Eigenvalue Decompositions of Kernels & the Hyperbolic Brachistochrone Equation

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Certificate of Original Authorship

I, Peter G. Morrison, declare that this thesis is submitted in fulfilment of the requirements for the award of Doctor of Philosophy, in the Department of Mathematics and Physical Sciences at the University of Technology Sydney. This thesis is wholly my own work unless otherwise referenced or acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis. This document has not been submitted for qualifications at any other academic institution. This research is supported by the Australian Government Research Training Program.

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Abstract

This research connects the fields of time optimal control, special function theory and spectral decompositions. We begin with a derivation of the quantum brachistochrone, and its extension to hyperbolic space. This allows the analysis of the basic unitary and pseudounitary operators which define this time optimal control problem, which we use to derive the geometry of these spaces via the Fubini-Study metric. Further, through use of curvilinear formula for the Laplacian, we are able to define eigenfunctions associated with particular geometries related to the solutions, which are given by unitary or pseudounitary operators and associated metrics. We use a sophisticated methodology for analysis of the heat kernels, wave kernels and EPD equivalents that is based on a geometric understanding of the eigenfunctions that define the spectral decompositions. We apply these new methods in the context of the hyperbolic heat kernel, gaining new insight into some older, unestablished results of McKean. Following this, we detail a systematic technique based on a hyperbolic equivalent of the projection-slice theorem for analysis of index integrals, closing with further generalisation of spectral theories to Radon and Euler-Poisson-Darboux equations and Weierstrass transformations.

Part I

Foreword

0.0.1 Fundamental Propositions

We outline the logical foundations for the following program of analysis contained in this work. Our starting point is to make the observation that the concept of measure is fundamentally related to the ideas of probability and geometry. This is a relationship that is bound by physical necessity. It is readily derived through simple examples, and in an empirical sense may be determined through a close study of experiments in fluid flow using ink droplets or coloured smoke in air.

Consider the simple proposition of a single drop of ink placed on a piece of paper; one can easily see that the process of diffusion occurs via capillary action at the surface of the paper, and the ink spreads out from where it initially hits. An example of a three dimensional diffusion might be if we injected a drop of ink through a very thin tube into a large tank full of still water, far from the surface. One would see that the ink spreads out from the source, as the ink mixes with the water, the concentration of ink decreasing as it radiates away from the source.

In analysing these types of systems, we are forced to resolve the process whereby diffusion occurs. In this thesis we shall take a probabilistic viewpoint and thereby state that diffusion processes are the result of taking statistical averages of microscopic irreversible processes; these individual probabilistic events constitute an approximate realisation of a continuous probability density which describes the state of the system. A gas is said to diffuse if its concentration is high at one point in a system and it proceeds to a situation where it is mixed with other species, or an ink drop in a glass of water spreads out to form a uniform concentration. This process is a consequence of the random interactions of the constituent atoms in both situations, and we use differential equations to model these types of scenarios in the approximation of continuous change.

To understand the interaction between models of probability and geometry, we might ask the question as to how an alteration of the geometry results in a change in the probability density of the system. Analysis of these types of models can be achieved through modification of the state of flow of the system through a changing of the velocity field as is achieved in a fluid flow tank. Another simple example might be if, instead of dropping the ink on to a flat piece of paper, we used a paper surface with non-zero curvature. In this case, the ink would run at a different rate depending on the curvature, as it would tend to run in the direction of steepest descent or least resistance, and we would expect a difference between this situation and the ink drop we observed when dropping ink on to a flat piece of paper. If we carefully cut out the area obtained from the curved surface, for example on a paper globe, we would find that the ink spreads differently depending on the geometry.

The principal aim of this work is provide quantitative understanding of this relationship through a series of calculations involving the hyperbolic plane. The hyperbolic plane is a place in which curvature plays an important role in generalising our understanding of diffusion, and it is an active branch of mathematics with many links to other branches of pure and applied research. This proposal links the concepts of group representation to the geometry and probability density functionals through a systematic program of analysis based on the quantum brachistochrone principle and time optimal control theory.

In terms of geometric constructs, this research has developed from the premises of projective geometry. This powerful way of understanding the intersection between the theories of matrix calculus and those of continuous differential operators forms the nexus between many other different fields of analysis. As the following document shows, there is a connection between the world of differential operators which define the diffusion on the space, the curvature of the geometry, and the matrix groups defined through a certain optimisation principle that lies on the manifold. This principle is one of least time under bounded isotropic constraints, which essentially restricts the motion of the matrix group to certain spherical or hyperbolic spaces. The connection is a metric known as the Fubini-Study metric, which is the natural metric associated with projective state space. The use of concepts related to the metric and Laplacian in a curved space allow direct translation of the hyperbolic equivalent of the Fubini-Study metric of projective state space into classes of operators related to hyperbolic special functions.

On a deeper level, the question of solutions to systems of PDEs (partial differential equations) can be developed in terms of the measure theory of the probability density functional. This is given through functionals of the distance measure, in particular the kernel or transition probability density. We develop a theory of this using concepts from the field of positive definite functions and group representations to demonstrate the utility of this method. In particular, a close examination of positive bounded functions and group representations leads naturally to the development of integral operators equivalent to probability density functionals. As a demonstration of this fact, we shall consider some new examples of these operators and kernels using spherical zonal decompositions over the distance function, and relate this directly to aspects of the hyperbolic heat kernel.

The constitution of the theory must therefore rest upon the concepts of groups, functions and representations. It is simple to observe that the order of matrix multiplication is important. One can see directly that the action of a function must also preserve this order in some way, if the function acts upon the state, then as the state is transformed, so is the argument of the function. By calculating the symmetry groups that act upon the state, one can therefore obtain the symmetry group of the function, and this is the foundation of the theory of group representations. Group representations give one way in which we can find sets of special functions that have properties associated with matrix groups. In this thesis we shall demonstrate a simpler way in which we can derive groups of special functions using some insights from differential geometry.

The root of the hypothesis expressed herein is that the functions we call special are associated to the geometry of the system they exist in, as only certain geometries can be defined that are internally self-consistent these functions appear again and again in e.g. the equations of mathematical physics, quantitative finance, diffusion. By understanding the process of the generation of special functions from the projective geometry on the state space, and combining this with a detailed understanding of the differential geometry, we can approach problems that are extremely difficult using the methods of functional analysis and solve them directly using eigenfunction decompositions of the kernel. This kernel is the positive-valued, measure-preserving function which represents the action of the group, which we define using the hyperbolic extension of the brachistochrone principle.

0.1 Statement of the Problems

The following section outlines the major questions that are raised and solved in the following thesis. The principal findings stem from a close analysis of the hyperbolic heat kernel, and its derivation through use of the brachistochrone principles and differential geometry. We take a theoretical stimulus from some comments issued by McKean, Feynman, Kac and Bracewell. Combination of these methods delivers an axiomatic approach for solutions to the following technical problems, and is sufficient to derive a number of index transforms and other important integrals that appear in physical and financial applications.

0.1.1 Hyperbolic Heat Kernel

The following result, known in the field of hyperbolic analysis as the McKean heat kernel, is given by:

$$K\left(\cosh d\right) = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}} \int_d^\infty \frac{be^{-b^2/4t}}{\sqrt{\cosh b - \cosh d}} db \tag{1}$$

where $d = d(x, y)$ is the hyperbolic distance function. It is stated in Davies [1], without proof, that this is the solution to the hyperbolic heat equation:

$$\frac{\partial K}{\partial t} = \frac{\partial^2 K}{\partial \rho^2} + \coth \rho \frac{\partial K}{\partial \rho} \tag{2}$$

We shall briefly outline the history of this formula as it is in this context that the justification for further analysis appears. This result is quoted in the papers of McKean and Singer [2], and various other results related to curvature and heat kernels in hyperbolic space that we shall touch on in [3]. Now, although this result is quoted, and a later paper by McKean quotes an earlier reference by Robin c. 1957 [4], none of these papers actually go into the detail of proving this result. Indeed, although the work by Robin is excellent, much of the reproduction of the body of work is based on methods of special functions, and the result is again quoted rather than proved, with an emphasis on the heavy use of power series identities of hypergeometric functions. In many ways, the work of Robin builds on earlier work in special function theory with a direction towards earlier results from the French school as reported by Legendre in terms of the famous multipole expansion. This is to be considered in historical context, particularly due to Robin's employ as Telecommunications Director one would expect that the many calculations related to line transmission would end in Legendre functions of one kind or another.

A tracing of the results given in Robin [4] gives the earlier paper of Hobson c. 1889, further research in similar areas turns up results quoted by Mehler [5] c. 1866, where similar results are touched on. Hobson in [6,7], claim the following solution for Laplace's equation in toroidal harmonics:

$$K_p^m(\cosh \phi) = C_{mn}(\phi) \int_0^\phi \frac{\cos pu}{(\cosh \phi - \cosh u)^{1/2-m}} du \tag{3}$$

where $C_{mn}(\phi)$ are normalisation constants, known in classical analysis as the Dirichlet-Murphy formula (see e.g. [8], eqs. 14.1.2-4). In Robin [4], the author claims an equivalent result (Tome III pp. 154, also Tome II pp. 42) but does not prove it. The relationship between the cosine transform and the heat kernel seems intuitive, as expressed through the denominator of the integrand, but it has not been solved directly. Other works of interest include the early investigations of Heine [9], who examined conical functions. Indeed, there is much relation between the subject matter of all of these works, as they were working towards the expression of special function theory in terms of hypergeometric analysis. This appears to have developed prior to the development of group theory by Mackey [10], and it is important to understand that in many ways the use of the theory of kernels and Legendre functions is powerful way to tackle some difficult problems both known and unknown in modern mathematics.

Other authors quote the work of McKean, and those who have examined similar topics related to the study of the hyperbolic Laplace operator seem to have primarily quoted the original paper, which does not contain a complete proof of the heat kernel.

The first proof of the hyperbolic heat kernel which does not directly reference McKean or McKean-Singer [2,3] to the author's knowledge appears in the works of Chavel [11]. This important but somewhat difficult to find resource published c. 1984 contains an exposition on the heat kernel associated to the Laplacian in a hyperbolic manifold from the direction of group theory. In particular, the author obtains an expression for the hyperbolic heat kernel equivalent to that of McKean by use of a convolution theorem, composition laws and product formulae for the Legendre functions of toroidal type.

Other results that are available that follow in a similar thread include those in [12], who has analysed the hyperbolic heat kernel. The authors in this paper claim that an unpublished result has supplied a proof, but the reference is unknown, and they erroneously cite McKean-Singer [2] as the first reported author to present a proof of the hyperbolic heat kernel. From a different direction, through use of path integral methods, Grosche, also Grosche and Steiner report related kernels in [13–16]. The tabulated results for many different related physical systems may be found in their compendium [17]. In particular, the results obtained for hyperbolic space imply certain relationships between modified Bessel functions, Whittaker functions and the toroidal harmonics. However, the dense mathematical content implied in the analysis of path integrals in hyperbolic space prohibits direct engagement with the kernel.

In this work we present an alternate methodology for arriving at the major results for hyperbolic heat kernels that borrows from the new and developing science of time optimal control theory. We shall show how both of the formulae shown above may be directly related to the construction of a brachistochrone in hyperbolic space, and derive all major formulae related to the kernel by utilising the techniques of differential geometry and projective geometry of quantum states and their hyperbolic equivalents. We demonstrate that this is a powerful technique for gaining insight into these types of complicated systems, and discuss the physical aspects of such axiomatic views of analysis in the hyperbolic domain, in particular by deriving metrics and composition laws analogous to those of special and general relativity, in the latter for the Robertson-Walker metric.

0.1.2 Analysis of Some Arguments of Feynman and Kac

We discuss some comments of Feynman and Kac that relate to the hyperbolic heat kernel and eigenfunction decompositions. In particular, the completeness relationship is shown using an argument of Feynman [18] to be a result of the Plancherel identity and some simple kernel theorems. As this is essential to many of the arguments contained in the prior proofs, where we have essentially exploited a decomposition over the completeness relationship, we reproduce some selected arguments from the original work of Feynman as in [18–20] and Kac [21] as it pertains to our question of time dispersive systems.

0.1.2.1 Statements of Kac

Kac, in [21] makes the following argument.

Example 0.1.1. Kac (1949) [21] states that the fundamental solution of the partial differential equation given by diffusion with potential:

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} - uV(x)f \tag{4}$$

is related to the solution of the differential equation:

$$\frac{1}{2}\frac{d^2\psi}{dx^2} - (s + uV(x))\psi = 0 \tag{5}$$

by Laplace transform.

He furthermore goes on to state that “on formal grounds one might.. expect that the inversion with respect to s will yield the fundamental solution...We do not pursue this connection in a general and rigorous manner because we feel that not enough would be gained by doing so. In fact, in most cases, the treatment of (this expression) is anyway reduced to a treatment of a corresponding Sturm-Liouville problem” [21]. Kac chose not to calculate a detailed theory of the Sturm-Liouville problem. In this thesis, we show that much is to be gained through the use of this methodology in understanding the solutions to various hyperbolic partial differential equations, specifically those directly related to the hyperbolic heat kernel. In particular, we consider the application of change of variables to reduce many different PDEs to the form 0.1.1 through the Bose invariant [22], potential theory and the speed and scale measures. Using such methods, we have extracted eigenfunctions, derived kernels and been able to understand the basic structure of the differential equation described through 0.1.1 for a large class of second-order PDEs related to diffusion problems on spherical and hyperbolic spaces.

We state a modification of the argument of Kac, that he does not seem to have realised; i.e. the possibility of taking a Fourier transform or Laplace transform in x results in the following differential-convolution equation.

Example 0.1.2. The Fourier transform in space of the diffusion equation with potential defines a convolution equation:

$$\frac{\partial \mathcal{F} f}{\partial t} = \mathcal{F}\left[\frac{1}{2}\frac{\partial^2 f}{\partial x^2} - uV(x)f\right] \tag{6}$$

$$\frac{\partial g}{\partial t} = -\frac{k^2}{2}g - u\mathcal{F}\left[V(x)f\right] = -\frac{k^2}{2}g - u\bar{V}(k) \star g(k,t) \tag{7}$$

then formal inversion of this gives a solution via:

$$f(x,t) = \mathcal{F}^{-1}\left[g(k,t)\right] \tag{8}$$

This is true for $x \in \mathbb{R}$. In the case of the positive plane $x > 0$, the Fourier convolution is replaced by the Laplace equivalent. As we shall see, this has important consequences for the analysis of the hyperbolic plane and kernel solutions to PDEs on this geometry. In this work we shall prove that the method based on analysis of the convolution theorem and Sturm-Liouville theory is a powerful way to derive the hyperbolic heat kernel, contrary to the claims of Kac. We link this with the axiomatic basis provided by the hyperbolic brachistochrone principle, and use this to establish a novel method for deriving this functional.

0.1.2.2 Statements of Feynman

Feynman, in [18], makes the following observation regarding the kernel solution. He states that "it is fundamental to our definition of ψ as probability amplitude that the integral of $\psi^*\psi$ is constant. In terms of the kernel this means that if f is the wave function at time t_a, then at time t_b it has the same square integral". He then goes on to prove this in the following way. Taking the equation for the kernel/propagator, he writes the propagated solution in terms of that of the past:

Example 0.1.3. The kernel $K(b,a)$ can be written in terms of an integral transform that takes us from a state $f(a)$, with some probability measure dx_a to state $\psi(b)$:

$$\psi(b) = \int_{-\infty}^{+\infty} K(b,a) f(a) dx_a \tag{9}$$

This is analogous to the Fourier transform, where we obviously have $K(b,a)dx_a = e^{iab}da$, which takes us from the time domain to the frequency domain and back again. In this work, we shall consider more general examples of this hypothesis, including the use of similar transform methods to solve boundary value problems, where a represents some point in the past, expressed through the boundary value $f(a)$, and the propagated solution is given by $\psi(b)$.

Remark 1. For this section we shall use the notation and original comments of Feynman.

The major result then follows directly [18].

Theorem 0.1.4. *The completeness relationship can be written in terms of the kernel:*

$$\int_{-\infty}^{+\infty} K(b,a')K^*(b,a)dx_b = \delta(x'_a - x_a) \tag{10}$$

where $\delta(.)$ is the Dirac delta function, x'_a, x_a any two points on the space of possible states. This may be derived from the Plancherel theorem, which in the context of this brief calculation takes the form of the square integrability condition:

$$\int_{-\infty}^{+\infty} \psi^*(b)\psi(b)dx_b = \int_{-\infty}^{+\infty} f^*(a)f(a)dx_a \tag{11}$$

which is satisfied by any reasonably well-behaved set of harmonic functions, and additionally is required for the probability amplitudes to be well-defined between any two set of states, as per Feynman's comments quoted above.

Proof. We shall briefly replicate the proof as it appears in the original arguments of Feynman [18]. Assuming the Plancherel theorem as in 0.1.4, he writes:

$$\int_{-\infty}^{+\infty} \psi^*(b)\psi(b)dx_b = \int_{-\infty}^{+\infty} f^*(a)f(a)dx_a \tag{12}$$

whereupon inserting the expression for the state $\psi(a)$ from 0.1.3, used multiple times and rearranging the integrals Feynman obtained:

$$\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} K(b,a')f(a')dx'_a\right]\left[\int_{-\infty}^{+\infty} K^*(b,a)f^*(a)dx_a\right]dx_b = \int_{-\infty}^{+\infty} f^*(a)f(a)dx_a \tag{13}$$

or the expression:

$$\int_{-\infty}^{+\infty} f^*(a)f(a)dx_a = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[K(b,a')K^*(b,a)f(a')f^*(a)\right] dx'_a dx_b dx_a \tag{14}$$

whereupon the expression is readily evaluated:

$$f^*(a)f(a) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[K(b,a')K^*(b,a)f(a')f^*(a)\right] dx'_a dx_b \tag{15}$$

Rearranging the integrals, Feynman found:

$$f^*(a)f(a) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[K(b,a')K^*(b,a)dx_b\right] f(a')f^*(a)dx'_a \tag{16}$$

which implies the inner integral is a delta function:

$$\int_{-\infty}^{+\infty} K(b,a')K^*(b,a)dx_b = \delta(x'_a - x_a) \tag{17}$$

i.e. the completeness relation. □

This is an important relationship, as it essentially shows that the properties of the Plancherel theorem, as given by the Wiener-Khinchin lemma, quantum mechanics or otherwise, is not independent of the property of completeness. In fact, by using a completeness property, we naturally imply that a Plancherel formula exists, by inversion of the argument. This is an interesting point, and this property may be extended to many other spaces using the Bose invariant [22] theory or otherwise.

In this work, we shall show how one may construct the equivalent of eigenvalue decomposition formulae for the kernel using the Sturm-Liouville method. This gives a general example of completeness on the space of hyperbolic states. We shall explicitly show how the problem of Feynman can be stated in terms of completeness relationships that apply on the hyperbolic plane, and use this to evaluate the hyperbolic heat kernel.

0.1.3 Projection-Slice Theorem

The projection-slice theorem gives a connection between the Fourier, Abel and Hankel transforms and is of invaluable use in the theory of Fourier transforms, see Bracewell [23–26] for the development of this concept. This theorem states that the Fourier-Abel-Hankel transforms compose a cycle, called the FAH cycle. For a radial function $f(r)$, these transforms are related by:

$$(\mathcal{F}\mathcal{A})[f(r)] = (\mathcal{H})[f(r)] \tag{18}$$

where $\mathcal{F}$ is the Fourier transform, $\mathcal{A}$ is the Abel transform, and $\mathcal{H}$ is the Hankel transform.

In this thesis we extend this theorem to equivalents in the hyperbolic plane using Sturm-Liouville theory of the hyperbolic heat kernel and some integral transform methods. This allows us to generalise the FAH cycle to a MKW cycle, where the Mehler-Fock or associated Legendre functions play the role of the Fourier transform in hyperbolic space. This uncovers a deep connection between systems of hyperbolic eigenfunctions, as expressed through the Whittaker, modified Bessel and associated Legendre functions. We employ this theorem to derive a number of index integration formulae related to integral transforms in the hyperbolic plane, some of which were previously listed without proof in tabulated data in the work of Oberhettinger, see [27–31]. Index integration is a relatively new topic in transform theory, which allows generalisation of the properties of integration over a functional index as opposed to the function argument. This was outlined in the original PhD thesis of Yakubovich [32,33], where a number of similar results to those obtained in this thesis were explored. In Brychov, Glaeske et. al [34] (c.1983), the authors treated aspects of integral transforms of convolution type using similar methodology to that utilised in this work. Other works in a similar vein may be found in Russian/Belarusian literature dating to the 1980s, see for example Tuan, Marichev and Yakubovich (c. 1986) [35], where the authors treated some aspects of the Kontorovich-Lebedev transform via a hypergeometric method.

Further extensions of the hypergeometric method of kernel transforms may be found in Yakubovich and Luchko (c. 1994) [36], where the authors treated the composition formulae of hypergeometric functions via a convolution transform method. An updated current work related to the convolution theory of the index Whittaker transformation is the reference text of Sousa and Yakubovich [37], where the authors used a generalised form of convolution identity to identify product formulae for the confluent hypergeometric (Whittaker W) function.

0.1.4 Bose Theory of Transformation of the Potential

In this work we shall discuss some advanced topics related to the PDEs in 0.1.1. One place in which to start is with the potential theory of the heat equation or its quantum equivalent, as expressed through the time dependent Schrödinger equation. As we shall demonstrate throughout this work, the use of the Bose invariant potential and scale factors allows us to bring any second order partial differential equation into the diffusion plus potential form. We might write this as:

Example 0.1.5.

$$i\frac{\partial u}{\partial t} = -\frac{1}{2}\nabla^2 u + V(x)u \tag{19}$$

for the space dependent potential $V(x)$, and an equivalent equation for e.g. a time dependent potential which appears in the Euler-Poisson-Darboux equation.

Using the scale factor of the PDE, we can always find a transformation such that any second order PDE fits the right-hand side of this expression; the major difference between the wave, EPD and heat equations is the order of the derivative and any additional terms on the left-hand side.

It is important therefore to consider any additional transformations that can be applied to this type of equation to bring it into a manageable form. One such transformation is given through the concept of supersymmetry, see [38] for a detailed explanation of the topic. As it suits our purpose, the major result is as follows. For the quantum equation as written, we may write this as the equivalent Riccati equation using a convolution star product, as described in Curtright [38], see also [39]:

$$-\frac{1}{2}\nabla^2 u + V(x)u = \hat{H}u \tag{20}$$

They write the star-decomposition in the form:

$$\hat{H} = \hat{Q}^* \star \hat{Q} = \left(\frac{\hat{p}}{\sqrt{2}} + iW(x)\right) \star \left(\frac{\hat{p}}{\sqrt{2}} - iW(x)\right) \tag{21}$$

where $W(x)$ is a function of x chosen to match the potential, and $\hat{p}$ is the momentum operator $\hat{p} = -i\partial_x$. A simple expansion of the terms shows that we must have:

$$\hat{H} = \hat{Q}^* \star \hat{Q} = \frac{1}{2}\hat{p} \star \hat{p} + \frac{i}{\sqrt{2}}\left[W(x), \hat{p}\right]_\star + \frac{1}{2}\hat{W}(x) \star \hat{W}(x) \tag{22}$$

which collapses to:

$$\hat{H} = \hat{Q}^* \hat{Q} = \frac{1}{2}\hat{p}^2 + \frac{i}{\sqrt{2}}\left[W(x), \hat{p}\right] + \frac{1}{2}[\hat{W}(x)]^2 \tag{23}$$

in the diagonal reference frame, where the star convolution simply becomes the product of operators. Using the standard formula for the action of the momentum operator, we may write:

$$[W(x), \hat{p}] = -i\frac{\partial W(x)}{\partial x} \tag{24}$$

hence, for equivalence to occur, we must have:

$$W^2 - \frac{1}{\sqrt{2}}\frac{\partial W}{\partial x} = V(x) \tag{25}$$

This can be seen from the perspective of the Darboux transform as the potential equation:

$$W(x) = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x}\left(\ln \psi_0\right) \tag{26}$$

$$-\frac{1}{2}\frac{\partial^2 \psi_0}{\partial x^2} + V(x)\psi_0 = 0 \tag{27}$$

Craddock [40], also Craddock and Lennox in [41,42] have shown that a differential equation of the form:

$$\frac{\partial u}{\partial t} = \sigma x^\gamma \frac{\partial^2 u}{\partial x^2} + f(x)\frac{\partial u}{\partial x} - g(x)u \tag{28}$$

may be solved by analysis of the Riccati-type ODE:

$$\sigma x \frac{\partial h}{\partial x} - \sigma h + \frac{h^2}{2} + 2\sigma x^{2-\gamma} g(x) = b(x) \tag{29}$$

Using the Bose invariant [22], we easily show that one may transform any differential equation of the type above using the scale factor as follows:

$$\Phi(x) = \exp\left(\int^x \frac{b(w')}{2a(w')}dx'\right) = \exp\left(2\sigma \int^x s^{-\gamma} f(s)ds\right) \tag{30}$$

which can be seen as a sort of Fourier-Mellin transform of the drift function. We find the Bose invariant potential equal to:

$$V(x) = -\frac{x^{-\gamma}}{4\sigma^2}\left(2\sigma\frac{df(x)}{dx} + 4\sigma g(x) - \frac{2\sigma\gamma f(x)}{x} + \frac{[f(x)]^2}{x^\gamma}\right) \tag{31}$$

In this thesis, we shall use a number of similar techniques to find decompositions for unitary matrices and operators. These methods share many of the properties of the simple example above, we use some sort of additional construct to transform a difficult problem into something more tractable.

Craddock and Lennox [41,42] carried out an in-depth computation of the Lie groups associated with this family of PDEs. In this calculation, we shall show that the development of a theory of the symmetries of a large group of similar PDEs can be carried out using precepts and axioms from projective geometry, differential geometry, and the theory of groups.

Probing the relationships between these types of differential systems is important to the development of the analysis of special functions, stochastic processes and probability theory. The relationships familiar in mathematics and physics between special functions, PDEs and group representations are commonplace in their usage, quoted often, and taken for granted to a certain extent. Examples of this include composition formulae, from the simple precepts of spherical geometry to the hyperbolic geometry discussed herein this work, eigenfunctions, kernels on higher dimensional spaces, product formulae and convolutions. All these topics can be related through the concept of group homomorphism as expressed through the product relations and matrix decompositions. Hopefully this investigation is of use in shining some light on this often obscure realm of calculus, as special function theory has proven time and again to be an elegant way in which to describe the solutions to problems in differential analysis, physics and mathematical finance. Our calculations shall show that the understanding of potential equations and their symmetries is one way in which we can derive many results familiar to mathematical physics. The application of further transformation laws such as those of Darboux as detailed above is a program of analysis we exploit thoroughly in this work, and it is hoped that these types of techniques will continue to yield insight into the field of special functions.

CHAPTER 1

Introduction

Transform theory of functions forms the basis for much of the understanding that has been accumulated within mathematics over the last centuries, and it is important to find new ways in which the families of special functions can be related to one another. One way in which this may be achieved is through the use of spectral theory and the basic relationships of hyperbolic geometry. In this thesis we shall discuss some methods in which various well known index transforms may be related to each other using completeness relationships and kernel identities. In doing so, we establish a deeper understanding of the mappings of the hyperbolic plane, and how conformal transforms can be used to turn groups of special functions into one another.

Diffusion problems in the hyperbolic plane have a known association with automorphic forms see e.g. [43,44]. We shall be considering some systems of diffusions in the positive half-plane [45]. Analogous problems have been calculated at length in [46]. We shall be pursuing a different type of analysis to these works, as we shall focus solely on the nature of the kernel as it relates to various distance functions in the hyperbolic plane.

The topic of hyperbolic diffusion and conical functions has a long history, dating back to the seminal works of Mehler [5] and Fock [47]. This work has utilised results on hyperbolic geometry that may be found in the papers of Chavel [11] and also the notes of Canzani [48]. In particular, the insight in [48] is of particular utility in understanding the meaning behind diffeomorphisms between isospectral systems. We note that the convolution theorem argument as presented for the associated Legendre function may be found in [11], but we reproduce it here for completeness.

We shall require several novel results from the theory of the quantum brachistochrone; of historical interest are the papers of Carlini, Okudaira et. al [49–51]. These works calculate various symmetry groups which are time optimal on the complex projective space of states. More recent works by the author [52–55] utilise a methodology which is free of the restrictive boundary conditions found in [49–51].

Grosche & Steiner [13–17] present a number of formulae that are closely related to the calculation in this thesis. In particular, we are able to replicate the results of the calculation contained in [13] without the difficulties of calculating path integrals and complex algebra. The formulae obtained relate to their observations regarding quantum diffusion problems on various hyperbolic systems, which are related to the Kontorovich-Lebedev, Mehler-Fock and other index transforms. We note that these papers, specifically [14] give completeness and orthogonality relationships for the conical functions, and relate these sets of associated Legendre functions to a composition formula for the KL transform.

Other papers of relevance with regard to the science of stochastic processes include the works of Sousa & Yakubovich [56], who analyse the nature of various types of index transformations considered in this thesis from the direction of the Titchmarsh-Kodaira theorem [57]. The second author has also calculated a number of related transformation kernels in [58]. Other works in this field that are of interest include those of Rodrigues

[59] who analysed the nature of the composition kernel for the conical functions; these calculations have subsequently been extended to the Whittaker kernels in [60]. The works of Albanese et. al [61–63] also provide a useful reference of the technique of Bose invariants and their use in solving diffusion kernels. The original paper by A. K. Bose may be found in [22].

With reference to the topic of associated Legendre functions and Mehler-Fock transforms, the papers of Lenz [64,65] provide a thorough overview of the topic from the direction of image processing techniques. We shall use the insights contained within these papers to examine the homomorphism between the matrix groups defined by the quantum brachistochrone and various stereographic projections of said operators.

1.1 Research Aims

We shall briefly outline the principal aims and methods applied within this thesis. We propose a direct link between some new and innovative ways of understanding matrix groups and time optimisation principles that results in a readily understood unification of the theories of diffusion and differential geometry. The link is, of course, related to representation theory, and as is well known from the studies of Langlands, Mackey and many others dating back to Laplace, Gauss and perhaps even earlier, that the concept of a group operation and representation is central to many different topics in mathematics.

The question of how one goes about constructing sets of functions that represent matrix operators, and the definition of the matrix groups themselves is indeed one of the central problems in modern mathematics.

One difficulty involved in the theory of special functions and group representations is the complicated nature of many central principles and the key assumptions in showing how the different groups arise. Indeed, one might be able to argue that there might exist a theory within which these groups may be the natural consequence of some other lemma. The prohibitive and non-intuitive methods of calculation have the distinct disadvantage of offering little hope for a greater theory at first glance.

It is for this reason that we turn to a different approach, and show that by implementing a certain principle of least time under linear constraints we are able to derive some fundamental unitary matrices. By using projective geometry, we then are able to define the differential geometry and invariants such as the second fundamental form, which gives us access to the continuous dynamical operators. This acts as a shortcut to the often tedious methods of standard group representation theory, and has the benefit of offering substantial intuition into the process that lies behind the curtain.

Further, we are able to take this technique and show that it is still valid not only in a spherical, isotropic regime, but also in the situation where the state inhabits a hyperbolic space. To our knowledge, this is the first working example of a hyperbolic brachistochrone, and the first analysis of the representation theory of the matrix brachistochrone equation, and the related diffusion theory as given by this synthesis is consistent with that of the Mehler-Fock functions. In doing so, we highlight a number of different representations for the kernel through analysis of the distance function which exists on the metric space. We borrow from the concepts of group homomorphism to find representations of the special groups derived herein. This allows development of a coherent body of theory using spectral/eigenfunction decomposition, and the solution of certain index integrals by a new technique.

So, in summary, the key point is to demonstrate the relationship between the hyperbolic symmetries given by pseudounitary operators, the different forms of geometry on the hyperbolic plane, and differential equations with eigenstates represented by special functions. Although similar forms of representation theory may be found in standard references, the use of differential geometry and the principle of least time to achieve this outcome is novel. Note also that this is the first primary application examining the links between the brachistochrone principles of quantum mechanics and representation theory in this way. In some ways, we can think of this as linking the old world, the modern world and the post-modern world of mathematics together, the various spheres of influence being projective and differential geometry, quantum mechanics and representation theory. In doing so, we pay homage to the unbroken chain of ideas that originates with Fermat's principle of least time, the theories of Riemannian geometry, and the work of Stone and von Neumann on unitary representations. That such a link exists is natural when one considers that a brachistochrone is related to the geodesic, and it is this observation that underlies the results considered herein.

1.2 Structure of the Thesis

We briefly outline the structure of the work, and give a brief reference to the basic methods, sources and materials that are covered in each chapter. Wherever possible we have given attribution within the thesis to references when used, and we highlight below the original results and approaches that follow in the calculations contained herein.

1.2.1 Theory of the Brachistochrone

This chapter serves as a basic introduction to the methods and techniques of least-time paths and extremal functions. We give a brief derivation of the major results of classical mechanics through use of the Poisson bracket, then proceed to calculate known results such as Hamilton's equations, Lagrangians and the Euler-Lagrange equations.

We demonstrate some applications of the principle of least action in both classical and quantum mechanics, and comment on the correspondence between the theories in each domain. In particular, we analyse the familiar territory of flux and continuity equations to derive the heat equation.

Following this, we review some historical examples that introduce the concept of least-time paths; in order, we address the classical cycloid problem of Bernoulli, Snell's law of refraction and ray optics, and the Pontryagin principle of classical optimal control.

1.2.2 Mathematical Details

This chapter introduces the main methods and techniques used in this work. In particular, we analyse the theory of hyperbolic groups using a novel method developed by the author. This is an original result utilising techniques from a previous stream of research in [52–55] that generalised the theory of time optimal control using the connection to decompositions of unitary matrices. To set the framework under which we are able to obtain this result, we introduce the concept of the state, and the operators which act upon it that will be used in the main calculation, such as projections, transforms and the basic Hamiltonian and time evolution operators which act in the projective space.

We then focus on the development of the differential geometry we associate to the projective space. We take an axiomatic approach to the problem, and prove the quantum brachistochrone equations from a least time action principle by developing an approach to time optimal control theory, and show how this is related to the projective measure on the space given by the time of the system. This is shown to be directly related to the von Neumann equation and Ehrenfest theorem. We then proceed to derive the formulae for the metric in the complex projective space. This metric may be found in the pseudounitary space by modification of results that may be found originating in the work of Kuzmak [66], where the authors show that it is possible to define the Fubini-Study metric using a matrix representation of the state. We then proceed to show how this metric, as is well known from classical differential geometry, defines a form of the Laplacian operator in the curvilinear space. Concluding, we examine a series of examples that relate the various representations of the hyperbolic plane together, and evaluate the different forms of metrics, closing with the Cayley transform, the Poincare plane and the hyperbolic Laplacian operator.

In doing so, we develop an original approach to finding the associated eigenfunctions and kernels for the hyperbolic space. The coupling of differential geometry and matrix optimisation with the group theory of pseudounitary operators is a powerful new method of functional analysis. We take a basis in the theories of positive definite kernels, which can be understood broadly in terms of Mercer's theorem [67] and Stone's theory of positive definite functions [68]. We derive Mercer's theorem [67] for positive homomorphisms, and show how positive definite functions are related to the homomorphisms of the group. Using some techniques from image processing [64,65] we find representations for the kernel; also by utilising results from [69] in the theory of group representations, we find expressions for the distance formula on hyperbolic space.

The primary aim is to link the theories of distance functions, positive definite functions, group representations, homomorphisms and spherical zonal functions all through the common thread of the kernels that exist on such spaces. Distance is intimately tied to the concept of measure, and through utilisation of methods that come from group theoretic concepts we are able to derive many of the kernels that appear in Stone's and Mercer's theorem by appealing to the concept of distance. Spherical functions are another powerful way in which to view the structure of a group, and many of the results that follow in the later chapters rely on use of concepts from this area of research. An excellent description of the topic may be found in the works of Dieudonne [70] and Godement [71]. We discuss the major results from the theory of spherical functions and derive basic formulae that are repeatedly applied in the following chapters.

We give a listing of the major types of special functions that appear in the thesis, their notation and basic classification according to the potential theory of second order differential equations. Closing, we derive the Bose invariant and Sturm-Liouville theory through a series of worked examples that demonstrate the basic principles. These concepts allow many of the different PDE systems encountered in this thesis to be transformed into one another. We derive the Feynman-Kac formula for a killed process using some simple stochastic calculus, and show how this is related to a Sturm-Liouville problem given by the solution to a separable equation. We then derive the basic transformation formulae, including speed and scale measures and the Bose invariant potential for the system. We conclude with a series of small lemmas related to the speed and scale measures, and give a brief calculation of the Feller form of any second order PDE.

1.2.3 Hyperbolic Heat Kernel

In this chapter we apply the method of the brachistochrone equation to derive the basic hyperbolic pseudounitary operators using analytical continuation. These are then applied specifically to the problem of the hyperboloid; we derive the Fubini-Study metric on this space and show that it is equivalent to the various forms of the Poincare metric. This is a novel result, showing that a direct linkage between the functional representations of groups may be found by analysis of the differential geometry as given by time optimisation principles. In doing so, we directly tackle the calculation of the hyperbolic heat kernel as given in the thesis problems, and show that much of the axiomatic basis that has been missed in this problem can be derived from this perspective. We show how this heat kernel can be seen as a function of the hyperbolic distance, and exploit this relationship by using the formula for the Laplacian in a curvilinear space. This allows direct derivation of the eigenfunctions via the Helmholtz equation, which we then employ to look at various relations of the Mehler-Fock functions and convolution theory on this space. This convolution theory, as given in the work of Chavel [11], allows the hyperbolic heat kernel to be derived through appeal to the laws of transformation of the integral on the group. We give a concise axiomatic basis to these calculations through use of the derivation of results from the pseudounitary operators that stem from the brachistochrone principle, and the use of the Fubini-Study metric to derive the Laplacian on the hyperbolic space.

We obtain other important functionals that relate to the hyperbolic heat kernel including composition formulae, Green's functions and addition formulae. These are shown to relate to the distance formulae. In deriving the solution to the hyperbolic heat kernel as appearing in McKean [2], and earlier in Robin [4], this section outlines a general theory of the special functions that underlie the hyperbolic space. These are a special type of associated Legendre functions, which we refer to as Mehler-Fock functions, which have many useful properties that generalise from this first major application of this theory.

We apply differential geometry on the hyperbolic space to find various distance measures and addition formulae, demonstrating the connections with relativity theory in the hyperbolic plane with several examples. For completeness, we identify the known result from the theory of rapidity with that following from the composition law of the hyperbolic distance function, and derive the addition formula of special relativity. This leads to a broader discussion related to other connections to general relativity, and the implications for singular states appropriate for such models. McKean [2] was able to derive generalised models for solutions in spaces with curvature; other excellent discussions of the relationship between curvature and heat kernels may be found in Canzani [48]. In this direction we consider a generalised example of the hyperbolic heat kernel that is also singular, and possesses curvature. Although this falls outside of the definition of the hyperbolic heat kernel, the theory of the Fubini-Study metric presents some consequences for particular solutions of singular metrics which we calculate using methods of projective geometry and matrix calculus.

To further demonstrate the broader applicability of the techniques and methods, the concluding sections of the chapter focus on the composition laws from the perspective of index transforms, in particular some integrals from Oberhettinger stated without proof are verified through application of the techniques developed in this chapter (see collated results in [27–30]). This sets the framework for the results that follow in the remainder of the thesis.

1.2.4 Index Integrals and the Projection-Slice Theorem

The projection-slice theorem is of major utility in the field of Fourier analysis, following the initial investigations of Bracewell (CSIRO, c. 1956, see [23]), who proved the existence of the Fourier-Abel-Hankel cycle. In this chapter we generalise this theorem to the hyperbolic equivalent, from the perspective of index transforms that follow from the analysis of the hyperbolic heat kernel. In many ways the Mehler-Fock transform plays the role of the Fourier equivalent in the hyperbolic plane, with the Whittaker and Kontorovich-Lebedev transforms as the other parts in the cycle of transformations. The major aim in this chapter is to show how the various forms of index integrals may all be resolved in similar ways by recourse to the projection-slice theorem in the hyperbolic plane.

Chavel, in [11] outlines a simple method to gain the basic Hankel transform (order 0), which relies on a theory of area integration in the complex plane. We show how this can be extended to the hyperbolic plane, and derive the generalised Hankel transform and Kontorovich-Lebedev transforms as an application of this modification of the basic case in [11]. This allows us to develop further the analysis of index integrals, and we show how some basic examples of Oberhettinger related to scattering theory are evaluated (see [72–74] for application to problems in diffraction and wave propagation). Other uses of index transforms are briefly reviewed, including the major utilisations in mathematical finance as related to pricing of contracts with time averaging features such as Asian options. A number of references including the work of Linetsky [75], Yor [76,77], Dufresne [78,79] have commented on the relationship between index transforms. More recently, the work of Sousa, Yakubovich see [56] has lead to a broader research effort towards better analysis of these systems. This aims at contributing a deeper understanding and better methods for solving these problems through use of the projection-slice theorem.

The remaining half of this chapter focuses on direct application of the methods of transformation, showing how various index integrals that give Asian option prices can be formulated in this way. We briefly discuss the convolution theorem for this system and some special cases of index transforms related to the Yor integral. Many of these results appeared in a paper by the author of this work as part of conference proceedings at MATRIX 2022 in the context of options pricing models and numerical implementation [80].

1.2.5 Radon Transforms and the EPD Equation

As a major application of the methods stemming from the Fubini-Study metric, this chapter focuses on the development of the theory of the Radon transform. The primary aim in doing so is to arrive at the Euler-Poisson-Darboux equation, and show how this is fundamentally related to the problems considered in the hyperbolic heat kernel. In doing so, we show how this powerful method of analysis can be directly applied to connect theories of eigenfunctions and kernels with the theories of differential geometry. The deeper relationships between transforms, kernels and operators are emphasised through the series of calculations in this chapter. The extension of the question and solution to the hyperbolic heat kernel is given additional context in this fashion; we emphasise more broadly how this solution generalises to other systems of hyperbolic equations through the use of the Radon transform.

Radon transforms give the tomographical solutions that arise in problems of imaging, and many would be familiar with their application in medical science and sensing. In this chapter, we analyse the Radon measure through the Fourier theorem which gives the group invariant solution. This approach, which may be found in the works of Helgason [81], and also the research of Deans [82,83], is a powerful methodological framework that essentially relies on the concepts of spherical transform theory. Other authors including Hafoud et. al [84–86] have commented on the relationship between the projective plane, the reduced sphere given by the Hopf fibration, and Gegenbauer polynomial expansions of the heat kernel. In this chapter we use the theory of the curvilinear Laplacian to relate the differential geometry to the Radon transform, which is shown to be given by a PDE system known classically as the Euler-Poisson-Darboux equation. This simple shortcut to the group representation theory allows rapid generalisation of the EPD system to a number of new equations with similar properties.

A tomography or tomographical representation is the composition of an object through slices taken along an axis. The Radon transform is the mathematical encapsulation of this slicing process. Helgason [81] has shown that the Radon transform is essentially the intertwining operator for the Laplacian on this special space, which is equivalent to the restricted sphere on the projective space. In the same way that the Fourier transform plays a special role as it intertwines with the Laplace operator, the Radon transform and the Euler-Poisson-Darboux equation define similar relationships on a form of hyperbolic space.

We show directly how the Fubini-Study metric gives the Euler-Poisson-Darboux equations using a generalisation of the methods in the preceding chapters, and derive kernels by a system based on eigenfunction decompositions and spectral analysis. The EPD equations are then generalised to give the both time-EPD and space-EPD equations. The primary modification is the question of kernels that exist between two times at a space point, as opposed to two space points at a particular time. This simple change allows the calculation of a wide range of time dispersive systems. We show how these sort of systems obey Lipschitz continuity and calculate the basic kernels, which are identified with certain types of Mehler-Fock and modified Bessel functions.

The potential theory of these types of EPD systems leads to a deeper discussion of the potential theory of a special type of Bessel function. This section engages with aspects of special functions, relating the time solutions of the Bessel/EPD equation with hypergeometric functions and the Jacobi polynomials. Further considerations of the potential theory allows generalisation of the EPD equation to a number of new types of eigenfunction systems, including Whittaker, Bessel, Laguerre and Legendre functions.

Craddock and Lennox (see the series of papers [40–42,46,87–92]) have developed the technique of essentially exact solutions to systems of PDEs. This method relies on the reduction of fundamental solutions to an integral form via use of Lie symmetry groups. An essentially exact solution gives the fundamental solution as a combination of two transforms which operate together. In this chapter we apply this method to derive EPD equivalents of essentially exact solutions involving the Whittaker, Mehler-Fock and Kontorovich-Lebedev transformations coupled to the EPD equation via the use of special types of Bessel eigenfunctions.

We close this chapter with derivation of two major types of classification for the systems of eigenfunctions. The first gives a quick derivation of a method known in the field of statistical analysis as the Pearson classification system c. 1893, which was originally employed in [93] in an attempt to classify probability density functions through their moments. We show how this can be constructed using some analysis of the potential theory of second-order differential equations, and relate this to the Sturm-Liouville theory explored throughout this work. This leads to a different proposed classification system utilising the Bose invariant which does not rely on the same approximations as the Pearson classification. Concluding, we apply this classification method to a number of systems of singular hyperbolic EPD eigenfunctions and tabulate the results.

1.2.6 The Weierstrass Transform

This chapter deals with some advanced concepts that follow from analysis of the hyperbolic heat kernel. In earlier sections we establish the connection between systems of eigenfunctions and decompositions of the kernel. This allows us to engage more directly with the transform theory of the heat kernel, which is given by the Weierstrass transformation.

The primary focus of this chapter is to demonstrate the connection between theories of the hyperbolic heat kernel as given by their integral representations and the Weierstrass transformation. This allows identification of the solution spaces for the wave, heat and EPD kernels to be mapped to one another. We further apply this to derive the kernel solutions for these equations using index transformations and eigenfunction decompositions.

The Weierstrass transformation [94–97] is the form of transform with the kernel argument given by the Gaussian function, i.e. the standard solution to the heat equation. However, there are not many tabulated results such as for e.g. the Fourier transform, so we derive the major equivalent properties that are required, such as inversion and symmetries.

This leads to calculation of the kernel and related Green's function through use of index integration methods based on the projection-slice theorem for hyperbolic space. We apply the techniques of Bose invariant potentials to recover the solutions, and show how there exists a fundamental equivalence between systems of EPD type, wave type and heat kernel type. This result, discussed in depth in Bragg and Dettman [98,99], is approached using eigenfunction decompositions of the kernel. We show how forms of the hyperbolic heat kernel are given through Weierstrass transforms and derive an integral related to the Yakubovich kernel. Craddock's result using a sine transform [46] is compared to this result and comment is given on the similarity of proof.

As is known from the results of classical functional analysis, transforms are related to group symmetries, and these mean that it is often possible to associate a transform with a particular group of special functions or polynomials. In the case of the Weierstrass transformation, this role is played by the Hermite transform, which gives a decomposition of the Weierstrass transformation over the set of Hermite polynomials. We develop further the theory of operator calculus, showing that many of these results can be obtained through use of the solution operator method.

Heaviside [100,101] was a major proponent and originator of the concept of solution operators in the theory of differential equations. As is well-known nowadays, the application of the Laplace transform changes differential equations into algebraic equations. In his original work in the field of electrical circuitry, Heaviside was able to exploit this equivalence to solve many problems that escaped the minds of those used to the concepts of fluid dynamics as prevailed at the time. The underlying premise was, however, somewhat stronger than this. In the same way that a Laplace transform takes an operator into a numerical function, one can think of the inverse process, in which the solutions are functions of operators. This is the method of the solution operator, and our analysis of the Weierstrass function shows that this is a valid and useful technique closely allied to the results derived in the previous chapters.

The solution operator method is systematically applied to the wave, heat and EPD equations, followed by a direct application of Sturm-Liouville theory and the Bose invariant. This leads to rapid identification of the symmetries of the EPD equation of Bessel type. We analyse the residue calculus of a simple example, showing that one can recover meaningful solutions through use of operator calculus and some Gaussian integrals. This leads to calculation of the major types of solution operators, the associated kernel functions, and the integral theorems which connect them to one another. This is then followed by a detailed computation of the kernel solutions for the Bessel type modification of the heat, wave and EPD equations. We close the chapter with a theorem relating the solutions of EPD equations with a certain hyperbolic PDE obeyed by the Green's function.

1.3 Addressing the Problems

We briefly outline the method of solution for the problems as stated in the foreword. The hyperbolic heat kernel, as noted in McKean [2], is derived from an axiomatic basis, where we apply the method of the hyperbolic brachistochrone and the Fubini-Study metric. This allows access to eigenfunction representations of the kernel, that we evaluate and compare with expressions obtained by Chavel [11] using the method of convolution. Both proofs are given context through the direct derivation of the group theory via the pseudounitary operators and the associated differential geometry implied by the metric. Utilising the powerful connections between matrix optimisation principles, projective geometry and group representation theory is a relatively robust and direct approach to solving these types of problems in functional analysis.

The general theory of the techniques required in the solution of the hyperbolic heat kernel is used to derive a form of the projection-slice theorem. This is expressed in terms of hyperbolic transforms and eigenfunctions associated to the hyperbolic heat kernel using some integrals in special function theory.

This is applied to a number of different examples of index integrals to demonstrate the concepts of index transform theory and hyperbolic functions. We show how many different index transforms can all be computed in a similar way using the projection-slice theorem. This is a major new application of the concepts that result from the solution of the hyperbolic heat kernel by the method of the brachistochrone principle. The methods and techniques of eigenfunction decompositions of the kernel allow for the generation of many index integration formulae previously difficult to compute.

The statements of Kac can therefore be seen in context; there is a consistent, useful theory of Sturm-Liouville operators for this type of system. We demonstrate this fact repeatedly through use of the Bose invariant potential theory, and index transformations. The convolution theory equivalent is addressed, as we generalise the approach given in Chavel [11] to give a consistent account for the various types of hyperbolic eigenfunctions.

As is known from probability theory and stochastic processes, there is a clear connection between the Feynman-Kac theorem and Sturm-Liouville theory. Feynman clearly knew that eigenfunction representations are essential for construction of the kernel; this is a fact known in at least some form dating to the work of Mehler c. 1866 [5]. The use of the Mehler formula is ubiquitous in Feynman's analysis of dynamic systems. What is less clear is if that Feynman was aware of the delicate issues that arise regarding the differences between discrete and continuous spectra in evaluating the kernel, We can see in the statements that he is keenly interested in evaluating these types of relations, and the connection between completeness, measure and probability is of primary importance.

In this work we show how many of the methods stemming from Feynman's initial researches in quantum mechanics are modified in this new context, with the discrete summation over eigenstates recast into index integration over the continuous spectrum. The notion of completeness in this domain is then defined through the kernel solution of the operator on the space. Characterising our understanding of how this solution behaves and is computed in various representations of the hyperbolic plane forms a major component of this research. We address, master, and apply the notion of completeness in the hyperbolic plane to define the solution space and transition probability density of the system.

Craddock and Lennox observe that there exist Lie symmetry groups on the level of PDE operators that define fundamental solutions for many of the important systems known to integral transform theory and special functions. This thesis resolves many of the difficulties that arise through the use of group representation theory through a direct appeal to the concepts of scale invariance. Much of the labour involved in evaluating Lie symmetries through prolongation methods obscures the intuition that such groups offer once obtained. We directly work to simplify the analysis of the symmetry groups with a series of concrete examples. In doing so, we encounter, employ and generalise invariance principles through use of ideas from potential theory. Through application of methods from Sturm-Liouville theory and gauge potentials, we successfully set out a theoretical framework for understanding many of the different symmetry groups in the hyperbolic plane. In doing so, we open connections to a number of different areas of special function theory and integral transforms.

In many ways, the problems as discussed are all different aspects of the same fundamental question, that of finding kernel solutions and transition probability densities which is indeed the crux of modern probability theory, quantitative finance and quantum physics. This work has shown that the methods of brachistochrone theories with group representations and differential geometry offers a ready access point to the underlying eigenfunctions and kernels, enabling a clarity of solution that is difficult to replicate through other methods.

1.3.1 Comments

We have endeavoured wherever possible to give references to other work which is used in this calculation. It is a broad topic of analysis, encompassing many different disciplines, and it is difficult to search exhaustively. In part, this work should be considered one of demonstration and synthesis, in which we shall go about constructing a method and a system by which we can analyse other groups. This is a new technique, taking as a starting point the action of unitary matrices and a principle of least time, and arriving at explanations for the theory of special functions and kernel transformations via differential geometry. By using each of the theories as outlined above we are able to derive many descriptive formulae for these systems.

In terms of a hierachy of ideas, we place the principle of least time and postulates of projective geometry at the core. The differential geometry follows as a natural consequence via the Fubini-Study metric. From this, we then generate distance functions, and hence the kernel and Green's functions for the diffusion operators, defined through the Laplacian. Various transformations between special functions allow us to derive integral formulae that appear formidable by other means. This is a novel approach to generating group representations, and it is the original contribution of the author. We must stress that the brachistochrone principle is in some ways a primary theory that can be seen to overlay all these different concepts. It is hoped that by showing the application of this technique it opens up the area to further development, particularly with regard to the representation theory of other special functions.

CHAPTER 2

Theory of the Brachistochrone

2.1 Introduction

The following chapter introduces the theory of optimisation and least-time paths in a historical context for the reader who may be unfamiliar with the topic. We cover a number of important examples that appear in classical and quantum mechanics from the perspective of least-action principles, with a particular focus on those of least time. The word brachistochrone ($\beta\rho\alpha\chi\iota\sigma\tau o\chi\rho o\nu\eta$) means "the curve of least time" (chronos: time, brachistos: least). This special curve has many useful properties, and as the history of science has shown, it has been a fruitful place of investigation for centuries already, and centuries yet to come. Amongst the illustrious names who should be mentioned, first and foremost must be Fermat, who can be considered the originator of the method of least time, when considering problems in ray optics. Huyghen's principle follows as a consequence of this when applied to the question of a spherical wavefront. Classical mechanics was pioneered through the understanding of more general problems in optimisation, through the efforts of Lagrange, Euler and Hamilton most prominently. In a modern context, Dirac, Schrödinger and Feynman applied this concept to quantum mechanics, while Pontryagin and Bellman used it within problems of optimal control.

2.2 Classical Mechanics and the Poisson Bracket

In this thesis, we shall take the definition of the state to be the specification of the system whereby all other dependent variables may be found. The state of the system allows complete resolution of all probability densities and measurable quantities. In classical mechanics, the basic notion is to take the set of positions and conjugate momenta via the $2n-$ space variables (q_i, p_i). The Liouville flow on the symplectic $2n-$ space + 1 time manifold is then governed by the Poisson bracket:

Definition 2.2.1. The Poisson bracket is given by the derivative formula:

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \tag{2.1}$$

and satisfies:

- The antisymmetric identity $\{f, g\} = -\{g, f\}$
- The Jacobi identity $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
- The Leibniz rule $\{fg, h\} = f\{g, h\} + g\{f, h\}$

2.2.1 Time Evolution of Classical Systems

The Poisson bracket is sufficient to define the dynamics of phase space, via the Hamiltonian operator, familiar in classical mechanics as the energy:

Definition 2.2.2. The Hamiltonian operator is defined by the energy function:

$$\mathscr{H}(q_i, p_i; t) = \frac{1}{2}m\dot{q_i}^2 + V(q_i, t) \tag{2.2}$$

where $\dot{q}_i$ is the conjugate momentum to the co-ordinate q_i, and the potential energy is $V(q_i, t)$. In a time independent potential, energy is a conserved quantity.

The dynamics of the phase space, which consists of the canonical position and conjugate momentum functions q_i, p_i is determined in classical statistical mechanics by the Liouville equation:

Definition 2.2.3. The time evolution of any continuous function f is defined through the Poisson bracket identity:

$$\frac{df}{dt} = \{\mathscr{H}(q_i, p_i), f\} + \frac{\partial f}{\partial t} \tag{2.3}$$

where $\{,\}$ is the Poisson bracket, and $\mathscr{H}(q_i, p_i)$ is the Hamiltonian of the system, defined by the positions and conjugate momenta. If the function is not an explicit function of time, then $\dfrac{\partial f}{\partial t} = 0$, and we have the restricted Liouville evolution:

$$\frac{df}{dt} = \{\mathscr{H}(q_i, p_i), f\} \tag{2.4}$$

Corollary 2.2.4. *For any conserved quantity, its value does not change in time.*

$$\frac{df}{dt} = 0 \tag{2.5}$$

This implies that under the action of the Poisson bracket there is a symmetry:

$$\{\mathscr{H}(q_i, p_i), f\} = 0 \tag{2.6}$$

Proof. This is elementary, as a consequence of the definition of the Poisson bracket 2.2.1 we have:

$$\frac{df}{dt} = \{\mathscr{H}(q_i, p_i), f\} = 0 \tag{2.7}$$

and hence the identity follows. Note that from the definition, one can readily recover:

$$\frac{d\mathscr{H}}{dt} = \{\mathscr{H}(q_i, p_i), \mathscr{H}(q_i, p_i)\} \tag{2.8}$$

$$= \sum_{i=1}^{n} \frac{\partial \mathscr{H}}{\partial p_i}\frac{\partial \mathscr{H}}{\partial q_i} - \frac{\partial \mathscr{H}}{\partial p_i}\frac{\partial \mathscr{H}}{\partial q_i} = 0 \tag{2.9}$$

and hence the energy, as given through the Hamiltonian function, is constant in time, and is a conserved quantity. □

2.2.2 Euler-Lagrange Equations

We shall now derive the basic result for the calculus of variations, as given by the Euler-Lagrange equations. We define an action principle as given by the optimisation:

Definition 2.2.5. The Lagrangian principle of least action is defined through the integral:

$$S = \int_{t_1}^{t_2} \mathscr{L}(q_i, \dot{q}_i) d\tau \tag{2.10}$$

where the action is minimised $\delta S = 0$ for the realised optimum path. This path is an extremal, known as the brachistochrone in optics, and the path of least action in mechanics.

Corollary 2.2.6. *The Euler-Lagrange equations for the action principle 2.2.5 are given by:*

$$\frac{\partial \mathscr{L}}{\partial q_i} - \frac{d}{d\tau}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right) = 0 \tag{2.11}$$

and give the extremal path that solves the action principle $\delta S = 0$.

Proof. If we differentiate the action principle directly, we have:

$$\frac{\partial}{\partial \alpha}\int_{t_1}^{t_2} \mathscr{L}(q_i, \dot{q}_i) d\tau = \int_{t_1}^{t_2} \frac{\partial}{\partial \alpha}\mathscr{L}(q_i, \dot{q}_i) d\tau = 0 \tag{2.12}$$

where we have differentiated under the integral sign. Then we expand with respect to the canonical position and momentum co-ordinates:

$$\int_{t_1}^{t_2} \left[\frac{\partial \mathscr{L}}{\partial q_i}\frac{\partial q_i}{\partial \alpha} + \frac{\partial \mathscr{L}}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \alpha}\right] d\tau = 0 \tag{2.13}$$

and indeed we can write:

$$\int_{t_1}^{t_2} \frac{\partial \tau}{\partial \alpha}\left[\frac{\partial \mathscr{L}}{\partial q_i} dq_i + \frac{\partial \mathscr{L}}{\partial \dot{q}_i} d\dot{q}_i\right] = 0 \tag{2.14}$$

Writing out the derivatives explicitly, we have:

$$\int_{t_1}^{t_2} \frac{\partial \tau}{\partial \alpha}\left[\frac{\partial \mathscr{L}}{\partial q_i} dq_i + \frac{\partial \mathscr{L}}{\partial \dot{q}_i} d\left(\frac{\partial q_i}{\partial \tau}\right)\right] = 0 \tag{2.15}$$

We may use integration by parts; we have:

$$\frac{\partial \mathscr{L}}{\partial \dot{q}_i} d\left(\frac{\partial q_i}{\partial \tau}\right) = v du \tag{2.16}$$

where $u = \dfrac{\partial q_i}{\partial \tau}, v = \dfrac{\partial \mathscr{L}}{\partial \dot{q}_i}$, and hence the integration by parts formula goes over into:

$$vdu = uv| - udv \tag{2.17}$$

or

$$\int_{t_1}^{t_2} \frac{\partial \tau}{\partial \alpha}\left[\frac{\partial \mathscr{L}}{\partial \dot{q}_i} d\left(\frac{\partial q_i}{\partial \tau}\right)\right] = \int_{t_1}^{t_2} \frac{\partial \tau}{\partial \alpha}\left[\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\frac{\partial q_i}{\partial \tau}\Bigg|_{t_1}^{t_2} - \frac{\partial q_i}{\partial \tau} d\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right)\right] \tag{2.18}$$

$$= \int_{t_1}^{t_2} \frac{\partial \tau}{\partial \alpha}\left[-dq_i \frac{d}{d\tau}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right)\right] \tag{2.19}$$

On the boundary, we necessarily have $\left.\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\frac{\partial q_i}{\partial \tau}\right|_{t_1}^{t_2} = 0$, and therefore:

$$\int_{t_1}^{t_2} \frac{\partial \tau}{\partial \alpha}\left[\frac{\partial \mathscr{L}}{\partial q_i}dq_i - dq_i\frac{d}{d\tau}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right)\right] = 0 \tag{2.20}$$

which can be rewritten as:

$$\int_{t_1}^{t_2} \frac{\partial \tau}{\partial \alpha}\left[\frac{\partial \mathscr{L}}{\partial q_i} - \frac{d}{d\tau}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right)\right] dq_i = 0 \tag{2.21}$$

implying that

$$\frac{\partial \mathscr{L}}{\partial q_i} - \frac{d}{d\tau}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right) = 0 \tag{2.22}$$

as required. □

The Euler-Lagrange equations 2.2.6 and the symmetries of the Hamiltonian under the Poisson bracket that define the conserved quantities or constants of motion are sufficient to derive all results that stem from classical mechanics and optics. For example, one can define the classical Lagrangian function given by the difference between the kinetic and potential energy as below.

Definition 2.2.7. The Lagrangian for classical mechanics is given by the difference of energies:

$$\mathscr{L} = T(\dot{q}_i) - V(q_i) = \frac{1}{2}m_i\dot{q}_i^{\,2} - V(q_i) \tag{2.23}$$

where $V(q_i)$ is a purely co-ordinate dependent potential, m_i is the mass associated to co-ordinate q_i, and the velocity is $\dot{q}_i$.

Corollary 2.2.8. *Under the principle of least action, the Euler-Lagrange equations 2.2.6 define Newton's law of motion:*

$$\frac{d}{d\tau}(m_i\dot{q}_i) = -\frac{\partial V(q_i)}{\partial q_i} \tag{2.24}$$

Proof. This is a straightforward application of 2.2.6 to 2.2.7. We have:

$$\mathscr{L} = T(\dot{q}_i) - V(q_i) = \frac{1}{2}m_i\dot{q}_i^{\,2} - V(q_i) \tag{2.25}$$

Calculating the partial derivatives, we find:

$$\frac{\partial \mathscr{L}}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}m_i\dot{q}_i^{\,2}\right) = m_i\dot{q}_i \tag{2.26}$$

and

$$\frac{\partial \mathscr{L}}{\partial q_i} = -\frac{\partial V(q_i)}{\partial q_i} \tag{2.27}$$

The result follows. □

So, it is clear that in classical mechanics that the definition of the canonical position variables and conjugate momenta, as given through $q_i, \dot{q}_i$ are a sufficient basis to begin one's calculation of the properties of the system. This is the basic notion of the state; from the microscopic picture generated from such variables, one can build up all dependent functions and physical characteristics of interest for the system under consideration.

2.2.3 Hamilton's Equations

We shall now briefly comment on the well-known extension of such a state-based description of reality to quantum mechanics. If we compute the Poisson bracket of the position and momentum co-ordinates against a suitable test function f, we find the following.

Lemma 2.2.9. *The Poisson bracket $\{,\}$ 2.2.1 of the position q_j and momentum p_j satisfies the following relations:*

$$\{f, q_j\} = -\frac{\partial f}{\partial p_j} \tag{2.28}$$

$$\{f, p_j\} = +\frac{\partial f}{\partial q_j} \tag{2.29}$$

Proof. Using the definition of the Poisson bracket from 2.2.1, we may write:

$$\{f, q_j\} = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial q_j}{\partial q_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial f}{\partial q_i} \tag{2.30}$$

$$= \delta_{ji}\frac{\partial f}{\partial p_i} - 0 \times \frac{\partial f}{\partial q_i} = \frac{\partial f}{\partial p_j} \tag{2.31}$$

where we have assumed independent variables through $\frac{\partial q_j}{\partial p_i} = 0$, $\frac{\partial q_j}{\partial q_i} = \delta_{ji} = \delta_{ij}$, the Kronecker delta used as per usual in this context, with obvious relations to symplectic geometry as is well-known. The second leg of the lemma follows as:

$$\{f, p_j\} = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial p_j}{\partial q_i} - \frac{\partial p_j}{\partial p_i}\frac{\partial f}{\partial q_i} = -\frac{\partial f}{\partial q_i} \tag{2.32}$$

thus establishing the lemma. □

Remark 2. This is a well-known result dating to the original research of Lagrange, Hamilton and Poisson.

We shall now use this to establish the following two corollaries, which form an alternative basis for understanding classical mechanics.

Corollary 2.2.10. *Hamilton's equations for classical particle dynamics may be written in the form:*

$$\{\mathscr{H}, p_j\} = -\frac{\partial \mathscr{H}}{\partial q_i} \tag{2.33}$$

and

$$\{\mathscr{H}, q_j\} = +\frac{\partial \mathscr{H}}{\partial q_i} \tag{2.34}$$

Corollary 2.2.11. *Under the conditions of Liouville evolution 2.2.9, we may write the previous corollary 2.2.10 as:*

$$\frac{dp_j}{dt} = \{\mathscr{H}, p_j\} = -\frac{\partial \mathscr{H}}{\partial q_j} \tag{2.35}$$

$$\frac{dq_j}{dt} = \{\mathscr{H}, q_j\} = +\frac{\partial \mathscr{H}}{\partial p_j} \tag{2.36}$$

i.e. the classical Hamilton's equations.

Remark 3. These imply Newton's law of motion, as has been established in a historical context by Hamilton, Lagrange and others.

Proof. This follows from a simple substitution of a time independent function $\mathscr{H}(q_i, p_i)$ for f into 2.2.10, and use of the relation 2.2.9 as is well-known. □

2.2.4 Correspondence Principles

We shall now show one simple way in which these relations can be used to understand the corresponding formulae in quantum mechanics. This sort of correspondence principle should be taken as an intuitive guide, but not a direct and binding relationship, in the sense of Bohr and Heisenberg, we approach first quantisation. One notes that from the Poisson bracket, we have naturally derived the formulae:

Lemma 2.2.12.

$$\{p_i, q_j\} = -\frac{\partial p_i}{\partial p_j} = -\delta_{ij} \tag{2.37}$$

$$\{q_i, p_j\} = +\frac{\partial q_i}{\partial q_j} = \delta_{ij} \tag{2.38}$$

where q_i, p_j are the canonical position and momentum variables, and δ_{ij} the Kronecker delta, with $\{,\} =$ the Poisson bracket 2.2.1.

Proof. This follows from 2.2.1 and substitution of the variables into the left-hand side of the identity. □

Quantum mechanics may be seen as asking the question as to whether there exist brackets which have analogous properties to the Poisson bracket, as defined in 2.2.1, but which are non-commutative. For example, one might write the following correspondence:

Proposition 2.2.13.

$$\{p_i, q_j\} = -\{q_i, p_j\} \tag{2.39}$$

i.e. $\{\} \to [,]$ giving the correspondence between one set of bracket relations and another. The basic antisymmetric identity is implied:

$$[\hat{a}, \hat{b}] = -[\hat{b}, \hat{a}] \tag{2.40}$$

If we take the real-valued bracket, we might write:

$$[\hat{x}, \hat{p}_x] = \delta_{xx} = 1 \tag{2.41}$$

$$[\hat{p}_x, \hat{x}] = -[\hat{x}, \hat{p}_x] = -1 \tag{2.42}$$

which is satisfied by differential operators in the following way:

$$-\frac{\partial}{\partial x}(xf) + x\frac{\partial f}{\partial x} = xf' - f - xf' = -f \tag{2.43}$$

$$-x\frac{\partial f}{\partial x} + \frac{\partial}{\partial x}(xf) = +f \tag{2.44}$$

implying the pair $\hat{p}_x = -\frac{\partial}{\partial x}, \hat{x} = x$ satisfies this condition.

Remark 4. We shall go into greater detail regarding the construction of Hermitian operators and so on for the quantum and hyperbolic spaces considered in this thesis, this basic introduction is simply for completeness, to show the direct relationship between the ideas of classical optimisation theory, mechanics, and Lagrangian principles. To truly derive quantum mechanics using the rough argument in proposition 2.2.13 requires the insertion of a factor of i, which cannot be justified from classical arguments alone. However, as the following brief calculation shows, the use of action principles can be another valid direction to understand these types of spaces.

2.3 Examples of Action Principles

One further way in which to comprehend the connection between classical and quantum mechanics is via the action principle, as originally discussed in [102]. If we consider the basic form of the action, for a non-relativistic quantum space we may write:

Definition 2.3.1. The quantum action principle is given by the Lagrangian:

$$S(\psi,\psi^*) = \int dt \int d^3x \mathscr{L}(\psi,\psi^*) \tag{2.45}$$

where ψ is the wavefunction, the $*$ denotes complex conjugation, the integration being over all space and time paths between the initial and terminal points.

Lemma 2.3.2. *The Lagrangian defined in the quantum action principle can be decomposed into symmetric and antisymmetric parts:*

$$\mathscr{L}(\psi,\psi^*) = \mathscr{L}_{s.}(\psi,\psi^*) + \mathscr{L}_{a.s.}(\psi,\psi^*) \tag{2.46}$$

$$\mathscr{L}_{a.s.}(\psi,\psi^*) = -\mathscr{L}_{a.s.}(\psi^*,\psi) \tag{2.47}$$

$$\mathscr{L}_{s.}(\psi,\psi^*) = \mathscr{L}_{s.}(\psi^*,\psi) \tag{2.48}$$

The symmetric part of this function is real, and the antisymmetric part is purely imaginary.

Proof. Any function may be decomposed into its symmetric and antisymmetric parts. For example, in this instance, one may take the following combinations:

$$\mathscr{L}(\psi,\psi^*) = \frac{1}{2}\left(\mathscr{L}(\psi,\psi^*) + \mathscr{L}(\psi^*,\psi)\right) + \frac{1}{2}\left(\mathscr{L}(\psi,\psi^*) - \mathscr{L}(\psi^*,\psi)\right) \tag{2.49}$$

$$= \mathscr{L}_{s.}(\psi,\psi^*) + \mathscr{L}_{a.s.}(\psi,\psi^*) \tag{2.50}$$

We have, by inspection, $\mathscr{L}_{s.}(\psi,\psi^*) = \mathscr{L}_{s.}(\psi^*,\psi)$ and $\mathscr{L}_{a.s.}(\psi,\psi^*) = -\mathscr{L}_{a.s.}(\psi^*,\psi)$, which establishes the first part of the lemma. Defining then the complex conjugation operation via:

$$\bar{\mathscr{L}}(\psi,\psi^*) = \mathscr{L}(\psi^*,\psi) \tag{2.51}$$

we have immediately that:

$$\bar{\mathscr{L}}_{s.} = \mathscr{L}_{s.}, \bar{\mathscr{L}}_{a.s.} = -\mathscr{L}_{a.s.} \tag{2.52}$$

implying the second leg of the lemma. □

Remark 5. Although this may seem like a sketch definition, we shall show in the following that one may establish a particular form of the action principle through some very simple analysis. This will give the explicit form of the Lagrangian for this form of quantum mechanics. For the following calculation, and for the rest of the document we take natural units ($\hbar = c = m = 1$) except when directly specified, for simplicity of analysis without loss of generality.

2.3.1 Quantum Action Principle

Lemma 2.3.3. *A general antisymmetric function that is first order in space and time derivatives is given by the linear combination:*

$$\mathscr{L}_{a.s.}(\psi,\psi^*) = \frac{i}{2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) + \frac{i}{2}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right) = \mathscr{L}_0^-(\psi,\psi^*) + \mathbf{j}(\psi,\psi^*) \tag{2.53}$$

and the symmetric equivalent:

$$\mathscr{L}_{s.}(\psi,\psi^*) = -\frac{1}{2}\nabla\psi\cdot\nabla\psi^* - V\psi^*\psi \tag{2.54}$$

These functions satisfy the symmetric and antisymmetric relations 2.3.2.

Proof. By inspection, both of these functions are anti-symmetric or symmetric. However, with the first, we note that the current term ($\mathbf{j}(\psi, \psi^*)$) is dimensionally inconsistent with the function, and we therefore reject it from the Lagrangian formulation. $\square$

With this ansatz, we can then write the quantum action principle in the form given below, and the Euler-Lagrange equations may be solved.

Lemma 2.3.4. *The quantum action principle 2.3.1, with Lagrangian 2.3.3 may be written:*

$$\mathscr{L}(\psi, \psi^*) = \frac{i}{2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) - \frac{1}{2}\nabla\psi \cdot \nabla\psi^* - V\psi^*\psi \tag{2.55}$$

The Euler-Lagrange equations for this Lagrangian define the Schrödinger equation for the non-relativistic quantum system:

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\nabla^2\psi + V\psi \tag{2.56}$$

where V is the potential function of the system.

Proof. For this problem, we have two canonical momentum co-ordinates, related to the function ψ and its complex conjugate. The Euler-Lagrange equations 2.2.6 then generalise to yield:

$$\frac{\partial \mathscr{L}}{\partial \psi^*} - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial(\dot{\psi^*})}\right) - \nabla \cdot \left(\frac{\partial \mathscr{L}}{\partial(\nabla\psi^*)}\right) = 0 \tag{2.57}$$

$$\frac{\partial \mathscr{L}}{\partial \psi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial(\dot{\psi})}\right) - \nabla \cdot \left(\frac{\partial \mathscr{L}}{\partial(\nabla\psi)}\right) = 0 \tag{2.58}$$

where we have assumed that the function ψ and its complex conjugate are independent variables. Computing the differentials, we have immediately that:

$$\frac{i}{2}\left(\frac{\partial \psi}{\partial t}\right) - \frac{\partial}{\partial t}\left(-\frac{i}{2}\psi\right) - \frac{1}{2}\nabla \cdot \nabla\psi - V\psi = 0 \tag{2.59}$$

and the result follows. The second equation follows as the complex conjugate of the first, or may be evaluated from the other Euler-Lagrange equation. $\square$

2.3.2 Current, Flux and Continuity

We shall now discuss some established facts related to the current term ($\mathbf{j}(\psi, \psi^*)$), which was rejected from the Lagrangian on the basis of dimensional consistency. We shall prove the following known fact from quantum physics, and establish the connection between the probabilistic viewpoint of matter and the dynamics as expressed through the function ψ, which we tentatively term the state of the system.

Lemma 2.3.5. *For the current vector defined through:*

$$\mathbf{j} = \mathbf{j}(\psi, \psi^*) = \frac{i}{2}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right) \tag{2.60}$$

then this vector satisfies the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{2.61}$$

where $\rho = \psi\psi^* = |\psi|^2$ is the probability density of the system. This implies that probability is conserved for quantum systems.

Proof. Differentiating the probability density directly, we have for $\rho = \psi\psi^*$:

$$\frac{\partial \rho}{\partial t} = \dot{\psi}^* \psi + \psi^* \dot{\psi} \tag{2.62}$$

If we assume the time evolution equation, we have for the function ψ and its complex conjugate:

$$i\frac{\partial \psi}{\partial t} = \hat{H}\psi, -i\frac{\partial \psi^*}{\partial t} = \left(\hat{H}\psi\right)^* \tag{2.63}$$

and hence:

$$\frac{\partial \rho}{\partial t} = i\left(\psi^* \hat{H}\psi - \psi\left(\hat{H}\psi\right)^*\right) \tag{2.64}$$

Substituting the result from 2.3.4, we have the decomposition of the Hamiltonian into the free operator and potential:

$$\hat{H} = -\frac{1}{2}\nabla^2 + V = \hat{H}_0 + V \tag{2.65}$$

Assuming that the potential is real, as it must be under the conditions of unitary evolution, we have:

$$\psi^* V \psi - \psi\left(V\psi\right)^* = 0, V^* = V \tag{2.66}$$

and we can therefore write:

$$\frac{\partial \rho}{\partial t} = i\left(-\frac{1}{2}\psi^*\nabla^2\psi + \frac{1}{2}\psi\nabla^2\psi^*\right) \tag{2.67}$$

$$= \frac{i}{2}\left(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right) \tag{2.68}$$

Using identities from vector calculus, it is not too hard to show:

$$\nabla \cdot \left[\psi\nabla\psi^* - \psi^*\nabla\psi\right] = \nabla\psi^* \cdot \nabla\psi + \psi\nabla^2\psi^* - \nabla\psi^* \cdot \nabla\psi - \psi^*\nabla^2\psi \tag{2.69}$$

$$= \psi\nabla^2\psi^* - \psi^*\nabla^2\psi \tag{2.70}$$

and we arrive at:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{2.71}$$

where $\mathbf{j} = \frac{i}{2}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right]$, as required.

□

From this result, one can immediately derive the heat equation with some simple assumptions. We have:

Corollary 2.3.6. *Assume that the current is linearly dependent on the gradient of the density, via Fick's law:*

$$\mathbf{j} = \kappa\nabla\rho \tag{2.72}$$

This implies the heat equation via the continuity law:

$$-\frac{\partial \rho}{\partial t} = \kappa\nabla^2\rho \tag{2.73}$$

where κ is some constant. The sign can be changed by reversing the direction of flux.

Proof. This is trivial, given 2.3.5. We simply substitute the formula for the gradient to obtain:

$$-\frac{\partial \rho}{\partial t} = \kappa\nabla^2\rho \tag{2.74}$$

□

2.4 Least Time as an Action Principle

As we have shown, it is therefore possible to derive much of classical and quantum mechanics from the perspective of an action principle of one type or another. The interested reader is directed to Feynman [103], where the author discussed the various applications of these types of optimisations to problems in physics. There is one commonality between these results, as calculated thus far in 2.2.7, 2.3.1. In a sense, these are problems in energy conservation, but with the energy functional, whether it is the Lagrangian or Hamiltonian, expressed in terms of canonical momentum and position co-ordinates. This focus on the relationship between momentum and position is one that may be traced back in quantum mechanics to the original questions of Einstein, Bohr and Sommerfeld, who asked about the complementarity of variables as given through the Heisenberg uncertainty principles.

However, there is an alternative formulation of a different type of action principle that may be applied. In this, the variational principle and optimisation is over time paths, and is broadly known as the principle of least time, or Fermat's principle of optics. This basic premise states that the physical path is the one which minimises the time taken between the source and the receiver, given constraints, and is familiar from optics as the basic ray-tracing algorithm which determines the path of a light ray. In the following calculations, we identify a number of simple, basic problems that relate to minimisation principles over time or equivalently path length. These investigations have a long and illustrious history, dating to the original researches of Fermat and Bernoulli. Although these problems are well-known, we include them for the reader who is perhaps unfamiliar with the principles of brachistochrones (paths of least time) and their applications. In order, we shall address the connection of least time paths under constant gravity (Bernoulli, the cycloid) and the path of a light ray through two media with different refractive indices (Snell's Law). Following this, we shall show how the Pontryagin maximisation principle can be used to derive control operators which drive a classical system from one state to another in least time. This is very similar in nature to the calculations that follow in the main body of this thesis, and offers a simple entry point to the broader discussion of brachistochrones, which we shall generalise in later chapters to hyperbolic spaces. In demonstrating these simple examples, we shall identify and discuss the physical and philosophical motivation for basing the theory applied in this thesis from within the principle of least time, as this is the primary focus of this work.

2.4.1 The Cycloid as a Brachistochrone

Example 2.4.1. For a particle, initially at rest, and travelling under the influence of constant gravitational acceleration, the curve between any two points which minimises the time taken is a cycloid (Bernoulli, 1697). This special curve which minimises time is called a brachistochrone.

Proof. This first example of a brachistochrone may be solved using standard methods of variational calculus, in particular we shall employ the Euler-Lagrange equations 2.2.6 for a suitable action functional which gives the time of transit between the initial and final points. We have:

$$T = \int \frac{ds}{v} \tag{2.75}$$

where v is the velocity, T is the time taken for the path. We suppress the initial and final points given by the limits of the integral for simplicity without loss of generality. If we examine the arc length in the Cartesian space, we have increment given by:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}dx \tag{2.76}$$

Additional information which is provided is that the particle is initially at rest. The energy is then given by:

$$E = \frac{1}{2}mv^2 - mgy = 0 \tag{2.77}$$

as initially the kinetic energy is zero, the potential energy given by $-mgy$, and finally the potential energy is zero and kinetic energy $\frac{1}{2}mv^2$. Then the velocity is easily found to be:

$$v = y' = \sqrt{2gy} \tag{2.78}$$

We may therefore write the action principle in the form:

$$T = \int \frac{ds}{v} = \int \mathscr{L} dx = \int \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dx \tag{2.79}$$

and the Lagrangian as:

$$\mathscr{L} = \mathscr{L}(y, y') = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} \tag{2.80}$$

The Euler-Beltrami relation states that for a Lagrangian with a redundant variable, in this case x, we have:

$$\mathscr{L} - y' \frac{\partial \mathscr{L}}{\partial y'} = C \tag{2.81}$$

Applying this to the Lagrangian, we have immediately that:

$$\mathscr{L}(y, y') = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} \tag{2.82}$$

$$\frac{\sqrt{1+y'^2}}{\sqrt{2gy}} - y' \frac{1}{2\sqrt{2gy}\sqrt{1+y'^2}} \times 2y' = C \tag{2.83}$$

$$\frac{\sqrt{1+y'^2}}{\sqrt{2gy}} - \frac{y'^2}{\sqrt{2gy}\sqrt{1+y'^2}} = C \tag{2.84}$$

$$\frac{1}{\sqrt{2gy}} \frac{\left(1+y'^2 - y'^2\right)}{\sqrt{1+y'^2}} = C \tag{2.85}$$

which results in the non-linear differential equation:

$$\frac{1}{\sqrt{2gy}\sqrt{1+y'^2}} = C \tag{2.86}$$

or

$$\frac{1}{y(1+y'^2)} = 2gC^2 \tag{2.87}$$

The solution to this problem is well-known, since at least Bernoulli and Newton's time. If we take the parametrisation given by:

$$x(\theta) = k(\theta - \sin\theta) \tag{2.88}$$

$$y(\theta) = -k(1 - \cos\theta) \tag{2.89}$$

then it is simple to show that the differential elements are related by:

$$dx = k(1 - \cos\theta) = -yd\theta \tag{2.90}$$

and

$$dy = -k\sin\theta d\theta \tag{2.91}$$

Then we have the differential relationship:

$$y' = \frac{dy}{dx} = -\frac{k \sin\theta d\theta}{k(1-\cos\theta)} = -\frac{\sin\theta}{(1-\cos\theta)} \tag{2.92}$$

$$y'^2 + 1 = \frac{\sin^2\theta + (1-\cos\theta)^2}{(1-\cos\theta)^2} = \frac{2}{(1-\cos\theta)} \tag{2.93}$$

$$\left(y'^2 + 1\right) y = \frac{2}{(1-\cos\theta)} \times -k(1-\cos\theta) = -2k = 2gC^2 \tag{2.94}$$

implying that for equivalence, we should simply choose:

$$C^2 = -\frac{k}{g} \tag{2.95}$$

or

$$C = \sqrt{-\frac{k}{g}} \tag{2.96}$$

which establishes the proof as required. □

Remark 6. We note that this simple example provides all the basic elements of the underlying philosophy of calculating the brachistochrone. The original inspiration for this stems from the questions of J. Bernoulli, who posed the question, and received 5 responses, from another J. Bernoulli, Newton, Leibniz, L'Hospital and Tschirnhaus.

We find the time path by understanding the arc length and velocity, and using this to construct the interval for the space. Once this is done, it is readily evaluated using variational calculus via the Euler-Lagrange equations. Later parts of this thesis extend this basic schemata to a series of more complicated spaces, including finite quantum systems and hyperbolic realms, while still remaining close to the concept of minimising time taken between two points, given constraints on allowed explorable degrees of freedom.

This first problem is essentially a free system, in that there are no external boundary conditions or changes at an interface requiring continuity.

2.4.2 Snell's Law of Refraction

The next example shows that by a judicious application of the concept of a time path under constraints, we can identify the brachistochrone and use this to derive some physical properties known in ray optics as Snell's law of refraction.

Example 2.4.2. A particle travels in two dimensions from an initial point to a final point. The refractive index for $x \leq 0$ is n_1, and for $x \geq 0$ it is n_2. The path of least time is given by Snell's law of refraction:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{2.97}$$

where the angles θ_1, θ_2 are the angles of incidence and refraction.

Remark 7. This phenomenon forms one primary axiom of ray and electron optics. From the basis of the principle of least time, one may understand many different problems in ray optics in a similar fashion. For example, Bragg diffraction and the question of thin films may be computed in an analogous fashion.

