Finance Research Letters 67 (2024) 105522

Contents lists available at ScienceDirect

Finance Research Letters

journal homepage: www.elsevier.com/locate/frl

A comment on the relationship between operating leverage and financial leverage

Kristoffer Glover

University of Technology Sydney, PO Box 123 Broadway, NSW 2007, Australia

A R T I C L E I N F O

JEL classification:
G3
G13

Keywords:
Operating leverage
Financial leverage
Leverage substitution
Real options
Abandonment option
Default option

A B S T R A C T

Using a real options model, Sarkar (2020) recently demonstrated that operating and financial leverage are not necessarily substitutes. Once a firm is allowed to optimize their operating capacity (hence operating leverage), an increase in one could in fact lead to an increase in the other. Contrary to the claims made in Sarkar (2020), however, we demonstrate in this note that appealing to a firm's capacity decision is not necessary to produce such results. Specifically, we show that if a firm's embedded abandonment option is sufficiently valuable, the operating–financial leverage relationship can also become positive, irrespective of the firm's operating choices.

From an accounting perspective, a firm's fixed operating costs and their financing costs have the same impact on their bottom line. It is for this reason that operating leverage and financial leverage are commonly considered as substitutes (Van Horne, 1979; Dotan and Ravid, 1985). In other words, firms subject to higher fixed costs would naturally have less capacity to take on debt than firms subject to lower fixed costs (all else being equal). However, more recent empirical and theoretical literature has brought such folklore into question. Empirically, while a negative relationship between operating and financial leverage has been documented in many studies (Lev, 1974; Mandelker and Rhee, 1984; Kahl et al., 2019; Chen et al., 2019), inconclusive results and even positive relationships in certain situations have also been documented (see Lord, 1996; Ho et al., 2004, for example).

In an attempt to explain such inconclusive empirical evidence, Sarkar (2020) examined a real options model in which either (or both) types of leverage can be endogenized. Such a model thus relates to the literature on production flexibility and its effect on financial leverage, since a firm's operating leverage is determined to some degree by their decisions on production capacity and investment (see Mauer and Triantis, 1994; Reinartz and Schmid, 2016; Kumar and Yerramilli, 2018).[1] Li et al. (2020) also highlighted the importance of allowing operational flexibility when investigating the relationship between operating and financial leverage. Demonstrating that allowing a firm to optimally reduce production can have a significant effect on the relationship between the two types of leverage.

While there is no question that production flexibility and the ability to control operating leverage will have an impact on the relationship between operating and financial leverage, the simple aim of this note is to demonstrate that an ambiguous relationship between the two types of leverage is also present *without* appealing to such operational flexibility. The key economic driver of this ambiguity is the effect of earnings volatility and the value of the firm's embedded abandonment/default option.

Our key insights come from the fact that, for a given and fixed operating leverage, the model investigated by Sarkar (2020) is equivalent to the model proposed in Glover and Hambusch (2014), which introduced fixed operating cost into the classical

E-mail address: kristoffer.glover@uts.edu.au.

[1] It should be noted that Sarkar (2020) does not model production flexibility in the sense that production can be altered dynamically, therefore links are only implicit to channels such as those exposed in Mauer and Triantis (1994) (that production flexibility may increase the tax benefits of debt financing and reduce the expected cost of financial distress by lowering default risk) and the results in Reinartz and Schmid (2016), who demonstrate empirically that, in the context of power producers, more production flexibility is related to an increased financial leverage.

https://doi.org/10.1016/j.frl.2024.105522
Received 2 April 2024; Received in revised form 9 May 2024; Accepted 10 May 2024
Available online 2 July 2024
1544-6123/© 2024 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

model of Leland (1994). The inclusion of operating leverage into Leland's model effectively introduces an abandonment option into the value of the unlevered firm. Correctly incorporating the value of such flexibility was subsequently found to induce a negative relationship between a firm's profitability and its financial leverage; thus reconciling the trade-off theory with the negative relationship between these two variable observed in the data (Titman and Wessels, 1988; Rajan and Zingales, 1995).[2] Revisiting the model of Glover and Hambusch (2014) in the context of the operating–financial leverage relationship, a richer dynamic between the two types of leverage is revealed and exposed in this note.

