HETEROGENEOUS BELIEFS AND ADAPTIVE BEHAVIOUR IN A CONTINUOUS-TIME ASSET PRICE MODEL

XUE-ZHONG HE* AND KAI LI

Finance Discipline Group, UTS Business School

University of Technology, Sydney

PO Box 123, Broadway, NSW 2007, Australia

tony.he1@uts.edu.au, kai.li@uts.edu.au

ABSTRACT. This paper extends the analysis of the seminal work of Brock and Hommes (1997, 1998) on heterogeneous beliefs and rational routes to randomness in discrete-time models to a continuous-time model of asset pricing. The resulting model characterized mathematically by a system of stochastic delay differential equations provides a unified approach to deal with adaptive behaviour of heterogeneous agents and market stability impact of lagged price used by chartists to form their expectations. For the underlying deterministic model, we show not only that the result of Brock and Hommes on rational routes to market instability in discrete-time holds in continuous-time but also a double edged effect of an increase in lagged price used by the chartists on market stability. For the stochastic model, we demonstrate that the interaction and boundedly rational behaviour of heterogeneous agents can generate various market phenomena such as bubbles and crashes and replicate stylized facts including volatility clustering, and long range dependence in volatility.

Key words: Heterogeneous beliefs, bounded rationality, adaptiveness, fundamentalists, chartists, stability, stochastic delay differential equations.

JEL Classification: G12, G14, E32

Acknowledgement: Financial support for He from the Australian Research Council (ARC) under Discovery Grant (DP110104487) and a UTS Business School research grant is gratefully acknowledged. We are grateful to valuable comments from Cars Hommes (the editor) and three anonymous referees.

**Corresponding author.*

1. Introduction

It is well recognized that the traditional view of homogeneity and perfect rationality in financial markets faces a number of theoretical limitations and empirical challenges. Over the last two decades, there is a growing research on heterogeneity and bounded rationality in financial markets. With different groups of traders having different expectations about future prices, asset price fluctuations can be caused by an endogenous mechanism. For instance, by considering two types of traders, typically fundamentalists and chartists, Beja and Goldman (1980), Day and Huang (1990), Chiarella (1992) and Lux (1995) amongst many others have shown that interaction of agents with heterogeneous expectations may lead to market instability. More significantly, Brock and Hommes (1997, 1998) introduce the concept of an adaptively rational equilibrium. A key aspect of their models is that they exhibit expectations feedback. Agents adapt their beliefs over time by choosing from different predictors or expectation functions based upon their past performance (such as realized profits). They show that such boundedly rational behaviour of agents can lead to market instability and the resulting nonlinear dynamical system is capable of generating complex behaviour from local stability to high order cycles and chaos as the intensity of choice to switch predictors increases.

Following the seminal work of Brock and Hommes, various heterogeneous agent models (HAMs) have been developed to incorporate adaptation, evolution, heterogeneity, wealth effect, and even learning with both Walrasian and market maker market clearing scenarios[1]. Those models have successfully explained various market behaviour (such as market booms and crashes, long deviations of the market price from the fundamental price), the stylized facts (such as skewness, kurtosis, volatility clustering and fat tails of returns), and power laws behaviour, including

[1]For example, both asset price and wealth dynamics with heterogeneous beliefs are considered in Chiarella and He (2001) and Anufriev and Dindo (2010); the framework has been extended to a market maker scenario in Farmer and Joshi (2002) and Chiarella and He (2003*b*); Chiarella and He (2002, 2003*a*) consider the impact of heterogeneous risk aversion and learning; Chiarella et al (2006) examine the dynamics of moving averages; Westerhoff (2004), Chiarella et al (2005) and Westerhoff and Dieci (2006) show that complex price dynamics may also result within a multi-asset market framework.

the long range dependence in return volatility[2], observed in financial markets. We refer the reader to Hommes (2006), LeBaron (2006), Chiarella et al. (2009), Lux (2009), and Chen et al. (2011) for surveys of the recent development in this literature.

The framework of Brock and Hommes and its various extensions are in a discrete-time setup. The setup facilitates economic understanding of the role of heterogeneous expectations and mathematical analysis, it however faces a limitation when dealing with expectations formed from the lagged prices over different time horizons and a challenge to characterize the adaptive behaviour in a continuous-time. In discrete-time models, different time horizons used to form the expectations or trading strategies lead to different dimensions of the systems which need to be analyzed individually. In particular, when the time horizon of historical information used is long, the resulting models are high dimensional systems. Very often, a theoretical analysis of the impact of lagged prices over different time horizons is difficult when the dimension of the system is high[3]. The recent development of HAMs in continuous-time in He *et al* (2009) and He and Zheng (2010) overcomes this limitation in discrete-time. In the continuous-time HAM, the time horizon of historical price information used by chartists is simply presented by a time delay. The resulting model is characterized mathematically by a system of delay differential equations[4].

[2]For example, Alfarano, Lux and Wagner (2005), Gaunersdorfer and Hommes (2007) and He and Li (2007) have provided some insight into the underlying mechanism on volatility clustering and long range dependence in volatility.

[3]For example, to examine the role of different moving average rules used by chartists on market stability, Chiarella *et al* (2006) propose a discrete-time HAM whose dimension depends on the time horizon of chartists used in moving average.

[4]Although the applications of delay differential equation models to asset pricing and financial market modelling are relatively new, their applications to characterize fluctuation of commodity prices and cyclic economic behaviour have a long history, see, for example, Haldane (1932), Kalecki (1935), Goodwin (1951), Larson (1964), Howroyd and Russell (1984) and Mackey (1989). The development further leads to the studies on the effect of policy lag on macroeconomic stability, see, for example, Phillips (1954, 1957), Yoshida and Asada (2007), and on neoclassical growth model in Matsumoto and Szidarovszky (2011).

It provides a uniform treatment on various time horizons used in the discrete-time models.

Motivated by the continuous-time HAMs developed in He *et al* (2009) and He and Zheng (2010), this paper intends to characterize the switching mechanism of the adaptive behaviour of heterogeneous agents in a continuous-time asset pricing model under a market maker scenario, instead of the Walrasian scenario used in Brock and Hommes (1998). Within the proposed model, this paper has three aims. The first is to examine if the result of Brock and Hommes (1998) on rational routes to market instability still holds in a continuous-time setup. The second is to study the joint impact of the adaptive switching mechanism and the increase in time horizon on market stability. The third is to explore potential of the model to replicate various market behaviour, stylized facts and long range dependence observed in financial markets. In order to focus the analysis on the roles of time horizons, both He *et al* (2009) and He and Zheng (2010) do not consider adaptive behaviour of agents. In this paper, we follow Brock and Hommes (1998) to introduce adaptive behaviour of agents who switch their strategies in a boundedly rational way according to some 'performance' or 'fitness' measure such as cumulated profits of strategies over past time horizons. For the corresponding deterministic model, we first show that the result of Brock and Hommes on rational routes to market instability in discrete-time holds in continuous-time. That is, adaptive switching behaviour of agents can lead to market instability as the switching intensity increases, generating excess volatility. We then show a double edged effect of an increase in the lagged price information used by the chartists on market stability, meaning that an increase in time delay can not only destabilize the market but also stabilize the market, a very different feature of the continuous-time HAM from the discrete-time HAMs. This phenomenon is also observed in the continuous-time model in He *et al* (2009) and He and Zheng (2010) without switching, implying that this phenomenon is not due to the switching mechanism. However, the switching affects the price dynamics significantly when market becomes unstable. By including noise traders and imposing a stochastic process on fundamental price, we demonstrate that the model is able to generate various market phenomena, such as long deviations of the market price from the fundamental price, bubbles, crashes, and the stylized facts, including non-normality in asset returns, volatility clustering, and long range dependence of high-frequency returns, observed in financial markets. In particular, we show that the switching can generate more realistic long range dependence in volatility.