Proof. If we take the time of flight of an arbitrary path between the initial and final points, noting that the velocity is altered by the refractive index, we may write:

$$T = \frac{n_1 d(\mathbf{x}_1, \mathbf{x}_0)}{c} + \frac{n_2 d(\mathbf{x}_0, \mathbf{x}_2)}{c} \tag{2.98}$$

where T is the time of flight, and $d(,)$ is the Cartesian distance between any two points in the two dimensional space. We may therefore write:

$$T = \frac{n_1}{c}\sqrt{(x_1 - x_0)^2 + y_1^2} + \frac{n_2}{c}\sqrt{(x_0 - x_2)^2 + y_2^2} \tag{2.99}$$

for the time length of the path, given the two different refractive indices. The extremal path is then given by the value where this functional is stationary, we must have:

$$\Delta T = \int \delta T = \int \frac{\delta T}{\delta x_0} dx_0 = \min \tag{2.100}$$

and therefore we find:

$$\frac{\delta S}{\delta x_0} = \frac{\partial T}{\partial x_0} = -\frac{2n_1(x_1 - x_0)}{c\sqrt{(x_1 - x_0)^2 + y_1^2}} + \frac{2n_2(x_0 - x_2)}{c\sqrt{(x_0 - x_2)^2 + y_2^2}} = 0 \tag{2.101}$$

This can be simplified to give:

$$\frac{n_1(x_0 - x_1)}{\sqrt{(x_1 - x_0)^2 + y_1^2}} = \frac{n_2(x_2 - x_0)}{\sqrt{(x_0 - x_2)^2 + y_2^2}} \tag{2.102}$$

and substituting the trigonometric values, we readily conclude that:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{2.103}$$

□

2.4.3 Pontryagin's Maximum Principle

For the remainder of this chapter, we shall discuss some more advanced topics in control theory that may not be familiar to those with a physics or mathematics background. We direct the interested reader to the reference text [104], see 19.2.4.30-39 pp. 187 for details regarding the Pontryagin maximisation principle. In this formulation of control theory, one creates a Hamiltonian objective function which is composed of system and control variables, and uses this with a figure of merit that is then optimised against. In the context of this thesis, we shall show how one can use the time taken as a constraint variable, and use this to derive some results which are very similar to that explored in more detail throughout the rest of this work.

2.4.3.1 Real Control System

The basic concept is thus, as set forth originally by Pontryagin:

Lemma 2.4.3. *Assume we have system variables $\mathbf{x}(t)$, control variables $\mathbf{u}(t)$, with dynamics following the autonomous regime:*

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right) \tag{2.104}$$

where the initial condition is $\mathbf{x}(0) = \mathbf{x}_0$ and time is in the range $t \in [0, T]$. Define further the cost functional, given by the terminal cost and the Lagrangian:

$$J = \Psi\left[\mathbf{x}(T)\right] + \int_0^T \mathscr{L}\left(\mathbf{x}(t), \mathbf{u}(t)\right) dt \tag{2.105}$$

where $\Psi\left[\mathbf{x}(T)\right]$ is the terminal cost, and the Lagrangian function to be optimised is purely in terms of the system and control variables, with any explicit time dependence suppressed. Then the augmented Hamiltonian function is given by the formula [104]:

$$\mathscr{H}\left(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t\right) = \lambda^T \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right) + \mathscr{L}\left(\mathbf{x}(t), \mathbf{u}(t)\right) \tag{2.106}$$

and satisfies the optimisation principle for any set of admissable controls:

$$\mathscr{H}\left(\tilde{\mathbf{x}}(t), \tilde{\mathbf{u}}(t), \tilde{\lambda}(t), t\right) \leq \mathscr{H}\left(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t\right) \tag{2.107}$$

Remark 8. We shall illustrate with several examples the practice and application of this technique in modern control theory. In particular, we shall choose the action in this formula to be given by time, which leads a simple reduction via $\mathscr{L}\left(\mathbf{x}(t), \mathbf{u}(t)\right) = 1$. This will give us access to a different type of least-time problems, in this case related to control and optimisation.

2.4.3.2 Complex Control System

Example 2.4.4. Assume the autonomously controlled system of variables, defined through the first derivative as in 2.4.3, we have for a two-dimensional system:

$$\frac{d\mathbf{x}(t)}{dt} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \tag{2.108}$$

Assume further that the control variables satisfy:

$$\frac{d^2\mathbf{x}(t)}{dt^2} = \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \cos z \\ \sin z \end{bmatrix} \tag{2.109}$$

For the Lagrangian action principle given by least time:

$$S = \int_0^T 1 dt = T = \min \tag{2.110}$$

where we neglect any terminal cost from the analysis, then the time-optimising control is given by:

$$z(t) = \tan^{-1}\left(\frac{d - bt}{c - at}\right) + \pi \tag{2.111}$$

Proof. The state and control variables as written can be compacted into augmented form:

$$\mathbf{X}(t) = \begin{bmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{bmatrix}, \frac{d\mathbf{X}(t)}{dt} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} u \\ v \\ \cos z \\ \sin z \end{bmatrix} \tag{2.112}$$

and the Hamiltonian function is therefore given by the linear combination as expressed in 2.4.3:

$$\mathscr{H} = 1 + p_x\dot{x} + p_y\dot{y} + p_u\dot{u} + p_v\dot{v} \tag{2.113}$$

where the Lagrangian in this case is unity, representing least time under the action principle. Then we have, upon substituting the assumed form of the control fields:

$$\mathscr{H} = 1 + p_x u + p_y v + p_u \cos z + p_v \sin z \tag{2.114}$$

We have proven in 2.2.10, 2.2.2, 2.2.11 that Hamilton's equations give an alternative form of the extremal path of the action principle. In this context, we may write these equations explicitly and derive the optimum control fields directly. Then we have:

$$\dot{p}_i = -\frac{\partial \mathscr{H}}{\partial q_i} \tag{2.115}$$

and applying this to the co-ordinates x, y, u, v in the augmented state-control system, we have:

$$\dot{p}_x = -\frac{\partial \mathscr{H}}{\partial x} = 0 \Rightarrow p_x = a \tag{2.116}$$

$$\dot{p}_y = -\frac{\partial \mathscr{H}}{\partial y} = 0 \Rightarrow p_y = b \tag{2.117}$$

$$\dot{p}_u = -\frac{\partial \mathscr{H}}{\partial u} = -p_x = -a \Rightarrow p_u = c - at \tag{2.118}$$

$$\dot{p}_v = -\frac{\partial \mathscr{H}}{\partial v} = -p_y = -b \Rightarrow p_v = d - bt \tag{2.119}$$

and the Hamiltonian function to be maximised is then:

$$\mathscr{H} = 1 + au + bv + (c - at)\cos z + (d - bt)\sin z \tag{2.120}$$

Writing this in the form:

$$\mathscr{H} = \mathscr{H}_i + \mathscr{H}_d \tag{2.121}$$

we have both

$$\mathscr{H}_i = 1 + au + bv \tag{2.122}$$

and

$$\mathscr{H}_d = (c - at)\cos z + (d - bt)\sin z = \mathscr{H}_d(z) \tag{2.123}$$

Alternatively, we can write this in terms of the momentum functions:

$$\mathscr{H}_d(z) = p_u \cos z + p_v \sin z = A\cos(z - \phi) \tag{2.124}$$

from which we find that:

$$A\cos(z - \phi) = A\left(\cos z \cos\phi + \sin z \sin\phi\right) \tag{2.125}$$

and consequently it is simple to see that we must have the relations:

$$A\cos\phi = p_u, A\sin\phi = p_v \tag{2.126}$$

where $A = \sqrt{p_u^2 + p_v^2}$. We arrive then at the following:

$$\tan\phi = \frac{p_v}{p_u}, \phi = \tan^{-1}\left(\frac{p_v}{p_u}\right) \tag{2.127}$$

For a minimum, we must have:

$$A\cos(z - \phi) = \min \tag{2.128}$$

and therefore we conclude that

$$z(t) - \phi = \pi \tag{2.129}$$

implying that we must have:

$$z(t) = \tan^{-1}\left(\frac{p_v}{p_u}\right) + \pi = \tan^{-1}\left(\frac{d - bt}{c - at}\right) + \pi \tag{2.130}$$

as required. □

For a second example of this type of least-time control problem, we extend the analysis to complex functions instead of cosine/sine for the control functions.

Example 2.4.5. Assume as in 2.4.4, however, modify the control functions to be complex. Then we will have: $\frac{d\mathbf{x}(t)}{dt} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$ as before, and the control variables will satisfy:

$$\frac{d^2\mathbf{x}(t)}{dt^2} = \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \xi(z) \\ \xi^*(z) \end{bmatrix} \tag{2.131}$$

in an analogous way to the previous problem. As before, we take the Lagrangian to be unity, and the action principle is again one of least time:

$$S = \int_0^T 1dt = T = \min \tag{2.132}$$

Then the solution for the optimal control field $z(t)$ satisfies:

$$z = \phi + \frac{1}{2}\cos^{-1}\left(\frac{(p_u^2 + p_v^2)}{2p_u p_v}\right) - \frac{\pi}{2} \tag{2.133}$$

$$= \phi + \frac{1}{2}\cos^{-1}\left(\frac{1}{2}\left(\frac{c-at}{d-bt} + \frac{d-bt}{c-at}\right)\right) - \frac{\pi}{2} \tag{2.134}$$

Proof. In a similar way to the preceding example 2.4.4, we can immediately write down the Hamiltonian function to be optimised for the control fields, which drive the system from one reachable point to another in least time. We have:

$$\mathscr{H} = 1 + p_x u + p_y v + p_u \xi(z) + p_v \xi^*(z) \tag{2.135}$$

Again, applying Hamilton's equations to find the extremal 2.2.10, 2.2.2, 2.2.11, we have:

$$\dot{p}_i = -\frac{\partial \mathscr{H}}{\partial q_i} \tag{2.136}$$

and the conjugate momenta may be derived for the variables x, y, u, v as:

$$\dot{p}_x = -\frac{\partial \mathscr{H}}{\partial x} = 0, \dot{p}_y = -\frac{\partial \mathscr{H}}{\partial y} = 0 \tag{2.137}$$

$$p_x = a, p_y = b \tag{2.138}$$

$$\dot{p}_u = -\frac{\partial \mathscr{H}}{\partial u} = -p_x = -a \tag{2.139}$$

$$\dot{p}_v = -\frac{\partial \mathscr{H}}{\partial v} = -p_y = -b \tag{2.140}$$

$$p_u = c - at, p_v = d - bt \tag{2.141}$$

Hence we find the Hamiltonian function has the form:

$$\mathscr{H} = 1 + au + bv + p_u \xi(z) + p_v \xi^*(z) = \mathscr{H}_i + \mathscr{H}_d \tag{2.142}$$

where

$$\mathscr{H}_d(z) = p_u \xi(z) + p_v \xi^*(z) = (c - at)\xi(z) + (d - bt)\xi^*(z) \tag{2.143}$$

As the control fields are complex, we may write $\xi(z) = e^{i(z-\phi)}$, and we find the linear combination to be given by:

$$p_u \xi(z) + p_v \xi^*(z) = R \exp(i\vartheta) \tag{2.144}$$

where

$$R = \sqrt{p_u^2 + p_v^2 + 2p_u p_v \cos\left((z-\phi) - (-(z-\phi))\right)} \tag{2.145}$$

or

$$R = \sqrt{p_u^2 + p_v^2 + 2p_u p_v \cos\left(2(z-\phi)\right)} \tag{2.146}$$

We therefore find that the argument (angle of complex function $\xi(z)$) may be written:

$$\vartheta = \tan^{-1}\left(\frac{p_u \sin(z-\phi) + p_v \sin(-(z-\phi))}{p_u \cos(z-\phi) + p_v \cos(-(z-\phi))}\right) \tag{2.147}$$

$$= \tan^{-1}\left(\frac{p_u \sin(z-\phi) - p_v \sin(z-\phi)}{p_u \cos(z-\phi) + p_v \cos(z-\phi)}\right) = \tan^{-1}\left(\left(\frac{p_u - p_v}{p_u + p_v}\right)\tan(z-\phi)\right) \tag{2.148}$$

To find the extremal point, and therefore the acceptable control, we are therefore faced with the problem:

$$R = \sqrt{p_u^2 + p_v^2 + 2p_u p_v \cos\left(2(z-\phi)\right)} = \min \tag{2.149}$$

which is solved by:

$$p_u^2 + p_v^2 + 2p_u p_v \cos\left(2(z-\phi)\right) = 0 \tag{2.150}$$

Inverting this relationship, we find immediately that:

$$\cos\left(2(z-\phi)\right) = -\frac{(p_u^2 + p_v^2)}{2p_u p_v} \tag{2.151}$$

and

$$-(z-\phi) = \frac{1}{2}\cos^{-1}\left(-\frac{(p_u^2 + p_v^2)}{2p_u p_v}\right) + \frac{\pi}{2} \tag{2.152}$$

hence the control parameter $z(t)$ is:

$$z = \phi + \frac{1}{2}\cos^{-1}\left(\frac{(p_u^2 + p_v^2)}{2p_u p_v}\right) - \frac{\pi}{2} \tag{2.153}$$

$$= \phi + \frac{1}{2}\cos^{-1}\left(\frac{1}{2}\left(\frac{c-at}{d-bt} + \frac{d-bt}{c-at}\right)\right) - \frac{\pi}{2} \tag{2.154}$$

□

Remark 9. Although this is a simple, contrived example, in later sections of this work the parameter derived in the control fields for the time optimal path:

$$\frac{1}{2}\cos^{-1}\left(\frac{(p_u^2 + p_v^2)}{2p_u p_v}\right) \tag{2.155}$$

appears in many different contexts. For example, an appropriate substitution will reduce this to a type of hyperbolic distance function which will be covered at length in subsequent chapters.

2.5 Comments

The historical examples encountered in this chapter demonstrate that calculation of extremal paths through use of the principle of least time gives much insight into a wide variety of different systems. We shall now emphasise some aspects of this brief review that are important in the remainder of the document, that may not be obvious to the reader who is unfamiliar with this type of analysis.

It is clear that in the situation of classical mechanics, the specification of the position and momentum co-ordinates of the particles and the interactions between them is sufficient and necessary to give the state of the system. We can see this reflected in the form of the Hamiltonian and Lagrangian functions for classical mechanics. However, in the quantum mechanical scenario, we are reduced to the question of specifying the state of the system through a complex function, and the probability density is given by the modulus of this function squared. It is also straightforward to see that the quantum action principle defined in 2.3.1 depends on the energy, and not the time of the path between any two reachable states.

In quantum mechanics, the connection between operators, eigenfunctions and their associated eigenvalues naturally arose from a thorough analysis of the dynamical equations that drive the system. We therefore propose a working ansatz, that there should be an equivalent methodology to the established techniques based on energetic considerations and least action, but using the principles of brachistochrones and least time. This is the main result of this thesis, to explain and widen the application of this idea to a new domain within hyperbolic spaces.

CHAPTER 3

Mathematical Details

We shall state the major results we shall require from the literature, any axioms, notations and definitions we shall require in the following proofs and calculations.

3.1 Basic Concepts

The calculation methods in this thesis use a number of concepts and mathematical notations that may be familiar only to those who have studied quantum physics or group theory, so we shall describe our system of analysis and any relevant notation for completeness.

3.1.1 The State

Consider a continuous function ψ, which describes the state of a system. This system might be any situation where there is an internal state, evolving through the application of a continuous dynamical operator as per the standard arguments of calculus. For example, in quantum mechanics, one is interested in the distribution of internal configurations of the system, in quantitative finance and stochastic processes, one goes about calculating the probability distribution of the prices at certain times, and evaluating the expectation values. Both are essentially the same task, we wish to determine the values at some later time or other space co-ordinate, given the dynamics of the system. Take the state variable to be given by q, some continuous operator which exists on the space, with canonical conjugate defined by differential operator ∂_q. By the state of the system, we imply that there exists some basic differential operator which describes the dynamics at large, which we may write as:

$$\hat{H}(q, \partial_q, t)\psi = -\frac{\partial \psi}{\partial t} \tag{3.1}$$

for the heat equation, or even more generally:

$$\hat{H}(q, \partial_q, \partial_q^2, ...)\psi = \hat{\mathfrak{L}}(t, \partial_t, \partial_t^2, ...)\psi \tag{3.2}$$

For example, the heat equation is defined through the system:

$$\hat{H}(q, \partial_q, \partial_q^2, ...) = \partial_q^2 \tag{3.3}$$

$$\hat{\mathfrak{L}}(t, \partial_t, \partial_t^2, ...) = -\partial_t \tag{3.4}$$

whereas the wave equation is given by the same Hamiltonian $\hat{H}$, but the time dependent term is modified to $\hat{\mathfrak{L}} = \partial_t^2$. The function ψ which solves this expression for the system dynamics is termed the state. Knowledge of the state is sufficient to describe the system at any other configuration of space and time, given some initial conditions and/or boundary conditions on the problem. Our interest in the state description of reality is that one may break down the various fundamental operators into eigenvalue decompositions. These formulae relate to kernels which we discuss below. An eigenstate ψ_n of a particular operator $\hat{A}$ is given through the expression:

$$\hat{A}\psi_n = \lambda_n \psi_n \tag{3.5}$$

where λ_n is the eigenvalue we associate with the operator. Many of the results in the following proof depend on properties of eigenfunctions and the solutions to partial differential equations through the use of composition formulae, as prescribed through the methods of kernels and group representation theory. Occasionally we shall have recourse to the brevity and succinct formalism as expressed through the Dirac bra-ket notation, briefly, the state in any configuration x may be written:

$$\psi(x) = \langle x|\psi\rangle \tag{3.6}$$

The following sections outline various applications of Dirac notations as they occur in the calculations.

3.1.2 Operator Theory

In this thesis we must distinguish between commutative and non-commutative quantities. A commutative quantity is naturally given through the multiplication of any two functions, we know that this is independent of order. A non-commutative quantity in this context takes the form of either a continuous differential operator, or a matrix. In general, we shall denote a matrix or a differential operator by the notation $\hat{L}$, with the caret denoting non-commutativity. Whether something is either a matrix or a differential operator is then to be given through the content of the calculation and we shall specify the exact form of every operator wherever necessary. When we write an equation of the form:

$$\hat{L}u = f(u) \tag{3.7}$$

we say that $\hat{L}$ acts on u. The standard notation for the commutation bracket of matrices:

$$[\hat{L}_1, \hat{L}_2] = \hat{L}_1\hat{L}_2 - \hat{L}_2\hat{L}_1 \tag{3.8}$$

and the anticommutation relation:

$$\{\hat{L}_1, \hat{L}_2\} = \frac{1}{2}\left(\hat{L}_1\hat{L}_2 + \hat{L}_2\hat{L}_1\right) \tag{3.9}$$

are assumed. We also abuse notation and write the following commutation and anticommutation relationships for continuous differential operators as given by:

$$[\hat{L}_1, \hat{L}_2]f = \hat{L}_1\hat{L}_2 f - \hat{L}_2\hat{L}_1 f \tag{3.10}$$

A note on the operator notation. An operator acting on a function may be written as:

$$\hat{L}u = \hat{L}(u) \tag{3.11}$$

In the case of the simplest differential operator, we have $\hat{L}_1 = \partial_x$ for example. Then the commutation relationship is:

$$[\hat{L}_1, \hat{L}_2]f = \partial_x(\hat{L}_2 f) - \hat{L}_2(\partial_x f) \tag{3.12}$$

It is important to realise that although it appears as multiplication, in the case of the continuous differential operators, the commutation relations are given by nested operations.

The link between the two types of non-commutative objects, given by matrix groups and differential operators, is of course representation theory as considered at length in the work of Vilenkin [69].

The following thesis contains detailed calculations involving the method of the solution operator, see for example , 7.4.4, 7.4.5, 7.4.18 in this work. The solution operator extends the argument of functionals to operators, which in practice is the observation that one may represent the solution to a differential equation by assuming the differential operator is a constant, and then solve the differential equation with it on this proviso. Although not always correct, such an approach can give useful insight into the solution space. It has the added benefit that, by combining this technique with that of Lie/commutator brackets, one may arrive at solutions to PDEs in a few lines vs a long and involved calculation in many instances. In this work we shall use this technique to gain insight into many different systems and the transformations between them.

3.1.3 Matrix Analysis

In this following proofs contained in this thesis, we shall require both operators and matrices at certain times. Both have non-commutative properties in their respective domains, i.e. $\hat{A}\hat{B} \neq \hat{B}\hat{A}$, or alternatively $\left[\hat{A}, \hat{B}\right] \neq 0$ and we shall emphasise in context whether we are working with operators or matrices where appropriate. We shall overload the notation to a certain extent, and use the caret over a letter to denote either a non-commutative operator or a matrix. Matrices, in an analogous way to the previously considered operators shall be denoted by $\hat{A}$; if these matrices have an underlying parameter dependence, they shall be written as $\hat{A}(t)$, and two parameter dependence shall be written as e.g. $\hat{A}(t, s)$. If a matrix is tracefree, we shall denote this by a tilde instead of a hat for notational convenience, via $\tilde{A}$. The basic operation on matrices is defined through the commutator:

Definition 3.1.1. The commutator bracket of two operators $\hat{A}$ and $\hat{B}$ is:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{3.13}$$

and the fundamental matrix dynamics is defined via the von-Neumann equation (see e.g J. von Neumann, [105] c. 1932):

$$i\frac{d\hat{A}}{dt} = [\hat{H}, \hat{A}] \tag{3.14}$$

where the matrix $\hat{H}$ is termed the Hamiltonian matrix, it is Hermitian $\hat{H}^\dagger = \hat{H}$.

Definition 3.1.2. The traceless part of the Hamiltonian matrix denoted $\hat{H}$:

$$\tilde{H} = \hat{H} - \frac{\hat{\mathbf{1}}\mathrm{Tr}[\hat{H}]}{N} \tag{3.15}$$

where N is the length of the diagonal of the square matrix $\tilde{H}$.

Although in standard quantum mechanics the dimensionality of the vectors defined through the bra-ket formalism may be infinite, in this thesis we shall often use the summary notation:

Definition 3.1.3. The ket vector of the state is defined by:

$$|\Psi\rangle = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{C}^n \tag{3.16}$$

where $c_j \in \mathbb{C}$, this shall be used to denote a finite state vector.

This notation results in a great deal of simplification of many of the matrix differential equations we shall encounter. Where necessary, we shall denote a complex vector that is an explicit function of some parameter via $|\psi(t)\rangle$.

Definition 3.1.4. The Hermitian conjugate or adjoint is defined through the complex conjugate transpose:

$$\langle\Psi| = [c_1^*, c_2^*, ..., c_n^*] = (|\Psi\rangle^*)^T = \left(|\Psi\rangle^T\right)^* \in \bar{\mathbb{C}}^n \tag{3.17}$$

Definition 3.1.5. The state vector is normalised to unity, such that:

$$\langle\Psi|\ \Psi\rangle = |c_1|^2 + ... + |c_n|^2 = 1 \tag{3.18}$$

Definition 3.1.6. The pure state projection operator is defined by the outer product matrix:

$$\hat{P} = |\Psi\rangle\,\langle\Psi| \tag{3.19}$$

Lemma 3.1.7. *The pure state projection operator $\hat{P}$ is idempotent.*

Proof. Elementary. Proceed using the properties of the state 3.1.5.

$$\hat{P}^2 = |\Psi\rangle\,\langle\Psi|\ \Psi\rangle\,\langle\Psi| = |\Psi\rangle\,\langle\Psi| = \hat{P} \tag{3.20}$$

This is a result that, for our purposes is sufficient. However, more generalised states exist that may not satisfy such expressions. We can see that the value of the trace will be preserved in the case of a pure state. □

Definition 3.1.8. Expectation values are then calculated in this notation via

$$\mathbb{E}\left[\hat{A}\right] = \left\langle\Psi\left|\hat{A}\right|\Psi\right\rangle = \left\langle\hat{A}\right\rangle \tag{3.21}$$

for any Hermitian operator

$$\hat{A}^\dagger = (\hat{A}^T)^* = \hat{A} \tag{3.22}$$

In this thesis the principal types of important matrices are projections $\hat{P}$, given above; the Hamiltonian matrix denoted $\tilde{H}(t)$, and the unitary matrix defined through the matrix exponential $\hat{U}(t,s)$, defined below.

Definition 3.1.9. The time evolution operator is defined by:

$$\hat{U}(t,s) = \mathcal{T}\left[\exp\left(-i\int_s^t \tilde{H}(\tau)d\tau\right)\right] \tag{3.23}$$

where $\mathcal{T}$ is the time-ordering (Dyson) operator. In the context of this thesis, we shall often rely on eigenvalue decompositions that allow us to identify this special operator directly. As such, we shall denote this in the form:

$$\hat{U}(t,s) = \exp\left(-i\int_s^t \tilde{H}(\tau)d\tau\right) \tag{3.24}$$

Note that this is not generally true, but is a useful shorthand notation for the unitary operators which shall be encountered in this work only.

This operator appears at many critical parts of the matrix analysis. We shall now prove some small but useful lemmas related to the time evolution operator defined by this unitary matrix.

Lemma 3.1.10. *At $s = t$, the time evolution operator $\hat{U}(t,s)$ as given in 3.1.9 satisfies $\hat{U}(t,t) = \hat{\mathbf{1}}$, where $\hat{\mathbf{1}}$ is the identity matrix.*

Proof. We have $\hat{U}(t,s) = \exp\left(-i\int_s^t \tilde{H}(\tau)d\tau\right)$, taking the limit of the right hand side we obtain the result. Alternatively, we have $e^{-i\int_s^s \tilde{H}(\tau)d\tau} = e^{\mathbf{0}} = \mathbf{1}$, where $\mathbf{0}$ is the matrix of zeroes. □

Lemma 3.1.11. *The time evolution operator satisfies a time reversal condition, given by the Hermitian adjoint:*

$$\hat{U}^\dagger(t,s) = \hat{U}(s,t) \tag{3.25}$$

Proof.

$$\hat{U}^\dagger(t,s) = \exp\left(i\int_s^t \tilde{H}^\dagger(\tau)d\tau\right) = \exp\left(-i\int_t^s \tilde{H}(\tau)d\tau\right) = \hat{U}(s,t) \tag{3.26}$$

We have used the Hermitian nature of $\tilde{H}$ in this elementary proof. □

Lemma 3.1.12. *The time evolution operator $\hat{U}(t,s)$ is unitary.*

Proof. Using 3.1.10, 3.1.11 we may write:

$$\hat{U}(t,s)\hat{U}^\dagger(t,s) = \hat{U}(t,s)\hat{U}(s,t) = \hat{U}(t,t) = \hat{\mathbf{1}} \tag{3.27}$$

The other side of the expression proceeds in an analogous fashion with the time direction reversed. □

Lemma 3.1.13. *The time evolution operator $\hat{U}(t,s)$ satisfies the Schrödinger equation:*

$$\frac{d\hat{U}(t,s)}{dt} = -i\tilde{H}(t)\hat{U}(t,s) \tag{3.28}$$

Proof. Proceed in the sense of a Frechet derivative:

$$\frac{d\hat{U}(t,s)}{dt} = \lim_{dt\to 0}\frac{\hat{U}(t+dt,s) - \hat{U}(t,s)}{dt} = \lim_{dt\to 0}\frac{\hat{U}(t+dt,t)\hat{U}(t,s) - \hat{U}(t,s)}{dt} \tag{3.29}$$

$$= \lim_{dt\to 0}\frac{(\hat{U}(t+dt,t) - \hat{\mathbf{1}})}{dt}\hat{U}(t,s) \tag{3.30}$$

$$= \lim_{dt\to 0}\frac{((\hat{\mathbf{1}} - i\tilde{H}(t)dt + ..) - \hat{\mathbf{1}})}{dt}\hat{U}(t,s) \tag{3.31}$$

$$= -i\tilde{H}(t)\hat{U}(t,s) \tag{3.32}$$

as required. We have used the time additive condition, which follows from the relations 3.1.11, 3.1.12. Alternatively, one may employ the fundamental theorem of calculus:

$$\frac{d}{dt}\left(\hat{U}(t,s)\right) = -i\left(\frac{d}{dt}\int_s^t \tilde{H}(\tau)d\tau\right)\hat{U}(t,s) = -i\tilde{H}(t)\hat{U}(t,s) \tag{3.33}$$

to obtain the same result. □

We shall now prove a lemma which will be of some use in simplification of the unitary matrix given by the time evolution operator. This relationship is proved using the diagonal representation of the Hamiltonian operator $\tilde{H}$. In this representation we have the following:

Definition 3.1.14. The diagonal representation of the Hamiltonian is:

$$\tilde{H}(t) = \hat{W}(t)\hat{L}\hat{W}^{-1}(t) \tag{3.34}$$

where the diagonal matrix $\hat{L}$ is:

$$\hat{L} = \begin{bmatrix} E_1 & & & & \\ & E_2 & & & \\ & & \ddots & & \\ & & & E_{n-1} & \\ & & & & E_n \end{bmatrix} \tag{3.35}$$

where the E_k are the eigenvalues of the matrix $\tilde{H}(t)$, the matrix of eigenvectors given through $\hat{W}(t)$.

In this work, we shall only consider simple systems where the eigenvalues E_k are constants in the diagonal reference frame. This means that there exists a special transformation, given by the matrix of eigenvectors, which we can use to move to this particularly simple form to evaluate the dynamics. We shall now prove a simple relationship between the Hamiltonian, the matrix of eigenvectors and the unitary operator $\hat{U}(t,s)$.

Lemma 3.1.15. *Assume the Hamiltonian matrix evolves unitarily, via*

$$\hat{U}(t,s)\tilde{H}(s)\hat{U}^{\dagger}(t,s) = \tilde{H}(t) \tag{3.36}$$

and further, assume that there exists a transformation that takes the Hamiltonian matrix into a constant diagonal matrix. Then the unitary operator given by $\hat{U}(t,s)$ has the matrix decomposition:

$$\hat{U}(t,s) = \hat{W}(t)\hat{W}^{-1}(s) \tag{3.37}$$

where $\hat{W}(t)$ is the matrix of eigenvectors of $\tilde{H}$.

Proof. We assume that there exists a diagonal representation of $\tilde{H}$, such that $\tilde{H}(t) = \hat{W}(t)\hat{L}\hat{W}^{-1}(t)$, where $\tilde{L}$ is a diagonal matrix, $\hat{W}(t)$ is the matrix of eigenvectors associated to the matrix $\tilde{H}$. Then we may write:

$$\hat{L} = \hat{W}(t)^{-1}\tilde{H}(t)\hat{W}(t) = \hat{W}^{-1}(s)\tilde{H}(s)\hat{W}(s) \tag{3.38}$$

Inserting this into the expression for unitary evolution, we obtain:

$$\tilde{H}(t) = \hat{W}(t)\hat{W}^{-1}(s)\tilde{H}(s)\hat{W}(s)\hat{W}(t)^{-1} = \hat{U}(t,s)\tilde{H}(s)\hat{U}^{-1}(t,s) \tag{3.39}$$

from which follows $\hat{U}(t,s) = \hat{W}(t)\hat{W}^{-1}(s)$ on identification of the components with the unitary evolution. Note that this entire proof hinges on the constancy of the diagonal matrix $\hat{L}$ in time. Although there may be other examples where this assumption does not apply, for this thesis we shall only work with simple problems that can always be reduced to this case.

□

Remark 10. Although the time evolution operator solves the unitary evolution equation for the Hamiltonian:

$$\tilde{H}(t) = \hat{U}(t,s)\tilde{H}(s)\hat{U}^{\dagger}(t,s) \tag{3.40}$$

this solution is only defined up to a transformation that leaves the Hamiltonian matrix invariant. We can see this, if it is invariant, it commutes, writing the linear algebra:

$$\hat{R}\tilde{H}(t)\hat{R}^{\dagger} = \tilde{H}(t) \Leftrightarrow \hat{R}\tilde{H}(t) = \tilde{H}(t)\hat{R} \tag{3.41}$$

where $\hat{R}\hat{R}^\dagger = \mathbf{1}$ is a unitary transformation leaving the Hamiltonian matrix unchanged. Then we have:

$$\hat{R}\tilde{H}(t)\hat{R}^\dagger = \hat{U}(t,s)\hat{R}\tilde{H}(s)\hat{R}^\dagger\hat{U}^\dagger(t,s) \tag{3.42}$$

implying that the evolution of the Hamiltonian is then given by:

$$\tilde{H}(t) = \hat{R}^\dagger\hat{U}(t,s)\hat{R}.\tilde{H}(s).\hat{R}^\dagger\hat{U}^\dagger(t,s)\hat{R} \tag{3.43}$$

$$= \hat{U}'(t,s)\tilde{H}(s)\hat{U}'^\dagger(t,s) \tag{3.44}$$

i.e. the unitary given by $\hat{U}'(t,s) = \hat{R}^\dagger\hat{U}(t,s)\hat{R}$ also solves this expression.

To distinguish a unitary matrix from its pseudounitary counterpart, we use the notation $\hat{\mathcal{U}}(t,s)$, the italicised script indicating the pseudounitary. These expressions shall be applied to various types of 2×2 matrices, but it is important to note that this theory is of a general nature and may be applied to any finite matrix group in a similar way, as long as the eigenvalues are constant in time. In this thesis, this is the only type of Hamiltonian matrix we shall encounter, but it is possible that more generalised systems exist external to our purpose. Our particular time dependence will have constancy of the eigenvalues, leading to a discrete spectrum with clear separation. In the later sections of this document, we encounter a generalisation of this prospect using analytic continuation of the time variable within the time evolution operator. As the original problem has an eigenmatrix with constant eigenvalues, multiplicity 1, the hyperbolic extension shares this property with modification of the unitary properties to a pseudounitary equivalent.

3.1.4 Notations for Transforms

In this thesis we shall encounter a number of different transforms, including Mellin, Fourier, Laplace, Hankel and more related to shift transforms on groups. We note that transforms act on function in a very similar way to how the differential operators act on the state, and we shall denote them accordingly, for example, writing the derivative as $\partial_x u$, and the Fourier transform in shorthand as $\mathcal{F}u$. For consistency, when referring to a known integral transform, as in this example with the Fourier transform, we shall use a cursive script such as $\mathcal{F}u = \mathcal{F}[u(x)](s)$ to represent the transformation of the co-ordinate x by the transform $\mathcal{F}$. Often we shall intermix the idea of a transform with that of a non-commutative operator, as we shall see such ideas have much application in the understanding of special functions and the hyperbolic plane.

3.1.5 Notations for Kernels

In this thesis we shall be almost exclusively concerned with kernel solutions to partial differential equations of one type or another. The simplest form of kernel comes from diffusion problems, for example writing the PDE for the heat equation $u_t = u_{xx}$, we can specify the solution through the initial value problem $u(x,0) = f(x)$ via the integral transform:

$$u(x,t) = \int K(x,y;t)f(y)dy \tag{3.45}$$

where the term $K(x,y;t)$ is referred to as the kernel. In other works, this may be called the fundamental solution, transition probability density or propagator. This function has the special property of pushing our initial condition forward in time, and represents the solution to the PDE system. We shall write the kernel solution of a PDE system in the form:

$$K(x,t;y,t') \tag{3.46}$$

and shall read this as the probability functional that takes us from the state y at time t' to the state x at time t. The first part of this thesis is mostly confined to the question of the heat equation, in which case we only need to consider systems which are time translation invariant, via:

$$K(x,y;t-t') = K(x,y;\tau) = K(x,t;y,t') \tag{3.47}$$

In terms of the evolution of the system, the fundamental property of all kernel solutions is that they satisfy the following:

Definition 3.1.16. The kernel solution $K(x,y;t)$ to a PDE system, e.g. for the heat equation IVP with $u(x,0) = \phi(x)$, $u_t = u_{xx}$ is given by:

$$u(x,t) = \int K(x,y;t)\phi(y)d\mathfrak{m} \tag{3.48}$$

where $d\mathfrak{m}$ is the measure.

Remark 11. This formula naturally generalises the heat equation $u_t = \mathcal{L}u$ to parabolic operators of the form $\mathcal{L} = a(x)\partial_x^2 + b(x)\partial_x + c(x)$. The measure in this case is given by the speed or scale measures, which we analyse in detail in 3.5.8.

This can be seen as a special type of integral transform taking the past solution given by the initial condition $\phi(y)$ into $u(x,t)$. The integration measure can be arrived at from a variety of directions; valid ways in which to do so include speed and scale measures. A lengthy discussion on the topic and the relation to Titchmarsh-Kodaira theorems may be found in Sousa and Yakubovich [56]. In terms of operator calculus, we shall often use the equivalent expressions e.g. for the heat equation, we have:

$$\frac{\partial u}{\partial t} = \nabla^2 u \tag{3.49}$$

where ∇^2 is the Laplacian operator associated to the space, which is not necessarily Cartesian. In this case, the solution operator may be evaluated via the rules of functional calculus, where the Laplacian is treated as an effective constant. Exponentiation of the Laplacian gives the solution operator as defined through the operator theoretic kernel:

$$u(x,t) = \exp\left(t\nabla^2\right)u(x,0) = e^{t\nabla^2}\phi(y) \tag{3.50}$$

which implies the equivalence between these two representations of kernel solutions through relations such as:

$$\exp\left(t\nabla^2\right) \sim \int K(x,y;t)d\mathfrak{m} \tag{3.51}$$

We shall require the following spectral theorem, which relies on the decomposition of the kernel solution to the heat equation.

Definition 3.1.17. If we have that the spectrum of eigenvalues is imiscid, we mean that the spectrum has continuous and discrete parts which do not overlap except at the boundary point.

Remark 12. The definition of an imiscid spectrum relates to the physical situation of immiscible fluids. Immiscible fluids do not mix states, and hence form a distinct boundary, as with oil and water. In our context, we mean that the spectra of the continuous and discrete parts of the system do not mix. This is an important and necessary property that allows us to separate the behaviour of the discrete and continuous components, and essentially decouple their dynamics via a direct sum. In practice, this means that we can treat the dynamics of the continuous and discrete parts of the system independently, and solve for each in their respective domain.

Theorem 3.1.18. *The eigenvalue decomposition of the kernel solution of the heat equation $u_t = \nabla^2 u$ for a system with an imiscid spectrum 3.1.17 may be written as:*

$$K(x, y; t) = \sum_n |c_n|^2 e^{-E_n t} \psi_n^*(y)\psi_n(x) + \int e^{-E_\lambda t} \phi_\lambda^*(y)\phi_\lambda(x) d\mathfrak{m}(\lambda) \tag{3.52}$$

where in the discrete spectrum we have eigenfunction $\psi_n(x)$ with eigenvalue $-E_n$, conversely in the continuous spectrum eigenfunction $\phi_\lambda(x)$ with continuous eigenvalue $-E_\lambda$. The integration measure in the continuous spectrum $d\mathfrak{m}(\lambda)$ and discrete weightings $|c_n|^2$ may be recovered from the completeness relationship.

Proof. Assume that the heat equation is given by:

$$\frac{\partial u}{\partial t} = \nabla^2 u \tag{3.53}$$

and that we have eigenfunctions $\psi_n(x)$, $\phi_\lambda(x)$ as above. We shall show only the discrete component, the proof of the continuous spectrum follows in an analogous fashion. The eigenfunction system satisfies:

$$\nabla^2 \psi_n(x) = -E_n \psi_n(x) \tag{3.54}$$

Then the kernel in the discrete spectrum may be specified through the eigenfunction decomposition:

$$K_{disc.}(x, y; t) = \sum_n |c_n|^2 e^{-E_n t} \psi_n^*(y)\psi_n(x) \tag{3.55}$$

which satisfies the heat equation:

$$\frac{\partial}{\partial t}\left(\sum_n |c_n|^2 e^{-E_n t} \psi_n^*(y)\psi_n(x)\right) = \sum_n -E_n |c_n|^2 e^{-E_n t} \psi_n^*(y)\psi_n(x) \tag{3.56}$$

$$\nabla^2\left(\sum_n |c_n|^2 e^{-E_n t} \psi_n^*(y)\psi_n(x)\right) = \sum_n |c_n|^2 e^{-E_n t} \psi_n^*(y)\nabla^2\left(\psi_n(x)\right) \tag{3.57}$$

$$= -\sum_n E_n |c_n|^2 e^{-E_n t} \psi_n^*(y)\psi_n(x) \tag{3.58}$$

As $E_n^* = E_n$, i.e. the eigenvalue spectrum is real. The completeness relation 0.1.4 for the kernel in the discrete spectrum then gives:

$$\delta(y - x) = K_{disc.}(x, y; 0) = \sum_n |c_n|^2 \psi_n^*(y)\psi_n(x) \tag{3.59}$$

For the continuous states, the proof proceeds in an analogous fashion while using differentiation under the integral sign, which is valid as the measure is not a function of the time or space parameters, only of the continuous eigenvalue λ. The result is

$$K_{cont.}(x, y; t) = \int e^{-E_\lambda t} \phi_\lambda^*(y)\phi_\lambda(x) d\mathfrak{m}(\lambda) \tag{3.60}$$

is the kernel solution in the continuous spectrum. Together, these components solve the differential equation in the imiscid spectrum, as defined in 3.1.17, given by the union of the discrete and continuous components:

$$\frac{\partial K}{\partial t} = \nabla^2 K \tag{3.61}$$

$$K(x, y; 0) = \delta(y - x) \tag{3.62}$$

We can show this very simply, using a different method. If the spectrum is imiscid, then the continuous and discrete parts separate, and we may write:

$$⨍ K(x, y; t) \begin{pmatrix} \psi_n(y) \\ \phi_\lambda(y) \end{pmatrix} d\mathbf{m}(y) \tag{3.63}$$

$$= \sum_j \psi_j(y) \sum_n |c_n|^2 e^{-E_n t} \psi_n^*(y) \psi_n(x) + \int dy.\phi_{\lambda'}(y) \int e^{-E_\lambda t} \phi_\lambda^*(y) \phi_\lambda(x) d\mathfrak{m}(\lambda) \tag{3.64}$$

$$= \sum_n |c_n|^2 e^{-E_n t} \delta_{n,j} \psi_n(x) + \int e^{-E_\lambda t} \delta_{\lambda,\lambda'} \phi_\lambda(x) d\mathfrak{m}(\lambda) \tag{3.65}$$

$$= \sum_n |c_n|^2 e^{-E_n t} \psi_n(x) + \int e^{-E_\lambda t} \phi_\lambda(x) d\mathfrak{m}(\lambda) \tag{3.66}$$

$$= \begin{pmatrix} \psi_n(x, t) \\ \phi_\lambda(x, t) \end{pmatrix} \tag{3.67}$$

where we have imposed the orthogonality conditions:

$$\delta_{\lambda,\lambda'} = \int dy.\phi_\lambda^*(y) \phi_{\lambda'}(y) \tag{3.68}$$

and

$$\sum_j \psi_j(y) \psi_n^*(y) = \delta_{n,j} \tag{3.69}$$

This means we have constructed the integral transform which takes us from the IVP into the solution at time t to the PDE given by the heat equation. Of course, this is on the proviso of the existence of suitable orthogonality or completeness relationships, which is to be established. □

This shows that completeness relationships, eigenvalue decompositions and the theory of the kernel are all interrelated concepts. In this thesis, we shall use this decomposition as implied by the spectral theory of the kernel to derive fundamental solutions to the heat kernel on hyperbolic space. Further, we extend this to other spaces as given by wave and EPD equations outside of the formalism of the heat kernel. Although we have mainly solved the forwards heat equation in this thesis, it is simple to see that basic modifications of the spectral decomposition formulae can be utilised to provide answers for the corresponding backwards Kolmogorov problem.

3.2 Differential Geometry

The following two subsections outline the basic hypotheses of the differential geometry of projective state space $\mathbb{CP}^{n-1}$, as defined through the action of the state, and the fundamental formulae related to the hyperbolic plane and stereographic projections of the pseudosphere. We first calculate a detailed framework using some advanced methods from quantum control theory and projective geometry, that enables us to analyse the necessary matrix symmetries and unitary and pseudounitary operators.

3.2.1 Quantum Brachistochrone and Projective Geometry

We shall begin by constructing some unitary matrix operators using the principle of least time under linear constraints. Our starting point is the von Neumann equation, familiar from quantum mechanics as derived in [105]. We shall state this as the following:

Theorem 3.2.1. *The von Neumann equation [105] may be derived as follows. Assume the existence of a unitary operator described by the exponential of a Hamiltonian matrix*

$$\hat{U}(t,0) = \exp\left(-i\int_0^t \tilde{H}(s)ds\right) \tag{3.70}$$

Given a Hermitian matrix $\hat{A}(t)$ which defines an observable operator via $\hat{A}^\dagger = \hat{A}$, if the Hamiltonian matrix is $\tilde{H} \doteq \tilde{H}(t)$, both $\hat{A}, \tilde{H} \in \mathbb{M}^{n\times n}$, with $\tilde{H}$ tracefree, the unitary evolution of the explicitly time dependent operator $\hat{A}$ satisfies the matrix differential equation:

$$i\frac{d\hat{A}}{dt} = [\tilde{H}, \hat{A}] = \tilde{H}\hat{A} - \hat{A}\tilde{H} \tag{3.71}$$

where unitary evolution is given by:

$$\hat{A}(t) = \hat{U}(t,0)\hat{A}(0)\hat{U}^\dagger(t,0) \tag{3.72}$$

Proof. The exponential form of the time evolution operator may be differentiated directly to yield the Schrödinger equation:

$$i\frac{d\hat{U}(t,0)}{dt} = \tilde{H}(t)\hat{U}(t,0) \tag{3.73}$$

Taking the Hermitian adjoint, we have:

$$-i\frac{d\hat{U}^\dagger(t,0)}{dt} = \left(\tilde{H}(t)\hat{U}(t,0)\right)^\dagger = \hat{U}^\dagger(t,0)\tilde{H}^\dagger(t) = \hat{U}^\dagger(t,0)\tilde{H}(t) \tag{3.74}$$

The unitary evolution of an operator is written:

$$\hat{A}(t) = \hat{U}(t,0)\hat{A}(0)\hat{U}^\dagger(t,0) \tag{3.75}$$

where we have taken the time dependence as explicit within the operator. Differentiating this with respect to time, we have:

$$\frac{d\hat{A}(t)}{dt} = \frac{d}{dt}\left(\hat{U}(t,0)\hat{A}(0)\hat{U}^\dagger(t,0)\right) \tag{3.76}$$

$$= \frac{d\hat{U}(t,0)}{dt}\hat{A}(0)\hat{U}^\dagger(t,0) + \hat{U}(t,0)\hat{A}(0)\frac{d\hat{U}^\dagger(t,0)}{dt} \tag{3.77}$$

Substitution of the expressions for the derivatives of the time evolution operator yields:

$$\frac{d\hat{A}(t)}{dt} = -i\tilde{H}(t)\hat{U}(t,0)\hat{A}(0)\hat{U}^\dagger(t,0) + i\hat{U}(t,0)\hat{A}(0)\hat{U}^\dagger(t,0)\tilde{H}(t) \tag{3.78}$$

$$= -i\left[\tilde{H}(t), \hat{A}(t)\right] \tag{3.79}$$

as required. Consider now the expression for a matrix with a trace part, we have:

$$\hat{H} = \tilde{H} + \frac{\hat{\mathbf{1}}\mathrm{Tr}[\hat{H}]}{n} \tag{3.80}$$

as can be seen by taking traces of both sides. Evaluating the commutator, we may write:

$$\left[\tilde{H} + \frac{\hat{\mathbf{1}}\mathrm{Tr}[\hat{H}]}{n}, \hat{A}\right] = \left[\tilde{H}, \hat{A}\right] \tag{3.81}$$

as the commutator of the trace part with any operator is zero, since it is a multiple of the identity matrix:

$$\left[\frac{\hat{\mathbf{1}}\mathrm{Tr}[\hat{H}]}{n}, \hat{A}\right] = \frac{\mathrm{Tr}[\hat{H}]}{n}\left[\hat{\mathbf{1}}, \hat{A}\right] = 0 \tag{3.82}$$

This means that the contribution to the time evolution that is given by the trace part is a phase factor, and the only meaningful contribution to the dynamics comes from the traceless part of the Hamiltonian matrix. One may see this directly by substitution of the trace decomposition into the unitary operator. □

Corollary 3.2.2. *The Ehrenfest theorem [106]. For an operator $\hat{A}$ as above, with expectation value defined through the bra-ket of the state: $\mathbb{E}[f(\hat{A})] = \left\langle f(\hat{A})\right\rangle = \langle\Psi| f(\hat{A})|\Psi\rangle$, the expectation form of the time evolution is given by:*

$$\left\langle \frac{d\hat{A}}{dt}\right\rangle = -i\left\langle [\tilde{H}, \hat{A}]\right\rangle \tag{3.83}$$

Proof. See Ehrenfest in [106] for the original proof with regards to continuous Hamiltonian operators. For our proof, all we require is to take the bra-ket of the von Neumann equation. □

Corollary 3.2.3. *The Schrödinger equation for the dynamics of the state vector $|\Psi(t)\rangle \in \mathbb{CP}^{n-1}$ is given by:*

$$i\frac{d|\Psi\rangle}{dt} = \tilde{H}|\Psi\rangle = \tilde{H}(t)|\Psi\rangle \tag{3.84}$$

Proof. The time evolution operator applied to the state has the action:

$$\hat{U}(t,0)|\Psi(0)\rangle = |\Psi(t)\rangle \tag{3.85}$$

Differentiating directly with respect to time, we have:

$$\frac{d\hat{U}(t,0)}{dt}|\Psi(0)\rangle = \frac{d}{dt}|\Psi(t)\rangle \tag{3.86}$$

Substituting the relationship for the derivative of the time evolution operator, we obtain the result. □

We shall now state several results that we shall rely on repeatedly in this thesis. These theorems are several conditions on the form of the Hamiltonian operator which constrain the dynamics in a form so as to allow solution of the dynamic evolution of the state, as given by the Schrödinger equation. Our method of derivation shall differ from that used originally in Carlini et. al (see e.g. [49–51,107,108] for a series of works on the topic), in that we shall proceed from the direction of the von Neumann equation in order to resolve the matrix dynamics of the Hamiltonian. It is important to note that this is an equivalent formulation of the theory of the quantum brachistochrone, which allows identification of shortest time paths for state-to-state transitions on the complex projective manifold as given originally in [49,50], however we modify the formalism so as to examine the unitary operators in a way which is independent of the initial and terminal boundary conditions. We shall discuss the link between the two representations in the conclusion of this subsection. Our aim in deriving the major results in this fashion is to allow us to find metrics associated to the unitary operators on the complex projective space, and identify these metrics with the underlying Laplacian operators that define the eigenfunctions and kernels. This process will allow us to shortcut many of the complications of group representation theory; as we shall show, this is a powerful insight that gives us full access to the nature of the state geometry and solution space of the heat kernel. The relevant metric towards which we aim is, of course, the principle of least time under constraints. This optimisation principle in this case takes the form of an Euler-Lagrange equation in which the Hamiltonian variable appears as a variable, and not an assumed form appearing from the physical specifications of the system. This is an important modification of the approach towards understanding symmetry and unitary operators on the state space which defines the probability density of the system. We shall implement a program towards identification of the metrics that one can associate with the unitary operators and kernels found using the quantum brachistochrone equation. Note that many of the elements in this procedure have already been assembled in the series of papers by the author in [52–55], which built on earlier co-operative work performed in collaboration with the lead author from [49], resulting in [52]. We seek to push this methodology to its limit and logical conclusion and see what the differential geometry and kernels implied by such a theory give for group representations and the theory of special functions. This is, as far as the author is aware, a novel insight that is the major contribution of this work. Elements addressing similar problems may be found in the work of Kuzmak [66], however the authors did not proceed to identify the group representations in the way in which we shall. Further, we shall show that analytical continuation of the state from the complex projective manifold defined by the sphere to the hyperboloid enables us to derive many results that were previously difficult to access due to the restrictive nature of calculations in group representation theory.

A constrained optimisation means that there are some degrees of freedom which are accessible, and some which are not. In this sense, we must take into account the nature of the possible evolutions that may occur while respecting this property. One way to understand what constitutes a constraint matrix is by thinking of it as everything that excludes the system Hamiltonian. The most general constraint is therefore a linear combination of arbitrary time dependent functions multiplied by the set of generators for the Lie algebra which are not in the Hamiltonian. For any finite Lie group, this may be evaluated, up to the specification of the explicit time dependence of the functions which give the linear combination. By reducing the degrees of freedom over which the system may evolve, we can determine the dynamic properties of the state. In doing so, we must specify the constraint laws on the Hamiltonian matrix in order to evaluate the Lagrangian optimisation, which determines the solution to the quantum brachistochrone.

Essentially, this acts as a Lagrange multiplier in the optimisation of least time for state-to-state transfer via the Fubini-Study metric. As we shall show, this addition to the basic action principle allows solution of the Euler-Lagrange equations in the form of the quantum brachistochrone equations.

Theorem 3.2.4. *Assume some finite Lie algebra, with orthonormality expressed through the trace inner product:*

$$\langle \hat{g}_i, \hat{g}_j \rangle = \mathrm{Tr}\left[\hat{g}_i^\dagger \hat{g}_j\right] = \delta_{ij} \tag{3.87}$$

Assume further that the generators are Hermitian, satisfying $\hat{g}_i^\dagger = \hat{g}_i$, and are also tracefree $\mathrm{Tr}\left[\hat{g}_j\right] = 0$. Let there exist a Hamiltonian matrix $\tilde{H}$, tracefree and Hermitian as before, existing in this set. Then there exists a constraint matrix $\tilde{F}$, such that $\mathrm{Tr}\left(\tilde{H}\tilde{F}\right) = 0$.

Proof. Take a finite partition of some Lie group G, which we denote H such that $G = H \bigcup H^c$. Let $\tilde{H} = \sum_{i \in H} \lambda_i \hat{g}_i$, where $\lambda_i \in \mathbb{R}, \hat{g}_j = \hat{g}_j^\dagger$, and the generators are the tracefree matrices $\mathrm{Tr}\hat{g}_j = 0$. It is trivial to see that the linear combination of tracefree matrices is tracefree, hence the Hamiltonian matrix is tracefree. Taking a linear combination in the complementary set, we have $\tilde{F} = \sum_{j\notin H} \alpha_j \hat{g}_j$, evaluating the trace we have:

$$\mathrm{Tr}\left(\tilde{H}\tilde{F}\right) = \mathrm{Tr}\left(\sum_{i\in H} \lambda_i \sum_{j\notin H} \alpha_j \hat{g}_i \hat{g}_j\right) = \sum_{i\in H}\sum_{j\notin H} \lambda_i \alpha_j \mathrm{Tr}\left(\hat{g}_i \hat{g}_j\right) = 0 \tag{3.88}$$

where we have used the linearity of the trace, orthonormality of the generating set and the fact that by assumption there exists no $i \in G$ and $i \notin G$ as these sets are mutually exclusive. □

Although at this stage we have not specified the exact form of the time dependent functions which are given by the formula for the Hamiltonian $\tilde{H}(t) = \sum_{i\in H} \lambda_i(t)\hat{g}_i$ or constraint $\tilde{F}(t) = \sum_{j\notin H} \alpha_j(t)\hat{g}_j$, we are free to write these as matrix combinations of these arbitrary functions. As we shall see, the addition of one more simple observation to the Lagrangian action principle is sufficient to specify the form of all these functions. This means that, although at face value in this general form it may appear that these functions and dependent matrices are not unique, once we add further conditions, they are completely determined in terms of time.

We shall call the matrix complementary to the Hamiltonian matrix in this way the constraint, and shall denote it generally in the form $\tilde{F}(t)$. The constraint $\tilde{F}$ represents the degrees of freedom that are inaccessible to the system via the dynamic operator $\tilde{H}$. As we shall see, this is very important in understanding how the form of the Lagrangian principle we shall derive comes about. For the sequel, we assume such properties as have been used in the prior calculations. The symmetry principles and constraints are related to the statistics of the probability, as described through the state vector. We shall state an important formula that demonstrates the necessity for identifying the correct frame of reference, which in this case is given by the trace independent part of the Hamiltonian matrix.

Theorem 3.2.5. *Given a finite Hamiltonian matrix which is a function of time $\tilde{H} \doteq \tilde{H}(t)$, The variance function is independent of the trace part of the Hamiltonian matrix, $\Delta E^2 = \Delta\tilde{H}^2 = \Delta\hat{H}^2$, where $\tilde{H} = \hat{H} - \frac{1}{n}\mathbf{1}\mathrm{Tr}(\tilde{H})$ is the traceless part.*

Proof. Without loss of generality, we drop the time dependence as it is not essential to the proof. Recall the definition of the variance as given by the expectation value:

$$\Delta A^2 = \left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2 \tag{3.89}$$

If we take the trace form of the Hamiltonian matrix, we may write:

$$\hat{H} = \tilde{H} + \frac{\hat{\mathbf{1}}}{n}\mathrm{Tr}\hat{H} \tag{3.90}$$

Evaluating the variance, we may write:

$$\left\langle \hat{H}^2 \right\rangle - \left\langle \hat{H} \right\rangle^2 = \left\langle \left(\tilde{H} + \frac{\hat{\mathbf{1}}}{n}\mathrm{Tr}\hat{H}\right)^2 \right\rangle - \left\langle \left(\tilde{H} + \frac{\hat{\mathbf{1}}}{n}\mathrm{Tr}\hat{H}\right) \right\rangle^2 \tag{3.91}$$

Expanding the terms, and using the orthonormality of the state $\langle\psi|\,\psi\rangle = 1$, which is unchanged by a unit matrix multiplication inside the bra-ket, we arrive at the result:

$$\left\langle \hat{H}^2 \right\rangle - \left\langle \hat{H} \right\rangle^2 = \left\langle \tilde{H}^2 \right\rangle - \left\langle \tilde{H} \right\rangle^2 \tag{3.92}$$

which we denote as the energy variance ΔE^2. This establishes that the variance relation is independent of the trace degrees of freedom, which is a form of gauge invariance. □

We shall now state the second constraint which we shall use in formulating the quantum brachistochrone problem.

Definition 3.2.6. [Isotropic Constraint] We state a condition of isotropy in terms of the energy overhead of the system. This is given by the constraint on the allowable Hamiltonian operators, tracefree and Hermitian as before such that:

$$\mathrm{Tr}\left(\frac{\tilde{H}^2}{2}\right) = \frac{1}{2}\sum_n E_n^2 < \infty \tag{3.93}$$

Remark 13. This is a practical working ansatz that we require as without it, it might be possible to move from one point to another in zero time by applying infinite energy. This is discussed in depth in [49,50]. In effect, this takes the form of an energy overhead being some finite number. In practical terms, we can consider this to be constraining the Hamiltonian matrix to have eigenvalues that are finite, and countable. As in this thesis we only consider examples of this type, in particular small dimensional matrices with time independent eigenvalue spectra, this is satisfied by all our constructs within this work. The cyclic property of the trace and multiplication by an identity transformation given by a unitary operator $\hat{Q}^\dagger\hat{Q} = 1$ gives:

$$\mathrm{Tr}\left(\frac{\tilde{H}^2}{2}\right) = \mathrm{Tr}\left(\frac{\tilde{H}^2}{2}\hat{Q}^\dagger\hat{Q}\right) = \frac{1}{2}\mathrm{Tr}\left(\hat{Q}\frac{\tilde{H}^2}{2}\hat{Q}^\dagger\right) = \frac{1}{2}\mathrm{Tr}\left(\hat{Q}\tilde{H}\hat{Q}^\dagger.\hat{Q}\tilde{H}\hat{Q}^\dagger\right) \tag{3.94}$$

and therefore the isotropic condition is invariant under unitary transformations, the linear constraint law invariant in an identical way. In the diagonal representation, we therefore have:

$$\tilde{H}(t) \sim \hat{Q}\tilde{H}(t)\hat{Q}^\dagger = \sum_i E_j \left|j\right\rangle \left\langle j\right| = \begin{bmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & E_n \end{bmatrix} \tag{3.95}$$

Evaluating the trace of the squared Hamiltonian in the diagonal representation, which is permissible as the trace is invariant, we find:

$$\mathrm{Tr}\left(\frac{\tilde{H}^2}{2}\right) = \frac{1}{2}\sum_n E_n^2 < \infty \tag{3.96}$$

This demonstrates how this type of constraint comes about for the Lagrangian optimisation we seek. For a finite Hamiltonian matrix of odd or even dimensions, the eigenvalues may be rearranged such that they are symmetric about the centre, with net energy zero. For this reason, the isotropic condition is important, as it represents a square mean or L^2 expectation value in the absence of other values coming from the trace and linear constraint.

Note that in quantum mechanics, for the reader who may not be familiar with such concepts, any global phase is represented by the trace part of the Lie algebra, which is the diagonal identity matrix. This element is the only element in the algebra which commutes with all other elements in the scenarios we shall consider. For this reason, we may remove it, as it will only contribute a global phase. This is a type of gauge invariance which we shall comment on later in this section. For our aims, we only need to know that the Hamiltonian and associated constraint are arbitrary tracefree Hermitian matrices that together form the total Lie algebra, and the identity component plays no role in this due to the gauge invariance aspect.

We shall now derive a major theorem, which we shall call the quantum brachistochrone equation. This theorem, originally appearing in [49,50], can be derived through application of the von Neumann equation as previously derived. We shall state this in the following form:

Theorem 3.2.7. *Given a Hamiltonian and associated constraint matrix $\tilde{H}(t), \tilde{F}(t) \in \mathbb{M}^{n\times n}$, both Hermitian, tracefree and of finite dimension such that $\tilde{H}(t) = \tilde{H}^\dagger(t)$ and $\tilde{F}(t) = \tilde{F}^\dagger(t)$, then the sum of these operators $\hat{A}(t) = \tilde{H}(t) + \tilde{F}(t)$ is Hermitian and tracefree. Assuming unitary evolution of the constraint and Hamiltonian, under the conditions of the von Neumann equation (Hermiticity, unitary evolution), we have:*

$$i\frac{d}{dt}(\tilde{H}(t) + \tilde{F}(t)) = [\tilde{H}(t), \tilde{F}(t)] \tag{3.97}$$

.

Proof. The sum of any two Hermitian operators is obviously Hermitian, $\tilde{H}^\dagger(t) + \tilde{F}^\dagger(t) = \tilde{H}(t) + \tilde{F}(t)$, it is elementary to show that the sum of any tracefree operators is also tracefree. The conditions of the von Neumann equation are satisfied, in that we have a Hermitian observable which evolves unitarily, the evolution described by the transformation:

$$\hat{A}(t) = \hat{U}(t,0)\hat{A}(0)\hat{U}^\dagger(t,0) \tag{3.98}$$

where we have $\hat{A}(t) = \hat{A}^\dagger(t)$. The von Neumann equation for the time dependent operator $\hat{A}(t)$ reads as:

$$i\frac{d\hat{A}}{dt} = [\tilde{H}, \hat{A}] = \tilde{H}\hat{A} - \hat{A}\tilde{H} \tag{3.99}$$

substitution of the operator $\tilde{H}(t) + \tilde{F}(t)$ gives the result, as any operator commutes with itself $\left[\tilde{H}(t), \tilde{H}(t)\right] = 0$. □

We shall now demonstrate the central principle that shall dominate a crucial part of this thesis. This shall consist of a form of least-action principle that expresses the motion of the state on the complex projective state space, given as a principle of least time. These types of optimisation procedures are familiar in classical optics since the times of Fermat, however, this problem differs in that the Hamiltonian matrix appears as a variable in the Euler-Lagrange equations.

Theorem 3.2.8. *The principle of least time under bounded constraints defines a quantum brachistochrone problem. It is written:*

$$\delta t = \int 1dt + \text{constraints} \tag{3.100}$$

$$= \int \frac{\sqrt{\left\langle d\Psi \left| \left(1 - \hat{P}\right) \right| d\Psi \right\rangle}}{\Delta E} + \text{constraints} = \min \tag{3.101}$$

where $\hat{P}$ is the projection operator $\hat{P} = |\Psi\rangle\langle\Psi|$, the state vector $|\Psi\rangle \in \mathbb{CP}^n$ obeying normalisation $\langle\Psi|\,\Psi\rangle = 1$, where the state vector evolves unitarily according to $i\frac{d|\Psi\rangle}{dt} = \tilde{H}(t)|\Psi(t)\rangle$. These two expressions are equal using the expression for the energy variance in terms of the Hamiltonian operator $\tilde{H}$. The minimum is to be taken in terms of the time taken between any initial point and terminus, while obeying the constraints.

This optimisation principle expresses the same qualitative information as Fermat's principle of least time in optics, where the ray travels at different speeds and with different paths depending on the conditions inside and outside the media. In our context, we are optimising the time taken between any two accessible points described on the space of states, and are solving to find the optimal Hamiltonian matrix that joins these via the time evolution operator. The objective function in this case is given by the first part of the action principle, which is equal to the path time δt.

Proof. The energy variance is given by:

$$\Delta E = \sqrt{\left\langle \tilde{H}^2 \right\rangle - \left\langle \tilde{H} \right\rangle^2} = \sqrt{\left\langle \Psi \middle| \tilde{H}^2 \middle| \Psi \right\rangle - \left\langle \Psi \middle| \tilde{H} \middle| \Psi \right\rangle^2} \tag{3.102}$$

as shown in the variance condition, or by consideration of the expectation value in the Hamiltonian matrix $\tilde{H}$ via $\mathbb{E}\left(\tilde{H}(t)\right) = \langle\psi| \tilde{H}(t) |\psi\rangle$. This can be rewritten in terms of the projection operator:

$$\Delta E = \sqrt{\left\langle \Psi \middle| \tilde{H} \left(1 - |\Psi\rangle \langle\Psi|\right) \tilde{H} \middle| \Psi \right\rangle} \tag{3.103}$$

and further

$$\Delta E = \sqrt{\left\langle \Psi \middle| \tilde{H} \left(1 - \hat{P}\right) \tilde{H} \middle| \Psi \right\rangle} = \frac{1}{dt}\sqrt{\left\langle d\Psi \middle| \left(1 - \hat{P}\right) \middle| d\Psi \right\rangle} \tag{3.104}$$

where the final line is obtained by the expression for unitary evolution of the state $i\frac{d|\Psi\rangle}{dt} = \tilde{H}(t)|\Psi(t)\rangle$ and its Hermitian adjoint equivalent. Dividing through by ΔE and rearranging, we find the result as:

$$\int dt = \int \frac{\sqrt{\left\langle d\Psi \middle| \left(1 - \hat{P}\right) \middle| d\Psi \right\rangle}}{\Delta E} \tag{3.105}$$

The constraints in this thesis are to be specified through the constraint matrix and the isotropic condition. □

Corollary 3.2.9. *The energy dispersion is given by the Aharanov-Anandan relation:*

$$\Delta E dt = \sqrt{\left\langle d\Psi \middle| \left(1 - \hat{P}\right) \middle| d\Psi \right\rangle} \tag{3.106}$$

a result originally obtained in by Aharanov and Anandan in [109].

We shall comment on the construction of the optimisation problem in our formalism and the addition of constraints. The constrained optimisation may be written with the additional implicitly assumed condition that the evolution be unitary on the state as given by the following.

Corollary 3.2.10. *The quantum brachistochrone problem 3.2.8 is defined through the Lagrangian optimisation principle which minimises the time taken for state-to-state transfer, given unitary dynamics and the Lagrange multipliers given by the linear and isotropic constraints:*

$$\delta t = \int 1 dt + \int \lambda_1 \left(\mathrm{Tr}\left(\tilde{H}\tilde{F}\right) - 0\right) dt + \int \lambda_2 \left(\mathrm{Tr}\left(\frac{\tilde{H}^2}{2}\right) - k\right) dt = \min \tag{3.107}$$

where the term $\int 1dt$ *is the time taken throughout the quantum evolution, to be minimised, and the constraints, given variously by* $\mathrm{Tr}\left(\tilde{H}\tilde{F}\right) = 0$ *and* $\mathrm{Tr}\left(\frac{\tilde{H}^2}{2}\right) = k$ *force the Hamiltonian to be only in allowed degrees of freedom. Further, we take implicitly the dynamics through* $i\frac{d\hat{U}(t,0)}{dt} = \tilde{H}(t)\hat{U}(t,0)$ *which defines the evolution of the state via the time evolution equation and unitary operator* $\hat{U}(t,0)\left|\Psi(0)\right\rangle = \left|\Psi(t)\right\rangle$*, the projection operator form being given by the replacement of* $1dt$ *as in the principle of least time in 3.2.10. The state vector as before is an* n*-dimensional complex vector with unit normalisation. The solution to this problem is an optimal Hamiltonian, which defines an optimal path on the state space via the time evolution operator.*

Proof. Addition of the constraints given by Thms. 3.2.4, 3.2.6 to the optimisation problem 3.2.8 yields the result. □

We shall now show how the Euler-Lagrange equations for the system define the quantum brachistochrone equation, as derived through the von Neumann equivalent in 3.2.1, 3.2.7. The argument for this originates in the papers of Carlini, Okudaira et. al [49,50], however, as we have modified the derivation through use of matrix mechanics methods we shall demonstrate this for completeness. If we take the variation of the time part with respect to the Hamiltonian matrix, Hermitian and tracefree, the maximum is defined on the surface. We shall take the following definition for the derivative of a trace identity, which we shall require to establish the necessary equivalence between this method of time optimisation and some more conventional techniques from quantum mechanics.

Definition 3.2.11. The matrix derivative is denoted $\frac{\partial}{\partial \hat{X}}$, which reads as the derivative with respect to the matrix $\hat{X}$. This derivative satisfies the trace identities:

$$\frac{\partial}{\partial \hat{X}}\left(\mathrm{Tr}\left[\hat{A}\hat{X}\right]\right) = \frac{\partial}{\partial \hat{X}}\left(\mathrm{Tr}\left[\hat{X}\hat{A}\right]\right) = \hat{A} \tag{3.108}$$

where $\mathrm{Tr}\left[.\right]$ is the trace.

Remark 14. Although there is a rich theory of matrix derivatives, in the following calculation we shall not require more than this basic identity and some rearrangements using the trace.

Theorem 3.2.12. *The variation of the time part of the Lagrangian principle given by 3.2.10* $\int 1dt$ *defines the control problem:*

$$\frac{\partial}{\partial \tilde{H}}\left(\frac{\left\langle \Psi \left| \tilde{H}\left(1-\hat{P}\right)\tilde{H}\right| \Psi\right\rangle}{\Delta E^2}\right) \tag{3.109}$$

$$= 2\frac{\sqrt{\left\langle \Psi \left| \tilde{H}\left(1-\hat{P}\right)\tilde{H}\right| \Psi\right\rangle}}{\Delta E}\frac{\partial}{\partial \tilde{H}}\left(\frac{\sqrt{\left\langle \Psi \left| \tilde{H}\left(1-\hat{P}\right)\tilde{H}\right| \Psi\right\rangle}}{\Delta E}\right) = 0 \tag{3.110}$$

where $\tilde{H}$ *is the traceless component of the Hamiltonian matrix,* $\hat{P}$ *is the projection matrix, the matrix derivative as in 3.2.11, and the variance given by the term* ΔE^2*, which we assume positive and constant under the conditions 3.2.5, 3.2.6.*

Proof. Using 3.2.9, we have $S = \int 1dt$, with:

$$dt = \frac{\sqrt{\left\langle d\Psi \left| \left(1 - \hat{P}\right) \right| d\Psi \right\rangle}}{\Delta E} \tag{3.111}$$

$$= \frac{\sqrt{\left\langle \dot{\Psi} \left| \left(1 - \hat{P}\right) \right| \dot{\Psi} \right\rangle}}{\Delta E} ds \tag{3.112}$$

where we write $\left|\dot{\Psi}\right\rangle = \frac{d}{ds} \left|\Psi(s)\right\rangle$. Inserting the expression for the time evolution of the state from the definition 3.2.3, we have:

$$dt = \frac{1}{\Delta E} \sqrt{\left\langle \Psi \left| \tilde{H} \left(1 - \hat{P}\right) \tilde{H} \right| \Psi \right\rangle} ds \tag{3.113}$$

The optimisation problem is then determined by the solution to the Lagrangian $\int L ds$, where we consider the Lagrangian from the quantum brachistochrone problem 3.2.10 free of constraints. As can be seen by inspection, they are equivalent. However, this is also satisfied by the solution to the problem $\int L^2 ds$, as can be seen by differentiating under the integral sign. We can therefore write the derivative with respect to $\tilde{H}$ of this Lagrangian in the form:

$$\frac{\partial}{\partial \tilde{H}} \int L^2 ds = \int \frac{\partial L^2}{\partial \tilde{H}} ds = \int 2L \frac{\partial L}{\partial \tilde{H}} ds \tag{3.114}$$

and therefore the problems are equivalent. On the extremal surface, we must therefore have:

$$\frac{\partial}{\partial \tilde{H}} \left(\left\langle \Psi \left| \tilde{H} \left(1 - \hat{P}\right) \tilde{H} \right| \Psi \right\rangle \right) = 0 \tag{3.115}$$

as required. □

We have the freedom to choose the tracefree representation $\tilde{H}$, as the variational principle shows that the energy variance is independent of the trace of the Hamiltonian matrix $\hat{H}$. As the Lagrangian is independent of the time derivative of the Hamiltonian matrix, the Euler-Lagrange equations read as:

Lemma 3.2.13.

$$\frac{\partial S}{\partial \tilde{H}} = \frac{d}{dt} \frac{\partial S}{\partial \dot{\tilde{H}}} = 0 \tag{3.116}$$

We shall now prove a result that we shall require several times in the following calculations.

Theorem 3.2.14. *Assume the constraints in the Lagrangian optimisation 3.2.10 are zero, and only the time component remains. Assume further that the state evolves unitarily via 3.2.3. The Euler-Lagrange equations 3.2.13 applied to the time component of the action define the matrix system of equations:*

$$\frac{\partial}{\partial \tilde{H}} \left(\left\langle \Psi \left| \tilde{H} \left(1 - \hat{P}\right) \tilde{H} \right| \Psi \right\rangle \right) = \left\{ \tilde{H}, \hat{P} \right\} - \left\langle \tilde{H} \right\rangle \hat{P} \tag{3.117}$$

where $\tilde{H}$ is the $n \times n$ traceless Hermitian Hamiltonian matrix $\tilde{H} = \tilde{H}^\dagger$, $\hat{P} = \left|\Psi\right\rangle \left\langle\Psi\right|$ is the projection operator, and the state $\left|\Psi\right\rangle \in \mathbb{CP}^{n-1}$.

Proof. Without loss of generality, we consider the condition whereby the constraints are zero, as they may be added as extra terms on the left hand side of the derivative. Assume further that the energy variance is some constant under the conditions of 3.2.5, in this case we may simplify the optimisation problem of the time variable to evaluation of the matrix derivative:

$$\frac{\partial}{\partial \tilde{H}}\left(\left\langle \Psi \middle| \tilde{H}\left(1-\hat{P}\right)\tilde{H} \middle| \Psi \right\rangle\right) = 0 \tag{3.118}$$

This can be evaluated using the rules for matrix differentiation as in 3.2.11. Writing the matrix derivative as a trace, we have:

$$\frac{\partial}{\partial \tilde{H}}\left(\left\langle \Psi \middle| \tilde{H}\left(1-\hat{P}\right)\tilde{H} \middle| \Psi \right\rangle\right) = \frac{\partial}{\partial \tilde{H}}\mathrm{Tr}\left(\tilde{H}\left(1-\hat{P}\right)\tilde{H}\hat{P}\right) \tag{3.119}$$

$$= \frac{1}{2}\frac{\partial}{\partial \tilde{H}}\mathrm{Tr}\left(\hat{P}\tilde{H}\left(1-\hat{P}\right)\tilde{H}\right) + \frac{1}{2}\frac{\partial}{\partial \tilde{H}}\mathrm{Tr}\left(\tilde{H}\left(1-\hat{P}\right)\tilde{H}\hat{P}\right) \tag{3.120}$$

$$= \frac{1}{2}\left(\hat{P}\tilde{H}\left(1-\hat{P}\right) + \left(1-\hat{P}\right)\tilde{H}\hat{P}\right) \tag{3.121}$$

$$= \frac{1}{2}\left(\hat{P}\tilde{H} + \tilde{H}\hat{P}\right) - \left(\hat{P}\tilde{H}\hat{P}\right) \tag{3.122}$$

$$= \left\{\tilde{H}, \hat{P}\right\} - |\Psi\rangle\langle\Psi|\,\tilde{H}\,|\Psi\rangle\langle\Psi| \tag{3.123}$$

$$= \left\{\tilde{H}, \hat{P}\right\} - \left\langle\tilde{H}\right\rangle\hat{P} \tag{3.124}$$

where $\hat{P} = |\Psi\rangle\langle\Psi|$ is the projection operator. We have evaluated the left and right matrix derivatives by using the expressions:

$$\frac{\partial}{\partial \tilde{H}}\mathrm{Tr}\left(\hat{A}\tilde{H}\right) = \frac{\partial}{\partial \tilde{H}}\mathrm{Tr}\left(\tilde{H}\hat{A}\right) = \hat{A} \tag{3.125}$$

and therefore

$$\frac{\partial}{\partial \tilde{H}}\left(\frac{\left\langle \Psi \middle| \tilde{H}\left(1-\hat{P}\right)\tilde{H} \middle| \Psi \right\rangle}{\Delta E^2}\right) = \frac{1}{\Delta E^2}\left(\left\{\tilde{H}, \hat{P}\right\} - \left\langle\tilde{H}\right\rangle\hat{P}\right) \tag{3.126}$$

as required. □

Remark 15. Note that this differs slightly from the expression in [49] as we have defined the anticommutation relation via $\left\{\hat{A}, \hat{B}\right\} = \frac{1}{2}\left(\hat{A}\hat{B} + \hat{B}\hat{A}\right)$.

Corollary 3.2.15. *The Euler-Lagrange equations for the constraint equations defined by 3.2.4, 3.2.6 are given by:*

$$\frac{d}{dt}\frac{\partial S_{const.}}{\partial \dot{\tilde{H}}} = \frac{\partial}{\partial \dot{\tilde{H}}}\frac{dS_{const.}}{dt} = \lambda_1\tilde{H} + \lambda_2\tilde{F} \tag{3.127}$$

Proof. The solution to the time optimisation problem 3.2.10 without constraints may be written:

$$\frac{\partial}{\partial \tilde{H}}\left(S_{time}\right) = 0 \tag{3.128}$$

With the addition of time dependence in the use of the linear and isotropic constraints 3.2.4, 3.2.6, this becomes:

$$\frac{\partial}{\partial \tilde{H}}\left(S_{time} + S_{cons.}\right) = \frac{d}{dt}\frac{\partial S_{const.}}{\partial \dot{\tilde{H}}} \tag{3.129}$$

$$\frac{d}{dt}\left(\frac{\lambda_1}{2}\mathrm{Tr}\tilde{H}^2+\lambda_2\mathrm{Tr}\tilde{H}\tilde{F}\right)=\frac{\lambda_1}{2}\mathrm{Tr}\tilde{H}\dot{\tilde{H}}+\lambda_2\mathrm{Tr}\dot{\tilde{H}}\tilde{F} \tag{3.130}$$

$$\frac{d}{dt}\frac{\partial S_{const.}}{\partial\dot{\tilde{H}}}=\frac{\partial}{\partial\dot{\tilde{H}}}\frac{dS_{const.}}{dt}=\frac{\partial}{\partial\dot{\tilde{H}}}\frac{d}{dt}\left(\frac{\lambda_1}{2}\mathrm{Tr}\tilde{H}^2+\lambda_2\mathrm{Tr}\tilde{H}\tilde{F}\right) \tag{3.131}$$

$$=\frac{\partial}{\partial\dot{\tilde{H}}}\left(\lambda_1\mathrm{Tr}\tilde{H}\dot{\tilde{H}}+\lambda_2\mathrm{Tr}\dot{\tilde{H}}\tilde{F}+\lambda_2\mathrm{Tr}\tilde{H}\dot{\tilde{F}}\right) \tag{3.132}$$

$$=\lambda_1\tilde{H}+\lambda_2\tilde{F} \tag{3.133}$$

□

Corollary 3.2.16. *The solution to the optimisation problem with constraints 3.2.10 is defined through the matrix equation:*

$$\frac{1}{\Delta E^2}\left(\left\{\tilde{H},\hat{P}\right\}-\left\langle\tilde{H}\right\rangle\hat{P}\right)-\lambda_1\tilde{H}-\lambda_2\tilde{F}=0 \tag{3.134}$$

Proof. We have the Euler-Lagrange equations 3.2.15, 3.2.13:

$$\frac{\partial}{\partial\tilde{H}}\left(S_{time}+S_{cons.}\right)=\frac{d}{dt}\frac{\partial S_{const.}}{\partial\dot{\tilde{H}}} \tag{3.135}$$

Using the results for each of these variational equations, we find:

$$\frac{1}{\Delta E^2}\left(\left\{\tilde{H},\hat{P}\right\}-\left\langle\tilde{H}\right\rangle\hat{P}\right)-\lambda_1\tilde{H}-\lambda_2\tilde{F}=0 \tag{3.136}$$

□

Remark 16. In [49], the authors obtained a matrix equation given by:

$$\frac{1}{\Delta E^2}\left(\left\{\tilde{H},\hat{P}\right\}-\left\langle\tilde{H}\right\rangle\hat{P}\right)-\lambda_1\tilde{H}+\left|\Psi\right\rangle\left\langle\Phi\right|+\left|\Psi\right\rangle\left\langle\Phi\right|=0 \tag{3.137}$$

We can see that we can force equivalence between this method and that given by 3.2.16 through the relation:

$$-\lambda_2\tilde{F}=\left|\Psi\right\rangle\left\langle\Phi\right|+\left|\Psi\right\rangle\left\langle\Phi\right| \tag{3.138}$$

We shall now show briefly how this representation of the optimal solution results in a certain form of the constraint. Taking traces of both sides of this equation:

$$\frac{1}{\Delta E^2}\left(\mathrm{Tr}\tilde{H}-\left\langle\tilde{H}\right\rangle\mathrm{Tr}\hat{P}\right)-\lambda\mathrm{Tr}\tilde{H}+\left\langle\Psi\right|\left.\Phi\right\rangle+\left\langle\Phi\right|\left.\Psi\right\rangle=0 \tag{3.139}$$

and using the fact that the expectation and trace of the Hamiltonian operator are equal (to zero), and the trace of the projection operator is unity 3.2.20 we find:

$$\left\langle\Psi\right|\left.\Phi\right\rangle+\left\langle\Phi\right|\left.\Psi\right\rangle=0 \tag{3.140}$$

Using the properties of bra-ket notation, we then conclude that:

$$\left\langle\Psi\right|\left.\Phi\right\rangle=-\left\langle\Phi\right|\left.\Psi\right\rangle=\left(\left\langle\Psi\right|\left.\Phi\right\rangle\right)^* \tag{3.141}$$

$$\left\langle\Psi\right|\left.\Phi\right\rangle=iW, W\in\mathbb{R} \tag{3.142}$$

Remark 17. The key difference between this methodology and that of Carlini et. al is that the authors in [49,50] force the time optimal evolution through the application of extraneous Lagrange multipliers, that constrain the evolution to only unitary evolutions via the time evolution equation (see 3.2.3). Our method is implicit, in that we shall take the evolution operator, and von Neumann equations as derived (3.2.1), and use this to demonstrate consistency without the difficulty of evaluating these complicated Euler-Lagrange equations, which upon application of the respective methods from [49,52] result in the quantum brachistochrone equation 3.2.7. We shall now show through a simple argument some ways in which they may be obtained and their equivalence with the von Neumann equation 3.2.1.

We recall the result from the previous theorem:

Theorem 3.2.17. *Given a Hermitian, tracefree Hamiltonian matrix $\tilde{H}$, existing on $\mathbb{M}^{n\times n}$, some state vector $|\Psi\rangle \in \mathbb{CP}^{n-1}$, and projection matrix given by the outer product $\hat{P} = |\Psi\rangle\langle\Psi|$, the matrix derivative of the time component of the least time action principle is given by:*

$$\frac{\partial}{\partial\tilde{H}}\left(\frac{\left\langle\Psi\left|\tilde{H}\left(1-\hat{P}\right)\tilde{H}\right|\Psi\right\rangle}{\Delta E^2}\right) = \frac{1}{\Delta E^2}\left(\left\{\tilde{H},\hat{P}\right\} - \left\langle\tilde{H}\right\rangle\hat{P}\right) \tag{3.143}$$

where $\Delta E^2 = \left\langle\tilde{H}^2\right\rangle - \left\langle\tilde{H}\right\rangle^2$ is the variance of the Hamiltonian operator, also $\left\{\hat{A},\hat{B}\right\} = \frac{1}{2}\left(\hat{A}\hat{B}+\hat{B}\hat{A}\right)$, expectation given by $\left\langle\hat{A}\right\rangle = \left\langle\Psi\left|\hat{A}\right|\Psi\right\rangle = \mathrm{Tr}\left(\hat{A}\hat{P}\right)$.

We shall now prove a number of lemmas that will demonstrate the equivalence of the formulations expressed in [49,52]. We shall require the following:

Definition 3.2.18. The expectation value is defined through:

$$\left\langle\hat{A}\right\rangle = \left\langle\Psi\left|\hat{A}\right|\Psi\right\rangle \tag{3.144}$$

where $|\Psi\rangle \in \mathbb{CP}^{n-1}$.

Definition 3.2.19. The projection operator is given by the outer product of the state with itself $\hat{P} = |\Psi\rangle\langle\Psi|$.

Lemma 3.2.20. *The trace of the projection operator is unity.*

Proof.

$$\mathrm{Tr}\hat{P} = \langle\Psi|\,\Psi\rangle = 1 \tag{3.145}$$

as this is the normalisation of the state vector. □

Definition 3.2.21. The matrix operator $\hat{G}$

$$\hat{G} = \hat{A} - \left\langle\hat{A}\right\rangle\hat{P} \tag{3.146}$$

where $\hat{A}$ is a Hermitian matrix, $\hat{A} = \hat{A}^\dagger$, $\hat{P}$ is the projection operator as in 3.2.19. In this proof we shall take the special set of matrices which satisfy $\mathrm{Tr}\hat{A} = \left\langle\hat{A}\right\rangle$.