The value of a firm's abandonment option has also been seen to have important implications in other real options settings. For example, Guthrie (2011) demonstrates the importance of a firm's abandonment option in explaining the relationship between the firm's operating leverage and its systematic risk. Specifically, demonstrating a non-monotonic relationship between the two. In addition, Wong (2009) considers the effect a firm's abandonment option on its investment decisions. Both papers point to the importance of earnings volatility, and hence the value of the embedded abandonment option, on model outcomes.

More broadly, operating leverage has also been shown to be a plausible explanation for the value premium in the cross section of stock returns (see Carlson et al., 2004; Zhang, 2005; Novy-Marx, 2020), and has been linked to other asset pricing anomalies, such as the profitability premium (see, for example, Novy-Marx, 2013; Kogan et al., 2021). Thus, gaining a deeper insight into the interplay between operating and financial leverage may yield further insights into the asset-pricing implications of leverage.

The rest of this note is structured as follows. We first present the model studied by Sarkar (2020) in Section 1 and then explore the true relationship between operating and financial leverage (when operating leverage is given exogenously) in Section 2. Concluding remarks can be found in Section 3 and more technical details of our arguments found in the Appendices.

1. The model

Full details of the model can be found in Sarkar (2020) (or equivalently Glover and Hambusch, 2014), however we highlight here the features of the model sufficient to understand our subsequent comments. The model was introduced by Sarkar (2020) to address various shortcomings in the models of Chen et al. (2019) and Kumar and Yerramilli (2018) and allows for both operating leverage and financial leverage choices to be endogenized in a dynamic, yet tractable, model.

A firm invests in a project with productive capacity Q, that generates cash flows equal to xQ^{γ} per unit of time. Here, $\gamma \in (0,1)$ represents diminishing returns to capital, and x models an exogenous uncertainty process that is assumed to follow a standard geometric Brownian motion,

$$dx(t) = \mu x(t)dt + \sigma x(t)dZ(t),$$

where μ and σ are the drift and volatility of the process, respectively, and $Z(t)$ is a standard Brownian motion. The project has fixed operating costs of fQ per unit time and it is also assumed that the firm has outstanding (perpetual) debt that pays a coupon of c per unit time. The firm is also subject to the tax rate $\tau > 0$ and hence the after-tax profit of the levered firm is given by

$$(1-\tau)(xQ^{\gamma} - fQ - c).$$

Should the firm's profits become sufficiently negative the firm will declare bankruptcy and default on its debt obligation. Specifically, such default occurs when the process x falls below an endogenously determined level x_b (cf. Leland, 1994). Upon bankruptcy, debtholders receive the value of the unlevered firm, minus a proportional bankruptcy cost $\alpha \in (0,1]$.

Importantly, the inclusion of a fixed operating cost in the model (via fQ) introduces operating leverage and the presence of the coupon payment c introduces financial leverage, allowing for the relationship between the two to be investigated.

Expressions for the value of the firm's risky debt $D(x)$, and equity $E(x)$, are derived in Sarkar (2020, Section 3.2), along with the endogenous default boundary x_b (these expressions can also be found in Section 2 below). Sarkar (2020) then considers the three cases in which the level of operating leverage (via Q), the level of financial leverage (via c), or both, are endogenized. This is done by maximizing the total firm value (debt plus equity) with respect to Q, c, or Q and c, respectively. Within each scenario, the relationship between the Degree of Operating Leverage (DOL)–defined as $\frac{xQ^{\gamma}}{xQ^{\gamma}-fQ}$–and the Degree of Financial Leverage (DFL)–defined as $\frac{D(x)}{D(x)+E(x)}$–is investigated.