The paper is organized as follows. We first introduce a stochastic HAM of asset pricing in continuous-time with heterogeneous agents who are allowed to switch among two types of strategies, fundamentalists and chartists, based on accumulated profits of the strategies in Section 2. In Section 3, we apply stability and bifurcation theory of delay differential equations, together with numerical analysis of the nonlinear system, to examine the impact of switching and time horizon used by the chartists on the market stability. Section 4 provides some numerical simulation results of the stochastic model in exploring the impact of switching and the potential of the model to generate various market behavior and the stylized facts. Section 5 concludes.

2. The Model

Consider a financial market with a risky asset (such as stock market index) and let $P(t)$ be the (cum dividend) price of the risky asset at time t. The modelling of the dynamics of the risky asset follows closely to the current HAMs. However, instead of using a discrete-time setup and Walrasian scenario, we consider a continuous-time setup and a market maker scenario (as in Beja and Goldman 1980, Chiarella and He 2003*b*, Hommes *et al* 2005 and Chiarella et al 2006). The market consists of fundamentalists who trade according to fundamental analysis, chartists who trade based on price trend calculated from weighted moving averages of historical prices over a time horizon, and a market maker who clears the market by providing liquidity. The behaviour of the fundamentalists and chartists is modelled as in He *et al* (2009) and He and Zheng (2010). For completeness, we introduce the demand functions of the fundamentalists and the chartists briefly and refer the reader to He *et al* (2009) and He and Zheng (2010) for details.

The fundamentalists believe that the market price $P(t)$ is mean-reverting to the fundamental price $F(t)$ that can be estimated based on various types of fundamental information. They buy (sell) the stock when the current price $P(t)$ is below (above) the fundamental price $F(t)$. For simplicity, the demand of the fundamentalists, $Z_f(t)$ at time t, is assumed to be proportional to the price deviation from the fundamental price, namely,

$$Z_f(t) = \beta_f[F(t) - P(t)], \tag{2.1}$$

where $\beta_f > 0$ is a constant parameter, measuring the speed of mean-reversion of the market price to the fundamental price, which may be weighted by a risk aversion coefficient of the fundamentalists, and $F(t)$ is the fundamental price of an exogenous random process to be specified in Section 4.

The chartists are modelled as trend followers. They believe that the future market price follows a price trend $u(t)$. When the current price is above the trend, the trend followers believe the price will rise and they like to hold a long position of the risky asset; otherwise, the trend followers take a short position. We assume that the demand of the chartists is given by

$$Z_c(t) = \tanh\big(\beta_c[P(t) - u(t)]\big). \tag{2.2}$$

The S-shaped demand function capturing the trend following behavior is well documented in the HAM literature (see, for example, Chiarella et al. (2009)), where the parameter β_c represents the extrapolation rate of the trend followers on the future price trend when the price deviation from the trend is small. However, they limit their positions when the deviation is large. Among various price trends used in practice, we assume that the price trend $u(t)$ at time t is calculated by an exponentially decaying weighted average of historical prices over a time interval $[t-\tau, t]$,

$$u(t) = \frac{k}{1 - e^{-k\tau}} \int_{t-\tau}^{t} e^{-k(t-s)} P(s) ds, \tag{2.3}$$

where time delay $\tau \in (0, \infty)$ represents a price history used to calculate the price trend, and $k > 0$ is a decay rate. Equation (2.3) implies that, when forming the price trend, the trend followers believe the more recent prices contain more information about the future price movement so that the weights associated to the historical prices decay exponentially with a decay rate k. In particular, when $k \to 0$, the price trend $u(t)$ in equation (2.3) is simply given by the standard moving average with equal weights,

$$u(t) = \frac{1}{\tau}\int_{t-\tau}^{t} P(s)ds. \tag{2.4}$$

When $k \to \infty$, all the weights go to the current price so that $u(t) \to P(t)$. For the time delay, when $\tau \to 0$, the trend followers regard the current price as the price trend. When $\tau \to \infty$, they use all the historical prices to form the price trend

$$u(t) = \frac{1}{k}\int_{-\infty}^{t} e^{-k(t-s)}P(s)ds. \tag{2.5}$$

In general, for $0 < k < \infty$, equation (2.3) can be expressed as a delay differential equation with time delay τ

$$du(t) = \frac{k}{1-e^{-k\tau}}\big[P(t) - e^{-k\tau}P(t-\tau) - (1-e^{-k\tau})u(t)\big]dt. \tag{2.6}$$

In the spirit of Brock and Hommes (1997, 1998) and Chiarella *et al*, (2006), we now introduce the evolution of market population of agents. Let $N_f(t)$ and $N_c(t)$ be the numbers of agents who use the fundamental and chartist strategies, respectively, at time t. Assume that market population of agents $N_f(t)+N_c(t) = N$ is a constant. Denote by $n_f(t) = N_f(t)/N$ and $n_c(t) = N_c(t)/N$ the market fractions of agents who use the fundamental and trend following strategies, respectively. The net profits of the fundamental and trend following strategies over a short time interval $[t-dt, t]$ are measured by, respectively,

$$\pi_f(t)dt = Z_f(t)dP(t) - C_f dt, \qquad \pi_c(t)dt = Z_c(t)dP(t) - C_c dt, \tag{2.7}$$

where $C_f,\ C_c \geq 0$ are constant costs of the strategies per unit time. The performances of the strategies are measured by cumulated and weighted net profits over time intervals $[t-\tau_i, t]$[5],

$$U_i(t) = \frac{\eta_i}{1-e^{-\eta_i\tau_i}}\int_{t-\tau_i}^{t} e^{-\eta_i(t-s)}\pi_i(s)ds, \qquad i = f, c, \tag{2.8}$$

[5]The time delays used to measure the performances can be different from the delay used by the chartists to calculate the price trend in general. In addition, comparing to the trend followers, the fundamentalists use historical prices over a longer time horizon τ_f with lower decaying rate η_f. The impact of different time horizons and decay rates in the performance is discussed in footnote 11.