Lemma 3.2.22. *The operator $\hat{G}$ as in 3.2.21 has trace and expectation zero.*

Proof. Assume the conditions such as in 3.2.21. Taking traces of both sides, we have:

$$\mathrm{Tr}\hat{G} = \mathrm{Tr}\hat{A} - \left\langle \hat{A} \right\rangle \mathrm{Tr}\hat{P} = \mathrm{Tr}\hat{A} - \left\langle \hat{A} \right\rangle \tag{3.147}$$

The trace and expectation of $\hat{A}$ are equal under 3.2.21, hence we have $\mathrm{Tr}\hat{G} = 0$. Taking expectations with the state vector on both sides, we find:

$$\left\langle \hat{G} \right\rangle = \left\langle \hat{A} - \left\langle \hat{A} \right\rangle \hat{P} \right\rangle = \left\langle \hat{A} \right\rangle - \left\langle \hat{A} \right\rangle \left\langle \hat{P} \right\rangle = 0 \tag{3.148}$$

where we used 3.2.20. □

Lemma 3.2.23. *Under the conditions of 3.2.21, the trace and expectation can be written:*

$$\mathrm{Tr}\hat{A} = \mathrm{Tr}(\hat{A}\hat{P}) = \left\langle \hat{A} \right\rangle \tag{3.149}$$

The expectation and trace can be seen in terms of equivalence classes under the unitary transformation $\hat{Q}(.)\hat{Q}^\dagger$.

Proof. Under 3.2.21, we have:

$$\mathrm{Tr}\hat{A} = \left\langle \hat{A} \right\rangle \tag{3.150}$$

which, using 3.2.18 gives:

$$\mathrm{Tr}(\hat{A}\hat{P}) = \mathrm{Tr}(\hat{A}\hat{P}.\hat{\mathbf{1}}) = \mathrm{Tr}(\hat{A}\hat{P}\hat{Q}^\dagger\hat{Q}) = \mathrm{Tr}(\hat{Q}\hat{A}\hat{P}\hat{Q}^\dagger) \tag{3.151}$$

where we used the cyclic identity of the trace, and the unitary decomposition of the unit matrix $\hat{Q}^\dagger\hat{Q} = \hat{\mathbf{1}}$. This implies the equivalence class $\hat{A} \sim \hat{Q}\hat{A}\hat{P}\hat{Q}^\dagger$. Note that further insertions of the decomposition of identity can transform the projection operator. This is a demonstration of the known fact that the trace is an invariant operator on the space of matrices. □

Lemma 3.2.24. *For an operator $\hat{A}$, with trace equal to expectation as in 3.2.21, the anticommutator $\left\{\hat{A}, \hat{B}\right\} = \frac{1}{2}\left(\hat{A}\hat{B} + \hat{B}\hat{A}\right)$ of the matrix $\hat{G}$ with the projection operator $\hat{P}$ satisfies:*

$$\left\{\hat{G}, \hat{P}\right\} = \hat{G} \tag{3.152}$$

Proof. Assume the form of $\hat{G}$, as in 3.2.21. Evaluating the anticommutator as written, we have:

$$\left\{\hat{A} - \left\langle \hat{A} \right\rangle \hat{P}, \hat{P}\right\} = \left\{\hat{A}, \hat{P}\right\} - \left\langle \hat{A} \right\rangle \left\{\hat{P}, \hat{P}\right\} = \left\{\hat{A}, \hat{P}\right\} - \left\langle \hat{A} \right\rangle \hat{P} \tag{3.153}$$

where we have used $\hat{P}^2 = \hat{P}$, which is straightforward to derive from 3.2.19. Using the result for the equivalence classes in the operator $\hat{A}$ as in 3.2.23, and assuming the identity transformation:

$$\left\{\hat{A}, \hat{P}\right\} = \frac{1}{2}\left(\hat{A}\hat{P} + \hat{P}\hat{A}\right) = \frac{1}{2}(\hat{A} + \hat{A}) = \hat{A} \tag{3.154}$$

we obtain the result. □

Lemma 3.2.25. *The operator $\hat{G}$ is tracefree and Hermitian.*

Proof. We have discussed the tracefree condition as in 3.2.22. To show that it is Hermitian, we take the conjugate transpose directly:

$$\hat{G}^\dagger = \hat{A}^\dagger - \left\langle \hat{A} \right\rangle \hat{P}^\dagger = \hat{A} - \left\langle \hat{A} \right\rangle \hat{P} = \hat{G} \tag{3.155}$$

where we have used $\hat{P}^\dagger = \hat{P}$, which is trivial from the definition of the projection operator 3.2.19. □

Lemma 3.2.26. *The projection operator $\hat{P}$ satisfies the von Neumann equation 3.2.1:*

$$i\frac{d\hat{P}}{dt} = \left[\tilde{H}(t), \hat{P}\right] \tag{3.156}$$

Proof. Taking the definition of the projection operator directly from 3.2.19, differentiation with respect to time yields:

$$i\frac{d}{dt}\left|\Psi\right\rangle\left\langle\Psi\right| = i\frac{d\left|\Psi\right\rangle}{dt}\left\langle\Psi\right| + i\left|\Psi\right\rangle\frac{d\left\langle\Psi\right|}{dt} \tag{3.157}$$

$$= \tilde{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\tilde{H} \tag{3.158}$$

where we used the unitary evolution of the state given by 3.2.3, hence the result follows. □

We can now state the major result and prove equivalence between the schemes expressed in [49,52], given by the quantum brachistochrone equation 3.2.7 using this form of projective theory, thereby demonstrating equivalence of the explicit and implicit schemes with the principles of von Neumann matrix mechanics.

Theorem 3.2.27. *For the operator $\hat{G}$, given as in 3.2.21, the dynamics in time are given by the von Neumann equation:*

$$i\frac{d\hat{G}}{dt} = \left[\tilde{H}, \hat{G}\right] \tag{3.159}$$

Further, for any operator $\hat{A}$ with expectation value that is constant in time we have the von Neumann equation:

$$i\frac{d\hat{A}}{dt} = \left[\tilde{H}(t), \hat{A}\right] \tag{3.160}$$

Proof. We assume, as in 3.2.21 that $\hat{G} = \hat{A} - \left\langle\hat{A}\right\rangle\hat{P}$, and the operators defined as in the previous theorems. Under the conditions of 3.2.1, the operator must be Hermitian and of finite dimension. We satisfy these conditions as given by 3.2.25, therefore we may write:

$$i\frac{d}{dt}\left(\hat{A} - \left\langle\hat{A}\right\rangle\hat{P}\right) = \left[\tilde{H}(t), \hat{A} - \left\langle\hat{A}\right\rangle\hat{P}\right] \tag{3.161}$$

Expanding the commutator brackets and evaluating the derivatives, we may write:

$$i\frac{d\hat{A}}{dt} - i\frac{d}{dt}\left(\left\langle\hat{A}\right\rangle\hat{P}\right) = \left[\tilde{H}(t), \hat{A}\right] - \left\langle\hat{A}\right\rangle\left[\tilde{H}(t), \hat{P}\right] \tag{3.162}$$

$$i\frac{d\hat{A}}{dt} - \left[\tilde{H}(t), \hat{A}\right] = i\hat{P}\frac{d\left\langle\hat{A}\right\rangle}{dt} + i\left\langle\hat{A}\right\rangle\frac{d\hat{P}}{dt} - \left\langle\hat{A}\right\rangle\left[\tilde{H}(t), \hat{P}\right] \tag{3.163}$$

Substitution of the von Neumann equation for the projection operator $\hat{P}$ as in 3.2.26 collapses this to:

$$i\frac{d\hat{A}}{dt} - \left[\tilde{H}(t), \hat{A}\right] = i\hat{P}\frac{d\left\langle\hat{A}\right\rangle}{dt} \tag{3.164}$$

For an operator with expectation value constant in time, we must therefore have:

$$i\frac{d\hat{A}}{dt} = \left[\tilde{H}(t), \hat{A}\right] \tag{3.165}$$

as required. □

Corollary 3.2.28. *For any tracefree, independent matrices $\tilde{H}, \tilde{F} \in \mathbb{M}^{n\times n}$ given by the constraint equations 3.2.4, 3.2.6, the unitary evolution of the variable $\hat{A} = \tilde{H} + \tilde{F}$ is given by:*

$$i\frac{d}{dt}\left(\tilde{H}+\tilde{F}\right) = \left[\tilde{H}, \tilde{F}\right] \tag{3.166}$$

Proof. The operator $\hat{A} = \tilde{H} + \tilde{F}$ is Hermitian, and on the condition that the consituent operators are trace and expectation equivalent as in 3.2.21, 3.2.27 applies. See also 3.2.7 in this work for an equivalent proof. This is the quantum brachistochrone equation. □

Remark 18. If the expectation of the operator $\hat{A}$ is zero, as is the trace, it might seem that the whole formalism could collapse to a trivial equation of the form $0 = 0$. However, this is not the case. As we can see, this is a bounded optimisation, and indeed we have added constraints that are of this type 3.2.4.

We shall now show how one may derive the other major results that differ in structure between [49] and [52] on the one hand, and more advanced techniques as developed in this thesis. Recall the imposed constraint condition from [49] where the authors stated the Lagrangian variable:

Definition 3.2.29.

$$S_{c.} = i\left(\left\langle \dot{\Phi} \,|\Psi \right\rangle - \left\langle \Psi \middle| \dot{\Phi} \right\rangle\right) + \left\langle \Phi \middle| \tilde{H} \,|\Psi \right\rangle + \left\langle \Psi \middle| \tilde{H} \,|\Phi \right\rangle \tag{3.167}$$

where both $|\Psi\rangle, |\Phi\rangle \in \mathbb{CP}^{n-1}$ are complex finite state vectors.

Lemma 3.2.30. *The Euler-Lagrange equations for the action defined by 3.2.29 in the variables $\tilde{H}$ and $|\Psi\rangle$ are*

$$\frac{d}{dt}\frac{\partial S_c}{\partial \left\langle \dot{\Phi} \right|} = \frac{\partial S_c}{\partial \langle \Phi|} \Rightarrow i\frac{d}{dt}|\Psi\rangle = \tilde{H}\,|\Psi\rangle \tag{3.168}$$

and

$$\frac{\partial S_c}{\partial \tilde{H}} = |\Psi\rangle\langle\Phi| + |\Psi\rangle\langle\Phi| \tag{3.169}$$

Proof. Details are elementary using the rules of matrix differentiation as in 3.2.11. We rearrange using the cyclic trace identity and differentiate using the formula as in 3.2.11. □

Remark 19. We can see the difference in methodologies here, in that this constraint, which is not imposed in the action principle 3.2.8, but implicitly assumed in 3.2.3 and 3.2.1 appears directly inside the action principle originally appearing in [49]. This is essentially equivalent to explicit and implicit schemata for calculating the optimal solution, on the one hand we might assume the dynamics ipso facto, in the other we constrain the solution space to only solutions with this property. Both are obviously equivalent formulations of the same problem as we have shown.

In the paper [49], the authors impose the condition:

Definition 3.2.31.

$$S_{c.} = i\left(\left\langle \dot{\Phi} \,|\Psi \right\rangle - \left\langle \Psi \middle| \dot{\Phi} \right\rangle\right) + \left\langle \Phi \middle| \tilde{H} \,|\Psi \right\rangle + \left\langle \Psi \middle| \tilde{H} \,|\Phi \right\rangle \tag{3.170}$$

It is at this point, we shall depart from this analysis and demonstrate some truly novel techniques within this field of application. Our key divergence in methodology is to work in what essentially amounts to working with the unitary matrices directly as opposed to setting boundary conditions on the state as in [49,50]. We shall briefly discuss some results that hinge on our prior analysis, and their connection with the work of Kuzmak in [66]. The major aim in this work is identification of the projective systems given by the Fubini-Study metrics. In this way, we can identify the families of special functions which define the eigenfunction decompositions of the kernel. This is one form of group representation theory, in this case a simple method that allows us to cut through the difficulties of hyperbolic space and deal with the differential geometry directly. Kuzmak, in [66], has calculated the Fubini-Study metric associated to the Heisenberg group, SO(3) and an arbitrary Lie group. This method relies on unitary operators, and, importantly, does not use differential geometry to analyse the eigenfunctions. We shall discuss the primary results and show how they connect with our method as defined through 3.2.10, and some simple ways in which the findings in this research can be extended. We begin this stage of the calculation with a short review of the basic structures of projective and differential geometry that we shall require. These concepts are necessary for our development of a detailed theory of distance and measure, which we shall use to establish the solution for the hyperbolic heat kernel in later sections of this work.

Definition 3.2.32. The distance measure between any two states $|\Phi\rangle, |\Psi\rangle \in \mathbb{C}^n$ is given by the Bures measure:

$$d(\Psi,\Phi) = \cos^{-1}\left(\sqrt{\frac{\langle\Psi|\,\Phi\rangle\,\langle\Phi|\,\Psi\rangle}{\langle\Phi|\,\Phi\rangle\,\langle\Psi|\,\Psi\rangle}}\right) \tag{3.171}$$

The Fubini-Study metric ds^2_{FS} is defined on the space of states, given through the projection of the complex space $\mathbb{C}^n \to \mathbb{C}^{n-1}$ to the unit sphere in the space of states. This is given by the infinitesmal form of the Bures distance:

$$\cos^2\theta \sim \frac{1}{4}ds^2_{FS} = 1 - |\langle d\Psi\,|\Psi\,\rangle|^2 = \left\langle d\Psi \left| \left(1-\hat{P}\right) \right| d\Psi\right\rangle \tag{3.172}$$

where $\theta = d(\Psi, d\Psi)$ is the infinitesmal distance for a state, and $\hat{P} = |\Psi\,\rangle\langle\,\Psi|$ is the projection operator.

Proof. We can show how the Fubini-Study metric is derived in the following way. Taking the Bures distance as in the definition, we have:

$$d(\Psi,d\Psi) = \cos^{-1}\left(\sqrt{\frac{\langle\Psi|\,d\Psi\rangle\,\langle d\Psi|\,\Psi\rangle}{\langle d\Psi|\,d\Psi\rangle\,\langle\Psi|\,\Psi\rangle}}\right) \tag{3.173}$$

and therefore

$$\cos\theta = \sqrt{\frac{\langle\Psi|\,d\Psi\rangle\,\langle d\Psi|\,\Psi\rangle}{\langle d\Psi|\,d\Psi\rangle\,\langle\Psi|\,\Psi\rangle}} = \sqrt{\langle\Psi|\,d\Psi\rangle\,\langle d\Psi|\,\Psi\rangle} \tag{3.174}$$

where the angle is the infinitesmal distance $\theta = d(\Psi, d\Psi)$. Assuming unit normalisation $\langle d\Psi|\,d\Psi\rangle = 1, \langle\Psi|\,\Psi\rangle = 1$, we have immediately, using a Taylor series expansion and the projection operator $\hat{P} = |\Psi\,\rangle\langle\,\Psi|$:

$$\cos^2\theta \sim 1-\theta^2 = 1 - |\langle d\Psi\,|\Psi\,\rangle|^2 \tag{3.175}$$

$$= 1 - \langle d\Psi\,|\Psi\,\rangle\,\langle\Psi\,|d\Psi\,\rangle \tag{3.176}$$

$$= |\langle d\Psi\,|d\Psi\,\rangle|^2 - \langle d\Psi\,|\Psi\,\rangle\,\langle\Psi\,|d\Psi\,\rangle \tag{3.177}$$

$$= \left\langle d\Psi \left| \left(1-\hat{P}\right) \right| d\Psi\right\rangle \tag{3.178}$$

which gives for the Fubini-Study metric, the formula:

$$\cos^2\theta \sim \frac{1}{4}ds^2_{FS} = 1 - |\langle d\Psi\,|\Psi\,\rangle|^2 = \left\langle d\Psi \left| \left(1-\hat{P}\right) \right| d\Psi\right\rangle \tag{3.179}$$

as required. □

Remark 20. In our context, this means that for quantum states and their hyperbolic equivalents, we are constrained to a sphere or pseudosphere, which reduces the co-ordinates by one. This gives the necessary projection as required under the Fubini-Study metric.

Theorem 3.2.33. *The Fubini-Study metric associated with the principle of least time for the quantum brachistochrone 3.2.8 is the real part of the matrix tensor:*

$$F_{\alpha\beta} = \left\langle \bar{\Psi}_\alpha \right| \left. \Psi_\beta \right\rangle - \left\langle \bar{\Psi}_\alpha \right| \left. \Psi \right\rangle \left\langle \Psi \right| \left. \Psi_\beta \right\rangle = \left\langle \bar{\Psi}_\alpha \right| \left(\mathbf{1} - \hat{P} \right) \left| \Psi_\beta \right\rangle \tag{3.180}$$

$$\left| \psi_\beta \right\rangle = \frac{\partial}{\partial x_\beta} \left| \Psi \right\rangle \tag{3.181}$$

where $\left|\Psi\right\rangle \in \mathbb{CP}^{n-1}$ *as in 3.1.3, the projection operator* $\hat{P} = \left|\Psi\right\rangle \left\langle\Psi\right|$ *as in 3.2.19.*

Proof. We note the time part of the action principle 3.2.10, 3.2.9, 3.2.32 takes the form:

$$\mathcal{I} = \sqrt{\left\langle d\Psi \middle| \left(1 - \hat{P}\right) \middle| d\Psi \right\rangle} \tag{3.182}$$

Expanding the derivative term with respect to an underlying basis set, we may write $\left|d\Psi\right\rangle = \left|\Psi_\alpha\right\rangle dx^\alpha$, hence we have $\mathcal{I} = F_{\alpha\beta} dx^\alpha dx^\beta$, and we may calculate the Fubini-Study metric, finding:

$$F_{\alpha\beta} = \left\langle \bar{\Psi}_\alpha \right| \left. \Psi_\beta \right\rangle - \left\langle \bar{\Psi}_\alpha \right| \left. \Psi \right\rangle \left\langle \Psi \right| \left. \Psi_\beta \right\rangle = \left\langle \bar{\Psi}_\alpha \right| \left(\mathbf{1} - \hat{P}(\tau, \varphi, \psi) \right) \left. \Psi_\beta \right\rangle \tag{3.183}$$

and therefore

$$\mathfrak{Re}\left[\mathcal{I}\right] = g_{\alpha\beta} dx^\alpha dx^\beta = ds^2 \tag{3.184}$$

i.e. $g_{\alpha\beta} = \mathfrak{Re}\left[F_{\alpha\beta}\right]$. □

We shall now state the central hypothesis that we shall use to develop our theory of the state and eigenfunctions we may associate with the principles of the quantum brachistochrone and Lagrangians. The central premise is:

Theorem 3.2.34. *A Laplacian operator* ∇^2 *may be associated to the Fubini-Study metric* $g_{\alpha\beta} = \mathfrak{Re}\left[F_{\alpha\beta}\right]$*, through use of curvilinear geometry. The Laplacian is computed using the classical formula of differential geometry:*

$$\nabla^2 f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_\alpha} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial f}{\partial x_\beta} \right) \tag{3.185}$$

where $g = \det(g_{\alpha\beta})$ *is the determinant of the Fubini-Study metric tensor.*

Proof. See Canzani's notes on spectral geometry [48] for an in-depth analysis. For our purposes, we take the divergence as given by the exterior derivative of the form:

$$\operatorname{div}(\boldsymbol{X})\omega = d\omega \diamond \boldsymbol{X} + d(\omega \diamond \boldsymbol{X}) \tag{3.186}$$

where $\diamond$ is the exterior product. The volume is given by the determinant:

$$\omega = \sqrt{\det g_{\alpha\beta}} dx_1 \wedge \ldots \wedge dx_n \tag{3.187}$$

and hence the volume 1-form is constant, i.e. $d\omega = 0$. We therefore can say that the divergence may be written:

$$\operatorname{div}(\boldsymbol{X})\omega = d(\omega \diamond \boldsymbol{X}) \tag{3.188}$$

the exterior product being given by:

$$\omega \diamond \boldsymbol{X} = \sum (-1)^{j+1} X_j \sqrt{\det g_{\alpha\beta}} dx_1 \wedge ... \wedge dx_j \wedge ... \wedge dx_n \tag{3.189}$$

Computing the exterior derivative, we then have:

$$d(\omega \diamond \boldsymbol{X}) = \sum (-1)^{j+1} \frac{\partial}{\partial x_j} \left((-1)^{j+1} X_j \sqrt{\det g_{\alpha\beta}} \right) dx_1 \wedge ... \wedge dx_n \tag{3.190}$$

$$= \sum \frac{\partial}{\partial x_j} \left(X_j \sqrt{\det g_{\alpha\beta}} \right) dx_1 \wedge ... \wedge dx_n \tag{3.191}$$

$$= \sum_j \frac{1}{\sqrt{\det g_{\alpha\beta}}} \frac{\partial}{\partial x_j} \left(X_j \sqrt{\det g_{\alpha\beta}} \right) \omega \tag{3.192}$$

We therefore conclude that the divergence is given by the formula:

$$\mathrm{div}(\boldsymbol{X}) = \sum_j \frac{1}{\sqrt{\det g_{\alpha\beta}}} \frac{\partial}{\partial x_j} \left(X_j \sqrt{\det g_{\alpha\beta}} \right) \tag{3.193}$$

To prove the equivalent formula for the gradient, observe that we may write the metric in the form:

$$g_{ij} = \sum_k \frac{\partial x^k}{\partial q^i} \frac{\partial x^k}{\partial q^j} \tag{3.194}$$

and therefore the incremental change in some function dependent on this parameter set is given by:

$$\sum_{i,j} \frac{\partial f}{\partial x_i} g^{ij} \partial x_j = \sum_{i,j} \partial f g^{ij} \frac{\partial x_j}{\partial x_i} = \sum_{i,j} \partial f g^{ij} g_{ij} = df = \nabla f . d\boldsymbol{x} \tag{3.195}$$

and therefore the gradient is:

$$\nabla f = \mathrm{grad} f = \sum_i \frac{\partial f}{\partial x_i} g^{ij} \tag{3.196}$$

Combining these two relations, we then have the Laplacian:

$$\mathrm{div}(\mathrm{grad} f) = \nabla^2 f = \sum_{i,j} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_j} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x_i} \right) \tag{3.197}$$

where $g = \det g_{\alpha\beta}$ is the determinant of the metric. □

Remark 21. We shall often write the expression for the arc length or interval on the space in the form:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{3.198}$$

Note that the theory of the second fundamental form for the metric may be written in terms of this arc length as:

$$ds^2 = E du^2 + 2F du dv + G dv^2 \tag{3.199}$$

and further, this implies the symmetric metric defined through the matrix:

$$g_{ij} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \tag{3.200}$$

and surface element through $dA = \sqrt{EG - F^2} du dv$. This thesis shall show how to construct the Fubini-Study metric, and then through these relationships in differential geometry, how one may associate various sets of eigenfunctions through the Laplacian operator in curvilinear co-ordinates.

3.2.2 Differential Geometry of the Hyperboloid

The following theorems relate to the basic hyperbolic geometries used in the calculations in this thesis. The various different representations are considered in turn, we begin with a review of the basic differential geometry of the hyperboloid.

Theorem 3.2.35. *The metric in hyperbolic geometry is given by the arc length formula of the fundamental form:*

$$ds^2 = dx^2 + dy^2 - dw^2 \tag{3.201}$$

In hyperbolic coordinates this metric may be written:

$$ds^2 = -dr^2 + r^2 \sinh^2 \tau d\phi^2 + r^2 d\tau^2 \tag{3.202}$$

where $w = r\cosh\tau$*,* $x = r\sinh\tau\cos\phi$ *and* $y = r\sinh\tau\sin\phi$*.*

Proof. Let $\mathbb{H} = \left\{(x, y, w)|x^2 + y^2 - w^2 = -r^2\right\}$ be the hyperbola of two sheets, i.e. $x, y, z \in SO(2, 1)$. Take as chart the hyperbolic coordinates, where we have the parametrisation:

$$w = r\cosh\tau, x = r\sinh\tau\cos\phi, y = r\sinh\tau\sin\phi \tag{3.203}$$

The metric in the space $\mathbb{H}$ is given by $ds^2 = dx^2 + dy^2 - dw^2 = g_{\alpha\beta}dx^\alpha dx^\beta$, whence using the expressions for the hyperbolic co-ordinates, we find

$$ds^2 = -dr^2 + r^2 \sinh^2 \tau d\phi^2 + r^2 d\tau^2 \tag{3.204}$$

as required. □

Corollary 3.2.36. *For a pseudosphere of unit radius, the metric is given by:*

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = \sinh^2 \tau d\phi^2 + d\tau^2 \tag{3.205}$$

Proof. Using 3.2.35, we take $dr = 0$, $r = 1$, obtaining the result. □

We shall now prove a number of small lemmas that relate the geometric structure of the various representations of hyperbolic spaces, given by the disk, plane and pseudosphere.

Lemma 3.2.37. *The area element for the pseudosphere of unit radius is:*

$$dA = \sinh\tau d\phi d\tau \tag{3.206}$$

where the angles are defined as in 3.2.35.

Proof. Consider the theorem of the fundamental form, for the metric

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \tag{3.207}$$

and we will have surface element $dA = \sqrt{EG - F^2}dudv$. Applying this to the metric above, we have $E = 1$, $G = \sinh^2\tau$, $F = 0$, hence we obtain

$$dA = \sqrt{\sinh^2 \tau} d\phi d\tau = \sinh\tau d\phi d\tau \tag{3.208}$$

thereby establishing the claim in the lemma. □

Remark 22. Using the notation from geometric algebra to give the surface an orientation, we have $\mathbf{n}dA = d\mathbf{A} = \sinh\tau d\phi \wedge d\tau$.

Definition 3.2.38. The stereographic projection is defined by the mapping $z = e^{i\phi}\tanh\dfrac{\tau}{2}$ of the hyperbolic plane, given by the pseudosphere geometry 3.2.36.

We shall now prove some representations of the hyperbolic plane that are related to the geometric structures encountered in this thesis.

Theorem 3.2.39. *For the mapped coordinates as in 3.2.38, the metric is given by the formula:*

$$\frac{4dz.dz^*}{\left(1-|z|^2\right)^2} = \sinh^2\tau d\phi^2 + d\tau^2 \tag{3.209}$$

Proof. Using the mapping from 3.2.38, we can calculate the infinitesmals, obtaining:

$$dz = e^{i\phi}(ud\tau + ivd\phi) \tag{3.210}$$

$$dz^* = e^{-i\phi}(ud\tau - ivd\phi) \tag{3.211}$$

where $u = \frac{1}{2}(1-\tanh^2\frac{\tau}{2})$, $v = \tanh\frac{\tau}{2}$. Computing $dz.dz^* = |dz|^2$ we find

$$dz.dz^* = v^2 d\phi^2 + u^2 d\tau^2 \tag{3.212}$$

Noting that $|z| = \tanh\frac{\tau}{2} = v$, we have

$$dz.dz^* = |z|^2 d\phi^2 + \frac{1}{4}\left(1-|z|^2\right)^2 d\tau^2 \tag{3.213}$$

which may be rearranged to give

$$\frac{4dz.dz^*}{\left(1-|z|^2\right)^2} = \frac{4|z|^2}{\left(1-|z|^2\right)^2}d\phi^2 + d\tau^2 \tag{3.214}$$

Using the expression for $|z|$, we can write

$$\frac{|z|^2}{\left(1-|z|^2\right)^2} = \frac{\tanh^2\frac{\tau}{2}}{\left(1-\tanh^2\frac{\tau}{2}\right)^2} = \frac{\tanh^2\frac{\tau}{2}}{\operatorname{sech}^4\frac{\tau}{2}} = \sinh^2\frac{\tau}{2}\cosh^2\frac{\tau}{2} = \frac{1}{4}\sinh^2\tau \tag{3.215}$$

where we used $1-\tanh^2 x = \operatorname{sech}^2 x$. Substituting this into the expression for the metric in the complex plane, we find

$$\frac{4dz.dz^*}{\left(1-|z|^2\right)^2} = \sinh^2\tau d\phi^2 + d\tau^2 \tag{3.216}$$

as required. □

Remark 23. In real coordinates, this metric can be defined through the formula:

$$ds^2 = \frac{4(dX^2+dY^2)}{(1-(X^2+Y^2))^2} \tag{3.217}$$

which is the left-hand side of the identity given through 3.2.39.

This work shall focus on the application of the method of the principle of least time to determine sets of eigenfunctions that define metrics, similar or equivalent to the metrics described by 3.2.36, 3.2.35, 3.2.39. These eigenfunctions are given through the differential equations of the Laplacian in curved space 3.2.34, the metric being defined by the Fubini-Study projection 3.2.32.

3.2.3 Cayley Transform

The following lemmas are required in our calculations in this thesis. We shall show some simple ways in which one may apply the formula for the Poincare metric 3.2.39 to define further representations of hyperbolic space. These representations are important, as they are related to an important family of eigenfunctions as given under the Laplace operator in curved space 3.2.34.

Definition 3.2.40. The Cayley transform gives a mapping of the complex plane onto itself. For our purposes, it gives us further representations of the hyperbolic geometry. We take the coordinate $z = X + iY \in \mathbb{C}$ in the complex plane. The Cayley transform given by the stereographic projection 3.2.38 is:

$$\varrho = \frac{z+1}{i(z-1)} \tag{3.218}$$

The inverse transform is given by the formula:

$$z = \frac{\varrho - i}{\varrho + i} \tag{3.219}$$

Lemma 3.2.41. *The metric in the hyperbolic plane defined by the Cayley transform 3.2.40 is given by:*

$$ds^2 = \frac{4(dX^2 + dY^2)}{(1-(X^2+Y^2))^2} \tag{3.220}$$

or

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \tag{3.221}$$

These are two forms of the metric associated to the hyperbolic plane. In the first, this is given the raw solution for the hyperbolic metric as given through co-ordinates $z = X + iY$. The second is derived from the transformed co-ordinates under the Cayley transform $\rho = x + iy$.

Proof. We begin with the expression for the Cayley transform 3.2.40. Expanding in real and imaginary parts, we may write:

$$X + iY = \frac{x + iy - i}{x + iy + i} = \frac{[x^2 + (y^2 - 1) - 2ix]}{x^2 + (y+1)^2} \tag{3.222}$$

from which we obtain

$$X = \frac{[x^2 + (y^2 - 1)]}{x^2 + (y+1)^2} \tag{3.223}$$

$$Y = \frac{-2x}{x^2 + (y+1)^2} \tag{3.224}$$

We have shown in the previous calculation 3.2.39 that

$$ds^2 = \frac{4(dX^2 + dY^2)}{(1-(X^2+Y^2))^2} \tag{3.225}$$

Translating this into the variables (X, Y), we have the expression:

$$dX^2 + dY^2 = \left(\frac{\partial X}{\partial x}dx + \frac{\partial X}{\partial y}dy\right)^2 + \left(\frac{\partial Y}{\partial x}dx + \frac{\partial Y}{\partial y}dy\right)^2 \tag{3.226}$$

$$= \frac{4(dx^2 + dy^2)}{(x^2 + (1+y)^2)^2} \tag{3.227}$$

$$(1 - (X^2 + Y^2))^2 = \frac{16y^2}{(x^2 + (1+y)^2)^2} \tag{3.228}$$

and therefore we have:

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \tag{3.229}$$

as stated in the lemma. □

We shall now show how this form of the metric can be used in conjunction with the theory of the Laplace operator in a curved space to derive the basic equations of hyperbolic diffusion theory. Note that we have the formula provided by 3.2.34, and we may read off the coordinates directly from the hyperbolic metric as in 3.2.41.

Theorem 3.2.42. *The metric associated to the hyperbolic plane, with geometry as in 3.2.41 is:*

$$g_{\alpha\beta} = \begin{bmatrix} y^{-2} & 0 \\ 0 & y^{-2} \end{bmatrix} \tag{3.230}$$

Proof. As obtained in 3.2.41, we have:

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \tag{3.231}$$

Using the equation for the metric, we may write:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{3.232}$$

the result follows from identification of the coefficients. □

Corollary 3.2.43. *The Laplacian on the hyperbolic geometry is:*

$$\nabla^2 f = y^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \tag{3.233}$$

Proof. For the metric

$$g_{\alpha\beta} = \begin{bmatrix} y^{-2} & 0 \\ 0 & y^{-2} \end{bmatrix} \tag{3.234}$$

we can construct the Laplace operator through

$$\nabla^2 f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_\alpha} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial f}{\partial x_\beta} \right) \tag{3.235}$$

The determinant is readily evaluated:

$$g = \det g_{\alpha\beta} = \frac{1}{y^4} \tag{3.236}$$

$$\nabla^2 f = y^2 \frac{\partial}{\partial x_\alpha} \left(\frac{1}{y^2} g^{\alpha\beta} \frac{\partial f}{\partial x_\beta} \right) \tag{3.237}$$

and hence we find the contravariant metric:

$$g_{\alpha\beta} = \begin{bmatrix} y^2 & 0 \\ 0 & y^2 \end{bmatrix} \tag{3.238}$$

Calculating the Laplacian, we find:

$$\nabla^2 f = y^2 \frac{\partial}{\partial x_\alpha} \left(\delta_{\alpha\beta} \frac{\partial f}{\partial x_\beta} \right) = y^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \tag{3.239}$$

□

Remark 24. This metric and related Laplacian is an important part of our analysis of the hyperbolic plane, and will be used to calculate the hyperbolic heat kernel in the following chapter. We note that this appears in Davies [1], as a summary analysis of the hyperbolic plane, and in Bump [43], where the author employs a similar Laplacian with an added magnetic potential to derive Maass forms and other special functions related to automorphic forms.

3.3 Kernels, Groups and Positive Definite Functions

We shall now prove some basic theorems in the theory of kernels. The function referred to as a kernel in this thesis is a distance preserving measure that represents the structure of the group. Generally the system, which is described by the algebra or differential geometry, will obey partial differential equations that describe the dynamics, which in this work shall be of the heat, wave or Euler-Poisson-Darboux type. The kernel is a special solution to these PDEs, with particular properties at the initial condition. Kernels are special amongst the set of solutions, as they give completeness relationships for the state. As Mercer [67] and Stone [68] showed in their work, there is a deep connection between the group structure and the types of positive definite functions that can be associated with them. This is a principal result we shall exploit to find the hyperbolic heat kernel. The following subsections relate to the derivation of the group laws that give positive definite functions. In particular, we give some simple proofs of Stone's theorem and Mercer's theorem, and further analyse the group structure through a strict consideration of the convolution laws, kernel structure equations and the theory of spherical functions. A major part of the new results that follow shall utilise these methods, in tandem with the calculations we have carried out to establish the appropriate differential geometry. Our aim and objective shall be to find the correct way to find the kernel given by the quantum brachistochrone in the hyperbolic geometry. This will form the basis of our identification of the hyperbolic heat kernel and serve as a primary example of this technique.

3.3.1 Theory of Positive Definite Functions

We shall begin with a brief review of Stone's and Mercer's theorem, see the original papers in [67,68] for reference, and Sugiura [110] for a modern review of Stone's theorem. These two theorems form an important milestone in developing the analysis of the connections between kernels, groups and positive definite functions. As the following calculations show, kernels preserve the structure of the group through the regular representations. We emphasise the various forms of functional analysis that flow from these results; in particular, knowledge of the group then implies the existence of completeness relationships via 0.1.4. We show that the kernel, shift operators, positive definite functions and projective unitary operators are all mutually interchangeable concepts.

3.3.1.1 Stone's Theorem and Positive Definite Functions

We shall discuss some results covered in Sugiura [110] that are of use in understanding the following proofs in a group-theoretic setting. In [110], the author discusses Stone's theorem [68] in the context of positive functions.

Notation 1. In the following definitions and lemmas, we shall denote f, g as continuous functions that map groups to scalars. We shall denote an arbitrary group by the character G, and the measure space associated with this group shall be written as $d\mathfrak{m}$. A non-commutative element will be labelled in bold to emphasise that the group element appears as the argument of a function in this context, we shall write group elements as $\mathbf{g}$. When we require indexation of the group elements, we shall write $\mathbf{g}_i$ to denote a particular group element in order to write summation formulae and so on. On the integral space of f, g, the group element is given by $\mathbf{k}$, which serves as a dummy variable to label the integration over the measure. In the following, the variables $\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{k}$ are non-commutative elements of the group G.

Definition 3.3.1. A continuous function on a group is called positive definite if for any finite set of elements $\mathbf{g}_i \in G$ and any real numbers $\xi_1, .., \xi_n \in \mathbb{R}$ we have the inequality:

$$\sum_{i,j=1}^{n} \varphi\left(\mathbf{g}_i^{-1}\mathbf{g}_j\right)\xi_i\xi_j \geq 0 \tag{3.240}$$

as given in [110].

Remark 25. We note that this is strictly true for sets of discrete eigenfunctions and eigenvalues. In the case where there is only a single continuous eigenvalue, these expressions will generalise with replacement of sums by integrals, or in the case of an imiscid spectrum as in 3.1.17, the addition of an integral part. Also, depending on the nature of the inner product defined through:

$$\sum_{i=1}^{n} \xi_i\xi_i = \langle \xi, \xi \rangle \tag{3.241}$$

we can have different definitions for positivity. For example, we can have the complex equivalent:

$$\sum_{i,j=1}^{n} \varphi\left(\mathbf{g}_i^{-1}\mathbf{g}_j\right)\xi_i^*\xi_j \geq 0 \tag{3.242}$$

with inner product:

$$\sum_{j=1}^{n} \xi_j^*\xi_j = \langle \xi, \xi \rangle \tag{3.243}$$

where we have complex numbers $\xi_1, .., \xi_n \in \mathbb{C}$. A group-theoretic definition of a positive definite function in a modern context is presented in Helgason, where it is stated that a positive definite function is such that for all possible choices of group elements $\mathbf{x}_i, \mathbf{x}_j$ in G, we have:

$$\sum_{j=1}^{n} \phi\left(\mathbf{x}_i^{-1}\mathbf{x}_j\right)\xi_j^*\xi_i \geq 0 \tag{3.244}$$

where $\xi_1, .., \xi_n \in \mathbb{C}$. All of these definitions are highly dependent on correct selection of the inner product. Helgason further gives the identities for a positive definite function of Hermitian type $\phi(\mathbf{1}) = \phi(\delta_{ij}) \geq 0$, $\phi\left(\mathbf{x}^{-1}\right) = [\phi(\mathbf{x})]^*$, $|\phi(\mathbf{x})| \leq |\phi(\mathbf{1})|$. As we shall see, the second of these identities can be used to define a group homomorphism, which we can use to define the kernel in a different way which we consider later in this section.

Definition 3.3.2. The left regular representation $\pi(.)$ of G is given by:

$$\left(\pi(\mathbf{u})f\right)(\mathbf{v}) = f(\mathbf{u}^{-1}\mathbf{v}) \tag{3.245}$$

where $f(.)$ is a function that accepts group members as an argument. A representation is a homomorphism from the group to a family of operators on a vector space. We consider a function given by the convolution:

$$\varphi(\mathbf{x}) = (f^* \star f)(\mathbf{x}) \tag{3.246}$$

where the conjugate function on a group is defined through $f^*(\mathbf{u}) = \bar{f}(\mathbf{u}^{-1})$.

Remark 26. See e.g. [110] for a modern application of these techniques in a topological group setting. In the following, we replicate the principal points of analysis in order to derive Stone's theorem, i.e. the connection between positive definite functions, group representations and the convolution theory of groups. Our aim in doing so is to work up our understanding of the functions that preserve group structure, through representation formulae, and then use this to establish the analytic regime which governs special functions. This will act as a guiding analytic umbrella under which we will formulate an complementary mode using the quantum brachistochrone principle 3.2.10, the Lagrangian in a curvilinear space 3.2.34, and the Fubini-Study metric 3.2.32. We shall directly implement the convolution theory shown below in order to derive the hyperbolic heat kernel in the following chapter, and use these methods to understand further aspects of spherical functions and other special representations of groups.

Definition 3.3.3. The left convolution product is given by:

$$(f \star g)(\mathbf{x}) = \int_G f(\mathbf{x}\mathbf{k}^{-1})g(\mathbf{k})d\mathfrak{m} \tag{3.247}$$

The corresponding definition of the right convolution is equal to:

$$(f \star g)(\mathbf{x}) = \int_G f(\mathbf{k}^{-1})g(\mathbf{k}\mathbf{x})d\mathfrak{m} \tag{3.248}$$

Remark 27. See Dieudonne, also Godement for earlier analysis [70,71] for an extensive set of formulae that cover this basic topic in group theory of spherical functions. The reference text of Chavel in Riemannian geometry and Laplacians in curved space also contains reference to similar calculations, see [11] for further details.

Definition 3.3.4. The right convolution product as in 3.3.3 of a function f with itself is:

$$\varphi(\mathbf{x}) = (f^* \star f)(\mathbf{x}) = \int_G \bar{f}(\mathbf{k})f(\mathbf{k}\mathbf{x})d\mathfrak{m} \tag{3.249}$$

where $\mathfrak{m}$ is the measure of the group.

We shall now review a modern proof of Stone's theorem (see [110]), a known result in the theory of positive definite kernels that we shall utilise to understand connections between the theory of unitary operators, convolution products and positive definite functions. This can be seen as one entry point into the broader theory of group representations.

Theorem 3.3.5. *Assume the existence of a function $f(.)$ which accepts arguments from a group, and further a convolution product of this function with itself via the relations above. Any convolution product may be associated with a kernel $\varphi(.)$ via the inner product:*

$$\varphi(\mathbf{x}) = (f^* \star f)(\mathbf{x}) = \left\langle \hat{U}_{\mathbf{x}} f, f \right\rangle \tag{3.250}$$

where the convolution $\star$ is given by 3.3.3, the group element is represented by $\mathbf{x}$, a unitary operator by $\hat{U}_{\mathbf{x}}$, acting on f, and the inner product by the brackets $\langle .,. \rangle$. This kernel is positive definite.

Proof. We shall take the modern proof from [110]. Sugiura takes the definition of a positive definite operator via the inner product homomorphism:

$$\varphi(\mathbf{x}) = (f^* \star f)(\mathbf{x}) = \left\langle \hat{U}_{\mathbf{x}} f, f \right\rangle \tag{3.251}$$

As unitary transformations are length preserving, it shall be positive definite, as lengths are positive definite. To show this, we take the following argument adapted from [110]:

$$\sum_{i,j=1}^{n} \varphi\left(\mathbf{g}_i^{-1}\mathbf{g}_j\right)\xi_i\xi_j = \sum_{i,j=1}^{n}\left\langle \hat{U}_{\mathbf{g}_i^{-1}\mathbf{g}_j} f, f\right\rangle \xi_i\xi_j \tag{3.252}$$

$$= \sum_{i,j=1}^{n}\left\langle \hat{U}\mathbf{g}_i^{-1}\mathbf{g}_j f, f\right\rangle \xi_i\xi_j = \sum_{i,j=1}^{n}\left\langle \xi_j\hat{U}\mathbf{g}_j f, \xi_i\hat{U}\mathbf{g}_i f\right\rangle \tag{3.253}$$

$$= \left\langle \sum_{j=1}^{n}\xi_j\hat{U}\mathbf{g}_j f, \sum_{i=1}^{n}\xi_i\hat{U}\mathbf{g}_i f\right\rangle = \sum_{j=1}^{n}\left|\xi_j\hat{U}\mathbf{g}_j f\right|^2 \geq 0 \tag{3.254}$$

as required. The middle step follows as a consequence of length preservation, a general property of unitary operators. Note that one must carefully identify the necessary inner product structure in order to characterise the nature of the eigenvectors. If the space is unitary/Hermitian, one can rely on a standard Hilbert space inner product, but in this thesis we shall encounter different types of systems which do not share all the same properties. In this instance, we have used reality of the coefficients, but is simple to see that using the correct modification of the inner product structure to deal with complex coefficients one can recover essentially the same result. □

Remark 28. This simple discussion illustrates the deep connections between the theory of convolutions, positive definite functions and unitary transformations. It is this insight that we shall apply in order to resolve the correct representations associated to the hyperbolic plane, through the differential geometry and Laplacian. The function $\varphi(.)$ shall be shown to be equal to another positive definite function, which we shall use as our basic definition of a positive definite symmetric kernel.

3.3.1.2 Mercer's Theory of Positive Definite Kernels

We shall now prove a basic result that is known from the theory of positive symmetric functions. This will be a modern retake of Mercer's theorem (c. 1909) [67], which states the following:

Theorem 3.3.6. *Assume the existence of an operator $\hat{T}$, satisfying the integral identity:*

$$u(x,t) = \hat{T}f = \int K(x,y;t)f(y)dy \tag{3.255}$$

with boundary condition $u(x,0) = f(x)$, satisfying the PDE:

$$\frac{\partial u}{\partial t} = \mathcal{L}(x,\partial_x)u \tag{3.256}$$

Then we have the following:

$$K(x,y;0) = \delta(y-x) \tag{3.257}$$

$$K(x,y;0) = \sum_n \phi_n^*(x)\phi_n(y) = \sum_n \phi_n^*(y)\phi_n(x) = K(y,x;0) \tag{3.258}$$

where $\phi_n(x)$ is the eigenfunction of the operator $\mathcal{L}(x,\partial_x)$, with eigenvalue $-\lambda_n$. The function inside the integral, which we call the kernel $K(x,y;t)$ satisfies an identical PDE, with different boundary conditions:

$$K(x,y;t) = \sum_n e^{-\lambda_n t}\phi_n^*(y)\phi_n(x) \tag{3.259}$$

and

$$K(x,y;0) = \delta(y-x) \tag{3.260}$$

as before. This function may be used to define a new solution to the PDE via $\xi = -\dfrac{\partial K}{\partial t}$, *given by the spectral expansion:*

$$\xi(x,y;0) = \xi(x,y) = \sum_n \lambda_n \phi_n^*(y)\phi_n(x) \tag{3.261}$$

which satisfies the positivity condition:

$$\int\int \theta(x)\xi(x,y)\theta(y)dxdy \geq 0 \tag{3.262}$$

for all suitably continuous functions $\theta(.)$ *with expansions on the set of real valued eigenfunctions with positive eigenvalues.*

Proof. We will recall a modern proof of this known fact, as it illustrates clearly the connection between spectral analysis, completeness and orthogonality relationships. Taking the precepts of the theorem, we have the integral operator:

$$u(x,t) = \hat{T}f = \int K(x,y;t)f(y)dy \tag{3.263}$$

which collapses to $u(x,0) = f(x) = \int K(x,y;0)f(y)dy$ for $t = 0$. We therefore find immediately that $K(x,y;0) = \delta(y-x)$, and using the symmetry property of the delta function, we must have $K(x,y;0) = K(y,x;0)$, which means that initially, this kernel is symmetric. We prove the spectral expansion in the following way. We have the orthogonality relationship assumed for the eigenfunctions:

$$\int \phi_n(x)\phi_m^*(x)dx = \delta_{nm} \tag{3.264}$$

which satisfy the operator equation $\mathcal{L}(x,\partial_x)\phi_n(x) = -\lambda_n\phi_n(x)$. Further, the boundary condition may be expanded as:

$$g(x) = \sum_n c_n\phi_n(x) \tag{3.265}$$

Inverting the transformation, and using the orthogonality relationship, we have:

$$c_n = \int g(x)\phi_n^*(x)dx \tag{3.266}$$

Consider now the specific case given by the function:

$$g(x) = \delta(x-y) = \delta(y-x) \tag{3.267}$$

where we have used the even-ness property of the delta function. We have then the expansion:

$$c_n = \int \delta(x-y)\phi_n^*(x)dx = \phi_n^*(y) \tag{3.268}$$

and therefore $\delta(x-y) = \sum_n c_n\phi_n(x) = \sum_n \phi_n^*(y)\phi_n(x) = K(y,x;0)$. Using once more the even-ness property of the delta function, we obtain the symmetric kernel in the form:

$$K(x,y;0) = \sum_n \phi_n^*(x)\phi_n(y) = \sum_n \phi_n^*(y)\phi_n(x) = K(y,x;0) \tag{3.269}$$

Note we recover the solution to the PDE via:

$$u(x,0) = f(x) = \int \sum_n \phi_n^*(y)\phi_n(x)f(y)dy \tag{3.270}$$

$$= \sum_n \phi_n(x) \int \phi_n^*(y)f(y)dy \tag{3.271}$$

$$= \sum_n c_n \phi_n(x) \tag{3.272}$$

We now derive the time-dependent solution to the PDE, using the kernel. Writing the eigenfunction equation, we have $\mathcal{L}(x,\partial_x)\phi_n(x) = -\lambda_n\phi_n(x)$, hence:

$$\mathcal{L}(x,\partial_x)\left[e^{-\lambda_n t}\phi_n(x)\right] = -\lambda_n e^{-\lambda_n t}\phi_n(x) = \left(\frac{\partial}{\partial t}\right)\left[e^{-\lambda_n t}\phi_n(x)\right] \tag{3.273}$$

and we can write the spectral expansion for the kernel as:

$$K(x,y;t) = \sum_n e^{-\lambda_n t}\phi_n^*(y)\phi_n(x) \tag{3.274}$$

which obviously satisfies:

$$\frac{\partial K}{\partial t} = \mathcal{L}(x,\partial_x)K \tag{3.275}$$

with delta function initial condition.

To recover the final part of the theorem, we note that by differentiating with respect to time, and observing that the operator $\mathcal{L}(x,\partial_x)$ is independent of time by assumption and therefore is commutative with the time derivative, then a further solution is given by:

$$\xi = -\frac{\partial K}{\partial t} \tag{3.276}$$

which satisfies, again, the PDE $\frac{\partial \xi}{\partial t} = \mathcal{L}(x,\partial_x)\xi$. Writing the spectral expansion, which we may derive from the kernel $K(x,y;t)$, we obtain:

$$\xi = \xi(x,y;t) = \sum_n \lambda_n e^{-\lambda_n t}\phi_n^*(y)\phi_n(x) \tag{3.277}$$

and hence taking $t=0$, we find:

$$\xi(x,y;0) = \xi(x,y) = \sum_n \lambda_n \phi_n^*(y)\phi_n(x) \tag{3.278}$$

To prove positive definiteness, it is straightforward to show that we have $\xi(x,x) = \sum_n \lambda_n|\phi_n(x)|^2 \geq 0$, and also as $|\phi_n(x)|^2 \geq 0$, we must have that the eigenvalues are positive definite $\lambda_n \geq 0$. As in the theorem, we then take the special case of real-valued eigenfunctions with positive definite eigenvalues, in which case $\phi_n^*(x) = \phi_n(x)$ and $\xi(x,y) = \sum_n \lambda_n\phi_n(y)\phi_n(x)$. Evaluating the double integral, we may write:

$$\int\int \theta(x)\xi(x,y)\theta(y)dxdy = \int\int \sum_{k,k',n} c_k\phi_k(x)\lambda_n\phi_n(y)\phi_n(x)c_{k'}\phi_{k'}(y)dxdy \tag{3.279}$$

$$= \sum_{k,k',n} c_k c_{k'}\lambda_n \int \phi_n(y)\phi_{k'}(y)dy \int \phi_k(x)\phi_n(x)dx \tag{3.280}$$

$$= \sum_{k,k',n} c_k c_{k'} \lambda_n \delta_{nk'} \delta_{nk} \tag{3.281}$$

$$= \sum_n |c_n|^2 \lambda_n \tag{3.282}$$

So, we therefore conclude that:

$$\int \int \theta(x)\xi(x,y)\theta(y)dxdy = \sum_n |c_n|^2 \lambda_n \geq 0 \tag{3.283}$$

which is true given the positivity of the modulus squared of the coefficients $|c_n|^2 \geq 0$ (as the coefficients and eigenfunctions are real), and the positivity of the eigenvalues $\lambda_n \geq 0$. We have expanded as $\theta(x) = \sum_k c_k \phi_k(x)$ in the eigenfunction basis. This concludes the proof.

□

Remark 29. We note that this is a simple introductory proof of this theorem, and many other ways exist to prove this result, for systems with e.g. complex valued eigenfunctions, or negative definite spectra the theorem may be modified accordingly.

There is a close analogy here between the calculus of group representations as given by Stone's theorem 3.3.5 and Mercer's theorem. In many ways these are overlapping concepts, that point at the decomposition of the kernel, and eigenfunction expansions. Mercer's theorem differs from the direct approach of Stone's theorem in that, instead of assuming a unitary representation via the inner product, one proceeds directly from the kernel representation as a decomposition over the eigenfunctions. In the following simple calculation, we shall show that very basic assumptions on the nature of functions which act on the group level are sufficient to guarantee that this function is positive definite. This is a necessary property that we shall require in order to describe the kernel; as it is a transition probability density, it must be positive, bounded and integrate out to unity. In establishing the positivity of certain functions, which arise from Stone's theorem in the convolution context, we wish to emphasise that in many systems, including that described by the heat equation, this is a necessary condition on the solution space for any real problem.

Notation 2. In the following, the group members are denoted $\mathbf{u}, \mathbf{v}$, and are generally non-commutative as in the previous section regarding Stone's theorem. We write $\psi(\mathbf{u})$ as an eigenfunction, which accepts group elements as argument. In the same sense that we can write Mercer's decomposition of a positive definite kernel as:

$$K(x,y) = \sum_n \lambda_n \psi_n(y)\psi_n(x) = \langle \psi(x), \psi(y) \rangle \tag{3.284}$$

we say that this is the eigenfunction, under the inner product bracket.

Definition 3.3.7. A unital homomorphism $\Phi(.)$ on a group G is defined by:

$$\Phi(\mathbf{u}^{-1}) = \Phi^*(\mathbf{u}) \tag{3.285}$$

where $\mathbf{u} \in G$ some element in the group, with $\Phi : G \to \mathbb{C}$. This is equivalent to 3.3.2. $\Phi(.)$ is a function which maps group elements into the complex numbers. For example, one such function is the Möbius transformation, which maps objects from the 2×2 matrices into complex numbers.

Definition 3.3.8. A group homomorphism is defined by the product law:

$$\Phi(\mathbf{gh}) = \Phi(\mathbf{g})\,\Phi(\mathbf{h}) \tag{3.286}$$

The left regular representation is defined by the group action $\pi(.)$:

$$(\pi(\mathbf{u})\Phi)(\mathbf{v}) = \Phi(\mathbf{u}^{-1}\mathbf{v}) \tag{3.287}$$

A special case of this is given by the identity:

$$\Phi(\mathbf{u}^{-1}\mathbf{v}) = \Phi\left(\mathbf{u}^{-1}\right)\Phi(\mathbf{v}) = \Phi^*(\mathbf{u})\Phi(\mathbf{v}) \tag{3.288}$$

Definition 3.3.9. It is possible to define a unity or identity element. We define this identity element by the symbol $\hat{P}$. This element can be written in terms of the unital homomorphism $\Phi(.)$:

$$\hat{P} = \Phi^*(\mathbf{u})\Phi(\mathbf{u}) \tag{3.289}$$

where $\mathbf{u}$ is the group element. This is simple to see as:

$$\hat{P} = \Phi^*(\mathbf{u})\Phi(\mathbf{u}) = \Phi(\mathbf{u}^{-1})\Phi(\mathbf{u}) = \Phi\left(\mathbf{u}^{-1}\mathbf{u}\right) = \Phi(\mathbf{1}) = 1 \tag{3.290}$$

Definition 3.3.10. Assume we have a symmetric kernel $K(\mathbf{u},\mathbf{v})$ defined by the left regular representation $\pi(.)$:

$$K(\mathbf{u},\mathbf{v}) = (\pi(\mathbf{u})F)(\mathbf{v}) = F\left(\mathbf{u}^{-1}\mathbf{v}\right) \tag{3.291}$$

where the function $F(.)$ is given by $F(\mathbf{a}) = \hat{P}\Phi(\mathbf{a})$, $\hat{P}$ the identity element as in 3.3.9. We write:

$$K(\mathbf{u},\mathbf{v}) = F\left(\mathbf{u}^{-1}\mathbf{v}\right) = \hat{P}\Phi\left(\mathbf{u}^{-1}\mathbf{v}\right) \tag{3.292}$$

We shall now prove that this kernel is positive definite, in line with Mercer's theory of positivity. This is the same form as the function used in Stone's theorem. Here we will show how this may be used in a different context to prove positivity under very basic conditions.

Theorem 3.3.11. *Under the inner product bracket $\langle , \rangle$ the kernel $K(\mathbf{u},\mathbf{v})$ is positive definite, i.e.*

$$\sum_{\mathbf{u},\mathbf{v}\in G} \left\langle F(\mathbf{u}^{-1}\mathbf{v})\psi(\mathbf{v}), \psi(\mathbf{u})\right\rangle \geq 0 \tag{3.293}$$

where the kernel has representation $K(\mathbf{u},\mathbf{v}) = (\pi(\mathbf{u})F)(\mathbf{v}) = F\left(\mathbf{u}^{-1}\mathbf{v}\right)$, and the function $F(.)$ is the projection $F(\mathbf{a}) = \hat{P}\Phi(\mathbf{a})$, $\hat{P}$ as defined in 3.3.9, the unital homomorphism $\Phi(.)$ as in 3.3.8.

Proof. We can take the kernel to be given by the projective representation $F(v) = \hat{P}\Phi(v)$, with $\hat{P}$ a projection operator:

$$F(\mathbf{u}^{-1}\mathbf{v}) = \hat{P}\Phi(\mathbf{u}^{-1}\mathbf{v}) \tag{3.294}$$

Resolving the positive definite function, we may write:

$$\sum_{\mathbf{u},\mathbf{v}\in G} \left\langle F(\mathbf{u}^{-1}\mathbf{v})\psi(\mathbf{v}), \psi(\mathbf{u})\right\rangle = \sum_{\mathbf{u},\mathbf{v}\in G} \left\langle \hat{P}\Phi(\mathbf{u}^{-1}\mathbf{v})\psi(\mathbf{v}), \psi(\mathbf{u})\right\rangle \tag{3.295}$$

Now, a projection operator may be written in the form $\hat{P} = \Phi^*(\mathbf{u})\Phi(\mathbf{u})$ 3.3.9, we also have the group homomorphic identity $\Phi(\mathbf{u}^{-1}\mathbf{v}) = \Phi\left(\mathbf{u}^{-1}\right)\Phi(\mathbf{v}) = \Phi^*(\mathbf{u})\Phi(\mathbf{v})$ 3.3.8, finding:

$$\left\langle \hat{P}\Phi(\mathbf{u}^{-1}\mathbf{v})\psi(\mathbf{v}), \psi(\mathbf{u})\right\rangle = \left\langle \Phi^*(\mathbf{u})\Phi(\mathbf{u})\Phi(\mathbf{u}^{-1})\Phi(\mathbf{v})\psi(\mathbf{v}), \psi(\mathbf{u})\right\rangle \tag{3.296}$$

$$= \left\langle \Phi(\mathbf{u})\Phi(\mathbf{u}^{-1})\Phi(\mathbf{v})\psi(\mathbf{v}), \Phi(\mathbf{u})\psi(\mathbf{u}) \right\rangle \tag{3.297}$$

$$= \langle \Phi(\mathbf{v})\psi(\mathbf{v}), \Phi(\mathbf{u})\psi(\mathbf{u}) \rangle \tag{3.298}$$

and therefore demonstrating that the function is positive definite as required:

$$\sum_{\mathbf{u},\mathbf{v}\in G} \left\langle F(\mathbf{u}^{-1}\mathbf{v})\psi(\mathbf{v}), \psi(\mathbf{u}) \right\rangle = \langle \Phi(\mathbf{v})\psi(\mathbf{v}), \Phi(\mathbf{u})\psi(\mathbf{u}) \rangle \tag{3.299}$$

$$= \left\langle \sum_{\mathbf{v}\in G} \Phi(\mathbf{v})\psi(\mathbf{v}) \, , \sum_{\mathbf{u}\in G} \Phi(\mathbf{u})\psi(\mathbf{u}) \right\rangle \geq 0 \tag{3.300}$$

We have used the property that the group representation be faithful, i.e. it is injective, and one-to-one, and therefore it maps identity to identity via $\Phi\left(\mathbf{u}\right)\Phi(\mathbf{u}^{-1}) = \Phi(\mathbf{1}) = \mathbf{1}$. This further implies that the kernel has structure:

$$K(\mathbf{u},\mathbf{v}) = \left(\pi(\mathbf{u})F\right)(\mathbf{v}) = F\left(\mathbf{u}^{-1}\mathbf{v}\right) = \Phi^*(\mathbf{u})\Phi\left(\mathbf{u}\right)\Phi\left(\mathbf{u}^{-1}\mathbf{v}\right) \tag{3.301}$$

$$= \Phi^*(\mathbf{u})\Phi\left(\mathbf{u}\right)\Phi(\mathbf{u}^{-1})\Phi\left(\mathbf{v}\right) = \Phi^*(\mathbf{u})\Phi\left(\mathbf{v}\right) \tag{3.302}$$

This is a somewhat familiar result, that we can see as a decoupling of the kernel solution into eigenfunctions of the group elements $\mathbf{u}, \mathbf{v}$, given by the product of the unital homomorphism of the group elements. □

In this sense, we can see the relationship between unital homomorphisms, positive definite kernels and representation theory as developing out of our precepts of what constitutes the kernel. These two theorems show the deeper connections between the unitary representations and positive definite kernels. The link is, of course, the existence of convolution structures. This is a theme on which we shall draw heavily in our analysis, using convolution as expressed through composition formulae to derive a number of kernels related to hyperbolic diffusion.

3.3.2 Distance Functions and Kernels for Hyperbolic Space

We shall now review some more advanced concepts in kernel theory that we shall require in this thesis. The reader is referred to the papers of Lenz [64,65,111] where the author outlines a method to identify kernels on the hyperbolic space related to image reconstruction. We take the major results from these papers.

Definition 3.3.12. Take the hyperbolic metric for the Poincare disk as given in the derivation 3.2.41, we write:

$$ds^2 = \frac{4(dX^2 + dY^2)}{\left(1 - (X^2 + Y^2)\right)^2} \tag{3.303}$$

Making the substitutions $dr^2 = dX^2 + dY^2$, $r^2 = X^2 + Y^2$, we arrive at the hyperbolic infinitesmal in radial form:

$$ds^2 = \frac{4dr^2}{\left(1 - r^2\right)^2} \tag{3.304}$$

or $ds = \dfrac{2dr}{1 - r^2}, r > 0$. Define now the Möbius transformation:

$$T_w(z) = \frac{z - w}{1 - \bar{w}z} \tag{3.305}$$

This maps complex numbers into the hyperbolic disk, we have the properties $T_w(w) = 0$, $|T_w(z)| = |T_z(w)|$. The hyperbolic distance to any radial point is then:

$$d(0, u) = \int_0^{|u|} ds = \int_0^{|u|} \frac{2dr}{1 - r^2} = 2\tanh^{-1}(|u|) \tag{3.306}$$

Further, as the transformation takes us to any point in the disk, we can write the distance function as:

$$d(w,z) = d(0, T_w(z)) = 2\tanh^{-1}\left(\left|\frac{z-w}{1-\bar{w}z}\right|\right) \tag{3.307}$$

where we satisfy the basic distance properties $d(w,w) = 0$, $d(w,z) = d(z,w)$.

Definition 3.3.13. The kernel, as defined in the sense of a positive definite operator on the hyperbolic disk 3.3.5, 3.3.11 depends only on the hyperbolic distance between x and y. In the following, we take only the simple example where time dependence is zero. The kernel $K(x,y)$ is given by the formula:

$$K(x,y) = K(d(x,y)) \tag{3.308}$$

where $d(x,y)$ is the hyperbolic distance as in 3.3.12.

Remark 30. This appears in Davies [1], where the author states some results for the hyperbolic plane. Although the following simple argument is only for time independent kernels, in this thesis we shall show how this readily extends to problems with explicit time dependent features. The result for the kernel as a function of the distance only follows from the property of group invariance of the distance function. The kernel is the group invariant function which is symmetric in the parameters x, y. This naturally follows from the properties of the distance function. One can see that any measure preserving, group invariant function will then be necessarily be given by the above formula.

Lemma 3.3.14. *The kernel is symmetric in x and y, i.e. $K(x,y) = K(y,x)$*

Proof. We have that the distance is symmetric in x, y as distance is an unsigned measure. Therefore $d(x,y) = d(y,x)$, which is easy to derive from 3.3.12. Inserting this into 3.3.13 we obtain the result. □

Following Lenz [64,65,111], the natural generalisation of this type of symmetric kernel is given through the kernel on a group, defined by $K(\mathbf{x},\mathbf{y})$.

Definition 3.3.15. An isotropic kernel on a group with elements $\mathbf{x}, \mathbf{y}$ obeys the invariance principle:

$$K(\hat{\Omega}\mathbf{x}, \hat{\Omega}\mathbf{y}) = K(\mathbf{x},\mathbf{y}) \tag{3.309}$$

where $\hat{\Omega}$ is an invertible (unitary) transform that operates on the group elements, $\hat{\Omega} : G \to G$.

Lemma 3.3.16. *The group invariant kernel defined through 3.3.15 depends only on the distance from the identity element of the group.*

Proof. For the particular instance where $\hat{\Omega} = \mathbf{g}^{-1}$, we have by basic substitution that:

$$K(\mathbf{x},\mathbf{y}) = K(\mathbf{x}^{-1}\mathbf{x}, \mathbf{x}^{-1}\mathbf{y}) = K(\mathbf{1}, \mathbf{x}^{-1}\mathbf{y}) = K(\mathbf{x}^{-1}\mathbf{y}, \mathbf{1}) = K(d(\mathbf{x}^{-1}\mathbf{y}, \mathbf{1})) = F(\mathbf{x}^{-1}\mathbf{y}) \tag{3.310}$$

□

Remark 31. Note the similarity in form between this presentation of the kernel and that derived in Stone's and Mercer's theorem in this work 3.3.5, 3.3.11. In particular, the functional form derived in this manner is simply the group representation as given by $\pi(\mathbf{x})$. In our case, we shall be interested in the hyperbolic geometry through the stereographic transformation. In this instance, the group elements that exist on the matrix space of unitary operators are mapped to group invariant functions that preserve the representation theory. This is the simple relationship $\hat{g}_l, \hat{g} \in SU(1,1) \to \mathfrak{su}(1,1)$.

These basic results and definitions from the theory of symmetric kernels, when coupled with the insights provided by Mercer's and Stone's theorem as in 3.3.11, 3.3.5 provide a powerful formulation for deriving the group representations associated with the unitary and pseudounitary operators. We shall now prove a simple lemma that demonstrates the connection between this theory of symmetric kernels and that of the fundamental solution. The results for the spectral theory of the heat equation give the kernel via eigenvalue decomposition as in 3.1.18. We shall take the kernel to be defined by the solution to the initial value problem.

Definition 3.3.17. The heat kernel is given by the solution to the heat equation:

$$u_t = \nabla^2 u \tag{3.311}$$

and is the special solution with initial condition given by the delta function $K(\mathbf{x}, \mathbf{y}; 0) = \delta(\mathbf{y} - \mathbf{x})$. For general initial conditions, the kernel solution satisfies:

$$u(\mathbf{x}, t) = \int K(\mathbf{x}, \mathbf{y}; t) u(\mathbf{y}, 0) d\mathfrak{m}(\mathbf{y}) = \int K(\mathbf{x}, \mathbf{y}; t) f(\mathbf{y}) d\mathfrak{m}(\mathbf{y}) \tag{3.312}$$

and the initial condition is defined through $u(\mathbf{x}, 0) = f(\mathbf{x})$, and the integration measure is given through $d\mathfrak{m}(\mathbf{y})$.

Definition 3.3.18. The kernel $K(\mathbf{x}, \mathbf{y}; 0) = K(\mathbf{x}, \mathbf{y})$ in the sense of the positive definite kernels from Stone's and Mercer's theorem 3.3.5, 3.3.11 is the solution to the Helmholtz problem:

$$K(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{y} - \mathbf{x}) \tag{3.313}$$

$$\nabla^2 K(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{y} - \mathbf{x}) \tag{3.314}$$

where we use the Dirac delta function. This is the restriction of the heat kernel solution where the time dependence is taken as t approaches zero.

From this we can immediately prove the following connection between the heat kernel solution and the symmetric, positive definite kernels from Stone's and Mercer's theorem.

Theorem 3.3.19. *Assume the initial condition is given by the eigenfunction expansion:*

$$f(\mathbf{y}) = \sum_j l_j \psi_j(\mathbf{y}) \tag{3.315}$$

where the eigenfunctions solve the Helmholtz equation:

$$\nabla^2 \psi_j(\mathbf{x}) = -E_j \psi_j(\mathbf{x}) \tag{3.316}$$

and have normalisation:

$$\int \psi_n^*(\mathbf{y}) \psi_j(\mathbf{y}) d\mathfrak{m}(\mathbf{y}) = \delta_{nj} |c_j|^2 \tag{3.317}$$

The solution to the integral equation:

$$\int K(\mathbf{x}, \mathbf{y}; 0) f(\mathbf{y}) d\mathfrak{m}(\mathbf{y}) = f(\mathbf{x}) \tag{3.318}$$

is a delta function, and can be expressed through the kernel:

$$K(\mathbf{x}, \mathbf{y}; 0) = \sum_n |c_n|^2 \psi_n^*(\mathbf{y}) \psi_n(\mathbf{x}) \tag{3.319}$$

with $|c_j|^2 = 1$. We keep the normalisation constants for convenience. The symmetric kernel as in 3.3.14 is given by the real part of the time dependent spectral decomposition with $t = 0$ 3.1.18.

$$K(\mathbf{x}, \mathbf{y}) = K(\mathbf{x}, \mathbf{y}; 0) \tag{3.320}$$

In the discrete spectrum, this is given by the eigenfunction decomposition:

$$K(\mathbf{x}, \mathbf{y}) = \sum_n |c_n|^2 \psi_n^*(\mathbf{x}) \psi_n(\mathbf{y}) \tag{3.321}$$

where the eigenfunctions are solutions of the Helmholtz equation, i.e. $-E_n\psi_n(\mathbf{x}) = \nabla^2\psi_n(\mathbf{x})$, and the c_n are normalisation constants of the eigenfunctions of the Helmholtz equation, and are of modulus unity according to the previous result of this theorem.

Proof. We begin with the proof of the delta function property. If we take the initial condition as given by the eigenfunction expansion $f(\mathbf{y}) = \sum_j l_j \psi_j(\mathbf{y})$, writing the kernel as the integral over the initial condition:

$$u(\mathbf{x}, t) = \int K(\mathbf{x}, \mathbf{y}; t) u(\mathbf{y}, 0) d\mathfrak{m}(\mathbf{y}) = \int K(\mathbf{x}, \mathbf{y}; t) f(\mathbf{y}) d\mathfrak{m}(\mathbf{y}) \tag{3.322}$$

From the eigenfunction expansion, in the discrete case:

$$K(\mathbf{x}, \mathbf{y}; 0) = \sum_n |c_n|^2 \psi_n^*(\mathbf{y}) \psi_n(\mathbf{x}) \tag{3.323}$$

with the continuous and imiscid 3.1.17 cases following as a ready generalisation of the sum to a continuous integral over the continuous eigenvalue. We therefore may write:

$$\int K(\mathbf{x}, \mathbf{y}; 0) f(\mathbf{y}) d\mathfrak{m}(\mathbf{y}) = \sum_{n,j} |c_n|^2 l_j \psi_n(\mathbf{x}) \int \psi_n^*(\mathbf{y}) \psi_j(\mathbf{y}) d\mathfrak{m}(\mathbf{y}) \tag{3.324}$$

$$= \sum_{n,l} |c_n|^2 l_j \psi_n(\mathbf{x}) \delta_{nj} = \sum_j |c_j|^2 l_j \psi_j(\mathbf{x}) \tag{3.325}$$

which implies that for $|c_j|^2 = 1$, we will have:

$$\int K(\mathbf{x}, \mathbf{y}; t) f(\mathbf{y}) d\mathfrak{m}(\mathbf{y}) = \sum_j l_j \psi_j(\mathbf{x}) = f(\mathbf{x}) \tag{3.326}$$

where we used the orthogonality condition:

$$\int \psi_n^*(\mathbf{y}) \psi_j(\mathbf{y}) d\mathfrak{m}(\mathbf{y}) = \delta_{nj} |c_j|^2 = \delta_{nj} \tag{3.327}$$

Using 3.1.18, we have:

$$K(\mathbf{x}, \mathbf{y}; t) = \int_{\Omega_c} e^{-E_p t} \phi_p^*(\mathbf{x}) \phi_p(\mathbf{y}) d\mathfrak{m}(p) + \sum_{\Omega_d} |c_n|^2 e^{-E_n t} \psi_n^*(\mathbf{x}) \psi_n(\mathbf{y}) \tag{3.328}$$

We shall take only the discrete part of the kernel, as the spectrum is imiscid 3.1.17 and separates into discrete and continuous parts, the continuous part follows automatically. In this case, we obtain the second part of the theorem:

$$K(\mathbf{x}, \mathbf{y}; 0) = \sum_{\Omega_d} |c_n|^2 \psi_n^*(\mathbf{x}) \psi_n(\mathbf{y}) = K(\mathbf{x}, \mathbf{y}) \tag{3.329}$$

To obtain the symmetric kernel, note that we have the relation:

$$K^*(\mathbf{y},\mathbf{x}) = \sum_{\Omega_d} |c_n|^2 (\psi_n^*(\mathbf{y})\psi_n(\mathbf{x}))^* = \sum_{\Omega_d} |c_n|^2 \psi_n^*(\mathbf{x})\psi_n(\mathbf{y}) = K(\mathbf{x},\mathbf{y}) \tag{3.330}$$

Writing the kernel as a sum of real and complex parts via

$$K(\mathbf{x},\mathbf{y}) = K_s(\mathbf{x},\mathbf{y}) + iK_{as}(\mathbf{x},\mathbf{y}) \tag{3.331}$$

Substituting, we therefore have:

$$K^*(\mathbf{y},\mathbf{x}) = K_s(\mathbf{y},\mathbf{x}) - iK_{as}(\mathbf{y},\mathbf{x}) = K_s(\mathbf{x},\mathbf{y}) + iK_{as}(\mathbf{x},\mathbf{y}) \tag{3.332}$$

which gives upon separation into real and complex parts the symmetric and antisymmetric kernels:

$$K_s(\mathbf{y},\mathbf{x}) = K_s(\mathbf{x},\mathbf{y}) \tag{3.333}$$

$$K_{as}(\mathbf{y},\mathbf{x}) = -K_{as}(\mathbf{x},\mathbf{y}) \tag{3.334}$$

We identify the symmetric part of the kernel as in 3.3.14 with the real part of the spectral decomposition and vice versa. □

Corollary 3.3.20. *There exists a completeness relation given by the symmetric kernel:*

$$f(\mathbf{x}) = \int K(\mathbf{x},\mathbf{y}) f(\mathbf{y}) d\mathfrak{m} \tag{3.335}$$

where $d\mathfrak{m}$ is the measure space of the kernel $K(\mathbf{x},\mathbf{y})$, $F(.), f(.)$ two continuous functions. This implies $K(\mathbf{x},\mathbf{y};0) = \delta(\mathbf{y}-\mathbf{x})$. If an eigenfunction expansion exists, then the kernel can be written:

$$K(\mathbf{x},\mathbf{y};0) = \sum_n \psi_n^*(\mathbf{y})\psi_n(\mathbf{x}) \tag{3.336}$$

with orthogonality relationship $\int \psi_j^*(\mathbf{x})\psi_n(\mathbf{x}) d\mathfrak{m}(\mathbf{x}) = \delta_{jn}$. This is the expansion in the discrete case, the continuous case following by replacement of sums with integrals over the continuous eigenvalue. This kernel defines an integral transform:

$$\int K_k(\mathbf{x},0;0) f(\mathbf{x}) d\mathfrak{m}(\mathbf{x}) = \int \psi_k(\mathbf{x}) f(\mathbf{x}) d\mathfrak{m}(\mathbf{x}) = F(k) \tag{3.337}$$

There exists an inverse transform:

$$f(\mathbf{x}) = \int \psi_k^*(\mathbf{x}) F(k) d\mathfrak{m}'(k) \tag{3.338}$$

under the assumption of completeness:

$$\int \psi_k^*(\mathbf{x})\psi_k(\mathbf{y}) d\mathfrak{m}'(k) = \delta(\mathbf{y}-\mathbf{x}) \tag{3.339}$$

Proof. We have the spectral expansion of the kernel in the continuous case, assuming unit normalisations we can write:

$$K(\mathbf{x},\mathbf{y};t) = \sum_n e^{-E_n t} \psi_n^*(\mathbf{y})\psi_n(\mathbf{x}) \tag{3.340}$$

$$K(\mathbf{x},\mathbf{y};0) = \sum_n \psi_n^*(\mathbf{y})\psi_n(\mathbf{x}) \tag{3.341}$$

At time equal to zero, we have the integral equation:

$$\int K(\mathbf{x},\mathbf{y};0)f(\mathbf{y})d\mathfrak{m}(\mathbf{y}) = f(\mathbf{x}) \tag{3.342}$$

which implies we have $K(\mathbf{x},\mathbf{y};0) = \delta(\mathbf{y}-\mathbf{x})$. To prove the second part of the identity, take the kernel as given by the sum in the discrete case:

$$K(\mathbf{x},\mathbf{y};0) = \sum_k \psi_k^*(\mathbf{y})\psi_k(\mathbf{x}) = \sum_k K_k(\mathbf{x},\mathbf{y};0) \tag{3.343}$$

We have the orthogonality relationship:

$$\int \psi_j^*(\mathbf{x})\psi_n(\mathbf{x})d\mathfrak{m}(\mathbf{x}) = \delta_{jn} \tag{3.344}$$

Writing the transform, which takes us into the eigenvalue spectrum in the same way that a Fourier transform takes input data into frequency space, we have:

$$\int K_k(\mathbf{x},0;0)f(\mathbf{x})d\mathfrak{m}(\mathbf{x}) = \int \psi_k(\mathbf{x})f(\mathbf{x})d\mathfrak{m}(\mathbf{x}) = F(k) \tag{3.345}$$

Now, consider the invertibility argument. We can write:

$$\int \psi_k^*(\mathbf{y})F(k)d\mathfrak{m}'(k) = \int \psi_k^*(\mathbf{x})\left(\int \psi_k(\mathbf{y})f(\mathbf{y})d\mathfrak{m}(\mathbf{y})\right)d\mathfrak{m}'(k) \tag{3.346}$$

$$= \int f(\mathbf{x})\left(\int \psi_k^*(\mathbf{x})\psi_k(\mathbf{y})d\mathfrak{m}'(k)\right)d\mathfrak{m}(\mathbf{x}) \tag{3.347}$$

$$= \int f(\mathbf{x})\delta(\mathbf{y}-\mathbf{x})d\mathfrak{m}(\mathbf{x}) = f(\mathbf{y}) \tag{3.348}$$

We therefore conclude that:

$$f(\mathbf{x}) = \int \psi_k^*(\mathbf{x})F(k)d\mathfrak{m}'(k) \tag{3.349}$$

where we used the completeness relationship:

$$\int \psi_k^*(\mathbf{x})\psi_k(\mathbf{y})d\mathfrak{m}'(k) = \delta(\mathbf{y}-\mathbf{x}) \tag{3.350}$$

and the integral is over the relevant measure in the dual space k. □

Remark 32. We can use the properties of a symmetric kernel to prove a simple form of the Chapman-Kolmogorov equation, a known result for heat kernels and diffusion processes. If we take the heat equation, we have $u_t = \nabla^2 u$, the kernel given by the integral relation:

$$u(\mathbf{x},t) = \int K(\mathbf{x},\mathbf{y};t)u(\mathbf{y},0)d\mathfrak{m} = \int K(\mathbf{x},\mathbf{y};t)f(\mathbf{y})d\mathfrak{m} \tag{3.351}$$

We have the initial condition $K(\mathbf{x},\mathbf{y};0) = K(\mathbf{x},\mathbf{y}) = \delta(\mathbf{y}-\mathbf{x})$. Composing the integral, with the initial condition given by the kernel $K(\mathbf{x},\mathbf{y})$, we find the Chapman-Kolmogorov equation: gives:

$$\int K(\mathbf{x},\mathbf{z};t)K(\mathbf{z},\mathbf{y})d\mathfrak{m}(\mathbf{z}) = \int K(\mathbf{x},\mathbf{z};t)\delta(\mathbf{y}-\mathbf{z})d\mathfrak{m}(\mathbf{z}) = K(\mathbf{x},\mathbf{y};t) \tag{3.352}$$

Although this a simplistic derivation, in this thesis we shall use similar relationships to derive product and composition laws in order to analyse the hyperbolic heat kernel.

3.3.3 Shift Theorem of the Kernel

We shall now prove several small theorems that relate to the representation theory of groups. We take as reference the seminal work of Vilenkin, [69], who analysed the representation theory of unitary groups using a theory of shift operators and invariant kernels. As the follow brief calculation shows, this is quite closely allied with the theory of positive definite functions as given through Stone's and Mercer's theorems, and directly equivalent to the theory of symmetric kernels. In many ways, these seemingly different schemata point towards a common set of operational axioms underpinning this infrastructure. In our case, this is given by the spectral theory of the kernel 3.1.18, and the diffusion problem involving the Fubini-Study metric 3.2.10 and the curved Laplacian 3.2.34. Following [69] we take the following basic definitions in the theory of shift operators and group representations. Throughout the following subsection $\mathbf{x}$ denotes a group element, which is generally non-commutative as given by bold emphasis, consistent with this section.

Definition 3.3.21. The left regular representation is defined through the action on a function:

$$(\pi(\mathbf{g})f)(\mathbf{x}) = f(\mathbf{g}^{-1}\mathbf{x}) \tag{3.353}$$

where $\mathbf{g}, \mathbf{x}$ are group elements, $\pi(.)$ is called the shift operator of the representation of the group. The action of the representation takes us from the group G into the representation space V.

Remark 33. Note that this definition stems from the basic definition of the shift operator in Fourier theory $f(x+a) = Tf(x)$. This can be seen from the equation for the Fourier transform:

$$\int [e^{i\mathbf{k}\cdot\mathbf{a}}\hat{f}(\mathbf{k})]e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k} = \int \hat{f}(\mathbf{k})e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{a})} = f(\mathbf{x}+\mathbf{a}) \tag{3.354}$$

Definition 3.3.22. An operator is called invariant if it is permutable with the shift operator, i.e. the action of the left regular representation:

$$[\pi(\mathbf{g}), \hat{A}]f(\mathbf{x}) = 0 \tag{3.355}$$

where $[,]$ is the commutation bracket, which in this case operates on a function from the left. This is equivalent to the transformation of the operator $\hat{A}$:

$$\hat{A} = [\pi(\mathbf{g})]^{-1}\hat{A}[\pi(\mathbf{g})] \tag{3.356}$$

i.e. identical to relationships for the time evolution operator and unitary transformations more generally, as in 3.1.15.

Proof. We can establish that these definitions are equivalent. Take the commutation bracket as written, we have:

$$[\pi(\mathbf{g}), \hat{A}]f(\mathbf{x}) = 0 \tag{3.357}$$

Writing out the action of the operator $\hat{A}$ and the regular representation, we may write $(\pi(\mathbf{g})\hat{A}f)(\mathbf{x}) = \hat{A}(\pi(\mathbf{g})f)(\mathbf{x})$. Inverting the operator for the representation, we have:

$$(\hat{A}f)(\mathbf{x}) = ([\pi(\mathbf{g})]^{-1}\hat{A}[\pi(\mathbf{g})]f)(\mathbf{x}) \tag{3.358}$$

which establishes that the actions of the operators are equivalent via $\hat{A} = [\pi(\mathbf{g})]^{-1}\hat{A}[\pi(\mathbf{g})]$. In this sense, commutative operators, regular representations and the theory of unitary invariance are closely related. □

To our ends, we take the symmetric kernel induced by the solution to the heat equation 3.3.19. As before, we have as in 3.3.15:

Definition 3.3.23. A kernel is called invariant under the action of the group $\mathbf{g}$ if we have the relationship:

$$K(\mathbf{gx}, \mathbf{gy}) = K(\mathbf{x}, \mathbf{y}) \tag{3.359}$$

where $\mathbf{g}$ is a group element.

We shall now prove some simple small lemmas to show how the convolution theory of Stone's and Mercer's theorem as in [67,68], can be recovered (see 3.3.11, 3.3.5 for brief calculations in this work). We recall the definition of the integral transform from the symmetric kernel 3.3.20, which we may use to define an operator as in Vilenkin [69]:

Definition 3.3.24. Let $K(\mathbf{x}, \mathbf{y})$ be an invariant kernel, and consider the integral operator $\hat{A}$, acting on a function f, with measure $\mathfrak{m}(.)$:

$$\hat{A}f(\mathbf{x}) = \int K(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathfrak{m}(\mathbf{y}) \tag{3.360}$$

Lemma 3.3.25. *The integral operator $\hat{A}$ commutes with the shift operator as given through the left regular representation $\pi(\mathbf{g})$. This operator is invariant under 3.3.22.*

Proof. This proof appears in Vilenkin [69], Thm. I.3.4.2, pp. 41. Vilenkin states "...(a)n operator is called invariant relative to the group G if it is permutable with the shift operators". We observe the following, that the group measure is invariant, and the kernel is invariant, under the assumptions of the lemma. Then we have:

$$\hat{A}f(\mathbf{x}) = \int K(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathfrak{m}(\mathbf{y}) \tag{3.361}$$

for the integral operator. Composing this with the shift operator, we write:

$$[\hat{A}\pi(\mathbf{g})]f(\mathbf{x}) = [\hat{A}]f(\mathbf{g}^{-1}\mathbf{x}) = \int K(\mathbf{x}, \mathbf{y}) f(\mathbf{g}^{-1}\mathbf{y}) d\mathfrak{m}(\mathbf{y}) \tag{3.362}$$

Now, assume we have some element from the group, and changing the integration variable, we have $\mathbf{g}^{-1}\mathbf{y} = \mathbf{z}, \mathbf{y} = \mathbf{gz}$. Therefore the composition of the integral operator with the shift gives:

$$[\hat{A}\pi(\mathbf{g})]f(\mathbf{x}) = \int K(\mathbf{x}, \mathbf{gz}) f(\mathbf{z}) d\mathfrak{m}(\mathbf{gz}) \tag{3.363}$$

Using the group invariance property of the kernel, as assumed in the lemma, we have:

$$K(\mathbf{x}, \mathbf{gz}) = K(\mathbf{g}^{-1}\mathbf{x}, \mathbf{g}^{-1}\mathbf{gz}) = K(\mathbf{g}^{-1}\mathbf{x}, \mathbf{z}) \tag{3.364}$$

where the measure is also invariant:

$$d\mathfrak{m}(\mathbf{gz}) = d\mathfrak{m}(\mathbf{z}) \tag{3.365}$$

Then we have:

$$[\hat{A}\pi(\mathbf{g})]f(\mathbf{x}) = \int K(\mathbf{g}^{-1}\mathbf{x}, \mathbf{z}) f(\mathbf{z}) d\mathfrak{m}(\mathbf{z}) \tag{3.366}$$

Calculating the other side of the identity, we find:

$$[\pi(\mathbf{g})\hat{A}]f(\mathbf{x}) = \pi(\mathbf{g}) \int K(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathfrak{m}(\mathbf{y}) \tag{3.367}$$

$$= \int \pi(\mathbf{g}) K(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathfrak{m}(\mathbf{y}) \tag{3.368}$$

$$= \int K(\mathbf{g}^{-1}\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathfrak{m}(\mathbf{y}) \tag{3.369}$$

which is equal to the integral we previously derived. The lemma is proved. $\square$

Under the conditions of Mercer [67], any positive definite kernel defines a homomorphism through the action of the group as in 3.3.7, 3.3.8, 3.3.11. This is stated in Vilenkin [69] as the following group law for the kernel.