A central result of Sarkar (2020) is that the introduction of the firm's capacity decision (determining optimal operating leverage) induces an ambiguous (and possibly non-monotonic) relationship between DOL and DFL. However, it is claimed that "[w]hen DOL is exogenously specified and the company chooses DFL optimally, there is indeed a negative relationship between the two, consistent with the 'leverage substitution' literature".[3] However, upon further investigation, while a negative relationship between DOL and DFL is observed for some parameter values (including those chosen by Sarkar, 2020), the relationship can actually become positive for others. In fact, the parameters do not need to be perturbed far from Sarkar's base case for the relationship to reverse. In particular, increasing the volatility of the process x from $\sigma = 20\%$ to 30% is sufficient.

The next section explores the true relationship between DOL and DFL when the firm's operating leverage is given exogenously.

[2] In the absence of operating leverage (as in Leland, 1994), profitability and financial leverage are independent.

[3] The specifics of this result are presented in Section 4.2 of Sarkar (2020) and summarized there in *Result 1*: "When DOL is exogenously specified, optimal DFL is a decreasing function of DOL".

2. Optimal financial leverage for a given level of operating leverage/fixed costs

Since we are considering the case where operating leverage is fixed, we can set $Q \equiv 1$ in the model of Sarkar (2020) without loss of generality.[4] Indeed, the case $Q = 1$ corresponds to the seminal model of Leland (1994), with the addition of a fixed cost—as proposed by Glover and Hambusch (2014). With this simplifying assumption, the value of debt and equity in the model can be summarized as follows:

$$D(x; f, c_*) = \frac{c_*}{r} + \left[(1-\alpha)u(f, c_*) - \frac{c_*}{r}\right]\left(\frac{x}{x_b(f, c_*)}\right)^{\beta}, \quad \text{for } x \geq x_b(f, c_*), \tag{1}$$

$$E(x; f, c_*) = (1-\tau)\left[\frac{x}{r-\mu} - \frac{f + c_*}{r} - \frac{x_b(f, c_*)}{\beta(r-\mu)}\left(\frac{x}{x_b(f, c_*)}\right)^{\beta}\right], \quad \text{for } x \geq x_b(f, c_*), \tag{2}$$

$$u(f, c_*) = \frac{1-\tau}{r(\beta-1)}\left[f + \beta c_* - f^{1-\beta}(f + c_*)^{\beta}\right], \tag{3}$$

$$x_b(f, c_*) = (f + c_*)\frac{(r-\mu)\beta}{r(\beta-1)}, \tag{4}$$

$$\beta = 0.5 - \mu/\sigma^2 - \sqrt{(0.5 - \mu/\sigma^2)^2 + 2r/\sigma^2} < 0, \tag{5}$$

where $r > \mu$ denotes the risk-free rate, x_b the level of the uncertainty process $x(t)$ at which equityholders would optimally default on their debt, and u the value of the firm's unlevered assets upon default at x_b. We have also highlighted the dependence of values on the fixed cost parameter f and the optimal coupon level chosen c_*, since these are the parameters that will change in response to changes in operating leverage (DOL). Moreover, the optimal coupon c_* is determined via the firm-value-maximizing condition $\frac{\partial}{\partial c}\big(D(x; f, c) + E(x; f, c)\big) = 0$ which yields

$$\tau + \left(\frac{x}{x_b(f, c_*)}\right)^{\beta}\left[\frac{(1-\epsilon)\beta c_*}{f + c_*} - \tau\right] = 0, \tag{6}$$

where $\epsilon := (1-\alpha)(1-\tau)$. Eq. (6)—equivalent to Eq. (13) in Sarkar (2020)—can be solved easily via standard root-finding algorithms.

Next, Sarkar (2020) defines the degree of financial leverage as $\text{DFL} = D/(D+E)$, or simply the debt-to-asset ratio. As such, the relationship between DOL and DFL can be analysed by considering the partial derivative

$$\frac{\partial \text{DFL}}{\partial \text{DOL}} = \frac{\partial}{\partial \text{DOL}}\left(\frac{D}{D+E}\right) = \frac{DE}{(D+E)^2}\left[\frac{1}{D}\frac{\partial D}{\partial \text{DOL}} - \frac{1}{E}\frac{\partial E}{\partial \text{DOL}}\right]. \tag{7}$$

Since $D \geq 0$ and $E \geq 0$, the sign of the relationship is determined by the sign of the difference between the elasticities of the firm's debt and equity values to DOL.