where $\eta_i > 0$ and $\tau_i > 0$ for $i = f, c$ represent the decay parameter and time horizon respectively used to measure the performance of the fundamentalists and trend followers. Consequently,

$$dU_i(t) = \eta_i \left[\frac{\pi_i(t) - e^{-\eta_i \tau_i} \pi_i(t - \tau_i)}{1 - e^{-\eta_i \tau_i}} - U_i(t) \right] dt, \qquad i = f, c. \tag{2.9}$$

By using the replicator dynamics (see, for example, Chapter 7 in Hofbauer and Sigmund, 1998), the evolution dynamics of the market populations are governed by

$$dn_i(t) = \beta n_i(t)[dU_i(t) - d\bar{U}(t)], \qquad i = f, c, \tag{2.10}$$

where $d\bar{U}(t) = n_f(t)dU_f(t) + n_c(t)dU_c(t)$ is the change of the average performance (over a time interval $[t, t + dt]$) of the two strategies and $\beta > 0$ is a constant, measuring the switching intensity of agents who change their strategy to a better performing strategy. In particular, if $\beta = 0$, there is no switching among agents, while for $\beta \to \infty$ all agents immediately switch to the better strategy.

It can be verified that the above switching mechanism in continuous-time setup is consistent with the one used in discrete-time HAMs. In fact, the dynamics of the market fraction $n_f(t)$ satisfies

$$dn_f(t) = \beta n_f(t)(1 - n_f(t))[dU_f(t) - dU_c(t)], \tag{2.11}$$

leading to

$$n_f(t) = \frac{e^{\beta U_f(t)}}{e^{\beta U_f(t)} + e^{\beta U_c(t)}}, \tag{2.12}$$

which is the discrete choice model used in Brock and Hommes (1997, 1998). In addition, when $\tau_i \to 0$, $U_i(t) \approx \pi_i(t)$, defining the performance by the current profit. When $\tau_i \to \infty$, $U_i(t + dt) \approx U_i(t) + \delta_i \pi_i(t)$ with $\delta_i = \eta_i dt$, defining the performance as cumulated historical profits that decay geometrically at a rate of δ_i.

Finally, the price $P(t)$ at time t is adjusted by the market maker according to the aggregate market excess demand, that is,

$$dP(t) = \mu \big[n_f(t)Z_f(t) + n_c(t)Z_c(t) \big] dt + \sigma_M dW_M(t),$$

where $\mu > 0$ represents the speed of the price adjustment by the market maker, $W_M(t)$ is a standard Wiener process capturing the random excess demand process either driven by unexpected market news or noise traders, and $\sigma_M > 0$ is a constant.

To sum up, the market price of the risky asset is determined according to the following stochastic delay differential system with three different time delays and two noise processes[6]

$$\begin{cases} dP(t) = \mu\Big[n_f(t)Z_f(t) + \big(1-n_f(t)\big)Z_c(t)\Big]dt + \sigma_M dW_M(t), \\ du(t) = \dfrac{k}{1-e^{-k\tau}}\Big[P(t) - e^{-k\tau}P(t-\tau) - (1-e^{-k\tau})u(t)\Big]dt, \\ dU_f(t) = \dfrac{\eta_f}{1-e^{-\eta_f\tau_f}}\Big[\pi_f(t) - e^{-\eta_f\tau_f}\pi_f(t-\tau_f) - (1-e^{-\eta_f\tau_f})U_f(t)\Big]dt, \\ dU_c(t) = \dfrac{\eta_c}{1-e^{-\eta_c\tau_c}}\Big[\pi_c(t) - e^{-\eta_c\tau_c}\pi_c(t-\tau_c) - (1-e^{-\eta_c\tau_c})U_c(t)\Big]dt, \end{cases} \tag{2.13}$$

where $n_f(t)$ is defined by (2.12), $Z_f(t)$ and $Z_c(t)$ are defined by (2.1) and (2.2), respectively, and $\pi_i(t)$ is defined by (2.7) for $i = f, c$.

In summary, we have established an adaptively heterogeneous belief model of asset price in a continuous-time. The resulting model is characterized by a five dimensional system of nonlinear stochastic delay differential equations, which can be difficult to analyze directly. To understand the interaction of the deterministic dynamics and noisy processes, we first study the dynamics of the corresponding deterministic model in Section 3. The stochastic model (2.13) is then analyzed in Section 4.

3. Dynamics of the Deterministic Delay Model

By assuming that the fundamental price is a constant $F(t) \equiv \bar{F}$ and there is no market noise $\sigma_M = 0$, the system (2.13) becomes a deterministic differential system with three time delays

$$\begin{cases} \dfrac{dP(t)}{dt} = \mu\Big[n_f(t)\beta_f(\bar{F} - P(t)) + \big(1-n_f(t)\big)\tanh\Big(\beta_c\big(P(t)-u(t)\big)\Big)\Big], \\ \dfrac{du(t)}{dt} = \dfrac{k}{1-e^{-k\tau}}\Big[P(t) - e^{-k\tau}P(t-\tau) - (1-e^{-k\tau})u(t)\Big], \\ \dfrac{dU_f(t)}{dt} = \dfrac{\eta_f}{1-e^{-\eta_f\tau_f}}\Big[\pi_f(t) - e^{-\eta_f\tau_f}\pi_f(t-\tau_f) - (1-e^{-\eta_f\tau_f})U_f(t)\Big], \\ \dfrac{dU_c(t)}{dt} = \dfrac{\eta_c}{1-e^{-\eta_c\tau_c}}\Big[\pi_c(t) - e^{-\eta_c\tau_c}\pi_c(t-\tau_c) - (1-e^{-\eta_c\tau_c})U_c(t)\Big], \end{cases} \tag{3.1}$$

[6]Note that $Z_f(t)$ is a stochastic process depending on the stochastic fundamental process $F(t)$ specified later in Eq. (4.1).

where

$$\pi_i(t) = \mu Z_i(t)\Big[n_f(t)Z_f(t) + \big(1 - n_f(t)\big)Z_c(t)\Big] - C_i, \qquad i = f, c.$$

It is easy to see that $(P, u, U_f, U_c) = (\bar{F}, \bar{F}, -C_f, -C_c)$ is a unique steady state of the system (3.1), which consists of the constant fundamental price and the costs of the strategies per unit time. We therefore call $(P, u, U_f, U_c) = (\bar{F}, \bar{F}, -C_f, -C_c)$ the *fundamental steady state*. We now study the dynamics of the deterministic model (3.1), including the stability and bifurcation of the fundamental steady state.