Lemma 3.3.26. *Assume that the kernel is positive definite, in the sense of Mercer and Stone's theorem 3.3.2 and 3.3.5. We have shown that any positive definite kernel may be written in terms of a function* $K(\mathbf{x},\mathbf{y}) = (\pi(\mathbf{y})F)(\mathbf{x}) = F\left(\mathbf{y}^{-1}\mathbf{x}\right)$*, under the conditions of positivity of this function on the group. Assume further that this function is Hermitian,* $F\left(\mathbf{g}^{-1}\right) = (F(\mathbf{g}))^*$*. Then the kernel possesses the symmetry:*

$$K(\mathbf{x},\mathbf{y}) = K^*(\mathbf{y},\mathbf{x}) \tag{3.370}$$

The symmetric part of the kernel is real.

Proof. Note that this appears as a result in Vilenkin [69], Thm. I.4.11, pp. 47. We have:

$$K^*(\mathbf{x},\mathbf{y}) = \left[F\left(\mathbf{y}^{-1}\mathbf{x}\right)\right]^* = F\left(\left[\mathbf{y}^{-1}\mathbf{x}\right]^{-1}\right) = F\left(\mathbf{x}^{-1}\mathbf{y}\right) = K(\mathbf{y},\mathbf{x}) \tag{3.371}$$

The result follows. We have used the Hermitian symmetry of the function. To see that the symmetric part is real, we must have $K(\mathbf{y},\mathbf{x}) = K(\mathbf{x},\mathbf{y})$ in this case, this implies that $K^*(\mathbf{x},\mathbf{y}) = K(\mathbf{x},\mathbf{y})$. □

Remark 34. The intersection between the theories of functions operating on groups, positive definite kernels and eigenvalue decompositions is where this thesis aims to make a contribution. As we can see, these are technically involved idioms, where it is not always clear which approach is the optimum way in which to find the kernel. We can see that there is a clear link between the structure of the functions which define the kernel, positive definite operators which exist on groups,and the various representations one can construct through the action of the shift operator. Analysis of commutative group structure and construction of representations is one well known route for resolving this difficulty. In this thesis, we shall take these concepts as our basic ideas, but using some more advanced techniques, we will sidestep many of the difficulties that are associated with the conventional approach.

Other related notions of commutativity of functions acting on operators include that of central functions, and centralisers more generally.

Definition 3.3.27. A central function $f(.)$ on a group G satisfies:

$$f(\mathbf{gh}) = f(\mathbf{hg}) \tag{3.372}$$

or

$$f(\mathbf{g}) = f(\mathbf{h}^{-1}\mathbf{gh}) \tag{3.373}$$

where $\mathbf{g},\mathbf{h}$ are two elements in the group.

Remark 35. The second form is reached by inserting the element $\mathbf{g}' = \mathbf{h}^{-1}\mathbf{g}$ into the first identity. This is identical to the character equation:

$$\chi(\mathbf{g}) = \chi(\mathbf{h}^{-1}\mathbf{gh}) \tag{3.374}$$

meaning that any central function may be derived as a sum over the character elements:

$$f(\mathbf{g}) = \sum c_a\chi_a(\mathbf{g}) \tag{3.375}$$

These are the basic concepts in the theory of group representations and symmetric kernels required in the calculation of the hyperbolic heat kernel. We note the connection given through the group representation equation 3.3.26 are equivalent to the positive definite decomposition over the eigenfunctions as in Stone's or Mercer's theorem 3.3.5, 3.3.11, or the distance theory of the hyperbolic kernel 3.3.13. This offers major insight into some ways in which we can construct the hyperbolic heat kernel directly using the group theory of the hyperbolic space. The group theory in this method is not, however, specified. It is exogenous to the theory of the kernel, we are only able to construct the kernel once it is determined. For that purpose, we shall use the constructions of the quantum brachistochrone to find the metric and unitary operators. We shall now show a further representation of the theory of functions on groups that relates to the composition formulae and special function theory. This will form our response to the questions of completeness and orthonormality posed by Feynman in our thesis problem 0.1.4.

3.3.4 Spherical Functions

We wish to use some deep properties of spherical integration theory to analyse the nature of the representations of special functions, and how this relates to the structure of the groups that generate them. Some useful references are contained in [56,70,71,112–114]. In [71], the authors outline a mechanism by which one can associate integral measure theory with certain decomposition laws over functions which are zonal, i.e. have a spherical representation. Further, in Dieudonne [70], the Gelfand pairs of transforms are analysed from this perspective. We also note the discussions contained in the works of Sousa and Yakubovich [37,115] contain many similar questions in analysis, in particular the convolution formula associated to the index Whittaker function. We take our reference point in the following brief calculations as the group homomorphism, given generally by the Fourier convolution:

Theorem 3.3.28.

$$\mathcal{F}[f \star g] = \mathcal{F}[f]\mathcal{F}[g] \tag{3.376}$$

where $\mathcal{F}$ is the Fourier transform.

Notation 3. We write this symbolically as

$$\mathcal{F}[f \star g] = \hat{f}.\hat{g} \tag{3.377}$$

$$\hat{f} = \mathcal{F}[f] \tag{3.378}$$

$$\hat{g} = \mathcal{F}[g] \tag{3.379}$$

We shall prove a number of small lemmas that relate to the convolution theory of special functions. Our aim is to build this into a coherent description of the spherical identities that must exist under the group representation theory. We wish to apply this to the eigenfunctions and build out a theory of the convolution as it applies to the various systems of equations we are familiar with as given through the theory of the hyperbolic kernel as in the previous subsections. We shall begin with a simple example as given in Godement [71], the following definitions shall apply in force throughout this subsection.

Definition 3.3.29. Let G be a connected Lie group. A harmonic function $f()$ is defined through the mean value property (see Helgason, [114] 2.2.2.5, pp. 403), originally in [71]:

$$f(\mathbf{x} \cdot \mathbf{z}) = \int_K f(\mathbf{x}\mathbf{k}\mathbf{y} \cdot \mathbf{z}) d\mathfrak{m}(\mathbf{k}) \tag{3.380}$$

where $\mathbf{x}, \mathbf{y} \in G$, f being harmonic on G/K, K being a compact subgroup represented through the projective homomorphism $() \cdot \mathbf{z}$, G the group itself. By harmonic, we imply that there exists some Laplacian operator such that $\nabla^2 f = 0$.

Example 3.3.30. A prototypical example of this form of harmonic function is the following, known as Gauss's mean value transform. If we have a complex function, which is analytic on some region $0 < |z - z_0| < R$, we take $z = z_0 + re^{i\theta}$. As it is analytic, it obeys the Cauchy-Riemann equations, and we have $\nabla^2 f = 0$, i.e. the function is harmonic. Then the average value of the function on a circle is the value at the centre of the circle.

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0 \tag{3.381}$$

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta \tag{3.382}$$

This is Cauchy's integral theorem, rewritten in a different way. This is the qualitative information that the theory of spherical functions attempts to generalise. As we shall show later in the thesis, there are many different aspects of special function theory that can be understood using these techniques.

Remark 36. Generally K for our particular examples will be some subset of rotations which we can factorise out of the group decomposition via $G = KAK$ or otherwise. We will use a stereographic transformation to reduce the action of the group to a homomorphism.

We shall quote some useful theorems from Helgason [114] which relate to the spherical theory of special functions. These theorems form the basis for interpretation of the representation theory of special functions through zonal decompositions of the group. In many ways this is similar to the methodology explored in Vilenkin [69], and shares many common principles with the theory of positive definite operators as given through Mercer's and Stone's theorems 3.3.11, 3.3.5.

Definition 3.3.31. If V is a Hilbert space, and for each $\mathbf{x} \in G$ the representation $\pi(\mathbf{x})$ on V is unitary, then $\pi(.)$ is a unitary representation. If there exists a left and right invariant measure $d\mathfrak{m}(\mathbf{x})$, and the space V is complete, then there exists for each $f \in \mathcal{C}(G)$ the continuous linear transformation:

$$\pi(f)\mathbf{v} = \int_G f(\mathbf{x})\pi(\mathbf{x})\mathbf{v}d\mathfrak{m}(\mathbf{x}) \tag{3.383}$$

for $\mathbf{v} \in V$, some test function. This acts as:

$$(\mathbf{x}, \mathbf{v}) \to \pi(\mathbf{x})\mathbf{v} \tag{3.384}$$

where $\pi(\mathbf{x})$ is as in 3.3.34. This is a type of Fourier transform, as shown in the examples below. Note that this is taken verbatim from [114], [116] (see Helgason, pp. 386, Ch. IV, Lemma 1.1, also Bourbaki, VIII, 2.2).

Example 3.3.32. We shall show briefly how this formula reduces to the Fourier transform. If we take the action of the group:

$$\pi(\mathbf{x}) = e^{-i\mathbf{a}\cdot\mathbf{x}}, d\mathfrak{m}(\mathbf{x}) = d\mathbf{x} \tag{3.385}$$

then application of the formula reduces to:

$$\pi(f) = \int_G f(\mathbf{x})e^{-i\mathbf{a}\cdot\mathbf{x}}d\mathbf{x} \tag{3.386}$$

In the instance that we have associated Legendre function defined through the group representation, then we have:

$$\pi(\mathbf{x}) = \mathcal{P}_{i\rho-1/2}\left(y\right), d\mathfrak{m}(\mathbf{x}) = \frac{\left|\Gamma\left(i\rho + 1/2\right)\right|^2 \sinh\left(\pi\rho\right)}{\pi}d\rho \tag{3.387}$$

This is one way to derive the Mehler-Fock transform:

$$\pi(f) = \int_0^\infty f(\rho)\mathcal{P}_{i\rho-1/2}\,(y)\,|\Gamma\,(i\rho+1/2)|^2 \sinh(\pi\rho)\,d\rho = F(y) \tag{3.388}$$

We will cover these types of special functions in depth in following sections of this work. Although we will not focus on the development of the Fourier transform from this perspective, it is important to note that it is a useful technique for deriving the transformations that exist on the space. However, using the kernel and eigenfunction decompositions, we can essentially shortcut this process by identifying the eigenfunction and evaluating the equivalent of the Fourier transform directly.

Definition 3.3.33. We recall another definition of a unitary representation. Let G be a group, $\pi : G \to U(\mathscr{H})$ be a representation. where $U(\mathscr{H})$ are the unitary operators on the Hilbert space $\mathscr{H}$. If we have the relationship:

$$\pi(\mathbf{gh})f = \pi(\mathbf{g})\pi(\mathbf{h})f \tag{3.389}$$

for $f \in \mathscr{H}$, then $\{\pi(.), \mathscr{H}\}$ is a unitary representation of G.

Lemma 3.3.34. *Let G be a locally compact group, two elements $\mathbf{x}, \mathbf{y} \in G$, V a Frechet space. Let $\pi : G \to \mathrm{Aut}(V)$ be a mapping from the group G to V. If the mapping satisfies:*

$$\pi(\mathbf{xy}) = \pi(\mathbf{x})\pi(\mathbf{y}) \tag{3.390}$$

and $\mathbf{v} \in V, \mathbf{x} \to \pi(\mathbf{x})\mathbf{v}$ is continuous in $\mathbf{x}$, the mapping $\pi(.)$ is a homomorphism of G on V. The left regular representation, also from G onto V satisfies this relation.

Proof. We note that we have already constructed the regular representations in 3.3.10. Writing out the formula explicitly, we have:

$$(\pi(\mathbf{xy})f)(\mathbf{u}) = f\left((\mathbf{xy})^{-1}\mathbf{u}\right) = f\left(\mathbf{y}^{-1}\mathbf{x}^{-1}\mathbf{u}\right) \tag{3.391}$$

and for the other side of the equality, we have:

$$(\pi(\mathbf{x})\pi(\mathbf{y})f)(\mathbf{u}) = (\pi(\mathbf{y})f)(\mathbf{x}^{-1}\mathbf{u}) = f\left(\mathbf{y}^{-1}\mathbf{x}^{-1}\mathbf{u}\right) \tag{3.392}$$

where the second step follows as the composition rule is left-regular composing. We therefore satisfy the composition law given by the formula. To show that it is an automorphism, note that for an invariant member of the group, we have $\mathbf{xyx}^{-1} = \mathbf{y}$, and hence

$$\pi(\mathbf{xyx}^{-1}) = \pi(\mathbf{y}) = \pi(\mathbf{x})\pi(\mathbf{y})\pi(\mathbf{x}^{-1}) \tag{3.393}$$

Consequently, we have:

$$\pi(\mathbf{xx}^{-1}) = \pi(\mathbf{1}) = \mathbf{1} = \pi(\mathbf{x})\pi(\mathbf{x}^{-1}) \tag{3.394}$$

i.e the simple rule for inversion $[\pi(\mathbf{x})]^{-1} = \pi(\mathbf{x}^{-1})$, which is obvious from the composition rule. We then have the automorphic relationship:

$$\pi(\mathbf{y}) = \pi(\mathbf{x})\pi(\mathbf{y})\pi(\mathbf{x}^{-1}) = \pi(\mathbf{x})\pi(\mathbf{y})\left[\pi(\mathbf{x})\right]^{-1} \tag{3.395}$$

We note that the definition is equivalent to the hypothesis as expressed in 3.3.8. $\square$

Further, we take the following definition for the convolution product on a group, which is equivalent to the definitions used in establishing the representation theory of the positive definite kernel in 3.3.5, 3.3.4.

Definition 3.3.35. The convolution product $\star$ on a group G is given by:

$$(f \star g)(\mathbf{u}) = \int_G f(\mathbf{y}) g(\mathbf{y}^{-1}\mathbf{u}) d\mathfrak{m}(\mathbf{y}) \tag{3.396}$$

where the integral is over the group G, $f, g \in \mathcal{C}(G)$ some functions which act on elements of the group $\mathbf{x}, \mathbf{y} \in G$, the measure of the group given by $\mathfrak{m}(\mathfrak{x})$.

We shall now prove a theorem related to the representation of the convolution of two functions f, g, which is analogous to the Fourier product law.

Theorem 3.3.36. *Consider a continuous linear transform of a function f acting in G, specified through the conditions of 3.3.31 and given by $\pi(f)$. The action of the left regular representation on the convolution product $f \star g$ is a homomorphism:*

$$\pi\left[(f \star g)\right] = \pi(f)\pi(g) \tag{3.397}$$

Proof. We take the definitions of the continuous linear transform of a function f and the convolution product as in 3.3.35. We have for the right hand side:

$$\pi(f)\pi(g)\mathbf{v} = \int_G f(\mathbf{x})\pi(\mathbf{x}) \left[\int_G g(\mathbf{y})\pi(\mathbf{y})\mathbf{v} d\mathfrak{m}(\mathbf{y})\right] d\mathfrak{m}(\mathbf{x}) \tag{3.398}$$

$$= \int_{G\times G} d\mathfrak{m}(\mathbf{x}) d\mathfrak{m}(\mathbf{y}).f(\mathbf{x})\pi(\mathbf{x})g(\mathbf{y})\pi(\mathbf{y})\mathbf{v} \tag{3.399}$$

$$= \int_{G\times G} d\mathfrak{m}(\mathbf{y}) d\mathfrak{m}(\mathbf{x}).f(\mathbf{x})g(\mathbf{x}^{-1}\mathbf{y})\pi(\mathbf{y})\mathbf{v} \tag{3.400}$$

$$= \int_G d\mathfrak{m}(\mathbf{y}).\varphi(\mathbf{y})\pi(\mathbf{y})\mathbf{v} \tag{3.401}$$

$$= \pi(\varphi)\mathbf{v} = \pi\left(f \star g\right)\mathbf{v} \tag{3.402}$$

as required. We have invoked the Fubini-Tonelli theorem to rearrange the integral. □

Remark 37. This is equivalent to the Fourier product law 3.3.28 which takes the convolution into multiplication of Fourier transforms.

Example 3.3.37. This example shall show how one can derive the area and length measures for several simple systems. We shall show that these are invariant measures under some very simple conditions. Consider the map of a complex number to a vector:

$$z = (x, iy) = x + iy \tag{3.403}$$

The arc length follows the standard formula:

$$dl = \sqrt{|dz|^2} = \sqrt{\dot{x}^2 + \dot{y}^2}dt = \sqrt{d\mathbf{x}^\dagger d\mathbf{x}} \tag{3.404}$$

Now, if we write the differential change in the complex vector and its conjugate, this can be packed into a matrix:

$$\hat{u} = \begin{bmatrix} dz \\ dz^* \end{bmatrix} = \begin{bmatrix} dx & idy \\ dx & -idy \end{bmatrix} \tag{3.405}$$

Now, it is the fundamental proposition of geometric algebra that the exterior product of two infinitesmals gives the area element, with outward orientation. In this context, the antisymmetric product is given by the determinant, which has the same antisymmetric property of the cross product and exterior product. We write:

$$\det \hat{u} = \left\| \begin{matrix} dx & idy \\ dx & -idy \end{matrix} \right\| = -2idxdy = -2idA(z) \tag{3.406}$$

It is plain to see, using the determinant rule for products of matrices, that we must have $\det\left(\hat{R}\right) = 1$, for $dA(\hat{R}z) = dA(z)$ to preserve the area element, and for the length, we must have $\hat{R}^{\dagger}\hat{R} = \mathbf{1}$. This is a unitary transformation. This will generalise to different inner products depending on the scenario. For our second example of this type of relationship, consider now the complex variable $z = re^{i\theta}$, with arc length formula:

$$dl = \sqrt{|dz|^2} = \sqrt{|z_t|}dt \tag{3.407}$$

It is simple to show that this is given by:

$$|dz|^2 = dr^2 + r^2 d\theta^2 \tag{3.408}$$

which reduces to the function:

$$dl = \sqrt{|dz|^2} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}dt \tag{3.409}$$

If we construct the matrix as before, with the co-ordinates stacked, we will we have:

$$\hat{u} = \left[\begin{array}{c} dz \\ dz^* \end{array}\right] = \left[\begin{array}{cc} dr & ird\theta \\ dr & -ird\theta \end{array}\right] \tag{3.410}$$

and, again, the area element in the form:

$$\det\hat{u} = \left\|\begin{array}{cc} dr & ird\theta \\ dr & -ird\theta \end{array}\right\| = -2irdrd\theta = -2idA(z) \tag{3.411}$$

which we write as $dA(z) = -\frac{1}{2i}\det\hat{u}$. Generally, for an invariant area measure in the complex plane with this parametrisation, we must have:

$$dA(\hat{R}z) = -\frac{1}{2i}\det\left(\hat{R}\hat{u}\right) = -\frac{1}{2i}\det\hat{u}.\det\left(\hat{R}\right) \tag{3.412}$$

which implies that for $dA(\hat{R}z) = dA(z)$, hence $\det\left(\hat{R}\right) = 1$, and the unitary condition following from the inner product. Finally, if we decompose the hyperbolic metric, we may write:

$$ds^2 = d\tau^2 + \sinh^2\tau d\phi^2 = (d\tau + i\sinh\tau d\phi)\,.\,(d\tau - i\sinh\tau d\phi) \tag{3.413}$$

The matrix of the differentials is written out in identical fashion:

$$\hat{u} = \left[\begin{array}{c} dz \\ dz^* \end{array}\right] = \left[\begin{array}{cc} d\tau & i\sinh\tau d\phi \\ d\tau & -i\sinh\tau d\phi \end{array}\right] \tag{3.414}$$

and obtaining the area element:

$$\det\hat{u} = \left\|\begin{array}{cc} d\tau & i\sinh\tau d\phi \\ d\tau & -i\sinh\tau d\phi \end{array}\right\| = -2i\sinh\tau d\phi d\tau \tag{3.415}$$

We can see that this implies once more that for the area element to be invariant, we must have $dA(\hat{R}z) = dA(z)$, and this gives that:

$$\det\left(\hat{R}\right) = 1 \tag{3.416}$$

and the unitary conditions as before. This can be seen as the substitution $\phi \to \phi + \sigma$ into the area element. We now analyse what it means for a function to be radially dependent in this situation. For example, in the simplest instance of Cartesian co-ordinates, we would have:

$$f(\mathbf{x}) = f\left(\sqrt{\mathbf{x}^T\mathbf{x}}\right) = f(r) \tag{3.417}$$

with overloading of the notation for simplicity of argument. It is straightforward to see that this reduces to a similar result if we take a transformation of the input element:

$$f(\hat{R}\mathbf{x}) = f\left(\sqrt{\mathbf{x}^T\left(\hat{R}^T\hat{R}\right)\mathbf{x}}\right) = f\left(\sqrt{\mathbf{x}^T\mathbf{x}}\right) \tag{3.418}$$

i.e. we recover $f(\hat{R}\mathbf{x}) = f(\mathbf{x}) = f(r)$ for $\hat{R}^T\hat{R} = \mathbf{1}$, the transformation in this case being a rotation matrix. Similar generalisations exist for the hyperbolic case and we will not go into this further. The basic concept is that if the area element and line element are transformed, then only special transformations preserve these infinitesmals. These special transformations are the rotations, if we are dealing with radial functions. Geometrically, a radial function will only depend on the distance from the origin, and will be symmetric under a rotation about this axis. Finally, we consider the Möbius transformation:

$$T_w \cdot z = \frac{z+w}{1+\bar{w}z} \tag{3.419}$$

which is different in sign of w from that in 3.3.12. However, we can write rotation by an angle by using the complex exponential:

$$\kappa_\sigma \cdot z = e^{i\sigma}z \tag{3.420}$$

Any point in the space will therefore be equal to a rotation, and a radial displacement, we therefore have:

$$g_w \cdot z = (\kappa_\sigma T_R) \cdot z \tag{3.421}$$

Note that we have the homomorphism mapping identity to identity:

$$T_1 \cdot z = 1 \tag{3.422}$$

These and many other results that we shall exploit may be found in Chavel, Ch. X.2, pp. 243 [11].

Definition 3.3.38. From Chavel, Ch. X.2, pp. 243 [11], we have the following. Assume a complex function in the disk as defined above, also see 3.3.12. Consider a transformation which rotates a complex number, we can write:

$$\kappa_\sigma \cdot z = e^{i\sigma}z, \kappa_\sigma^{-1} \cdot z = e^{-i\sigma}z \tag{3.423}$$

Then for the line measure, this is invariant under rotations:

$$|dz^2| = |d(\kappa_\sigma^{-1} \cdot z)|^2 = |d(\kappa_\sigma \cdot z)|^2 \tag{3.424}$$

The area measure is invariant under rotations by an identical argument to the preceding examples, we will have $dA(z) = dA(\kappa_\sigma^{-1} \cdot z) = dA(\kappa_\sigma^{-1} \cdot z)$, and also:

$$\det\left[\hat{u}\left(\kappa_\sigma^{-1} \cdot z\right)\right] = \left\| \begin{array}{cc} e^{-i\sigma}dx & ie^{-i\sigma}dy \\ e^{i\sigma}dx & -ie^{+i\sigma}dy \end{array} \right\| = -2idxdy = \det\left[\hat{u}\left(z\right)\right] \tag{3.425}$$

as an example of this type of symmetry. A radially symmetric function obeys the equation:

$$f\left(\kappa_\sigma^{-1} \cdot z\right) = f\left(e^{-i\sigma}z\right) = f(z) \tag{3.426}$$

This function must be a function of the absolute value of the complex number z:

$$f(z) = g(|z|) \tag{3.427}$$

These definitions are all we require to show that the convolution theorem in the hyperbolic plane is invariant to rotations.

Lemma 3.3.39. *Assume a radial function, invariant under rotations via* $\varphi(\kappa_\sigma^{-1} \cdot w) = \varphi(w) = \varphi(|w|)$*, and further, an area measure in the hyperbolic plane given by:*

$$dA(\kappa_\sigma^{-1} \cdot w) = dA(w) \tag{3.428}$$

where $\kappa_\sigma^{-1} \cdot w = e^{-i\sigma} w$ *is a rotation. Then the convolution theorem given by the area integral:*

$$(\varphi \star \rho)(z) = \int_{\mathbb{H}^2} \varphi(w)\psi(g_w^{-1} \cdot z) dA(w) \tag{3.429}$$

is invariant under rotations $(\varphi \star \rho)(\kappa_\sigma \cdot z) = \int_{\mathbb{H}^2} \varphi(w)\psi(g_w^{-1} \cdot z)dA(w) = (\varphi \star \rho)(z) = (\varphi \star \rho)(|z|)$*, and is a function only of the radial distance.*

Proof. To establish that the area measure is invariant under rotations, we may write:

$$\det\left[\hat{u}\left(\kappa_\sigma^{-1} \cdot z\right)\right] = \begin{Vmatrix} e^{-i\sigma}dx & ie^{-i\sigma}dy \\ e^{i\sigma}dx & -ie^{+i\sigma}dy \end{Vmatrix} = -2idxdy = \det\left[\hat{u}\left(z\right)\right] \tag{3.430}$$

which is true for $dx = d\tau, dy = \sinh\tau d\phi$. The area measure is rotationally invariant for the hyperbolic metric. Evaluating the convolution, we may write:

$$(\varphi \star \rho)(\kappa_\sigma \cdot z) = \int_{\mathbb{H}^2} \varphi(w)\psi(g_w^{-1} \cdot (\kappa_\sigma \cdot z))dA(w) \tag{3.431}$$

$$= \int_{\mathbb{H}^2} \varphi(\kappa_\sigma^{-1} \cdot w)\psi(g_w^{-1} \cdot \kappa_\sigma \cdot (\kappa_\sigma^{-1} \cdot z))dA(\kappa_\sigma^{-1} \cdot w) \tag{3.432}$$

As the function is radial, we have:

$$\varphi(\kappa_\sigma^{-1} \cdot w) = \varphi(w) = \varphi(|w|) \tag{3.433}$$

and as we have shown the measure is invariant to rotations, then:

$$dA(\kappa_\sigma^{-1} \cdot w) = dA(w) \tag{3.434}$$

Consequently, we conclude:

$$(\varphi \star \rho)(\kappa_\sigma \cdot z) = \int_{\mathbb{H}^2} \varphi(w)\psi(g_w^{-1} \cdot z)dA(w) = (\varphi \star \rho)(z) = (\varphi \star \rho)(|z|) \tag{3.435}$$

and establish the proof of the lemma. □

We shall extend this lemma to groups with an invariant rotation. The generalisation follows as a natural consequence.

Theorem 3.3.40. *Assume a left-multiplicative group, acting on some element* $\mathbf{z} \in G$*, and further an rotationally invariant measure and a radial function in the sense of the preceding lemma. Then we have a rotation matrix* $\hat{R}$*, and the radial function and measure are:*

$$\varphi(\hat{R}^{-1}\mathbf{z}) = \varphi(\mathbf{z}) = \varphi(|\mathbf{z}|) \tag{3.436}$$

$$d\mathfrak{m}\left(\hat{R}^{-1}\mathbf{z}\right) = d\mathfrak{m}\left(\hat{R}\mathbf{z}\right) = d\mathfrak{m}(\mathbf{z}) \tag{3.437}$$

Under these conditions, the convolution product is rotationally invariant, i.e. radial:

$$(\varphi \star \vartheta)\left(\hat{R}\mathbf{z}\right) = (\varphi \star \vartheta)(\mathbf{z}) \tag{3.438}$$

and the convolution is therefore a function of the distance only.

Proof. We can see that this is trivially true from $\hat{R}\mathbf{z} = \mathbf{z}$, which is a fixed point theorem for the rotation. Any radial vector will be invariant to rotation. To prove it using the convolution, we take the definitions 3.3.39 and 3.3.35:

$$(\varphi \star \vartheta)\left(\hat{R}\mathbf{z}\right) = \int_G \varphi(\mathbf{y})\vartheta(\mathbf{y}^{-1}\hat{R}\mathbf{z})d\mathfrak{m}(\mathbf{y}) \tag{3.439}$$

$$= \int_G \varphi(\hat{R}\mathbf{y})\vartheta(\mathbf{y}^{-1}\hat{R}^{-1}\hat{R}\mathbf{z})d\mathfrak{m}(\hat{R}\mathbf{y}) \tag{3.440}$$

$$= \int_G \varphi(\mathbf{y})\vartheta(\mathbf{y}^{-1}\mathbf{z})d\mathfrak{m}(\mathbf{y}) \tag{3.441}$$

$$= (\varphi \star \vartheta)(\mathbf{z}) \tag{3.442}$$

as required, where we have used the rotational invariance of the functions to evaluate the convolution in the transformed reference frame. □

Example 3.3.41. We will show a basic example of this type of result for the convolution theorem. Consider the Laplace transform on n-dimensional Euclidean space, then we have the convolution:

$$(f * g)(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{y})g(\mathbf{x}-\mathbf{y})d\mathbf{y} \tag{3.443}$$

If we write the convolution of the rotated input, we have:

$$(f * g)(\hat{R}\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{y})g(\hat{R}\mathbf{x}-\mathbf{y})d\mathbf{y} \tag{3.444}$$

$$= \int_{\mathbb{R}^n} f(\hat{R}\mathbf{y}')g(\hat{R}\mathbf{x}-\hat{R}\mathbf{y}')d(\hat{R}\mathbf{y}') \tag{3.445}$$

where we rescaled the integral using $\mathbf{y} = \hat{R}^{-1}\mathbf{y}'$. If the measure is rotationally invariant, and the functions are radial, then we have $f(\hat{R}\mathbf{y}') = f(\mathbf{y}'), d(\hat{R}\mathbf{y}') = d\mathbf{y}'$, and also:

$$g(\hat{R}\mathbf{x} - \hat{R}\mathbf{y}') = g(\hat{R}(\mathbf{x}-\mathbf{y}')) = g(\mathbf{x}-\mathbf{y}') \tag{3.446}$$

We conclude that the convolution is also invariant to these rotations:

$$(f * g)(\hat{R}\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{y}')g(\mathbf{x}-\mathbf{y}')d\mathbf{y}' = (f * g)(\mathbf{x}) \tag{3.447}$$

This can be shown more directly using differential geometry. If we look at the metric for e.g. hyperbolic geometry, we have:

$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & \sinh^2\tau \end{bmatrix} \tag{3.448}$$

Then for the rotation:

$$\hat{R}_z(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \tag{3.449}$$

we have the invariance $\hat{R}_z(\theta)g_{\alpha\beta}\hat{R}_z^\dagger(\theta) = g_{\alpha\beta}$. This invariance under rotations carries through into the distance function and any radially dependent functions.

We now close this section with a brief review of useful formulae from spherical functions. Special formula, which we refer to as spherical zonal formulae, give these relations.

Definition 3.3.42. When a function $f(.)$ satisfies the group integral relation:

$$\int_{\mathbf{k}\in K} f(\mathbf{xky}) d\mathfrak{m}\left(\mathbf{k}\right) = f(\mathbf{x})f(\mathbf{y}) \tag{3.450}$$

where K is a group, $\mathbf{x}, \mathbf{y} \in G$ any two members of a group G (not necessarily equal to K), the group integral measure is $d\mathfrak{m}\left(\mathbf{k}\right)$ with integration variable $\mathbf{k}$, we say $f(.)$ is a spherical function, or that a spherical (zonal) decomposition exists.

Remark 38. See e.g. Helgason [114] for a thorough description of the topic.

Example 3.3.43. We shall construct a simple form of homeomorphism, that will be used to construct a map that takes matrices into the complex numbers. The standard formula for the Möbius transformation:

$$\mathbf{g} \cdot \mathbf{x} = (\mathbf{gx})_1 = \frac{ax+b}{cx+d}, \mathbf{g} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \mathbf{x} \sim \begin{bmatrix} x \\ 1 \end{bmatrix} \tag{3.451}$$

maps any 2×2 matrix into the complex numbers. We shall now show how this type of mapping can be used for hyperbolic systems. We have the Cayley transforms:

$$\hat{M}_1 \rho = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & +i \end{bmatrix} \begin{bmatrix} \rho \\ 1 \end{bmatrix} \sim \frac{1}{\sqrt{2}} \left(\frac{\rho - i}{\rho + i} \right) \tag{3.452}$$

and

$$\hat{M}_1 \mathbf{z} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} \sim \frac{1}{\sqrt{2}} \left(\frac{z+1}{i(z-1)} \right) \tag{3.453}$$

which satisfy the invertibility condition $\left(\hat{M}_1 \hat{M}_2\right) \rho = \left(\hat{M}_2 \hat{M}_1\right) \rho = \rho$. We can use this transformation to map 2×2 matrices into the complex numbers in a smooth, continuous way. In the context of the Cayley transform, we can clearly see how there is a direct connection between the different models of hyperbolic space and unitary matrices.

Remark 39. Using the definition of the spherical projection via the homeomorphism, we can define a series of other related functions that give composition formulae for the group as in the spherical decomposition. Note the fine difference here between homomorphisms, automorphisms and homeomorphisms.

Definition 3.3.44. We take the following definition verbatim from Helgason, [114], Ch. IV, 2, Prop. 2.4, pp. 402. Assume a group with decomposition $G = KAN$, or $G = KAK$. We call the quotient group the group $X = G/K$. Assume there exists a Laplacian on the quotient group, then we have a set of eigenvalues:

$$E_\mu = \{f \in \mathcal{E}\left(G/K\right) : Df = \mu(D) f\} \tag{3.454}$$

for the eigenvalue function $\mu : D \in \boldsymbol{D}\left(G/K\right) \to \mathbb{C}$ which is defined for each quotient space of G. Then for each non-zero eigenvalue $E_\mu \neq 0$, the joint eigenspace contains exactly one spherical function $\phi(.)$. The members of E_μ given by functions $f(.)$ which are eigenfunctions of the Laplace-type operator are characterised by the integral equation:

$$\int_{\mathbf{k}\in K} f(\mathbf{g} \cdot [\mathbf{xky}]) d\mathfrak{m}\left(\mathbf{k}\right) = f(\mathbf{g} \cdot \mathbf{x}) \phi(\mathbf{g} \cdot \mathbf{y}) \tag{3.455}$$

The group homeomorphism is labelled $\mathbf{g} \cdot (.)$, the group element in this case designated by $\mathbf{xky}$ as in the KAN decomposition, and $\mathbf{k} \in K$.

Definition 3.3.45. Prop 2.4, Ch. IV.2, pp. 402, (Helgason, [114]). The spherical composition law 3.3.42 is the special case $\phi(\mathbf{g}\cdot\mathbf{y}) = f(\mathbf{g}\cdot\mathbf{y})$, with the function operating via homeomorphism.

Definition 3.3.46. Prop 2.5, Ch. IV.2, pp. 403, (Helgason, [114]). The harmonic functions are given by the special case $\phi(\mathbf{g}\cdot\mathbf{y}) = 1$. The harmonic functions are functions defined on the quotient group $u \in G/K$, and are the solution to $Du = 0$ for all $D \in \boldsymbol{D}(G/K)$. Helgason further states that if the space is two-point homogeneous, this collapses to the solutions of Laplace's equation:

$$Lu = 0 \tag{3.456}$$

where L is the Laplace operator. In our spaces, this will be the case.

See Helgason [114] for a theoretical exposition on spherical groups. It is this formula we are primarily interested in, as the right hand side relates to the homomorphism in the convolution product.

We shall now prove the central theory of spherical functions that are related to this work. They follow as a simple consequence of the definitions 3.3.45, 3.3.29, 3.3.44.

Lemma 3.3.47. *A spherical function on a group $KAN = G$ given by the group element* $\mathbf{xky}$ *is given by the Haar integral:*

$$\int_{\mathbf{k}\in K} f(\mathbf{xky})d\mathfrak{m}(\mathbf{k}) = f(\mathbf{x})f(\mathbf{y}) \tag{3.457}$$

where the integral is over the group, the measure specified by $d\mathfrak{m}(\mathbf{k})$*,*

Proof. Simply replace $\tilde{f}(\mathbf{u}) = f(\mathbf{g}\cdot\mathbf{u})$ in 3.3.45. □

Remark 40. This is essentially the same as 3.3.45. In this case, the function operates on the level of a group. We note that for this to be true, the function must be appropriately continuous in the group as in a Frechet derivative or otherwise.

We shall now prove a simple theorem that shall simplify our calculations regarding spherical functions.

Theorem 3.3.48. *The composition formula for a spherical function:*

$$\int_{\mathbf{k}\in K} f(d_K(\mathbf{x},\mathbf{y}))d\mathfrak{m}(\mathbf{k}) = f(\mathbf{x})f(\mathbf{y}) \tag{3.458}$$

where $d_K(\mathbf{x},\mathbf{y})$ is the distance function on the group K, $\mathbf{x},\mathbf{y}$ two group elements in the KAN decomposition, the measure specified by $d\mathfrak{m}(\mathbf{k})$ with integration variable on the group $\mathbf{k}$.

Remark 41. Note that $d_K(\mathbf{x},\mathbf{y})$ is implicitly a function of the generic element $\mathbf{k} \in K$.

Proof. We identify the function $f(.)$ with the kernel $K(\mathbf{x},\mathbf{y})$ using 3.3.26. Take the group decomposition law:

$$\int_{\mathbf{k}\in K} f(\mathbf{xky})d\mathfrak{m}(\mathbf{k}) = f(\mathbf{x})f(\mathbf{y}) \tag{3.459}$$

If we take the kernel formula 3.3.13, we have the right hand side as:

$$K(\mathbf{x},\mathbf{y}) = f(\mathbf{x})f(\mathbf{y}) \tag{3.460}$$

and left-hand side equal to $K(\mathbf{x},\mathbf{y}) = \int_{\mathbf{k}\in K} f(d_K(\mathbf{x},\mathbf{y}))d\mathfrak{m}(\mathbf{k})$. Obviously the kernel is symmetric, we have automatically from the distance formula $d_K(\mathbf{x},\mathbf{y}) = d_K(\mathbf{y},\mathbf{x})$, also $f(\mathbf{x})f(\mathbf{y}) = f(\mathbf{y})f(\mathbf{x})$ as $f(.)$ is a scalar function, and multiplication of scalars is commutative therefore we can identify each distance function with an associated product formula. Note that we can also take the unital decomposition of the symmetric kernel via 3.3.11, where we found $K(\mathbf{x}, \mathbf{y}) = \Phi^*(\mathbf{x})\Phi(\mathbf{y})$, and for the symmetric component $\Phi^*(\mathbf{x}) = \Phi(\mathbf{x})$. □

We can make one further extension to this theorem, which we shall require in order to evaluate the hyperbolic heat kernel. Note that this result is stated without proof in [1] (see pp. 178, Thm. 5.7.1).

Corollary 3.3.49. *Assume that there exists a distance function $d_K(\mathbf{x}, \mathbf{y})$ which describes a positive definite kernel $K_{\mathbf{k}}(\mathbf{x}, \mathbf{y}; t)$, giving the solution to the heat equation on K, some group, $u(\mathbf{x}, t) = \int K_{\mathbf{k}}(\mathbf{x}, \mathbf{y}; t)u(\mathbf{y}, 0)d\mathfrak{m}(\mathbf{y})$ with $u_t = \nabla^2 u$, then we have:*

$$K_{\mathbf{k}}(\mathbf{x}, \mathbf{y}; t) = f(d_K(\mathbf{x}, \mathbf{y}), t) \tag{3.461}$$

i.e. the kernel $K(.,.)$ is a function of the distance metric only.

Proof. This follows readily by replacement of $f(d_K(\mathbf{x}, \mathbf{y}))$ with $f(d_K(\mathbf{x}, \mathbf{y}), t)$ in the previous theorem. Note we can write:

$$K(\mathbf{x}, \mathbf{y}; t) = \int_{\mathbf{k}\in K} f(d_K(\mathbf{x}, \mathbf{y}), t)d\mathfrak{m}(\mathbf{k}) = \int_{\mathbf{k}\in K} K_{\mathbf{k}}(\mathbf{x}, \mathbf{y}; t)d\mathfrak{m}(\mathbf{k}) \tag{3.462}$$

and the completeness relation:

$$K_{\mathbf{k}}(\mathbf{x}, \mathbf{y}; 0) = \delta(\mathbf{y} - \mathbf{x}) \tag{3.463}$$

The second relation should be seen as an expansion in terms of the eigenfunctions as in 3.3.18. It follows from the delta function relation $u(\mathbf{x}, 0) = \int \delta(\mathbf{y} - \mathbf{x})u(\mathbf{y}, 0)d\mathfrak{m}(\mathbf{y})$. □

We have established a workable theory of kernels, that draws deeply from the ideas of representations, positive definite functions and convolutions. In the following chapter, we will implement many of these concepts in order to reach a detailed understanding of the structure of the hyperbolic heat kernel and eigenfunctions related to the product composition law as expressed through the spherical theorems.

3.4 Special Functions

In this thesis, we shall have recourse to the following types of special functions.

Definition 3.4.1. The associated Legendre functions $\mathcal{P}_\nu^{-\mu}(x), \mathcal{Q}_\nu^\mu(x)$ are given by the special cases of the Gauss hypergeometric function:

$$\mathcal{P}_\nu^{-\mu}(x) = \left(\frac{x-1}{x+1}\right)^{\mu/2} {}_2F_1\left([\nu+1, -\nu]\,;[\mu+1]\,, \frac{1}{2} - \frac{x}{2}\right) \tag{3.464}$$

$$\mathcal{Q}_\nu^\mu(x) = \frac{2^\nu\Gamma(\nu+1)(x+1)^{\mu/2}}{(x-1)^{\mu/2+\nu+1}} {}_2F_1\left([\nu+1, \nu+\mu+1]\,;[2\mu+2]\,, \frac{2}{1-x}\right) \tag{3.465}$$

where the Gauss hypergeometric function is defined through the power series:

$${}_2F_1\left([a, b]\,;[c]\,, z\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{s=0}^{\infty} \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)s!} z^s \tag{3.466}$$

see DLMF 14.3.9, 14.3.19, 15.2.1 [8]. These functions have integral representation:

$$\mathcal{P}_{\nu}^{-\mu}(z) = \frac{(z^2-1)^{\mu/2}}{2^{\nu}\Gamma(\mu-\nu)\Gamma(\nu+1)} \int_0^{\infty} \frac{(\sinh t)^{2\nu+1}}{(z+\cosh t)^{\mu+\nu+1}} dt \tag{3.467}$$

valid where $\Re\mu > \Re\nu > -1$, and

$$\mathcal{Q}_{\nu}^{\mu}(z) = \frac{\sqrt{\pi}(z^2-1)^{\mu/2}}{2^{\mu}\Gamma\left(\mu+1/2\right)\Gamma\left(\nu-\mu+1\right)} \int_0^{\infty} \frac{(\sinh t)^{2\nu+1}}{(z+(z^2-1)^{1/2}\cosh t)^{\mu+\nu+1}} dt \tag{3.468}$$

where $\Re(\nu+1) > \Re\mu > -\dfrac{1}{2}$, on the positive interval $1 < z < \infty$, and the functions are continuous on $\mathbb{C}\backslash(-\infty, 1]$ and extended by analytical continuation to the excluded points and regions. See DLMF 14.25.1,14.25.2 [8]. The solution which is regular, i.e. continuous and non-singular on the interval $1 < z < \infty$ is $\mathcal{P}_{\nu}^{-\mu}(z)$. These functions have limiting behaviour as $\mu \to \infty, \nu$ fixed:

$$\mathcal{P}_{\nu}^{-\mu}(z) = \frac{1}{\Gamma(\mu+1)} \left(\frac{2\mu u}{\pi}\right)^{1/2} K_{\nu+1/2}\left(\mu u\right) \left(1 + \mathcal{O}\left(\frac{1}{\mu}\right)\right) \tag{3.469}$$

and

$$\mathcal{Q}_{\nu}^{\mu}(z) = \frac{1}{\mu^{\nu+1/2}} \left(\frac{\pi u}{2}\right)^{1/2} I_{\nu+1/2}\left(\mu u\right) \left(1 + \mathcal{O}\left(\frac{1}{\mu}\right)\right) \tag{3.470}$$

(see DLMF 14.15.2-3, [8]) and also the complementary limits for $\mu \to \infty, \nu$ fixed:

$$\mathcal{P}_{\nu}^{-\mu}(\cosh\xi) = \frac{1}{\nu^{\mu}} \left(\frac{\xi}{\sinh\xi}\right)^{1/2} I_{\mu}\left(\left(\nu+1/2\right)\xi\right) \left(1 + \mathcal{O}\left(\frac{1}{\nu}\right)\right) \tag{3.471}$$

and

$$\mathcal{Q}_{\nu}^{\mu}(\cosh\xi) = \frac{\nu^{\mu}}{\Gamma(\nu+\mu+1)} \left(\frac{\xi}{\sinh\xi}\right)^{1/2} K_{\mu}\left(\left(\nu+1/2\right)\xi\right) \left(1 + \mathcal{O}\left(\frac{1}{\nu}\right)\right) \tag{3.472}$$

where $I_{\mu}(.), K_{\mu}(.)$ are the modified Bessel functions, defined below. The limits may be found in Olver (see DLMF 14.15.13-14, [8]). In this thesis we shall employ the following subset of the associated Legendre functions, known as the Mehler-Fock or toroidal hermonics. These are specified through a modified form of the associated Legendre differential equation, given by:

$$(x^2-1)\frac{\partial^2 f}{\partial x^2} + 2x\frac{\partial f}{\partial x} + \left(\frac{(\nu^2+1)}{4} - \frac{k^2}{(x^2-1)}\right) f = 0 \tag{3.473}$$

where solution is given by the sum:

$$f(x) = C_1 \mathcal{P}_{i\nu-1/2}^{k}(x) + C_2 \mathcal{Q}_{i\nu-1/2}^{k}(x) \tag{3.474}$$

The solution which is regular on the interval $[1, \infty)$ is $\mathcal{P}_{i\nu-1/2}^{k}(z)$, and this is more commonly known as the Mehler-Fock function. The special case $\mathcal{P}_{i\nu-1/2}(x) = \mathcal{P}_{i\nu-1/2}^{0}(x)$.

The second type of special functions that we shall encounter are the Macdonald or modified Bessel functions. These may be developed from the basic theory of Bessel functions, defined as the infinite series familiar in classical analysis.

Definition 3.4.2. The Bessel function $J_\alpha(x)$ has series representation:

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(n+\alpha+1\right)} \left(\frac{x}{2}\right)^{2n+\alpha} \tag{3.475}$$

where $\Gamma(.)$ is the gamma function. Further, we have integral representation:

$$J_\alpha(x) = \frac{1}{\pi}\int_0^\pi \cos\left(\alpha\tau - x\sin\tau\right) d\tau - \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-x\sinh t-\alpha t} dt \tag{3.476}$$

and generating function for integer values m:

$$e^{z(t-t^{-1})/2} = \sum_{m=-\infty}^{+\infty} t^m J_m(z) \tag{3.477}$$

from DLMF 10.12.1. The modified Bessel functions of the first and second kind $I_\alpha(.), K_\alpha(.)$ are defined through the transformations of the Bessel function:

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n+\alpha} \tag{3.478}$$

and

$$K_\alpha(x) = \frac{\pi}{2}\left(\frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\pi\alpha)}\right) \tag{3.479}$$

where we have integral representations:

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos\left(\nu\theta\right) d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t} dt \tag{3.480}$$

and

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh(\nu t) dt \tag{3.481}$$

where $|\arg(z)| < \frac{\pi}{2}$, see DLMF 10.32.4-9 [8]. A generating function exists for the function $I_m(.)$ for integer values m and is given by:

$$e^{z(t+t^{-1})/2} = \sum_{m=-\infty}^{+\infty} t^m I_m(z) \tag{3.482}$$

DLMF 10.31.1 [8]. In this thesis we will repeatedly encounter an altered type of modified Bessel functions, which are given by the functions $K_{i\nu}(x), I_{i\nu}$, i.e. modified Bessel functions of complex index. In the positive half plane $0 < z < \infty$, the solution with regular behaviour such that it is continuous and non-singular is the function $K_{i\nu}(x)$. Note the asymptotic expansions as $\nu \to \infty$:

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2\nu}}\left(\frac{ez}{2\nu}\right)^{-\nu} \tag{3.483}$$

$$I_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}}\left(\frac{ez}{2\nu}\right)^{\nu} \tag{3.484}$$

which may be found in [8], 10.41.1-2.

One other type of important special function related to the modified Bessel and Mehler-Fock type which appears in the following calculations is given by the Whittaker or confluent hypergeometric function. In many ways this set of functions can be seen as a transformation of either the Bessel or associated Legendre (toroidal) functions.

Definition 3.4.3. The Whittaker or confluent hypergeometric functions $M_{\kappa,\mu}(z), W_{\kappa,\mu}(z)$ are the Mellin transforms of the modified Bessel functions $I_{2\mu}(x), K_{2\mu}(x)$ multiplied by e^{-t}:

$$\frac{1}{\Gamma(1+2\mu)}M_{\kappa,\mu}(z) = \frac{\sqrt{z}e^{z/2}}{\Gamma\left(\mu+\kappa+1/2\right)}\int_0^\infty e^{-t}t^{-\kappa-1/2}I_{2\mu}(2\sqrt{zt})dt \tag{3.485}$$

where $\Re(\kappa-\mu)-1/2>0$, and

$$W_{\kappa,\mu}(z) = \frac{2\sqrt{z}e^{-z/2}}{\Gamma\left(1/2+\mu-\kappa\right)\Gamma\left(1/2-\mu-\kappa\right)}\int_0^\infty e^{-t}t^{-\kappa-1/2}K_{2\mu}(2\sqrt{zt})dt \tag{3.486}$$

respectively $\Re(\mu-\kappa)+1/2>0$, see 13.16.3-4,8 [8]. There exists a connection formula:

$$W_{\kappa,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}M_{\kappa,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}M_{\kappa,-\mu}(z) \tag{3.487}$$

and a symmetry $W_{\kappa,\mu}(z) = W_{\kappa,-\mu}(z)$ which follows. In the limit that the first index approaches zero, we have the following limiting forms that connect the Whittaker functions to the modified Bessel functions:

$$W_{0,\nu}(2z) = \sqrt{\frac{2z}{\pi}}K_\nu(z) \tag{3.488}$$

and

$$M_{0,\nu}(2z) = 2^{2\nu+1/2}\Gamma(1+\nu)\sqrt{z}I_\nu(z) \tag{3.489}$$

see 13.18.8-9, [8]. The Whittaker functions solve the differential equation:

$$\frac{d^2f}{dx^2} + \left(-1/4+\frac{\mu}{x}+\frac{1/4-\nu^2}{x^2}\right)f = 0 \tag{3.490}$$

$$f(x) = C_1W_{\mu,\nu}(x) + C_2M_{\mu,\nu}(x) \tag{3.491}$$

The non-singular function of the pair in this case is given by $W_{\mu,\nu}(x)$. In this thesis, this function appears as the index Whittaker transform.

Remark 42. Although it might seem that the levels of concatenation involved in computing the Whittaker functions are remarkable, it turns out that various integral relations give simple, intuitive connections that relate these sets of special functions. Together, these three functions are deeply involved in the notion of index integration, especially the Mehler-Fock, Kontorovich-Lebedev and index Whittaker transformations. This topic is covered extensively in the following chapters, especially those related to the hyperbolic heat kernel and projection-slice theorem.

3.4.1 Tables of Special Functions Appearing in the Text

The defining differential equation for each of the following special functions may be written in the form:

Definition 3.4.4.

$$a(x)\frac{\partial^2 u}{\partial x^2} + b(x)\frac{\partial u}{\partial x} + c(x)u(x) = 0 \tag{3.492}$$

The following page presents the major types of special functions that appear in the text, along with their associated differential equation.

Name	Symbol	$a(x)$	$b(x)$	$c(x)$	Notes
Hermite Polynomial	$H_n(x)$	1	$-2x$	$2n$	
Parabolic Cylinder Function	$D_\mu(x)$	1	0	$\left(-\frac{x^2}{4} + a + \frac{1}{2}\right)$	Equivalent to Weber function.
Legendre Function	$\mathcal{P}_\nu(x), \mathcal{Q}_\nu(x)$	$(1-x^2)$	$-2x$	$\nu(\nu+1)$	
Associated Legendre Function	$\mathcal{P}_\nu^\mu(x), \mathcal{Q}_\nu^\mu(x)$	$(1-x^2)$	$-2x$	$\left(\nu(\nu+1) - \frac{\mu^2}{1-x^2}\right)$	
Gegenbauer Polynomial	$C_n^\mu(x), \mathcal{C}_\nu^\mu(x)$	$(1-x^2)$	$-(2\mu+1)x$	$\nu(\nu+2\mu)$	If we have $n = -1/2 + \mu + \nu$, the solution is given by a polynomial.
Laguerre Function	$L_\nu(x)$	x	$(1-x)$	ν	If $\nu = n$, some integer, the solution is a polynomial.
Associated Laguerre Function	$L_\nu^\mu(x)$	x	$(\mu+1-x)$	ν	If $\nu = n$, some integer, the solution is a generalised polynomial.
Bessel Function	$J_\nu(x)$	x^2	x	$x^2 - \nu^2$	
Modified Bessel Function	$K_\nu(x), I_\nu(x)$	x^2	x	$-(x^2+\nu^2)$	
Whittaker Function	$W_{\mu,\nu}(x), M_{\mu,\nu}(x)$	x^2	0	$\left(-\frac{x^2}{4} + \mu x + \frac{1}{4} - \nu^2\right)$	Equivalent to the confluent hypergeometric function of Kummer.
Bessel Polynomial	$y_n(x;a)$	x^2	$(ax+2)$	$-n(n+a-1)$	
Associated Bessel Polynomial	$J_n^\mu(x)$	x^2	$1+2x(1+\mu)$	$-n(n+2\mu+1)$	

3.5 Bose Invariant and Sturm-Liouville Theory

We shall now prove a number of theorems related to the invariant or normal form of the ODE given by 3.4.4. The reader is directed to the works of Albanese et. al [61–63] where the authors have derived a number of formulae that we shall use in the following work. We shall begin with a simple argument from stochastic processes.

Definition 3.5.1. Assume an Ito process:

$$dX_t = \mu dt + \sigma dW_t \tag{3.493}$$

where $\{W_t : t \geq 0\}$ is a standard Brownian motion. We call μ the drift and σ the variance of the stochastic process.

Lemma 3.5.2. *For the process:*

$$\int_0^t k(x_s)ds \tag{3.494}$$

the process depends only on the values on the boundary x_0, x_t.

Proof. Using the fundamental theorem of calculus, we have:

$$\int_0^t k(x_s)ds = F(x_t) - F(x_0) \tag{3.495}$$

where $F(x)$ is an antiderivative of $k(x)$, from which the property follows. □

We also automatically have the following condition.

Corollary 3.5.3.

$$\frac{\partial}{\partial t}\int_0^t k(x_s)ds = k(x_t) \tag{3.496}$$

We shall now prove a simplified version of the Feynman-Kac lemma. We shall take the following definitions as our axioms.

Definition 3.5.4. Assume the following function given by the expectation of the killed measure:

$$u(x,t) = \mathbb{E}\left[e^{-\int_0^t k(x_s)ds}\right] \tag{3.497}$$

This can be written as the stochastic integral:

$$u(x,t) = \int e^{-\int_0^t k(x_s)ds} p(x,t)dx \tag{3.498}$$

where $x_t = x$, and we drop the subscripts where convenient.

Lemma 3.5.5. *Take a function $u(x,t)$ function as in 3.5.4, and further the process with boundary condition $x_t = x$, with stochastic dynamics 3.5.1. Assume further that we have stochastic differential equation defined through the PDE relation:*

$$df = \frac{\partial f}{\partial t}dt + \mu\frac{\partial f}{\partial x}dx + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\langle dx, dx\rangle \tag{3.499}$$

in the sense of Ito, where $\langle dx, dx\rangle = dx^2 = \mathcal{O}(dt)$. The stationary point of the expectation value equation $\mathbb{E}[du] = 0$ is given by the solution to the differential equation:

$$-k(x)p(x,t) + \frac{\partial p(x,t)}{\partial t} + \mu\frac{\partial p(x,t)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p(x,t)}{\partial x^2} = 0 \tag{3.500}$$

This is the Feynman-Kac formula [18,21].

Remark 43. We have shown this only for time independent systems as given through the parameter dependence of the killing measure $k(.)$. This property is easily extended to fully time dependent potentials. As this thesis is focused on the eigenvalue representations, we shall not require this generalisation to determine the solutions herein. Although the solution to this problem is well-known historically, (see the original works by Feynman [19] and Kac [21]), we illustrate this connection for completeness, as in the following lemma we shall show how any diffusion problem of this type may be recast into a particularly simple form through use of the Bose invariant.

Proof. Taking 3.5.4, 3.5.1, $x_t = x$,

$$u(x,t) = \mathbb{E}\left[e^{-\int_0^t k(s)ds}\right] \tag{3.501}$$

and applying the Ito lemma:

$$du = \frac{\partial}{\partial t}\int e^{-\int_0^t k(x_s)ds} p(x,t)dxdt \tag{3.502}$$

$$+\frac{\partial}{\partial x}\int e^{-\int_0^t k(x_s)ds} p(x,t)dxdx + \frac{1}{2}\frac{\partial^2}{\partial x^2}\int e^{-\int_0^t k(x_s)ds} p(x,t)dx(dx)^2 \tag{3.503}$$

$$= \int e^{-\int_0^t k(x_s)ds}\left(-k(x)p(x,t)dt + \frac{\partial p(x,t)}{\partial t}dt + \frac{\partial p(x,t)}{\partial x}dx + \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2}(dx)^2\right) \tag{3.504}$$

$$= \int e^{-\int_0^t k(x_s)ds}\left(-k(x)p(x,t)dt + \frac{\partial p(x,t)}{\partial t}dt + \mu\frac{\partial p(x,t)}{\partial x}dt + \frac{\sigma^2}{2}\frac{\partial^2 p(x,t)}{\partial x^2}dt\right) \tag{3.505}$$

$$+\int e^{-\int_0^t k(x_s)ds}\left(\sigma\frac{\partial p(x,t)}{\partial x}dW_t\right) \tag{3.506}$$

where we used 3.5.1. Applying $\mathbb{E}\left[du\right] = 0$, we have as the killing measure and distribution are t-measurable functions:

$$\mathbb{E}[\int e^{-\int_0^t k(x_s)ds}\left(\sigma\frac{\partial p(x,t)}{\partial x}dW_t\right)] = \int e^{-\int_0^t k(x_s)ds}\sigma\frac{\partial p(x,t)}{\partial x}\mathbb{E}\left(dW_t\right) = 0 \tag{3.507}$$

from which we obtain:

$$\int e^{-\int_0^t k(x_s)ds}\left[-k(x)p(x,t) + \frac{\partial p(x,t)}{\partial t} + \mu\frac{\partial p(x,t)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p(x,t)}{\partial x^2}\right]\mathbb{E}[dt] = 0 \tag{3.508}$$

Using $\mathbb{E}[dt] = dt$, as t is not a stochastic variable, from the integral we find:

$$\int \eta(x,t)dt = 0 \tag{3.509}$$

implying that $\eta(x,t) = 0$, and we obtain the result. □

We shall now use this lemma to derive the basic formula for the eigenfunction systems encountered in this thesis, in particular those that are of the form 3.4.4.

Lemma 3.5.6. *There exists a separable solution to the PDE expressed through the Feynman-Kac formula as in 3.5.5. This is given by the decomposition $p(x,t) = w(t)f(x)$. This defines an eigenvalue problem:*

$$\frac{dw(t)}{dt} = -Ew(t) \tag{3.510}$$

$$\left(k(x) + E\right)f(x) + \frac{\sigma^2}{2}\frac{d^2 f}{dx^2} + \mu\frac{df}{dx} = 0 \tag{3.511}$$

Proof. Using the form of the separable solution, we have:

$$\frac{1}{w(t)}\frac{dw(x,t)}{dt} = \frac{1}{f(x)}\left(k(x)f(x) + \frac{\sigma^2}{2}\frac{d^2 f}{dx^2} + \mu\frac{df}{dx}\right) = -E \tag{3.512}$$

where the constant E follows as the only function of x and t that satisfies this is a constant. We therefore have the result. □

We can now state the following corollary.

Corollary 3.5.7. *The generator of the killed process 3.5.1 is:*

$$\frac{\sigma^2(x)}{2}f'' + \mu(x)f' + \hat{k}(x)f = 0 \tag{3.513}$$

where σ is the variance, μ is the drift and the process is killed at rate $\hat{k}(x)$. We associate this generator with the set of eigenfunctions which solve the implied ODE.

Proof. Trivial, without loss of generality we take $\hat{k}(x) = k(x) + E$ in 3.4.4, alternately the ground state $E = 0$ yields the equivalent result. □

Remark 44. It is known from statistical mechanics considerations that the free energy of the system is given by the ground state, which in this instance is given by $E = 0$. From the free energy, one can construct the relevant distribution functions. In the same way, knowing the distribution of eigenstates shifted by a constant is the same as the unshifted distribution, up to a phase or scaling factor depending on whether the system is in real or imaginary time, in this way we simply take $k(x) + E \to k(x)$, and backsubstitute after solving the equations.

We shall now take some definitions from [56,61,62,75,117] regarding the speed and scale distributions.

Definition 3.5.8. The speed and scale measures $\mathfrak{m}(x), \mathfrak{s}(x)$ of the differential equation defined by $\dfrac{\sigma^2(x)}{2}f'' + \mu(x)f' + k(x)f = 0$ (e.g. 3.5.7) are the functions of the drift and variance:

$$\mathfrak{m}(x) = \frac{2}{\sigma^2(x)}\exp\left(\int^x \frac{2\mu(s)}{\sigma^2(s)}ds\right) \tag{3.514}$$

and

$$\mathfrak{s}'(x) = \exp\left(-\int^x \frac{2\mu(s)}{\sigma^2(s)}ds\right) \tag{3.515}$$

We shall now prove that the speed and scale measures are invariant measures on the solution space given by the unkilled problem.

Lemma 3.5.9. *The generator of the unkilled process defined by the eigenvalue problem 3.5.7 is:*

$$\mathcal{L}f = a(x)\frac{\partial^2 f}{\partial x^2} + b(x)\frac{\partial f}{\partial x} \tag{3.516}$$

Proof. Taking $\mathcal{L}f = -Ef$, with $k(x) = 0$ in 3.5.7, and substituting $a(x) = \dfrac{\sigma^2}{2}, b(x) = \mu$ we immediately obtain the result. □

Lemma 3.5.10. *The generator $\mathcal{L}$ is invariant under the following transformation:*

$$a^{-1}[h(x)]^{-1}\mathcal{L}\left[h(x)f(x)\right] = \frac{\partial^2 f}{\partial x^2} + \mathcal{I}(x)f(x) \tag{3.517}$$

where

$$h(x) = \exp\left(-\int^x \frac{b(s)}{2a(s)} ds\right) \tag{3.518}$$

is the scale factor of the system.

Proof. Take the scale factor equal to:

$$h(x) = \exp\left(-\int^x \frac{b(s)}{2a(s)} ds\right) \tag{3.519}$$

Substituting, we obtain:

$$a^{-1}[h(x)]^{-1}\mathcal{L}\left[h(x)f(x)\right] = \frac{\partial^2 f}{\partial x^2} + \left[\frac{a'b - b'a}{2a^2} - \frac{1}{4}\left(\frac{b}{a}\right)^2\right] f(x) \tag{3.520}$$

$$= \frac{\partial^2 f}{\partial x^2} + \mathcal{I}(x) f(x) \tag{3.521}$$

as required. □

We shall now prove a theorem that shall show the killed process may be solved in a similar way.

Theorem 3.5.11. *The killed process 3.4.4 defined by the generator:*

$$\mathcal{L} = a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x} + c(x) \tag{3.522}$$

has normal invariant form:

$$a^{-1}[h(x)]^{-1}\mathcal{L}\left[h(x)f(x)\right] = \frac{\partial^2 f}{\partial x^2} + \left[\frac{2(a'b - b'a) + 4ac - b^2}{4a^2}\right] f(x) \tag{3.523}$$

We call the function:

$$\mathcal{I}(x) = \frac{2(a'b - b'a) + 4ac - b^2}{4a^2} \tag{3.524}$$

the Bose invariant of the generator given by the killed process.

Proof. Transforming 3.5.10 by $\mathcal{L} \to \mathcal{L} + c(x)$, we obtain the generator in 3.5.11. We therefore have:

$$a^{-1}[h(x)]^{-1}\left(\mathcal{L} + c(x)\right)\left[h(x)f(x)\right] = a^{-1}[h(x)]^{-1}\mathcal{L}\left[h(x)f(x)\right] + \frac{c(x)}{a(x)} f(x) \tag{3.525}$$

$$= \frac{\partial^2 f}{\partial x^2} + \left[\frac{2(a'b - b'a) + 4ac - b^2}{4a^2}\right] f(x) \tag{3.526}$$

as required. □

Remark 45. A thorough discussion of the application of Bose invariants to various stochastic processes may be found in [61,62], and the original research by A. K. Bose in [22] (c. 1964), who tried to examine all different types of potentials for second order DEs with solvable eigenfunctions. The principal utility is in reducing the differential equations for the eigenfunctions to a more analytically tractable form. They are also fundamental symmetries of the differential operators in question. By putting this together with the kernel formulae, many different identities can be developed.

We shall now show how the speed and scale measures 3.5.8 are related to the scale factor and Bose invariant method as in 3.5.11, 3.5.10.

Lemma 3.5.12. *The speed and scale measures are defined as:*

$$\mathfrak{m}(x) = \frac{2}{\sigma^2(x)} \exp\left(\int^x \frac{2b(s)}{\sigma^2(s)} ds\right) = \frac{1}{a} \exp\left(\int^x \frac{b(s)}{a(s)} ds\right) \tag{3.527}$$

$$\mathfrak{s}'(x) = \exp\left(-\int^x \frac{b(s)}{a(s)} ds\right) \tag{3.528}$$

Proof. Substitution of $a(x) = \dfrac{\sigma^2}{2}$, $b(x) = \mu$ into 3.5.8 produces the result. □

Lemma 3.5.13. *For the ODE $\mathcal{L} = a(x)\partial_x^2 + b(x)\partial_x$ as in 3.5.10, the scale measure has eigenvalue zero and solves:*

$$\mathcal{L}\mathfrak{s}(x) = 0 \tag{3.529}$$

Corollary 3.5.14. *Taking the definition of the speed and scale measures from 3.5.8, the invariant distributions are related by:*

$$\mathfrak{s}(x) = \int \frac{1}{a(y)\mathfrak{m}(y)} dy \tag{3.530}$$

Proof. Taking 3.5.10, we have:

$$\mathcal{L}f = a(x)\frac{\partial^2 f}{\partial x^2} + b(x)\frac{\partial f}{\partial x} \tag{3.531}$$

Assume we solve $\mathcal{L}f = 0$, then we may write:

$$a(x)\partial_x^2 f + b(x)\partial_x f = 0 \tag{3.532}$$

which may be rewritten:

$$\frac{\partial_x^2 f}{\partial_x f} = -\frac{b(x)}{a(x)} \tag{3.533}$$

or

$$\frac{\partial}{\partial x}\left(\ln\left(\frac{\partial f}{\partial x}\right)\right) = -\frac{b(x)}{a(x)} \tag{3.534}$$

Integrating and taking exponentials, we find:

$$\frac{\partial f}{\partial x} = \exp\left(-\int_0^x \frac{b(s)}{a(s)} ds\right) \tag{3.535}$$

$$f(x) = \int^x \exp\left(-\int^y \frac{b(s)}{a(s)} ds\right) dy \tag{3.536}$$

i.e. $f(x) = s(x)$, as in 3.5.8. To prove the corollary, simply substitute the definitions from 3.5.8 into the result. □

To conclude, we shall prove the following.

Lemma 3.5.15. *Define the integrated speed and scale measures by the formulae:*

$$\mathfrak{s}(x) = \int^x s(y) dy \tag{3.537}$$

$$\mathfrak{m}(x) = \int^x m(y) dy \tag{3.538}$$

$$\mathfrak{s}(x) = \int^x \exp\left(-\int^y \frac{b(s)}{a(s)} ds\right) dy \tag{3.539}$$

$$\mathfrak{m}(x) = \int^x \frac{1}{a} \exp\left(\int^y \frac{b(s)}{a(s)} ds \right) dy \tag{3.540}$$

Then the following expressions are true:

$$m(x) = \frac{1}{a(x)s(x)} \tag{3.541}$$

$$\frac{s'}{s} = -\frac{b}{a} \tag{3.542}$$

$$\mathfrak{m}'(x)\mathfrak{s}'(x) = \frac{1}{a} \exp\left(-\int^x \frac{b(s)}{a(s)} ds \right) \exp\left(\int^x \frac{b(s)}{a(s)} ds \right) = \frac{1}{a(x)} \tag{3.543}$$

Proof. The first identity follows immediately from the the definition of the integral, as does the second. The third identity is readily obtained:

$$\mathfrak{m}'(x)\mathfrak{s}'(x) = \frac{1}{a} \exp\left(-\int^x \frac{b(s)}{a(s)} ds \right) \exp\left(\int^x \frac{b(s)}{a(s)} ds \right) = \frac{1}{a(x)} \tag{3.544}$$

as required. □

We shall now prove a theorem that shows how the speed and scale distributions are intimately related to the diffusion problem specified through 3.4.4, 3.5.9. We shall require the following definition:

Lemma 3.5.16. *The L'Hospital derivative is given by:*

$$[\hat{D}_g f](x) = \frac{f'(x)}{g'(x)} \tag{3.545}$$

The Feller form of the PDE system $\mathcal{L} = a(x)\partial_x^2 + b(x)\partial_x$ is given by:

$$[\hat{D}_\mathfrak{m} \hat{D}_\mathfrak{s} f](x) = a(x)\frac{\partial^2 f}{\partial x^2} + b(x)\frac{\partial f}{\partial x} \tag{3.546}$$

where $\mathfrak{s}, \mathfrak{m}$ are the speed and scale measures as in 3.5.8.

Proof. Using the definition of the L'Hospital derivative, we have:

$$[\hat{D}_\mathfrak{m} \hat{D}_\mathfrak{s} f](x) = \frac{1}{\mathfrak{m}'(x)} \frac{\partial}{\partial x} \left(\frac{f'(x)}{\mathfrak{s}'(x)} \right) = \frac{1}{\frac{\partial \mathfrak{m}(x)}{\partial x}} \frac{\partial}{\partial x} \left(\frac{1}{\frac{\partial \mathfrak{s}(x)}{\partial x}} \frac{\partial f(x)}{\partial x} \right) \tag{3.547}$$

$$= \frac{d}{d\mathfrak{m}(x)} \frac{d}{d\mathfrak{s}(x)} f(x) \tag{3.548}$$

$$= a(x)\frac{\partial^2 f}{\partial x^2} + b(x)\frac{\partial f}{\partial x} \tag{3.549}$$

as required. □

This concludes this brief description of the transform theory of second-order PDEs. We note the long and illustrious history that this subject enjoys, and emphasise that in many ways this can be traced to the investigations of Sturm and Liouville on the topic. This thesis shall borrow heavily from this technique of transformation and invariance, and as we shall show that the method of Bose invariance as given by the formulae derived in 3.5.11, 3.5.10 is of particular use in determining solutions to PDEs.

We have shown in this part of the thesis a number of different techniques which we shall use to determine the hyperbolic heat kernel. We have derived formulae that will enable us to determine, from the basic constraints of the system, the fundamental unitary operators that apply on the space. We are able to associate that with a metric, given by the Fubini-Study metric of projective geometry, and through that method derive the Laplace operator.

In tandem with this investigation, we have shown how the use of positive definite kernels can be related to the group invariance properties via the shift theorems of Vilenkin [69]. We have demonstrated that this technique is formally equivalent to the method of Mercer's theorem and Stone's theorem. Consequently, from knowledge of the distance functions that operate on the space we were able to determine the kernel as a product or composition formula, which defines a convolution. The way in which this was achieved was through the use of spherical decompositions over certain groups of special functions. This powerful technique, when coupled with the spectral theory 3.1.18 gives us much insight into the behaviour of hyperbolic systems.

CHAPTER 4

Hyperbolic Heat Kernel

4.1 Introduction and Review

This chapter develops the theory of the quantum brachistochrone, and extends it to the hyperbolic plane. In the same way that the trigonometric functions can be analytically continued to give hyperbolic equivalents, the following calculation shows how the quantum brachistochrone still maintains validity in the hyperbolic realm. The major modification is then from unitary to pseudounitary operators. To the author's knowledge, this is original work and should be considered the extension of the initial problem calculated in the quantum brachistochrone hypothesis, stated in one form in the work of Carlini [49] and extended by the author in the series of works [52–55]. We shall show how the synthesis given by the Fubini-Study metric results in a metric known in classical analysis as the Poincare metric for the hyperbolic disk. This is an important stepping stone in our analysis of the hyperbolic heat kernel, as we must determine this in order to find the associated Laplace operator. Following this initial analysis, we discuss the various systems of eigenfunctions on the Poincare disk, including Mehler-Fock, Macdonald and Whittaker functions. The relationship between these systems of differential equations is exploited using an Laplace transform method on the hyperbolic plane, which allows us to transform the solutions to the Helmholtz equation from one space to another. Discussion of further applications of this technique is given with particular reference to diffusion systems on alternate forms of hyperbolic and projective geometry.

4.2 Outline of Chapter

As we have seen in the preceding chapter, the quantum brachistochrone principle can be cast as a least action optimisation with linear and isotropic constraints, as shown in 3.2.10. This leads naturally to consideration of the Fubini-Study metric more generally and a detailed analysis of the relationship between the differential geometry implied by unitary operators and the associated Laplacian in curvilinear spaces as given through this metric. The quantum brachistochrone method is a technique for finding paths between states which optimise time of flight given constraints. This uses the quantum brachistochrone equation 3.2.7 to derive unitary operators which give these optimum trajectories on the complex projective space.

In the following calculations, we shall derive the hyperbolic equivalent of the quantum brachistochrone equation 3.2.7. This shall then be used to give a formal basis to the Fubini-Study metric 3.2.33, and allow us to define the Laplacian 3.2.34. As we shall prove, the Fubini-Study metric for the hyperbolic disk is given by the Poincare metric, as in 3.2.39. This gives us a simple entry point into a discussion of the eigenfunctions, some related groups and composition formulae associated with the hyperbolic space. By computing the brachistochrone equation on the hyperbolic surface, the metrics and differential geometry developed are then used to define a diffusion problem; this is then solved to produce the hyperbolic heat kernel. By examining the various different transformations between groups of eigenstates, it is then shown that a number of seemingly disparate systems are effectively transmutable. This is a theme that will feature heavily in following chapters, where we consider the analogues of the projection-slice theorem and the Euler-Poisson-Darboux equation.

Note that although Vilenkin [69], using the method of shift operators as in 3.3.21, 3.3.26 gives valid constructions for the kernel for a wide class of unitary, pseudounitary and affine transformation groups, there is no underlying method for generating these groups themselves other than geometric postulates. We propose an alternate schemata, where these classes of matrices are derived from the brachistochrone equation 3.2.10, and the representation theory follows as a consequence of the geometry as given through the Laplacian in the curved space.

4.3 Hyperbolic Brachistochrone

The quantum brachistochrone principle seeks to find optimal operators for state-to-state transfer on the complex projective space of states, given linear constraints and the condition of isotropy. This is an optimisation problem that is similar in many respects to Fermat's principle of least time in ray optics, where the physical path is given by the solution to the least time path, given the source and the observer. This known equivalence appears here in a different context, where the Hamiltonian and associated unitary operator define the state transfer via the projective geometry given by the Fubini-Study metric.

In the following section, we shall show how one may derive and implement many of the principles that follow from the results calculated in 3.2.8, 3.2.10. This shall be referred to as the quantum brachistochrone method, where we begin with a constrained time optimisation problem that determines the unitary matrices that operate upon the state. This optimisation problem, equivalent to the physical regime of least time under bounded constraints, yields solutions for the unitary operators, and the differential geometry. This forms the cornerstone for our further investigations into the representation theory and structure of the convolution groups we shall require in order to construct the hyperbolic heat kernel.

The quantum brachistochrone method depends entirely on the existence of unitary operators, and we shall exploit many of their properties to evaluate the matrix algebra that is associated to the complex projective space of states. We note that the quantum brachistochrone equation, as expressed through the matrix differential equation 3.2.7 can be taken as the basic concept from which we may develop our understanding of the principles of this complex state space. As the state space is unitary, for Hermitian generators, the representation theory being unital via the homomorphism as expressed through the structure of the positive definite kernels. However, as is well-known from classic hyperbolic analysis, the geometric constructions of hyperbolic planes are not unitary, only pseudounitary. To illustrate the differences, consider the basic norm defined by the inner product:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\dagger \mathbf{y} \tag{4.1}$$

The norm of any vector in this space may be written as $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ and is invariant under the substitution of a unitary transformation, via:

$$\left\|\hat{U}\mathbf{x}\right\| = \sqrt{\left\langle \hat{U}\mathbf{x}, \hat{U}\mathbf{x} \right\rangle} = \sqrt{(\hat{U}\mathbf{x})^\dagger \hat{U}\mathbf{x}} = \sqrt{\mathbf{x}^\dagger \hat{U}^\dagger \hat{U}\mathbf{x}} = \|\mathbf{x}\| \tag{4.2}$$

where $\hat{U}^{\dagger}\hat{U} = \mathbf{1}$ is the unitary property. We write this generally with the dagger representing the Hermitian transpose of the vector. In the case of a real inner product, we will have a transpose instead of a Hermitian transpose, and this will have a real inner product. However, in the case of a hyperbolic system, we will have an indefinite inner product, which means we must generalise this situation to one where we have $\langle \mathbf{x}, \mathbf{y} \rangle = \bar{\mathbf{x}}\hat{L}\mathbf{y}$, where $\hat{L}$ is a matrix is a constant matrix with trace signature containing positive and negative parts. For example, in the simplest case of $2 \times\ 2$ matrices, we have:

$$\hat{L} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.3}$$

and therefore the pseudonorm in this instance is given through:

$$\langle \mathbf{x}, \mathbf{x} \rangle = \begin{bmatrix} \bar{x_1}, & \bar{x_2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = |x_1|^2 - |x_2|^2 \tag{4.4}$$

where the bar represents an adjoint, generally consisting of inversion of some, but not all, co-ordinates, transposition and conjugation in some form. The equivalent transformations to the unitary operators still satisfy a type of identity:

$$\langle \mathbf{x}, \mathbf{x} \rangle = \langle Q\mathbf{x}, Q\mathbf{x} \rangle = \bar{\mathbf{x}}\bar{Q}\hat{L}Q\mathbf{x} \tag{4.5}$$

$$\bar{Q}\hat{L}Q = \hat{L} \tag{4.6}$$

but in this case, two vectors which are not identical can have a norm of zero together (hence the naming of this object a pseudonorm). This requires much more care in treating this space. When we say pseudounitary, we mean that the norm is no longer positive definite, and the matrix unitaries are replaced by pseudounitaries. In practice, this means replacing the Hermitian transpose or real transpose operation by a matrix inverse which performs a similar operation. As can be seen in this simple example, an indefinite inner product can be seen as a hyperbolic system through $\langle \mathbf{x}, \mathbf{x} \rangle = r^2$ for an indefinite inner product space, and we shall utilise this type of relationship to understand the structure of the hyperbolic heat kernel.

For this reason, we extend the method of the quantum brachistochrone to the hyperbolic plane, and hyperbolic spaces more generally. As is well known, SU(2) is associated to a spherical geometry; we are interested in how the SU(2)↔SU(1,1) equivalence may be exploited. In general terms, we expect that it will take the form of a rotation into imaginary time, in the same way that a Wick rotation [118] allows us to take a complex path integral for a quantum space and use this to generate a real path integral for a stochastic space, in imaginary time.

4.3.1 Quantum Brachistochrone Equation

We begin with a brief review of the basic solution on SU(2) for the quantum brachistochrone equation 3.2.7. We shall take the conditions 3.2.6, 3.2.4 as our axioms, along with the von Neumann equation 3.2.1 and the unitary operator 3.1.9. Our proof begins with a replication of the results of [49,52] for the 2×2 matrices. We take the following basic definition from linear algebra for the Pauli matrices.

Definition 4.3.1. The Pauli matrices are defined as the generators of SU(2):

$$\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \hat{\sigma}_x = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.7}$$

We now define the inputs for the basic time optimal quantum control problem, through the Hamiltonian matrix and associated constraint, which are to be determined by the optimisation procedure as given by the quantum brachistochrone equation.

Definition 4.3.2. Take the Hamiltonian operator:

$$\tilde{H}(t) = \lambda_x(t)\hat{\sigma}_x + \lambda_y(t)\hat{\sigma}_y = \begin{bmatrix} 0 & \epsilon(t) \\ \epsilon^*(t) & 0 \end{bmatrix} \tag{4.8}$$

and constraint:

$$\tilde{F}(t) = \omega_z(t)\hat{\sigma}_z = \begin{bmatrix} \omega_z(t) & 0 \\ 0 & -\omega_z(t) \end{bmatrix} \tag{4.9}$$

where $\epsilon(t), \epsilon^*(t) \in \mathbb{C}$ and $\omega_z(t) \in \mathbb{R}$ are arbitrary functions of time to be determined by the solution to the quantum brachistochrone equation 3.2.7. These matrices form a partition of the generators of SU(2) into two subgroups, up to linear combination of the multiplying functions.

Lemma 4.3.3. *Under the condition that the function $\epsilon(t)$ be bounded, the Hamiltonian $\tilde{H}(t)$ and constraint $\tilde{F}(t)$ satisfy the isotropic and linear conditions, 3.2.6, 3.2.4, and are tracefree and Hermitian.*

Proof. That these matrices are tracefree and Hermitian follows from the matrix representations in 4.3.2. To evaluate the linear constraint, we have from 3.2.4:

$$\mathrm{Tr}\left[\tilde{H}\tilde{F}\right] = \mathrm{Tr}\left[\begin{bmatrix} 0 & \epsilon(t) \\ \epsilon^*(t) & 0 \end{bmatrix}\begin{bmatrix} \omega_z(t) & 0 \\ 0 & -\omega_z(t) \end{bmatrix}\right] = 0 \tag{4.10}$$

For the isotropic constraint:

$$\mathrm{Tr}\left[\frac{1}{2}\tilde{H}^2\right] = \frac{1}{2}\mathrm{Tr}\left[\begin{bmatrix} 0 & \epsilon(t) \\ \epsilon^*(t) & 0 \end{bmatrix}\begin{bmatrix} 0 & \epsilon(t) \\ \epsilon^*(t) & 0 \end{bmatrix}\right] \tag{4.11}$$

$$= \frac{1}{2}\left(|\epsilon(t)|^2 + |\epsilon(t)|^2\right) = |\epsilon(t)|^2 < \infty \tag{4.12}$$

as required. □

We shall now apply the theorem given by the quantum brachistochrone equation 3.2.7.

Theorem 4.3.4. *The matrices given by the constraint $\tilde{F}(t)$, the Hamiltonian matrix $\tilde{H}(t)$ and their sum $\tilde{H}(t) + \tilde{F}(t)$ are Hermitian for all times t. Their evolution is unitary, and governed by the equations:*

$$\hat{U}(t,0)\left[\tilde{H}(0) + \tilde{F}(0)\right]\hat{U}^\dagger(t,0) = \tilde{H}(t) + \tilde{F}(t) \tag{4.13}$$

$$\hat{U}(t,0)\tilde{H}(0)\hat{U}^\dagger(t,0) = \tilde{H}(t) \tag{4.14}$$

$$\hat{U}(t,0)\tilde{F}(0)\hat{U}^\dagger(t,0) = \tilde{F}(t) \tag{4.15}$$

If the constraint is constant in time, $\tilde{F}(t) = \tilde{F}(0)$, as determined by the solution to the quantum brachistochrone equation 3.2.10, this information is sufficient to determine the form of the unitary transformation which solves the Schrödinger equation.

