{(0)}$. As in the one-dimensional case, the vertex of the simplex corresponding to the current principal approximation is fixed and generalised Farey dissections are applied iteratively to edges with this vertex in common until a value of $\Gamma(1, \mathbf{x})$ is found that is larger than that found in the previous step. The preceding approximation then yields a new principal approximation. Each time a principal approximation is found the corresponding vertex of the simplex is fixed in succeeding iterations and dissections are applied to a new subset of edges. The rule for selecting the next edge to be dissected in each iteration is designed to reduce the volume of the simplex in a way which avoids the construction of a needle-shaped region, in order not to miss the identification of good approximations. For this reason the selection rule, denoted by μ, does not depend explicitly on the vector $\mathbf{x}$—rather, it depends on a measure of the size of the simplex.

With this type of construction, a multidimensional continued fraction is determined by an initial choice of simplex and the vector $\mathbf{x}$ to be approximated. Successive approximation vectors are determined by a simple recursion based on the selection rule μ. The algorithm produces a sequence $\{\epsilon_n\}$ of binary values indicating whether in the n'th iteration a new principal approximation has been determined ($\epsilon_{n+1} = 1$) or not ($\epsilon_{n+1} = 0$). Equivalently, this may be specified by the sequence $\{s_m\}$ defined by

$$s_0 = 0, \quad s_m = \min\{n : \epsilon_n = 1, n > s_{m-1}\}, \ (m > 0) \tag{3}$$

or the sequence $\{b_m\}$:

$$b_1 = s_1, \quad b_m = s_m - s_{m-1} \ (m > 1). \tag{4}$$

The sequence $\{b_m\}$ is precisely the sequence of partial quotients of an ordinary continued fraction in the one-dimensional case. Szekeres [Szekeres, 1970] proved that every vector continued fraction constructed in this way converges to a vector $\mathbf{x} \in \mathbb{R}^t$. On the other hand, in [Langtry, 1996] it was pointed out that, as in the one-dimensional case, an initial choice of simplex and a given sequence $\{b_m\}$ determine a particular vector $\mathbf{x}$ to which the algorithm converges. This was then exploited in a search for generators of good rank 1 lattices rules amongst good approximations to poorly approximable vectors.

4 A matrix continued fraction algorithm

We seek to extend Szekeres' approach to the construction of good approximations to a $t_1 \times t_2$ matrix X. We shall do this essentially by approximating each column vector of the matrix using a method based on that of [Szekeres, 1970]. However, the method requires a number of modifications. In the following we shall denote the columns of X by $\mathbf{x}_p$, $p = 1, \ldots, t_2$. For each $\mathbf{x}_p$ we choose

an initial simplex and corresponding matrix $M_p^{(0)}$ as in §3. In describing the algorithm we shall require the following definitions.

$$\begin{aligned}
\mathcal{M}^{(0)} &= \left\{ M_p^{(0)} : p = 1, \ldots, t_2 \right\}, \\
\mathcal{J} &= \{ (j_1, \ldots, j_{t_2}) : 0 \le j_p \le t_1, 1 \le p \le t_2 \}, \\
A_{\mathbf{j}}^{(0)} &= \left(M_{p,i,j_p}^{(0)} \right)_{\substack{i=1,\ldots,t_1 \\ p=1,\ldots,t_2}}, \\
B_{\mathbf{j}}^{(0)} &= \left(M_{p,0,j_p}^{(0)} \right)_{p=1,\ldots,t_2}, \\
A_{\mathbf{j}}^{(0)} / B_{\mathbf{j}}^{(0)} &= \left(M_{p,i,j_p}^{(0)} / M_{p,0,j_p}^{(0)} \right)_{\substack{i=1,\ldots,t_1 \\ p=1,\ldots,t_2}},
\end{aligned}$$

where $M_{p,i,j}^{(0)}$ represents the element in the i'th row and j'th column of $M_p^{(0)}$.

The $t_1 + 1$ columns of $M_p^{(0)}$ represent the nearest integer vectors to $\mathbf{x}_p$. By selecting one column (say the j_p'th) from each matrix $M_p^{(0)}$ we can form a total of $(t_1 + 1)^p$ integer matrices $A_{\mathbf{j}}^{(0)} / B_{\mathbf{j}}^{(0)}$ approximating X. We specify as an initial approximation to X the matrix $A_{\mathbf{0}}^{(0)} / B_{\mathbf{0}}^{(0)}$.