At the fundamental steady state, the market fractions of the fundamentalists and the chartists become $n_f^* := 1/(1 + e^{\beta C})$ and $n_c^* := 1/(1 + e^{-\beta C})$ respectively, where $C = C_f - C_c$ measures the disparity of the strategy cost rates. Obviously, when $C = 0$, $n_f^* = n_c^* = 0.5$, meaning that the market fractions at the fundamental steady state is independent of the switching intensity parameter β. However, if it costs agents more to use the fundamental strategy, that is $C > 0$, then there are more chartists than the fundamentalists at the fundamental steady state, that is $n_c^* > n_f^*$. Furthermore, when $C > 0$, an increase in β decreases the steady state market fraction n_f^* of the fundamentalists.

It is known (see Gopalsamy 1992) that[7] the stability is characterized by the eigenvalues of the characteristic equation of the system at the steady state. Denote $\gamma_f = \mu n_f^* \beta_f$ and $\gamma_c = \mu(1 - n_f^*)\beta_c$. Then the characteristic equation of the system (3.1) at the fundamental steady state $(P, u, U_f, U_c) = (\bar{F}, \bar{F}, -C_f, -C_c)$ is given by[8]

$$\Delta(\lambda) := (\lambda + \eta_f)(\lambda + \eta_c)\widetilde{\Delta}(\lambda) = 0, \tag{3.2}$$

where

$$\widetilde{\Delta}(\lambda) = \lambda^2 + (k + \gamma_f - \gamma_c)\lambda + k\gamma_f - k\gamma_c + \frac{k\gamma_c}{1 - e^{-k\tau}} - \frac{k\gamma_c e^{-(\lambda+k)\tau}}{1 - e^{-k\tau}}. \tag{3.3}$$

[7]For a general theory of functional differential equations, we refer the reader to Hale (1997).

[8]Interestingly, the time delays τ_f, τ_c and decaying rates η_f, η_c introduced in the performance measures in (2.8) do not appear in the characteristic equation, hence they do not affect the local stability and bifurcation analysis. This is due to the fact that they are in higher order terms and they affect the nonlinear dynamics, rather than the dynamics of the linearized system. Their impact on the nonlinear dynamics is addressed there in footnote 11.

Note that equation (3.3) has the same form as the characteristic equation of the model studied in He *et al* (2009) and He and Zheng (2010) except that γ_f and γ_c are defined differently. Hence we can apply Theorems 3.2, 3.3 and 3.4 in He *et al* (2009) and Proposition 3.5 in He and Zheng (2010) to system (3.1). For completeness, we summarize the results as follows and refer the details to He *et al* (2009) and He and Zheng (2010).

Firstly, the stability of the steady state do not change for time delay $\tau > \tilde{\tau}$, where

$$\widetilde{\tau} = \frac{1}{k}\ln\left[1 + \frac{2k\gamma_c}{(k+\gamma_f-\gamma_c)^2 + 2 \mid k+\gamma_f-\gamma_c \mid \sqrt{k\gamma_f}}\right].$$

That is, there is an upper bound on the time delay for stability change. Secondly, the change in stability happens only as there is a $\tau \in (0, \widetilde{\tau}]$ and a non-negative integer n such that $S_n^+(\tau) = 0$ or $S_n^-(\tau) = 0$ defined by[9]

$$S_n^{\pm}(\tau) = \tau - \frac{\theta_\pm(\tau) + 2n\pi}{\omega_\pm(\tau)}, \qquad \tau \in (0, \widetilde{\tau}], \qquad n = 0, 1, 2, \cdots,$$

where

$$\omega_\pm = \left(\frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}\right)^{\frac{1}{2}}, \quad \theta_\pm(\tau) = \begin{cases} \arccos(a_{4\pm}), & \text{for } a_{3\pm} \geq 0; \\ 2\pi + \arcsin(a_{3\pm}), & \text{for } a_{3\pm} < 0,\ a_{4\pm} \geq 0; \\ 2\pi - \arccos(a_{4\pm}), & \text{for } a_{3\pm} < 0,\ a_{4\pm} < 0 \end{cases}$$

and

$$a_1 = k^2 + \gamma_f^2 + \gamma_c^2 - 2\gamma_f\gamma_c - \frac{2k\gamma_c}{1-e^{-k\tau}}, \qquad a_2 = k^2\gamma_f^2 + \frac{2k^2\gamma_f\gamma_c e^{-k\tau}}{1-e^{-k\tau}},$$

$$a_{3\pm} = \frac{-\omega_\pm(\tau)(1-e^{-k\tau})(k+\gamma_f-\gamma_c)}{k\gamma_c e^{-k\tau}}, \qquad a_{4\pm} = 1 - \frac{(1-e^{-k\tau})(\omega_\pm^2(\tau) - k\gamma_f)}{k\gamma_c e^{-k\tau}}.$$

Denote

$$\tau_0 = \inf\Big\{\{\widetilde{\tau}\}\bigcup\{\tau \in (0, \widetilde{\tau}] \mid \exists n \in \{0, 1, 2, \cdots\},\ S_n^+(\tau) = 0 \text{ or } S_n^-(\tau) = 0\}\Big\}.$$

Then the local stability and bifurcation of the fundamental steady state with respect to the time delay of system (3.1) are summarized in the following proposition.

Proposition 3.1. *The fundamental steady state of system* (3.1) *is*

(i) *asymptotically stable for* $\tau \in [0, \tau_0)$*;*

(ii) *asymptotically stable for* $\tau > \widetilde{\tau}$ *when* $\gamma_f > \gamma_c - k$*;*

[9]We refer to Theorem 3.3 in He *et al* (2009) for the properties of functions $S_n^{\pm}(\tau)$.

(iii) *unstable for* $\tau > \widetilde{\tau}$ *when* $\gamma_f < \gamma_c - k$.

In addition, the system (3.1) *undergoes Hopf bifurcations at the zero solutions of functions* $S_n^{\pm}(\tau)$.

(a) Function $S_n^{\pm}$

(b) Price bifurcation

(c) $\tau = 3$

(d) $\tau = 16$

FIGURE 3.1. (a) The plots of $S_n^{\pm}$ as functions of τ; (b) the corresponding bifurcation diagram of the market prices with respect to τ; and the market price for (c) $\tau = 3$ and (d) $\tau = 16$. Here $k = 0.05$, $\mu = 1$, $\beta_f = 1.4$, $\beta_c = 1.4$, $\beta = 2$, $C_f = 0.05$, $C_c = 0.03$, $\eta_f = 0.5$, $\eta_c = 0.6$, $\tau_f = 17$, $\tau_c = 16$, and $\bar{F} = 1$.