Proof. We shall show one form of the unitary. Assume the first formula, we have:

$$\hat{U}(t,0)\left[\tilde{H}(0) + \tilde{F}(0)\right]\hat{U}^\dagger(t,0) = \tilde{H}(t) + \tilde{F}(t) \tag{4.16}$$

Left multiplying by $\hat{U}(t,0)$, we have:

$$\hat{U}(t,0)\left[\tilde{H}(0) + \tilde{F}(0)\right] = \tilde{H}(t)\hat{U}(t,0) + \tilde{F}(t)\hat{U}(t,0) \tag{4.17}$$

Further, under the assumptions of the theorem, we have constancy of the constraint $\tilde{F}(t) = \tilde{F}(0)$, hence we find:

$$\hat{U}(t,0)\left[\tilde{H}(0)+\tilde{F}(0)\right]-\tilde{F}(0)\hat{U}(t,0)=\tilde{H}(t)\hat{U}(t,0)=i\frac{d}{dt}\hat{U}(t,0) \tag{4.18}$$

where we used the Schrödinger equation. Obviously this can be written as:

$$\hat{U}(t,0)=\exp\left(i\tilde{F}(0)t\right)\exp\left(-i\left[\tilde{H}(0)+\tilde{F}(0)\right]t\right)=\hat{U}_1(t,0)\hat{U}_2(t,0) \tag{4.19}$$

We show further some other ways we can derive the unitary operator. If the constraint is constant, we must have $\tilde{F}(t) = \tilde{F}(0)$, which implies the symmetry:

$$\hat{U}(t,0)\tilde{F}(0)\hat{U}^\dagger(t,0)=\tilde{F}(0) \tag{4.20}$$

which we can rewrite as the commutation relation $[\tilde{F}(0), \hat{U}(t,0)] = 0$. Further, writing the unitary as the product of two matrices, we have $\hat{U}(t,0) = \hat{U}_1(t,0)\hat{U}_2(t,0)$, and this is equivalent to the commutation relationship:

$$[\tilde{H}(0)+\tilde{F}(0),\hat{U}_2(t,0)]=0 \tag{4.21}$$

which implies that, as the transformations are unitary, we must have:

$$\hat{U}_2(t,0)=\exp\left(-it\left(\tilde{H}(0)+\tilde{F}(0)\right)\right) \tag{4.22}$$

and further, that the other unitary matrix must obey:

$$\hat{U}_1(t,0)\tilde{H}(0)\hat{U}_1^\dagger(t,0)=\tilde{H}(t) \tag{4.23}$$

and

$$\hat{U}_1(t,0)\tilde{F}(0)\hat{U}_1^\dagger(t,0)=\tilde{F}(t)=\tilde{F}(0) \tag{4.24}$$

Again, we arrive at the same decomposition:

$$\hat{U}(t,0)=\exp\left(it\tilde{F}(0)\right)\exp\left(-it\left(\tilde{H}(0)+\tilde{F}(0)\right)\right) \tag{4.25}$$

which satisfies the unitary evolution of the Hamiltonian plus constraint:

$$\hat{U}(t,0)\left[\tilde{H}(0)+\tilde{F}(0)\right]\hat{U}^\dagger(t,0)=\tilde{H}(t)+\tilde{F}(t)=\tilde{H}(t)+\tilde{F}(0) \tag{4.26}$$

where we have used the assumption that the constraint is constant in time. In this thesis, we shall only be working with very simple contrived forms of the constraint where this is true. Finally, we can evaluate the unitary matrices in the decomposition via:

$$\tilde{H}(0)+\tilde{F}(0)=\hat{P}\hat{D}\hat{P}^\dagger \tag{4.27}$$

where $\hat{P}$ is a unitary matrix which brings the sum of the initial constraint and Hamiltonian into diagonal form $\hat{D}$, being a matrix of constant eigenvalues. We can evaluate the unitary operator explicitly:

$$\hat{U}_2(t,0)=\hat{P}\exp(-it\hat{D})\hat{P}^\dagger=\hat{Q}(t)\hat{Q}^\dagger(0) \tag{4.28}$$

where $\hat{Q}(t) = \hat{P}\exp(-it\hat{D})$. This may also be derived through use of the eigenvalue decomposition directly of $\tilde{H}(0) + \tilde{F}(0)$ and $\tilde{H}(t) + \tilde{F}(0)$. Other decompositions exist for $\hat{U}_1(t,0)$. For example, if there exists a diagonalising transform, taking the Hamiltonian matrix into a matrix of constant values, we may write:

$$\tilde{H}(t)=\hat{W}(t)\hat{L}\hat{W}^\dagger(t) \tag{4.29}$$

In particular, we have:

$$\hat{L} = \hat{W}^{\dagger}(t)\tilde{H}(t)\hat{W}(t) = \hat{W}^{\dagger}(0)\tilde{H}(0)\hat{W}(0) \tag{4.30}$$

implying that the unitary relationship can be recovered through:

$$\tilde{H}(t) = \hat{W}(t)\hat{W}^{\dagger}(0)\tilde{H}(0)\hat{W}(0)\hat{W}^{\dagger}(t) \tag{4.31}$$

and hence $\hat{U}_1(t,0) = \hat{W}(t)\hat{W}^{\dagger}(0)$. Of course, we must verify that:

$$\hat{U}_1(t,0)\tilde{F}(0)\hat{U}_1^{\dagger}(t,0) = \tilde{F}(t) = \tilde{F}(0) \tag{4.32}$$

□

Remark 46. Note that for any chosen constraint and Hamiltonian, the part given by the unitary matrix $\exp\left(-i\left[\tilde{H}(0)+\tilde{F}(0)\right]t\right) = \hat{U}_2(t,0)$ will be similar for each system we choose, it will be a constant rotation in the Hilbert space in the plane of the initial values of the Hamiltonian and constraint. Further differences in dynamics will come from the first part of the time evolution operator, which encodes the behaviour of the state under the Hamiltonian matrix for different choices of the constraint. This will be given by the eigenvalue decomposition.

Theorem 4.3.5. *The quantum brachistochrone problem, specified by the solution to the matrix differential equation 3.2.7, and for the case 4.3.2 is:*

$$i\frac{d}{dt}\left(\tilde{H}(t)+\tilde{F}(t)\right) = \left[\tilde{H}(t),\tilde{F}(t)\right] \tag{4.33}$$

This equation is equivalent to the condition that the sum of the constraint and Hamiltonian evolves unitarily, as it must because it is Hermitian:

$$\tilde{H}(t)+\tilde{F}(t) = \hat{U}(t,0)\left[\tilde{H}(0)+\tilde{F}(0)\right]\hat{U}^{\dagger}(t,0) \tag{4.34}$$

Proof. Take the expression, and substitute in the unitary evolution, we find:

$$i\frac{d}{dt}\left(\tilde{H}(t)+\tilde{F}(t)\right) = i\frac{d\hat{U}(t,0)}{dt}\left[\tilde{H}(0)+\tilde{F}(0)\right]\hat{U}^{\dagger}(t,0)+\hat{U}(t,0)\left[\tilde{H}(0)+\tilde{F}(0)\right]i\frac{d\hat{U}^{\dagger}(t,0)}{dt} \tag{4.35}$$

Using the Schrödinger equation, we have the dynamics for the time evolution operator via:

$$i\frac{d\hat{U}(t,0)}{dt} = \tilde{H}(t)\hat{U}(t,0) \tag{4.36}$$

and for the adjoint:

$$-i\frac{d\hat{U}^{\dagger}(t,0)}{dt} = \hat{U}^{\dagger}(t,0)\tilde{H}(t) \tag{4.37}$$

Substituting, we obtain:

$$i\frac{d}{dt}\left(\tilde{H}(t)+\tilde{F}(t)\right) = -\tilde{H}(t)\hat{U}(t,0)\left[\tilde{H}(0)+\tilde{F}(0)\right]\hat{U}^{\dagger}(t,0)+\hat{U}(t,0)\left[\tilde{H}(0)+\tilde{F}(0)\right]\hat{U}^{\dagger}(t,0)\tilde{H}(t) \tag{4.38}$$

$$= -\tilde{H}(t)\left[\tilde{H}(t)+\tilde{F}(t)\right]+\left[\tilde{H}(t)+\tilde{F}(t)\right]\tilde{H}(t) \tag{4.39}$$

$$= \left[\tilde{H}(t),\tilde{F}(t)\right] \tag{4.40}$$

as required. These two conditions are equivalent. Of course, one might proceed from the starting point of the von Neumann equation, in which case this is trivially true. □

Theorem 4.3.6. *The solution to the quantum brachistochrone problem:*

$$i\frac{d}{dt}\left(\tilde{H}(t)+\tilde{F}(t)\right)=\left[\tilde{H}(t),\tilde{F}(t)\right] \tag{4.41}$$

with choices of constraint and Hamiltonian as given by the linear combinations of the Pauli matrices as in 4.3.2

$$\tilde{H}(t)=\lambda_x(t)\hat{\sigma}_x+\lambda_y(t)\hat{\sigma}_y=\begin{bmatrix}0 & \epsilon(t)\\ \epsilon^*(t) & 0\end{bmatrix} \tag{4.42}$$

and constraint:

$$\tilde{F}(t)=\omega_z(t)\hat{\sigma}_z=\begin{bmatrix}\omega_z(t) & 0\\ 0 & -\omega_z(t)\end{bmatrix} \tag{4.43}$$

is given by the Hamiltonian matrix:

$$\tilde{H}(t)=r\begin{bmatrix}0 & e^{-i(2\omega_z t+\phi)}\\ e^{i(2\omega_z t+\phi)} & 0\end{bmatrix} \tag{4.44}$$

and constraint:

$$\tilde{F}(t)=\tilde{F}(0)=\begin{bmatrix}\omega_z & 0\\ 0 & -\omega_z\end{bmatrix} \tag{4.45}$$

For the particular choice of initial condition:

$$\tilde{H}(0)=\begin{bmatrix}0 & r\\ r & 0\end{bmatrix} \tag{4.46}$$

i.e. $\phi=0$ *we have the time dependence explicitly through the Hamiltonian matrix:*

$$\tilde{H}(t)=r\begin{bmatrix}0 & e^{-2i\omega t}\\ e^{2i\omega t} & 0\end{bmatrix} \tag{4.47}$$

Corollary 4.3.7. *The time evolution operator is given by the solution to the unitary evolution of the Hamiltonian matrix plus constraint:*

$$\tilde{H}(t)+\tilde{F}(t)=\hat{U}(t,0)\left[\tilde{H}(0)+\tilde{F}(0)\right]\hat{U}^\dagger(t,0) \tag{4.48}$$

As the constraint matrix is constant, the conditions $\tilde{F}(t)=\tilde{F}(0)$ *hold and the theorem 4.3.4 applies, i.e. we will have:*

$$\hat{U}(t,0)=\exp\left(i\tilde{F}(0)t\right)\exp\left(-i\left[\tilde{H}(0)+\tilde{F}(0)\right]t\right)=\hat{U}_1(t,0)\hat{U}_2(t,0) \tag{4.49}$$

as well as the decomposition from the theorem $\hat{U}(t,0)=\hat{Q}(t)\hat{Q}^\dagger(0)$.

Proof. Note that the first part of the proof originally appears in [49], where the authors used the method of state-to-state transfer to derive essentially the same result. We shall depart from their analysis, and first implement 3.2.7. We find the matrix equation:

$$i\frac{d}{dt}\left(\tilde{H}(t)+\tilde{F}(t)\right)=\left[\tilde{H}(t),\tilde{F}(t)\right]=\begin{bmatrix}0 & -2\omega_z\epsilon\\ 2\omega_z\epsilon^* & 0\end{bmatrix} \tag{4.50}$$

from which we easily conclude that $\dot{\omega}_z=0$, $\epsilon(t)=e^{2i\omega_z t}\epsilon(0)$, the result following upon substitution. We choose $\phi=0$ for convenience without loss of generality in the following discussion. The constraint is therefore a constant matrix, we can therefore apply the theorem 4.3.4, and write the decomposition of the time evolution operator as:

$$\hat{U}(t,0)=\exp\left(i\tilde{F}(0)t\right)\exp\left(-i\left[\tilde{H}(0)+\tilde{F}(0)\right]t\right)=\hat{U}_1(t,0)\hat{U}_2(t,0) \tag{4.51}$$

or

$$\hat{U}(t,0) = \hat{U}_1(t,0)\hat{P}\exp(-it\hat{D})\hat{P}^\dagger \tag{4.52}$$

The matrix $\hat{P}$ satisfies:

$$\hat{P} = i\sqrt{\frac{r}{2\sqrt{\omega^2+r^2}}}\begin{bmatrix} -\dfrac{r}{\sqrt{\omega^2+r^2}+\omega} & \dfrac{r}{\sqrt{\omega^2+r^2}-\omega} \\ 1 & 1 \end{bmatrix} \tag{4.53}$$

and is unitary $\hat{P}\hat{P}^\dagger = \hat{P}^\dagger\hat{P} = \mathbf{1}, \det\hat{P} = 1$. From the decomposition, we obtain:

$$\hat{U}_2(t,0) = \hat{P}\exp(-it\hat{D})\hat{P}^\dagger = \exp\left(-it\left[\tilde{H}(0)+\tilde{F}(0)\right]\right) \tag{4.54}$$

$$= \begin{bmatrix} \cos\Omega t - \dfrac{i\omega}{\Omega}\sin\Omega t & -\dfrac{ir}{\Omega}\sin\Omega t \\ -\dfrac{ir}{\Omega}\sin\Omega t & \cos\Omega t + \dfrac{i\omega}{\Omega}\sin\Omega t \end{bmatrix} \tag{4.55}$$

with $\Omega = \sqrt{\omega^2+r^2}$. The other unitary matrix is readily evaluated, as it is a constant diagonal matrix we have:

$$\hat{U}_1(t,0) = \exp\left(i\omega\hat{\sigma}_z\right) = \begin{bmatrix} e^{-i\omega t} & 0 \\ 0 & e^{+i\omega t} \end{bmatrix} \tag{4.56}$$

The other diagonal operator from the first decomposition is given by:

$$\exp(-it\hat{D}) = \begin{bmatrix} e^{i\Omega t} & 0 \\ 0 & e^{-i\Omega t} \end{bmatrix} \tag{4.57}$$

and the unitary transform $\hat{P}$ satisfies:

$$\hat{P} = i\sqrt{\frac{r}{2\sqrt{\omega^2+r^2}}}\begin{bmatrix} -\dfrac{r}{\sqrt{\omega^2+r^2}+\omega} & \dfrac{r}{\sqrt{\omega^2+r^2}-\omega} \\ 1 & 1 \end{bmatrix} \tag{4.58}$$

with $\det\hat{P} = 1$. We shall now show how the method of eigenvalue decomposition gives a simple answer to this problem, by applying the remaining parts of the theorem. Writing the unitary evolution, and using the time constancy of the constraint, we have:

$$\hat{U}(t,0)\left[\tilde{H}(0)+\tilde{F}(0)\right]\hat{U}^\dagger(t,0) = \tilde{H}(t)+\tilde{F}(t) = \tilde{H}(t)+\tilde{F}(0) \tag{4.59}$$

which we write explicitly in our instance as:

$$\tilde{H}(t)+\tilde{F}(t) = \tilde{H}(t)+\tilde{F}(0) = \tilde{H}(t)+\omega\hat{\sigma}_z \tag{4.60}$$

We can write the eigenvalue decomposition of this matrix using the formula:

$$\hat{Q}(t) = \hat{P}\exp(-it\hat{D}) = \sqrt{\frac{r}{2\Omega}}e^{i\omega t}\begin{bmatrix} \dfrac{re^{-2i\omega t}}{\Omega-\omega} & -\dfrac{re^{-2i\omega t}}{\Omega+\omega} \\ 1 & 1 \end{bmatrix} \tag{4.61}$$

This solves the problem, by inspection we have the unitary operator as given by:

$$\hat{Q}(t)\hat{Q}^\dagger(s) = \begin{bmatrix} e^{-i\omega(t-s)} & 0 \\ 0 & e^{i\omega(t-s)} \end{bmatrix} = \hat{U}(t,s) \tag{4.62}$$

which satisfies:

$$\hat{U}(t,s)\left[\tilde{H}(s)+\tilde{F}(s)\right]\hat{U}^\dagger(t,s) = \tilde{H}(t)+\tilde{F}(t) = \begin{bmatrix} \omega & re^{-2i\omega t} \\ re^{+2i\omega t} & -\omega \end{bmatrix} \tag{4.63}$$

and the constraint matrix is constant in time. Finally, we may arrive at the same matrix via eigenvalue decomposition of the Hamiltonian, we have:

$$\tilde{H}(t) = \hat{W}(t)\hat{W}^\dagger(s)\tilde{H}(s)\hat{W}(s)\hat{W}^\dagger(t) \tag{4.64}$$

However, this is not the complete solution. We can see this by examining the transformations of the Hamiltonian and constraint. The unitary evolution of this operator is:

$$\hat{U}(t,s)\left[\tilde{H}(s) + \tilde{F}(s)\right]\hat{U}^\dagger(t,s) = \tilde{H}(t) + \tilde{F}(t) \tag{4.65}$$

The unitary matrix produced from the eigenvalue decomposition in this instance satisfies:

$$\hat{U}(t,s)\tilde{F}(s)\hat{U}^\dagger(t,s) = \tilde{F}(0) \tag{4.66}$$

and

$$\hat{U}(t,s)\tilde{H}(s)\hat{U}^\dagger(t,s) = \tilde{H}(t) \tag{4.67}$$

which follows from the time-constancy of the constraint, and the unitary evolution of the Hamiltonian. However, the time evolution operator in this instance is not uniquely defined in this way. For example, if we can find a matrix such that:

$$\hat{R}(t,s)\left[\tilde{H}(s) + \tilde{F}(s)\right]\hat{R}^\dagger(t,s) = \tilde{H}(s) + \tilde{F}(s) \tag{4.68}$$

with $\hat{R}(t,s)$ assumed unitary, and taking the new unitary matrix $\hat{U}'(t,s) = \hat{U}(t,s)\hat{R}(t,s)$, we obviously will still satisfy the unitary evolution of the Hamiltonian and constraint via:

$$\hat{U}'(t,s)\left[\tilde{H}(s) + \tilde{F}(s)\right]\hat{U}'^\dagger(t,s) = \tilde{H}(t) + \tilde{F}(t) \tag{4.69}$$

In particular, at the initial time, the symmetry is given by:

$$\hat{R}(t,0)\left[\tilde{H}(0) + \tilde{F}(0)\right]\hat{R}^\dagger(t,0) = \tilde{H}(0) + \tilde{F}(0) \tag{4.70}$$

which implies that we must have:

$$\hat{R}(t,0) = \exp\left(-it\left[\tilde{H}(0) + \tilde{F}(0)\right]\right) \tag{4.71}$$

This function is time-translation invariant, and we recover the previously derived unitary evolution of the Hamiltonian and constraint via:

$$\hat{U}(t,s) = \exp\left(+it\tilde{F}(0)\right)\exp\left(-it\left[\tilde{H}(0) + \tilde{F}(0)\right]\right) \tag{4.72}$$

as required. This is the same decomposition as arrived at through the other methods. □

We shall now analyse the rotation part of the unitary matrix, given by the formula $\hat{U}_2(t,s) = \exp\left(-it\left[\tilde{H}(0) + \tilde{F}(0)\right]\right)$, where this unitary is solely a function of the initial values of the Hamiltonian and constraint matrices.

Theorem 4.3.8. *Assume that the Hamiltonian matrix consists of some combination of two Pauli matrices, with the constraint given by the orthogonal complement, a linear multiple of the remaining element. Further, take an initial condition such that the Hamiltonian is a scalar multiplied by a single generator, given by a Pauli matrix such as $\tilde{H}(0) = r\hat{\sigma}_j$ where j is any one of x, y or z. The rotation matrix part of the unitary, which commutes with the sum of the Hamiltonian and constraint matrices via:*

$$\hat{R}(t,0)\left[\tilde{H}(0) + \tilde{F}(0)\right]\hat{R}^\dagger(t,0) = \tilde{H}(0) + \tilde{F}(0) \tag{4.73}$$

can be expressed as a rotation in one of 6 planes, given by the xy, yx, yz, zy or xz, zx planes. These rotations have analytic matrix form where for example:

$$\hat{R}_{xy}(t,0) = \exp\left(-it\left[\tilde{H}(0) + \tilde{F}(0)\right]\right) = \exp\left(-it\left[r\hat{\sigma}_y + \omega\hat{\sigma}_x\right]\right) \tag{4.74}$$

These rotations do not play a role in the dynamic evolution of the system, i.e. for each set of constraints and Hamiltonian matrices given by decompositions over the Pauli matrices as in the preceding examples, we have the condition that under the rotation the sum of the Hamiltonian and constraint does not change as stated above. The dynamic part of the unitary operator is given by the evolution of the Hamiltonian:

$$\tilde{H}(t) = \hat{U}(t,s)\tilde{H}(s)\hat{U}^{\dagger}(t,s) \tag{4.75}$$

For the 3 situations where the constraint is given by a linear multiple of a single Pauli matrix, and is therefore constant in time, this implies the existence of unitary matrices which satisfy the time evolution equation for the Hamiltonian via:

$$\tilde{H}(t) = \exp\left(+i(t-s)\tilde{F}(0)\right)\tilde{H}(s)\exp\left(-i(t-s)\tilde{F}(0)\right) \tag{4.76}$$

These matrices have explicit form:

$$\hat{U}_z(t,0) = \exp\left(i\omega t\hat{\sigma}_z\right) = \begin{bmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{bmatrix} \tag{4.77}$$

$$\hat{U}_y(t,0) = \exp\left(i\omega t\hat{\sigma}_y\right) = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \tag{4.78}$$

$$\hat{U}_x(t,0) = \exp\left(i\omega t\hat{\sigma}_x\right) = \begin{bmatrix} \cos\omega t & i\sin\omega t \\ i\sin\omega t & \cos\omega t \end{bmatrix} \tag{4.79}$$

The quantum brachistochrone problem on SU(2) for a constraint given by a single generator has solution in terms of these unitary operators, multiplied by a rotation.

Proof. The form of the solution to the quantum brachistochrone equation will not change, as long as the constraint only has a single generator, and the constraint will always be a constant in this instance for SU(2). We are free to take our preferred direction in either the x, y or z axes, which defines the constraint, and more generally along any rotation axis. If we were to change the situation such that we were dealing with a constraint with two generators, the Hamiltonian would be constant, which is simply equivalent to switching the role of the Hamiltonian and constraint. In this case, for SU(2), there will exist transformations such that we can permute the indices of one time optimal control problem into another.

The rotation matrices are readily evaluated using the Euler formula for the matrix exponential:

$$\exp\left(-i\theta\mathbf{n}\cdot\sigma\right) = \mathbf{1}\cos\theta - i(\mathbf{n}\cdot\sigma)\sin\theta \tag{4.80}$$

Writing the time evolution operator for a particular choice of constraint, in this case the Hamiltonian matrix we take to be $\tilde{H}(t) = \lambda_y(t)\hat{\sigma}_y + \lambda_z(t)\hat{\sigma}_z$ and $\tilde{F}(t) = \omega\hat{\sigma}_x$ as constraint matrix, and initial condition $\tilde{H}(0) = r\hat{\sigma}_y$, we must have rotation given by:

$$\hat{R}_{xy}(t,0) = \exp\left(-it\left[\tilde{H}(0) + \tilde{F}(0)\right]\right) = \exp\left(-it\left[r\hat{\sigma}_y + \omega\hat{\sigma}_x\right]\right) \tag{4.81}$$

and dynamic part of the unitary:

$$\hat{U}_x(t,0) = \exp\left(it\tilde{F}(0)\right) = \exp\left(i\omega t\hat{\sigma}_x\right) = \begin{bmatrix} \cos\omega t & i\sin\omega t \\ i\sin\omega t & \cos\omega t \end{bmatrix} \tag{4.82}$$

Further rotations are readily evaluated using an identical method. We have for x, z, z, x:

$$\hat{R}_{xz}(t,0) = \begin{bmatrix} \cos\Omega t - \frac{i\omega}{\Omega}\sin\Omega t & -\frac{ir}{\Omega}\sin\Omega t \\ -\frac{ir}{\Omega}\sin\Omega t & \cos\Omega t + \frac{i\omega}{\Omega}\sin\Omega t \end{bmatrix} \tag{4.83}$$

$$\hat{R}_{zx}(t,0) = \begin{bmatrix} \cos\Omega t - \frac{ir}{\Omega}\sin\Omega t & -\frac{i\omega}{\Omega}\sin\Omega t \\ -\frac{i\omega}{\Omega}\sin\Omega t & \cos\Omega t + \frac{ir}{\Omega}\sin\Omega t \end{bmatrix} \tag{4.84}$$

For y, z and z, y:

$$\hat{R}_{yz}(t,0) = \begin{bmatrix} \cos\Omega t - \frac{i\omega}{\Omega}\sin\Omega t & -\frac{r}{\Omega}\sin\Omega t \\ \frac{r}{\Omega}\sin\Omega t & \cos\Omega t + \frac{i\omega}{\Omega}\sin\Omega t \end{bmatrix} \tag{4.85}$$

$$\hat{R}_{zy}(t,0) = \begin{bmatrix} \cos\Omega t - \frac{ir}{\Omega}\sin\Omega t & -\frac{\omega}{\Omega}\sin\Omega t \\ \frac{\omega}{\Omega}\sin\Omega t & \cos\Omega t + \frac{ir}{\Omega}\sin\Omega t \end{bmatrix} \tag{4.86}$$

and finally for x, y and y, x:

$$\hat{R}_{xy}(t,0) = \begin{bmatrix} \cos\Omega t & -\frac{(i\omega + r)}{\Omega}\sin\Omega t \\ -\frac{(i\omega - r)}{\Omega}\sin\Omega t & \cos\Omega t \end{bmatrix} \tag{4.87}$$

$$\hat{R}_{yx}(t,0) = \begin{bmatrix} \cos\Omega t & -\frac{(ir + \omega)}{\Omega}\sin\Omega t \\ -\frac{(ir - \omega)}{\Omega}\sin\Omega t & \cos\Omega t \end{bmatrix} \tag{4.88}$$

where $\Omega = \sqrt{\omega^2 + r^2}$. With each of these matrices we associate the relevant quantum brachistochrone problem. We can show a simple way to reach the unitary which transforms the Hamiltonian matrix. We have the time evolution equation:

$$\tilde{H}(t) = \hat{U}(t,s)\tilde{H}(s)\hat{U}^\dagger(t,s) \tag{4.89}$$

We know that there exists an eigenvalue decomposition for the unitary operator. We may write:

$$\hat{U}(t,s) = \hat{W}(t)\hat{W}^\dagger(s) \tag{4.90}$$

However, if we transform unitarily to a new reference frame such that:

$$\tilde{H}'(t) = \hat{X}\tilde{H}(t)\hat{X}^\dagger = \hat{U}'(t,s)\tilde{H}'(t)\hat{U'}^\dagger(t,s) \tag{4.91}$$

we must have $\hat{U}'(t,s) = \hat{X}\hat{U}(t,s)\hat{X}^\dagger$. Therefore, the new eigenvalue decomposition will be given by:

$$\hat{U}'(t,s) = \hat{X}\hat{W}(t)\hat{X}^\dagger\hat{X}\hat{W}^\dagger(s)\hat{X}^\dagger = \hat{W}'(t)\hat{W}'^\dagger(s) \tag{4.92}$$

We illustrate with a concrete example. Take the set of conditions which we have proved before, we have:

$$\tilde{H}(0) = r\hat{\sigma}_x, \tilde{F}(t) = \tilde{F}(0) = \omega\hat{\sigma}_z \tag{4.93}$$

Then we know the optimal Hamiltonian matrix is given by:

$$\tilde{H}(t) = \begin{bmatrix} 0 & re^{-2i\omega t} \\ re^{+2i\omega t} & 0 \end{bmatrix} \tag{4.94}$$

The unitary matrix of eigenvectors of the Hamiltonian is given by:

$$\hat{W}(t) = \frac{e^{i\omega t}}{\sqrt{2}} \begin{bmatrix} e^{-2i\omega t} & -e^{-2i\omega t} \\ 1 & 1 \end{bmatrix} \tag{4.95}$$

Take now the further unitary transformation given by the matrix:

$$\hat{X} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -i & i \\ -1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -i & i \end{bmatrix} \tag{4.96}$$

This matrix is Hermitian, and unitary:

$$\hat{X}^{-1} = \hat{X}^{\dagger} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & i \\ -1 & -i \end{bmatrix} \tag{4.97}$$

From the equation for the optimal Hamiltonian matrix, it is not too hard to see that we must have:

$$\tilde{H}(-t) = \hat{\sigma}_x \tilde{H}(t) \hat{\sigma}_x^{\dagger} \tag{4.98}$$

In terms of the matrix $\hat{X}$, we satisfy the following identities:

$$\hat{X}\tilde{H}(t)\hat{X}^{\dagger} = r \begin{bmatrix} \cos 2\omega t & \sin 2\omega t \\ \sin 2\omega t & -\cos 2\omega t \end{bmatrix} \tag{4.99}$$

$$\hat{X}^{\dagger}\tilde{H}(t)\hat{X} = r \begin{bmatrix} \sin 2\omega t & -i\cos 2\omega t \\ i\cos 2\omega t & -\sin 2\omega t \end{bmatrix} \tag{4.100}$$

for the transformation of the Hamiltonian, and:

$$\hat{X}\tilde{F}(t)\hat{X}^{\dagger} = \hat{X}\tilde{F}(0)\hat{X}^{\dagger} = \begin{bmatrix} 0 & -i\omega \\ i\omega & 0 \end{bmatrix} \tag{4.101}$$

$$\hat{X}^{\dagger}\tilde{F}(t)\hat{X} = \hat{X}^{\dagger}\tilde{F}(0)\hat{X} = \begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix} \tag{4.102}$$

The matrix of eigenvectors transforms accordingly, and we have:

$$\hat{W}'^{\dagger}(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -i & i \end{bmatrix} \frac{e^{i\omega t}}{\sqrt{2}} \begin{bmatrix} e^{-2i\omega t} & -e^{-2i\omega t} \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & i \\ -1 & -i \end{bmatrix} \tag{4.103}$$

$$= \sqrt{i} \begin{bmatrix} \cos \omega t & \sin \omega t \\ -i\sin \omega t & i\cos \omega t \end{bmatrix} \tag{4.104}$$

$$\hat{W}'(t) = (\sqrt{-i})^* \begin{bmatrix} \cos \omega t & i\sin \omega t \\ \sin \omega t & -i\cos \omega t \end{bmatrix} \tag{4.105}$$

and the unitary matrix satisfying the decomposition and symmetry formula:

$$\hat{U}'(t,s) = \hat{X}\hat{U}(t,s)\hat{X}^{\dagger} = \hat{W}'(t)\hat{W}'(s)^{\dagger} \tag{4.106}$$

$$= \begin{bmatrix} \cos[\omega(t-s)] & i\sin[\omega(t-s)] \\ i\sin[\omega(t-s)] & \cos[\omega(t-s)] \end{bmatrix} \tag{4.107}$$

This method is sufficient to recover all the necessary unitary matrices, the theorem is proved. We discuss this further in the calculations that follow. □

Remark 47. Although in this theorem we have used the unitary properties of the matrix of eigenvectors, we can relax this property to simply an invertible matrix. This will be analysed in detail in calculating various other unitary and pseudounitary operators for SU(2) and SU(1,1), where we must change from a strict use of unitary transforms, to a situation where the pseudounitary equivalent is scale dependent, and the matrices are only invertible and not Hermitian.

Lemma 4.3.9. *We may write the general solution for the unitary operator on SU(2) under the conditions of the quantum brachistochrone problem or equivalent unitary transformations as:*

$$\hat{U}(t,0) = \exp\left(it\mathbf{m}\cdot\sigma\right)\exp\left(-it(\mathbf{n}+\mathbf{m})\cdot\sigma\right) \tag{4.108}$$

where the vectors defining the spinor expansion of the matrices satisfy:

$$\mathbf{n}\cdot\mathbf{m} = 0 \tag{4.109}$$

with $\mathbf{m} = \omega\hat{\mathbf{e}}_j$ *and* $\mathbf{n} = r\hat{\mathbf{e}}_k$*. This matrix is given by the formula:*

$$\hat{U}(t,0) = \left(\cos\omega t\cos\Omega t + \frac{\omega}{\Omega}\sin\omega t\sin\Omega t\right)\mathbf{1} \tag{4.110}$$

$$+i\left(\sin\omega t\cos\Omega t - \frac{\omega}{\Omega}\sin\Omega t\cos\omega t\right)\hat{\sigma}_j \tag{4.111}$$

$$+\frac{r}{\Omega}\sin\omega t\sin\Omega t.\hat{\sigma}_j\hat{\sigma}_k \tag{4.112}$$

The function multiplying the identity matrix in this expansion is the spherical law of cosines:

$$\cos d = \cos\omega t\cos\Omega t + \cos u\sin\omega t\sin\Omega t \tag{4.113}$$

with $\cos u = \dfrac{\omega}{\Omega}$.

Proof. Using the Euler decomposition, we may write:

$$\exp\left(-i\theta\mathbf{n}\cdot\sigma\right) = \mathbf{1}\cos\theta - i(\mathbf{n}\cdot\sigma)\sin\theta \tag{4.114}$$

We have, using the assumptions in the lemma, the following:

$$\exp\left(it\mathbf{m}\cdot\sigma\right) = \cos\omega t\mathbf{1} + i\sin\omega t\hat{\sigma}_j \tag{4.115}$$

and

$$\exp\left(-it(\mathbf{n}+\mathbf{m})\cdot\sigma\right) = \exp\left(-it\sqrt{r^2+\omega^2}(\mathbf{a})\cdot\sigma\right) \tag{4.116}$$

$$= \cos\Omega t\mathbf{1} - i\mathbf{a}\cdot\hat{\sigma}\sin\Omega t \tag{4.117}$$

where the vector $|\mathbf{a}| = 1$ is of unit modulus, and satisfies:

$$\mathbf{a} = \frac{1}{\Omega}\left(\omega\hat{\mathbf{e}}_j + r\hat{\mathbf{e}}_k\right) \tag{4.118}$$

with $\Omega = \sqrt{r^2+\omega^2}$. Writing the unitary, we have:

$$\exp\left(it\mathbf{m}\cdot\sigma\right)\exp\left(-it(\mathbf{n}+\mathbf{m})\cdot\sigma\right) \tag{4.119}$$

$$= \left(\cos\omega t\mathbf{1} + i\sin\omega t\hat{\sigma}_j\right)\left(\cos\Omega t\mathbf{1} - i\left(\frac{1}{\Omega}\left(\omega\hat{\sigma}_j + r\hat{\sigma}_k\right)\right)\sin\Omega t\right) \tag{4.120}$$

from which the result follows. □

We shall now state some more general observations which follow from the structure of the time optimisation principle and the quantum brachistochrone equation 3.2.10. We have the following simple lemma which relates the solution of the unitary evolution to other equivalent solutions.

Lemma 4.3.10. *For the unitary evolution of the Hamiltonian and constraint, we can write:*

$$\hat{U}(t,s)\left(\tilde{H}(s)+\tilde{F}(s)\right)\hat{U}^{\dagger}(t,s)=\tilde{H}(t)+\tilde{F}(t) \tag{4.121}$$

which solves:

$$i\frac{d}{dt}\left(\tilde{H}(t)+\tilde{F}(t)\right)=\left[\tilde{H}(t),\tilde{F}(t)\right] \tag{4.122}$$

These expressions are invariant under the substitution $\tilde{H}'(t)=\tilde{H}(t)+\kappa\mathbf{1}$, with $\kappa\in\mathbb{R}$, some real constant. Under the transformation:

$$\left(\tilde{H}(t),|\psi\rangle\right)\rightarrow\left(\tilde{H}(t)+\frac{d\phi}{dt}\mathbf{1},e^{-i\phi}|\psi\rangle\right) \tag{4.123}$$

the equations defining the action principle for least time 3.2.8 are invariant. Further, under any unitary transformation the action principle is invariant, as are the constraint equations, where we take the action principle:

$$\delta t=\int 1dt+\int\lambda_1\left(\mathrm{Tr}\left(\tilde{H}\tilde{F}\right)-0\right)dt+\int\lambda_2\left(\mathrm{Tr}\left(\frac{\tilde{H}^2}{2}\right)-k\right)dt=\min \tag{4.124}$$

where $\hat{R}\hat{R}^{\dagger}=\hat{R}^{\dagger}\hat{R}=\mathbf{1}$ is the unitary transformation.

Proof. The first part is equivalent to the statement that the unitary evolution is only defined up to a global phase change. If we assume the unitary evolution is described through $\hat{X}'=\hat{U}\hat{X}\hat{U}^{\dagger}$. and further, take the global phase such that

$$\hat{U}'(t,s)=\exp\left(-i\int_s^t\alpha(s)\mathbf{1}\right)\hat{U}(t,s)=e^{-i\phi(s,t)}\hat{U}(t,s) \tag{4.125}$$

then it is simple to see that this does not change the essential dynamical expressions for the unitary evolution. However, the time evolution of the unitary operator changes to:

$$i\frac{d\hat{U}'(t,s)}{dt}=\left(\tilde{H}(t)+\alpha(t)\mathbf{1}\right)\hat{U}'(t,s)=\left(\tilde{H}(t)+\frac{d\phi}{dt}\mathbf{1}\right)\hat{U}'(t,s) \tag{4.126}$$

This is gauge invariance and proves the first part of the lemma, as we can easily see that by adding a negative phase to the ket-vector which represents the input, we can remove the phase that comes from the global phase change to the unitary or additive scalar component to the Hamiltonian, giving global phase invariance. The second part is readily established using trace identities, it suffices to show that each part of 3.2.10 is invariant. Beginning with the formula from 3.2.10, we have:

$$\delta t=\int 1dt+\int\lambda_1\left(\mathrm{Tr}\left(\tilde{H}\tilde{F}\right)-0\right)dt+\int\lambda_2\left(\mathrm{Tr}\left(\frac{\tilde{H}^2}{2}\right)-k\right)dt=\min \tag{4.127}$$

The identity matrix is obviously invariant under any unitary transformation. To evaluate the other parts, note that the tracefree property is preserved under unitary transformation:

$$\tilde{F}_R=\hat{R}\tilde{F}\hat{R}^{\dagger} \tag{4.128}$$

$$\mathrm{Tr}\left(\tilde{F}\right)=\mathrm{Tr}\left(\tilde{F}\hat{R}^{\dagger}\hat{R}\right)=\mathrm{Tr}\left(\hat{R}\tilde{F}\hat{R}^{\dagger}\right)=\mathrm{Tr}\left(\tilde{F}_R\right)=0 \tag{4.129}$$

and similarly for the Hamiltonian $\tilde{H}$, where we have inserted the identity matrix as a decomposition over a unitary $\hat{R}^{\dagger}\hat{R}=\hat{\mathbf{1}}$ and used the cyclic property of the trace. The linear constraint transforms as follows:

$$\mathrm{Tr}\left(\tilde{H}\tilde{F}\right)=\mathrm{Tr}\left(\tilde{H}\tilde{F}\hat{R}^{\dagger}\hat{R}\right)=\mathrm{Tr}\left(\hat{R}\tilde{H}\tilde{F}\hat{R}^{\dagger}\right) \tag{4.130}$$

$$= \mathrm{Tr}\left(\hat{R}\tilde{H}\hat{R}^\dagger\hat{R}\tilde{F}\hat{R}^\dagger\right) = \mathrm{Tr}\left(\tilde{H}_R\tilde{F}_R\right) = 0 \tag{4.131}$$

Finally, the isotropic condition 3.2.6 evaluates simply as:

$$\mathrm{Tr}\left(\frac{\tilde{H}^2}{2}\right) = \frac{1}{2}\mathrm{Tr}\left(\tilde{H}^2\right) = \frac{1}{2}\mathrm{Tr}\left(\hat{R}\tilde{H}\hat{R}^\dagger\hat{R}\tilde{F}\hat{R}^\dagger\right) = \mathrm{Tr}\left(\frac{\tilde{H}_R^2}{2}\right) < \infty \tag{4.132}$$

which proves the statement. □

For the remainder of this section, we shall be concerned with the unitary matrices which arise from the symmetries of the Hamiltonian, in the exclusion of the rotation component, which does not encode any change to the dynamics. This is because the sum of the Hamiltonian and constraint is invariant under the action of the unitary matrix which forms the rotation part of the time evolution operator. As can be easily seen, the invariance principle extends to an equivalence in the unitary decomposition of the time evolution operator.

Corollary 4.3.11. *Assume the unitary decomposition of the time optimal quantum control problem on SU(2), where we consider a single generator constraint as per the calculations of this section. This is given by the matrix defined by:*

$$\hat{U}(t,0) = \exp\left(it\tilde{F}(0)\right)\exp\left(-it\left[\tilde{H}(0)+\tilde{F}(0)\right]\right) = \hat{U}_1(t,0)\hat{U}_2(t,0) \tag{4.133}$$

If we transform $\hat{U}(t,s)$ by a further unitary transformation $\hat{U}(t,s) \to \hat{R}\hat{U}(t,s)\hat{R}^\dagger$, this is equivalent to the transformation of the control problem:

$$\tilde{F}(0) \to \hat{R}\tilde{F}(0)\hat{R}^\dagger \tag{4.134}$$

$$\tilde{H}(0) \to \hat{R}\tilde{H}(0)\hat{R}^\dagger \tag{4.135}$$

and the transformation of the unitary via:

$$\hat{U}(t,s) \to \hat{R}\hat{U}(t,s)\hat{R}^\dagger = \exp\left(-i\int_s^t \hat{R}\tilde{H}(\tau)\hat{R}^\dagger d\tau\right) \tag{4.136}$$

Proof. Taking the time evolution operator as the product of two unitaries, and expanding as per the precepts of this section, we have:

$$\hat{U}(t,s) \to \hat{R}\hat{U}(t,s)\hat{R}^\dagger = \hat{R}\hat{U}_1(t,s)\hat{U}_2(t,s)\hat{R}^\dagger \tag{4.137}$$

$$= \hat{R}\hat{U}_1(t,s)\hat{R}^\dagger\hat{R}\hat{U}_2(t,s)\hat{R}^\dagger \tag{4.138}$$

Using the expressions for the unitaries in terms of the initial conditions of the Hamiltonian and constraint, where we assume that we have a constant constraint under the conditions of this section, we may write:

$$\hat{U}_1(t,s) \to \hat{R}\hat{U}_1(t,s)\hat{R}^\dagger = \exp\left(i\hat{R}\tilde{F}(0)\hat{R}^\dagger t\right) \tag{4.139}$$

$$\hat{U}_2(t,s) \to \hat{R}\hat{U}_2(t,s)\hat{R}^\dagger = \exp\left(-i\hat{R}\left[\tilde{H}(0)+\tilde{F}(0)\right]\hat{R}^\dagger t\right) \tag{4.140}$$

where we used the constancy of the unitary transform $\hat{R}$ to take it inside the exponential. This implies that we must have:

$$\tilde{F}(0) \to \hat{R}\tilde{F}(0)\hat{R}^\dagger \tag{4.141}$$

and

$$\tilde{H}(0) \to \hat{R}\tilde{H}(0)\hat{R}^\dagger \tag{4.142}$$

as the transformed constraint and initial Hamiltonian matrix, and the unitary transforms as:

$$\hat{U}(t,s) \to \hat{R}\hat{U}(t,s)\hat{R}^{\dagger} = \exp\left(-i\int_s^t \hat{R}\tilde{H}(\tau)\hat{R}^{\dagger}d\tau\right) \tag{4.143}$$

This proves the corollary. We shall use this to identify the transformations between the different time optimal control problems for a simple linear constraint on SU(2).

Assume a constant, i.e. time independent, invertible transform $\hat{R}$, $\hat{R}\hat{R}^{\dagger} = \hat{R}^{\dagger}\hat{R} = \mathbf{1}$. Consider the transform of the Magnus expansion of the time dependent Hamiltonian matrix:

$$\hat{U}(t,s) \to \hat{R}\hat{U}(t,s)\hat{R}^{\dagger} = \exp\left(-i\int_s^t \hat{R}\tilde{H}(\tau)\hat{R}^{\dagger}d\tau\right) \tag{4.144}$$

$$= \hat{R}\left(\hat{\mathbf{1}} - i\hat{A}_1 + \frac{1}{2!}(-i)^2\hat{A}_2 + ...\right)\hat{R}^{\dagger} \tag{4.145}$$

where the terms in the Magnus series are given by the repeated commutator identities:

$$\hat{A}_1 = \int_0^{t-s} \tilde{H}(t_1)dt_1,\ \hat{A}_2 = \int_0^{t-s}\int_0^{t_1}\left[\tilde{H}(t_1), \tilde{H}(t_2)\right]dt_1dt_2, ... \tag{4.146}$$

Under the unitary transformation $\hat{R}$ we have:

$$\hat{A}_n \to \hat{R}\hat{A}_n\hat{R}^{\dagger} = \int_0^{t-s}\int_0^{t_1}..\int_0^{t_{n-1}} \hat{R}\left[\left[\tilde{H}(t_1), \tilde{H}(t_2)\right], ...\right]_{n-1}\hat{R}^{\dagger}dt_1dt_2...dt_n \tag{4.147}$$

$$= \int_0^{t-s}\int_0^{t_1}..\int_0^{t_{n-1}} \left[\left[\hat{R}\tilde{H}(t_1)\hat{R}^{\dagger}, \hat{R}\tilde{H}(t_2)\hat{R}^{\dagger}\right], ...\right]_{n-1} dt_1dt_2...dt_n \tag{4.148}$$

which collapses to:

$$\hat{U}(t,s) \to \hat{R}\hat{U}(t,s)\hat{R}^{\dagger} = \exp\left(-i\int_s^t \hat{R}\tilde{H}(\tau)\hat{R}^{\dagger}d\tau\right) \tag{4.149}$$

and proves the statement once more. □

We shall now show through some simple transformations that this solution is able to be mapped to other problems of similar dimensions. This shall be our method of identifying the relevant unitary and pseudounitary operators that exist on this space. We shall rely on the following result.

Theorem 4.3.12. *The unitary matrix which solves the time evolution equation:*

$$\hat{U}_1(t,s)\tilde{H}(s)\hat{U}_1{}^{\dagger}(t,s) = \tilde{H}(t) \tag{4.150}$$

specifies the solution to the quantum brachistochrone equation, up to a transformation which leaves the Hamiltonian matrix and the constraint unchanged (a rotation as discussed earlier in this section). This problem, defined in 3.2.7, with Hamiltonian $\tilde{H}(t)$, constraint $\tilde{F}(t)$ as in 4.3.3 is given through this special unitary matrix multiplied by a rotation. Choose the particular example

$$\tilde{H}(t) = \begin{bmatrix} 0 & re^{-2i\omega t} \\ re^{+2i\omega t} & 0 \end{bmatrix} = \lambda_x(t)\hat{\sigma}_x + \lambda_y(t)\hat{\sigma}_y, \tilde{H}(0) = r\hat{\sigma}_x, \tilde{F}(t) = \tilde{F}(0) = \omega\hat{\sigma}_z \tag{4.151}$$

Then the evolution equation for the Hamiltonian matrix is solved using 3.1.15. This unitary matrix may be specified through the eigenvector transformation from 3.1.15, i.e. $\hat{U}_1(t,s) = \hat{W}(t)\hat{W}^{\dagger}(s)$ where the matrices giving the eigenvalue decomposition are:

$$\hat{L} = r\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.152}$$

and

$$\hat{W}(t) = \frac{e^{i\omega t}}{\sqrt{2}} \begin{bmatrix} e^{i\omega t} & -e^{i\omega t} \\ e^{-i\omega t} & e^{-i\omega t} \end{bmatrix} \tag{4.153}$$

$$\hat{U}_1(t,s) = \begin{bmatrix} e^{-i\omega(t-s)} & 0 \\ 0 & e^{i\omega(t-s)} \end{bmatrix} \tag{4.154}$$

The systems described by the Hamiltonian operators:

$$\tilde{H}(t) = \lambda_x(t)\hat{\sigma}_x + \lambda_z(t)\hat{\sigma}_z \tag{4.155}$$

and

$$\tilde{H}(t) = \lambda_y(t)\hat{\sigma}_y + \lambda_z(t)\hat{\sigma}_z \tag{4.156}$$

are equivalent under constant unitary transforms $\hat{R}$ to the solution of the quantum brachistochrone 4.3.12. The unitary operators which satisfy:

$$\hat{U}(t,s)\tilde{H}(s)\hat{U}^\dagger(t,s) = \tilde{H}(t) \tag{4.157}$$

for these two examples are:

$$\hat{S}_1\hat{U}_1(t,s)\hat{S}_1^{-1} = \begin{bmatrix} \cos(\omega(t-s)) & -\sin(\omega(t-s)) \\ \sin(\omega(t-s)) & \cos(\omega(t-s)) \end{bmatrix} \tag{4.158}$$

and

$$\hat{S}_1^{-1}\hat{U}_1(t,s)\hat{S}_1 = \begin{bmatrix} \cos(\omega(t-s)) & i\sin(\omega(t-s)) \\ i\sin(\omega(t-s)) & \cos(\omega(t-s)) \end{bmatrix} \tag{4.159}$$

Proof. Note that elements of this proof originally appear in [49], where the authors used the method of state-to-state transfer to derive essentially the same result. We shall depart from their analysis, and first implement 3.2.7, and use some methods from linear algebra to achieve a similar conclusion. Taking the Hamiltonian and constraint:

$$\tilde{H}(t) = \begin{bmatrix} 0 & re^{-2i\omega t} \\ re^{+2i\omega t} & 0 \end{bmatrix} = \lambda_x(t)\hat{\sigma}_x + \lambda_y(t)\hat{\sigma}_y \tag{4.160}$$

$$\tilde{H}(0) = r\hat{\sigma}_x, \tilde{F}(t) = \tilde{F}(0) = \omega\hat{\sigma}_z \tag{4.161}$$

We find the matrix equation:

$$i\frac{d}{dt}\left(\tilde{H}(t) + \tilde{F}(t)\right) = \left[\tilde{H}(t), \tilde{F}(t)\right] = \begin{bmatrix} 0 & -2\omega_z\epsilon \\ 2\omega_z\epsilon^* & 0 \end{bmatrix} \tag{4.162}$$

from which we see that $\dot{\omega}_z = 0$, $\epsilon(t) = e^{2i\omega_z t}\epsilon(0)$, the result following from the initial condition. The unitary matrix which evolves the Hamiltonian may be recovered by the following simple calculation. The eigenvalues and matrix of eigenvectors of the Hamiltonian matrix $\tilde{H}(t)$ are:

$$\hat{L} = r\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.163}$$

and

$$\hat{W}(t) = \frac{e^{i\omega t}}{\sqrt{2}} \begin{bmatrix} e^{i\omega t} & -e^{i\omega t} \\ e^{-i\omega t} & e^{-i\omega t} \end{bmatrix} \tag{4.164}$$

and the unitary property of the matrix of eigenvectors is given by $\hat{W}^{-1}(t) = \hat{W}^T(-t) = \hat{W}^\dagger(t)$, $\det\hat{W}(t) = 1$. Applying 3.1.15, i.e. $\hat{U}_1(t,s) = \hat{W}(t)\hat{W}^\dagger(s)$ we have immediately:

$$\hat{U}_1(t,s) = \begin{bmatrix} e^{-i\omega(t-s)} & 0 \\ 0 & e^{i\omega(t-s)} \end{bmatrix} \tag{4.165}$$

which concludes the proof of the first part of the theorem. One can also check that this transformation solves the expressions:

$$\hat{U}(t,s)\tilde{H}(s)\hat{U}^{\dagger}(t,s) = \tilde{H}(t) \tag{4.166}$$

which is required under the conditions of 3.1.15. To establish the second part, take the two matrices:

$$\hat{S}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} i & -i \\ -1 & -1 \end{bmatrix} \tag{4.167}$$

and

$$\hat{S}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{4.168}$$

Then simple linear algebra gives for the transformed unitary and Hamiltonians:

$$\hat{X}_R = \hat{R}\hat{X}\hat{R}^{\dagger} \tag{4.169}$$

where $\hat{X}$ is either the Hamiltonian matrix or the unitary. We find:

$$\hat{S}_2\tilde{H}(t)\hat{S}_2^{-1} = \hat{S}_2^{-1}\tilde{H}(t)\hat{S}_2 = \begin{bmatrix} r\cos 2\omega t & ir\sin 2\omega t \\ -ir\sin 2\omega t & -r\cos 2\omega t \end{bmatrix} \tag{4.170}$$

with transformed unitary:

$$\hat{S}_2\hat{U}_1(t,s)\hat{S}_2^{-1} = \hat{S}_2^{-1}\hat{U}_1(t,s)\hat{S}_2 = \begin{bmatrix} \cos(\omega(t-s)) & -i\sin(\omega(t-s)) \\ -i\sin(\omega(t-s)) & \cos(\omega(t-s)) \end{bmatrix} \tag{4.171}$$

and similarly:

$$\hat{S}_1^{-1}\hat{H}(t)\hat{S}_1 = \begin{bmatrix} r\sin 2\omega t & ir\cos 2\omega t \\ -ir\cos 2\omega t & -r\sin 2\omega t \end{bmatrix} \tag{4.172}$$

$$\hat{S}_1\hat{H}(t)\hat{S}_1^{-1} = \begin{bmatrix} -r\cos 2\omega t & -r\sin 2\omega t \\ -r\sin 2\omega t & r\cos 2\omega t \end{bmatrix} \tag{4.173}$$

$$\hat{S}_1^{-1}\hat{U}_1(t,s)\hat{S}_1 = \begin{bmatrix} \cos(\omega(t-s)) & i\sin(\omega(t-s)) \\ i\sin(\omega(t-s)) & \cos(\omega(t-s)) \end{bmatrix} \tag{4.174}$$

$$\hat{S}_1\hat{U}_1(t,s)\hat{S}_1^{-1} = \begin{bmatrix} \cos(\omega(t-s)) & -\sin(\omega(t-s)) \\ \sin(\omega(t-s)) & \cos(\omega(t-s)) \end{bmatrix} \tag{4.175}$$

We can write the Hamiltonian operators over the Pauli matrices as:

$$\tilde{H}_{S_1} = -r(\hat{\sigma}_z\cos 2\omega t + \hat{\sigma}_x\sin 2\omega t) \tag{4.176}$$

$$\tilde{H}_{S_2} = r(\hat{\sigma}_z\cos 2\omega t - \hat{\sigma}_y\sin 2\omega t) \tag{4.177}$$

and establish the claim in the theorem. Note that these results are only unique up to a unitary transformation of the Hamiltonian and constraint which leaves their sum and constituent parts invariant. □

Theorem 4.3.13. *The time evolution operator is given by the product of unitary matrices:*

$$\hat{U}(t,s) = \exp\left(+it\tilde{F}(0)\right)\exp\left(-it\left[\tilde{H}(0)+\tilde{F}(0)\right]\right) \tag{4.178}$$

which for the case of a simple constraint given by a single generator is:

$$\hat{U}(t,s) = \exp\left(+it\omega\hat{\sigma}_i\right)\exp\left(-it\left[r\hat{\sigma}_j+\omega\hat{\sigma}_i\right]\right) \tag{4.179}$$

The rotation part of this unitary is the matrix:

$$\hat{R}_{ij}(t,0) = \exp\left(-it\left[\tilde{H}(0)+\tilde{F}(0)\right]\right) = \exp\left(-it\left[r\hat{\sigma}_j + \omega\hat{\sigma}_i\right]\right) \tag{4.180}$$

For the systems defined by:

$$\tilde{H}(t) = \begin{bmatrix} 0 & re^{-2i\omega t} \\ re^{+2i\omega t} & 0 \end{bmatrix} = \lambda_x(t)\hat{\sigma}_x + \lambda_y(t)\hat{\sigma}_y, \tilde{H}(0) = r\hat{\sigma}_x, \tilde{F}(t) = \tilde{F}(0) = \omega\hat{\sigma}_z \tag{4.181}$$

$$\tilde{H}(t) = r\begin{bmatrix} \cos 2\omega t & -i\sin 2\omega t \\ i\sin 2\omega t & -\cos 2\omega t \end{bmatrix} = \lambda_z(t)\hat{\sigma}_z + \lambda_y(t)\hat{\sigma}_y, \tilde{H}(0) = r\hat{\sigma}_z, \tilde{F}(t) = \tilde{F}(0) = \omega\hat{\sigma}_x \tag{4.182}$$

$$\tilde{H}(t) = r\begin{bmatrix} \cos 2\omega t & \sin 2\omega t \\ \sin 2\omega t & -\cos 2\omega t \end{bmatrix} = \lambda_z(t)\hat{\sigma}_z + \lambda_x(t)\hat{\sigma}_x, \tilde{H}(0) = r\hat{\sigma}_z, \tilde{F}(t) = \tilde{F}(0) = \omega\hat{\sigma}_y \tag{4.183}$$

the rotation part of the time evolution operator is specified by the matrices:

$$R_{ij}(t,0) = \exp\left(-it\left[\tilde{H}(0)+\tilde{F}(0)\right]\right) = \exp\left(-it\left[r\hat{\sigma}_j + \omega\hat{\sigma}_i\right]\right) \tag{4.184}$$

$$R_{zx}(t,0) = \begin{bmatrix} \cos\Omega t - \dfrac{i\omega}{\Omega}\sin\Omega t & -\dfrac{ir}{\Omega}\sin\Omega t \\ -\dfrac{ir}{\Omega}\sin\Omega t & \cos\Omega t + \dfrac{i\omega}{\Omega}\sin\Omega t \end{bmatrix} \tag{4.185}$$

$$R_{xz}(t,0) = \begin{bmatrix} \cos\Omega t - \dfrac{ir}{\Omega}\sin\Omega t & -\dfrac{i\omega}{\Omega}\sin\Omega t \\ -\dfrac{i\omega}{\Omega}\sin\Omega t & \cos\Omega t + \dfrac{ir}{\Omega}\sin\Omega t \end{bmatrix} \tag{4.186}$$

$$R_{yz}(t,0) = \begin{bmatrix} \cos\Omega t - \dfrac{ir}{\Omega}\sin\Omega t & -\dfrac{\omega}{\Omega}\sin\Omega t \\ \dfrac{\omega}{\Omega}\sin\Omega t & \cos\Omega t + \dfrac{ir}{\Omega}\sin\Omega t \end{bmatrix} \tag{4.187}$$

Together with the unitaries $\hat{U}_1$ which we have derived in 4.3.12, these define a complete set of solutions to the quantum brachistochrone problem on SU(2). In the same way that the different forms of the time evolution operator were transmutable, these rotation matrices also transform in a similar way, for example:

$$\hat{S}_2 R_{zx}(t,0)\hat{S}_2^{-1} = R_{xz}(t,0) = \hat{S}_2^{-1}R_{zx}(t,0)\hat{S}_2 \tag{4.188}$$

$$\hat{S}_1 R_{zx}(t,0)\hat{S}_1^{-1} = \exp\left(-it\left[-r\hat{\sigma}_z + \omega\hat{\sigma}_y\right]\right) \tag{4.189}$$

$$\hat{S}_1^{-1} R_{zx}(t,0)\hat{S}_1 = \exp\left(it\left[r\hat{\sigma}_z + \omega\hat{\sigma}_y\right]\right) \tag{4.190}$$

where $\hat{S}_1, \hat{S}_2$ are the invertible matrices from 4.3.12.

Proof. All of these formulae can be evaluated using the Euler decomposition of the SU(2) exponential:

$$\exp\left(-i\theta\mathbf{n}\cdot\sigma\right) = \mathbf{1}\cos\theta - i(\mathbf{n}\cdot\sigma)\sin\theta \tag{4.191}$$

and the transformations follow from an application of simple linear algebra. □

This ends our discussion of the different types of unitary operators and their relationship to the time evolution operator for the quantum brachistochrone problem. In the sections of this chapter that follow, we shall employ the properties of the matrices we have derived in order to examine the differential geometry that underlies this optimisation principle. We can see how that only a small, simple example where we take only a constraint in a single direction solves a number of seemingly different problems. This can be seen as an aspect of the isotropy principle, which states that there is no preferred direction, only convention. In this sense, one can expect that, in the absence of a directional bias, each of these problems where the constraint lies along either the x, y or z directions will be equivalent, up to permutation of the co-ordinates. This equivalence is expressed through the unitary transformation of each of these problems into one another.

Remark 48. The unitary operator produced as a result of this calculation shall be used in the formation of a theory of hyperbolic operators. The three unitary operators given by 4.3.12 define three different representations of the same underlying problem. This can be seen easily through use of the eigenvalue decomposition, as each problem shares identical eigenvalues. The Hamiltonian matrices as in 4.3.12 solve a transformed version of the problem described by 4.3.2. They are the solution to the quantum brachistochrone equation with a transformed Hamiltonian and constraint, as in 3.2.10. We have derived the rotation matrices which multiplied together with these unitaries give the time evolution operator. Together, these give a complete description of the solutions that are possible for a unitary evolution that is described by the quantum brachistochrone equation. However, as we shall show in the following section, we are able to relax the strict unitary conditions and use matrix inverses in an entirely analogous fashion to derive the pseudounitary operators that are equivalent to the unitary matrices we have used in this section.

4.3.2 Hyperbolic Brachistochrone via Analytic Continuation

We shall now prove some new results that rely on the analytical continuation of these unitary operators to complex time. This shall require the use of a new description of the brachistochrone problem, specified through the eigenvalue decomposition of the unitary operator as in 3.1.15, the quantum brachistochrone equation 3.2.7 and the constraint laws 3.2.6, 3.2.4. In order to derive the differential geometry given by the Fubini-Study metric 3.2.34, we shall require the construction of pseudounitary equivalents to the unitary matrices derived in 4.3.12, 4.3.11. One way in which to do so is to continue the time variable t to a complex argument. We shall require the following.

Lemma 4.3.14. *The analytical continuation of the Hamiltonian matrices defined by the solution to 4.3.11 defines a hyperbolic control problem equivalent to 3.2.10, 3.2.7. This control problem is specified through the matrix equations:*

$$-\frac{d}{dt}(\tilde{\mathcal{H}}(t) + \tilde{\mathcal{F}}(t)) = [\tilde{\mathcal{H}}(t), \tilde{\mathcal{F}}(t)] \tag{4.192}$$

$$\mathrm{Tr}\left[\tilde{\mathcal{H}}(t)\tilde{\mathcal{F}}(t)\right] = \mathrm{Tr}\left[\tilde{\mathcal{H}}(t)\right] = \mathrm{Tr}\left[\tilde{\mathcal{F}}(t)\right] = 0 \tag{4.193}$$

and

$$\mathrm{Tr}\left[\frac{\tilde{\mathcal{H}}^2(t)}{2}\right] = R^2 < \infty \tag{4.194}$$

where $\tilde{\mathcal{H}}(t) = \tilde{H}(it)$ is the analytical continuation of the Hamiltonian, similarly for the constraint.

Proof. Assume, as in 3.2.4, the decompositions over the generators of the spaces, we then have $\tilde{H}(t) = \sum_{i\in G} \lambda_i(t)\hat{g}_i$ and $\tilde{F}(t) = \sum_{j\notin G} \beta_j(t)\hat{g}_j$. We therefore obtain the analytical continuations as the (non-Hermitian) matrices:

$$\tilde{\mathcal{H}}(t) = \tilde{H}(it) = \sum_{i\in G} \lambda_i(it)\hat{g}_i \tag{4.195}$$

and similarly for the analytical continuation of the constraint. Using the orthonormality of the generators $\hat{g}_i$, we then conclude that:

$$\mathrm{Tr}\left[\tilde{\mathcal{H}}(t)\tilde{\mathcal{F}}(t)\right] = \mathrm{Tr}\left[\tilde{\mathcal{H}}(t)\right] = \mathrm{Tr}\left[\tilde{\mathcal{F}}(t)\right] = 0 \tag{4.196}$$

To prove the matrix differential equation, note that the quantum brachistochrone equation 3.2.7 as written:

$$i\frac{d}{dt}\left(\tilde{H}(t) + \tilde{F}(t)\right) = \left[\tilde{H}(t), \tilde{F}(t)\right] \tag{4.197}$$

Substituting $t = it$ and analytically continuing, we have:

$$-\frac{d}{d(it)}\left(\tilde{H}(it) + \tilde{F}(it)\right) = \left[\tilde{H}(it), \tilde{F}(it)\right] \tag{4.198}$$

which yields the matrix differential equation from the theorem, We term this the hyperbolic brachistochrone equation, as it performs a similar role in hyperbolic state space to the quantum brachistochrone equation. The proof of the constraint equation is through construction. We shall take the following example of an analytically continued Hamiltonian matrix, such as that obtained in the solution 4.3.11. Take the Hamiltonian:

$$\tilde{H}(t) = R\begin{bmatrix} \cos(2\omega_z t) & i\sin(2\omega_z t) \\ -i\sin(2\omega_z t) & -\cos(2\omega_z t) \end{bmatrix} \tag{4.199}$$

as in 4.3.11, R is some constant. Analytic continuation to complex time gives:

$$\tilde{\mathcal{H}}(t) = \tilde{H}(it) = R\begin{bmatrix} \cosh(2\omega_z t) & -\sinh(2\omega_z t) \\ \sinh(2\omega_z t) & -\cosh(2\omega_z t) \end{bmatrix} \tag{4.200}$$

Taking the trace, and using 3.2.6:

$$\mathrm{Tr}\left[\frac{\tilde{\mathcal{H}}^2(t)}{2}\right] = R^2 < \infty \tag{4.201}$$

we obtain the result. □

Theorem 4.3.15. *The hyperbolic equivalent of the unitary operator 3.1.9 is given by:*

$$\hat{\mathcal{U}}(t,s) = \exp\left(+\int_s^t \tilde{\mathcal{H}}(s)ds\right) \tag{4.202}$$

where $\tilde{\mathcal{H}}(t) = \tilde{H}(it)$ is the analytical continuation of the Hamiltonian matrix. For the case of a constant constraint, as considered for SU(2), this extends to the analytical continuation of the unitary operator:

$$\mathcal{U}(t,0) = \hat{U}(it,0) = \exp\left(-t\tilde{F}(0)\right)\exp\left(t\left[\tilde{H}(0) + \tilde{F}(0)\right]\right) \tag{4.203}$$

The time evolution equation for the Hamiltonian matrix is:

$$\mathcal{U}_1(t,s)\tilde{\mathcal{H}}(t)\mathcal{U}_1^{-1}(t,s) = \tilde{\mathcal{H}}(s) \tag{4.204}$$

The eigenvalue decomposition derived in 3.1.15 which solves this is modified to:

$$\hat{\mathcal{U}}(t,s) = \hat{W}(it)\hat{W}^{-1}(is) \tag{4.205}$$

by analytic continuation of the time variable. We assume that there exists a constant matrix of eigenvalues which we can invertibly transform to. In this instance, as was the case in SU(2) for the quantum brachistochrone, this is always true.

Proof. We take the definition of the time evolution operator from 3.1.9. Continuing to complex time, we have:

$$\hat{\mathcal{U}}(t,s) = \hat{U}(it,is) = \exp\left(-i\int_{is}^{it} \tilde{H}(\tau)d\tau\right) = \exp\left(\int_{s}^{t} \tilde{H}(i\tau)d(i\tau)\right) \tag{4.206}$$

and hence:

$$\hat{\mathcal{U}}(t,s) = \exp\left(\int_{s}^{t} \tilde{\mathcal{H}}(s)ds\right) \tag{4.207}$$

as required. Assume now a constraint matrix that is constant and unchanged through the unitary evolution. We have already shown that for the unitary case, with a constant constraint matrix, the time evolution operator can be written as the product of matrix exponentials:

$$\hat{U}(t,0) = \exp\left(it\tilde{F}(0)\right)\exp\left(-it\left[\tilde{H}(0)+\tilde{F}(0)\right]\right) = \hat{U}_1(t,0)\hat{U}_2(t,0) \tag{4.208}$$

Analytically continuing to imaginary time, we have:

$$\mathcal{U}(t,0) = \hat{U}(it,0) = \exp\left(-t\tilde{F}(0)\right)\exp\left(t\left[\tilde{H}(0)+\tilde{F}(0)\right]\right) \tag{4.209}$$

Note that the unitary evolution for the Hamiltonian matrix given through 3.1.15:

$$\tilde{H}(t) = \hat{U}_1(t,s)\tilde{H}(s)\hat{U}_1^{\dagger}(t,s) \tag{4.210}$$

Proceeding by continuing this expression to complex time, we obtain:

$$\tilde{H}(it) = \hat{U}_1(it,is)\tilde{H}(is)\hat{U}_1^{\dagger}(it,is) \tag{4.211}$$

We also have the matrix decomposition over the diagonal representation:

$$\tilde{H}(it) = \hat{W}(it)\hat{L}\hat{W}^{-1}(it) \tag{4.212}$$

and

$$\tilde{H}(is) = \hat{W}(is)\hat{L}\hat{W}^{-1}(is) \tag{4.213}$$

We therefore have the identity:

$$\hat{L} = \hat{W}^{-1}(is)\tilde{H}(is)\hat{W}(is) = \hat{W}^{-1}(it)\tilde{H}(it)\hat{W}(it) \tag{4.214}$$

which upon rearrangement gives:

$$\tilde{H}(it) = \hat{W}(it)\hat{W}^{-1}(is)\tilde{H}(is)\hat{W}(is)\hat{W}^{-1}(it) \tag{4.215}$$

Note that we have:

$$\hat{W}(it)\hat{W}^{-1}(is)\hat{W}(is)\hat{W}^{-1}(it) = \hat{\mathbf{1}} \tag{4.216}$$

and

$$\hat{W}(is)\hat{W}^{-1}(it)\hat{W}(it)\hat{W}^{-1}(is) = \hat{\mathbf{1}} \tag{4.217}$$

and hence these two matrices are inverses of each other:

$$\left(\hat{W}(it)\hat{W}^{-1}(is)\right)^{-1} = \hat{W}(is)\hat{W}^{-1}(it) \tag{4.218}$$

Writing $\hat{\mathcal{U}}_1(t,s) = \hat{U}_1(it,is) = \hat{W}(it)\hat{W}^{-1}(is)$, we therefore obtain the result. This result hinges on the ability to be able to find an invertible transformation that takes the

Hamiltonian matrix into a constant matrix, which in this case we choose to be the matrix of eigenvalues given by $\hat{L}$. Further, we can explicitly calculate these matrices for the equivalent examples with a single constraint. Consider the matrix:

$$\exp\left(-t\tilde{F}(0)\right)\exp\left(t\left[\tilde{H}(0)+\tilde{F}(0)\right]\right)=\mathcal{U}_1(t,0)\mathcal{U}_2(t,0) \tag{4.219}$$

We shall calculate this for the various examples that shall appear. We have:

$$\mathcal{U}_1(t,0)=\exp\left(-t\omega\hat{\sigma}_z\right)=\begin{bmatrix} e^{-\omega t} & 0 \\ 0 & e^{+\omega t}\end{bmatrix} \tag{4.220}$$

$$\mathcal{U}_2(t,0)=\exp\left(t(\omega\hat{\sigma}_z+r\hat{\sigma}_x)\right)=\begin{bmatrix} \cosh\Omega t+\frac{\omega}{\Omega}\sinh\Omega t & \frac{r}{\Omega}\sinh\Omega t \\ \frac{r}{\Omega}\sinh\Omega t & \cosh\Omega t-\frac{\omega}{\Omega}\sinh\Omega t\end{bmatrix} \tag{4.221}$$

with boundary conditions $\tilde{H}(0)=r\hat{\sigma}_x, \tilde{F}(t)=\tilde{F}(0)=\omega\hat{\sigma}_z$, and

$$\mathcal{U}_1(t,0)=\exp\left(-t\omega\hat{\sigma}_y\right)=\begin{bmatrix} \cosh\omega t & i\sinh\omega t \\ -i\sinh\omega t & \cosh\omega t\end{bmatrix} \tag{4.222}$$

$$\mathcal{U}_2(t,0)=\exp\left(t(\omega\hat{\sigma}_y+r\hat{\sigma}_z)\right)=\begin{bmatrix} \cosh\Omega t+\frac{r}{\Omega}\sinh\Omega t & -\frac{i\omega}{\Omega}\sinh\Omega t \\ \frac{i\omega}{\Omega}\sinh\Omega t & \cosh\Omega t-\frac{r}{\Omega}\sinh\Omega t\end{bmatrix} \tag{4.223}$$

with boundary conditions:

$$\tilde{H}(0)=r\hat{\sigma}_z, \tilde{F}(t)=\tilde{F}(0)=\omega\hat{\sigma}_y \tag{4.224}$$

and finally:

$$\mathcal{U}_1(t,0)=\exp\left(-t\omega\hat{\sigma}_x\right)=\begin{bmatrix} \cosh\omega t & -\sinh\omega t \\ -\sinh\omega t & \cosh\omega t\end{bmatrix} \tag{4.225}$$

and

$$\mathcal{U}_2(t,0)=\exp\left(t(\omega\hat{\sigma}_x+r\hat{\sigma}_z)\right)=\begin{bmatrix} \cosh\Omega t+\frac{r}{\Omega}\sinh\Omega t & \frac{\omega}{\Omega}\sinh\Omega t \\ \frac{\omega}{\Omega}\sinh\Omega t & \cosh\Omega t-\frac{r}{\Omega}\sinh\Omega t\end{bmatrix} \tag{4.226}$$

with boundary conditions:

$$\tilde{H}(0)=r\hat{\sigma}_z, \tilde{F}(t)=\tilde{F}(0)=\omega\hat{\sigma}_x \tag{4.227}$$

By inspection, all of these formulae can be reached through use of analytical continuation of the time variable of the equations we derived in the quantum brachistochrone. We call this the hyperbolic brachistochrone. □

Remark 49. In the unitary groups as defined by the quantum brachistochrone problem as in 4.3.11, we proved that global phase is not important to the dynamics of the system under unitary transformations. In the extension of unitary to pseudounitary groups, we can see that this property is changed to scaling, as the constant is effectively complexified in the phase. Essentially, if we scale up and scale down any matrix by the same amount we arrive back at where we began. For many of the calculations that follow, we shall not be focusing on the rotation matrix part of the pseudounitary operator, as it does not play a role in the dynamics. In doing so, we are actually looking at the quotient group, where we divide the time evolution operator by the second unitary matrix $\hat{\mathcal{U}}/\mathcal{U}_2 \sim \mathcal{U}_1$. As we shall see in the following chapter, we are able to define the Fubini-Study metric in any instance where we are able to write down the specific structure of the state vector at any time, whether it is in a hyperbolic or spherical space is immaterial. What is important is knowing the evolution of the system, and this is given through the matrices which regulate this process. By understanding the different unitary and pseudounitary matrices, we can then move forward to calculate the differential geometry directly for the hyperbolic heat kernel using the Fubini-Study metric.

4.3.3 Pseudounitary Operators and the KAK Decomposition

We shall now apply this theorem to several examples to show how we can generate the pseudounitary operators that we shall need in order to define the Fubini-Study metric 3.2.33 on the hyperbolic space.

Corollary 4.3.16. *The pseudounitary operators given by analytical continuation of the time parameter in 4.3.11 are:*

$$\hat{\mathcal{U}}_1(\phi) = \begin{bmatrix} \cosh\phi & i\sinh\phi \\ -i\sinh\phi & \cosh\phi \end{bmatrix}, \hat{\mathcal{U}}_2(\phi) = \begin{bmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{bmatrix} \tag{4.228}$$

Proof. This follows by elementary substitution of $t = it$ into the formula from 4.3.11, we have:

$$\hat{U}_{S_1}(\phi) = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \to \hat{\mathcal{U}}_1(\phi) = \hat{U}_{S_1}(i\phi) = \begin{bmatrix} \cosh\phi & i\sinh\phi \\ -i\sinh\phi & \cosh\phi \end{bmatrix} \tag{4.229}$$

and also

$$\hat{U}_{S_2}(\phi) = \begin{bmatrix} \cos\phi & i\sin\phi \\ i\sin\phi & \cos\phi \end{bmatrix} \to \hat{\mathcal{U}}_2(\phi) = \hat{U}_{S_2}(i\phi) = \begin{bmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{bmatrix} \tag{4.230}$$

from which the result follows. □

We shall now state an important known result from the study of matrix groups. This can be traced back to the work of Cartan (see Helgason, [113] for a modern perspective). For our purposes, we require the following assertion.

Theorem 4.3.17. *The Cartan or KAK decomposition of the pseudounitary is given by the formula:*

$$\hat{\mathcal{U}}_j(\tau,\varphi,\psi) = \hat{U}_0\left(\frac{\varphi}{2}\right)\hat{\mathcal{U}}_j\left(\frac{\tau}{2}\right)\hat{U}_0\left(\frac{\psi}{2}\right) \tag{4.231}$$

where $j \in 1,2$ in this instance, and $\hat{U}_0\,(.)$ is the diagonal unitary matrix:

$$\hat{U}_0\,(\varphi) = \begin{bmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{bmatrix} \tag{4.232}$$

Further, we take the $\hat{\mathcal{U}}_j\left(\frac{\tau}{2}\right)$ as either of the two pseudounitary matrices:

$$\hat{\mathcal{U}}_1(\phi) = \begin{bmatrix} \cosh\phi & i\sinh\phi \\ -i\sinh\phi & \cosh\phi \end{bmatrix} \tag{4.233}$$

$$\hat{\mathcal{U}}_2(\phi) = \begin{bmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{bmatrix} \tag{4.234}$$

where $0 \leq \varphi \leq 2\pi$ and $0 < \phi < \infty$, i.e. the first matrix group is periodic, and the second is hyperbolic.

Proof. The Cartan decomposition for SU(1,1) is known, see e.g. Vilenkin [69], (pp.293, VI.1.2.5) for a simple discussion of the extension of the KAK equivalent for the pseudounitary operators. We merely extend the analysis to consider both cases implied by the pseudounitary operators given by the two cases in the analytical continuation as in 4.3.17. □

We shall now apply this to the two examples as given in 4.3.17. As we shall see, these two different decompositions imply symmetry properties on the pseudounitary operators. We shall require these results in order to analyse the form of the Fubini-Study metric, which we shall calculate using the projection operator on the state.

Example 4.3.18. Taking

$$\hat{\mathcal{U}}_1(\phi) = \hat{U}_{S_1}(i\phi) = \begin{bmatrix} \cosh\phi & i\sinh\phi \\ -i\sinh\phi & \cosh\phi \end{bmatrix} \tag{4.235}$$

and the diagonal element:

$$\hat{U}_0(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix} \tag{4.236}$$

and applying 4.3.17 we have, taking $0 < \tau < \infty, \varphi, \psi \in [0, 2\pi]$:

$$\hat{\mathcal{U}}_1(\tau,\varphi,\psi) = \hat{U}_0\left(\frac{\varphi}{2}\right)\hat{\mathcal{U}}_1\left(\frac{\tau}{2}\right)\hat{U}_0\left(\frac{\psi}{2}\right) \tag{4.237}$$

$$= \begin{bmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{bmatrix} \begin{bmatrix} \cosh\frac{\tau}{2} & i\sinh\frac{\tau}{2} \\ -i\sinh\frac{\tau}{2} & \cosh\frac{\tau}{2} \end{bmatrix} \begin{bmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{bmatrix} \tag{4.238}$$

and hence:

$$\hat{\mathcal{U}}_1(\tau,\varphi,\psi) = \begin{bmatrix} \cosh\frac{\tau}{2}e^{i(\varphi+\psi)/2} & i\sinh\frac{\tau}{2}e^{i(\varphi-\psi)/2} \\ -i\sinh\frac{\tau}{2}e^{-i(\varphi-\psi)/2} & \cosh\frac{\tau}{2}e^{-i(\varphi+\psi)/2} \end{bmatrix} \tag{4.239}$$

This is one form of the basic pseudounitary matrix we shall associate to SU(1,1) and the hyperbolic brachistochrone.

Example 4.3.19. The second type of pseudounitary matrix we shall require in order to calculate the Fubini-Study metric is associated with the other type of non-diagonal element in the KAK decomposition as in 4.3.17. Writing the decomposition explicitly as in the previous example, we have:

$$\hat{\mathcal{U}}_2(\tau,\varphi,\psi) = \hat{U}_0\left(\frac{\varphi}{2}\right)\hat{\mathcal{U}}_2\left(\frac{\tau}{2}\right)\hat{U}_0\left(\frac{\psi}{2}\right) \tag{4.240}$$

and hence:

$$\hat{\mathcal{U}}_2(\tau,\varphi,\psi) = \begin{bmatrix} \cosh\frac{\tau}{2}e^{i(\varphi+\psi)/2} & -\sinh\frac{\tau}{2}e^{i(\varphi-\psi)/2} \\ -\sinh\frac{\tau}{2}e^{-i(\varphi-\psi)/2} & \cosh\frac{\tau}{2}e^{-i(\varphi+\psi)/2} \end{bmatrix} \tag{4.241}$$

Remark 50. The symmetry properties of these two matrices differ in that we have:

$$\hat{\mathcal{U}}_1^{-1}(\tau,\varphi,\psi) = \hat{\mathcal{U}}_1^T(\tau,-\varphi,-\psi) \tag{4.242}$$

and

$$\hat{\mathcal{U}}_2^{-1}(\tau,\varphi,\psi) = \hat{\mathcal{U}}_2^\dagger(-\tau,\varphi,\psi) = \hat{\mathcal{U}}_2^T(-\tau,-\varphi,-\psi) \tag{4.243}$$

respectively. This is easily shown using the matrix inversion formula for 2×2 matrices. Note that these properties imply that these spaces are not unitary, only pseudounitary. One could take this as an alternative place in which to develop the analysis of these types of extension of the standard types of hyperbolic spaces through use of inner product norm relations as discussed in the introduction to this chapter.

4.4 Fubini-Study Metric for the Hyperboloid

We shall now show how these two formulae for the pseudounitary operators can be used to construct two representations of the Fubini-Study metric in the hyperbolic space. We shall take as reference the formulae from 3.2.33, see also [66], where the author has calculated the following problem for SU(2) and the Heisenberg group. We shall extend the analysis by calculating the relevant projection operators in the hyperbolic space.

4.4.0.1 The Case of the Space SU(1,1)

In the following, we shall depart from the method of the quantum and hyperbolic brachistochrone, and simply use the unitary matrices that we have derived from the KAK decomposition and so on. However, the Fubini-Study metric is the underlying principle that defines this, so we should expect that the differential geometry will be somehow related to models of classical geometrical analysis. The power in using the Fubini-Study metric lies in that it may be applied to any unitary operator or pseudounitary operator whatsoever, as long as it is invertible in the right way and possesses a valid adjoint, then it will be possible to define the metric by this method. This should be considered to be a related theory, in that it shares many of the same necessities, such as finding unitary operators or pseudounitary equivalents, examining the action of these operators on a state, and the fundamental formula is identical to the least time minimisation action sans constraints.

Our advantage in understanding the mechanics of the quantum brachistochrone is that we have developed an understanding of how these unitaries come about, and using a very small number of working assumptions on the nature of the constraint, we can derive the unitary matrices and time dependence of the optimal Hamiltonian. These are necessary inputs for calculating the Fubini-Study metric, which we shall employ in the following calculation.

This section shall use the pseudounitary operator stemming from the results of 4.3.18. We shall apply a number of small lemmas in order to derive the Fubini-Study metric associated with this space. In particular, we shall use the sequence of theorems given by 3.2.19, 3.2.33 with a simple definition of the state to derive the metric, which is equivalent to a representation of the Poincare metric as derived in 3.2.39.

Lemma 4.4.1. *The projection operator on $SU(1,1)$ is given by the formula:*

$$\hat{P}(\tau,\varphi,\psi) = \begin{bmatrix} \cosh^2\dfrac{\tau}{2} & -\dfrac{ie^{i\varphi}}{2}\sinh\tau \\ -\dfrac{ie^{-i\varphi}}{2}\sinh\tau & -\sinh^2\dfrac{\tau}{2} \end{bmatrix} \tag{4.244}$$

where $0 \leq \tau < \infty$, $0 \leq \varphi, \psi \leq 2\pi$.

Proof. We take the definition of the projection operator from 3.2.19, and therefore we have:

$$\hat{P}(\tau,\varphi,\psi) = \left|\bar{\Psi}(\tau,\varphi,\psi)\right\rangle \left\langle \Psi(\tau,\varphi,\psi)\right| \tag{4.245}$$

The pseudounitary operator 4.3.18 gives:

$$\left|\Psi(\tau,\varphi,\psi)\right\rangle = \hat{\mathcal{U}}_1(\tau,\varphi,\psi)\left|\Psi(0)\right\rangle \tag{4.246}$$

and for the adjoint, we have 4.3.18:

$$\langle \bar{\Psi}(\tau,\varphi,\psi) \rvert = \langle \bar{\Psi}(0) \rvert \hat{\mathcal{U}}_1^{-1}(\tau,\varphi,\psi) = \langle \bar{\Psi}(0) \rvert \hat{\mathcal{U}}_1^{T}(\tau,-\varphi,-\psi) \tag{4.247}$$

$$= \langle \Psi(\tau,-\varphi,-\psi) \rvert \tag{4.248}$$

Evaluating the projection operator using the initial state $|\Psi(0)\rangle = [1,0]^T$, and the pseudounitary operator:

$$\hat{\mathcal{U}}_1(\tau,\varphi,\psi) = \begin{bmatrix} \cosh\frac{\tau}{2} e^{i(\varphi+\psi)/2} & i\sinh\frac{\tau}{2} e^{i(\varphi-\psi)/2} \\ -i\sinh\frac{\tau}{2} e^{-i(\varphi-\psi)/2} & \cosh\frac{\tau}{2} e^{-i(\varphi+\psi)/2} \end{bmatrix} \tag{4.249}$$

we then have:

$$\hat{P}(\tau,\varphi,\psi) = \begin{bmatrix} \cosh^2\frac{\tau}{2} & -\frac{ie^{i\varphi}}{2}\sinh\tau \\ -\frac{ie^{-i\varphi}}{2}\sinh\tau & -\sinh^2\frac{\tau}{2} \end{bmatrix} \tag{4.250}$$

thereby establishing the claim. □

We shall also require the following simple result that can be found using simple differentiation.