However, DOL (defined as $x/(x-f)$ in our simplified model[5]) does not explicitly appear as a parameter in the debt and equity expressions in (1)–(5). Instead, DOL varies directly with the value of the fixed cost parameter f (for a fixed x). Intuitively, a higher fixed cost f will lead to a higher DOL. We therefore set $f = f(\text{DOL}) = x(1 - 1/\text{DOL})$, and consider the effect of DOL on debt and equity values *through* changes in f.

In addition, the value of f, and hence DOL, will not only affect the value of debt and equity for a fixed coupon level c, it will also affect the value of the *optimal* coupon chosen by the firm; as can be seen from Eq. (6). Hence, the degree of operating leverage affects the value of the firm's debt and equity via two channels. A direct channel via the parameter f (and its impact on the profitability of the company) and an *indirect* channel via the optimal coupon choice c_* (which is itself affected by the parameter f, and hence DOL). Noting that $D = D\big(f(\text{DOL}), c_*(f(\text{DOL}))\big)$ and $E = E\big(f(\text{DOL}), c_*(f(\text{DOL}))\big)$, we see from the chain rule that

$$\begin{aligned}\frac{\partial \text{DFL}}{\partial \text{DOL}} &= \frac{DE}{(D+E)^2}\frac{df}{d\text{DOL}}\left[\frac{1}{D}\left(\frac{\partial D}{\partial f} + \frac{\partial D}{\partial c_*}\frac{dc_*}{df}\right) - \frac{1}{E}\left(\frac{\partial E}{\partial f} + \frac{\partial E}{\partial c_*}\frac{dc_*}{df}\right)\right] \\ &= \frac{DE}{(D+E)^2}\frac{df}{d\text{DOL}}\left[\frac{1}{D}\frac{\partial D}{\partial f} - \frac{1}{E}\frac{\partial E}{\partial f} + \frac{dc_*}{df}\left(\frac{1}{D}\frac{\partial D}{\partial c_*} - \frac{1}{E}\frac{\partial E}{\partial c_*}\right)\right] \\ &= \frac{DE}{(D+E)^2}\frac{df}{d\text{DOL}}\Bigg[\underbrace{\frac{1}{D}\frac{\partial D}{\partial f} - \frac{1}{E}\frac{\partial E}{\partial f}}_{\text{Valuation Effect } (A)} + \underbrace{\frac{dc_*}{df}\left(\frac{1}{D} + \frac{1}{E}\right)\left(-\frac{\partial E}{\partial c_*}\right)}_{\text{Financing Effect } (B)}\Bigg],\end{aligned} \tag{8}$$

where we have used the firm-value-maximizing condition to substitute $\frac{\partial D}{\partial c_*} = -\frac{\partial E}{\partial c_*}$ in the third equality above. Since $\frac{df}{d\text{DOL}} = x/\text{DOL}^2 > 0$, the sign of the relationship between DFL and DOL is thus determined by the sign of $A + B$, the sum of the valuation and financing effects in (8). We will find below that both the Valuation Effect (A) and the Financing Effect (B) can be negative or positive, depending on the parameter regime chosen. Therefore, the relationship between DOL and DFL, even with exogenous operating leverage, is far from clear.

[4] The original problem can be recovered by setting $x \to xQ^{-\gamma}$, $x_b \to x_b Q^{-\gamma}$ and $f \to fQ^{-1}$ in expression (1)–(4) below.

[5] This corresponds to the standard definition of DOL, namely the ratio of the contribution margin (sales − variable costs) to EBIT (sales − variable costs − fixed costs). Here, the contribution margin is equivalent to x and fixed costs equivalent to f.

Specifically, it can be shown from the definitions of debt and equity in Eqs. (1)–(6) that the following, intuitive, inequalities hold.[6]

$$\frac{\partial D}{\partial f} \leq 0 \text{ and } \frac{\partial E}{\partial f} = \frac{\partial E}{\partial c_*} \leq 0. \tag{9}$$

Hence, all else being equal, the value of both debt and equity will decrease if the amount of fixed costs, hence DOL, increase. Therefore, the sign of the Valuation Effect (A) is ambiguous, and determined by the *difference* in elasticities of debt and equity to fixed costs. Numerical investigation of the value of A reveals that it can indeed be either positive or negative, depending on the parameters chosen.