Proposition 3.1 implies that the fundamental steady state is stable for either small or large time delay when the market is dominated by the fundamentalists (in the sense of $\gamma_f + k > \gamma_c$). Otherwise, when the trend followers become more active comparing with the fundamentalists (in the sense of $\gamma_c > \gamma_f + k$), the fundamental steady state becomes unstable through Hopf bifurcations when time delay increases. Meanwhile, when the trend followers put more weights to the most recent historical prices (so that k is large), the fundamental price is stabilized. This result is in line with the results obtained in discrete-time HAMs. In fact, when the time horizon is small, the insignificant price trend, resulting in weak trading signals, limits the destabilizing activity of the chartists. Consequently, the fundamentalists dominate the market and the market becomes stable. However, Proposition 3.1 also indicates a very interesting phenomenon of the continuous-time model that is not easy to obtain in discrete-time model, which is the stability switching[10]. That is, the system becomes unstable as time delay increases initially, but the stability can be recovered when the time delay becomes large enough. Intuitively, when time horizon is large, the price trend becomes significant, resulting in strong trading signals. However, the activity of the trend followers, measured by γ_c and k, is limited by the activity of the fundamentalists, measured by γ_f. Therefore, the market is dominated by the fundamentalists, leading to a stable market. Fig. 3.1 illustrates such interesting stability switching phenomenon[11]. Fig. 3.1 (a) indicates two Hopf bifurcation values in τ, say $\tau_0 < \tau_1$, determined by two zero solutions of $S_0^{\pm}(\tau)$. The first one occurs when $S_0^+(\tau)$ crosses 0 at $\tau = \tau_0 \approx 7.45$ and the second one occurs when $S_0^-(\tau)$ crosses 0 at $\tau = \tau_1 \approx 31.09$. Fig. 3.1 (b) plots the corresponding bifurcation diagram of the market price with respect to τ showing that the fundamental steady state is stable for $\tau \in [0, \tau_0) \cup (\tau_1, \infty)$ and Hopf bifurcations occur at $\tau = \tau_0$ and $\tau = \tau_1$. Figs 3.1 (c) and (d) illustrate that the fundamental steady state is asymptotically

[10]This phenomenon is also observed in the continuous-time model in He *et al* (2009) and He and Zheng (2010) without switching, implying that this is not crucially due to the switching mechanism introduced in this paper. However, the switching affects the price dynamics significantly when the steady state becomes unstable and/or when the stochastic model is considered. This is demonstrated by Figs 3.3, 4.1 and 4.3 and the related discussions there.

[11]All the numerical results in this paper are based on $k = 0.05$, $\mu = 1$, $\beta_f = 1.4$, $\beta_c = 1.4$, $\beta = 2$, $C_f = 0.05$, $C_c = 0.03$, $\eta_f = 0.5$, $\eta_c = 0.6$, $\tau_f = 17$, $\tau_c = 16$ and $\bar{F} = 1$, unless specified otherwise. In particular, we choose $\eta_f = 0.5$, $\eta_c = 0.6$, $\tau_f = 17$ and $\tau_c = 16$ to take into account that the fundamentalists calculate the weighted cumulated profit over longer time horizons with small decaying rate in weights comparing to the trend followers. As we indicated earlier, they do not affect the local stability and bifurcations. However, simulations (not reported here) show that an increase in η_f (or a decrease in η_c) can increase the fluctuations in price and population switching, but τ_f and τ_c appear to have marginal effect on the fluctuations.

stable for $\tau = 3\,(< \tau_0)$ and unstable for $\tau = 16\,(\in (\tau_0, \tau_1))$. Numerical simulations for $\tau > \tau_1$ (not reported here) verify the stability of the fundamental steady state. The difference of the stability between small τ ($\tau < \tau_0$) and large τ ($\tau > \tau_1$) is that the speed of the convergence is high for small delays and low for large delays. We can see that it is the continuous-time model that facilitates such analysis on the stability effect of lagged price information and stability switching, an advantage of the continuous-time model over the discrete-time model.

FIGURE 3.2. The bifurcation of price with respect to β, here $\tau = 8$.

In the discrete-time Brock and Hommes framework, the rational routes to complicated price dynamics are characterized as the switching intensity β increases. For the continuous-time model developed in this paper, this result also holds. Fig. 3.2 plots the price bifurcation diagram with respect to the switching intensity parameter β. It shows that the steady state is stable when the switching intensity β is low, but becomes unstable as the switching intensity increases, bifurcating to stable periodic price with increasing fluctuations. The periodic fluctuations of the market prices are associated with periodic fluctuations of the market fractions. To illustrate this feature, Fig. 3.3 plots the time series of prices $P(t)$ and the market fraction of the fundamentalists $n_f(t)$, a phase plot of the price, and the distribution of the market fraction $n_f(t)$ of the fundamentalists for time delay $\tau = 16$. Based on the bifurcation diagram in Fig. 3.1 (b), the steady state is unstable for $\tau = 16$. Fig 3.3 (a) shows the periodic fluctuations in both the market fraction and the market price of the switching model (3.1). To better understand the impact of agents' adaptive switching behaviour when the fundamental steady state becomes unstable, we also

(a) Time series of the prices P with and without switching and market fraction n_f

(b) Phase plot of (P, n_f)

(c) The density of n_f

FIGURE 3.3. (a) The time series of the market prices $P(t)$ with switching (the blue solid line with high volatility) and without switching (the red dotted line with low volatility) and the market fraction $n_f(t)$ of fundamentalists (the green dash dot line); (b) the phase plot of $(P(t), n_f(t))$; (c) the density distribution of the market fraction $n_f(t)$ of the fundamentalists.

plot in Fig. 3.3 (a) the market price of the no-switching model in He *et al* (2009)[12]. One can see that the switching increases the price fluctuations. The phase plot in Fig. 3.3 (b) shows that price and fraction converge to a *figure-eight shaped* attractor, a phenomenon which is also observed in the discrete-time model in Chiarella

[12]The parameters in model (3.1) are chosen in such a way so that the steady state population fractions n_f^* and n_c^* are the same in the two models. Previous stability analysis demonstrates that when the market price of the switching model is unstable, so is the no-switching model.

et al (2006). More interestingly, the period of the fluctuation of the market price is twice as much as that of the market fraction and the market prices are close to the fundamental prices whenever the market fractions of the fundamentalists are high. The corresponding distribution of the market fraction $n_f(t)$ of the fundamentalists illustrated in Fig. 3.3 (c) shows clearly the switching of agents' trading strategies over the time. Further simulations (not reported here) show that the fluctuations in both price and population fraction increase as the switching intensity increases.

(a) The bifurcation value τ_0 as a function of n_f^* (b) The bifurcation value τ_0 as a function of β

FIGURE 3.4. The relationships of the first bifurcation value τ_0 with n_f^* and β.

Regarding the joint impact of the time delay, the switching, and the steady state market fractions on market stability, we have shown that an initial increase in time delay destabilizes the fundamental price, however a high steady state market fraction of the fundamentalists stabilizes the price. Also, an increase in switching intensity destabilizes the fundamental price. Hence, with respect to the stability of the steady state, a positive relation between the market fraction of the fundamentalists and the time delay and a negative relation between the switching intensity and the time delay are expected. The above intuition is verified in Fig. 3.4 which plots the first bifurcation value τ_0 with respect to the market fraction of the fundamentalists at the fundamental steady state n_f^* in Fig. 3.4 (a) and the intensity β in Fig. 3.4 (b).