Lemma 4.4.2. *For the initial state $|\Psi(0)\rangle = [1,0]^T$, with pseudounitary operator defined through 4.3.18, the differential components:*

$$|\Psi_\rho\rangle = \frac{\partial}{\partial\rho} |\Psi(\tau,\varphi,\psi)\rangle \tag{4.251}$$

satisfy:

$$|\Psi_\tau(\tau,\varphi,\psi)\rangle = \frac{1}{2}\begin{bmatrix} e^{i(\varphi+\psi)/2}\sinh\frac{\tau}{2} \\ -ie^{-i(\varphi-\psi)/2}\cosh\frac{\tau}{2} \end{bmatrix} \tag{4.252}$$

$$|\Psi_\varphi(\tau,\varphi,\psi)\rangle = \frac{1}{2}\begin{bmatrix} ie^{i(\varphi+\psi)/2}\cosh\frac{\tau}{2} \\ -e^{-i(\varphi-\psi)/2}\sinh\frac{\tau}{2} \end{bmatrix} \tag{4.253}$$

$$|\Psi_\psi(\tau,\varphi,\psi)\rangle = \frac{1}{2}\begin{bmatrix} ie^{i(\varphi+\psi)/2}\cosh\frac{\tau}{2} \\ e^{-i(\varphi-\psi)/2}\sinh\frac{\tau}{2} \end{bmatrix} \tag{4.254}$$

Proof. Using the definition of the pseudounitary operator 4.3.18, and the initial state $|\Psi(0)\rangle = [1,0]^T$, we have:

$$|\Psi(\tau,\varphi,\psi)\rangle = \hat{\mathcal{U}}_1(\tau,\varphi,\psi)\,|\Psi(0)\rangle \tag{4.255}$$

$$= \begin{bmatrix} \cosh\frac{\tau}{2} e^{i(\varphi+\psi)/2} & i\sinh\frac{\tau}{2} e^{i(\varphi-\psi)/2} \\ -i\sinh\frac{\tau}{2} e^{-i(\varphi-\psi)/2} & \cosh\frac{\tau}{2} e^{-i(\varphi+\psi)/2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.256}$$

$$= \begin{bmatrix} \cosh\frac{\tau}{2} e^{i(\varphi+\psi)/2} \\ -i\sinh\frac{\tau}{2} e^{-i(\varphi-\psi)/2} \end{bmatrix} \tag{4.257}$$

Differentiating the vectors term by term, we obtain the result. □

We now state the first major new result of this thesis in the following theorem.

Theorem 4.4.3. *The Fubini-Study metric on the hyperbolic space, defined through the formula 3.2.33, and associated with the form of the pseudounitary operator on SU(1,1) 4.3.18 is given by the Poincare metric, i.e.:*

$$ds^2 = \frac{1}{4}\left(-d\tau^2 + \sinh^2\tau d\varphi^2\right) \tag{4.258}$$

Proof. We shall take the definition of the Fubini-Study metric from 3.2.33, see also [66] for analysis on SU(2) and the Heisenberg group. The formula for the Fubini-Study metric reads as:

$$F_{\alpha\beta} = \left\langle \bar{\Psi}_\alpha \right| \left. \Psi_\beta \right\rangle - \left\langle \bar{\Psi}_\alpha \right| \left. \Psi \right\rangle \left\langle \Psi \right| \left. \Psi_\beta \right\rangle \tag{4.259}$$

and the Fubini-Study metric is given by the real part of this tensor:

$$g_{\alpha\beta} = \mathfrak{Re}\left[F_{\alpha\beta}\right] \tag{4.260}$$

We have calculated the differential components in 4.4.2, subsitution gives

$$F_{\alpha\beta} = \frac{1}{4}\begin{bmatrix} -1 & i\sinh\tau & 0 \\ i\sinh\tau & \sinh^2\tau & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.261}$$

or

$$g_{\alpha\beta} = \frac{1}{4}\begin{bmatrix} -1 & 0 & 0 \\ 0 & \sinh^2\tau & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.262}$$

We therefore derive the Fubini-Study metric to be equal to:

$$ds^2 = \frac{1}{4}(-d\tau^2 + \sinh^2\tau d\varphi^2) \tag{4.263}$$

which is equal to the metric for the hyperbolic plane model of Poincare, as discussed in the calculation following 3.2.39. □

Remark 51. We are able to use this relationship between the hyperbolic brachistochrone and the differential geometry of the projective state to define the Laplacian. The following section shall replicate this calculation for the alternative representation of the hyperbolic plane given by the second pseudounitary transformation 4.3.19.

4.4.0.2 The Case of the Space SU*(1,1)

We can see that there are two representations for the unitary matrices that cover the hyperbolic space. This is to be expected, as our model basically depends on the existence of a choice of one side of a hyperbola of two sheets. As we shall show, the use of the second type of pseudounitary matrix induces a metric that is of a very similar type to that calculated in the previous section (with a twist). To distinguish this from the first space, we term this to be SU*(1,1). Although it is still SU(1,1), it is different as the metric will show, and we use the second parameterisation of the KAK decomposition to derive it.

In a directly analogous way to the method used to establish 4.4.3, we assert the following:

Theorem 4.4.4. *For the pseudounitary operator given by 4.3.19, the Fubini-Study metric 3.2.33 is given by the fundamental form:*

$$ds^2 = \frac{1}{4}(d\tau^2 + \sinh^2\tau d\varphi^2) \tag{4.264}$$

with metric tensor:

$$g_{\alpha\beta} = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sinh^2\tau & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.265}$$

where $0 \leq \tau < \infty$, $0 \leq \varphi, \psi \leq 2\pi$ *as before.*

Proof. Proceeding using the pseudounitary operator, we have:

$$\hat{\mathcal{U}}_2(\tau,\varphi,\psi) = \begin{bmatrix} \cosh\frac{\tau}{2}e^{i(\varphi+\psi)/2} & -\sinh\frac{\tau}{2}e^{i(\varphi-\psi)/2} \\ \\ -\sinh\frac{\tau}{2}e^{-i(\varphi-\psi)/2} & \cosh\frac{\tau}{2}e^{-i(\varphi+\psi)/2} \end{bmatrix} \tag{4.266}$$

$$\hat{\mathcal{U}}_2^{-1}(\tau,\varphi,\psi) = \hat{\mathcal{U}}_2^{\dagger}(-\tau,\varphi,\psi) = \hat{\mathcal{U}}_2^{T}(-\tau,-\varphi,-\psi) \tag{4.267}$$

Following the same method as 4.4.1 4.4.2, we have the state:

$$|\Psi(\tau,\varphi,\psi)\rangle = \hat{\mathcal{U}}_2(\tau,\varphi,\psi)\,|\Psi(0)\rangle \tag{4.268}$$

$$= \begin{bmatrix} \cosh\frac{\tau}{2}e^{i(\varphi+\psi)/2} & -\sinh\frac{\tau}{2}e^{i(\varphi-\psi)/2} \\ \\ -\sinh\frac{\tau}{2}e^{-i(\varphi-\psi)/2} & \cosh\frac{\tau}{2}e^{-i(\varphi+\psi)/2} \end{bmatrix}\begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} \tag{4.269}$$

$$= \begin{bmatrix} \cosh\frac{\tau}{2}e^{i(\varphi+\psi)/2} \\ \\ -\sinh\frac{\tau}{2}e^{-i(\varphi-\psi)/2} \end{bmatrix} \tag{4.270}$$

Calculating the projection operator, we therefore have the adjoint state:

$$\left\langle\bar{\Psi}(\tau,\varphi,\psi)\right| = \langle\Psi(-\tau,-\varphi,-\psi)| = \begin{bmatrix} \cosh\frac{\tau}{2}e^{-i(\varphi+\psi)/2}, & \sinh\frac{\tau}{2}e^{i(\varphi-\psi)/2} \end{bmatrix} \tag{4.271}$$

and resulting projection operator as in 3.2.19, 4.4.1:

$$\hat{P}(\tau,\varphi,\psi) = \left|\bar{\Psi}(\tau,\varphi,\psi)\right\rangle\langle\Psi(\tau,\varphi,\psi)| \tag{4.272}$$

$$= \begin{bmatrix} \cosh\frac{\tau}{2}e^{i(\varphi+\psi)/2} \\ \\ -\sinh\frac{\tau}{2}e^{-i(\varphi-\psi)/2} \end{bmatrix}\begin{bmatrix} \cosh\frac{\tau}{2}e^{-i(\varphi+\psi)/2}, & \sinh\frac{\tau}{2}e^{i(\varphi-\psi)/2} \end{bmatrix} \tag{4.273}$$

$$= \begin{bmatrix} \cosh^2\frac{\tau}{2} & \frac{e^{i\varphi}}{2}\sinh\tau \\ -\frac{e^{-i\varphi}}{2}\sinh\tau & -\sinh^2\frac{\tau}{2} \end{bmatrix} \tag{4.274}$$

Calculating the derivatives as in 4.4.2, we readily obtain similar expressions by differentiation of the matrices, and evaluating as in 4.4.3 we have the Fubini-Study metric:

$$|\Psi_\tau(\tau,\varphi,\psi)\rangle = \frac{1}{2}\begin{bmatrix} e^{i(\varphi+\psi)/2}\sinh\frac{\tau}{2} \\ -e^{-i(\varphi-\psi)/2}\cosh\frac{\tau}{2} \end{bmatrix} \tag{4.275}$$

$$|\Psi_\varphi(\tau,\varphi,\psi)\rangle = \frac{1}{2}\begin{bmatrix} ie^{i(\varphi+\psi)/2}\cosh\frac{\tau}{2} \\ ie^{-i(\varphi-\psi)/2}\sinh\frac{\tau}{2} \end{bmatrix} \tag{4.276}$$

$$|\Psi_{\psi}(\tau,\varphi,\psi)\rangle = \frac{1}{2}\begin{bmatrix} ie^{i(\varphi+\psi)/2}\cosh\frac{\tau}{2} \\ -ie^{-i(\varphi-\psi)/2}\sinh\frac{\tau}{2} \end{bmatrix} \tag{4.277}$$

Reading off the components of the metric tensor, we then have:

$$g_{\alpha\beta} = \mathfrak{Re}\left[\langle\bar{\psi}_{\alpha}|\, \psi_{\beta}\rangle - \langle\bar{\psi}_{\alpha}|\, \Psi\rangle\, \langle\Psi|\, \psi_{\beta}\rangle\right] \tag{4.278}$$

$$|\psi_{\beta}\rangle = \frac{\partial}{\partial x_{\beta}}\, |\Psi\rangle \tag{4.279}$$

and hence using the derivatives obtained above, we may write:

$$F_{\alpha\beta} = \frac{1}{4}\begin{bmatrix} 1 & -2i\sinh\frac{\tau}{2}\cosh\frac{\tau}{2} & 0 \\ 2i\sinh\frac{\tau}{2}\cosh\frac{\tau}{2} & 4\sinh^2\frac{\tau}{2}\cosh^2\frac{\tau}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.280}$$

and Fubini-Study metric equal to:

$$g_{\alpha\beta} = \mathfrak{Re}\left[F_{\alpha\beta}\right] = g_{\alpha\beta} = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sinh^2\tau & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.281}$$

Writing this in terms of the metric, we then have the result:

$$4ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta} = d\tau^2 + \sinh^2\tau d\varphi^2 \tag{4.282}$$

as required to establish the claim. □

Remark 52. We have shown in these theorems how one can develop an understanding of the differential geometry through use of the projective state and the formula for the Fubini-Study metric as in 4.4.3, 4.4.4, and the Poincare metric as given by the results derived in 3.2.39.

4.5 Special Functions and Representation Theory

We shall now utilise the results for the differential geometry of the Poincare metric to construct the representation theory for the hyperbolic brachistochrone. The following subsections shall demonstrate that the formulae for the metrics given by 4.4.3, 4.4.4 imply a direct connection to the associated Legendre functions, or Mehler-Fock functions as given in 3.4.1. We shall depart from matrix analysis for now, having established the existence of the Fubini-Study metric we shall take the differential geometry as axiomatic in the following computations.

4.5.1 Hyperbolic Laplacian

The following calculation links the hyperbolic geometry as given by the Fubini-Study metric 4.4.3, 4.4.4 with the Laplacian operator, with an aim to establishing a link between sets of functions and sets of eigenstates. In this case, our states are described by the symmetry group of the pseudounitary operators, that are defined by the solution to the hyperbolic brachistochrone equation. By using the Fubini-Study metric, we show how this metric may be explicitly computed for a number of important examples related to the hyperbolic plane.

Our principal objective in deriving this metric is to isolate the different forms of the hyperbolic Laplacian. In doing so, we are able to calculate eigenfunctions for various forms of basic PDE systems that employ the Laplace operator in their fundamental definition. This enables us to characterise the eigenfunctions related to the hyperbolic state space which we arrive at from our calculations of the hyperbolic brachistochrone and Fubini-Study metric.

As is well known from analysis, the Laplacian operator describes the eigenfunctions, which give the harmonics of the space. As we have the metrics defined through either 4.4.3, or alternatively 4.4.4, we may apply the theory of the Laplacian in a curved space. We state the following simple application of 3.2.34:

Lemma 4.5.1. *For the hyperbolic metric given as a result of the pseudounitary operator 4.3.18, the Fubini-Study tensor as derived in 4.4.3:*

$$ds^2 = \frac{1}{4}\left(-d\tau^2 + \sinh^2\tau d\varphi^2\right) \tag{4.283}$$

Then the Laplacian in the hyperbolic space is given by the formula:

$$\nabla^2 f = -\frac{\partial^2 f}{\partial \tau^2} - \coth\tau\frac{\partial f}{\partial \tau} + \frac{1}{\sinh^2\tau}\frac{\partial^2 f}{\partial\varphi^2} \tag{4.284}$$

Proof. The metric as derived from 4.4.3 is:

$$4ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \tag{4.285}$$

$$g_{\alpha\beta} = \begin{bmatrix} -1 & 0 \\ 0 & \sinh^2\tau \end{bmatrix} \tag{4.286}$$

where we have removed the non-dynamic part of the metric, i.e. deleted the zero rows and columns from the metric tensor as they will not contribute to the Laplacian. The Laplace-Beltrami operator in a curved space is given by 3.2.34:

$$\nabla^2 f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\alpha}\left(\sqrt{g}g^{\alpha\beta}\frac{\partial f}{\partial x_\beta}\right) \tag{4.287}$$

We have e.g. $g = \det g_{\alpha\beta} = -\sinh^2\tau$, and contravariant metric given by:

$$g^{\alpha\beta} = \begin{bmatrix} -1 & 0 \\ 0 & \dfrac{1}{\sinh^2\tau} \end{bmatrix} \tag{4.288}$$

so we conclude that the Laplace operator is given the formula:

$$\nabla^2 f = \frac{1}{i\sinh\tau}\sum_{\alpha,\beta}\frac{\partial}{\partial x_\alpha}\left(i\sinh\tau g^{\alpha\beta}\frac{\partial f}{\partial x_\beta}\right) = \frac{1}{\sinh\tau}\sum_{\alpha,\beta}\frac{\partial}{\partial x_\alpha}\left(\sinh\tau g^{\alpha\beta}\frac{\partial f}{\partial x_\beta}\right) \tag{4.289}$$

The only non-zero components are $g^{\tau\tau}$ and $g^{\varphi\varphi}$, hence we find:

$$\nabla^2 f = \frac{1}{\sinh\tau}\frac{\partial}{\partial\tau}\left(\sinh\tau . g^{\tau\tau}\frac{\partial f}{\partial\tau}\right) \tag{4.290}$$

$$+\frac{1}{\sinh\tau}\frac{\partial}{\partial\varphi}\left(\sinh\tau . g^{\varphi\varphi}\frac{\partial f}{\partial\varphi}\right) \tag{4.291}$$

$$= -\frac{1}{\sinh\tau}\frac{\partial}{\partial\tau}\left(\sinh\tau\frac{\partial f}{\partial\tau}\right) + \frac{1}{\sinh^2\tau}\frac{\partial^2 f}{\partial\varphi^2} \tag{4.292}$$

$$= -\frac{\partial^2 f}{\partial\tau^2} - \coth\tau\frac{\partial f}{\partial\tau} + \frac{1}{\sinh^2\tau}\frac{\partial^2 f}{\partial\varphi^2} \tag{4.293}$$

as required. □

We immediately obtain the following corollary by extension of the Poincare metric to the alternative representation of the space given by 4.4.4.

Corollary 4.5.2. *For the Poincare metric given by 4.4.4:*

$$ds^2 = \frac{1}{4}(d\tau^2 + \sinh^2 \tau d\varphi^2) \tag{4.294}$$

$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & \sinh^2 \tau \end{bmatrix} \tag{4.295}$$

the Laplacian on the hyperbolic space given by the Fubini-Study metric, and pseudounitary operator as in 4.3.19 is:

$$\nabla^2 f = \frac{\partial^2 f}{\partial \tau^2} + \coth \tau \frac{\partial f}{\partial \tau} + \frac{1}{\sinh^2 \tau} \frac{\partial^2 f}{\partial \varphi^2} \tag{4.296}$$

Proof. Proceed as in 4.5.1, with the replacement of the metric by the formula as above, the only change is in the first component of the metric and the sign of the determinant i.e. $g = \sinh^2 \tau$, the result follows. □

4.5.2 The Pseudosphere and SO(2,1) Metric

We now consider the following simple extension of the hyperbolic metrics as derived in the previous calculations. It is known that the hyperboloid permits a representation given by the mapping from the disk to an object with properties similar to a sphere, i.e. a pseudosphere. Although we shall not go into it in greater depth than our brief sojourn into differential geometry, it is a fact well established in classic geometry that the only surfaces with constant curvature are the plane, sphere and pseudosphere. We shall now show with some simple calculations how we may recover this representation of the hyperbolic space for completeness.

Lemma 4.5.3. *The metric described by $SO(2,1)$ symmetry is:*

$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -r^2 & 0 \\ 0 & 0 & -r^2 \sinh^2 \tau \end{bmatrix} \tag{4.297}$$

The Laplacian associated to this metric is:

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) - \frac{1}{r^2 \sinh \tau} \frac{\partial}{\partial \tau} \left(\sinh \tau \frac{\partial f}{\partial \tau} \right) - \frac{1}{r^2 \sinh^2 \tau} \frac{\partial}{\partial \varphi} \left(\frac{\partial f}{\partial \varphi} \right) \tag{4.298}$$

Proof. The parametrisation of the coordinates on the hyperbolic surface with $SO(2,1)$ symmetry is such that it lies on the surface of a hyperboloid, in Lobachevsky space as opposed to a spherical or Euclidean type space. In such a situation, the surface will be parametrised such that: $z^2 - x^2 - y^2 = r^2$, in which case we may take the expression of a point on the surface as:

$$\begin{aligned} x &= r \sinh \tau \cos \varphi \\ y &= r \sinh \tau \sin \varphi \\ z &= r \cosh \tau \end{aligned} \tag{4.299}$$

following which it is a simple exercise in differentiation to show the metric may be written:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = dz^2 - dx^2 - dy^2 = dr^2 - r^2 d\tau^2 - r^2 \sinh^2 \tau d\varphi^2 \tag{4.300}$$

and we conclude that the metric is given by:

$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -r^2 & 0 \\ 0 & 0 & -r^2 \sinh^2 \tau \end{bmatrix} \tag{4.301}$$

establishing the first part of the theorem. The Laplace-Beltrami operator in a curved space is given by 3.2.34:

$$\nabla^2 f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_\alpha} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial f}{\partial x_\beta} \right) \tag{4.302}$$

We have determinant $g = \det g_{\alpha\beta} = r^4 \sinh^2 \tau$, and contravariant metric given by:

$$g^{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & -\frac{1}{r^2 \sinh^2 \tau} \end{bmatrix} \tag{4.303}$$

so we conclude that the Laplace operator is given the formula:

$$\nabla^2 f = \frac{1}{r^2 \sinh \tau} \sum_{\alpha,\beta} \frac{\partial}{\partial x_\alpha} \left(r^2 \sinh \tau g^{\alpha\beta} \frac{\partial f}{\partial x_\beta} \right) \tag{4.304}$$

The only non-zero components are g^{rr}, $g^{\tau\tau}$ and $g^{\varphi\varphi}$, hence we find:

$$\nabla^2 f = \frac{1}{r^2 \sinh \tau} \left(\frac{\partial}{\partial r} \left(r^2 \sinh \tau \frac{\partial f}{\partial r} \right) - \frac{\partial}{\partial \tau} \left(\frac{r^2 \sinh \tau}{r^2} \frac{\partial f}{\partial \tau} \right) - \frac{\partial}{\partial \varphi} \left(\frac{r^2 \sinh \tau}{r^2 \sinh^2 \tau} \frac{\partial f}{\partial \varphi} \right) \right) \tag{4.305}$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) - \frac{1}{r^2 \sinh \tau} \frac{\partial}{\partial \tau} \left(\sinh \tau \frac{\partial f}{\partial \tau} \right) - \frac{1}{r^2 \sinh^2 \tau} \frac{\partial}{\partial \varphi} \left(\frac{\partial f}{\partial \varphi} \right) \tag{4.306}$$

as required. □

Remark 53. In the following subsection, we consider the Helmholtz equation which is defined by the eigenvalue problem of the Laplacian $\nabla^2 f = Ef$. As our problem relates to the hyperbolic plane, and not higher order pseudospheres, it is appropriate to take the variable r in the above definition of the hyperbolic Laplacian as a constant, in which case the terms containing derivatives in r vanish. This leads to a type of hyperbolic Laplace operator which we discuss in the following subsection.

We shall now employ these expressions for the Laplace operator in order to derive the sets of eigenfunctions associated to the hyperbolic space. This is necessary in order to develop a formulation of the hyperbolic heat kernel as given by the spectral theory 3.1.18, or the convolution method as given by Stone's and Mercer's theory of positive definite kernels.

4.5.3 Helmholtz Equation Associated to the Laplacian

The calculations in this subsection shall show that, knowing the differential geometry as given by the Poincare metric, the special functions are readily ascertained by application of the Helmholtz equation. We take the following:

Theorem 4.5.4. *The hyperbolic Laplacian ∇^2 as derived in 4.5.1 defines an eigenfunction problem in the hyperbolic disk $\mathcal{D}$, specified by the Helmholtz equation $\nabla^2 f = Ef$, where f is the eigenfunction and E is the eigenvalue. We assume Dirichlet boundary conditions*

$\Omega \in \mathcal{D}$, $f(x) = \phi(x)$ and $x = \cosh\tau$ where $\phi(x)$ is some known function, generally zero on the boundary Ω. The hyperbolic Laplacian 4.5.1 has formula:

$$\nabla^2 f = -\frac{\partial^2 f}{\partial \tau^2} - \coth\tau \frac{\partial f}{\partial \tau} + \frac{1}{\sinh^2\tau}\frac{\partial^2 f}{\partial \varphi^2} \tag{4.307}$$

as derived from the Fubini-Study metric 4.4.3. The solution to the Helmholtz equation $\nabla^2 f = Ef$ is specified through the separable solution:

$$f(\tau,\varphi) = \sum_k c_k e^{-ik\varphi} g(\tau) \tag{4.308}$$

where the function $g(\tau)$ is given through the associated Legendre PDE:

$$Eg = -\frac{\partial^2 g}{\partial \tau^2} - \coth\tau \frac{\partial g}{\partial \tau} - \frac{k^2}{\sinh^2\tau} g \tag{4.309}$$

which has solution:

$$g(\tau) = \frac{1}{\sqrt{\sinh\tau}}\left(C_1 \mathcal{P}^{\xi}_{ik-1/2}(\coth\tau) + C_2 \mathcal{Q}^{\xi}_{ik-1/2}(\coth\tau)\right) \tag{4.310}$$

where the functions $\mathcal{P}^{\xi}_{ik-1/2}, \mathcal{Q}^{\xi}_{ik-1/2}$ are the Mehler-Fock functions 3.4.1 and the eigenvalue is given by $E = -\frac{1}{4} + \xi^2$. There exists a second form of the solution given by:

$$g(\tau) = C_1 \mathcal{P}^{ik}_{\xi-1/2}(\cosh\tau) + C_2 \mathcal{Q}^{ik}_{\xi-1/2}(\cosh\tau) \tag{4.311}$$

where $1 < \tau < \infty$, and C_1, C_2 are constants, to be specified by the boundary condition.

Remark 54. Note that the formulae for the solution in terms of the associated Legendre (Mehler-Fock) functions must be handled very carefully depending on the interval used for solution. In this case we have $x = \cosh\tau$, and hence the domain we are considering is $[1,\infty)$. Although the boundary conditions are left open in this case, this is so we can apply Whipple's formula with maximum generality.

Proof. We begin with substitution of the separable solution into the PDE, we have:

$$\frac{\partial^2 f}{\partial \varphi^2} = -\sum_k k^2 c_k e^{-ik\varphi} g(\tau) \tag{4.312}$$

$$\frac{\partial f}{\partial \tau} = \sum_k c_k e^{-ik\varphi} \frac{\partial g(\tau)}{\partial \tau} \tag{4.313}$$

etc., hence obtaining the Helmholtz equation:

$$Eg = -\frac{\partial^2 g}{\partial \tau^2} - \coth\tau \frac{\partial g}{\partial \tau} - \frac{k^2}{\sinh^2\tau} g \tag{4.314}$$

i.e. the associated Legendre DE, with solution given by the Mehler-Fock functions 3.4.1:

$$g(\tau) = \frac{1}{\sqrt{\sinh\tau}}\left(C_1 \mathcal{P}^{\sqrt{4E+1}/2}_{ik-1/2}(\coth\tau) + C_2 \mathcal{Q}^{\sqrt{4E+1}/2}_{ik-1/2}(\coth\tau)\right) \tag{4.315}$$

Assume now the eigenvalue, which we denote $E = -\frac{1}{4} + \xi^2$. In this case, we only have a single continuous eigenvalue which defines the spectrum of the system. We may rewrite the solution in the form:

$$g(\tau) = \frac{1}{\sqrt{\sinh\tau}}\left(C_1 \mathcal{P}^{\xi}_{ik-1/2}(\coth\tau) + C_2 \mathcal{Q}^{\xi}_{ik-1/2}(\coth\tau)\right) \tag{4.316}$$

as required, where C_1, C_2 are constants. To obtain the second part of the theorem, note Whipple's formulae for the associated Legendre functions [119], DLMF ref. 14.9.16-7 [8]:

$$\mathcal{Q}_\nu^\mu(x) = \sqrt{\frac{\pi}{2}}(x^2-1)^{-1/4}\mathcal{P}_{-\mu-1/2}^{-\nu-1/2}\left(\frac{x}{\sqrt{x^2-1}}\right) \tag{4.317}$$

$$\mathcal{P}_\nu^\mu(x) = \sqrt{\frac{2}{\pi}}(x^2-1)^{-1/4}\mathcal{Q}_{-(\mu+1/2)}^{\nu+1/2}\left(\frac{x}{\sqrt{x^2-1}}\right) \tag{4.318}$$

Substitution gives:

$$\frac{x}{\sqrt{x^2-1}} = \frac{\cosh\tau}{\sqrt{\cosh^2\tau-1}} = \coth\tau \tag{4.319}$$

and also:

$$(x^2-1)^{-1/4} = \frac{1}{\sqrt{\sinh\tau}} \tag{4.320}$$

for $x = \cosh\tau$. Application of Whipple's formulae gives the relationship between the Legendre-P and Legendre-Q functions of reversed indices via:

$$\mathcal{P}_{ik-1/2}^{\xi}(\coth\tau) = \sqrt{\frac{2}{\pi}}\sqrt{\sinh\tau}\,\mathcal{Q}_{-\xi-1/2}^{ik}(\cosh\tau) \tag{4.321}$$

$$\mathcal{Q}_{ik-1/2}^{\xi}(\coth\tau) = \sqrt{\frac{\pi}{2}}\sqrt{\sinh\tau}\,\mathcal{P}_{-\xi-1/2}^{-ik}(\cosh\tau) \tag{4.322}$$

and we obtain the solution:

$$g(\tau) = C_1\mathcal{P}_{-\xi-1/2}^{-ik}(\cosh\tau) + C_2\mathcal{Q}_{-\xi-1/2}^{ik}(\cosh\tau) \tag{4.323}$$

To obtain the second form of the solution, as stated in the theorem, we take the following symmetries, listed in 14.9.11-15,14.9.12-13 from DLMF [8]. We have $\mathcal{P}_\nu^{\pm\mu}(x) = \mathcal{P}_{-\nu-1}^{\pm\mu}(x)$, also $\mathcal{Q}_\nu^\mu(x) = \mathcal{Q}_\nu^{-\mu}(x)$. We also have the following relationships which relate to parity changes of the index:

$$P_\nu^{-m}(x) = \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}P_\nu^m(x) \tag{4.324}$$

where $\nu \neq m-1, m-2, ..$ and $\mathcal{Q}_\nu^\mu(x) = \mathcal{Q}_\nu^{-\mu}(x)$. We have then that:

$$\mathcal{P}_{\nu-1/2}^{\pm\mu}(x) = \mathcal{P}_{-(\nu+1/2)}^{\pm\mu}(x) \tag{4.325}$$

and hence the solution may be written:

$$g(\tau) = C_1\mathcal{P}_{\xi-1/2}^{ik}(\cosh\tau) + C_2\mathcal{Q}_{\xi-1/2}^{ik}(\cosh\tau) \tag{4.326}$$

Note the complementary form of Whipple's formula for the toroidal harmonics implies:

$$\mathcal{Q}_{\xi-1/2}^{-ik}(\cosh\tau) = \sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{\sinh\tau}}\mathcal{P}_{ik-1/2}^{-\xi}(\coth\tau) \tag{4.327}$$

$$\mathcal{P}_{\xi-1/2}^{-ik}(\cosh\tau) = \sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{\sinh\tau}}\mathcal{Q}_{-ik-1/2}^{-\xi}(\coth\tau) \tag{4.328}$$

which gives the form for the solution as expressed in the theorem, upon employing the symmetries of the associated Legendre functions. □

Remark 55. We shall take this as our primary result in establishing the formula for the hyperbolic heat kernel. This result is known, but we have derived it in a novel way, free of the use of shift transforms and the like as explored in the introduction. As the following brief series of lemmas shall demonstrate, the hyperbolic heat kernel follows as an application of either the spectral theory via 3.1.18, or the convolution theory of positive definite kernels as given by Stone's or Mercer's theorem as discussed in 3.3.5, 3.3.11.

4.6 Hyperbolic Heat Kernel

We shall now discuss one major result of this thesis, in that we shall be able to provide a fairly direct derivation of the hyperbolic heat kernel using the results we have established for the differential geometry as in 4.4.3, and associated Laplace operator in the hyperbolic space as derived in 4.5.1. This is important, as we shall show using a number of different techniques how the answer may be derived. In particular, we shall apply the distance function representation of the kernel and show how this is associated to the composition formula in the hyperbolic space. In the following argument, we take the hyperbolic heat kernel as defined by the diffusion problem given by the backwards heat equation:

Definition 4.6.1. Consider the Cauchy problem for diffusion in the hyperbolic disk $\mathcal{D}$. The hyperbolic heat equation is:

$$\frac{\partial u}{\partial t} = \nabla^2 u \tag{4.329}$$

where ∇^2 is the hyperbolic Laplacian as in 4.5.1 with initial value problem $u(\mathbf{x}, 0) = f(\mathbf{x})$. The solution of this equation can be expressed in terms of the integral kernel $K(\mathbf{x}, \mathbf{y}; t)$:

$$u(\mathbf{x}, t) = \int K(\mathbf{x}, \mathbf{y}; t) u(\mathbf{y}, 0) d\mathfrak{m}(\mathbf{y}) \tag{4.330}$$

where $d\mathfrak{m}(\mathbf{y})$ is the integration measure on the hyperbolic space and $K(.)$ is the heat kernel for ∇^2, the hyperbolic Laplace operator.

4.6.1 Distance formula on Hyperbolic Space

The following brief calculation shall apply the methods of Vilenkin [69], see also the discussions of Lenz in [64,65,111] where the author has analysed the decomposition theory of the pseudounitary operator in order to extract the kernel via the distance metric. We shall state the following lemma, which we shall require in order to apply the results from 3.3.29, 3.3.13.

We give the following plausibility argument for the structure of the kernel we shall derive. The kernel of the system represents the probability that a state changes from one configuration at one time to another configuration at another time. The system that we began with, using the hyperbolic brachistochrone, had an isotropic constraint. We should expect that the kernel will behave in a similar way, as the probability between any two states should only depend on the relative values of these states, specifically the difference between them, and not their values. The kernel should also be symmetric, or at least have a property that respects the symmetry when we interchange input and output configurations. Finally, the kernel should also be homogenous, which implies that there if there is some transformation under which the input and output values are invariant, then the kernel should not change. These operational axioms in many instances are sufficient to determine the form of the kernel, in particular the argument of the function.

If we examine how these axioms might be written out, we might have:

$$K(\mathbf{x}, \mathbf{y}) = K(\mathbf{y}, \mathbf{x}) \tag{4.331}$$

for symmetry, and

$$K(\hat{U}\mathbf{x}, \hat{U}\mathbf{y}) = K(\mathbf{x}, \mathbf{y}) \tag{4.332}$$

for invariance, and lastly:

$$K(\mathbf{x}, \mathbf{y}) = K(\mathbf{x} - \mathbf{y}, 0) \tag{4.333}$$

for homogeneity. Not all these axioms will be true for all kernels, but we will show with a simple example how these relate to the types of kernels we can expect from the heat equation more generally, and the hyperbolic heat equation specifically.

Example 4.6.2. For the un-normalised heat kernel $\in \mathbb{R}^n$, we have the simple expression:

$$K(\mathbf{x},\mathbf{y}) = \exp\left(-\frac{1}{2\sigma^2}\left\|\mathbf{x}-\mathbf{y}\right\|^2\right) = K(d(\mathbf{x},\mathbf{y})) \tag{4.334}$$

we have the distance function:

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = d(\mathbf{x}-\mathbf{y},0) \tag{4.335}$$

This kernel is symmetric, as is the distance function $d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{x})$, via $K(d(\mathbf{x},\mathbf{y})) = K(d(\mathbf{y},\mathbf{x}))$. Under a transformation $K(\hat{R}\mathbf{x},\hat{R}\mathbf{y}) = K(d(\hat{R}\mathbf{x},\hat{R}\mathbf{y})) = K(d(\mathbf{x},\mathbf{y}))$ we must have:

$$\left\|\hat{R}(\mathbf{x}-\mathbf{y})\right\|^2 = (\mathbf{x}-\mathbf{y})^T\,\hat{R}^T\hat{R}(\mathbf{x}-\mathbf{y}) \tag{4.336}$$

which implies the unitarity condition:

$$\hat{R}^T\hat{R} = \mathbf{1} \tag{4.337}$$

This is an example of a symmetric, translation invariant, homogenous kernel. Any kernel which has an argument that is of the form of a radial distance function will respect these properties. More generally, if there exists a symmetric function obeying the basic axioms:

$$d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{x}) \tag{4.338}$$

$$d(\mathbf{x},\mathbf{y}) = d(\mathbf{x}-\mathbf{y},0) \tag{4.339}$$

$$d(\hat{U}\mathbf{x},\hat{U}\mathbf{y}) = d(\mathbf{x},\mathbf{y}) \tag{4.340}$$

$$K(\mathbf{x},\mathbf{y}) = K(d(\mathbf{x},\mathbf{y})) \tag{4.341}$$

then this kernel will be respectively symmetric, translation invariant, or invariant under the action of $\hat{U}$, i.e. homogeneous. Not every kernel will have all of these properties, but by analysing the nature of the implied distance function on the hyperbolic space we can understand the hyperbolic heat kernel. Note that the definition of the origin is also important, and we can expect some differences to arise solely due to the idiosyncracities of the hyperbolic space.

Example 4.6.3. Assume that for the purposes of this example we have a kernel that exists on the hyperbolic space, defined through the action of the pseudounitary operators. Obviously a distance function exists, and we have shown how the metric is related to the unitary operators via the Fubini-Study method. If this kernel is symmetric, translation invariant, and homogenous under the action of the pseudounitary operator, it must be a function of the distance on the hyperbolic space. In the simplest instance of a hyperbolic geometry, we may take the hyperbolic cosine law:

$$\cosh d(\mathbf{x},\mathbf{y}) = \cosh\tau = \cosh\tau_1\cosh\tau_2 - \sinh\tau_1\sinh\tau_2\cos(\varphi_1-\varphi_2) \tag{4.342}$$

which we derived in 4.3.9, as the rotation component of the unitary or pseudounitary operator. We also have the Bures measure (see 3.2.32), which gives the projective distance measure for the Fubini-Study metric:

$$d(\Psi,\Phi) = \cos^{-1}\left(\sqrt{\frac{\langle\Psi|\,\Phi\rangle\,\langle\Phi|\,\Psi\rangle}{\langle\Phi|\,\Phi\rangle\,\langle\Psi|\,\Psi\rangle}}\right) \tag{4.343}$$

In the pseudounitary or hyperbolic case, this will be modified to:

$$d(\mathbf{x},\mathbf{y}) = \cosh^{-1}\left(\sqrt{\frac{(\bar{\mathbf{x}}\mathbf{y})(\bar{\mathbf{y}}\mathbf{x})}{(\bar{\mathbf{x}}\mathbf{x})(\bar{\mathbf{y}}\mathbf{y})}}\right) \tag{4.344}$$

We can see from this perspective that there will be an distance measure given by a hyperbolic cosine. If the kernel is then to be symmetric, invariant and homogeneous, we must therefore have:

$$K(\mathbf{x},\mathbf{y}) = K(d(\mathbf{x},\mathbf{y})) \tag{4.345}$$

and using the Bures metric, it must also be invariant under the transformation:

$$(\mathbf{x},\mathbf{y}) \to (\mathcal{R}\mathbf{x},\mathcal{R}\mathbf{y}) \tag{4.346}$$

with $\bar{\mathcal{R}}\mathcal{R} = \mathcal{R}\bar{\mathcal{R}} = \mathbf{1}$. Now, we must justify that the solution to the hyperbolic heat equation will have the same properties. In analogy with the solution to the heat equation as discussed in the previous example, we can see that the kernel is necessarily symmetric for the case that we interchange the input and output configurations. This is only true for the case of a first order equation in time which is real. In the complex case, we will have more complicated symmetries, which depending on the situation will lead to different behaviour. This is well known from studies into the relationships between statistical and quantum mechanics, see [18], [120]. So, as the heat equation does not contain any complex or imaginary parts, we can see that the eigenfunctions of this equation will be purely real, and the kernel will be symmetric.

If we write the heat equation, we will have:

$$\nabla^2_{(\mathbf{x})} u(\mathbf{x},t) = \frac{\partial u(\mathbf{x},t)}{\partial t} \tag{4.347}$$

where we denote the Laplacian operator with the vector index to illustrate that this is a function of the space co-ordinate parametrisation we are using. If we alter the co-ordinate basis by applying some transformation, we will have:

$$\nabla^2_{(\mathcal{R}\mathbf{x})} u(\mathcal{R}\mathbf{x},t) = \frac{\partial u(\mathcal{R}\mathbf{x},t)}{\partial t} \tag{4.348}$$

and substituting $\mathcal{R}\mathbf{x} = \mathbf{x}'$, these equations will have substantially the same form:

$$\nabla^2_{(\mathbf{x}')} u(\mathbf{x}',t) = \frac{\partial u(\mathbf{x}',t)}{\partial t} \tag{4.349}$$

However, for these to be consistent, we must also have the norms be of the same dimension:

$$\|\mathbf{x}'\|^2 = \|\mathbf{x}\|^2 \Rightarrow \mathcal{R}\bar{\mathcal{R}} = \bar{\mathcal{R}}\mathcal{R} = \mathbf{1} \tag{4.350}$$

which implies a type of orthogonality relation on the transform as can be seen. Our system, being hyperbolic, will not be strictly unitary, but only pseudounitary. Now, if we further take the kernel formula for the fundamental solution to the heat equation, we will have:

$$u(\mathbf{x},t) = \int K(\mathbf{x},\mathbf{y};t)u(\mathbf{y},0)d\mathbf{y} \tag{4.351}$$

We shall now show how the invariance of the kernel implies the invariance of this fundamental solution. We may write, as the transformation is invertible:

$$u(\bar{\mathcal{R}}\mathcal{R}\mathbf{x},t) = u(\mathbf{x},t) \tag{4.352}$$

$$= u(\bar{\mathcal{R}}\mathbf{x}',t) = \int K(\bar{\mathcal{R}}\mathbf{x}',\mathbf{y};t)u(\mathbf{y},0)d\mathbf{y} \tag{4.353}$$

$$= \int K(\bar{\mathcal{R}}\mathcal{R}\mathbf{x}',\mathcal{R}\mathbf{y};t)u(\mathcal{R}\mathbf{y},0)d[\mathcal{R}\mathbf{y}] \tag{4.354}$$

$$= \int K(\mathbf{x}', \mathcal{R}\bar{\mathcal{R}}\mathbf{y}'; t) u(\mathcal{R}\bar{\mathcal{R}}\mathbf{y}', 0) d[\mathcal{R}\bar{\mathcal{R}}\mathbf{y}'] \tag{4.355}$$

where we used the invariance of the kernel. In such case, we conclude:

$$u(\bar{\mathcal{R}}\mathbf{x}', t) = u(\mathbf{x}', t) = \int K(\mathbf{x}', \mathbf{y}'; t) u(\mathbf{y}', 0) d[\mathbf{y}'] \tag{4.356}$$

However, $\mathbf{x}'$ is arbitrary, we may therefore take $\mathbf{x}' = \mathbf{x}$, and we find:

$$u(\bar{\mathcal{R}}\mathbf{x}, t) = u(\mathbf{x}, t) \tag{4.357}$$

The solution is therefore invariant to a transformation of co-ordinates which leaves the distance, as expressed by the norm, unchanged. A simple example of this is a polar type system, where every point at a certain radius has an equivalent distance from the centre. In such a co-ordinate system, a rotation of all points by some angle does not change the distance of any point from the origin. So we can see that the heat equation will possess invariant solutions, which we can express by the kernel:

$$K(\mathcal{R}\mathbf{x}, \mathcal{R}\mathbf{y}; t) = K(\mathbf{x}, \mathbf{y}; t) \tag{4.358}$$

So, we have shown that there exists a symmetric, invariant kernel for the heat equation, and specifically this must be true for the hyperbolic heat equation. In the following sections, we shall use these primitive observations to develop a more extensive, detailed theory of this important object.

We shall now state more formally the results regarding the structure of the kernel on hyperbolic space, as described through the pseudounitary operators we have derived in the previous sections.

Lemma 4.6.4. *The hyperbolic heat kernel is a function of the hyperbolic distance, i.e.*

$$K(\mathbf{g}, \mathbf{g}_l) = K(\cosh\tau) = K(\cosh\tau_1 \cosh\tau_2 - \sinh\tau_1 \sinh\tau_2 \cos(\varphi_1 - \varphi_2)) \tag{4.359}$$

where $\mathbf{g}, \mathbf{g_l}$ *are two elements in the pseudounitary group which defines the hyperbolic space.*

Proof. We begin with application of the results from 3.3.16. Assume a distance function of the form $d(\mathbf{g}, \mathbf{g}_l)$. This may not always exist, but as the examples above show, this should exist for the hyperbolic heat kernel, and specifically, there should be an invariant, homogeneous and symmetric kernel on this space. Assume that this is the case, then we may write:

$$d(\mathbf{g}, \mathbf{g}_l) = d(\mathbf{g}^{-1}\mathbf{g}, \mathbf{g}^{-1}\mathbf{g}_l) = d(\mathbf{g}^{-1}\mathbf{g}, \mathbf{1}) \tag{4.360}$$

where we have used invariance of the distance function under the action of the group element $\mathbf{g}^{-1}$. This is to be interpreted in the sense of measuring the distance of a group element from the identity. We have immediately, following from this:

$$K(\mathbf{g}, \mathbf{g}_l) = K(d(\mathbf{g}, \mathbf{g}_l)) = K(d(\mathbf{g}_l, \mathbf{g})) = K(d(\mathbf{g}^{-1}\mathbf{g}_l, \mathbf{1})) \doteq K(d(\mathbf{g}^{-1}\mathbf{g}_l)) \tag{4.361}$$

which we understand to be a working assumption, which will be used under the proviso of invariance, homogeneity and symmetry. In the following, we shall be examining the result of choosing different group elements, and seeing what the effect is on the distance function. This will give an insight into the arguments of the kernel function.

We shall now evaluate this for the group elements given by the pseudounitary operators given in e.g. 4.3.18, 4.3.19 and show that this reduces to the distance formula representation of the kernel by a simple argument. Consider the matrix element given by $\mathbf{g}^{-1}\mathbf{g}_l$. If we take two elements in SU(1,1) which lie in the disk, we will have to force these each to have only two angular parameters. We have derived a formula for the generic element given by 4.3.18:

$$\mathbf{g}(\tau,\varphi,\psi)=\begin{bmatrix}\cosh\frac{\tau}{2}e^{i(\varphi+\psi)/2} & \sinh\frac{\tau}{2}e^{i(\varphi-\psi)/2}\\ \sinh\frac{\tau}{2}e^{-i(\varphi-\psi)/2} & \cosh\frac{\tau}{2}e^{-i(\varphi+\psi)/2}\end{bmatrix}\tag{4.362}$$

This will be sufficient to specify the elements $\mathbf{g}^{-1}$ and $\mathbf{g}_l$ which will lie in the hyperbolic disk.

Remark 56. Note that this is the matrix inverse of the pseudounitary operator given through 4.3.18. This is important, as in the construction of the regular representations as defined in [69], this thesis 3.3.4 the order of multiplication whether from the left or right makes a difference in the construction. We have shown in 3.3.1 - 3.3.26 how the convolution product is related to the action on a function via the representation law $(\pi(\mathbf{u})f)(\mathbf{v})=f(\mathbf{u}^{-1}\mathbf{v})$. The reader is referred to the work in [69] where a lengthy discussion is given regarding the left and right regular representations on SU(1,1) and quasiunitary matrices as given through an analysis of the shift operators.

As the hyperbolic heat kernel is the solution to the diffusion problem in the hyperbolic disk, in order to construct the kernel we take two distinct elements in the space such that for each we have $\mathbf{g}(\tau,\varphi,0)$, i.e. $\psi=0$ in the third argument, i.e. constrained to a disk, as the other co-ordinates are polar or hyperbolic. The generic element described by $\mathbf{g}^{-1}\mathbf{g}_l$ is then:

$$\mathbf{g}^{-1}\mathbf{g}_l=\mathbf{g}^{-1}(\tau_1,\varphi_1,0)\mathbf{g}_l(\tau_2,\varphi_2,0)=\mathbf{g}(\tau,\varphi,0)\tag{4.363}$$

Following the argument in Vilenkin [69](pp.293 VI.3.5-7, also pp.100 III. 1.2-7, see also Lenz [111] for an analogous calculation) we then take the decomposition:

$$\mathbf{g}(\tau,\varphi,\psi)=\mathbf{g}(-\tau_1,0,0)\mathbf{g}(\tau_2,\varphi_2-\varphi_1,0)=\mathbf{g}^{-1}(\tau_1,\varphi_1,0)\mathbf{g}(\tau_2,\varphi_2,0)\tag{4.364}$$

where we have applied 3.1.15 to decompose the pseudounitary operator. Applying the fundamental form of SU(1,1) matrix, i.e. the most general pseudounitary, we have:

$$\begin{bmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{bmatrix}=\begin{bmatrix}\cosh\frac{\tau}{2}e^{i(\varphi+\psi)/2} & \sinh\frac{\tau}{2}e^{i(\varphi-\psi)/2}\\ \sinh\frac{\tau}{2}e^{-i(\varphi-\psi)/2} & \cosh\frac{\tau}{2}e^{-i(\varphi+\psi)/2}\end{bmatrix}\tag{4.365}$$

where $|\alpha|^2-|\beta|^2=1$. We now write the matrix equation explicitly, we must have:

$$\begin{bmatrix}\cosh\frac{\tau}{2}e^{i(\varphi+\psi)/2} & \sinh\frac{\tau}{2}e^{i(\varphi-\psi)/2}\\ \sinh\frac{\tau}{2}e^{-i(\varphi-\psi)/2} & \cosh\frac{\tau}{2}e^{-i(\varphi+\psi)/2}\end{bmatrix}=\begin{bmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{bmatrix}\tag{4.366}$$

$$=\begin{bmatrix}\cosh\frac{\tau_1}{2}e^{-i\varphi_1/2} & -\sinh\frac{\tau_1}{2}e^{i\varphi_1/2}\\ \sinh\frac{\tau_1}{2}e^{-i\varphi_1/2} & \cosh\frac{\tau_1}{2}e^{i\varphi_1/2}\end{bmatrix}\begin{bmatrix}\cosh\frac{\tau_2}{2}e^{i\varphi_2/2} & \sinh\frac{\tau_2}{2}e^{i\varphi_2/2}\\ \sinh\frac{\tau_2}{2}e^{-i\varphi_2/2} & \cosh\frac{\tau_2}{2}e^{-i\varphi_2/2}\end{bmatrix}\tag{4.367}$$

Applying this to the group multiplication law given by the decomposition, we have:

$$\alpha=e^{-i\rho}\cosh\frac{\tau_1}{2}\cosh\frac{\tau_2}{2}-e^{i\rho}\sinh\frac{\tau_1}{2}\sinh\frac{\tau_2}{2}\tag{4.368}$$

which is obtained by multiplication of the group elements $\hat{g}^{-1}(\tau_1,\varphi_1,0)\hat{g}(\tau_2,\varphi_2,0)$, in this instance the upper left-hand element of the matrix equation. We have defined $\rho=$

$\frac{1}{2}(\varphi_1 - \varphi_2)$ for convenience. From the generic element $\mathbf{g}(\tau, \varphi, \psi)$, we can see that we must have $|\alpha|^2 = \cosh^2 \frac{\tau}{2}$ which gives us $2|\alpha|^2 - 1 = 2\cosh^2 \frac{\tau}{2} - 1 = \cosh \tau$. Substituting the first value of α into this second identity, and using trigonometric identities and double angle formulae, one obtains:

$$\cosh \tau = \cosh \tau_1 \cosh \tau_2 - \sinh \tau_1 \sinh \tau_2 \cos(\varphi_1 - \varphi_2) \tag{4.369}$$

which establishes the form of the distance function on the hyperboloid. Note the similarity to the known result for spherical triangles. To establish the kernel formula, we use 3.3.13, and assume a symmetric, invariant and homogeneous function as in the previous part of the proof, finding:

$$K(\mathbf{g}, \mathbf{g}_l) = K(\cosh \tau) = K(\cosh \tau_1 \cosh \tau_2 - \sinh \tau_1 \sinh \tau_2 \cos(\varphi_1 - \varphi_2)) \tag{4.370}$$

as required. It is simple to see that this kernel is symmetric under interchange of coordinates, as the argument is naturally so. It will naturally reflect the geometry we have defined through the pseudounitary operator we have constructed. We do not assume that we know the structure of this kernel, in that we have not specified the exact dependence on the distance function. This is to follow in the rest of this section. □

We immediately obtain the following corollary by an identical method.

Corollary 4.6.5. *Take the group element as defined by the multiplication law:*

$$\mathbf{g}(\tau, \varphi, \psi) = \mathbf{g}(\tau_1, 0, 0)\mathbf{g}(\tau_2, \varphi_2, 0) \tag{4.371}$$

with generic group element on $SU(1,1)$:

$$\mathbf{g}(\tau, \varphi, \psi) = \begin{bmatrix} \cosh \frac{\tau}{2} e^{i(\varphi+\psi)/2} & \sinh \frac{\tau}{2} e^{i(\varphi-\psi)/2} \\ \sinh \frac{\tau}{2} e^{-i(\varphi-\psi)/2} & \cosh \frac{\tau}{2} e^{-i(\varphi+\psi)/2} \end{bmatrix} \tag{4.372}$$

Then the kernel given by the distance formula on this type of hyperbolic space is:

$$K(\cosh \tau) = K(\cosh \tau_1 \cosh \tau_2 + \cos \varphi_2 \sinh \tau_1 \sinh \tau_2) \tag{4.373}$$

Proof. Proceeding in an identical fashion to 4.6.4, we have the group multiplication from the formula:

$$\mathbf{g}(\tau, \varphi, \psi) = \mathbf{g}(\tau_1, 0, 0)\mathbf{g}(\tau_2, \varphi_2, 0) \tag{4.374}$$

with:

$$\alpha = e^{i\varphi_2} \cosh \frac{\tau_1}{2} \cosh \frac{\tau_2}{2} + e^{-i\varphi_2} \sinh \frac{\tau_1}{2} \sinh \frac{\tau_2}{2} \tag{4.375}$$

Note the difference in sign here as opposed to the previous lemma. Calculating the hyperbolic cosine, which gives the distance on the hyperboloid, we may write:

$$\cosh \tau = 2|\alpha|^2 - 1 \tag{4.376}$$

$$= \cosh \tau_1 \cosh \tau_2 + \cos \varphi_2 \sinh \tau_1 \sinh \tau_2 \tag{4.377}$$

from which we may conclude as in [64,65,111]:

$$K(\cosh \tau) = K(\cosh \tau_1 \cosh \tau_2 + \cos \varphi_2 \sinh \tau_1 \sinh \tau_2) \tag{4.378}$$

This function $K(.)$ will be hyperbolic, lie in a disk, and be described by the pseudounitary which describes the distance function. We term this a hyperbolic heat kernel, as we shall show this is a static form of a more general solution to the hyperbolic heat equation. Again, we have not specified the exact dependence on the hyperbolic distance function. This shall follow using some results from special function theory that give a zonal decomposition of this function in terms of the hyperbolic distance. □

Remark 57. These two forms of the hyperbolic heat kernel are obviously equivalent through a change in the angular variable multiplying the hyperbolic sine term.

4.6.2 Composition Formula for Mehler-Fock Functions

The following calculation shall outline a brief schemata which allows derivation of the composition formulae for the Mehler-Fock functions via the distance function. We shall exploit this when combined with the expressions derived for the hyperbolic heat kernel as expressed through 4.6.4, 4.6.5.

Lemma 4.6.6. *The composition law for the distance formula as given by either 3.3.13, 4.6.5 can be found using the method of spherical functions. The zonal decomposition of the group gives the following composition law for the Mehler-Fock functions:*

$$\frac{1}{2\pi}\int_0^{2\pi} \mathcal{P}_{i\tau-1/2}(d(r,s))du = \mathcal{P}_{i\tau-1/2}(\cosh r)\mathcal{P}_{i\tau-1/2}(\cosh s) \tag{4.379}$$

where $\mathcal{P}_{i\tau-1/2}(\cosh r)$ *is the Legendre function of Mehler-Fock type, see [121]. The hyperbolic distance function is given by the formula:*

$$d(r,s) = \cosh r\cosh s + \cos u\sinh r\sinh s \tag{4.380}$$

and is a function of u*, which we omit for brevity. We have automatically by definition* $d(r,0) = \cosh r$ *and likewise for* s*.*

Proof. We shall extend some results for the associated Legendre function and polynomials that may be found in [114]. Consider the three dimensional pseudo-Euclidean space of rotations $G = SO(2,1), K = SO(2)$ the subgroup of two dimensional rotations which leave the vector $[0,0,1]^T$ fixed. The rotation vector under the Cartan decomposition is given by:

$$\mathbf{gkh}\cdot O = \begin{bmatrix} \cosh r & 0 & i\sinh r \\ 0 & 1 & 0 \\ -i\sinh r & 0 & \cosh r \end{bmatrix} \begin{bmatrix} \cos u & -\sin u & 0 \\ \sin u & \cos u & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.381}$$

$$\times \begin{bmatrix} \cosh s & 0 & i\sinh s \\ 0 & 1 & 0 \\ -i\sinh s & 0 & \cosh s \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.382}$$

$$= \begin{bmatrix} i(\cosh r\sinh s\cos u + \sinh r\cosh s) \\ i\sin u\sinh s \\ \cos u\sinh r\sinh s + \cosh r\cosh s \end{bmatrix} \tag{4.383}$$

$$\mathbf{gkh}\cdot O|_3 \doteq gkh = \cos u\sinh r\sinh s + \cosh r\cosh s \tag{4.384}$$

$$\mathbf{g}\cdot O|_3 \doteq g = \cosh r \tag{4.385}$$

and

$$\mathbf{h}\cdot O|_3 \doteq h = \cosh s \tag{4.386}$$

Evaluating the spherical zonal decomposition via the formula 3.3.45, 3.3.29, (see also [70,71,113,114]), we find:

$$\int_K f(\mathbf{gkh})dk = f(\mathbf{g})f(\mathbf{h}) \tag{4.387}$$

which yields:

$$\frac{1}{2\pi}\int_0^{2\pi} f(\cosh r\cosh s + \cos u\sinh r\sinh s)du = f(\cosh r)f(\cosh s) \tag{4.388}$$

The solution to this expression is given by the toroidal harmonics, i.e.:

$$f(\cosh r) = \mathcal{P}_{i\tau-1/2}(\cosh r) \tag{4.389}$$

This is the zonal decomposition we shall need in order to establish the form of the hyperbolic heat kernel. As we can see, the argument inside the integral is the hyperbolic distance, and it decouples into the relevant co-ordinates as is proper for a zonal spherical function. □

Remark 58. This proof is simply the analytic continuation of the arguments as given in [114] to derive the Legendre functions. A similar type of calculation may be given to derive Bessel related representations of the hyperbolic plane. The spherical zonal decomposition is a powerful method of analysis that enables many complex properties of eigenfunction products to be evaluated using the group laws.

4.7 Derivation of the Hyperbolic Heat Kernel

The following calculations concern the solution of the heat kernel we can associate to the hyperbolic plane, via the formula we have derived using the Laplacian and other methods 3.2.34, 4.5.3, 4.5.1. In the first, we use a formula for the Mehler-Fock transform to write down the eigenfunction decomposition, and invert the transform to obtain the kernel as a fundamental solution. For the second method, we note that a similar calculation appears in Chavel [11]; however, many of the details involved are missing from this reference. For completeness, we shall show how the convolution method complements the results established through use of the Mehler-Fock transform, and fill out the finer points of the calculation that may be found in [11].

Our aim in doing so is to develop solutions for the hyperbolic heat equation. Using some results from Davies [1], we shall then map the solutions for the expression given by:

$$u_t = \nabla^2 u = u_{rr} + u_r \coth r \tag{4.390}$$

where ∇^2 is the hyperbolic Laplacian, into solutions of the related system:

$$u_t = y^2(u_{xx} + u_{yy}) \tag{4.391}$$

We briefly outline how this is achieved. For the equation $u_t = y^2(u_{xx} + u_{yy})$, if we assume the separable solution, we may write $u = u(x, y, t) = h(x)g(y)\xi(t)$. The eigenvalue equations separate to give:

$$\psi_{p,k}(x, y, t) = Ce^{ikx}\sqrt{y}e^{-(p^2+1/4)t}K_{ip}(ky) \tag{4.392}$$

and using spectral decomposition, we can write the general solution as:

$$u(x, y, x_0, y_0, t) = C\sqrt{yy_0}\int_0^\infty e^{ik(x-x_0)}dk \int_0^\infty \exp\left(-[p^2 + 1/4]t\right) K_{ip}(ky)K_{ip}(ky_0)d\mathfrak{m}(p) \tag{4.393}$$

where $d\mathfrak{m}(p)$ is some integration measure, and p is the continuous eigenvalue, and we integrate over the eigenvalues to obtain the most general form. The inner part of this integral is given by the Yakubovich heat kernel:

$$\int_0^\infty e^{-p^2 t}K_{ip}(ky)K_{ip}(ky_0)d\mathfrak{m}(p) \tag{4.394}$$

The connection between the associated Legendre functions, which have a similar spectral integral, defining the Mehler-Fock kernel:

$$\int_0^\infty e^{-p^2 t}\mathcal{P}^{\mu}_{ip-1/2}(z)\mathcal{P}^{\mu}_{ip-1/2}(z')d\mathfrak{m}(p) \tag{4.395}$$

is that these systems of eigenfunctions are isospectral. More directly, using the Cayley transform and the various models of hyperbolic geometry, we showed that the metrics:

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \leftrightarrow \frac{4dz.dz^*}{(1-|z|^2)^2} \leftrightarrow \sinh^2 \tau d\phi^2 + d\tau^2 \tag{4.396}$$

are equivalent. Using the curvilinear Laplacian, it is straightforward to see that these metrics induce the following Laplace operators:

$$\nabla^2 f = y^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \tag{4.397}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial \tau^2} + \coth \tau \frac{\partial f}{\partial \tau} + \frac{1}{\sinh^2 \tau} \frac{\partial^2 f}{\partial \phi^2} \tag{4.398}$$

and

$$\nabla^2 f = \frac{(1-r^2)^2}{2} \left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} \right) \tag{4.399}$$

All of these problems are equivalent, and can be mapped into one another using various forms of the Cayley transform. A detailed calculation of this using path integrals may be found in [14]. For our purposes, we shall be using the second form of the Laplacian operator, and taking the radial part, we obtain:

$$\nabla^2 f(\tau) = \frac{\partial^2 f(\tau)}{\partial \tau^2} + \coth \tau \frac{\partial f(\tau)}{\partial \tau} \tag{4.400}$$

Alternatively, one may examine the separable solution in an analogous way, to find the differential equation:

$$\frac{\partial^2 f}{\partial \tau^2} + \coth \tau \frac{\partial f}{\partial \tau} + k^2 f + \frac{E f(\tau)}{\sinh^2 \tau} = 0 \tag{4.401}$$

which has a solution given in terms of an associated Legendre function. We shall use this representation of the solution for the hyperbolic heat kernel in the following sections; note that by taking $E = 0$, we recover the reduced hyperbolic Laplacian:

$$\frac{\partial^2 f}{\partial \tau^2} + \coth \tau \frac{\partial f}{\partial \tau} + k^2 f = 0 \tag{4.402}$$

which implies $\nabla^2 f = f_{\tau\tau} + \coth \tau f_\tau = \dfrac{\partial f}{\partial t}$ is the simplest example of a hyperbolic heat equation. We shall use this to develop a theory of the kernel solution for this instance, then derive a more sophisticated and generalised formula where $E \neq 0$ in the following chapters, where we shall employ the projection slice theorem. In terms of the solution, we can expect the simpler case to have a spectral decomposition over the Legendre functions with second index equal to zero, i.e. the normal Legendre functions, and the generalised formula to be given by a spectral decomposition over the associated Legendre functions.

4.7.0.1 Hyperbolic Heat Kernel via Mehler-Fock Transform

We shall now derive a simple representation of the hyperbolic heat kernel. This method uses the convolution representation of the kernel, as originally discussed in the work of Chavel [11]. In particular, we shall employ the results for the eigenfunction on the hyperbolic space via 4.5.4, alternatively via the spherical zonal decomposition law 4.6.6 to identify and solve the problem that is given by the heat kernel on the hyperbolic space. Many of the details important in the calculation contained in [11] are missing, so we shall perform a complete analysis of the problem in order to solve the McKean heat kernel. In the following, we shall be concerned with analysis of the differential equation given by the system:

Definition 4.7.1. The heat equation $u_t = \nabla^2 u$ on the hyperbolic plane is defined by the PDE system:

$$u_t = u_{rr} + u_r \mathrm{coth} r \tag{4.403}$$

This is the radial part of the system 4.5.2, which we have derived using analysis of the PDE system from the Laplace operator in a curved space 3.2.34, implied by the Fubini-Study metric 3.2.39.

We shall now show how this equation is solved in terms of the Mehler-Fock function. This is achieved through the following theorem:

Theorem 4.7.2. *The solution $u(r,t)$ to the PDE 4.7.1 is:*

$$u(r,t) = \int_0^\infty f(\xi) K(r,\xi;t) d(\cosh \xi) \tag{4.404}$$

where $u(r,0) = f(r)$, and the kernel $K(r,\xi;t)$ is given by the Mehler-Fock transform:

$$K(r,\xi;t) = \int_0^\infty d\rho e^{-(\rho^2+1/4)t} \rho \tanh(\pi\rho) \mathcal{P}_{i\rho-1/2}(\cosh \xi) \mathcal{P}_{i\rho-1/2}(\cosh r) \tag{4.405}$$

and $t > 0$, $\mathcal{P}_{i\rho-1/2}(.)$ is the Mehler-Fock function, i.e. associated Legendre function of toroidal type, defined in 3.4.1.

Proof. If we take the separable solution, we may write:

$$u(r,t) = w(r) e^{-(\lambda^2+1/4)t} \tag{4.406}$$

and the ODE is given by the eigenvalue problem:

$$w'' + w' \mathrm{coth} r + (\lambda^2 + 1/4) w = 0 \tag{4.407}$$

and we have:

$$w(r) = \mathcal{P}_{i\lambda-1/2}(\cosh r) \tag{4.408}$$

Thus the eigenfunction:

$$u_\lambda(r,t) = \mathcal{P}_{i\lambda-1/2}(\cosh r) e^{-(\lambda^2+1/4)t} \tag{4.409}$$

defines a family of solutions. Taking an integral over the continuous eigenvalue, we have:

$$u(r,t) = \int_0^\infty \varphi(\lambda) \mathcal{P}_{i\lambda-1/2}(\cosh r) e^{-(\lambda^2+1/4)t} d\lambda \tag{4.410}$$

and then:

$$u(r,0) = f(r) = \int_0^\infty \varphi(\lambda) \mathcal{P}_{i\lambda-1/2}(\cosh r) d\lambda \tag{4.411}$$

Thus we need to obtain the function $\varphi(\lambda)$. This is essentially a Mehler-Fock transform. We have the forward and inverse transforms 5.3.2:

$$F(x) = \int_0^\infty f(\rho) \mathcal{P}_{i\rho-1/2}(x) d\rho \tag{4.412}$$

$$f(\rho) = \rho \tanh(\pi\rho) \int_1^\infty F(x) \mathcal{P}_{i\rho-1/2}(x) dx \tag{4.413}$$

Consider the problem for $u(r,0)$. We put $z = \cosh r$. In this case, we must have:

$$f(\cosh^{-1} z) = F(z) = \int_0^\infty \varphi(\lambda) \mathcal{P}_{i\lambda-1/2}(z) d\lambda \tag{4.414}$$

and hence:

$$\varphi(\rho) = \rho \tanh(\pi\rho) \int_1^\infty F(z)\mathcal{P}_{i\rho-1/2}(z)\, dz \tag{4.415}$$

Reparametrising the integral, we have $\cosh^{-1} z = \xi \Leftrightarrow z = \cosh\xi$, $dz = \sinh\xi d\xi$, the interval changes over to $z = 1, \cosh^{-1}(1) = 0$, and therefore the function may be written:

$$\varphi(\rho) = \rho \tanh(\pi\rho) \int_0^\infty f(\xi)\mathcal{P}_{i\rho-1/2}(\cosh\xi) \sinh\xi d\xi \tag{4.416}$$

Hence the solution of the initial value problem is given by:

$$u(r,t) = \int_0^\infty \varphi(\rho)\mathcal{P}_{i\rho-1/2}(\cosh r)\, e^{-(\rho^2+1/4)t} d\rho \tag{4.417}$$

$$= \int_0^\infty d\rho.\rho \tanh(\pi\rho) \int_0^\infty d\xi.\sinh\xi.f(\xi)\mathcal{P}_{i\rho-1/2}(\cosh\xi)\,\mathcal{P}_{i\rho-1/2}(\cosh r)\, e^{-(\rho^2+1/4)t} \tag{4.418}$$

$$= \int_0^\infty d(\cosh\xi) f(\xi) \int_0^\infty d\rho e^{-(\rho^2+1/4)t} \rho \tanh(\pi\rho)\mathcal{P}_{i\rho-1/2}(\cosh\xi)\,\mathcal{P}_{i\rho-1/2}(\cosh r) \tag{4.419}$$

$$= \int_0^\infty f(\xi)K(r,\xi;t)d(\cosh\xi) \tag{4.420}$$

as required, where we have invoked Fubini-Tonelli to interchange the order of the integrals. The explicit form of the kernel is as in the theorem:

$$K(r,\xi;t) = \int_0^\infty d\rho e^{-(\rho^2+1/4)t} \rho \tanh(\pi\rho)\mathcal{P}_{i\rho-1/2}(\cosh\xi)\,\mathcal{P}_{i\rho-1/2}(\cosh r) \tag{4.421}$$

□

Remark 59. This is the Mehler-Fock kernel, which is known from stochastic processes and analysis of diffusion kernels, see [56].

4.7.0.2 Hyperbolic Heat Kernel via Convolution

We shall show a second way to find the hyperbolic heat kernel, using a convolution theorem. This calculation mirrors some results that can be found in Chavel [11]; however, our development of the geometric framework proceeds from the hyperbolic brachistochrone equation and the differential geometry. We have developed a workable theoretical framework in which we can understand this, by analysis of the differential geometry through the distance function and metric, and on a deeper level, through use of various decompositions of pseudounitary matrices we have derived. In this section, we shall apply the results that we have so far gleaned and apply them to analyse the differential equations that define the hyperbolic heat kernel. From this section forward, we depart from the differential geometry perspective, and focus solely on integral solutions to various types of PDEs that relate to the hyperbolic heat equation.

We take as our precepts the results established in 4.5.1, 4.5.4, where we developed the formulae for the Laplacian and the metric in the hyperbolic space, and the results for the different hyperbolic geometries we have analysed using the pseudounitary matrices we generated when applying the hyperbolic brachistochrone principle. This section aims to prove some small lemmas related to the invariant transformations of the Laplacian which we will apply to the hyperbolic system, which rely on the definition of an intertwining operator as outlined below. We shall use these results in the following section to evaluate an integral transform which will give us a tidy form of the solution to the hyperbolic heat equation.

Definition 4.7.3. For the following, assume that a transformation and its inverse exist, which we write as the operation $\mathcal{F}, \mathcal{F}^{-1}$, acting on some space of functions. For the Laplacian operator ∇^2, an invariant or intertwining operator $\mathcal{F}$ satisfies:

$$\nabla^2 = \mathcal{F}^{-1} \circ \nabla^2 \circ \mathcal{F} \tag{4.422}$$

or in functional terms:

$$\nabla^2 u = \mathcal{F}^{-1}[\nabla^2[\mathcal{F}u]] \tag{4.423}$$

where u is a suitable function on the space where ∇^2 operates.

Remark 60. This acts in a similar way to the unitary matrices 3.1.9 where we have $\hat{X} = \hat{U}^\dagger \hat{X} \hat{U}$, which defines the set of invariant transformations of $\hat{X}$, given by the unitary $\hat{U}$.

This implies the intertwining operator is commutative with the Laplacian, which we shall employ for the hyperbolic Laplacian.

Lemma 4.7.4. *The intertwining operator $\mathcal{F}$ satisfies the relationship:*

$$\mathcal{F}[\nabla^2 u] = \nabla^2[\mathcal{F}u] \tag{4.424}$$

where the Laplacian operator is given by ∇^2.

Proof. This is elementary using 4.7.3. We simply apply the transform $\mathcal{F}$ to both sides of the definition. □

Remark 61. This means that an invariant or intertwining operator is necessarily commutative with the Laplacian, and is unique up to phase. This property plays an important role in establishing the basis in group theory for Fourier analysis. In our context, we shall be examining the transforms which are invariant to the action of the hyperbolic Laplacian, which in this representation are related to the Mehler-Fock transform.

We shall illustrate the general technique that we will use, which dates to the original investigations of Fourier. In this case, we will examine how the fundamental solution comes about as a result of the kernel, using some simple spectral theory. The basic concept is one of decomposition into eigenvalues, in order to build up a solution. One way to do this is to look at the Fourier transform of the heat equation. The following example demonstrates how one does this for the basic heat equation. We shall then employ a very similar relationship to solve the hyperbolic heat equation.

Example 4.7.5. Consider the basic one dimensional heat equation, defined by:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{4.425}$$

which is solved by the integral kernel:

$$u(x,t) = \int K(x,y;t)u(y,0)dy \tag{4.426}$$

Given boundary conditions $f(x) = u(x,0)$, we have:

$$f(x) = \int K(x,y;0)u(y,0)dy \tag{4.427}$$

which implies that we must have the boundary condition $K(x,y;0) = \delta(y-x)$. Let us now write an eigenvalue decomposition for the kernel. If we take an eigenfunction that satisfies the ODE:

$$\frac{\partial^2 \phi_k}{\partial x^2} = -k^2 \phi_k \tag{4.428}$$

then we will satisfy the heat equation:

$$\frac{\partial^2 K}{\partial x^2} = \sum_{k=0}^{\infty} e^{-k^2 t}\phi_k^*(y)\frac{\partial^2 \phi_k(x)}{\partial x^2} = -\sum_{k=0}^{\infty}(k^2)e^{-k^2 t}\phi_k^*(y)\phi_k(x) = \frac{\partial K}{\partial t} \tag{4.429}$$

and indeed, we can write the kernel in spectral form as:

$$K(x,y;t) = \sum_{k=0}^{\infty} e^{-k^2 t}\phi_k^*(y)\phi_k(x) = \sum_{k=0}^{\infty} e^{-k^2 t}e^{iky}e^{-ikx} \tag{4.430}$$

which obviously satisfies the PDE. Further, we can write the kernel as a sum of sub-kernels, each of which are independent, as:

$$K(x,y;t) = \sum_{k=0}^{\infty} e^{-k^2 t}\phi_k^*(y)\phi_k(x) = \sum_{k=0}^{\infty} K_k(x,y;t) \tag{4.431}$$

We shall now show how this allows the solution of the heat equation. If we define the integral transform which takes the space parameter into the eigenvalue of the kernel, we might write:

$$\mathcal{F}f = \int_{-\infty}^{+\infty} K_k(x,0;0)f(x)dx = \bar{f}(k) \tag{4.432}$$

which in our case reduces to the Fourier transform.

$$\mathcal{F}f = \int_{-\infty}^{+\infty} e^{-ikx}f(x)dx = \bar{f}(k) \tag{4.433}$$

In this situation, we are guaranteed that:

$$\nabla^2 K_k(x,0;0) = 0, \frac{\partial}{\partial t}K_k(x,0;0) = 0 \tag{4.434}$$

and also that we can transform the Laplacian:

$$\mathcal{F}(\nabla^2 f) = \int_{-\infty}^{+\infty} K_k(x,0;0)\nabla^2 f(x)dx \tag{4.435}$$

$$\mathcal{F}(\nabla^2 f) = \int_{-\infty}^{+\infty} K_k(x,0;0)\nabla^2 f(x)dx \tag{4.436}$$

$$= -k^2\int_{-\infty}^{+\infty} K_k(x,0;0)f(x)dx \tag{4.437}$$

$$= -k^2\bar{f}(k) \tag{4.438}$$

This leaves us with the simple differential equation in time:

$$\frac{\partial \bar{f}(k)}{\partial t} = -k^2\bar{f}(k) \tag{4.439}$$

More formally, if we evaluate the Fourier transform of the second order derivative, we have:

$$\int_{-\infty}^{+\infty} e^{-ikx}\frac{\partial^2 f}{\partial x^2}dx = \int_{-\infty}^{+\infty} e^{-ikx}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)dx \tag{4.440}$$

$$= \frac{\partial f}{\partial x}e^{-ikx}\,|_{-\infty}^{+\infty} + ik\int_{-\infty}^{+\infty} e^{-ikx}\frac{\partial f}{\partial x}dx \tag{4.441}$$

and hence the Laplacian on this space goes over into:

$$\int_{-\infty}^{+\infty} e^{-ikx}\frac{\partial^2 f}{\partial x^2}dx = -k^2\int_{-\infty}^{+\infty} e^{-ikx}f(x)dx = -k^2\bar{f}(k) \tag{4.442}$$

Once more, we have the expression:

$$\mathcal{F}(\nabla^2 f) = \mathcal{F}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial t}\mathcal{F}f \tag{4.443}$$

$$\frac{\partial \bar{f}}{\partial t} = -k^2\bar{f}(k) \tag{4.444}$$

which is solved by $\bar{f}(k,t) = e^{-k^2 t}$. Inverting the transform, we write the solution to the heat equation as:

$$f(x,t) = \mathcal{F}^{-1}\left[e^{-k^2 t}\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-k^2 t}e^{ikx}dk \tag{4.445}$$

$$= \frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{x^2}{4t}\right) \tag{4.446}$$

This process demonstrates how useful it is to have an eigenfunction decomposition of the kernel, in order to find the solution to the heat equation. In the following segments, we shall use this in the context of the hyperbolic heat equation, where the role of the Fourier transform is played by the Mehler-Fock transform.

We also establish the following, in line with our previous calculations involving the symmetric kernel 3.3.14.

Lemma 4.7.6. *The solution to the heat equation $u_t = \nabla^2 u$ is defined by the integral kernel $K(\mathbf{x},\mathbf{y};t)$:*

$$u(\mathbf{x},t) = \int K(\mathbf{x},\mathbf{y};t)u(\mathbf{y},0)d\mathfrak{m}(\mathbf{y}) \tag{4.447}$$

where the initial condition is $u(x,0) = f(x)$, specified through the kernel as:

$$K(x,y;0) = \delta(y-x) \tag{4.448}$$

or $f(x) = \int K(x,y;0)f(y)dy$. The eigenvalue decomposition of the kernel defines the integral transform:

$$\mathcal{F}f = \int_{-\infty}^{+\infty} K_k(x,0;0)f(x)dx = \bar{f}(k) \tag{4.449}$$

The existence of this integral transform is sufficient to find the solution to the heat equation, as long as we can specify the form of the Laplace operator.

Proof. This proceeds in an identical fashion to the previous example, with the simple modification that under the curvilinear Laplacian, there exists an eigenfunction:

$$\nabla^2\phi_k = -\lambda_k\phi_k \tag{4.450}$$

where $-\lambda_k$, is some eigenvalue. In the case that the eigenvalues are discrete, the spectral decomposition will be a sum, as in the previous example. In the case that there exists only one continuous eigenvalue, the kernel will be given by an integral over the index, which we identify with a limit of the sum as the eigenvalues become infinitely dense. For an imiscid spectrum, i.e. one containing separated discrete and continuous components, we will have the standard decoupling of eigenvalues in the spectrum, as they do not mix. We shall show how we will apply the lemma to find a solution to the heat equation. Assume a discrete spectrum, the continuous and imiscid cases follow using identical logic. The eigenvalue decomposition of the kernel is:

$$K(x,y;t) = \sum_{k=0}^{\infty} e^{-\lambda_k t}\phi_k^*(y)\phi_k(x) = \sum_{k=0}^{\infty} K_k(x,y;t) \tag{4.451}$$

Then we will have a transform defined by:

$$K_k(x,0;0) = \phi_k(x) \tag{4.452}$$

$$\bar{f}(k,t) = \int \phi_k(x) f(x,t)dx \tag{4.453}$$

which satisfies $\mathcal{F}(\nabla^2 f) = \mathcal{F}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial t}\mathcal{F}f$. Writing the time derivative explicitly, we can write the integral:

$$\frac{\partial \bar{f}}{\partial t} = \int \phi_k(x)\frac{\partial f(x,t)}{\partial t}dx = \mathcal{F}\frac{\partial f(x,t)}{\partial t} \tag{4.454}$$

Assume further that we can write the function as an eigenfunction solely of k. Then we have $\nabla^2 f = -\lambda_k f$, and further:

$$\bar{f}(k,t) = e^{-\lambda_k t} \tag{4.455}$$

We must therefore have the following solution:

$$f(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\lambda_k t}\phi_k(x)dk \tag{4.456}$$

which can be seen by a simple calculation:

$$\frac{\partial f}{\partial t} = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \lambda_k e^{-\lambda_k t}\phi_k(x)dk \tag{4.457}$$

$$\nabla^2 f = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\lambda_k t}\nabla^2\phi_k(x)dk = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \lambda_k e^{-\lambda_k t}\phi_k(x)dk \tag{4.458}$$

$$= \frac{\partial f}{\partial t} \tag{4.459}$$

Note that depending on the domain of the problem, and the nature of the spectrum, the boundaries of the integrals may need to be altered in practice. For example, with the hyperbolic plane, care needs to be taken to ensure the limits are correct depending on whether one is using the strip, hyperboloid or pseudosphere model. □

Remark 62. We can use this type of result to guess at a form of the answer for the heat kernel, given what we already know of the eigenfunctions of the hyperbolic Laplacian. This simple proof demonstrates some results that we shall analyse in more detail in the remainder of the chapter. In hyperbolic geometry, all points that are along a horocycle have equivalent distance. In our context, the hyperbolic distance function has a factor of $\cos\varphi$:

$$x = \cosh\tau = \cosh\tau_1\cosh\tau_2 - \sinh\tau_1\sinh\tau_2\cos\varphi \tag{4.460}$$

If we take all the points up to this equivalence, the solution must involve an integral over the variable φ. Writing this more concisely:

$$f(x,t) = \frac{C}{2\pi}\int_0^{2\pi}\int_0^{+\infty} e^{-(\rho^2+1/4)t}\mathcal{P}_{i\rho-1/2}\left(\cosh\tau\right)d\mathfrak{m}(\rho)d\varphi \tag{4.461}$$

where, as we have shown, the Legendre functions are the appropriate representation for the hyperbolic plane as given through the spherical zonal decomposition of the hyperbolic distance function. For these functions the integration measure satisfies:

$$d\mathfrak{m}(\rho) = \rho \tanh \pi\rho \tag{4.462}$$

and the functions themselves satisfy the eigenfunction equation for the hyperbolic Laplacian:

$$\nabla^2 \mathcal{P}_{i\rho-1/2}\left(\cosh\tau\right) = -(\rho^2 + 1/4)\mathcal{P}_{i\rho-1/2}\left(\cosh\tau\right) \tag{4.463}$$

where the hyperbolic Laplacian is given by:

$$\nabla^2 = \frac{\partial^2}{\partial\tau^2} + \coth\tau\frac{\partial}{\partial\tau} \tag{4.464}$$

Using the spherical decomposition formula for the associated Legendre functions, we must have the equation:

$$\frac{1}{2\pi}\int_0^{2\pi} \mathcal{P}_{i\rho-1/2}\left(\cosh\tau\right) d\varphi = \mathcal{P}_{i\rho-1/2}\left(\cosh\tau_1\right) \mathcal{P}_{i\rho-1/2}\left(\cosh\tau_2\right) \tag{4.465}$$

Inserting this into the expression for the solution, and invoking Fubini-Tonelli to interchange the order of the integrals, we find:

$$f(x,t) = \frac{C}{2\pi}\int_0^{+\infty} e^{-(\rho^2+1/4)t}\int_0^{2\pi} \mathcal{P}_{i\rho-1/2}\left(\cosh\tau\right) d\varphi d\mathfrak{m}(\rho) \tag{4.466}$$

$$f(x,t) = C\int_0^{+\infty} e^{-(\rho^2+1/4)t}\mathcal{P}_{i\rho-1/2}\left(\cosh\tau_1\right) \mathcal{P}_{i\rho-1/2}\left(\cosh\tau_2\right) \rho\tanh\pi\rho d\rho \tag{4.467}$$

where C is a normalisation constant. This implies that the kernel solution for this system is given by the integral above. As the following sections of this work shall conclude, using more sophisticated techniques, this is indeed the case.

We then obtain the following corollary, as a result of the existence and uniqueness (up to phase) of both the invariant intertwining transform, and the transform from the symmetric kernel.

Corollary 4.7.7. *Assume the heat kernel $K(\mathbf{x},\mathbf{y};t)$ is given by the solution to the PDE:*

$$\frac{\partial K}{\partial t} = \nabla^2 K \tag{4.468}$$

with initial condition $K(\mathbf{x},\mathbf{y};0) = \delta(\mathbf{y}-\mathbf{x})$, ∇^2 the Laplacian operator for the system. This can be written as the operator theoretic solution:

$$K(\mathbf{x},\mathbf{y};t) = e^{t\nabla^2} \tag{4.469}$$

Then the transformation $\mathcal{F}$ defined in 4.7.6 intertwines with the Laplacian operator ∇^2, i.e. is invariant and commutative.

Proof. We establish this using the solution for the heat equation expressed in the form $K(\mathbf{x},\mathbf{y};t) = e^{t\nabla^2}$, which solves $\dfrac{\partial K}{\partial t} = \nabla^2 K$ by inspection. This solves the heat equation for general $u(\mathbf{x},t)$. Simply:

$$\nabla^2 u(\mathbf{x},t) = \nabla^2\int e^{t\nabla^2}u(\mathbf{y},0)d\mathfrak{m}(\mathbf{y}) = \int \nabla^2 e^{t\nabla^2}u(\mathbf{y},0)d\mathfrak{m}(\mathbf{y}) \tag{4.470}$$

$$= \int \frac{\partial}{\partial t}(e^{t\nabla^2})u(\mathbf{y},0)d\mathfrak{m}(\mathbf{y}) \tag{4.471}$$

$$= \frac{\partial}{\partial t}\int e^{t\nabla^2}u(\mathbf{y},0)d\mathfrak{m}(\mathbf{y}) \tag{4.472}$$

$$= \frac{\partial u(\mathbf{x},t)}{\partial t} \tag{4.473}$$

It is easy to see that the operators $e^{t\nabla^2}, \nabla^2$ commute by expanding in a Taylor series. Any operator satisfies

$$\lim_{t\to 0^+}\left[e^{t\hat{A}}, \hat{A}\right] = 0 \tag{4.474}$$

and the heat kernel $K(\mathbf{x},\mathbf{y};t) = e^{t\nabla^2}$ therefore commutes with ∇^2. Applying this to the integral of the initial condition, as specified by the problem $u(\mathbf{x},t) = \int K(\mathbf{x},\mathbf{y};t)u(\mathbf{y},0)d\mathfrak{m}(\mathbf{y})$ as t approaches zero, we obtain the result. Explicitly, we have that:

$$\left[e^{t\nabla^2}, \nabla^2\right] f = 0 \tag{4.475}$$

for the commutative relationship, then:

$$e^{t\nabla^2}(\nabla^2 f) - \nabla^2(e^{t\nabla^2} f) = 0 \tag{4.476}$$

and we may write:

$$e^{t\nabla^2}(\nabla^2 f) = \nabla^2(e^{t\nabla^2} f) \tag{4.477}$$

Wherever the operator function $e^{t\nabla^2}$ exists, it will be inverted by the function $e^{-t\nabla^2}$, and we obtain:

$$\nabla^2 f = e^{-t\nabla^2}\nabla^2(e^{t\nabla^2} f) \tag{4.478}$$

which is of the form $\nabla^2 f = \mathcal{F}^{-1}\nabla^2(\mathcal{F}f)$, i.e. an intertwining transform. $\square$

We shall show that two PDEs are equivalent via a co-ordinate transformation. This will allow us to completely identify the structure of the kernel using another method.