We also observe from (9) that, for a given level of fixed costs f, the equity value will decrease when the coupon level c_* increases. Thus, the sign of the Financing Effect (B) is entirely determined by the sign of $\frac{dc_*}{df}$, the sensitivity of the firm's optimal debt choice to the firm's level of fixed costs. If the firm optimally decreased the coupon level c_* by exactly the same amount as the fixed costs f increased (in other words, if f and c_* were perfect substitutes such that $\frac{dc_*}{df} = -1$), then from Eq. (8) we have

$$A + B = \frac{1}{D}\frac{\partial D}{\partial f} - \frac{1}{E}\frac{\partial E}{\partial f} - \left(\frac{1}{D} + \frac{1}{E}\right)\left(-\frac{\partial E}{\partial f}\right) = \frac{1}{D}\left(\frac{\partial D}{\partial f} + \frac{\partial E}{\partial f}\right) \leq 0, \tag{10}$$

upon recalling (9). Hence, a negative relationship would exist between DOL and DFL. However, firms do not optimally chose the coupon level in this way. In fact, we can show that $\frac{dc_*}{df} > -1$ for all parameter regimes.[7] Hence, all else being equal, if fixed costs increase by \$1 then the firm would optimally choose to reduce debt by *less* than \$1. This optimal financing decision has the potential to invert the negative relationship between operating and financial leverage, should the Financing Effect dominate the Valuation Effect.

Quite surprisingly, the optimal coupon level is not even guaranteed to be *decreasing* in f for all parameter regimes. For example, should the volatility σ be sufficiently high, an increase in fixed cost f can initially result in an *increased* optimal coupon (such that $\frac{dc_*}{df} > 0$), which then decreases beyond a critical level of fixed cost. This behaviour was first noted in Glover and Hambusch (2014)–see Figure 2(b), however its implication for the operating–financial leverage relationship was not investigated further.

Despite the ambiguous signs of both the valuation and financing effects described above, we find in our main result below that the relationship between DOL and DFL is still monotone. However, the sign of the relationship can be either positive or negative, which is determined exclusively by the value of β (which describes the relationship between the parameters μ, σ and r). This result can be summarized as follows.

Result 1. *For $\beta < -1$ (equivalently $\sigma^2 < \mu + r$), DFL and DOL exhibit a negative relationship. For $\beta = -1$ (equivalently $\sigma^2 = \mu + r$), DFL is independent of DOL, and for $\beta \in (-1, 0)$ (equivalently $\sigma^2 > \mu + r$), DFL and DOL exhibit a positive relationship.*

Proof. See Appendix C. □

Fig. 1 demonstrates how the firm's optimal leverage changes in response to a change in operating leverage (through a change in f). Importantly, the figure plots this relationship for different values of the firm's revenue uncertainty σ (hence β). A clear negative relationship can be seen for $\sigma = 20\%$ (the base case parameter used in Sarkar, 2020) but as the volatility increases the relationship becomes weaker and switches over to a positive relationship when σ increases above a critical threshold $\sigma_c = \sqrt{\mu + r}$. For the base case employed by Sarkar (2020) this corresponds to $\sigma_c = 28.28\%$.

2.1. Understanding the effect of volatility on the DOL–DFL relationship

Since the nature of the relationship between DOL and DFL is entirely determined by the parameter β, and this is the only parameter in which the firm's earnings volatility σ impacts debt and equity values, it is clear that the value of the equityholders' embedded abandonment/default option is the key economic driver of this result. We therefore end this note with a brief discussion on the mechanism through which a higher operating leverage could lead to higher financial leverage (without appealing to a firm's capacity decision as in Sarkar, 2020).