A sequence of further approximations is then determined iteratively by applying generalised Farey dissections to a subset of the simplexes at each step, yielding a corresponding new set $\mathcal{M}^{(n)} = \left\{ M_p^{(n)} : p = 1, \ldots, t_2 \right\}$ of matrices via the equations

$$\begin{aligned}
M_{p,i,0}^{(n+1)} &= \begin{cases} (1 - \epsilon_{n+1}) M_{p,i,0}^{(n)} + \epsilon_{n+1} M_{p,i,\mu_p(n)}^{(n)}, & \text{if } \mu_p(n) \neq 0, \\ M_{p,i,0}^{(n)}, & \text{if } \mu_p(n) = 0, \end{cases} \\
M_{p,i,\mu_p(n)}^{(n+1)} &= \begin{cases} M_{p,i,0}^{(n)} + M_{p,i,\mu_p(n)}^{(n)}, & \text{if } \mu_p(n) \neq 0, \\ M_{p,i,\mu_p(n)}^{(n)}, & \text{if } \mu_p(n) = 0, \end{cases} \\
M_{p,i,j}^{(n+1)} &= M_{p,i,j}^{(n)}, \quad \text{if } j \neq \mu_p(n),
\end{aligned} \tag{5}$$

where μ_p is defined below. Thus at the n'th iteration each column $\mathbf{x}_p$ is associated with a simplex and a corresponding $(t_1 + 1) \times t_1 + 1)$ representation matrix $M_p^{(n)}$. This in turn yields new approximations $A_{\mathbf{j}}^{(n)} / B_{\mathbf{j}}^{(n)}$, for $\mathbf{j} \in \mathcal{J}$, to X via the equations

$$\begin{aligned}
A_{\mathbf{j}}^{(n)} = \left(M_{p,i,j_p}^{(n)} \right)_{\substack{i=1,\ldots,t_1 \\ p=1,\ldots,t_2}}, \quad B_{\mathbf{j}}^{(n)} = \left(M_{p,0,j_p}^{(n)} \right)_{p=1,\ldots,t_2}, \\
A_{\mathbf{j}}^{(n)} / B_{\mathbf{j}}^{(n)} = \left(M_{p,i,j_p}^{(n)} / M_{p,0,j_p}^{(n)} \right)_{\substack{i=1,\ldots,t_1 \\ p=1,\ldots,t_2}}.
\end{aligned}$$

From this candidate set we select one matrix, corresponding to the multi-index $\mu(n) = (\mu_1(n), \ldots, \mu_{t_2}(n))$. Its p'th column is the rational vector stored in the $\mu_p(n)$'th column of the representation matrix $M_p^{(n)}$, and from these vectors we construct the next iteration of the algorithm via (5). The matrix $A_{\mu(n)}^{(n+1)} / B_{\mu(n)}^{(n+1)}$

then yields a new approximation that we shall call the $n+1$*'th approximation matrix.*

It remains to specify the selection rule μ and the sequence $\{\epsilon_n\}$. As in the case where $t_2 = 1$, the selection rule μ is intended to be independent of the matrix X—it is designed so that the simplexes corresponding to the columns of X reduce in a way that avoids the development of needle-shaped regions, and which allows the consideration of as many potential approximations to X as possible. In the next section we shall assume that X is not given *a priori*, but that the sequence $\{\epsilon_n\}$ is given, either directly or by (3) or (4). If, however, X is given, we specify the quality of the approximation in terms of values of the linear forms $\gamma = \Gamma_{\mathbf{c}}(\mathbf{1}, X)$ for suitable $\mathbf{c}$.