Lemma 4.7.8. *The differential operators defined by:*

$$\hat{\mathcal{L}}_1(x,\partial_x)u(x) = (x^2-1)\frac{\partial^2 u}{\partial x^2} + 2x\frac{\partial u}{\partial x} \tag{4.479}$$

and

$$\hat{\mathcal{L}}_2(r,\partial_r)u(\cosh r) = \frac{\partial^2 u}{\partial r^2} + \coth r\frac{\partial u}{\partial r} \tag{4.480}$$

are equivalent under a change of variables, i.e. $\hat{\mathcal{L}}_2(r,\partial_r)u(\cosh r) = \hat{\mathcal{L}}_1(x,\partial_x)u(x)$*, where* $x = \cosh r$*. The operators* $\hat{\mathcal{L}}_1(x,\partial_x), \hat{\mathcal{L}}_2(r,\partial_r)$ *define the solution to the hyperbolic heat equation:*

$$\frac{\partial u}{\partial t} = \hat{\mathcal{L}}_1(x,\partial_x)u(x) = \hat{\mathcal{L}}_2(r,\partial_r)u(\cosh r) \tag{4.481}$$

The eigenvalue equations may be written:

$$\hat{\mathcal{L}}_1(x,\partial_x)\phi_\rho(x) = -\left(\rho^2 + \frac{1}{4}\right)\phi_\rho(x) \tag{4.482}$$

$$\hat{\mathcal{L}}_2(r,\partial_r)\phi_\rho(\cosh r) = -\left(\rho^2 + \frac{1}{4}\right)\phi_\rho(\cosh r) \tag{4.483}$$

The solutions to these differential equations are the Mehler-Fock functions. Under the action of the Mehler-Fock transform, we have:

$$\mathcal{F}\nabla^2 u(\cosh r) = -(\rho^2 + 1/4)[\mathcal{F}u](\cosh r) \tag{4.484}$$

where ∇^2 *is the hyperbolic Laplacian, given in terms of either the operators* $\hat{\mathcal{L}}_1(x,\partial_x), \hat{\mathcal{L}}_2(r,\partial_r)$, $u(\cosh r)$ *is some eigenfunction, and the transformation satisfies the identity:*

$$[\mathcal{F}\nabla^2\mathcal{F}^{-1}(u)](\cosh r) = \nabla^2 u(\cosh r) = -(\rho^2+1/4)u(\cosh r) \tag{4.485}$$

Remark 63. Note that there is only a single continuous eigenvalue for this system.

Proof. To show equivalence of these two different operators, this amounts to a change of variables. We employ the change of parameter $x = \cosh r$, finding:

$$\frac{\partial u}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial u}{\partial r} = \frac{1}{\sinh r}\frac{\partial u}{\partial r} \tag{4.486}$$

and hence:

$$\hat{\mathcal{L}}_1(x,\partial_x)u(x) = \sinh r\frac{\partial}{\partial r}\left(\frac{1}{\sinh r}\frac{\partial u}{\partial r}\right) + 2\coth r\frac{\partial u}{\partial r} \tag{4.487}$$

$$= \frac{\partial^2 u}{\partial r^2} + \coth r\frac{\partial u}{\partial r} \tag{4.488}$$

i.e. $\hat{\mathcal{L}}_2(r,\partial_r)u(\cosh r) = \hat{\mathcal{L}}_1(x,\partial_x)u(x)$ as required. We have already shown in 4.7.3 that the Mehler-Fock transform is the intertwining operator for the hyperbolic heat kernel. We shall demonstrate directly that this is true using a formula from Craddock [46] (see pp. 40, 6.101-3). We have the result, obtained from integration by parts and boundary conditions $u'(1) = 0, \mathcal{P}_{i\rho-1/2}(1) = 0$:

$$\int_1^\infty \mathcal{P}_{i\rho-1/2}(x)[\hat{\mathcal{L}}_1(x,\partial_x)u](x)dx = \int_1^\infty u(x)\hat{\mathcal{L}}_1(x,\partial_x)\mathcal{P}_{i\rho-1/2}(x)dx \tag{4.489}$$

$$= \int_0^\infty u(\cosh r)\nabla^2\mathcal{P}_{i\rho-1/2}(\cosh r)d[\cosh r] \tag{4.490}$$

$$= -\left(\rho^2+\frac{1}{4}\right)\int_0^\infty u(\cosh r)\mathcal{P}_{i\rho-1/2}(\cosh r)\sinh r dr \tag{4.491}$$

We therefore obtain the result:

$$\mathcal{F}\nabla^2 u(r) = -(\rho^2+1/4)[\mathcal{F}u](r) \tag{4.492}$$

Substituting $u(r) = \mathcal{F}^{-1}\hat{u}(\rho)$, we find the diagonalising transformation of the hyperbolic Laplacian:

$$[(\mathcal{F}\nabla^2\mathcal{F}^{-1})\hat{u}](\rho) = -(\rho^2+1/4)[(\mathcal{F}\mathcal{F}^{-1})\hat{u}](\rho) \tag{4.493}$$

$$= -(\rho^2+1/4)\hat{u}(\rho) \tag{4.494}$$

We have already shown that $\nabla^2 u(\cosh r) = -(\rho^2+1/4)u(\cosh r)$ in 4.7.1, where $u(\cosh r)$ is an eigenfunction. We therefore conclude that:

$$\mathcal{F}\nabla^2\mathcal{F}^{-1} = \nabla^2 \tag{4.495}$$

as required to establish the lemma. □

We now employ these results for the intertwining transform to obtain a second formula for the hyperbolic heat kernel, which is akin to the method expressed in the thesis problems 0.1.1. We wish to recast the solution to the hyperbolic heat equation as a convolution transform. Recall the following PDE from 4.5.4:

Lemma 4.7.9. *The Helmholtz equation defined by the metric for the hyperbolic system in 4.5.4:*

$$ds^2 = \frac{1}{4}\left(-d\tau^2 + \sinh^2\tau d\varphi^2\right) \tag{4.496}$$

is simply $\nabla^2 f = Ef$, where ∇^2 is the curvilinear Laplacian as in 3.2.34. For this metric, the Helmholtz equation is the eigenvalue problem:

$$Ef = -\frac{\partial^2 f}{\partial\tau^2} - \coth\tau\frac{\partial f}{\partial\tau} + \frac{1}{\sinh^2\tau}\frac{\partial^2 f}{\partial\varphi^2} \tag{4.497}$$

For the following, we shall consider the special case whereby $E = \lambda, \varphi = \text{const.}$, hence arriving at the following problem, equivalent to the hyperbolic heat equation in the eigenfunction.

Corollary 4.7.10. *The eigenfunction problem associated with the Helmholtz equation for the hyperbolic system 4.5.4 satisfies:*

$$\frac{\partial^2 f}{\partial r^2} + \coth r\frac{\partial f}{\partial r} + \lambda f = 0 \tag{4.498}$$

with $f(r) = \mathcal{P}_{\sqrt{1/4-\lambda}-1/2}(\cosh r) = \mathcal{P}_{i\rho-1/2}(\cosh r)$ where $\rho^2 = \lambda - 1/4$, the Mehler-Fock functions as defined respectively in 3.4.1, 4.5.4, 4.6.6. These are the toroidal form of the associated Legendre functions of order zero defined on $[1,\infty)$.

Note that this is equivalent to the separable solution $u(r,t) = w(r)g(t)$ which satisfies the hyperbolic heat equation $u_t = u_{rr} + u_r \text{coth} r$ under our restrictions on the Helmholtz equation. We now state the following theorem [11]:

Theorem 4.7.11. *The solution to the heat problem defined by $u_t = \nabla^2 u$ where ∇^2 is the hyperbolic Laplace operator:*

$$u_t = u_{rr} + u_r \text{coth} r \tag{4.499}$$

or the equivalent eigenfunction problem $f_{rr} + \coth r f_r + \lambda f = 0$ is given by the convolution product identity:

$$u(\mathbf{x},t) = \mathcal{F}^{-1}\left[g(\rho,t)h(\rho)\right] = G \star H \tag{4.500}$$

where $\mathcal{F}$ is the operator that commutes with the Laplacian $\mathcal{F}(\nabla^2 f) = \nabla^2(\mathcal{F}f)$ as in 4.7.3, 4.7.4. $\mathcal{F}$ is the Mehler-Fock transform [56] [121] as shown in 4.7.1, i.e. is the transform given by the symmetric kernel which operates on the hyperbolic space and $\star$ the convolution product formula for the Mehler-Fock kernel.

Remark 64. The convolution product we shall be using here comes from the definition, 3.3.2, 3.3.35 for the convolution in terms of the kernel. Although there is a double integral formula for the Mehler-Fock convolution, all we shall require is that there exists some product which essentially obeys the Plancherel identity. In such a case, the Mehler-Fock transform of the convolution product will go over into the product of Mehler-Fock transforms, in an analogous way to e.g. the Fourier transform of a convolution product is the multiplication of the individual Fourier transforms. This is the type of convolution product formula we refer to in the following sections.

Proof. The Helmholtz equation for the hyperbolic plane:

$$Ef = -\frac{\partial^2 f}{\partial\tau^2} - \coth\tau\frac{\partial f}{\partial\tau} + \frac{1}{\sinh^2\tau}\frac{\partial^2 f}{\partial\varphi^2} \tag{4.501}$$

for the special case $\lambda = E, \varphi$ constant reduces to:

$$\frac{\partial^2 f}{\partial\tau^2} + \coth\tau\frac{\partial f}{\partial\tau} + \lambda f = 0 \tag{4.502}$$

To see that this is an equivalent problem to the hyperbolic heat equation, we take the PDE as in the lemma, and apply the results as derived in 4.7.1. Substitution of the trial solution $u(r,t) = e^{-\lambda t} f(r)$ into the hyperbolic heat equation gives:

$$\frac{\partial^2 f}{\partial r^2} + \coth r \frac{\partial f}{\partial r} + \lambda f = 0 \tag{4.503}$$

with solution we may write as $f(r) = \mathcal{P}_{\sqrt{1/4-\lambda}-1/2}(\cosh r) = \mathcal{P}_{i\rho-1/2}(\cosh r)$ where $\rho^2 = \lambda - 1/4$, so for $\tau = r$ we recover an equivalent differential systems for the Helmholtz and hyperbolic heat equation. We now consider the initial value problem given by the solution in 7.3.12. Radial initial data may be expressed through the separable solution $u(r,t) = e^{-\lambda t} f(r)$. We have the eigenfunction equation for the Mehler-Fock functions:

$$\nabla^2 \left[\mathcal{P}_{i\rho-1/2}(\cosh r)\right] = -\left(1/4 + \rho^2\right) \mathcal{P}_{i\rho-1/2}(\cosh r) \tag{4.504}$$

We can therefore write the solution to the hyperbolic heat equation for initial condition $u(r,0) = u(r)$ in the form $u(r,t) = e^{-(\rho^2+1/4)t} u(r)$, where $\nabla^2 u(r) = -(\rho^2 + 1/4)u(r)$. The Mehler-Fock transform of this solution is the integral:

$$\hat{u}(\rho,t) = \mathcal{F}\left[e^{-(\rho^2+1/4)t} u(r)\right] = \int_0^\infty \mathcal{P}_{i\rho-1/2}(\cosh r) e^{-(\rho^2+1/4)t} u(r) d[\cosh r] \tag{4.505}$$

which satisfies the hyperbolic heat equation in the following way. We have that $u(r)$ is an eigenfunction, therefore:

$$\nabla^2 u(r) = -(\rho^2 + 1/4)u(r) \tag{4.506}$$

Transforming, we find:

$$\mathcal{F}\nabla^2 u(r) = -(\rho^2 + 1/4)\mathcal{F}u(r) = -(\rho^2 + 1/4)\hat{u}(\rho) \tag{4.507}$$

we now substitute the inverse transform $u(r) = \mathcal{F}^{-1}\hat{u}(\rho)$ to obtain:

$$(\mathcal{F}\nabla^2\mathcal{F}^{-1})\hat{u}(\rho) = -(\rho^2 + 1/4)\hat{u}(\rho) = \nabla^2\hat{u}(\rho) \tag{4.508}$$

or $\nabla^2 = \mathcal{F}\nabla^2\mathcal{F}^{-1} \Rightarrow \mathcal{F}\nabla^2 = \nabla^2\mathcal{F}$, i.e. $\mathcal{F}$ commutes with the hyperbolic Laplacian. Evaluating the hyperbolic heat equation, we have:

$$\nabla^2\hat{u}(\rho,t) = e^{-(\rho^2+1/4)t}\nabla^2 \int_0^\infty \mathcal{P}_{i\rho-1/2}(\cosh r) u(r) d[\cosh r] \tag{4.509}$$

and using the commutation relation between the Mehler-Fock transform and Laplacian:

$$\nabla^2\hat{u}(\rho,t) = e^{-(\rho^2+1/4)t}(\nabla^2\mathcal{F})u(r) = e^{-(\rho^2+1/4)t}(\mathcal{F}\nabla^2)u(r) \tag{4.510}$$

and we therefore have $\nabla^2\hat{u}(\rho,t) = -(\rho^2 + 1/4)\hat{u}(\rho,t)$. The time derivative is readily evaluated as only the exponential factor has time dependency, and we see that this solves the hyperbolic heat equation. We therefore can extract the solution to the PDE using the inverse Mehler-Fock transform:

$$u(r,t) = \mathcal{F}^{-1}\left[e^{-(\rho^2+1/4)t} \int_0^\infty \mathcal{P}_{i\rho-1/2}(\cosh r) u(r) d[\cosh r]\right] \tag{4.511}$$

$$= \mathcal{F}^{-1}\left[g(\rho,t)h(\rho)\right] \tag{4.512}$$

and we can see that this expression solves the hyperbolic heat equation:

$$\frac{\partial u(r,t)}{\partial t} = \nabla^2 u(r,t) \tag{4.513}$$

as required. For the initial condition, we have:

$$u(r,0) = \int_0^\infty \hat{u}(\rho)\mathcal{P}_{i\rho-1/2}(\cosh r)\, d\rho = u(r) \tag{4.514}$$

i.e. the initial condition is given by a Mehler-Fock transform of the boundary in the eigenvalue ρ. We obtain the convolution by simple application:

$$\mathcal{F}^{-1}\left[g(\rho,t)h(\rho)\right] = G \star H \tag{4.515}$$

The following section will show how to construct this convolution, expanding on some results sketched in [11]. We shall employ a different approach relying on our constructions of convolution theorems, kernels and positive definite operators as in 3.3.4, 3.3.14, 3.3.1. □

Remark 65. We have explicitly relied on the transform commuting or intertwining with the Laplacian on the hyperbolic space. It is known in analysis that the Mehler-Fock function is appropriate for Fourier-type extensions to hyperbolic space. The reader is referred to [69] for exhaustive lists of these types of problems and their association with special function theory.

We shall now evaluate the convolution product using the representation derived in 3.3.3, 3.3.11, 3.3.26. The following argument is an adaptation of the discussion to be found in [11], see also [43] for a related calculation using the properties of automorphic forms, Haar measure and Whittaker functions.

Definition 4.7.12. Consider the action of a group on a particular element in the hyperbolic plane. All these actions will be given by some form of $SL(2,\mathbb{R})$, as can be derived using either the spherical function theory as in 3.3.42, or by directly appealing to the metric. For now, we will act in generality, and assume that a generic group element may be written as $g_z = \kappa_\theta T_r$, where $z = re^{i\theta}$, so the operator T_r scales an element, and the other rotates.

Lemma 4.7.13. *The solution of the hyperbolic heat equation $\nabla^2 u = u_t$ may be written as the integral:*

$$u(z,w;t) = \int_0^\infty d[\cosh w] K(z,w;t)u(w) \tag{4.516}$$

where we take the hyperbolic Laplacian as:

$$\nabla^2 = \frac{\partial^2}{\partial\tau^2} + \coth\tau\frac{\partial}{\partial\tau} \tag{4.517}$$

and the kernel $K(z,w;t)$ is given by:

$$K(z,w;t) = \int_0^\infty d\rho e^{-(\rho^2+1/4)t}\rho\tanh\pi\rho\mathcal{P}_{i\rho-1/2}(\cosh w).\mathcal{P}_{i\rho-1/2}(\cosh z) \tag{4.518}$$

The initial value is specified through $u(z,w;0) = u(w)$ and $\mathcal{P}_{i\rho-1/2}(\cosh z)$ are the associated Legendre functions of toroidal type, i.e. Mehler-Fock functions.

Proof. We shall identify several ways in which this may be proven. For example, we may apply the inverse Mehler-Fock transform directly, using the formula:

$$[\mathcal{F}^{-1}f](z) = \int_0^\infty d\rho.\rho\tanh\pi\rho\mathcal{P}_{i\rho-1/2}(\cosh z)f(\rho) \tag{4.519}$$

where we take the measure $\rho \tanh \pi\rho$ inside the integral. In this case, we may write out the solution explicitly as:

$$u(z,w;t) = \mathcal{F}^{-1}\left[g(\rho,t)h(\rho)\right] = \mathcal{F}^{-1}\left[e^{-(\rho^2+1/4)t}\int_0^\infty \mathcal{P}_{i\rho-1/2}(\cosh w)u(w)d[\cosh w]\right] \tag{4.520}$$

$$= \int_0^\infty d\rho.\rho \tanh \pi\rho \mathcal{P}_{i\rho-1/2}(\cosh z)\int_0^\infty d[\cosh w]e^{-(\rho^2+1/4)t}\mathcal{P}_{i\rho-1/2}(\cosh w)u(w) \tag{4.521}$$

$$= \int_0^\infty d\rho \int_0^\infty d[\cosh w]e^{-(\rho^2+1/4)t}\rho \tanh \pi\rho \mathcal{P}_{i\rho-1/2}(\cosh w).\mathcal{P}_{i\rho-1/2}(\cosh z)u(w) \tag{4.522}$$

where we have invoked Fubini-Tonelli and reversed the order of integration. We find:

$$u(z,w;t) = \int_0^\infty d[\cosh w]\int_0^\infty d\rho e^{-(\rho^2+1/4)t}\rho \tanh \pi\rho \mathcal{P}_{i\rho-1/2}(\cosh w).\mathcal{P}_{i\rho-1/2}(\cosh z)u(w) \tag{4.523}$$

or

$$u(z,w;t) = \int_0^\infty d[\cosh w]K(z,w;t)u(w) \tag{4.524}$$

as required, with the kernel given by:

$$K(z,w;t) = \int_0^\infty d\rho e^{-(\rho^2+1/4)t}\rho \tanh \pi\rho \mathcal{P}_{i\rho-1/2}(\cosh w).\mathcal{P}_{i\rho-1/2}(\cosh z) \tag{4.525}$$

Alternatively, one can implement the eigenfunction decomposition directly, as we know the eigenvalue. We have $\lambda(\rho) = (\rho^2+1/4)$, the eigenfunction given by $\psi_\rho(z) = \mathcal{P}_{i\rho-1/2}(\cosh z)$. The measure on the space is $d\mathfrak{m}(\rho) = \rho \tanh \pi\rho d\rho$. The kernel formula for a single continuous eigenvalue 3.1.18 is then given by:

$$K(z,w;t) = \int_0^\infty e^{-\lambda(\rho)t}\psi_\rho(z)\psi_\rho(w)d\mathfrak{m}(\rho) \tag{4.526}$$

which yields an identical result, where the eigenfunction is real on the domain hence conjugation is not required, and we have altered the sign of the eigenvalue, and used the symmetries of the Mehler-Fock function to account for the change in direction of time. Finally, we can employ the method of spherical functions and convolutions. If we consider the spherical product formula 3.3.42, 3.3.47, we have:

$$\int_K f(\mathbf{xky})d\mathfrak{m}(\mathbf{k}) = f(\mathbf{x})f(\mathbf{y}) \tag{4.527}$$

Writing the solution for this in the form $f(\mathbf{xky}) = f(d_K(\mathbf{x},\mathbf{y}))$, we have the Haar integral:

$$\int_K f(d_K(\mathbf{x},\mathbf{y}))d\mathfrak{m}(\mathbf{k}) = f(\mathbf{x})f(\mathbf{y}) \tag{4.528}$$

or

$$\int_K f(d_K(\mathbf{x},\mathbf{y}))d\mathfrak{m}(\mathbf{k}) = f(d_K(\mathbf{x},0))f(d_K(\mathbf{y},0)) \tag{4.529}$$

where we used $d_K(\mathbf{y},0) = d_K(0,\mathbf{y})$ as the distance measure is invariant under interchange of points. We also have the right hand side equal to the symmetric kernel, we therefore obtain:

$$K(\mathbf{x},\mathbf{y}) = \int_K f(d_K(\mathbf{x},\mathbf{y}))d\mathfrak{m}(\mathbf{k}) \tag{4.530}$$

and the kernel equal to:

$$K(\mathbf{x},\mathbf{y};t) = \int_K e^{-\lambda(k)t}f(d_K(\mathbf{x},\mathbf{y}))d\mathfrak{m}(\mathbf{k}) \tag{4.531}$$

We shall show how this can be derived using the composition law for the Mehler-Fock functions, as derived in 4.6.6. We have:

$$\frac{1}{2\pi}\int_0^{2\pi}\mathcal{P}_{i\rho-1/2}(d(r,s))du=\mathcal{P}_{i\rho-1/2}(\cosh r)\mathcal{P}_{i\rho-1/2}(\cosh s) \tag{4.532}$$

where $d(r,s)$ is the hyperbolic distance function. We know that the eigenfunction of the system is given by $f(d_K(\mathbf{x},\mathbf{y}))=\mathcal{P}_{i\rho-1/2}(d(r,s))$, the eigenvalue given by $\lambda=1/4+\rho^2$ the kernel can therefore be written:

$$K(z,w;t)=\int_0^\infty d\rho.\rho\tanh(\pi\rho)e^{-(\rho^2+1/4)t}\mathcal{P}_{i\rho-1/2}(\cosh d(z,w)) \tag{4.533}$$

where we used $d\mathfrak{m}\,(\mathbf{k})=\rho\tanh(\pi\rho)d\rho$ for the measure and $\mathbf{x}\to z,\mathbf{y}\to w$. Inserting the expression for the composition formula of the Mehler-Fock functions, we have:

$$\frac{1}{2\pi}\int_0^{2\pi}du.K(z,w;t)=\frac{1}{2\pi}\int_0^{2\pi}du.\int_0^\infty d\rho.\rho\tanh(\pi\rho)e^{-(\rho^2+1/4)t}\mathcal{P}_{i\rho-1/2}(\cosh d(z,w)) \tag{4.534}$$

$$=\int_0^\infty d\rho.\rho\tanh(\pi\rho)e^{-(\rho^2+1/4)}\frac{1}{2\pi}\int_0^{2\pi}du.\mathcal{P}_{i\rho-1/2}(\cosh d(z,w)) \tag{4.535}$$

$$=\int_0^\infty d\rho.\rho\tanh(\pi\rho)e^{-(\rho^2+1/4)}\mathcal{P}_{i\rho-1/2}(\cosh r)\mathcal{P}_{i\rho-1/2}(\cosh s) \tag{4.536}$$

$$=K(z,w;t) \tag{4.537}$$

where $z=\cosh r$, $w=\cosh s$. This implies the following result for the group action. As in the kernel integral above, we take only the angular component, and all group elements of the same radius to be equivalent. As we have $\mathbf{g}_z=\kappa_\theta\mathbf{T}_r$, where $z=re^{i\theta}$, this collapses to $\mathbf{g}_z=\kappa_\theta$. If we consider the convolution theorem for the Mehler-Fock transform, we have derived that the solution to the hyperbolic heat equation satisfies:

$$u(r,t)=u(d(z,w),t)=\mathcal{F}^{-1}\left[g(\rho,t)h(\rho)\right]=(g\star h)(z) \tag{4.538}$$

Under the invariant action of the group as in 3.3.2, we have the measure invariant to the group action via $d\mathfrak{m}[\kappa_\theta\cdot z]=d\mathfrak{m}[\kappa_\theta^{-1}\cdot z]=d\mathfrak{m}[z]$, the convolution satisfying:

$$(g\star h)(\kappa_\theta\cdot z)=(g\star h)(z) \tag{4.539}$$

Writing this out in full, we have the convolution integral given by the group action:

$$(g\star h)(z)=\int_K h(w)g(\mathbf{g}_z^{-1}\cdot z)d\mathfrak{m}[z] \tag{4.540}$$

However, we have already shown that:

$$h(\rho)=\int_0^\infty\mathcal{P}_{i\rho-1/2}(\cosh w)u(w)d[\cosh w] \tag{4.541}$$

and $g(\rho,t)=e^{-(\rho^2+1/4)t}$. If we take $g(\mathbf{g}_z^{-1}\cdot z,t)=e^{-(\rho^2+1/4)t}\mathcal{P}_{i\rho-1/2}(\cosh z)$, then as we know the measure:

$$\int_K d\mathfrak{m}[z]=\int_0^\infty d\rho.\rho\tanh(\pi\rho) \tag{4.542}$$

Computing the convolution, we have:

$$(g\star h)(z)=\int_K h(w)g(\mathbf{g}_z^{-1}\cdot z)d\mathfrak{m}[z] \tag{4.543}$$

$$= \int_0^\infty d\rho.\rho\tanh(\pi\rho)e^{-(\rho^2+1/4)t}\mathcal{P}_{i\rho-1/2}(\cosh z)\int_0^\infty d[\cosh w].\mathcal{P}_{i\rho-1/2}(\cosh w)u(w) \tag{4.544}$$

$$= \int_0^\infty d[\cosh w].\int_0^\infty d\rho.\rho\tanh(\pi\rho)e^{-(\rho^2+1/4)t}\mathcal{P}_{i\rho-1/2}(\cosh z)\mathcal{P}_{i\rho-1/2}(\cosh w)u(w) \tag{4.545}$$

$$= \int_0^\infty d[\cosh w].K(z,w;t) \tag{4.546}$$

where $K(z,w;t)$ is the hyperbolic heat kernel. We note that we have already proved this theorem using the initial value problem approach and the Mehler-Fock transform in 4.7.1. □

Theorem 4.7.14. *The solution to the hyperbolic heat kernel as given by the convolution transform 4.7.13 is:*

$$K(z,w;t) = \frac{1}{2\pi}\int_0^\infty \rho\tanh\pi\rho e^{-(\rho^2+1/4)t}\psi_\rho(g_w^{-1}\cdot z)d\rho \tag{4.547}$$

and explicitly, by the formula in this instance:

$$K(z,w;t) = \frac{1}{2\pi}\int_0^\infty \rho\tanh\pi\rho e^{-(\rho^2+1/4)t}\mathcal{P}_{i\rho-1/2}(\cosh d(z,w))d\rho \tag{4.548}$$

Remark 66. We have already proved this in the preceding lemma. The following proof shall use the inverse Mehler-Fock transform to achieve the same result.

Proof. Take the initial condition for the solution of the hyperbolic heat equation, we have $u(r,0) = u(r)$, and we also have the Mehler-Fock transform:

$$h(\rho) = \int_0^\infty \mathcal{P}_{i\rho-1/2}(\cosh w)u(w)d[\cosh w] \tag{4.549}$$

We have from the convolution equation:

$$u(z,w;t) = \mathcal{F}^{-1}\left[g(\rho,t)h(\rho)\right] = (g\star h)(z) \tag{4.550}$$

Consider the the integral transform given by $\int_K u(z)\psi_\rho(\mathbf{g}_w^{-1}\cdot z)d\mathfrak{m}(z)$. Then we have:

$$\int_K u(z)\psi_\rho(\mathbf{g}_w^{-1}\cdot z)d\mathfrak{m}(z) = \int_K u(z)\mathcal{P}_{i\rho-1/2}(\mathbf{g}_w^{-1}\cdot z)d\mathfrak{m}(z) \tag{4.551}$$

$$= \int_K u(z)\mathcal{P}_{i\rho-1/2}(d(\mathbf{g}_w^{-1}\cdot z,0))d\mathfrak{m}(z) \tag{4.552}$$

$$= \int_K u(z)\mathcal{P}_{i\rho-1/2}(d_K(w,z))d\mathfrak{m}(z) \tag{4.553}$$

Substituting the composition formula for the Mehler-Fock functions, as derived in 4.6.6, we have:

$$\frac{1}{2\pi}\int_0^{2\pi} du\mathcal{P}_{i\rho-1/2}(d_K(w,z)) = \mathcal{P}_{i\rho-1/2}(d_K(w,0))\mathcal{P}_{i\rho-1/2}(d_K(z,0)) \tag{4.554}$$

and hence:

$$\int_K u(z)\psi_\rho(\mathbf{g}_w^{-1}\cdot z)d\mathfrak{m}(z) = 2\pi\int_K u(z)\mathcal{P}_{i\rho-1/2}(d_K(w,0))\mathcal{P}_{i\rho-1/2}(d_K(z,0))d\mathfrak{m}(z) \tag{4.555}$$

$$= 2\pi\mathcal{P}_{i\rho-1/2}(d_K(w,0))\int_K u(z)\mathcal{P}_{i\rho-1/2}(d_K(z,0))d\mathfrak{m}(z) \tag{4.556}$$

which we write in the convolution form:

$$(u \star \psi_\rho(z))(w) = \int_K u(z)\psi_\rho(\mathbf{g}_w^{-1} \cdot z)d\mathfrak{m}(z) = 2\pi\psi_\rho(w)\hat{u}(\rho) \tag{4.557}$$

where we identify the Mehler-Fock transform of the initial condition:

$$\hat{u}(\rho) = \int_K u(z)\mathcal{P}_{i\rho-1/2}(d_K(z,0))d\mathfrak{m}(z) \tag{4.558}$$

and the Mehler-Fock function:

$$\psi_\rho(w) = \mathcal{P}_{i\rho-1/2}(d_K(w,0)) \tag{4.559}$$

We can write the solution to the hyperbolic heat kernel in convolution form, using the inverse Mehler-Fock transform in a way analogous to 0.1.1:

$$u(z,w;t) = \mathcal{F}^{-1}\left[g(\rho,t)h(\rho)\right] = \mathcal{F}^{-1}\left[\hat{u}(\rho)\psi_\rho(w,t)\right] \tag{4.560}$$

$$= \mathcal{F}^{-1}\left[\mathcal{F}u(z).\psi_\rho(w,t)\right] \tag{4.561}$$

$$= \mathcal{F}^{-1}\left[\mathcal{F}u(z).\psi_\rho(w,t)\right] \tag{4.562}$$

$$= \left[u \star \mathcal{F}^{-1}\psi_\rho(.,t)\right] \tag{4.563}$$

Writing this out in full, we have:

$$u(z,w;t) = \frac{1}{2\pi}\int d\mathfrak{m}(z)u(z)\int_0^\infty d\rho.\rho\tanh(\pi\rho)e^{-(\rho^2+1/4)t}\psi_\rho(\mathbf{g}_w^{-1} \cdot z) \tag{4.564}$$

which implies the integral equation:

$$u(z,w;t) = \frac{1}{2\pi}\int d\mathfrak{m}(z)u(z)K(z,w;t) \tag{4.565}$$

with integral kernel:

$$K(z,w;t) = \frac{1}{2\pi}\int_0^\infty d\rho.\rho\tanh(\pi\rho)e^{-(\rho^2+1/4)t}\psi_\rho(\mathbf{g}_w^{-1} \cdot z) \tag{4.566}$$

$$= \frac{1}{2\pi}\int_0^\infty d\rho.\rho\tanh(\pi\rho)e^{-(\rho^2+1/4)t}\mathcal{P}_{i\rho-1/2}(d_K(z,w)) \tag{4.567}$$

Note the formula for the eigenfunction in the disk, either by the spherical zonal representation 4.6.6, or 4.5.4, where we have essentially obtained an identical expression, but would have to exploit the composition formula in a reverse fashion. By multiplying by the exponential factor containing the eigenvalue through the composition formula for the Mehler-Fock function one can obtain similar results. We can see this directly by using the kernel we derived using the eigenfunction decomposition:

$$K(z,w;t) = C\int_0^\infty d\rho.\rho\tanh(\pi\rho)e^{-(\rho^2+1/4)t}\mathcal{P}_{i\rho-1/2}(\cosh z)\mathcal{P}_{i\rho-1/2}(\cosh w) \tag{4.568}$$

Using the composition formula for the Mehler-Fock functions 4.6.6, we have immediately:

$$\frac{1}{2\pi}\int_0^{2\pi} du\mathcal{P}_{i\rho-1/2}(d_K(w,z)) = \mathcal{P}_{i\rho-1/2}(d_K(w,0))\mathcal{P}_{i\rho-1/2}(d_K(z,0)) \tag{4.569}$$

and consequently:

$$K(z,w;t) = C\frac{1}{2\pi}\int_0^\infty d\rho.\rho\tanh(\pi\rho)e^{-(\rho^2+1/4)t}\mathcal{P}_{i\rho-1/2}(d_K(z,w)) \tag{4.570}$$

where the hyperbolic distance function contains the variable u via:

$$d_K(w,z) = \cosh w\cosh z + \cos u\sinh w\sinh z \tag{4.571}$$

and also $d_K(w,0) = \cosh w$ is independent of w, and likewise for $d_K(z,0)$. We have established this theorem for the hyperbolic heat kernel. As we shall see, this enables us to derive further results given by other integrals related to this formula. By choosing the correct normalisation constant C, which is obviously given by $C = \frac{1}{2\pi}\int_0^{2\pi} du = 1$, we have an equivalent formulation. $\square$

Remark 67. Indeed, we have identified the kernel given by the distance formula as in 3.3.13. These results are obviously equivalent as we have shown using different methods of analysis and group theory.

We now give the following result for the McKean heat kernel, as promised in the description of the thesis problem. This is an important milestone in understanding the nature of these types of complicated hyperbolic systems. We shall state the following theorems from Davies [1], who gives some results for the hyperbolic heat kernel without proof. We note that our calculations regarding the distance function and the hyperbolic heat kernel complement these results, which may be found in [1] (pp. 176-178, Thms. 5.7-5.7.1).

Theorem 4.7.15. *The Laplacian commutes with the group action g on $\mathbb{H}^2$. In particular, as in 3.3.15, we have the kernel invariance under g:*

$$K(z_1,z_2;t) = K(gz_1,gz_2;t) = k\left(d(z_1,z_2);t\right) \tag{4.572}$$

where $d(x,y)$ is the hyperbolic distance function. The metric ds^2 is given by the Cayley tranform of the Poincare metric, as derived in 3.2.40.

$$ds^2 = \frac{dx^2+dy^2}{y^2} \tag{4.573}$$

and the hyperbolic Laplacian satisfies:

$$\nabla^2 = y^2\left(\partial_x^2+\partial_y^2\right) \tag{4.574}$$

with fundamental action on polynomials $-\nabla^2 y^s = s(1-s)y^s$, and action on the distance measure $\nabla^2 d(a,z) = \cosh d(a,z)$. The hyperbolic heat equation may be written in the form:

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial\rho^2} + \coth\rho\frac{\partial k}{\partial\rho} \tag{4.575}$$

Proof. We have discussed kernel invariance and the different forms of the hyperbolic plane that make up the theorem in 3.3.15, 3.2.40, see also Bump, [43] for a lengthy discussion on the application of the hyperbolic Laplacian formula $-\nabla^2 y^s = s(1-s)y^s$ to automorphic forms of Maass type ([43], pp.103, 1.9-1.9.1). The group invariance of the kernel is reached through the application of the previous theorem 4.7.14, where we showed that the kernel is a Mehler-Fock transform, with argument given by the distance function. See also Davies (pp. 176, [1] 5.7-5.7.1) for discussions related to this theorem, where the author has listed results for some more general cases without proof. We can see through the Cayley transform 3.2.40 that the Poincare metric and this metric may be transformed into one another, that the Laplacians in the curvilinear space give equivalent representations of the hyperbolic plane follows, with the Laplacian given by 3.2.34. We have proved the various representations for the hyperbolic Laplacian 4.5.1, the heat equation in the hyperbolic plane follows as a result of $u_t = \nabla^2 u$. □

We shall now state the following result for the hyperbolic heat kernel, which is stated without proof in McKean [2], also Davies [1]. This is the formula from our original thesis question.

Theorem 4.7.16. *The hyperbolic heat kernel may be written in integral form as the truncated Weierstrass transform:*

$$K(z,w;t) = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}} \int_{d(z,w)}^{\infty} \frac{\beta \exp\left(-\frac{\beta^2}{4t}\right) d\beta}{\sqrt{\cosh\beta - \cosh d(z,w)}} \tag{4.576}$$

where $d(z,w)$ is the hyperbolic distance measure, see 3.3.12.

Proof. The Dirichlet-Murphy representation for the associated Legendre function is:

$$\mathcal{P}_{\nu}^{-\mu}(\cosh x) = \sqrt{\frac{2}{\pi}} \frac{\Gamma(\mu+1/2)\sinh^{\mu} x}{\Gamma(\mu+\nu+1)\Gamma(\mu-\nu)} \int_0^{\infty} \frac{\cosh\left((\nu+1/2)\,u\right)}{(\cosh x + \cosh u)^{\mu+1/2}} du \tag{4.577}$$

We have a formula for the conical function (see e.g. [122] GR 8.71.5 or a proof using toroidal co-ordinates in [123], eqs. 7.4.7, also 8.12.12-13, pp. 230):

$$\mathcal{P}_{i\rho-1/2}(\cosh d) = \frac{\sqrt{2}}{\pi} \coth\pi\rho \int_d^{\infty} \frac{\sin(\rho\beta)}{\sqrt{\cosh\beta - \cosh d}} d\beta \tag{4.578}$$

Inserting this into the kernel equation 4.7.14, we have:

$$K(z,w;t) = \frac{1}{2\pi} \int_0^{\infty} \rho \tanh\pi\rho e^{-(\rho^2+1/4)t} \mathcal{P}_{i\rho-1/2}(\cosh d(z,w)) d\rho \tag{4.579}$$

$$\begin{aligned} K(z,w;t) &= \frac{1}{2\pi} \int_0^{\infty} \rho \tanh\pi\rho e^{-(\rho^2+1/4)t} \frac{\sqrt{2}}{\pi} \coth\pi\rho d\rho \\ &\times \int_{d(z,w)}^{\infty} \frac{\sin(\rho\beta)}{\sqrt{\cosh\beta - \cosh d(z,w)}} d\beta \end{aligned} \tag{4.580}$$

Switching the order of integration, we obtain:

$$K(z,w;t) = \frac{1}{\sqrt{2}\pi^2} \int_{d(z,w)}^{\infty} \frac{d\beta}{\sqrt{\cosh\beta - \cosh d(z,w)}} \tag{4.581}$$

$$\times \int_0^{\infty} \rho \coth\pi\rho \tanh\pi\rho e^{-(\rho^2+1/4)t} \sin(\rho\beta) d\rho \tag{4.582}$$

$$= \frac{1}{\sqrt{2}\pi^2} \int_{d(z,w)}^{\infty} \frac{d\beta}{\sqrt{\cosh\beta - \cosh d(z,w)}} \int_0^{\infty} \rho e^{-(\rho^2+1/4)t} \sin(\rho\beta) d\rho \tag{4.583}$$

Using the integral $\int_0^{\infty} \rho e^{-(\rho^2+1/4)t} \sin(\rho\beta) d\rho = \frac{\sqrt{\pi}\beta}{4t^{3/2}} \exp\left(-\frac{\beta^2}{4t} - \frac{t}{4}\right)$ we find the result:

$$K(z,w;t) = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}} \int_{d(z,w)}^{\infty} \frac{\beta \exp\left(-\frac{\beta^2}{4t}\right) d\beta}{\sqrt{\cosh\beta - \cosh d(z,w)}} \tag{4.584}$$

which establishes the theorem. □

Corollary 4.7.17. *There exists a second integral representation for the hyperbolic heat kernel as in 4.7.16. We may write the kernel $K(z,w;t)$ in the form:*

$$K(z,w;t) = C\frac{e^{-t/4}e^{\pi^2/4t}}{4\pi^2 t^{3/2}}\int_0^\infty \frac{du}{\sqrt{(\cosh d(z,w)+\cosh u)}}e^{-u^2/(4t)}\left(\pi\cos\left(\frac{\pi u}{2t}\right) - u\sin\left(\frac{\pi u}{2t}\right)\right) \tag{4.585}$$

where C is a normalisation constant, and $d(z,w)$ is the hyperbolic distance.

Proof. We have the integral representation for the Mehler-Fock functions [123], eq. 7.4.7:

$$\mathcal{P}_{i\rho-1/2}(\cosh\alpha) = \frac{2}{\pi}\cosh\rho\pi\int_0^\infty du.\frac{\cos(\rho u)}{\sqrt{2(\cosh\alpha+\cosh u)}} \tag{4.586}$$

The kernel may be written:

$$K(z,w;t) = \frac{1}{2\pi}\int_0^\infty d\rho.\rho\tanh\pi\rho e^{-(\rho^2+1/4)t}\mathcal{P}_{i\rho-1/2}(\cosh d(z,w)) \tag{4.587}$$

as before. We insert the integral representation and obtain:

$$K(z,w;t) = \frac{1}{\sqrt{2}\pi^2}\int_0^\infty d\rho.\rho\tanh\pi\rho e^{-(\rho^2+1/4)t}\cosh\rho\pi\int_0^\infty du.\frac{\cos(\rho u)}{\sqrt{(\cosh d(z,w)+\cosh u)}} \tag{4.588}$$

$$= \frac{1}{\sqrt{2}\pi^2}\int_0^\infty d\rho.\rho\sinh(\pi\rho)e^{-(\rho^2+1/4)t}\int_0^\infty du.\frac{\cos(\rho u)}{\sqrt{(\cosh d(z,w)+\cosh u)}} \tag{4.589}$$

$$= \frac{e^{-t/4}}{\sqrt{2}\pi^2}\int_0^\infty \frac{du}{\sqrt{(\cosh d(z,w)+\cosh u)}}\int_0^\infty d\rho.\rho\sinh(\pi\rho)\cos(\rho u)\, e^{-(\rho^2+1/4)t} \tag{4.590}$$

Using the integral:

$$\int_0^\infty d\rho.\rho\sinh(\pi\rho)\cos(\rho u)\, e^{-\rho^2 t} = \frac{1}{(2t)^{3/2}}e^{(\pi^2-u^2)/(4t)}\left(\pi\cos\left(\frac{\pi u}{2t}\right) - u\sin\left(\frac{\pi u}{2t}\right)\right) \tag{4.591}$$

we find:

$$K(z,w;t) = \frac{e^{-t/4}e^{\pi^2/4t}}{4\pi^2 t^{3/2}}\int_0^\infty \frac{du}{\sqrt{(\cosh d(z,w)+\cosh u)}}e^{-u^2/(4t)}\left(\pi\cos\left(\frac{\pi u}{2t}\right) - u\sin\left(\frac{\pi u}{2t}\right)\right) \tag{4.592}$$

□

Remark 68. This proof can be contrasted with the results found in [124], where the method of spectral analysis in the hyperbolic plane yielded similar formulae. We shall discuss some approximation methods for the Gaussian transform in the conclusions. Following sections in our proof will show how the Mehler-Fock functions are related to the modified Bessel and Whittaker functions. In some ways, the kernel as written in this way can be taken as a central formula that is descriptive of all these different types of systems by a series of transformations. The Gauss or Weierstrass transformation that appears in the kernel formula in this representation can be thought of as the deformation of the Gaussian distribution into the hyperbolic space.

4.7.0.3 Derivation of the Green's Function

The following calculation shall show how the Green's function may be readily computed using the composition formula on the one hand, or alternatively some index transforms given by the Mehler-Fock functions in the hyperbolic distance. We direct the reader to the work of Grosche and Steiner, who used a path integral method to achieve essentially the same result in the series of papers [13–16]. We shall replicate their result using the formula for the hyperbolic kernel as given by 4.7.16. The Green's function of the kernel is an important transformation, which can be used to find identical properties of the system through use of a Laplace transform, where it exists. In the literature, this functional is often labelled as the resolvent. For our purposes, in this thesis, we shall use the following definitions for the Green's function interchangeably:

Definition 4.7.18. The Green's function is given by the Laplace transform of the kernel:

$$G(x,y;\sqrt{E}) = \int_0^\infty dt.e^{-Et}K(x,y;t) \tag{4.593}$$

where it exists. In our case, regarding the hyperbolic plane, we only have one continuous eigenvalue. We also have access to the Fredholm resolvent:

$$G(x,y;\sqrt{E}) = \int d\mathfrak{m}(\rho)\frac{\psi_\rho^*(x)\psi_\rho(y)}{\rho^2+E} \tag{4.594}$$

and the inverse Fourier transform:

$$\int_{-\infty}^{+\infty} e^{izt}G(x,y;\sqrt{iz})dz = K(x,y;t) \tag{4.595}$$

where they exist and are well defined.

Remark 69. Note that the form of the Green's function will depend on whether we are dealing with the forwards or backwards heat equation. In such a case, the sign of the eigenvalue changes, and the eigenvalue in the denominator will change sign accordingly. The basic formula remains the same for both cases, with appropriate modification.

Proof. We shall give a simple sketch proof that these are equivalent, using the spectral decomposition of the kernel. We have:

$$G(x,y;\sqrt{E}) = \int_0^\infty dt.e^{-Et}K(x,y;t) = \int_0^\infty dt.e^{-Et}\int d\mathfrak{m}(\rho)e^{-\lambda(\rho)t}\psi_\rho^*(x)\psi_\rho(y) \tag{4.596}$$

$$= \int d\mathfrak{m}(\rho)\psi_\rho^*(x)\psi_\rho(y)\int_0^\infty dt.e^{-Et}e^{-\lambda(\rho)t} \tag{4.597}$$

$$= \int d\mathfrak{m}(\rho)\frac{\psi_\rho^*(x)\psi_\rho(y)}{\lambda(\rho)+E} \tag{4.598}$$

as required. In our formulation, we shall often work with the reduced kernel, given by $e^{t/4}K(x,y;t)$, which has eigenvalue ρ^2. In such a case, the formula reduces to:

$$G(x,y;\sqrt{E}) = \int d\mathfrak{m}(\rho)\frac{\psi_\rho^*(x)\psi_\rho(y)}{\rho^2+E} \tag{4.599}$$

□

Theorem 4.7.19. *The Green's function for the hyperbolic heat kernel is given by the formula:*

$$G(x,y;s) = \frac{1}{\pi^2}\int_0^\infty \mathcal{Q}_{s-1/2}^{-i\rho}(\cosh a)\mathcal{Q}_{s-1/2}^{i\rho}(\cosh b)\cosh\left(\rho\left(\pi-|\alpha-\beta|\right)\right)d\rho \tag{4.600}$$

$$= \frac{1}{2\pi}\mathcal{Q}_{s-1/2}\left(\cosh d(x,y)\right) \tag{4.601}$$

where the hyperbolic distance is:

$$\cosh d(x,y) = \cosh\xi\cosh\varsigma - \sinh\xi\sinh\varsigma\cos\theta \tag{4.602}$$

Proof. We have established the existence of the hyperbolic distance formula in 3.3.12. Note the following composition formula for the associated Legendre polynomials of the second kind (see [125] eq. 2.58 for a derivation that stems from the formula [122] eq. 8.795.2):

$$\mathcal{Q}_\nu(wz - \sqrt{w^2-1}\sqrt{z^2-1}\cos\theta) = \int_0^\infty \cosh(\rho(\pi-\theta))\mathcal{Q}_\nu^{i\rho}(w)\mathcal{Q}_\nu^{-i\rho}(z)d\rho \tag{4.603}$$

Using the substitutions $z = \cosh a, w = \cosh b$, we may write:

$$\mathcal{Q}_\nu(\cosh\xi\cosh\varsigma - \sinh\xi\sinh\varsigma\cos\theta) = \mathcal{Q}_\nu(d(z,w)) \tag{4.604}$$

$$= \int_0^\infty \cosh(\rho(\pi-\theta))\mathcal{Q}_\nu^{i\rho}(w)\mathcal{Q}_\nu^{-i\rho}(z)d\rho \tag{4.605}$$

thereby obtaining the first leg of the theorem. To obtain the rest of the premise, note the kernel from 4.7.16 may be written:

$$\tilde{K}(x,y;t) = e^{t/4}K(x,y;t) = \frac{1}{2\pi}\int_0^\infty d\rho.\rho\tanh\pi\rho\mathcal{P}_{i\rho-1/2}(\cosh(d(x,y))e^{-\rho^2 t} \tag{4.606}$$

$$= \mathcal{F}^{-1}\left[e^{-E_\rho t}\right] \tag{4.607}$$

The Green's function is the Laplace transform of the kernel, i.e.:

$$\tilde{G}(x,y;s) = \mathcal{L}\left[\tilde{K}(x,y;t)\right] = \int_0^\infty dt.e^{-s^2 t}\tilde{K}(x,y;t) \tag{4.608}$$

$$= \frac{1}{2\pi}\int_0^\infty dt.e^{-s^2 t}\int_0^\infty d\rho.\rho\tanh\pi\rho\mathcal{P}_{i\rho-1/2}(\cosh(d(x,y))e^{-\rho^2 t} \tag{4.609}$$

$$= \frac{1}{2\pi}\int_0^\infty d\rho.\rho\tanh\pi\rho\mathcal{P}_{i\rho-1/2}(\cosh(d(x,y))\int_0^\infty dt.e^{-(s^2+\rho^2)t} \tag{4.610}$$

$$= \frac{1}{2\pi}\int_0^\infty d\rho\frac{\rho\tanh\pi\rho}{\rho^2+s^2}\mathcal{P}_{i\rho-1/2}(\cosh(d(x,y)) \tag{4.611}$$

$$= \frac{1}{2\pi}\mathcal{Q}_{s-1/2}(\cosh(d(x,y)) \tag{4.612}$$

where we have used formula 7.213 from [122]. ☐

Remark 70. In Grosche and Steiner [13], the authors compute the Green's function in a different but equivalent way. This is briefly discussed in the following calculation.

Theorem 4.7.20. *The Green's function for the hyperbolic heat kernel is given by the Laplace transform [13]:*

$$G(x,y;E) = \int_0^\infty e^{-Et}K(x,y;t)dt = \frac{m}{\sqrt{2\pi}}\int_0^\infty e^{-u\cosh d(x,y)}I_{ip}(u)\frac{du}{\sqrt{u}} \tag{4.613}$$

$$= \frac{m}{\pi}\mathcal{Q}_{ip-1/2}(\cosh d(x,y)) \tag{4.614}$$

where $I_{ip}(z)$ is the modified Bessel function 3.4.2 and $d(x,y)$ is the hyperbolic distance. Additional forms of the Green's function are provided by the Macdonald representation:

$$\mathcal{Q}_{i\nu-1/2}(\cosh d(x,y)) = \frac{2\sqrt{yy'}}{\pi^2}\int_{-\infty}^\infty dk e^{ik(x'-x)}\int_0^\infty \frac{dp.p\sinh\pi p}{p^2-\nu^2}K_{ip}(ky)K_{ip}(ky') \tag{4.615}$$

Proof. We briefly recall the proof from [13] as it has relevance to the following sections regarding various alternative representations of the hyperbolic disk. Taking the integral formulae for the associated Legendre functions from tables [122] (GR., 7.213):

$$\mathcal{Q}_{i\nu-1/2}(\cosh d(x,y)) = \int_0^\infty \frac{dp.p\tanh\pi p}{p^2-\nu^2}\mathcal{P}_{ip-1/2}(\cosh d(x,y)) \tag{4.616}$$

Further, the hyperbolic distance measure from 3.3.12:

$$\frac{y^2+y'^2+(x-x')^2}{2yy'} = \cosh d(x,y) \tag{4.617}$$

Additional to these postulates, Grosche and Steiner in [13] prove the eigenfunction decomposition of the Green's function:

$$\mathcal{P}_{ip-1/2}(\cosh d) = \frac{4\sqrt{yy'}}{\pi^2}\cosh\pi p\int_0^\infty \cos\left(k(x'-x)\right)K_{ip}(ky)K_{ip}(ky')dk \tag{4.618}$$

$$= \frac{2\sqrt{yy'}}{\pi^2}\cosh\pi p\int_{-\infty}^\infty e^{ik(x'-x)}K_{ip}(ky)K_{ip}(ky')dk \tag{4.619}$$

where the last step is due to the function overall being even in k. Inserting this expression into the other integral, invoking the Fubini-Tonelli theorem and interchanging the order of the integrals, we have:

$$\mathcal{Q}_{i\nu-1/2}(\cosh d(x,y)) = \int_0^\infty \frac{dp.p\tanh\pi p}{p^2-\nu^2}\mathcal{P}_{ip-1/2}(\cosh d(x,y)) \tag{4.620}$$

and hence:

$$\mathcal{Q}_{i\nu-1/2}(\cosh d(x,y)) = \frac{2\sqrt{yy'}}{\pi^2}\int_0^\infty \frac{dp.p\tanh\pi p}{p^2-\nu^2}\cosh\pi p\int_{-\infty}^\infty e^{ik(x'-x)}K_{ip}(ky)K_{ip}(ky')dk \tag{4.621}$$

$$= \frac{2\sqrt{yy'}}{\pi^2}\int_{-\infty}^\infty dk e^{ik(x'-x)}\int_0^\infty \frac{dp.p\sinh\pi p}{p^2-\nu^2}K_{ip}(ky)K_{ip}(ky') \tag{4.622}$$

as required. To prove the other representation of the Green's function, note the formula [122] eq. 6.622.3:

$$\int_0^\infty e^{-x\cosh\alpha}I_\nu(x)x^{\mu-1}dx = \sqrt{\frac{2}{\pi}}e^{-(\mu-1/2)\pi i}\frac{\mathcal{Q}^{\mu-1/2}_{\nu-1/2}(\cosh\alpha)}{\sinh^{\mu-1/2}\alpha} \tag{4.623}$$

where $\mathfrak{Re}\,(\mu+\nu)>0,\mathfrak{Re}(\cosh\alpha)>1$. Substituting $\nu=i\nu$

$$\int_0^\infty e^{-x\cosh\alpha}I_{i\nu}(x)\frac{dx}{\sqrt{x}} = \sqrt{\frac{2}{\pi}}\mathcal{Q}_{i\nu-1/2}(\cosh\alpha) \tag{4.624}$$

as claimed. □

Remark 71. We can see that this form of the solution for the Green's function is given by the Fourier transform of the Kontorovich-Lebedev transform of the resolvent, which we write in the form:

$$G(x,y;E) = \frac{m}{\pi}\mathcal{Q}_{i\nu-1/2}(\cosh d(x,y)) = \frac{2\sqrt{yy'}}{\pi^3}\mathscr{F}\left[\mathscr{K}\left[\frac{K_{ip}(ky')}{p^2-\nu^2}\right]\right] \tag{4.625}$$

In the radial system, we will obviously have

$$K(x,y;t) = \frac{2}{\pi^2}\int_0^\infty e^{-p^2t}dp.p\sinh\pi pK_{ip}(ky)K_{ip}(kx) \tag{4.626}$$

which is known as the Yakubovich kernel, see e.g. [56] for a detailed analysis of the topic.

We shall demonstrate one alternative way in which to derive the hyperbolic Green's function, using some integral formulae from [126], see Prudnikov, Vol. II, 2.16.52.11-12.

Lemma 4.7.21. *The hyperbolic Green's function can be written as the Laplace transform of the heat kernel, where it exists:*

$$G(x, y; s) = \int_0^\infty e^{-st} K(x, y; t) dt \tag{4.627}$$

For the system with eigenfunctions given by the modified Bessel function, the hyperbolic Green's function has form:

$$G(x, y; s) = \begin{cases} \frac{\pi^2}{2} I_{\sqrt{s}}(y) K_{\sqrt{s}}(x) & 0 < y < x \\ \frac{\pi^2}{2} I_{\sqrt{s}}(x) K_{\sqrt{s}}(y) & 0 < x < y \end{cases} \tag{4.628}$$

Proof. Taking the Laplace transform directly, and substituting the eigenfunctions, we have:

$$G(x, y; s) = \int_0^\infty e^{-st} K(x, y; t) dt \tag{4.629}$$

or

$$G(x, y; s) = \int_0^\infty dt.e^{-st} \int_0^\infty dp.p \sinh(\pi p) e^{-p^2 t} K_{ip}(x) K_{ip}(y) \tag{4.630}$$

$$= \int_0^\infty dp.p \sinh(\pi p) K_{ip}(x) K_{ip}(y) \int_0^\infty dt.e^{-(s+p^2)t} \tag{4.631}$$

$$= \int_0^\infty dp \frac{p \sinh(\pi p)}{p^2 + s} K_{ip}(x) K_{ip}(y) \tag{4.632}$$

Using the integral formula from Prudnikov, Vol. II, 2.16.52.11-12, we write:

$$\int_0^\infty \frac{x \sinh(\pi x)}{x^2 + z^2} K_{ix}(b) K_{ix}(c) dx = \begin{cases} \frac{\pi^2}{2} I_z(c) K_z(b) & 0 < c < b \\ \frac{\pi^2}{2} I_z(b) K_z(c) & 0 < b < c \end{cases} \tag{4.633}$$

which yields:

$$G(x, y; s) = \begin{cases} \frac{\pi^2}{2} I_{\sqrt{s}}(y) K_{\sqrt{s}}(x) & 0 < y < x \\ \frac{\pi^2}{2} I_{\sqrt{s}}(x) K_{\sqrt{s}}(y) & 0 < x < y \end{cases} \tag{4.634}$$

and proves the claim in the lemma. □

4.8 Relativity and the Composition Laws

We shall briefly discuss the relativistic consequences of the composition laws, given by the hyperbolic laws of addition 4.6.4. The following two calculations will show two facts of physics flow from the nature of the geometry, and that these can be directly traced to the differential geometry implied by the hyperbolic brachistochrone as demonstrated in the sequence of calculations used to produce the Poincare metric as in 4.4.3, 4.3.18. As the following calculation shows, the hyperbolic geometry can be used to derive a form of special and general relativity. The following proof is known for special relativity, but we include it so as to demonstrate the strength of the Fubini-Study technique in comparison as given through e.g. the Poincare metric as calculated previously.

4.8.1 Addition Formula of Special Relativity

It is known that the theory of special relativity is related to the hyperbolic plane. We can derive the law of addition for special relativity in the plane from our composition law 3.3.13.

Lemma 4.8.1. *Given the hyperbolic law of addition derived in 4.6.4:*

$$\cosh\tau = \cosh\tau_1\cosh\tau_2 - \cos\phi\sinh\tau_1\sinh\tau_2 \tag{4.635}$$

The law of addition of relativistic velocities takes the form:

$$V = \frac{\sqrt{u^2+v^2-2uv\cos\phi - \left(\frac{uv}{c}\sin\phi\right)^2}}{1-\frac{uv}{c^2}\cos\phi} \tag{4.636}$$

and in hyperbolic polar form:

$$\tanh\tau = \frac{\sqrt{\tanh^2\tau_1+\tanh^2\tau_2-2\tanh\tau_1\tanh\tau_2\cos\phi-(\tanh\tau_1\tanh\tau_2\sin\phi)^2}}{1-\tanh\tau_1\tanh\tau_2\cos\phi} \tag{4.637}$$

where $V/c = \tanh\tau$, i.e. $\frac{v}{c} = \tanh\tau_1, \frac{u}{c} = \tanh\tau_2$ is the rapidity for the velocities V, v, u, and ϕ is the angle between the two velocity vectors, ranging between $(0,\pi)$.

Proof. Taking the addition formula from 4.6.4, we have:

$$\cosh\tau = \cosh\tau_1\cosh\tau_2 - \cos\phi\sinh\tau_1\sinh\tau_2 \tag{4.638}$$

Using the formula for the hyperbolic circle:

$$\cosh^2 x - \sinh^2 x = 1 \tag{4.639}$$

we therefore have the identities

$$\frac{1}{\cosh^2 x} = 1-\tanh^2 x \Rightarrow \cosh x = \frac{1}{\sqrt{1-\tanh^2 x}} \tag{4.640}$$

$$\frac{1}{\sinh^2 x} = \frac{1}{\tanh^2 x} - 1 \Rightarrow \sinh x = \frac{1}{\sqrt{\frac{1}{\tanh^2 x}-1}} = \frac{\tanh x}{\sqrt{1-\tanh^2 x}} \tag{4.641}$$

where we are in the positive plane $x > 0$. Substituting these into the addition formula for relativistic velocities in hyperbolic form, we obtain:

$$\frac{1}{\sqrt{1-\tanh^2\tau}} = \frac{1}{\sqrt{1-\tanh^2\tau_1}}\frac{1}{\sqrt{1-\tanh^2\tau_2}} - \cos\phi\frac{\tanh\tau_1}{\sqrt{1-\tanh^2\tau_1}}\frac{\tanh\tau_2}{\sqrt{1-\tanh^2\tau_2}} \tag{4.642}$$

From relativity theory, we know we can use the rapidity via $\tanh\tau = V/c$ etc. Substituting, we find:

$$\frac{1}{\sqrt{1-\frac{V^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - \cos\phi\frac{v/c}{\sqrt{1-\frac{v^2}{c^2}}}\frac{u/c}{\sqrt{1-\frac{u^2}{c^2}}} \tag{4.643}$$

or

$$\sqrt{1-\frac{V^2}{c^2}} = \frac{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}}{1-\frac{uv}{c^2}\cos\phi} \tag{4.644}$$

Squaring, and solving for the total velocity, we find the addition formula:

$$1-\frac{V^2}{c^2}=\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u^2}{c^2}\right)}{(1-\frac{uv}{c^2}\cos\phi)^2} \tag{4.645}$$

$$\frac{V^2}{c^2}=1-\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u^2}{c^2}\right)}{(1-\frac{uv}{c^2}\cos\phi)^2} \tag{4.646}$$

$$V=c\sqrt{1-\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u^2}{c^2}\right)}{(1-\frac{uv}{c^2}\cos\phi)^2}}=\frac{\sqrt{u^2+v^2-2uv\cos\phi-\left(\frac{uv}{c}\sin\phi\right)^2}}{1-\frac{uv}{c^2}\cos\phi} \tag{4.647}$$

Substituting for the rapidity, we readily obtain the second form:

$$\tanh\tau=\frac{\sqrt{\tanh^2\tau_1+\tanh^2\tau_2-2\tanh\tau_1\tanh\tau_2\cos\phi-(\tanh\tau_1\tanh\tau_2\sin\phi)^2}}{1-\tanh\tau_1\tanh\tau_2\cos\phi} \tag{4.648}$$

Note that for the special case of parallel vectors, we have $\cos\phi=1$ and hence:

$$\tanh\tau=\frac{\sqrt{\tanh^2\tau_1+\tanh^2\tau_2-2\tanh\tau_1\tanh\tau_2}}{1-\tanh\tau_1\tanh\tau_2}=\frac{\tanh\tau_1-\tanh\tau_2}{1-\tanh\tau_1\tanh\tau_2} \tag{4.649}$$

which is the known addition formula for the hyperbolic tangent. □

Remark 72. This is the law of addition of velocities for special relativity, and in particular one can use this formula to perform many calculations of a physical nature. It is straightforward to see how this formula extends to Cartesian velocity vectors $\mathbf{u},\mathbf{v}\in\mathbb{R}^3$. For example, if we take the formula as written for the addition formula, we have:

$$V=\frac{\sqrt{u^2+v^2-2uv\cos\phi-\left(\frac{uv\sin\phi}{c}\right)^2}}{1-\frac{uv\cos\phi}{c^2}} \tag{4.650}$$

Using the classical formulae for vector cross and dot products, we have:

$$(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=|\mathbf{u}|^2+|\mathbf{v}|^2-2|\mathbf{u}||\mathbf{v}|\cos\phi \tag{4.651}$$

and

$$|\mathbf{u}\times\mathbf{v}|=|\mathbf{u}||\mathbf{v}|\sin\phi \tag{4.652}$$

and inserting these into the addition formula, we find:

$$V=\frac{\sqrt{u^2+v^2-2uv\cos\phi-\left(\frac{uv\sin\phi}{c}\right)^2}}{1-\frac{uv\cos\phi}{c^2}} \tag{4.653}$$

or

$$V=\frac{\sqrt{|\mathbf{u}-\mathbf{v}|^2-\frac{1}{c^2}|\mathbf{u}\times\mathbf{v}|^2}}{1-\frac{1}{c^2}\mathbf{u}\cdot\mathbf{v}} \tag{4.654}$$

This is the correct formula for the relative velocity, see Landau-Lifshitz (Vol. 3, pp.36, eq. 12.6) [127]; an old result calculated by Einstein c. 1905, but worth repeating here in a new context. It is apparent that the distance measure and the kernel are inextricably linked together, and it is this correspondence that is implied through the principles of projective geometry. We have shown in this thesis how the hyperbolic distance measure is derived, as an argument of the kernel. On a deeper level, we have shown with the calculations in this chapter how one may derive the hyperbolic distance formula from the pseudounitary operators that follow from the hyperbolic brachistochrone equation 3.3.12, 3.2.10. From this, we are able to find many different results related to the hyperbolic heat equation 4.7.1, and the velocity addition law in hyperbolic form for the rapidity as shown above. Although this is a simplistic model, it is readily extended by the addition of further factors involving non-zero curvature. It is known that the various models of the hyperbolic plane have constant, but non-zero curvature. It is an old result that the only surfaces with constant curvature are the sphere, the plane, and the pseudosphere, which is the SO(2,1) extension of the results explored in this thesis. The plane has zero curvature, and the sphere has positive curvature. We can see that modification of this basic model through changes to the state, as input into the brachistochrone problem 3.2.10, 3.2.7, will result in changes to the differential geometry.

As the following shall show, this is not the only way in which we may link the theory of the hyperbolic brachistochrone, and quantum variant, with the theories of differential geometry and relativity.

4.8.2 General Relativity and the Fubini-Study Metric

Given the previous discussion of the addition formula of special relativity as given by the hyperbolic distance formula, we shall develop some more general considerations of relativity theory, differential geometry and curvature in this segment. The interface of diffusion problems with these deeper questions of analysis are particularly relevant to the development of the science of hyperbolic PDEs. We can understand this link as an idealised ink droplet spreading on a manifold with curvature. The curvature causes it to 'run' at different rates, given by what amounts to the flow of the Ricci tensor, defining the diffusion. Although this thesis does not focus on the differential geometry as developed from the theory of Riemannian geometry, we shall require several basic concepts in order to illustrate the following analysis. This is necessary, in the spirit and ethos of of McKean, also McKean and Singer who in the series of papers [2], [3], [128] analysed the nature of diffusion in curved spaces via analysis of the heat kernel using this approach. This is very much in a similar direction to the investigations of Kac [21], who examined whether one could hear the shape of a drum through analysis of the curvature response profile. The negative answer to this question means that there exist isospectral systems, possibly for any spectrum. The following chapters will develop this concept further, using different formulations of the eigenfunction that defines the continuous spectrum of the system. For now, we take the following definitions from differential geometry for the various tensors that depend on the metric.

Notation 4. We assume Einstein summation notation for this subsection only, and shall write sums only when necessary for emphasis, with sums evaluated over repeated indices and differentiation indicated by the comma ','.

Definition 4.8.2. In local co-ordinates q^i, the metric g_{ij} is expressed by the relation:

$$g_{ij} = \frac{\partial x^k}{\partial q^i}\frac{\partial x^k}{\partial q^j} \tag{4.655}$$

and satisfies the inversion relation:

$$g_{ik}g^{kj} = \delta_k^j \tag{4.656}$$

where δ_k^j is the Kronecker delta and the metric inverts by raising of indices $[g_{ij}]^{-1} = g^{ij}$. The metric may be calculated through use of the expansion of the interval via $ds^2 = g_{ij}dq^i dq^j$.

Definition 4.8.3. The Christoffel symbols are given by the derivatives of the metric:

$$\Gamma^l_{jk} = \frac{1}{2}g^{lr}(g_{rj,k} + g_{rk,j} - g_{jk,r}) \tag{4.657}$$

where $g_{rj,k}$ indicates differentiation with respect to the parameter q^k. The Christoffel symbols define the covariant derivative (';'):

$$V^\alpha_{;\beta} = V^\alpha_{,\beta} + v^j\Gamma^k_{\beta j}V^\alpha_k \tag{4.658}$$

where v^j is tangent to V^α.

Definition 4.8.4. The Riemann curvature tensor is the combination of the Christoffel symbols:

$$R^\rho_{\sigma\mu\nu} = \Gamma^\rho_{\nu\sigma,\mu} - \Gamma^\rho_{\mu\sigma,\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{4.659}$$

The contractions of the Riemann tensor are given by traces over the repeated indices.

Definition 4.8.5. The Ricci curvature tensor is the contraction of the Riemann tensor:

$$R_{ij} = R^k_{ikj} \tag{4.660}$$

or:

$$R_{ij} = R^k_{ikj} = \Gamma^k_{ji,k} - \Gamma^k_{ki,j} + \Gamma^k_{k\lambda}\Gamma^\lambda_{ji} - \Gamma^k_{j\lambda}\Gamma^\lambda_{ki} \tag{4.661}$$

The trace of the Ricci tensor is the scalar curvature, or Ricci scalar:

$$R = g^{ij}R_{ij} \tag{4.662}$$

This is given by the formula:

$$R = g^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{k\lambda}\Gamma^\lambda_{ij} - \Gamma^k_{j\lambda}\Gamma^\lambda_{ki}\right) \tag{4.663}$$

Remark 73. The Ricci scalar is half the Gaussian curvature, for a surface. Note that the Christoffel symbol has symmetries $\Gamma^k_{ij} = \Gamma^k_{ji}$.

On a surface, we have the following result.

Lemma 4.8.6. *A surface is defined by the two-dimensional parametrisation:*

$$R_{abcd} = K(g_{ac}g_{db} - g_{ad}g_{bc}) \tag{4.664}$$

where R_{abcd} is the Riemann tensor 4.8.4, g_{ab} the metric tensor, and the curvature K is given by $K = R/2$, where R is the Ricci scalar 4.8.5. Further, the Ricci tensor R_{bd} on a surface is proportional to the metric:

$$R_{bd} = \frac{R}{2}g_{bd} \tag{4.665}$$

where R is the Ricci scalar.

Proof. Assume the form as given in the lemma for the curvature tensor, we have:

$$R_{abcd} = \frac{R}{2}(g_{ac}g_{db} - g_{ad}g_{bc}) \tag{4.666}$$

Raising the index with the contravariant metric, then contracting twice we have:

$$R^a_{bcd} = g^{ai}R_{ibcd} = \frac{R}{2}g^{ai}(g_{ic}g_{db} - g_{id}g_{bc}) \tag{4.667}$$

$$R^k_{bkd} = \frac{R}{2}g^{ki}(g_{ik}g_{db} - g_{id}g_{bk}) \tag{4.668}$$

$$R_{bd} = \frac{R}{2}g^{ki}g_{ik}g_{db} - \frac{R}{2}g^{ki}g_{id}g_{bk} \tag{4.669}$$

$$= \frac{R}{2}g^{ik}g_{ik}g_{db} - \frac{R}{2}g^{ik}g_{id}g_{bk} \tag{4.670}$$

$$= \frac{R}{2}\delta^k_k g_{db} - \frac{R}{2}\delta^k_d g_{bk} \tag{4.671}$$

For a surface, we know that we have $\delta^k_k = 2$, hence we find:

$$R_{bd} = \frac{R}{2}g_{db} = \frac{R}{2}g_{bd} \tag{4.672}$$

as required. □

Lemma 4.8.7. *On a surface, the metric is given by the fundamental form:*

$$g_{ij} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \tag{4.673}$$

There exists a transformation $\hat{X}$ such that the metric is diagonal in the co-ordinates $\mathbf{y} = \hat{X}^{-1}\mathbf{x}$ with $g_{ij} = \hat{X}^{-1}\tilde{g}_{ij}\hat{X}$ such that:

$$\tilde{g}_{ij} = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} \tag{4.674}$$

where $\lambda_\pm$ are eigenvalues of the metric:

$$\lambda_\pm = \frac{1}{2}\left(G + E \pm \sqrt{G^2 + E^2 + (4F^2 - 2EG)}\right) = \frac{1}{2}(G + E \pm \Delta) \tag{4.675}$$

Proof. We can construct the matrix of eigenvectors explicitly:

$$\hat{X} = \begin{bmatrix} -\dfrac{F}{\frac{1}{2}(E - \Delta - G)} & \dfrac{F}{\frac{1}{2}(E + \Delta - G)} \\ 1 & 1 \end{bmatrix} \tag{4.676}$$

$$\hat{X}^{-1} = \frac{1}{\Delta}\begin{bmatrix} -\dfrac{(-G + E - \Delta)(-G + E + \Delta)}{4F} & -\dfrac{1}{2}(-G + E - \Delta) \\ \dfrac{(-G + E - \Delta)(-G + E + \Delta)}{4F} & \dfrac{1}{2}(-G + E + \Delta) \end{bmatrix} \tag{4.677}$$

where $\Delta = \sqrt{G^2 + E^2 + (4F^2 - 2EG)}$. Matrix multiplication gives $\hat{X}\hat{X}^{-1} = \hat{X}^{-1}\hat{X} = \mathbf{1}$, and hence the result follows. □

We now establish the following properties of Einstein tensors as required for the following calculations.

Lemma 4.8.8. *An Einstein tensor satisfies the spherical decomposition of the Ricci tensor:*

$$E_{ij} = R_{ij} - \frac{1}{n} R g_{ij} \tag{4.678}$$

where R_{ij} is the Ricci tensor, $g_{ii} = n$ is the dimension of the space, as given by the trace of the metric, and R is the Ricci scalar. We have $E_{ii} = 0$, and further any system where the Einstein tensor is zero, which we call Einstein, has a Ricci tensor proportional to the metric, given by $R_{ij} = \lambda g_{ij}$.

Proof. The spherical decomposition of any matrix is given through:

$$\tilde{F}_{ij} = F_{ij} - \frac{\hat{\mathbf{1}}}{n} \mathrm{Tr}\left[F_{ij}\right] \tag{4.679}$$

where we have the identity matrix:

$$\hat{\mathbf{1}} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & 1 & & \ddots & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & \ddots & & 1 & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \tag{4.680}$$

and by inspection $\mathrm{Tr}\hat{\mathbf{1}} = n$, and hence $\mathrm{Tr}\tilde{F}_{ij} = 0$. We therefore construct the Einstein tensor:

$$E_{ij} = R_{ij} - \frac{1}{n} R g_{ij} \tag{4.681}$$

where in the metric space, we treat the identity as the metric tensor g_{ij}. This tensor is tracefree, as we have by definition $g_{ii} = n$ and therefore $E_{ii} = R_{ii} - R = 0$. If the Einstein tensor is zero, we have by construction the identity for the Ricci tensor:

$$E_{ij} = 0 \Rightarrow R_{ij} = \frac{1}{n} R g_{ij} \tag{4.682}$$

implying $R_{ij} = \lambda g_{ij}$. □

We now derive some basic lemmas in differential geometry that we shall require to establish claims in the following. We establish a further result for the Ricci tensor using tensor calculus.

Lemma 4.8.9. *The Ricci tensor has the symmetry property:*

$$R_{ij} = -g^{kl} R_{klij} \tag{4.683}$$

where g^{kl} is the inverse metric (contravariant) and R_{klij} is the Riemann curvature tensor. This is a trace form of the Ricci tensor.

Proof. From the definition of the Ricci tensor, we have:

$$R_{ij} = g^{kl} R_{ikjl} \tag{4.684}$$

We also have the symmetries of the Riemann tensor $R_{ikjl} = R_{ljki}$ and $R_{lkji} = -R_{klij}$. Writing the Ricci tensor, we have:

$$R_{ij} = g^{kl} R_{ikjl} \tag{4.685}$$

and using the symmetries of the Ricci tensor $R_{ij} = R_{ji}$, and we may write:

$$R_{ij} = R^c_{cij} = R^c_{cij} \Rightarrow R_{ij} = g^{cd} R_{icjd} = g^{kl} R_{ikjl} \tag{4.686}$$

$$= g^{kl} R_{lkji} = -g^{kl} R_{klij} \tag{4.687}$$

or

$$R_{ij} = -g^{kl} R_{klij} \tag{4.688}$$

as required. □

Furthermore, we have:

Lemma 4.8.10. *The Ricci tensor satisfies the formula (Kähler potential formula):*

$$R_{ij} = -\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} (\ln \det g_{ij}) = -\frac{\partial}{\partial x^j} \left(g^{kl} \frac{\partial}{\partial x^i} g_{kl} \right) \tag{4.689}$$

where g_{kl} is the metric, Einstein summation notation implied.

Proof. We have the definition of the Christoffel symbol 4.8.3:

$$\Gamma^l_{jk} = \frac{1}{2} g^{lr} (g_{rj,k} + g_{rk,j} - g_{jk,r}) \tag{4.690}$$

and hence we may write:

$$\Gamma^p_{ip} = \frac{1}{2} g^{pr} (g_{ri,p} + g_{rp,i} - g_{ip,r}) = \frac{1}{2} g^{pr} g_{rp,i} = \frac{1}{2} g^{pr} g_{pr,i} \tag{4.691}$$

where we used $g_{ri,p} = g_{ir,p} = g_{ip,r}$ as this is reference-frame free and does not depend on the co-ordinate indices. We find for the Riemann tensor:

$$R^p_{pji} = \Gamma^p_{ip,j} + \Gamma^p_{jp,i} + \Gamma^p_{j\lambda} \Gamma^\lambda_{ip} - \Gamma^p_{i\lambda} \Gamma^\lambda_{jp} \tag{4.692}$$

$$= 2\Gamma^p_{ip,j} \tag{4.693}$$

where we used $\Gamma^p_{j\lambda} \Gamma^\lambda_{ip} = \Gamma^p_{i\lambda} \Gamma^\lambda_{jp}$ and $\Gamma^p_{ip,j} = \Gamma^p_{jp,i}$, where again the use of Einstein summation notation enables us to conclude these sums are simplified in this way. On the other hand, we have the matrix identity:

$$\ln \det \hat{A} = \mathrm{Tr} \left[\hat{A} \right] \tag{4.694}$$

and hence:

$$\frac{\partial}{\partial x^i} \mathrm{Tr} \left[\hat{A} \right] = \hat{A}^{-1} \frac{\partial}{\partial x^i} \hat{A} \tag{4.695}$$

which may be rewritten as:

$$\frac{\partial}{\partial x^i} \ln \det \hat{A} = \frac{\partial}{\partial x^i} \mathrm{Tr} \left[\hat{A} \right] = \hat{A}^{-1} \frac{\partial}{\partial x^i} \hat{A} \tag{4.696}$$

Applying this to the metric, we may write:

$$\frac{\partial}{\partial x^i} \ln \det g_{kl} = [g_{kl}]^{-1} \frac{\partial}{\partial x^i} g_{kl} \tag{4.697}$$

$$= g^{kl} \frac{\partial}{\partial x^i} g_{kl} \tag{4.698}$$

We therefore can compute the formula for the Ricci tensor, finding:

$$R_{ij} = -\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}(\ln \det g_{ij}) = -\frac{\partial}{\partial x^j}\left(g^{kl}\frac{\partial}{\partial x^i}g_{kl}\right) \tag{4.699}$$

$$= -\frac{1}{2}\partial_j\Gamma^p_{ip} = -\frac{1}{2}\partial_j\Gamma^p_{pi} = -\frac{1}{2}.2R^p_{pji} \tag{4.700}$$

$$= -R^p_{pji} = R_{ij} \tag{4.701}$$

as required, demonstrating that these definitions are equivalent. We have used the symmetry of the Riemann curvature tensor via $R_{ij} = -g^{kl}R_{klij} = R^l_{ilj} = -R^l_{lij}$. □

Lemma 4.8.11. *Any system which is Einstein-Kähler has Ricci tensor R_{ij} proportional to the metric:*

$$R_{ij} = \lambda g_{ij} \tag{4.702}$$

where λ is a constant. The Fubini-Study metric is Einstein-Kähler.

Proof. For the following, we consider the augmented co-ordinates which carry the state vector, given by $\mathbf{Z}$, with adjoint state vector $\bar{\mathbf{Z}}$ into the vector with augmented state 1, which describes the fact that in the complex projective space, as considered throughout this thesis, the radius of the state is constrained to unity, as per the usual system of normalisation. We have:

$$\sum_{k=0}^{n}|z_k|^2 = \bar{\mathbf{Z}}\mathbf{Z} = 1, \mathbf{Z} \to \begin{bmatrix} z_k \\ 1 \end{bmatrix} = \begin{bmatrix} (n-1) \\ (1) \end{bmatrix} \tag{4.703}$$

and the basic relation:

$$1 + \sum_{k=1}^{n}|z_k|^2 = \sum_{k=0}^{n}|z_k|^2 \tag{4.704}$$

by construction. We may write the Kähler form of the potential as:

$$K = \ln\left(\sum_{k=0}^{n}|z_k|^2\right) = \ln\left(1 + \sum_{k=1}^{n}|z_k|^2\right) \tag{4.705}$$

where the Einstein-Kähler metric is defined as:

$$g_{ij} = \frac{\partial^2 K}{\partial z_i \partial \bar{z}_j} \tag{4.706}$$

Calculating, we have $K = \ln(1 + \sum_{k=1}^{n}\bar{z}_k z_k)$ and hence the potential is readily evaluated as:

$$\frac{\partial K}{\partial \bar{z}_j} = \frac{1}{1+\sum_{k=1}^{n}\bar{z}_k z_k}\frac{\partial}{\partial \bar{z}_j}\left(1 + \sum_{k=1}^{n}\bar{z}_k z_k\right) = \frac{\delta_{jk}z_k}{1+\sum_{k=1}^{n}\bar{z}_k z_k} \tag{4.707}$$

$$\frac{\partial^2 K}{\partial z_i \partial \bar{z}_j} = \frac{\partial}{\partial z_i}\left(\frac{\delta_{jk}z_k}{1+\sum_{k=1}^{n}\bar{z}_k z_k}\right) \tag{4.708}$$

$$= -\frac{1}{(1+\sum_{k=1}^{n}\bar{z}_k z_k)^2}\delta_{jk}z_k\delta_{ik'}\bar{z}_{k'} + \frac{\delta_{jk}\delta_{ik}}{1+\sum_{k=1}^{n}\bar{z}_k z_k} \tag{4.709}$$

Employing the identities for the Kronecker delta, and for the inner product, we have:

$$\delta_{jk}z_k\delta_{ik'}\bar{z}_{k'} = z_j\bar{z}_i = z_i\bar{z}_j \tag{4.710}$$

$$\delta_{jk}\delta_{ik} = \delta_{jk}\delta_{ki} = \delta_{ji} = \delta_{ij} \tag{4.711}$$

which combines to give the relationship:

$$\frac{\partial^2 K}{\partial z_i \partial \bar{z}_j} = -\frac{z_i \bar{z}_j}{\left(1+\sum_{k=1}^n \bar{z}_k z_k\right)^2} + \frac{\delta_{ij}}{1+\sum_{k=1}^n \bar{z}_k z_k} \tag{4.712}$$

$$= \frac{1}{\left(1+\sum_{k=1}^n \bar{z}_k z_k\right)^2}\left[\delta_{ij}\left(1+\sum_{k=1}^n \bar{z}_k z_k\right) - z_i \bar{z}_j\right] \tag{4.713}$$

and the metric as specified by:

$$g_{ij} = \frac{1}{\left(1+\sum_{k=1}^n \bar{z}_k z_k\right)^2}\left[\delta_{ij}\left(1+\sum_{k=1}^n \bar{z}_k z_k\right) - z_i \bar{z}_j\right] \tag{4.714}$$

To evaluate the Ricci tensor, we write it in Kähler form as:

$$R_{ij} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\left(\ln \det g_{ij}\right) \tag{4.715}$$

and employing the expression for the determinant, we have:

$$\det g_{ij} = \det\left(\frac{1}{\left(1+\sum_{k=1}^n \bar{z}_k z_k\right)^2}\left[\delta_{ij}\left(1+\sum_{k=1}^n \bar{z}_k z_k\right) - z_i \bar{z}_j\right]\right) \tag{4.716}$$

or

$$= \det\left(\hat{A} - |u\rangle\langle v|\right) = \det\left(\hat{A}\right)\det\left(\hat{\mathbf{1}} - \langle v|\hat{A}^{-1}|u\rangle\right) \tag{4.717}$$

$$= \det\left(\hat{A}\right)\left(1 - \langle v|\hat{A}^{-1}|u\rangle\right) \tag{4.718}$$

where the explicit form of the operators in bra-ket notation satisfies:

$$\hat{A} = \frac{\delta_{ij}\left(1+\sum_{k=1}^n \bar{z}_k z_k\right)}{\left(1+\sum_{k=1}^n \bar{z}_k z_k\right)^2} = \frac{\delta_{ij}}{\left(1+\sum_{k=1}^n \bar{z}_k z_k\right)} \tag{4.719}$$

and

$$|u\rangle\langle v| = \frac{z_i \bar{z}_j}{\left(1+\sum_{k=1}^n \bar{z}_k z_k\right)^2} \tag{4.720}$$

To evaluate the trace of the matrix $\hat{A}$, we write it as a scalar multiple of identity, finding:

$$\hat{A} = \lambda\hat{\mathbf{1}} \Rightarrow \det\left(\hat{A}\right) = \lambda^n = \frac{1}{\left(1+\sum_{k=1}^n \bar{z}_k z_k\right)^n} \tag{4.721}$$

For the inner product, the inverse matrix of $\hat{A}$ is evaluated using basic linear algebra:

$$\langle v|\hat{A}^{-1}|u\rangle = \langle v|\hat{A}^{-1}|u\rangle \tag{4.722}$$

and we have:

$$\hat{A}^{-1} = \delta_{ij}\left(1+\sum_{k=1}^n \bar{z}_k z_k\right) \tag{4.723}$$

and consequently we may write:

$$\langle v|\hat{A}^{-1}|u\rangle = \langle v|\delta_{ij}\left(1+\sum_{k=1}^n \bar{z}_k z_k\right)|u\rangle \tag{4.724}$$

$$= \left(1+\sum_{k=1}^n \bar{z}_k z_k\right)\langle v|\delta_{ij}|u\rangle = \left(1+\sum_{k=1}^n \bar{z}_k z_k\right)\langle v|\, u\rangle \tag{4.725}$$

which simplifies to:

$$\langle v|\, u\rangle = \frac{\sum_{k=1}^{n} z_k \bar{z}_k}{\left(1+\sum_{k=1}^{n} \bar{z}_k z_k\right)^2} \tag{4.726}$$

or

$$\langle v|\, \hat{A}^{-1}\, |u\rangle = \frac{\sum_{k=1}^{n} z_k \bar{z}_k}{\left(1+\sum_{k=1}^{n} \bar{z}_k z_k\right)} \tag{4.727}$$

Evaluating the determinant using the determinant-trace identity, we then may write the determinant of the metric as:

$$\det g_{ij} = \det\left(\hat{A}\right)\left(1 - \langle v|\, \hat{A}^{-1}\, |u\rangle\right) \tag{4.728}$$

$$= \frac{1}{\left(1+\sum_{k=1}^{n} \bar{z}_k z_k\right)^n}\left(1 - \frac{\sum_{k=1}^{n} z_k \bar{z}_k}{\left(1+\sum_{k=1}^{n} \bar{z}_k z_k\right)}\right) \tag{4.729}$$

$$= \frac{1}{\left(1+\sum_{k=1}^{n} \bar{z}_k z_k\right)^n}\left(\frac{\left(1+\sum_{k=1}^{n} \bar{z}_k z_k\right) - \sum_{k=1}^{n} z_k \bar{z}_k}{\left(1+\sum_{k=1}^{n} \bar{z}_k z_k\right)}\right) \tag{4.730}$$

$$= \frac{1}{\left(1+\sum_{k=1}^{n} \bar{z}_k z_k\right)^{n+1}} \tag{4.731}$$

Calculating the trace is now possible, we find:

$$\ln\det g_{ij} = -(n+1)\ln\left(1+\sum_{k=1}^{n} \bar{z}_k z_k\right) \tag{4.732}$$

and the Ricci tensor as given by:

$$R_{ij} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\left(\ln\det g_{ij}\right) = -(n+1)\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\left[\ln\left(1+\sum_{k=1}^{n} \bar{z}_k z_k\right)\right] \tag{4.733}$$

$$= (n+1) g_{ij} \tag{4.734}$$

where we have employed the relationship derived in the first part of this proof. This establishes the claim in the lemma. □

Lemma 4.8.12. *Hamilton's theory of Ricci flow (1982) [129], [130], [131] may be defined for a surface, where we have:*

$$\frac{\partial}{\partial t} g_{bd}(t) = 2R_{bd}(g(0)) \tag{4.735}$$

where g_{ab} is the metric, $R_{bd}(g)$ is the Ricci tensor as in 4.8.5. For the case of an Einstein metric, the Ricci flow takes the form:

$$\frac{\partial}{\partial t} g_{bd}(t) = \lambda g_{bd} \tag{4.736}$$

where λ is a constant.