As alluded to above, the firm's optimal debt choice—through the Financing Effect in (8)—has the potential to turn operating and financial leverage from substitutes to compliments. Moreover, a key determinant of the firm's optimal debt choice is the relative importance of the equityholders' embedded abandonment/default option, compared to their expected profit stream in perpetuity. Indeed, the equity value in (2) is often expressed as the sum of these two parts:

$$E(x; f, c_*) = \underbrace{(1-\tau)\left(\frac{x}{r-\mu} - \frac{f + c_*}{r}\right)}_{E_{perpetuity}} + \underbrace{(1-\tau)\frac{x_b(f, c_*)}{(-\beta)(r-\mu)}\left(\frac{x}{x_b(f, c_*)}\right)^{\beta}}_{E_{option}}. \tag{11}$$

Thus, as fixed costs f increase, the perpetuity component of the equity value ($E_{perpetuity}$) decreases due to a lower profitability. However, the option component (E_{option}) actually *increases* as fixed costs increase. Therefore, while the reduction in profitability

[6] See Appendix A for further details.

[7] See Appendix B for further details.

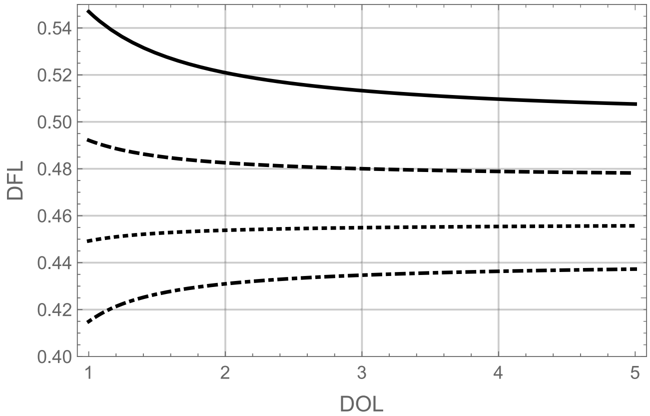

Fig. 1. A plot showing the DOL–DFL relationship when financial leverage is optimized given a fixed operating leverage, for varying values of revenue volatility σ. Solid line: $\sigma = 20\%$ (base case in Sarkar, 2020), dashed line: $\sigma = 25\%$, dotted line: $\sigma = 30\%$, and dot-dashed line: $\sigma = 35\%$. The remaining parameters are: $\tau = 0.15$, $\mu = 0.02$, $r = 0.06$, $\alpha = 0.35$, $Q = 1$ and $x_0 = 0.38$ (which corresponds to a situation in which $Q = 5$, $x_0 = 1$ and $\gamma = 0.6$, the base case used in Sarkar, 2020, see Footnote 4 of this paper).

provides an incentive for the firm to decrease financial leverage as fixed costs increase, the increased option value provides an incentive to increase financial leverage. As we have seen, for sufficiently high β (hence σ), it appears that the latter effect can dominate.

3. Conclusions

This note has attempted to shed a clearer light on the relationship between a firm's operating and financial leverage. While we have corrected an inaccuracy in the results presented in Sarkar (2020), this note has also further emphasized Sarkar's main argument; that the traditional view of operating and financial leverage as substitutes can easily be overthrown. Even when operating leverage is exogenously specified, we demonstrated that a positive, negative, or independent relationship between the two types of leverage can result from a firm's optimal capital structure decision. We also revealed that such ambiguity is driven by the value of the firm's earnings volatility relative to the expected growth in earnings and the risk-free rate, factors that influence the value of the firm's embedded abandonment/default option.

Specifically, while an increase in operating leverage reduces a firm's profitability, and hence capacity to take on debt, it *increases* the value of the equityholders' embedded option to default/abandon, which can provide incentive for the firm to *increase* financial leverage. Indeed, for sufficiently high levels of earnings volatility, the marginal increase in leverage due to the increased value of the option is enough to offset the marginal decrease in leverage due to lower profitability. Resulting in a positive relationship between operating and financial leverage.