Definition 1 *Let*

$$\gamma_{p,j}^{(n)} = \Gamma_{\widehat{\mathbf{c}}_{p,j}^{(n)}}(\mathbf{1}, X) = \left(\mathbf{1}, X^T\right) \widehat{\mathbf{c}}_{p,j}^{(n)},$$

where $\widehat{\mathbf{c}}_{p,j}^{(n)}$ *is the* j*'th column of* $\left(M_p^{(n)T}\right)^{-1}$*. Also, let* $\gamma_{p,\mathbf{j}}^{(n)} = \left(\gamma_{p,j}^{(n)}\right)_{j \in \mathbf{j}}$ *and*

$$g_{\mathbf{0}}^{(n)} = \min_{1 \le p \le t_2} \left\{ \left\| \gamma_{p,\mathbf{0}}^{(n)} \right\| \right\},$$
$$g_{\mu(n)}^{(n)} = \min_{1 \le p \le t_2} \left\{ \left\| \gamma_{p,\mu(n)}^{(n)} \right\| \right\}.$$

For $n \in \mathbb{N}$ *we shall say that the* n*'th approximation matrix* $A_{\mu(n-1)}^{(n)}/B_{\mu(n-1)}^{(n)}$ *is a* principal approximation matrix with respect to the norm $\|\cdot\|$ *if* $g_{\mathbf{0}}^{(n)} < g_{\mu(n)}^{(n)}$.

We specify the selection rule μ to choose that candidate approximation matrix $A_{\mathbf{j}}^{(n)}/B_{\mathbf{j}}^{(n)}$ which is farthest from the current principal approximation matrix.

Definition 2 *For* $\mathbf{j} \in \mathcal{J}$*, let*

$$U_{\mathbf{j}}^{(n)} = A_{\mathbf{j}}^{(n)}/B_{\mathbf{j}}^{(n)} - A_{\mathbf{0}}^{(n)}/B_{\mathbf{0}}^{(n)}$$

The selection rule $\mu(n)$ *with respect to the norm* $\|\cdot\|$ *is defined as the largest multi-index* $\mathbf{k}$ *(with respect to lexicographic order) such that*

$$\left\| U_{\mathbf{j}}^{(n)} \right\| \le \left\| U_{\mathbf{k}}^{(n)} \right\|.$$

For a given norm and selection rule, a particular continued fraction is specified by t_1, t_2, the initial simplex $\mathcal{M}^{(0)}$ and the sequence $\{b_m\}$. For convenience we shall denote such a continued fraction by $\left[\mathcal{M}^{(0)} : b_1, b_2, \ldots\right]_{t_1 \times t_2}$.

5 Lattice rules constructed from matrix continued fractions

As suggested in §2 we seek good rational approximations to matrices X having the property that the smallest values of $\Gamma_{\mathbf{c}}(\mathbf{1}, X)$, where $\mathbf{c} = (1, \mathbf{h})$, are large relative to the size of $\mathbf{h}$, or at least are large relative to the size of $\mathbf{c}$. Following the approach in [Langtry, 1996] we consider the approximation matrices of the matrix $\left[\mathcal{M}^{(0)} : 1, 1, \ldots\right]$. In particular, we consider the following two constructions, using the Euclidean vector norm.

1-*extended generator matrices.* Let $A^{(n)}_{\mu(n-1)}/B^{(n)}_{\mu(n-1)}$ be an approximation matrix for $\left[\mathcal{M}^{(0)} : 1, 1, \ldots\right]_{(s-1)\times t_2}$. Then the matrix $\left(\mathbf{1}, \left[A^{(n)}_{\mu(n-1)}/B^{(n)}_{\mu(n-1)}\right]^T\right)$ is a generator matrix for an s-dimensional lattice rule (see [Niederreiter, 1992, p. 131] for a review of generator matrices). Each generator has at least one component relatively prime to its denominator, $B^{(n)}_{\mu_j(n-1)}$ say. In the case that $t_2 = 1$ the matrix generates an s-dimensional rank 1 simple rule.

Direct generator matrices. The approximation matrices $A^{(n)}_{\mu(n-1)}/B^{(n)}_{\mu(n-1)}$ to the continued fraction $\left[\mathcal{M}^{(0)} : 1, 1, \ldots\right]_{s\times t_2}$ provide generator matrices for s-dimensional rules directly.

We note that the ranks of the rules so constructed may be less than t_2—the rank is determined by whether the cyclic groups corresponding to different generators have common non-trivial subgroups. This can occur only when two generators have denominators with a greatest common divisor exceeding 1.