We complete this section with an observation. The *twin-peak-shaped* density distribution in Fig. 3.3 (c) imply that, when the fundamental price is unstable, the market fractions tend to stay away from the steady state market fraction level most of the time and a mean of $n_f(t)$ below 0.5 clearly indicates the dominance of the chartist strategy. In summary, the analysis shows that the continuous-time HAM provides a better understanding of the market dynamics. Apart from providing some consistent results to the discrete-time HAMs on rational routes to market instability, we are able to study the impact of lagged prices used by the chartists on market stability. Also, the adaptive switching behaviour of agents can increase the price fluctuations.

4. Price Behavior of the Stochastic Model

In this section, through numerical simulations, we focus on the interaction between the dynamics of the deterministic model and noise processes and explore the potential power of the model to generate various market behavior and the stylized facts observed in financial markets. To complete the stochastic model (2.13), we introduce the stochastic fundamental price process,

$$dF(t) = \frac{1}{2}\sigma_F^2 F(t)dt + \sigma_F F(t)dW_F(t), \; F(0) = \bar{F}, \tag{4.1}$$

where $\sigma_F > 0$ represents the volatility of the fundamental return and $W_F(t)$ is a standard Wiener process. The market noise $W_M(t)$ and fundamental price noise $W_F(t)$ can be correlated and let ρ be their correlation. It follows from Eq. (4.1) that the fundamental return defined by $d\big(\ln(F(t))\big)$ is a pure white noise process following the normal distribution with mean of 0 and standard deviation of $\sigma_F\sqrt{dt}$. This ensures that any non-normality and volatility clustering of market returns that the model could generate are not carried from the fundamental returns.

Firstly, we explore the joint impact of the time horizon τ of the chartists and the two noise processes on the market price dynamics. For the corresponding deterministic model (3.1), Figs 3.1 (c) and (d) show that the fundamental steady state is stable for $\tau = 3$ and unstable for $\tau = 16$, leading to periodic fluctuations of the market price. For the stochastic model, with the same random draws of the fundamental price and the market noise processes, we plot the fundamental price (the blue dotted line) and the market prices of both the switching model (2.13) (the red solid line) and the no-switching model with population fractions being n_f^* and n_c^* (the green dash dot line) in Fig. 4.1 for two different values of τ. For $\tau = 3$, Fig. 4.1

(a) $\tau = 3$ (b) $\tau = 16$

FIGURE 4.1. The time series of the fundamental price (the blue dotted line) and the market prices of the switching model (the red solid line) and the no-switching model (the green dash dot line) for two delays (a) $\tau = 3$ and (b) $\tau = 16$. Here $\sigma_F = 0.12$, $\sigma_M = 0.05$ and $\rho = 0$.

(a) demonstrates that the market price[13] follows the fundamental price closely and there is no significant difference for the market prices with and without switching. For $\tau = 16$, Fig. 4.1 (b) indicates that the market price fluctuates around the fundamental price in cyclic way, which is underlined by the bifurcated periodic oscillation of the corresponding deterministic model. In addition, similar to the deterministic model, the price fluctuations of the stochastic model are high with switching.

Secondly, we explore the potential of the stochastic model in generating the stylized facts for daily data observed in financial markets. We choose $\tau = 3$ so that the steady state is stable[14], as illustrated in Fig. 3.1 (c). We study at first the case when the two stochastic processes are independent, that is $\rho = 0$. For the stochastic model with both noisy processes, Fig. 4.2 represents the results of a typical simulation. Fig. 4.2 (a) shows that the market price (the red solid line) follows the fundamental price (the blue dotted line) in general, but accompanied with large deviations from time to time. The returns of the market prices in Fig. 4.2 (b) show significant

[13]In the simulations in this section, time unit is a year, an annual volatility is given by $\sigma_F = 0.12$, and the time step of numerical simulations is 0.004, corresponding to one day.

[14]It appears that the stylized facts can also be obtained by choosing τ from its unstable interval. The implications of different choice of delays on the stylized facts would be interesting.

(a) The market price and the fundamental price

(b) The market returns (r)

(c) The density of the market returns

(d) The ACs of the market returns

(e) The ACs of the absolute returns

(f) The ACs of the squared returns

FIGURE 4.2. The time series of (a) the market price (red solid line) and the fundamental price (blue dotted line) and (b) the market returns; (c) the return distribution; the ACs of (d) the returns; (e) the absolute returns, and (f) the squared returns. Here $\sigma_F = 0.12$, $\sigma_M = 0.05$ and $\rho = 0$.

volatility clustering. Comparing to the corresponding normal distribution, the return distribution in Fig. 4.2 (c) displays high kurtosis. The returns show almost insignificant autocorrelations (ACs) in Fig. 4.2 (d), but the ACs for the absolute returns and the squared returns in Figs. 4.2 (e) and (f) are significant with strong decaying patterns as time lag increases, implying a long range dependence. These results demonstrate that the stochastic model established in this paper has a great potential to generate most of the stylized facts observed in financial markets.

We may argue that the above features of the stochastic model is a joint outcome of the interaction of the nonlinear HAM and the two stochastic processes similar to He and Zheng (2010). With the same random seeds, we report the simulation results in Figs. 4.3 and 4.4 when there is only one stochastic process involved. In Fig. 4.3, there is no market noise and the fundamental price is the only stochastic process. The time series, return density distribution, and the ACs of the returns, the absolute returns and the squared returns do not replicate these stylized facts demonstrated in Fig. 4.2. Alternatively, in Fig. 4.4 the market noise process is the only stochastic process. It shows that the return is basically described by a white noise process. Both Figs 4.3 and 4.4 indicate that the potential of the model in generating the stylized facts is not due to either one of the two stochastic processes, rather than to both processes. The underlying mechanism in generating the stylized facts, long range dependence, and the interplay between the nonlinear deterministic dynamics and noises are very similar to the one explored in He and Li (2007) for a discrete-time HAM. Economically, the fundamental noise can be very different from the market noise and consequently they affect the market price differently. Without the market noise, the market price are driven by the mean-reverting of the fundamentalists (to the fundamental price) and the trend chasing of the chartists; both contribute to building up market price trend. Due to the randomness of the fundamental price, there are persistent mean-reverting activities from the fundamentalists that provide the chartists opportunities to explore the price trend. Therefore, the significant ACs of the market returns, absolute returns and squared returns in Fig. 4.3 reflect the interaction of the fundamentalists and the chartists. However, with the market noise and a constant fundamental price, the price trend is less likely formed and

(a) The market price and the fundamental price

(b) The market returns (r)

(c) The density of the market returns

(d) The ACs of the market returns

(e) The ACs of the absolute returns

(f) The ACs of the squared returns

FIGURE 4.3. The time series of (a) the market price (red solid line) and the fundamental price (blue dotted line) and (b) the returns; (c) the return distribution; the ACs of (d) the returns; (e) the absolute returns, and (f) the squared returns. Here $\sigma_F = 0.12$, $\sigma_M = 0$ and $\rho = 0$.