CRediT authorship contribution statement

Kristoffer Glover: Writing – review & editing, Writing – original draft, Visualization, Software, Project administration, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Further details on Eq. (9)

Direct differentiation of Eq. (1) with respect to f yields, after some rearranging,

$$\frac{\partial D}{\partial f} = \frac{1}{r}\left(\frac{x}{x_b(f, c_*)}\right)^{\beta}\left[\frac{(1-\epsilon)\beta c_*}{f + c_*} - \epsilon\left(1 - \left(\frac{f}{f + c_*}\right)^{-\beta}\right)\right] \leq 0, \tag{A.1}$$

since $\epsilon = (1-\alpha)(1-\tau) \in [0,1)$ and $\beta < 0$; implying that $(f/(f+c_*))^{-\beta} \leq 1$.

Next, direct differentiation of Eq. (2) with respect to f, and noting that f and c_* are interchangeable (from the equityholders' perspective), yields

$$\frac{\partial E}{\partial f} = \frac{\partial E}{\partial c_*} = -\frac{(1-\tau)}{r}\left[1 - \left(\frac{x}{x_b(f,c_*)}\right)^{\beta}\right] \leq 0, \tag{A.2}$$

since $x \geq x_b(f,c_*)$ and $\beta < 0$; implying that $(x/x_b(f,c_*))^{\beta} \leq 1$.

Appendix B. Establishing that $\frac{dc_*}{df} > -1$

While there is no explicit expression available for $f \mapsto c_*(f)$, the mapping is defined implicitly via Eq. (6). Hence, the sensitivity of the optimal coupon to fixed costs can be computed by implicit differentiation of Eq. (6) to yield

$$\frac{dc_*}{df} = \frac{(1-\epsilon)(1+\beta)c_* - \tau(f+c_*)}{(1-\epsilon)(f-\beta c_*) + \tau(f+c_*)} = -1 + \underbrace{\frac{(1-\epsilon)(f+c)}{(1-\epsilon)(f-\beta c) + \tau(f+c)}}_{>0} > -1, \tag{B.1}$$

since $\epsilon \in [0,1)$, $\beta < 0$, and $\tau > 0$.

As an explicit example, we can take $\beta = -1$, in which case Eq. (6) yields the explicit expression

$$c_*(f)|_{\beta=-1} = \frac{\tau}{1-\epsilon+\tau}\left(\frac{2rx}{r-\mu} - f\right). \tag{B.2}$$

Hence $\frac{dc_*}{df}|_{\beta=-1} = -\tau/(1-\epsilon+\tau) > -1$.

Appendix C. Proof of Result 1

From Eq. (8), and recalling that $\frac{\partial E}{\partial f} = \frac{\partial E}{\partial c_*}$ from (9), we observe that the sign of the DOL–DFL relationship is the same as the sign of the following function G:

$$G\big(f, c_*(f)\big) := E\frac{\partial D}{\partial f} - \left(D + F\frac{dc_*}{df}\right)\frac{\partial E}{\partial f}, \tag{C.1}$$

where we have denoted the total firm value $F := D + E$. In other words, the task is to show that $G < 0$ for $\beta < -1$, $G = 0$ for $\beta = -1$, and $G > 0$ for $\beta \in (-1, 0)$.

First, we note that the function G above is defined over the interval $f \in [0, f_{max}]$, where f_{max} corresponds to the level of fixed costs beyond which it would be optimal for the equityholders to default on their debt immediately. It would also be optimal for the firm not to take on any debt at this point, and so the maximum fixed costs f_{max} is found by setting $x_b(f_{max}, 0) = x$ in (4), yielding

$$f_{max} = \frac{r(\beta-1)}{\beta(r-\mu)}x. \tag{C.2}$$

Hence, we observe from Eqs. (A.1), (A.2), and (B.1) that

$$\frac{\partial D}{\partial f}\Big|_{f=f_{max}} = \frac{\partial E}{\partial f}\Big|_{f=f_{max}} = 0 \quad \text{and} \quad \frac{dc_*}{df}\Big|_{f=f_{max}} = -\frac{\tau}{1-\epsilon+\tau}, \tag{C.3}$$

therefore $G\big(f_{max}, c_*(f_{max})\big) = 0$.