As an example we compare the performance with respect to P_2 of the four-dimensional rules constructed using the two methods, for $1 \leq t_2 \leq 3$. Tables 1–3 list the resulting rules in the order in which the corresponding approximation matrices occur in the relevant continued fractions. The order, rank and value of P_2 are given for each rule—full details are given in [Langtry, 1999]. Table 1 indicates that in the single-generator case **1**-extended rules (that is, rank 1 simple rules) appear to be superior with respect to this criterion. Tables 2–3 appear to suggest that this is not necessarily so when the generator set contains more than one generator.

6 Concluding remarks

This paper establishes the feasibility of constructing lattice rules of rank exceeding one by an extension of the methods of [Langtry, 1996]. Nevertheless, the results achieved are not competitive with the best results yielded by the constructions in [Langtry, 1996,Langtry, 1998], and further work is required

Table 1
P_2 values for **1**-extended (left) and direct (right) rules with $t_2 = 1$ and orders in the range 10^3 to 10^5.

order	rank	P_2
1254	1	8.41(-2)
1624	1	7.35(-2)
1486	1	6.94(-2)
2313	1	3.90(-2)
3110	1	3.39(-2)
2740	1	3.21(-2)
4364	1	2.29(-2)
5850	1	1.91(-2)
5053	1	1.92(-2)
8163	1	1.25(-2)
10214	1	6.27(-3)
9417	1	1.19(-2)
15267	1	4.16(-3)
19631	1	2.53(-3)
17580	1	3.49(-3)
27794	1	1.38(-3)
37211	1	8.60(-4)
32847	1	1.24(-3)
52478	1	6.32(-4)
70058	1	2.87(-4)
60641	1	4.77(-4)
97852	1	2.38(-4)

order	rank	P_2
1243	1	1.24(-1)
1732	1	7.94(-2)
1608	1	8.24(-2)
1787	1	6.29(-2)
2193	1	3.80(-2)
2851	1	4.49(-2)
3395	1	6.49(-2)
3519	1	3.48(-2)
3925	1	2.07(-2)
5044	1	3.87(-2)
6246	1	2.74(-2)
7320	1	6.08(-3)
10171	1	2.75(-2)
9765	1	4.47(-2)
8563	1	4.68(-3)
14809	1	2.50(-2)
18323	1	1.67(-3)
15883	1	2.02(-3)
22129	1	1.28(-3)
30692	1	1.78(-3)
26054	1	1.28(-3)
28499	1	8.74(-4)
44382	1	1.70(-3)
54553	1	6.22(-4)
59191	1	4.02(-4)
52821	1	3.15(-4)
66511	1	4.89(-4)

in order to assess the practicality of such rules.

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Table 2
P_2 values for **1**-extended (left) and direct (right) rules with $t_2 = 2$ and orders in the range 10^3 to 10^5.

order	rank	P_2
1189	1	4.36(-1)
1739	1	1.20(-1)
3420	2	3.87(-2)
3834	1	1.02(-1)
5695	1	2.27(-2)
12312	2	1.98(-2)
13938	1	1.22(-2)
16368	2	2.55(-3)
36270	2	1.45(-2)
56330	1	9.37(-3)
57072	2	4.72(-4)

order	rank	P_2
1665	1	1.25(-1)
2378	1	3.40(-2)
2205	1	4.15(-2)
3575	2	3.49(-2)
5475	1	1.34(-2)
6486	1	4.13(-2)
10168	2	4.72(-3)
14820	2	3.64(-3)
15142	1	3.48(-3)
27208	1	1.27(-3)
41888	1	8.45(-4)
43710	1	6.47(-4)
81216	2	9.20(-4)

Table 3
P_2 values for **1**-extended (left) and direct (right) rules with $t_2 = 3$ and orders in the range 10^3 to 10^5.

order	rank	P_2
1573	2	1.99(1)
1152	3	2.56(1)
1280	3	1.09(1)
11025	2	4.71(2)
27900	2	9.65(3)
48749	2	1.94(3)
64343	2	6.68(4)

order	rank	P_2
1764	2	7.41(-2)
1274	2	2.01(-1)
1815	2	7.74(-2)
4000	3	5.28(-2)
1512	3	1.87(-1)
3060	2	1.68(-1)
27744	3	5.15(-3)
20000	2	1.94(-2)
74925	2	1.04(-3)

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