(a) The market price and the fundamental price

(b) The market returns (r)

(c) The density of the market returns

(d) The ACs of the market returns

(e) The ACs of the absolute returns

(f) The ACs of the squared returns

FIGURE 4.4. The time series of (a) the market price (red solid line) and the fundamental price (blue dotted line) and (b) the returns; (c) the return distribution; the ACs of (d) the returns; (e) the absolute returns, and (f) the squared returns. Here $\sigma_F = 0$, $\sigma_M = 0.05$ and $\rho = 0$.

explored by the chartists. This limits the impact of the speculative behaviour of the chartists, which explains the insignificant ACs of the market returns, absolute returns and squared returns in Fig. 4.4. With both noise processes, the price trend is difficult to explore (due to the market noise) and consequently the returns become less predictable. However, the interaction of the fundamentalists and the chartists becomes intensive due to some large changes in the fundamental price from time to time, implying the significant ACs in return volatility, shown in Fig. 4.2 (d)-(f).

(a) The return distributions (b) The ACs of the absolute returns (c) The ACs of the squared returns

FIGURE 4.5. (a) The return distributions; the average ACs of (b) the absolute returns and (c) the squared returns based on 200 simulations for both the switching (the red solid line) and no-switching (the dash-dotted blue line) models . Here $\sigma_F = 0.12$, $\sigma_M = 0.05$ and $\rho = 0$.

To understand the impact of the adaptive switching behaviour of agents on the stylized facts and the AC patterns, based on 200 simulations with different random seeds, Fig. 4.5 reports the return distributions and the average ACs of the absolute returns and the squared returns of the switching model (2.13) (the red solid line) and the no-switching model (the blue dash-dotted line). Fig. 4.5 (a) shows that the switching model displays higher kurtosis than the no-switching model. In addition, Figs. 4.5 (b) and (c) show that the ACs of both the absolute returns and the squared returns are significantly. However, the ACs for the switching model decay quickly, which are more close to the AC patterns observed in financial time series[15].

[15]In a discrete-time model, He and Li (2007) show that the no-switching model is able to replicate the significant decaying AC patterns in the absolute and squared returns, but the speed

(a) The ACs of the absolute returns (b) The ACs of the squared returns

FIGURE 4.6. The average ACs of (a) the absolute returns and (b) the squared returns based on 200 simulations for $\rho = 0, \pm 0.5, \pm 1$. Here $\sigma_F = 0.12$ and $\sigma_M = 0.05$.

Given that the market noise may be correlated with the fundamental price noise, we now examine the impact of the correlated noises on the AC patterns. Based on 200 simulations for different ρ $(0, \pm 0.5, \pm 1)$, Fig. 4.6 compares the ACs of the absolute and squared returns, from which we have a number of interesting observations. Firstly, the ACs are significant and decaying for all correlations, implying that the mechanism in generating the long range dependence can be independent of the correlation of the two noise processes. Secondly, the ACs become more significant when the noise processes are negatively correlated, in particular, when $\rho = -1$; while they become less significant when they are positively correlated, in particular, when $\rho = 1$. Thirdly, not perfectly positively correlated noises lead to more realistic AC decaying patterns.

We conclude this section with a remark on the predictability of asset returns over different trading frequency. As one of the stylized facts, the insignificant ACs of daily returns imply that daily returns are not predictable. However, it is well documented (see for example Pesaran and Timmermann 1994, 1995) that weekly and monthly returns are predictable. Fig. 4.7 illustrates the ACs of (a) weekly and (b) monthly returns. The significant ACs indicate that weekly and monthly returns are predictable, showing that the model has potential to replicate the return

of the decaying is low comparing to the AC patterns observed in financial time series. Further statistic test would be useful to clarify such difference and we leave this to the future research.

(a) The ACs of the weekly market return (b) The ACs of the monthly market return

FIGURE 4.7. The ACs of the weekly and monthly market returns. Here $\tau_c = 16$, $\sigma_F = 0.12$, $\sigma_M = 0.05$ and $\rho = 0$.

predictability for different trading frequency. It would be interesting to explore this potential further.

5. Conclusion

This paper contributes to the development of financial market modelling and asset price dynamics with bounded rationality and heterogeneous agents. Most of the heterogeneous agent models developed in the literature are in discrete-time setup. Among various issues in this literature, the impact of adaptive behaviour on market stability has been well studied, while the impact of lagged prices (used by chartists to form their expectations) on market stability has not been well understood due to the problem of high dimensional systems. This paper develops a continuous-time framework to study the joint impact of lagged prices and adaptive behaviour of heterogeneous agents. By using the replicator dynamics in population evolution literature, we extend the discrete choice model used in discrete-time HAMs to a continuous-time model. The delay differential equations provide a uniform approach to study the impact of the lagged prices through a time delay parameter.

The continuous-time model developed in this paper studies a financial market consisting of adaptive and heterogeneous agents using fundamental and technical strategies. Agents change their strategies in a boundedly rational way according to a performance measure of the accumulated profits. The analysis of the model provides not only some consistent results to the discrete-time HAMs, such as stabilizing effect of fundamentalists, destabilizing effect of chartists, and the rational routes to market instability, but also a double edged effect of an increase in lagged prices on market stability. An increase in the using of lagged prices can not only destabilize, but also stabilize the market price. More importantly, the adaptive switching behaviour of agents can increase market price fluctuations. By introducing a market noise and imposing a stochastic process on fundamental price, we demonstrate that the model is able to generate long deviations of the market price from the fundamental price, bubbles, crashes, and most of the stylized facts, including non-normality in return, volatility clustering, and long range dependence of high-frequency returns, observed in financial markets. In addition, comparing to the no-switching model, the adaptive behaviour of agents can generate more realistic AC patterns in the absolute and squared returns.

The continuous-time framework developed in this paper has shown some advantages comparing to the discrete-time framework, in particular when dealing with the impact of lagged prices. The framework can be used to study the joint impact of many heterogeneous strategies based on different lagged prices on market stability. Also, the profitability of different trading strategies, including momentum and contrarian strategies, are well documented in empirical literature and it would be interesting to see if the continuous-time framework developed in this paper can be used to explore these empirical features. We leave these studies to the future research.

References

Alfarano, S., Lux, T. and Wagner, F. (2005), 'Estimation of agent-based models: The case of an asymmetric herding model', *Computational Economics* **26**, 19–49.

Anufriev, M. and Dindo, P. (2010), 'Wealth-driven selection in a financial market with heterogeneous agents', *Journal of Economic Behavior and Organization* **73**, 327–358.

Beja, A. and Goldman, M. (1980), 'On the dynamic behavior of prices in disequilibrium', *Journal of Finance* **35**, 235–247.