Next, we observe that when $\beta = -1$, the function G is not only zero at f_{max} but zero for all $f \in [0, f_{max}]$. Specifically, using the explicit expression for $c_*(f)|_{\beta=-1}$ obtained in Eq. (B.2), and direct substitution of $\beta = -1$ in (1)–(4) and (A.1)–(A.2), reveals from (C.1) that $G\big(f, c_*(f)\big)|_{\beta=-1} \equiv 0$.

Finally, it can also be shown via direct differentiation of (C.1) with respect to β that, for a fixed f, the mapping $\beta \mapsto G(\beta)$ is *increasing*. Hence, given that $G \equiv 0$ for $\beta = -1$, we conclude that $G < 0$ for $\beta < -1$ and $G > 0$ for $\beta \in (-1, 0)$; establishing the desired result.

References

Carlson, M., Fisher, A., Giammarino, R., 2004. Corporate investment and asset price dynamics: Implications for the cross-section of returns. J. Finance 59, 2577–2603.

Chen, Z., Harford, J., Kamara, A., 2019. Operating leverage, profitability and capital structure. J. Financ. Quant. Anal. 51, 369–392.

Dotan, A., Ravid, S.A., 1985. On the interaction of real and financial decisions of the firm under uncertainty. Rev. Finance 40, 501–517.

Glover, K., Hambusch, G., 2014. The trade-off theory revisited: On the effect of operating leverage. Int. J. Manag. Finance 10, 2–22.

Guthrie, G., 2011. A note on operating leverage and expected rates of return. Finance Res. Lett. 8, 88–100.

Ho, Y.K., Xu, Z., Yap, C.M., 2004. R&D investment and systematic risk. Account. Finance 44, 393–418.

Kahl, M., Lunn, J., Nilsson, M., 2019. Operating leverage and corporate financial policies. AFA 2012 Chicago Meetings Paper.

Kogan, L., Li, J., Zhang, H.H., Zhu, Y., 2021. Operating leverage and asset pricing anomalies. Working paper, MIT Sloan School of Management.

Kumar, P., Yerramilli, V., 2018. Optimal financial and operating leverage with real options. Rev. Financ. Stud. 31, 3452–3490.

Leland, H., 1994. Corporate debt value, bond covenants, and optimal capital structure. J. Finance 49, 1213–1252.

Lev, B., 1974. On the association between operating leverage and risk. J. Financial Quantiat. Anal. 9, 627–641.

Li, Z., Li, Q., Zeng, Y., 2020. Contraction flexibility, operating leverage, and financial leverage. J. Manag. Sci. Eng. 5, 43–56.
Lord, R.A., 1996. The impact of operating and financial risk on equity risk. J. Econ. Finance 20, 27–38.
Mandelker, G.N., Rhee, S.G., 1984. The impact of the degrees of operating and financial leverage on systematic risk of common stock. J. Financ. Quant. Anal. 19, 45–57.
Mauer, D.C., Triantis, A.J., 1994. Interactions of corporate financing and investment decisions: A dynamic framework. J. Finance 49, 1253–1277.
Novy-Marx, R., 2013. The other side of value: The gross profitability premium. J. Financ. Econ. 108, 1–28.
Novy-Marx, R., 2020. Operating leverage. Rev. Finance 15, 103–134.
Rajan, R.G., Zingales, L., 1995. What do we know about capital structure? Some evidence from international data. J. Finance 50, 1421–1460.
Reinartz, S.J., Schmid, T., 2016. Production flexibility, product markets, and capital structure decisions. Rev. Financ. Stud. 29, 1501–1548.
Sarkar, S., 2020. The relationship between operating leverage and financial leverage. Account. Finance 60, 805–826.
Titman, S., Wessels, R., 1988. The determinants of capital structure choice. J. Finance 43, 1–19.
Van Horne, J., 1979. Financial Management and Policy. Prentice-Hall, Englewood Cliffs, NJ.
Wong, K.P., 2009. The effects of abandonment options on operating leverage and investment timing. Int. Rev. Econ. Finance 18, 162–171.
Zhang, L., 2005. The value premium. J. Finance 60, 67–103.