Brock, W. and Hommes, C. (1997), 'A rational route to randomness', *Econometrica* **65**, 1059–1095.

Brock, W. and Hommes, C. (1998), 'Heterogeneous beliefs and routes to chaos in a simple asset pricing model', *Journal of Economic Dynamics and Control* **22**, 1235–1274.

Chen, S.-H., Chang, C. and Du, Y. R. (2011), 'Agent-based economic models and econometrics', *Knowledge Engineering Review*, forthcoming.

Chiarella, C. (1992), 'The dynamics of speculative behaviour', *Annals of Operations Research* **37**, 101–123.

Chiarella, C., Dieci, R. and Gardini, L. (2002), 'Speculative behaviour and complex asset price dynamics', *Journal of Economic Behavior and Organization* **49**, 173–197.

Chiarella, C., Dieci, R., and Gardini, L. (2005), 'The dynamic interaction of speculation and diversification', *Applied Mathematical Finance* **12**, 17–52.

Chiarella, C., Dieci, R. and He, X. (2009), *Heterogeneity, Market Mechanisms and Asset Price Dynamics*, Elsevier, pp. 277–344. in Handbook of Financial Markets: Dynamics and Evolution, Eds. Hens, T. and K.R. Schenk-Hoppe.

Chiarella, C. and He, X. (2001), 'Asset price and wealth dynamics under heterogeneous expectations', *Quantitative Finance* **1**, 509–526.

Chiarella, C. and He, X. (2002), 'Heterogeneous beliefs, risk and learning in a simple asset pricing model', *Computational Economics* **19**, 95–132.

Chiarella, C. and He, X. (2003*a*), 'Dynamics of beliefs and learning under a_l-processes – The heterogeneous case', *Journal of Economic Dynamics and Control* **27**, 503–531.

Chiarella, C. and He, X. (2003*b*), 'Heterogeneous beliefs, risk and learning in a simple asset pricing model with a market maker', *Macroeconomic Dynamics* **7**, 503–536.

Chiarella, C., He, X. and Hommes, C. (2006), 'A dynamic analysis of moving average rules', *Journal of Economic Dynamics and Control* **30**, 1729–1753.

Day, R. and Huang, W. (1990), 'Bulls, bears and market sheep', *Journal of Economic Behavior and Organization* **14**, 299–329.

Farmer, D. and Joshi, S. (2002), 'The price dynamics of common trading stratigies', *Journal of Economic Behavior and Organization* **49**, 149–171.

Gaunersdorfer, A. and Hommes, C. (2007), *A Nonlinear Structural Model for Volatility Clustering*, Springer, Berlin/Heidelberge, pp. 265–288. in Long Memory in Economics, Eds. Teyssiere, G. and A. Kirman.

Goodwin, R. (1951), 'The nonlinear accelerator and the persistence of business cycles', *Econometrica* **19**, 1–17.

Gopalsamy, K. (1992), *Stability and oscillations in delay differential equation of population dynamics*, Kluwer Academic Publisher, London.

Haldane, J. (1932), 'A contribution to the theory of price fluctuations', *Review of Economic Studies* **1**, 186–195.

Hale, J. (1997), *Theory of Functional Differential Equations*, Springer, New York.

He, X., Li, K., Wei, J. and Zheng, M. (2009), 'Market stability switches in a continuous-time financial market with heterogeneous beliefs', *Economic Modelling* **26**, 1432–1442.

He, X. and Li, Y. (2007), 'Power law behaviour, heterogeneity, and trend chasing', *Journal of Economic Dynamics and Control* **31**, 3396–3426.

He, X. and Zheng, M. (2010), 'Dynamics of moving average rules in a continuous-time financial market model', *Journal of Economic Behavior and Organization* **76**, 615–634.

Hofbauer, J. and Sigmund, K. (1998), *Evolutionary Games and Population Dynamics*, Cambridge University Press.

Hommes, C. (2006), *Heterogeneous Agent Models in Economics and Finance*, Vol. 2 of *Handbook of Computational Economics*, North-Holland, pp. 1109–1186. in *Agent-based Computational Economics*, Eds. Tesfatsion, L. and K.L. Judd.

Hommes, C., Huang, H. and Wang, D. (2005), 'A robust rational route to randomness in a simple asset pricing model', *Journal of Economic Dynamics and Control* **29**, 1043–1072.

Howroyd, T. and Russell, A. (1984), 'Cournol oligopoly models with time delays', *Journal of Mathematical Economics* **13**, 97–103.

Kalecki, M. (1935), 'A macroeconomic theory of the business cycle', *Econometrica* **3**, 327–344.

Larson, A. (1964), 'The hog cycle as harmonic motion', *Journal of Farm. Econ.* **46**, 375–386.

LeBaron, B. (2006), *Agent-based Computational Finance*, Vol. 2 of *Handbook of Computational Economics*, North-Holland, pp. 1187–1233. in *Agent-based Computational Economics*, Eds. Tesfatsion, L. and K.L. Judd.

Lux, T. (1995), 'Herd behaviour, bubbles and crashes', *Economic Journal* **105**, 881–896.

Lux, T. (2009), *Stochastic Behavioural Asset Pricing and Stylized Facts*, Elsevier, pp. 161–215. in *Handbook of Financial Markets: Dynamics and Evolution*, Eds. Hens, T. and K.R. Schenk-Hoppe.

Mackey, M. (1989), 'Commodity price fluctuations: Price dependent delays and nonlinearities as explanatory factors', *Journal of Economic Theory* **48**, 495–509.

Matsumoto, A. and Szidarovszky, F. (2011), 'Delay differential neoclassical growth model', *Journal of Economic Behavior and Organization* **78**, 272–289.

Pesaran, M. and Timmermann, A. (1994), 'Forecasting stock returns, an examination of stock market trading in the presence of transaction costs', *Journal of Forecasting* **13**, 335–367.

Pesaran, M. and Timmermann, A. (1995), 'Predictability of stock returns: Robustness and economic significance', *Journal of Finance* **50**, 1201–1228.

Phillips, A. (1954), 'Stabilization policy in a closed economy', *Economic Journal* **64**, 290–321.

Phillips, A. (1957), 'Stabilization policy and time forms of lagged responsesin a closed economy', *Economic Journal* **67**, 265–277.

Westerhoff, F. (2004), 'Multiasset market dynamics', *Macroeconomic Dynamics* **8**, 596–616.

Westerhoff, F. and Dieci, R. (2006), 'The effectiveness of Keynes-Tobin transaction taxes when heterogeneous agents can trade in different markets: a behavioral finance approach', *Journal of Economic Dynamics and Control* **30**, 293–322.

Yoshida, H. and Asada, T. (2007), 'Dynamic analysis of policy lag in a Keynes-Goodwin model: Stability, instability, cycles and chaos', *Journal of Economic Behavior and Organization* **62**, 441–469.