Remark 74. Note that the sign of time t in the equation above depends on the orientation of the surface and the sign of λ. For convenience, we take the case as defined above in the lemma.

Proof. This is a result of the property of an Einstein-Kähler metric. If we assume the Ricci flow form of the metric tensor, we have:

$$g_{bd}(t) = (1 + \lambda t) g_{bd} \tag{4.737}$$

which implies:

$$\frac{\partial}{\partial t} g_{bd}(t) = \lambda g_{bd} \tag{4.738}$$

and $g_{bd}(0) = g_{bd}$. Further, if the metric is Einstein, in which case we have $R_{bd}(g) = \lambda g_{bd}$ then:

$$\frac{\partial}{\partial t} g_{bd}(t) = \lambda g_{bd} = R_{bd}(g) \tag{4.739}$$

and the result follows. On a surface, we have already established that:

$$R_{bd} = \frac{R}{2} g_{bd} \tag{4.740}$$

which yields an identical result, with appropriate modification of the constant λ, where R is the Ricci scalar curvature. Note that the restriction that $\lambda = (n+1)$ for $n = 0, 1, 2, ..$ is equivalent to the restriction that the metric be not only Einstein, but Einstein-Kähler. □

We recall the complementary definition of the Ricci tensor using Lie theory.

Definition 4.8.13. The Ricci tensor may be expressed in terms of the Lie differentials via:

$$R_{ab} = \nabla_a \nabla_b - \nabla_b \nabla_a - \nabla_{[a,b]} \tag{4.741}$$

where ∇ is the Lie derivative $\nabla_X f = \sum_i X_i \frac{\partial f}{\partial x_i}$, and $[a, b] = ab - ba$ is the Lie commutator. It is known that this definition via the action of the curvature tensor is equivalent to the definition 4.8.5, via the Levi-Civita connection.

We state a result which may be found in [48].

Theorem 4.8.14. *Canzani [48] gives the following perturbative formula for the heat kernel $K(x, y; t)$ on a Riemannian manifold:*

$$K(x, y; t) = \frac{e^{-d^2(x,y)/4t}}{(4\pi t)^{n/2}} \left(\sum_{j=0}^{k} t^j u_j(x, y) + \mathcal{O}(t^{k+1}) \right) \tag{4.742}$$

where

$$u_0(x, x) = 1 \tag{4.743}$$

$$u_1(x, x) = \frac{1}{6} R_{ii}(x, x) \tag{4.744}$$

and $R_{ij}(x, x)$ is the Ricci tensor as defined in 4.8.13, $d(x, y)$ the distance function on the manifold.

Proof. See e.g. Canzani [48] for the proof. An earlier analysis may be found in the work of McKean and Singer [2], pp.51 eq. 4.4 for an earlier derivation of a similar formula. The question of the form of the fundamental solution as a function of the topological properties of the system has a long history dating back to previous investigations by Kac [21], where similar results can be found. □

We shall also take the following result from McKean and Singer [2], which expresses the small change limit of the metric under the influence of curvature effects.

Theorem 4.8.15. *In the presence of curvature, given by the Ricci tensor 4.8.13, and generalised appropriately to the Riemann tensor, the metric may be written as the small curvature expansion:*

$$g_{ij} = \delta_{ij} - \frac{1}{3}R_{ijkl}x^k x^l + ... \tag{4.745}$$

where g_{ij} is the Minkowski metric, R_{ijkl} the Riemann tensor. The volume measure for small curvature is approximated by:

$$dV_g = \left[1 - \frac{1}{6}R_{ij}x^k x^l + ...\right] dV_0 = \sqrt{\det g_{ij}} dV_0 \tag{4.746}$$

Proof. We take the expansion for the metric as derived in McKean and Singer [2]. Contracting the tensor, using $\det \delta_{ij} = 1$, $\det R_{ijkl} = R_{kl}$, i.e. the contraction of the Riemann tensor is the Ricci tensor, we obtain:

$$\sqrt{\det g_{ij}} = \left(1 - \frac{1}{3}R_{kl}x^k x^l\right)^{1/2} = 1 - \frac{1}{6}R_{kl}x^k x^l + ... \tag{4.747}$$

From the expression for the invariant volume 3.2.34 we have:

$$dV_g = \sqrt{\det g_{ij}} dV_0 \tag{4.748}$$

Substituting the expression for the square root of the determinant, we obtain the result. □

We shall now show how the Fubini-Study metric acts in a novel way which renders the expressions of geometrodynamics and differential geometry solvable in a particular simple fashion. We assert the following:

Theorem 4.8.16. *The Fubini-Study metric is proportional to the Ricci tensor:*

$$R_{\alpha\beta} = \Lambda_0 g_{\alpha\beta} \tag{4.749}$$

as proven in 4.8.11, where Λ_0 is a real valued constant of proportionality. Under this condition, the Einstein tensor $G_{\alpha\beta}$ is proportional to the metric, as is the stress tensor.

Proof. Assume the metric is proportional to the Ricci tensor as stated, we have proved this for the complex projective space in 4.8.11. We have then:

$$R_{\alpha\beta} = \Lambda_0 g_{\alpha\beta} \tag{4.750}$$

The Einstein tensor is the spherical decomposition of the Ricci tensor, as in 4.8.13. We may write:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \Lambda_0 g_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} \tag{4.751}$$

$$= (\Lambda_0 - \frac{1}{2}R)g_{\alpha\beta} \tag{4.752}$$

The Einstein equations of general relativity are written:

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta} \tag{4.753}$$

and hence

$$(\Lambda_0 + \Lambda - \frac{1}{2}R)g_{\alpha\beta} = \kappa T_{\alpha\beta} \tag{4.754}$$

i.e. the stress tensor is proportional to the metric, with an adjustment for curvature, thereby proving the statement. □

Remark 75. We direct the reader to [132] for further discussion of the equivalence between Fubini-Study and Einstein metrics.

We shall now list some results, derived using the Fubini-Study metric and the theory of fundamental forms and surfaces.

Theorem 4.8.17. *Gauss's theorems for the first and second fundamental forms may be written:*

$$ds^2 = g_{ij}dx^i dx^j = \langle d\mathbf{x}, \mathrm{I}(\mathbf{x},\mathbf{x})d\mathbf{x}\rangle \tag{4.755}$$

and

$$\langle d\mathbf{x}, \Pi(\mathbf{x},\mathbf{x})d\mathbf{x}\rangle = d\mathbf{x}^{\dagger}\Pi(\mathbf{x},\mathbf{x})d\mathbf{x} = \langle d\mathbf{x}, (\mathbf{n}\cdot H)\, d\mathbf{x}\rangle \tag{4.756}$$

where $\mathbf{n} = \dfrac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}$ *is the normal vector, H is the Hessian matrix of second order derivatives, the metric from the first fundamental form given by:*

$$g_{ij} = \mathrm{I}(\mathbf{x},\mathbf{x}) \tag{4.757}$$

The second fundamental form induces a metric which we write as $\Pi(\mathbf{x},\mathbf{x})$. *The Fubini-Study metric for a system described by the state vector:*

$$g_{ij} = \Re\left(F_{ij}\right) \tag{4.758}$$

where the Fubini-Study tensor is given through $F_{ij} = \left\langle \bar{\Psi}_i \left| \left(\hat{\mathbf{1}} - \hat{P}\right) \right| \Psi_j \right\rangle$*, the differentials evaluated as:*

$$|\Psi_j\rangle = \frac{\partial}{\partial x^j}|\Psi\rangle \tag{4.759}$$

where $\hat{P} = \left|\Psi\right\rangle\left\langle\bar{\Psi}\right|$ *is the projection operator on the state, satisfying* $\mathrm{Tr}\hat{P} = 1$*, and the state satisfies the inner product relation:*

$$\left\langle \bar{\Psi} \mid \Psi \right\rangle = \langle \Psi, \Psi\rangle \tag{4.760}$$

where the adjoint state is given through the bra-vector $\left\langle\bar{\Psi}\right|$.

Example 4.8.18. For the hyperbolic system described through the parametrisation:

$$\mathbf{x}(s) = \begin{bmatrix} \cosh u \sin v \\ \cosh u \cos v \\ \sinh u \end{bmatrix} \tag{4.761}$$

with inner product law:

$$\mathbf{x}\cdot\mathbf{y} = \langle \mathbf{x},\mathbf{y}\rangle = x_1y_1 + x_2y_2 - x_3y_3 \tag{4.762}$$

This system has first and second fundamental forms equal to the Fubini-Study metric.

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \cosh^2 u \end{bmatrix} \tag{4.763}$$

Proof. We shall show how in this space, all three definitions of the metric will agree. If we take the formula for the first fundamental form, we have:

$$ds^2 = \langle d\mathbf{x}, d\mathbf{x}\rangle = d\mathbf{x}\cdot d\mathbf{x} = (\mathbf{x}_u\cdot\mathbf{x}_u)du^2 + 2(\mathbf{x}_u\cdot\mathbf{x}_v)dudv - (\mathbf{x}_v\cdot\mathbf{x}_v)dv^2 \tag{4.764}$$

where we have taken the inner product inside the definition of the dot product, as in the example. We then have the components of the first fundamental form:

$$\mathbf{x}_u\cdot\mathbf{x}_u = \begin{bmatrix} \sinh u \sin v \\ \sinh u \cos v \\ \cosh u \end{bmatrix} \cdot \begin{bmatrix} \sinh u \sin v \\ \sinh u \cos v \\ \cosh u \end{bmatrix} \tag{4.765}$$

$$= \sinh^2 u - \cosh^2 u = -1 \tag{4.766}$$

$$\mathbf{x}_u \cdot \mathbf{x}_v = \begin{bmatrix} \sinh u \sin v \\ \sinh u \cos v \\ \cosh u \end{bmatrix} \cdot \begin{bmatrix} \cosh u \cos v \\ -\cosh u \sin v \\ 0 \end{bmatrix} = 0 \tag{4.767}$$

and

$$\mathbf{x}_v \cdot \mathbf{x}_v = \begin{bmatrix} \cosh u \cos v \\ -\cosh u \sin v \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \cosh u \cos v \\ -\cosh u \sin v \\ 0 \end{bmatrix} = \cosh^2 u \tag{4.768}$$

The interval of the first fundamental form may then be written:

$$ds^2 = -du^2 - \cosh^2 u dv^2 \tag{4.769}$$

which means that the metric from the first fundamental form is given through:

$$g_{ij} = -\begin{bmatrix} 1 & 0 \\ 0 & \cosh^2 u \end{bmatrix} \tag{4.770}$$

For the second fundamental form, we have the normal vector to the surface:

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} \tag{4.771}$$

where the norm is defined through the inner product:

$$|\mathbf{y}| = \sqrt{\mathbf{y} \cdot \mathbf{y}} = \sqrt{y_1^2 + y_2^2 - y_3^2} \tag{4.772}$$

In this space, the second fundamental form may be written using the inner product as:

$$\langle d\mathbf{x}, (\mathbf{n} \bullet H)\, d\mathbf{x} \rangle = \mathbf{n} \bullet \mathbf{x}_{uu} du^2 + 2\mathbf{n} \bullet \mathbf{x}_{uv} du dv - \mathbf{n} \bullet \mathbf{x}_{uu} dv^2 \tag{4.773}$$

where the large dot indicates the ordinary scalar product $\mathbf{a} \bullet \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$. In this case, the norm of the cross product is readily evaluated:

$$|\mathbf{x}_u \times \mathbf{x}_v| = \frac{1}{\sqrt{2\cosh^2 u - 1}} \tag{4.774}$$

and the normal vector to the surface:

$$\mathbf{n} = \begin{bmatrix} \cosh u \sin v \\ \cosh u \cos v \\ -\sinh u \end{bmatrix} \tag{4.775}$$

Evaluating the second fundamental form, we have the relations for the components $\mathbf{n} \bullet \mathbf{x}_{uu} = 1$, $\mathbf{n} \bullet \mathbf{x}_{uv} = 0$ and $\mathbf{n} \bullet \mathbf{x}_{vv} = -\cosh^2 u$. Substituting into the equation for the second fundamental form, we obtain:

$$\langle d\mathbf{x}, (\mathbf{n} \bullet H)\, d\mathbf{x} \rangle = du^2 + \cosh^2 u dv^2 \tag{4.776}$$

which implies the same metric for the system, up to sign. For the final part of the proof, we have the state vector equal to the parametrisation:

$$|\Psi\rangle = \begin{bmatrix} \cosh u \sin v \\ \cosh u \cos v \\ \sinh u \end{bmatrix} \tag{4.777}$$

The derivatives are evaluated using straightforward calculus:

$$|\Psi_u\rangle = \begin{bmatrix} \sinh u \sin v \\ \sinh u \cos v \\ \cosh u \end{bmatrix} \tag{4.778}$$

$$|\Psi_v\rangle = \begin{bmatrix} \cosh u \cos v \\ -\cosh u \sin v \\ 0 \end{bmatrix} \tag{4.779}$$

In this space, the adjoint vector is given through the relation:

$$\left\langle \bar{\Psi}(u,v) \right| = (|\Psi(-u,v)\rangle)^T \tag{4.780}$$

where we have:

$$\left\langle \bar{\Psi}(u,v) \right| \Psi(u,v)\rangle = 1 \tag{4.781}$$

In this case, the inner product as given through the bra-ket is defined as the ordinary dot product, with the adjoint taking the sign explicitly inside the vector. We have the adjoints of the derivatives of the state:

$$\left\langle \bar{\Psi}_u \right| = [-\sinh u \sin v, -\sinh u \cos v, \cosh u] \tag{4.782}$$

$$\left\langle \bar{\Psi}_v \right| = [\cosh u \cos v, -\cosh u \sin v, 0] \tag{4.783}$$

and the projection matrix:

$$\hat{P} = \left|\Psi\right\rangle\left\langle \bar{\Psi}\right| = \begin{bmatrix} \cosh^2 u \sin^2 v & \cosh^2 u \sin v \cos v & -\cosh u \sin v \sinh u \\ \cosh^2 u \sin v \cos v & \cosh^2 u \cos^2 v & -\cosh u \cos v \sinh u \\ \cosh u \sin v \sinh u & \cosh u \cos v \sinh u & -\sinh^2 u \end{bmatrix} \tag{4.784}$$

which satisfies $\mathrm{Tr}\hat{P} = 1$. Evaluating the Fubini-Study metric using the parts we have calculated, we find:

$$F_{ij} = g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \cosh^2 u \end{bmatrix} \tag{4.785}$$

Note that the third method, using the Fubini-Study metric, has the benefit of requiring specification of the correct inner product as part of the mechanics of producing the projection matrix, with trace unity. This avoids the complications of different inner products as required for either the first and second fundamental forms. □

Remark 76. Although this result is unlikely to be general, for various reasons related to the curvature which we shall discuss in the following, we are able to show a number of other related hyperbolic and spherical systems which share the same property as in the previous example.

The remainder of the following examples will be displayed with summary notes of the proof for brevity, as the calculation proceeds in an identical fashion to above.

Example 4.8.19. For the hyperboloid with parametrisation:

$$\mathbf{x}(s) = \begin{bmatrix} \cosh u \cos v \\ \cosh u \sin v \\ \sinh u \end{bmatrix} \tag{4.786}$$

with inner product $\mathbf{x} \cdot \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$ the metric is

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \cosh^2 u \end{bmatrix} \tag{4.787}$$

Other than a change of sign, the first and second fundamental forms are identical with the Fubini-Study metric.

Example 4.8.20. For the hyperboloid with parametrisation:

$$\mathbf{x}(s) = \begin{bmatrix} \cosh u \cos v \\ \sinh u \cos v \\ \sin v \end{bmatrix} \tag{4.788}$$

with inner product $\mathbf{x} \cdot \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_2 y_2 + x_3 y_3$ the metric is

$$g_{ij} = \begin{bmatrix} \cos^2 v & 0 \\ 0 & 1 \end{bmatrix} \tag{4.789}$$

Other than a change of sign, the first and second fundamental forms are identical with the Fubini-Study metric. The adjoint state is again specified through $\langle \bar{\Psi}(u,v) \rvert = (\lvert \Psi(-u,v) \rangle)^T$.

Example 4.8.21. For the hyperboloid with parametrisation:

$$\mathbf{x}(s) = \begin{bmatrix} \sinh u \cos v \\ \sinh u \sin v \\ \cosh u \end{bmatrix} \tag{4.790}$$

with inner product $\mathbf{x} \cdot \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle = -x_1 y_1 - x_2 y_2 + x_3 y_3$ the metric is

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & -\sinh^2 u \end{bmatrix} \tag{4.791}$$

Other than a change of sign, the first and second fundamental forms are identical with the Fubini-Study metric.

Example 4.8.22. For the spheroid with parametrisation:

$$\mathbf{x}(s) = \begin{bmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{bmatrix} \tag{4.792}$$

with inner product $\mathbf{x} \cdot \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$ the metric is

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \cos^2 u \end{bmatrix} \tag{4.793}$$

The first and second fundamental forms are identical with the Fubini-Study metric.

With the aid of these examples, we obtain the following corollary.

Corollary 4.8.23. *On $\mathbb{M}^{3\times 1} \sim (\mathbb{R}^3, \cdot)$, the space of 3×1 normalised column vectors with an inner product expressed through $\cdot$, the space can only be spherical or hyperbolic of one or two sheets, as in the previous examples, up to permutation of the indices and parameters. For all these spaces, the first and second fundamental form define tensors that are identical to the Fubini-Study metric. The signature of these spaces is given by the curvature, which distinguishes the Fubini-Study and second fundamental form. This system is Einstein, i.e. the metric tensor and the Ricci tensor are proportional.*

Proof. We have already established the various metrics and relationships in the examples above for the spherical and hyperbolic spaces in 3 dimensions. Obviously these are equivalent to any other spherical or hyperbolic spaces of the same dimension, up to permutation and a sign reflection, and permutation of internal parameters u, v. As these define surfaces, the Ricci theory applies, and we have:

$$R_{bd} = \frac{R}{2} g_{bd} \tag{4.794}$$

We also have the scalar Gaussian curvature K as given by:

$$K = \frac{R}{2} \tag{4.795}$$

where R is the Ricci scalar. The scalar curvature can be found using the Gauss Theorema Egregium:

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\det \mathrm{II}(\mathbf{x}, \mathbf{x})}{\det \mathrm{I}(\mathbf{x}, \mathbf{x})} \tag{4.796}$$

It is simple to see that there is an extra sign picked up from the inner product that gives us $K = \pm 1$ for all these systems. This further implies that one may distinguish these spaces on the basis of curvature, between spherical and hyperbolic. As the scalar curvature is a constant, the Ricci scalar is therefore a constant, the Ricci tensor then proportional to the metric as the Ricci theory of surfaces applies, and the system is Einstein. □

Remark 77. One may arrive at similar conclusions as above through use of the following argument. Each of the curves as given in the examples can be understood through the action of a unitary or pseudounitary operator. For the Fubini-Study metric, this unitary operator arises as the solution of a constrained optimisation as considered in the first part of this chapter. These operators define inner products, which differ for the various systems above, whether they are hyperbolic, of one or two sheets, or spherical. The unitary operators can be continued analytically to the complex plane with the usual replacement of trigonometric by hyperbolic functions and so on, with the solution remaining valid. This gives the change in curvature, which we can see through the Ricci tensor and second fundamental form.

However, on higher dimensions, we can expect several difficulties to arise that will spoil this happy concurrence. For example, it no longer becomes straightforward to define the parametrisation of the surface. Other problems involve the derivation of the nature of the inner product space. This necessarily requires that one be able to find the unitary and pseudounitary operators that exist, as these are the unique length, direction or scale preserving operators that define the inner product spaces in the sense of von Neumann. Although the theory of differential geometry appears to give similar answers in this contrived context, in higher dimensions the necessity of finding sensible parametrisations of the state vector in order to define the surfaces, and the difficulties of utilising cross products to find the normal vector and second fundamental form place the Fubini-Study method in good stead. This perspective of differential geometry has the relative advantage of giving consistent values for the metric, which contain the inner products within the structure and functional mechanisms of this technique. This contrasts with the often difficult and technical aspects of determining norms, inner product signatures, derivatives and other parts of differential geometry. On higher dimensions, the notion of a normal vector may become weakened, in that the cross product does not naturally render itself computable except via such formalisms as the wedge product. The Fubini-Study metric and the quantum brachistochrone technique offers a safe alternative to the older and established methods of differential geometry.

One other aspect of the utilisation of Fubini-Study metrics in general for understanding these types of spaces is the property that any such metric is necessarily Einstein, and in some cases Einstein-Kähler. This allows simplification of many different results in tensor calculus, as the Ricci tensor is proportional to the metric itself. The method of the quantum brachistochrone, and hyperbolic equivalent, is particularly useful is that the metric that is produced is guaranteed to be Einstein, and in many cases will be Einstein-Kähler, or have an equivalent representation as either a Riemannian, pseudo-Riemannian or sub-Riemannian manifold on some space. Many of the different solutions for the higher order tensor equations we have encountered in this section rely on the existence of a spherical decomposition, requiring one to find an Einstein tensor. Our calculations in this chapter have shown that understanding these forms of analysis can be achieved through use of the ideas of complex projective space, and some simple concepts from differential geometry. When combined with our other notions regarding kernels, groups and the structure of hyperbolic eigenfunctions, this gives us a powerful framework to answer many difficult problems.

Further questions in spectral analysis that arise from the calculations in this chapter include the problem of continuous and discrete states. In the case we have analysed, we required a discrete sum over a set of states in order to prove that the Fubini-Study metric was Einstein-Kähler. In the systems we have discussed, the degrees of freedom were constrained so as to give us a state vector that is a sum of a finite number of basis states. In such a system, the Einstein-Kähler equivalence for the Fubini-Study metric applies. However, much of the formalism relies on technical assumptions including the finiteness of the number of states. There are two extensions to this that should be included, which essentially amount to the same problem. In the first instance, one can consider the limit of the Kähler potential as the number of states increases to infinity. In the second instance, we consider the situation whereby we have a single state that is continuously divisible, with a single continuous eigenvalue. In both cases, the reader will be able to see that the sums over discrete states are replaced by integrals in the Kähler potential. It is possible that there exists a consistent method of calculating Ricci tensors and so on in the situation of continuous states through use of Lie derivatives and functional analysis. Many of the difficulties that one can expect to encounter arise through the definitions of measure. Through use of the correct formula for the determinant as the integral volume on the space, and a clever analysis of the measures that exist on a system, one might hope to circumvent many of these complications.

There are several other properties of the Fubini-Study metric related to the differential geometry of hyperbolic spaces that we will now consider. We have:

Lemma 4.8.24. *The imaginary part of the Fubini-Study tensor F_{ij} is related to the geometric or Pancharatnam-Berry phase.*

$$h_{ij} = \mathfrak{Im}[F_{ij}] \tag{4.797}$$

The operator h_{ij} is antisymmetric under conjugation:

$$h_{ij}^{\dagger} = -h_{ij} \tag{4.798}$$

The Pancharatnam-Berry phase gives the topological properties of the system through the Berry curvature. This is expressed through the exponential operator:

$$\hat{B} = \exp\left(\int_0^{\tau} h_{ij}(s)ds\right) \tag{4.799}$$

For the hyperbolic system with state:

$$|\Psi(s)\rangle = \begin{bmatrix} \cosh\dfrac{t}{2}e^{i(\varphi+\psi)/2} \\ \\ -i\sinh\dfrac{t}{2}e^{-i(\varphi-\psi)/2} \end{bmatrix} \tag{4.800}$$

the Berry operator satisfies:

$$\hat{B} = \begin{bmatrix} \cos\left(\cosh\tau - 1\right) & i\sin\left(\cosh\tau - 1\right) \\ i\sin\left(\cosh\tau - 1\right) & \cos\left(\cosh\tau - 1\right) \end{bmatrix} \tag{4.801}$$

Proof. To prove the lemma, first note that any tensor may be composed into its symmetric and antisymmetric part under the Hermitian transpose:

$$F_{ij} = \frac{1}{2}\left(F_{ij} + F_{ji}^*\right) + \frac{1}{2}\left(F_{ij} - F_{ji}^*\right) \tag{4.802}$$

The metric is the real part of this, it is symmetric:

$$g_{ij} = \frac{1}{2}\left(F_{ij} + F_{ji}^*\right) \tag{4.803}$$

$$g_{ij}^\dagger = (g_{ji})^* = \frac{1}{2}\left(F_{ji} + F_{ij}^*\right)^* = \frac{1}{2}\left(F_{ij} + F_{ji}^*\right) = g_{ij} \tag{4.804}$$

In an identical fashion, to prove antisymmetry, we have:

$$h_{ij} = \frac{1}{2}\left(F_{ij} - F_{ji}^*\right) \tag{4.805}$$

and we have:

$$h_{ij}^\dagger = (h_{ji})^* = \frac{1}{2}\left(F_{ji} - F_{ij}^*\right)^* = \frac{1}{2}\left(F_{ji}^* - F_{ij}\right) = -h_{ij} \tag{4.806}$$

The imaginary part of the Fubini-Study tensor therefore is antisymmetric under the Hermitian transpose, as is to be expected. To derive the Berry phase operator, note that we have already found the metric for the state vector in 4.4.3. In fact, we have shown that the Fubini-Study tensor may be written:

$$F_{ij} = \frac{1}{4}\begin{bmatrix} -1 & i\sinh t \\ i\sinh t & \sinh^2 t \end{bmatrix} \tag{4.807}$$

and hence the topological phase operator is given by the matrix:

$$h_{ij} \propto \begin{bmatrix} 0 & i\sinh t \\ i\sinh t & 0 \end{bmatrix} \tag{4.808}$$

Obviously, by inspection we have $h_{ij}^\dagger = -h_{ij}$ as is to be expected. For the following analysis we neglect the factor of 1/4 multiplying this matrix as it is may be taken as a rescaling at the end of the calculation. If we take the matrix h_{ij}, we may write the eigenvector decomposition as:

$$h_{ij} = \hat{W}\hat{L}\hat{W}^{-1}.\sinh t \tag{4.809}$$

where the matrix of eigenvectors $\hat{W}$ is the Hadamard matrix:

$$\hat{W} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{4.810}$$

and $\hat{L}$ is the Cartan element:

$$\hat{L} = i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.811}$$

To evaluate the matrix exponential, we write:

$$\hat{B} = \exp\left(\int_0^\tau h_{ij}(s)ds\right) = \exp\left(\int_0^\tau \hat{W}\hat{L}\hat{W}^{-1}.\sinh t dt\right) \tag{4.812}$$

To evaluate this, note that the matrix $\hat{W}$ telescopes outside the exponential, as it is constant it can be taken outside the integral:

$$\hat{B} = \hat{W}\exp\left(\hat{L}\int_0^\tau \sinh t dt\right)\hat{W}^{-1} \tag{4.813}$$

The time integral separates, and we have $\int_0^\tau \sinh t dt = \cosh\tau - 1$, the remaining calculation involves the exponential of the diagonal matrix:

$$\exp\left(\lambda\hat{L}\right) = \begin{bmatrix} e^{i\lambda} & 0 \\ 0 & e^{-i\lambda} \end{bmatrix} \tag{4.814}$$

We then compute the Berry curvature operator as:

$$\hat{B} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{i\lambda} & 0 \\ 0 & e^{-i\lambda} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \cos\lambda & i\sin\lambda \\ i\sin\lambda & \cos\lambda \end{bmatrix} \tag{4.815}$$

where $\lambda = \cosh\tau - 1$. To show that this solves the matrix exponential, it is simple to differentiate, we find:

$$\frac{d\hat{B}}{d\tau} = \begin{bmatrix} 0 & i\sinh\tau \\ i\sinh\tau & 0 \end{bmatrix}\begin{bmatrix} \cos\lambda & i\sin\lambda \\ i\sin\lambda & \cos\lambda \end{bmatrix} = h_{ij}(\tau)\hat{B} \tag{4.816}$$

as required. □

Remark 78. Although we shall not continue past this simple example with analysis of the topological phase term, we note that all the states with complex parts that we have considered in this paper have a corresponding Berry curvature tensor that forms, together with the Fubini-Study metric, the Fubini-Study tensor given through Kuzmak's formula [66].

We shall now demonstrate one way in which we may implement the theory of the Fubini-Study metric 3.2.33 in order to obtain systems and states that are consistent with the models of general relativity. In particular, the following known result is of historical interest in the context of our results derived in 4.8.19.

Lemma 4.8.25. *Define the system with SO(p,q) symmetry through the inner product:*

$$\mathbf{a}\cdot\mathbf{b} = \sum_{k=1}^{p} a_i b_i - \sum_{j=p+1}^{p+q} a_j b_j \tag{4.817}$$

then for the hyperbola:

$$z^2 - x^2 - y^2 = r^2 \tag{4.818}$$

we have an inner product with SO(1,2) symmetry. This is a system with an indefinite inner product. One parametrisation of the hyperboloid is given through:

$$\mathbf{x}(r,u,v) = r\begin{bmatrix} \sinh u\cos v \\ \sinh u\sin v \\ \cosh u \end{bmatrix} \tag{4.819}$$

In this view, the metric satisfies:

$$ds^2 = dr^2 - r^2\left(du^2 + \sinh^2 u.dv^2\right) = dr^2 + d\xi^2 + r^2 d\Omega^2 \tag{4.820}$$

which is equivalent to the space part of the Friedmann-Robertson-Walker metric via the radial surface $\xi = \xi(r)$.

Proof. It is simple to show that the parametrisation satisfies the inner product relation $\mathbf{x}\cdot\mathbf{x} = r^2$, and a lengthy but tedious calculation of the derivatives gives the equation for the metric:

$$ds^2 = dr^2 - r^2\left(du^2 + \sinh^2 u.dv^2\right) = dr^2 + d\xi^2 + r^2 d\Omega^2 \tag{4.821}$$

We have, by observation $d\xi^2 = -r^2 du^2$, and also $d\Omega^2 = -\sinh^2 u.dv^2$. If we take $\xi = \xi(r)$, we then satisfy:

$$ds^2 = \left(1 + \left(\frac{d\xi}{dr}\right)^2\right) dr^2 + r^2 d\Omega^2 \tag{4.822}$$

Assume the form of the FRW metric:

$$ds^2 = \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \tag{4.823}$$

then we have a solution if:

$$1 + \left(\frac{d\xi}{dr}\right)^2 = \frac{1}{1 - kr^2} \tag{4.824}$$

which defines a parabolic or hyperbolic surface depending on the sign of k via:

$$\xi(r) = A \pm \frac{\sqrt{k(1 - kr^2)}}{k} \tag{4.825}$$

We have, in terms of the original co-ordinates:

$$du^2 = -\frac{1}{r^2} d\xi^2 = -\frac{dr^2}{1 - kr^2} \tag{4.826}$$

$\square$

Remark 79. The Friedmann-Robertson-Walker (FRW) metric is well known in general relativity as a solution that describes expansion effects in a universe with non-negative curvature.

We shall now demonstrate a simple way in which the FRW metric may be derived from a projective state in the same way in which we were able to derive the Poincare metric from the hyperbolic state. This shall require the following lemma:

Lemma 4.8.26. *Assume there exists a state $|\Psi(t)\rangle$ such that under the generalised unitary transformation $|\Psi(t)\rangle = \hat{U}(t, 0)\,|\Psi(0)\rangle$, further the Fubini-Study metric given by 3.2.33:*

$$g_{jk} = \mathfrak{Re}\left[\langle\bar{\psi}_j|\,\psi_k\rangle - \langle\bar{\psi}_j|\,\Psi\rangle\,\langle\Psi|\,\psi_k\rangle\right] \tag{4.827}$$

$$|\psi_j\rangle = \frac{\partial}{\partial x^j}\,|\Psi\rangle \tag{4.828}$$

Then there exists a generalised state such that the Fubini-Study metric is equivalent to the FRW metric:

$$g_{jk} = \begin{bmatrix} r^2 \sin 2\phi & 0 \\ 0 & (1 - kr^2)^{-1} \end{bmatrix} \tag{4.829}$$

Proof. Take the projection operator 3.2.19, defined through the normalised state:

$$|\Psi\rangle = \frac{i}{\sqrt{kr^2}} \begin{bmatrix} \cos\phi \\ \sqrt{1 - kr^2} \\ \sin\phi \end{bmatrix} \tag{4.830}$$

$$\hat{\mathbf{1}} - |\Psi\rangle\,\langle\Psi| = \frac{1}{kr^2} \begin{bmatrix} kr^2 - \cos^2\phi & -\sigma\cos\phi & -\cos\phi\sin\phi \\ -\sigma\cos\phi & 1 & -i\sigma\sin\phi \\ -\cos\phi\sin\phi & i\sigma\sin\phi & kr^2 - \sin^2\phi \end{bmatrix} \tag{4.831}$$

$$\sigma = \sqrt{1 - kr^2} \tag{4.832}$$

Note that this state is singular at radius zero. We take $kr^2 > 1$ in order to recover the correct normalisation, other cases can be recovered using the correct definitions of the hyperbolic surface. In this case, we have the differential rates of change for the state vector as defined by:

$$|\psi_\phi\rangle = \frac{\partial}{\partial\phi}|\Psi\rangle = \frac{i}{\sqrt{k}r}\begin{bmatrix} -\sin\phi \\ 0 \\ \cos\phi \end{bmatrix} \tag{4.833}$$

$$|\psi_r\rangle = \frac{\partial}{\partial r}|\Psi\rangle = -\frac{i}{\sqrt{k}r^2}\begin{bmatrix} \cos\phi \\ \dfrac{1}{\sqrt{(1-kr^2)}} \\ \sin\phi \end{bmatrix} = -\frac{i}{\sqrt{k}r^2}\begin{bmatrix} \cos\phi \\ \dfrac{1}{\sigma} \\ \sin\phi \end{bmatrix} \tag{4.834}$$

We can then read off the components of the Fubini-Study metric as in 3.2.33, obtaining:

$$F_{jk} = \langle\bar{\psi}_j|\left(\mathbf{1}-\hat{P}\right)|\psi_k\rangle = \frac{1}{r^2}\begin{bmatrix} (kr^2-1)^{-1} & 0 \\ 0 & \dfrac{1}{k} \end{bmatrix} \tag{4.835}$$

Knowing that the only surfaces with constant curvature are the sphere, pseudosphere and plane, we can reproduce the spherical tensor:

$$R_{jk} = r^2 F_{jk} = \begin{bmatrix} (kr^2-1)^{-1} & 0 \\ 0 & 1/k \end{bmatrix} \tag{4.836}$$

which implies the metric

$$r^2 ds^2 = \frac{dr^2}{kr^2-1} + \frac{1}{k^2}d\phi^2 \tag{4.837}$$

We then are able to write in this metric

$$d\tau^2 = \frac{1}{k^2}d\phi^2 = \frac{dr^2}{1-kr^2} + r^2 ds^2 \tag{4.838}$$

which is the space part of the Schwartzchild metric, with the additional interpretation of the parameter s as a spherical angle. The links with the metric given by

$$g_{\alpha\beta} = \begin{bmatrix} (1-kr^2)^{-1} & 0 \\ 0 & r^2\sin^2\phi \end{bmatrix} \tag{4.839}$$

are then given by this sequence of operations. We establish the lemma through inversion of the diagonal elements, using the matrix transformation defined through:

$$\begin{bmatrix} 1 & 0 \\ 0 & r\sin\phi \end{bmatrix}\begin{bmatrix} (1-kr^2)^{-1} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & r\sin\phi \end{bmatrix} \tag{4.840}$$

This is sufficient to rotate the Robertson-Walker metric into the angularly dependent form suitable for general relativity as in the lemma. □

Remark 80. This is a very simplistic model, and the analogy should not be pushed too far; it is merely an illustrative example of how one can use the Fubini-Study metric to examine some properties in general relativity.

We close this section of the calculation with the following simple theorem.

Theorem 4.8.27. *The projective part of the Fubini-Study metric 3.2.33 satisfies the invariance principle:*

$$\langle\bar{\psi}_j|\left(\mathbf{1}-\hat{P}\right)|\psi_k\rangle = \langle\bar{\varphi}_j|\left(\mathbf{1}-\hat{P}_Q\right)|\varphi_k\rangle \tag{4.841}$$

Proof.

$$\langle\bar{\psi}_j|\left(\mathbf{1}-\hat{P}\right)|\psi_k\rangle = \langle\bar{\varphi}_j|\hat{Q}^\dagger\left(\mathbf{1}-\hat{P}\right)\hat{Q}|\varphi_k\rangle \tag{4.842}$$

$$= \langle\bar{\varphi}_j|\left(\mathbf{1}-\hat{P}_Q\right)|\varphi_k\rangle \tag{4.843}$$

where we have the unitary relationships

$$|\varphi_k\rangle = \hat{Q}|\psi_k\rangle \tag{4.844}$$

$$\hat{P}_Q = \hat{Q}^\dagger\hat{P}\hat{Q} \tag{4.845}$$

$$\hat{Q}^\dagger\hat{Q} = \mathbf{1} \tag{4.846}$$

as required. □

In this sense, one only needs to find the correct equivalence class of metrics in order to describe the problem correctly.

Remark 81. We have shown that this method is indeed a way in which to view a multitude of topics both in the harmonic analysis of groups and special functions, their matrix representations, and the differential geometries which define the eigenfunctions through the transform theory of kernels. It is hoped that this is a continuation in the spirit, if not the exactitude, of the question of Hilbert [133] as to the axiomatisation of physics.

4.9 Completeness Relations and Composition Formulae

We shall end this chapter with several types of integral formulae related to composition and completeness relations on the hyperbolic space. These examples are important descriptive cases of more complicated problems that we shall be examining in the following parts of this thesis.

4.9.0.1 Composition of Modified Bessel functions I

We begin with analysis of the following integral:

Theorem 4.9.1. *A composition formula for the modified Bessel functions may be written as a Kontorovich-Lebedev transform, i.e. an index transform over modified Bessel functions. This produces a function which depends on the hyperbolic distance:*

$$\int_0^\infty d\mu.\mu\sinh(\pi\mu)\Gamma(1/2+i\mu)\Gamma(1/2-i\mu)K_{i\mu}(x)\,K_{i\mu}(y) \tag{4.847}$$

$$= \frac{\pi^{3/2}}{2}\left(\frac{x+y}{2xy}\right)^{-1/2} K_{1/2}(x+y) \tag{4.848}$$

$$= \frac{\pi^2}{2}\frac{\sqrt{xy}}{(x+y)}e^{-(x+y)} \tag{4.849}$$

where the modified Bessel or Macdonald functions are as in 3.4.2, and the hyperbolic distance appears as the multiplying prefactor of this integral. This integral is symmetric under interchange of x and y.

Proof. Relevant tables of index transforms may be found in [29,122,123]. From Buchholz (see eq. 6.3α, pp. 85) [134] we have the formula for the product of two Whittaker functions as given by:

$$W_{\chi,\mu/2}(a_1 t)W_{\chi,\mu/2}(a_2 t) = \frac{2t\sqrt{a_1 a_2}}{\Gamma\left(\frac{1+\mu}{2}-\chi\right)\Gamma\left(\frac{1-\mu}{2}-\chi\right)}$$

$$\times \int_0^\infty \exp\left(-\frac{1}{2}(a_1+a_2)t\cosh\nu\right) K_\mu\left(t\sqrt{a_1 a_2}\sinh\nu\right)\left(\coth\frac{\nu}{2}\right)^{2\chi} d\nu \tag{4.850}$$

If we set the first index to zero, we find:

$$W_{0,i\mu}(a_1 t)W_{0,i\mu}(a_2 t) = \frac{t\sqrt{a_1 a_2}}{\pi} K_{i\mu}\left(\frac{a_1 t}{2}\right) K_{i\mu}\left(\frac{a_2 t}{2}\right) \tag{4.851}$$

$$= \frac{2t\sqrt{a_1 a_2}}{\Gamma(1/2+i\mu)\Gamma(1/2-i\mu)} \int_0^\infty \exp\left(-\frac{1}{2}(a_1+a_2)t\cosh\nu\right) K_{2i\mu}\left(t\sqrt{a_1 a_2}\sinh\nu\right) d\nu \tag{4.852}$$

which implies after rearrangement

$$\Gamma(1/2+i\mu)\Gamma(1/2-i\mu)K_{i\mu}\left(\frac{a_1 t}{2}\right) K_{i\mu}\left(\frac{a_2 t}{2}\right) \tag{4.853}$$

$$= 2\pi \int_0^\infty \exp\left(-\frac{1}{2}(a_1+a_2)t\cosh\nu\right) K_{2i\mu}\left(t\sqrt{a_1 a_2}\sinh\nu\right) d\nu$$

Using the substitutions $t = 1$, $a_1 = 2x$,$a_2 = 2y$, we find:

$$\Gamma(1/2+i\mu)\Gamma(1/2-i\mu)K_{i\mu}(x)\, K_{i\mu}(y) \tag{4.854}$$

$$= 2\pi \int_0^\infty \exp\left(-(x+y)\cosh\nu\right) K_{2i\mu}\left(2\sqrt{xy}\sinh\nu\right) d\nu \tag{4.855}$$

hence, after multiplying by the hyperbolic sine and integrating out, we find:

$$\int_0^\infty d\mu.\mu\sinh(\pi\mu)\Gamma(1/2+i\mu)\Gamma(1/2-i\mu)K_{i\mu}(x)\, K_{i\mu}(y) \tag{4.856}$$

$$= 2\pi \int_0^\infty d\mu.\mu\sinh(\pi\mu) \int_0^\infty \exp\left(-(x+y)\cosh\nu\right) K_{2i\mu}\left(2\sqrt{xy}\sinh\nu\right) d\nu \tag{4.857}$$

$$= 2\pi \int_0^\infty d\nu.\exp\left(-(x+y)\cosh\nu\right) \int_0^\infty d\mu.\mu\sinh(\pi\mu) K_{2i\mu}\left(2\sqrt{xy}\sinh\nu\right) \tag{4.858}$$

The inner integral may be calculated using the following formula from [122] (eq. 6.794.7, pp 751)

$$\int_0^\infty x\sinh\left(\frac{\pi x}{2}\right) K_{ix}(a)dx = \frac{\pi a}{2} \tag{4.859}$$

As this formula may seem divergent, we shall prove it by using some simple limit arguments from a known integral. The integral to be derived:

$$\int_0^\infty x\sinh\left(\frac{\pi x}{2}\right) K_{ix}(a)dx = \frac{\pi a}{2} \tag{4.860}$$

Taking the integral from Prudnikov, Vol. II, 2.16.48.20 [126]:

$$\int_0^\infty \sinh(ax)\sin(bx)K_{ix}(c)dx = \frac{\pi}{2}e^{-c\cos a\cosh b}\sin\left(c\sin a\sinh b\right) \tag{4.861}$$

Assume we take the limit as b approaches zero

$$b \to 0 \tag{4.862}$$

Now, although this is an artificial limit, we shall show that in the end the factor of b divides through and we are left with the desired result. If we take a small b approaching zero, and expand using a Taylor series, the various factors in the integral formula are approximated by:

$$\sin(bx) \sim bx - \frac{(bx)^3}{3!} + ... \approx bx \tag{4.863}$$

$$\sinh b \sim b + \frac{b^3}{3!} + ... \approx b \tag{4.864}$$

$$\cosh b \sim 1 + \frac{b^2}{2!} + .. \approx 1 \tag{4.865}$$

Taking the limit, we obtain:

$$\lim_{b\to 0}\int_0^\infty \sinh(ax)\sin(bx)K_{ix}(c)dx = \lim_{b\to 0}\int_0^\infty \sinh(ax)bxK_{ix}(c)dx \tag{4.866}$$

$$= \lim_{b\to 0}\frac{\pi}{2}e^{-c\cos a}\sin\left(bc\sin a\right) \tag{4.867}$$

$$= \lim_{b\to 0}\frac{\pi}{2}bce^{-c\cos a}\sin a \tag{4.868}$$

or

$$\lim_{b\to 0}\int_0^\infty \sinh(ax)bxK_{ix}(c)dx = \lim_{b\to 0}\frac{\pi}{2}bce^{-c\cos a}\sin a \tag{4.869}$$

Dividing through by b, we find:

$$\int_0^\infty \sinh(ax)xK_{ix}(c)dx = \frac{\pi}{2}ce^{-c\cos a}\sin a \tag{4.870}$$

To obtain the integral formula, we therefore take the parameter equal to:

$$a = \frac{\pi}{2} \tag{4.871}$$

hence

$$\cos\frac{\pi}{2} = 0, \sin\frac{\pi}{2} = 1 \tag{4.872}$$

or

$$\int_0^\infty x\sinh\left(\frac{\pi x}{2}\right)K_{ix}(c)dx = \frac{\pi}{2}c \tag{4.873}$$

which was to be obtained. Making the substitution $x \to 2x$, we obtain:

$$\int_0^\infty y\sinh\left(\pi y\right)K_{2iy}(a)dy = \frac{\pi a}{8} \tag{4.874}$$

and we find that the inner integral is given by:

$$\int_0^\infty d\mu.\mu\sinh(\pi\mu)K_{2i\mu}\left(2\sqrt{xy}\sinh\nu\right) = \frac{\pi}{8}2\sqrt{xy}\sinh\nu = \frac{\pi\sqrt{xy}}{4}\sinh\nu \tag{4.875}$$

and the final result as specified by:

$$\int_0^\infty d\mu.\mu\sinh(\pi\mu)\Gamma(1/2+i\mu)\Gamma(1/2-i\mu)K_{i\mu}\left(x\right)K_{i\mu}\left(y\right)$$

$$= 2\pi \int_0^\infty d\nu . \exp\left(-(x+y)\cosh\nu\right) \frac{\pi\sqrt{xy}}{4} \sinh\nu \tag{4.876}$$

$$= \frac{\pi^2\sqrt{xy}}{2} \int_0^\infty d\nu . \exp\left(-(x+y)\cosh\nu\right) \sinh\nu \tag{4.877}$$

Using the DLMF formula (eq. 10.32.7.3):

$$K_\nu(z) = \frac{\sqrt{\pi}\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+1/2\right)} \int_0^\infty e^{-z\cosh t}(\sinh t)^{2\nu} dt \tag{4.878}$$

valid for $\Re v > -\frac{1}{2}$, and $|\arg z| < \frac{\pi}{2}$ we find

$$\int_0^\infty e^{-z\cosh t} \sinh t . dt = K_{1/2}(z)\Gamma\left(1\right)\left(\frac{z}{2}\right)^{-1/2} \frac{1}{\sqrt{\pi}} \tag{4.879}$$

therefore the integral becomes:

$$\int_0^\infty d\nu . \exp\left(-(x+y)\cosh\nu\right)\sinh\nu = K_{1/2}(x+y)\left(\frac{x+y}{2}\right)^{-1/2} \frac{1}{\sqrt{\pi}} \frac{\pi^2\sqrt{xy}}{2} \tag{4.880}$$

$$= \frac{\pi^{3/2}}{2}\left(\frac{x+y}{2xy}\right)^{-1/2} K_{1/2}(x+y) \tag{4.881}$$

We therefore conclude that

$$\int_0^\infty d\mu . \mu \sinh(\pi\mu)\Gamma(1/2+i\mu)\Gamma(1/2-i\mu)K_{i\mu}\left(x\right)K_{i\mu}\left(y\right) \tag{4.882}$$

$$= \frac{\pi^{3/2}}{2}\left(\frac{x+y}{2xy}\right)^{-1/2} K_{1/2}(x+y) \tag{4.883}$$

To further simplify this integral, note that the integral is specified through:

$$\int_0^\infty d\mu . \mu \sinh(\pi\mu)\Gamma\left(\frac{1}{2}+i\mu\right)\Gamma\left(\frac{1}{2}-i\mu\right)K_{i\mu}(x)K_{i\mu}(y) \tag{4.884}$$

$$= \frac{\pi^{3/2}}{2}\left(\frac{x+y}{2xy}\right)^{-1/2} K_{1/2}(x+y) \tag{4.885}$$

The special case of the modified Bessel function may be written:

$$K_{1/2}(x+y) = \sqrt{\frac{\pi}{2(x+y)}} e^{-(x+y)} \tag{4.886}$$

$$I = \frac{\pi^2}{2}\frac{\sqrt{xy}}{(x+y)} e^{-(x+y)} \tag{4.887}$$

□

Remark 82. This is a special case of a generalised formula, which we shall calculate in the sequel. This appears as a tabulated result in Oberhettinger, without proof, as a restricted case of [28], pp. 14, and also as a formula in [122], GR 6.812.4. Note also the formula in Prudnikov [126], Vol. II 2.16.53.1:

$$\int_0^\infty ds . s \sinh(\pi n x)\left|\Gamma\left(\nu+irs\right)\right|^2 K_{is}(b)K_{is}(c) = I_{n,r,\nu} \tag{4.888}$$

For the special case $n = 1, \nu = 1/2, r = 1$

$$I_{1,1,1/2} = \frac{\pi^{3/2}}{\sqrt{2}}\left(\frac{bc}{b+c}\right) K_{1/2}(b+c) \tag{4.889}$$

which is identical to the result above.

4.9.0.2 Composition of Modified Bessel functions II

A composition formula for the modified Bessel functions can be evaluated using the Mehler-Fock functions and the completeness relation. Specifically:

Theorem 4.9.2. *The index integral given by the formula:*

$$\int_0^\infty \nu \sinh(\pi\nu)\Gamma(\lambda+i\nu)\Gamma(\lambda-i\nu)K_{i\nu}(a)K_{i\nu}(b)d\nu \tag{4.890}$$

$$= \frac{\pi^{3/2}\Gamma(\lambda+1/2)}{2}\left(\frac{a+b}{2ab}\right)^{-\lambda} K_\lambda(a+b) \tag{4.891}$$

where the special functions are given by the Macdonald and gamma functions as in [122], 3.4.2, 3.4.1.

Proof. From the formula for the Mehler-Fock kernel as in [56], we have the completeness relationship:

$$\pi\delta(x-y) = \int_0^\infty |\Gamma\left(ip+\mu\right)|^2 p\sinh(\pi p)\mathcal{P}_{ip-1/2}^{-\mu+1/2}\left(x\right)\mathcal{P}_{ip-1/2}^{-\mu+1/2}\left(y\right)dp \tag{4.892}$$

We also have the integral formula which converts a conical function into one of Macdonald type [122] (eq. 7.141.5, pp 776):

$$a^\lambda\sqrt{\frac{\pi}{2}}\int_1^\infty (x^2-1)^{\lambda/2-1/4}\mathcal{P}_{i\nu-1/2}^{-\lambda+1/2}(x)e^{-ax}dx = K_{i\nu}(a) \tag{4.893}$$

If we integrate both sides of this equation with respect to a modified Bessel function we may write:

$$\int_0^\infty \nu \sinh(\pi\nu)\Gamma(\lambda+i\nu)\Gamma(\lambda-i\nu)K_{i\nu}(a)K_{i\nu}(b)d\nu \tag{4.894}$$

$$= \int_0^\infty \nu\sinh(\pi\nu)\Gamma(\lambda+i\nu)\Gamma(\lambda-i\nu)(ab)^\lambda\frac{\pi}{2} \tag{4.895}$$

$$\times\int_1^\infty (x^2-1)^{\lambda/2-1/4}\mathcal{P}_{i\nu-1/2}^{-\lambda+1/2}(x)e^{-ax}dx\int_1^\infty (y^2-1)^{\lambda/2-1/4}\mathcal{P}_{i\nu-1/2}^{-\lambda+1/2}(y)e^{-by}dy$$

$$= (ab)^\lambda\frac{\pi}{2}\int_1^\infty (x^2-1)^{\lambda/2-1/4}e^{-ax}dx\int_1^\infty (y^2-1)^{\lambda/2-1/4}e^{-by}dy \tag{4.896}$$

$$\times\int_0^\infty \nu\sinh(\pi\nu)\Gamma(\lambda+i\nu)\Gamma(\lambda-i\nu)\mathcal{P}_{i\nu-1/2}^{-\lambda+1/2}(x)\mathcal{P}_{i\nu-1/2}^{-\lambda+1/2}(y)d\nu$$

$$= (ab)^\lambda\frac{\pi^2}{2}\int_1^\infty (x^2-1)^{\lambda/2-1/4}e^{-ax}dx\int_1^\infty \delta(x-y)(y^2-1)^{\lambda/2-1/4}e^{-by}dy \tag{4.897}$$

$$= (ab)^\lambda\frac{\pi^2}{2}\int_1^\infty dx.(x^2-1)^{\lambda-1/2}e^{-(a+b)x} \tag{4.898}$$

$$= (ab)^\lambda\frac{\pi^2}{2}\frac{\Gamma(\lambda+1/2)}{\sqrt{\pi}}\left(\frac{a+b}{2}\right)^{-\lambda} K_\lambda(a+b) \tag{4.899}$$

$$= \frac{\pi^{3/2}\Gamma(\lambda+1/2)}{2}\left(\frac{a+b}{2ab}\right)^{-\lambda} K_\lambda(a+b) \tag{4.900}$$

as required, see e.g. [122]. We have invoked the Fubini-Tonelli theorem to interchange the integrals as usual. $\square$

Remark 83. This can readily be used to derive the index transform in 4.9.1 through use of the special case $\lambda = 1/2$. Again, the reader is directed to [28], pp. 14, and also as a formula in [122], GR 6.812.4. In Prudnikov [126], Vol. II, 2.16.53.1,

$$\int_0^\infty ds.s \sinh(\pi nx) \left|\Gamma\left(\lambda + irs\right)\right|^2 K_{is}(b) K_{is}(a) = I_{n,r,\lambda} \tag{4.901}$$

which has special case:

$$I_{1,1,\lambda} = 2^{\lambda-1}\pi^{3/2}\left(\frac{b+a}{ba}\right)^{-\lambda}\Gamma\left(\lambda + 1/2\right) K_\lambda(b+a) \tag{4.902}$$

and is equivalent to the previous formula, as derived using the projection-slice theorem.

Remark 84. This is our second example of an index transform. The later parts of this thesis will cover an extensive list of these types of integral relations. Note that this hinges on the existence of completeness relationships and the conversion integral between the Mehler-Fock and modified Bessel or Macdonald functions.

4.10 Comments

This calculation has presented some novel methods for determining the relationships between some well known groups of special functions. We have shown that the theories of brachistochrones are well suited to the application of ideas from differential geometry and the calculus of diffusions; this is expected to be a straightforward way in which particle dynamics can be understood, even for complex systems as we have considered in the hyperbolic plane. We have shown directly how one can move from the group representation, given as a matrix with some symmetry, to the continuous operators familiar from differential calculus, via the methods of differential geometry. This is to be an area of future research, as the areas and topics developed in understanding this link will have direct bearing on questions of a physical nature. For now, we merely comment that the hyperbolic plane is not just a hypothetical construct. The differential systems we have developed an understanding of by using these techniques have direct bearing on such varied problems as oscillatory quantum systems, pricing of exotic options and various questions in hyperbolic flow. It will be interesting to see whether there are higher dimensional analogues of these types of problems. We have shown in this part of this work how one can begin with seemingly simple precepts related to symmetry groups on the hyperbolic plane and use the concept of homomorphism to relate this to various sets of eigenfunctions. The hyperbolic brachistochrone equation underlies this; using the homomorphism, which is given by Cayley-type transforms of functions, we are then able to show a number of interesting properties. The use of a brachistochrone to find geodesics and define the differential geometry is a novel technique, and allows us to avoid the lengthy calculations contained in [69] while obtaining essentially the same results. This fusion of geometry, functional analysis and matrix calculus is particularly profitable in such realms, as we do not have a readily available graphical way in which to interpret the results, in contrast to e.g. the Bloch sphere for SU(2), here we must deal with a pseudosphere. As such, it is necessary that the methodology be sound; it is pleasing that it is obviously an effective way in which to approach these systems. Given the results for the kernel in the form:

$$K(z,w;t) = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}} \int_{d(z,w)}^\infty \frac{\beta \exp\left(-\frac{\beta^2}{4t}\right) d\beta}{\sqrt{\cosh\beta - \cosh d(z,w)}} \tag{4.903}$$

in this thesis, also [11,124], we are brought to the question of whether there are other kernels that may be written in the above format, which essentially reduces to a Gaussian transform over some function of the distance. We can see directly how the perturbation series may be developed as given by this type of fundamental solution. Craddock [46] presents the following integral kernel for the PDE, which is related to the hyperbolic heat kernel in a different representation (see Theorem 3.1, pp. 353 in [46]):

$$K(z, w; t) = \frac{1}{4\sqrt{\pi}t^{3/2}} \int_{d(x,w)}^{\infty} \beta \exp\left(-\frac{\beta^2}{4t}\right) J_0\left(2xw \cosh\beta - x^2 - w^2\right) d\beta \tag{4.904}$$

where the hyperbolic distance function appears as:

$$d(x, w) = \cosh^{-1}\left(\frac{x^2 + w^2}{2xw}\right) \tag{4.905}$$

Sousa and Yakubovich [56] give the following Bougerol identity:

$$\frac{2}{\pi^2} \int_0^{\infty} \frac{dy}{y} \int_0^{\infty} d\tau . e^{-t\tau^2} K_{i\tau}(x) K_{i\tau}(y) \tau \sinh(\pi\tau) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{ix \sinh y} e^{-y^2/(4t)} dy \tag{4.906}$$

This expression appears in the context of evaluating certain expectation values of exponential Brownian motion, and is similarly linked to the analysis of the pricing of Asian options via heat-type PDEs on hyperbolic spaces, as in Craddock [46]. Note that the right hand side of this expression contains a type of integral transformation, which integrates an input function against the heat kernel. This is the common factor between all these different types of representations.

We can see how these different integral identities give a solution to the heat equation which is flat at small times, and deforms on the manifold in the correct way to reflect the underlying diffusion process. As we have shown in this work, there are deep interactions between systems of eigenfunctions with identical spectra, i.e. isospectrality, and further consideration of this question is of interest to the development of the science of hyperbolic analysis. Other groups should be directly amenable to the treatment we have prescribed to SU(1,1), amongst which the most obvious of these seems to be SU(2), with consequences for quantum mechanical systems. Further research directives would appear to be establishing the relevant sets of special functions for efficient computation of exotic options pricing using such methods, as well as determining whether such techniques can be used for pricing basic contracts using such things as CIR and OU processes. It would be hoped that a general theory of (pseudo)-spherical systems might be developed using such intuitions.

As another demonstration of this phenomenon, Hafoud et. al have derived the following formula for a Gegenbauer diffusion kernel via the Fubini-Study metric on quaternionic projective space [84–86]. For $t \geq 0$, the kernel admits the following representations:

$$Q_n(t; r) = \frac{1}{\pi^n} \sum_{l=0}^{\infty} (2l + n) \frac{(l + n - 1)!}{l!} e^{-4tl(l+n)} P_l^{(n-1,0)}(\cos 2r) \tag{4.907}$$

where $P_l^{(n-1,0)}(\cos 2r)$ are Jacobi polynomials. A second representation of this was shown to be:

$$Q_n(t; r) = \frac{e^{n^2 t}}{2^{n-2}\pi^{n+1}} \int_r^{\pi/2} \frac{d(-\cos u)}{\sqrt{\cos^2 r - \cos^2 u}} \hat{L}^n \Theta_{n+1}(t; u) \tag{4.908}$$

with

$$\Theta_{n+1}(t; u) = \sum_{l=0}^{\infty} e^{-4t(l+n/2)^2} \cos\left((2l + n)u\right) \tag{4.909}$$

and

$$\hat{L} = -\frac{1}{\sin u} \frac{d}{du} \tag{4.910}$$

The function $\Theta_{n+1}(t;u)$ is by inspection a form of Jacobi theta function. The differential operator $\hat{L}$ is related to the hyperbolic operators considered in this chapter, we have:

$$\frac{1}{\sin u}\frac{d}{du}\left(\sin u\frac{df}{du}\right)=\frac{d^2 f}{du^2}+\cot u\frac{df}{du} \tag{4.911}$$

which is a real form of the generator for the differential equation which we have analysed to find expressions for the hyperbolic heat kernel. On a more astute level, as examples of the direct connection between this representation for a kernel on a complex projective geometry and those considered in this thesis, the following formulae exist which bring together the Bessel, Gegenbauer, Legendre and Laguerre functions. In the following chapters, we shall exploit similar results to these in order to derive a number of integral formulae that were previously difficult to obtain. As a simple sketch of the type of connection we shall exploit, consider the following. We have the generating formula for the Gegenbauer functions:

$$\Gamma(\lambda+1/2)e^{z\cos\theta}\left(\frac{z}{2}\sin\theta\right)^{1/2-\lambda}J_{\lambda-1/2}\left(z\sin\theta\right)=\sum_{n=0}^{\infty}\frac{C_n^{\lambda}\left(\cos\theta\right)}{\left(2\lambda\right)_n}z^n \tag{4.912}$$

where $(.)_n$ is the rising factorial. We also have access to the integral representation for the Gegenbauer functions:

$$\frac{C_n^{\alpha+1/2}\left(\cos\theta\right)}{C_n^{\alpha+1/2}\left(1\right)}=\frac{2^{\alpha+1/2}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\left(\sin\theta\right)^{-2\alpha}\int_0^{\theta}\frac{\cos\left(\left[n+\alpha+1/2\right]\phi\right)}{\left(\cos\phi-\cos\theta\right)^{-\alpha+1/2}}d\phi \tag{4.913}$$

see e.g. DLMF eq. 18.10.1, [8], where we have $\alpha>-1/2$ and $0<\theta<\pi$. In a similar vein, the Dirichlet-Murphy integral representation for the Legendre polynomials is given by a special case of the Gegenbauer formula:

$$P_n(\cos\theta)=\frac{\sqrt{\pi}}{2}\int_0^{\theta}\frac{\cos\left(\left[n+1/2\right]\phi\right)}{\sqrt{\cos\phi-\cos\theta}}d\phi \tag{4.914}$$

see DLMF eq. 18.10.1 [8]. Finally, the Laguerre polynomials have an integral representation that is given by:

$$L_n^{\alpha}(x)=\frac{e^x x^{-\alpha/2}}{n!}\int_0^{\infty}e^{-t}t^{n+\alpha/2}J_{\alpha}\left(2\sqrt{xt}\right)dt \tag{4.915}$$

This formula can be viewed as a Mellin transform of a function similar to the generating function of the Gegenbauer polynomials, where the Bessel function appears in a different context, and alternatively as a generalised Hankel transform operating on a monomial. We can see from these formulae that the Gegenbauer functions have the Legendre functions as a special case; the Laguerre functions are a transformation of the Gegenbauer functions themselves. The connection is given by the Bessel functions in this instance, and the factors that appear in the denominator of these different formulations for the Gegenbauer and Legendre functions are familiar from our calculations in this chapter. Understanding the relationships between the groups of special functions we have encountered on a deeper level is the content of the next chapter, and we shall utilise many of the tools we have developed in analysis of the hyperbolic heat kernel to do so.

In other possible extensions, it might be possible to use these techniques to establish the nature of diffusion on higher dimensional hyperbolic groups of a physical nature. As both the equations of quantitative finance and physics are based around similar methods, theory and practice, it is possible to see that there may be new methods arising in this space. Other outstanding questions of interest might include the computation of Wigner functions using such methods for SU(1,1), see e.g. [135] for an approach based on creation and annihilation operators. Such functions have a natural interpretation in statistical mechanics, giving the Poisson bracket of the Hamiltonian flow.

This work has not discussed the problem of the discrete states, or the difference between the scattered and bound states. We note that for certain parts of the spectrum, different groups of special functions may be more important than the continuous groups we have calculated in this thesis, in particular the Laguerre functions. These expansions have been used to evaluate exponential Brownian motion [136,137] and the pricing of Asian options [138]. In particular, of interest is the development of kernel densities for these type of systems, and finding efficient ways to generate numerical values for the confluent hypergeometric function.

CHAPTER 5

Index Integrals & the Projection-Slice Theorem

5.1 Introduction and Review

The projection-slice theorem is a very useful way of binding together the Fourier, Abel and Hankel transforms, originally given in Bracewell [23–26]. In the following calculations, we introduce an extension of this formula, applied to the constructions given by various index transforms.

This extension takes the form of a modification to the principles and context of the projection-slice theorem, to account for our previous analysis of the hyperbolic plane. In this space, the roles of Fourier, Abel and Hankel transforms are replaced by corresponding formulae for other groups of special functions related to the hyperbolic plane. As we have shown in the previous chapter, there is a clear link between understanding the integral transforms that exist on a space, the nature of the group(s) that exist, and the eigenfunctions that describe the kernel through spectral decomposition. Developing from this, the projection-slice theorem offers us a solution to understanding the connections between these groups of special functions and the transforms that operate on the different representations of the hyperbolic space. In this chapter we shall show how many of the different integral transforms that arose in the analysis of the hyperbolic heat kernel may be evaluated. Although results exist in tables, the lack of referential material and centralised methodology for evaluation of the types of integrals that are encountered when attempting to solve these systems is prohibitive for research in this domain. For this reason, we tackle the solution of these problems directly to show applications of the projection-slice theorem.

Apart from the examples we have encountered in the analysis of the hyperbolic heat kernel in the previous chapter, index integrals have a varied field of application within the disciplines of the physical and quantitative sciences. In this chapter we prove various index integration formulae for Bessel, Whittaker and Mehler-Fock functions. Integrals of this type arise naturally in the pricing of various options formulae including Asian options, as well as scattering problems related to diffraction as originally considered by Oberhettinger in [72], [73], [74]. In particular, these types of integrals are useful for pricing certain exotic options, where the probability density function is given by a related type of index transform to the Mehler-Fock transform we derived for the hyperbolic heat kernel. In the realm of physics, when considering the problems of conduction and electromagnetics for two spheres that touch, similar problems emerge. The calculations in the following chapter shall demonstrate that there is a consistent way to evaluate these complicated integrals using the extension of the projection-slice theorem to hyperbolic space.

In the following calculations, we derive a number of interesting formulae using cosine Fourier transforms and relate these results to tabulated data of unknown methods contained in Oberhettinger [27,29] and Gradshteyn and Ryzik [122]. Discussion is given for the prospects of a more general theory of index integrals, the relationships between the groups of special functions considered, and some outstanding open questions that may be amenable to analysis in a similar fashion. The Whittaker, Kontorovich-Lebedev and Mehler-Fock transforms are applied in this context along with some formulae of Ramanujan and Bateman. We use this chapter as an opportunity to showcase the application of the concepts we have learnt through our analysis of the hyperbolic plane and heat kernel, and point to the greater utility and premise of the results presented herein beyond the examples we shall discuss in this thesis. Our task shall be to reduce the complicated world of index integration to a programmatic regime of analysis, which is amenable to our control and direction. This will facilitate our further understanding in the convoluted regimes of hyperbolic systems we have developed through computation of the heat kernel.

The concept of a index integral over a class of functions that depends continuously on some parameter has deep relations to group theory and various representations, with many practical uses in applied analysis. We note the known connection between the hypergeometric framework for kernels, the group theory of hypergeometric functions, and the theory of index integration. The original series of investigations in the literature related to index integrals and convolution theory may be found dating to the early 1980s, where for example we have Brychov, Glaeske et. al [34] utilising aspects of kernel theory and integral transforms of convolution type. Further, the case of convolution equations of hypergeometric type were treated for the Meijer-G and Fox-H function in greatest generality in the papers of Tuan, Marichev and Yakubovich (1986) [35]. The restricted case which covers the Kontorovich-Lebedev transform appears within [35], and similar results will be touched on in this chapter and following parts of this work. Aspects of similar topics to projection-slice theorems and the connection between different types of integral transforms may also be found in Yakubovich's original investigations [32,33], dating from 1987 and continuing.

Further extensions of the hypergeometric method of kernel transforms are found in Yakubovich and Luchko (c. 1994) [36], where the authors treated the convolution formula in a variety of settings. In particular, the cases of Kontorovich-Lebedev transforms (modified Bessel K/Macdonald function), Lebedev-Skalsaya (real/imaginary part of modified Bessel), Olevskii (hypergeometric type, Whittaker) and Mehler-Fock transforms were identified as of particular interest, and analysed using Parseval relations and the convolution theorem. These are all of course special cases of the Wimp (Meijer-G related) transform.

The series of works developed by Yakubovich et. al, in [58], [139], [140], also with Sousa in [56], [112], [141], has allowed for development of some techniques for analysis of the heat kernels which are associated to various different PDEs related to index transforms, in particular the Mehler-Fock, index Whittaker and Kontorovich-Lebedev transforms. The use of index Whittaker transforms can be traced to Wimp [142], c. 1964, who derived the following transform and inversion formula:

$$g(x) = \Gamma\left(\frac{1}{2} - \kappa - ix\right)\Gamma\left(\frac{1}{2} - \kappa + ix\right)\int_0^\infty W_{\kappa,ix}(t)f(t)dt \tag{5.1}$$

$$f(x) = \frac{1}{(\pi x)^2}\int_0^\infty t\sinh(2\pi t)W_{\kappa,ix}(t)g(t)dt \tag{5.2}$$

See, e.g. eqs. 4.9-4.10, [142].

However, the computational complexity involved in solving such problems in a systematic way has held back progress in this field. Many of the theorems familiar to the student or practitioner of Laplace and Fourier transforms are absent. This presents further difficulties; some might argue that there is no existing general framework in which these answers can be computed. Several of the primary resources in the field of this type of integration theory do not present their method of solution [27,29,122], only tabulated answers. This work is an attempt to decode the mathematical algorithms by which many of these previously obscured results may be obtained.

We note that Yakubovich, has derived the following integral formula [33], eq. 7.16, pp. 204:

$$\int_0^{\infty} \tau \sinh(2\pi\tau)\Gamma\left(\frac{1}{2} - \rho + i\tau\right)\Gamma\left(\frac{1}{2} - \rho - i\tau\right) K_{2i\tau}(y)W_{\rho,i\tau}(x)d\tau \tag{5.3}$$

$$= \pi^2 2^{2(\rho-1)} y^{1-2\rho} x^{\rho} \exp\left(\frac{-2x^2 + y^2}{4x}\right) \tag{5.4}$$

$$\Re(\rho) \leq 1/2 \tag{5.5}$$

which is a very similar result to those explored in this chapter via the projection-slice theorem in the hyperbolic plane.

An updated current work related to the convolution theory of the index Whittaker transformation may be found in Sousa and Yakubovich [37] (2022), where the authors used a generalised form of convolution identity to identify product formulae for the confluent hypergeometric function. These types of convolution theorems are extremely important in deriving composition formulae for the kernel, as we shall show repeatedly in the following parts of this thesis.

The applications for index integration are many, indeed they are useful in such varied fields as image processing [64,65,111], random media [143], diffraction and other electromagnetic/physical applications [74,144–146]. Many of the larger tabulated resources as computed by Oberhettinger seem to have been commissioned by the aerospace industry, so one would assume some use in aerodynamics. Unfortunately, as previously discussed, the method of solution is obscure or not available. Their utility in the field of quantitative finance has been of interest, with important work being carried out in the works contained in [37,56,112] where a number of complementary methodologies for solving diffusion kernels have been developed. Underlying the financial use case, the link between Yor integrals, Asian options and index integration and various other results may be found in the papers of Linetsky [75], Yor [77] and Dufresne [79]. In particular, the paper of Linetsky outlines a spectral decomposition over the eigenvalues for the kernel solution. Although we shall not go deeply into the structures of spectral theory in this work, it is important to emphasise that this powerful concept permeates at levels below and beyond what is exhibited in this chapter. Many of the different properties we determine clearly relate to spectral diffusion kernels, and we discuss the implications for other systems.

Index transforms have missed the same treatment as more prevalent results in transform theory. This class of integral transforms is more technically complex when compared to its more well-known sibling disciplines. As is well known, many physical systems, particularly those from quantum physics, have kernels with discrete eigenvalues. The kernels given through index transforms are different from this well-beaten path, as they contain eigenvalues that are continuous in the domain of the spectrum. Scattered states are often approximated in far-field at fixed angles, so the continuous nature of their dependencies on function indices can be easily overlooked. The use of index transforms allows many of the various different results available in integral calculus to be understood in the same framework of special function theory.

This chapter shall show that very little apparatus is required to derive a complex structure sufficient to describe many different groups important to the theory of special functions. However, a broad disclaimer must be made for any of the results contained herein. We have used results from integral tables at times, and much of the analysis depends on the use of different integral representations for the various types of special functions, their products, so on and so forth. As a result, we have endeavoured to reference sufficient information to track the equation to the resource material and ensure that all external results are properly able to be found. However, this is no guarantee of accuracy in the source material, and it may be possible that some integrals in tables are incorrect as written. As far as plausible, an effort has been made to identify any used formula(e); to determine any inconsistencies with said equations; and to state clearly any necessary modifications necessary to achieve our final results. Broadly speaking, this has been a successful program of analysis, and has enabled a number of different techniques to be developed in the field of kernels and diffusion theory which appear to have been previously lost to mathematical science.

5.2 Outline of Chapter

The following chapter is divided into four sections, broken down into derivation of the projection-slice theorem for special functions, methodology, application to special function theory and further computations associated with index transforms. The first and last of these are relatively free-standing, but the reader is advised to read the first two sections in order to avoid complication. We close the chapter with some discussion of some more advanced topics in diffusion theory that are opened up through the use of kernel transforms.

5.3 Extension of the Projection-Slice Theorem

We shall briefly show how the projection slice theorem is derived, and how this may be extended to the Mehler-Fock, Kontorovich-Lebedev and Whittaker transforms. In [23–26] the author gives the following derivation of FAH cycle which brings together the Fourier, Abel and Hankel transforms.

Theorem 5.3.1. *The Fourier-Abel-Hankel transforms compose a cycle, called the FAH cycle. For a radial function $f(r)$, these transforms are related by:*

$$(\mathcal{F}\mathcal{A})[f] = (\mathcal{H})[f] \tag{5.6}$$

where $\mathcal{F}$ is the Fourier transform, $\mathcal{A}$ is the Abel transform, and $\mathcal{H}$ is the Hankel transform.

Proof. This theorem originally appears in Bracewell c. 1956, (see the series of papers and works in [23–26]) . A modern proof proceeds using the following argument. The left-hand side of the identity may be written:

$$\mathcal{F}\mathcal{A}[f] = \int_{-\infty}^{+\infty} ds.e^{i2\pi sq}.2\int_{s}^{\infty} dx.\frac{xf(x)}{\sqrt{x^2-s^2}} \tag{5.7}$$

$$= 2\int_{0}^{\infty} dx.xf(x)\int_{-x}^{+x} ds.\frac{e^{i2\pi sq}}{\sqrt{x^2-s^2}} \tag{5.8}$$

where we have invoked the Fubini-Tonelli theorem, and altered the limits of the integrals to take into account the reparametrisation. The inner integral may be evaluated, using formula 8.414.10 GR from [122], we have the integral representation for the Bessel function:

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)}\int_{-1}^{+1} e^{izt}(1-t^2)^{\nu-1/2}dt \tag{5.9}$$

where $\Re\nu > -1/2$. The special case $\nu = 0$ then yields:

$$J_0(z) = \frac{1}{\Gamma(1/2)\Gamma(1/2)}\int_{-1}^{+1}\frac{e^{izt}}{\sqrt{1-t^2}}dt = \frac{1}{\pi}\int_{-1}^{+1}\frac{e^{izt}}{\sqrt{1-t^2}}dt \tag{5.10}$$

Changing the argument of the function, we find:

$$J_0(2\pi xq) = \frac{1}{\pi}\int_{-1}^{+1} \frac{e^{i2\pi xqt}}{\sqrt{1-t^2}}dt \tag{5.11}$$

which upon substitution of $t = \dfrac{\chi}{x}$ leads to the expression:

$$J_0(2\pi xq) = \frac{1}{\pi}\int_{-x}^{+x} \frac{e^{i2\pi q\chi}}{\sqrt{x^2-\chi^2}}d\chi \tag{5.12}$$

Subsequently, we find that we may write:

$$\mathcal{FA}[f] = 2\pi\int_0^\infty dx.xf(x)J_0(2\pi xq) \tag{5.13}$$

Writing this out in terms of the transforms, we have:

$$(\mathcal{FA})[f] = (\mathcal{H})[f] \tag{5.14}$$

as required. □

We shall now show how this theorem, which is of invaluable use in the field of computer tomography as well as many other applications, may be extended to the various types of index transforms. The fundamental concept is to extend the Fourier transform to the hyperbolic plane. As we have demonstrated in the previous sections of this work, the eigenfunctions associated to the hyperbolic plane and heat kernel are given by the Mehler-Fock functions. In the same way that the Fourier eigenfunctions are given by the diffusion on an interval, and associated with the kernel, the Mehler-Fock functions are defined by the corresponding problem in the hyperbolic domain. The functions we shall use to implement the following sets of transform pairs are defined in this thesis in 3.4.1, 3.4.2 and 3.4.3, where we have outlined the series and integral representations for the associated Legendre, Whittaker and modified Bessel functions.

Definition 5.3.2. The Mehler-Fock transform of a continuous function is defined by the integral:

$$\mathcal{M}f(y) = F(\nu) = \int_1^\infty \mathcal{P}^\mu_{i\nu-1/2}(y)f(y)dy \tag{5.15}$$

The inverse transform is given by the formula:

$$\mathcal{M}^{-1}F(\nu) = f(y) = \frac{1}{\pi}\int_0^\infty \nu\sinh(\pi\nu)\mathcal{P}^\mu_{i\nu-1/2}(y)\left|\Gamma\left(1/2-\mu+i\nu\right)\right|^2 d\nu \tag{5.16}$$

where $\mathcal{P}^\mu_{i\nu-1/2}(z)$ is the associated Legendre function of toroidal type, see [122]. This transform and its inverse exist for functions that decay like $f(y) = \mathcal{O}(y^\alpha)$ as $y \to \infty$, for some $\alpha < -1/2$, with domain $1 \le y < \infty$, that is, we have that the function $\dfrac{f(y)}{\sqrt{y}}$ is absolutely integrable on $[1,\infty)$. See Zayed [147], Thm. 25.2.

Definition 5.3.3. The Kontorovich-Lebedev transform of a continuous function is:

$$\mathcal{K}u(y) = U(\nu) = \int_0^\infty \frac{K_{i\nu}(y)}{y}u(y)dy \tag{5.17}$$

with inverse transform:

$$\mathcal{K}^{-1}U(\nu) = u(y) = \frac{2}{\pi^2}\int_0^\infty K_{i\nu}(y)U(\nu)\sinh(\pi\nu)\nu d\nu \tag{5.18}$$

where $K_{i\nu}(y)$ is the modified Bessel function of the second kind, also known as the Macdonald function. This transform and its inverse exist and are well-defined for $f(y) = \mathcal{O}(e^{cy})$ as $y \to \infty$ for some $0 \le c < 1$, with domain $0 < y < \infty$. See Zayed, [147], Thm. 24.2.

Definition 5.3.4. The Whittaker transform of a continuous function is:

$$(\mathcal{W}f)(\tau) = \bar{f}(\tau) = \int_0^\infty f(x)\frac{W_{\alpha,i\tau}(x)}{x^2}dx \tag{5.19}$$

with inverse:

$$(\mathcal{W}^{-1}\bar{f})(x) = f(x) = \frac{1}{\pi^2}\int_0^\infty \bar{f}(\tau)W_{\alpha,i\tau}(2x)\tau\sinh(2\pi\tau)\left|\Gamma\left(1/2-\alpha+i\tau\right)\right|^2 d\tau \tag{5.20}$$

where $W_{\alpha,i\tau}(.)$ is the Whittaker function of the second kind. This transform exists and is well defined where we have as $x \to \infty$, $f(x) = \mathcal{O}\left(e^{x/2}x^{2-\alpha}\right)$ and

$$\sum_{s=0}^\infty \frac{(1/2+i\tau-\alpha)_s\,(1/2-i\tau-\alpha)_s}{s!}(-x)^{-s} < \infty \tag{5.21}$$

where $(.)_s$ is the rising factorial (Pochammer symbol). This can be derived from the asymptotic expansion for large argument:

$$\frac{W_{\alpha,i\tau}(x)}{x^2} \sim e^{-x/2}x^{\alpha-2}\sum_{s=0}^\infty \frac{(1/2+i\tau-\alpha)_s\,(1/2-i\tau-\alpha)_s}{s!}(-x)^{-s} \tag{5.22}$$

which may be found in DLMF eq. 13.19.3 [8]. Note that the original definition of the index Whittaker transform as given by Wimp c. 1964 [142] differs from this by the factor of x^2 which is taken inside the function to be transformed.

Definition 5.3.5. The Laplace transform of a continuous function is:

$$(\mathcal{L}f)(s) = \int_0^\infty e^{-st}f(t)dt = \bar{f}(s) \tag{5.23}$$

where it exists and is well defined, i.e. for $f(t) = \mathcal{O}(e^{ct})$, where $c < s$. The inverse Laplace transform is given by the complex integral:

$$f(t) = \lim_{c\to 0^+}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)ds \tag{5.24}$$

Remark 85. We note that many other variations of these transforms exist, in particular the definitions of the Kontorovich-Lebedev and Mehler-Fock transforms have many alternative formulations. We take our definitions from Sousa and Yakubovich, where the authors have used the Titchmarsh-Kodaira theorem to reduce the transforms to the study of the solutions of the diffusion kernel, see the series of papers in [37,56,112].

We shall now prove some new results that are equivalent to the projection-slice theorem in the hyperbolic plane. We note the integral formulae from [122] (eqs. 7.141.5, 7.142.1):

Definition 5.3.6. The Whittaker form of the projection-slice theorem can be written using the integral:

$$\int_1^\infty e^{-yz}\left(\frac{z+1}{z-1}\right)^{\mu/2}\mathcal{P}^\mu_{i\nu-1/2}(z)dz = \frac{W_{\mu,i\nu}(2y)}{y} \tag{5.25}$$

Definition 5.3.7. The modified Bessel form of the projection-slice theorem can be written using a similar integral:

$$\int_1^\infty e^{-yz}\left(z^2-1\right)^{-\mu/2}\mathcal{P}^\mu_{i\nu-1/2}(z)dz = \sqrt{\frac{2}{\pi}}y^{\mu-1/2}K_{i\nu}(y) \tag{5.26}$$

Using these integrals, we can then derive a form of the projection-slice theorem that is valid on the hyperbolic plane, i.e. involves special functions associated to the different representations of this space as explored in the previous chapter. In particular, we have the following result.

Theorem 5.3.8. *The first form of the hyperbolic projection-slice theorem is given by:*

$$\mathcal{L}f(y) = \left(\frac{z+1}{z-1}\right)^{-\mu/2} \mathcal{M}^{-1}[\mathcal{W}\left[yf\left(\frac{y}{2}\right)\right]] \tag{5.27}$$

where $\mathcal{L}$ is the Laplace transform, $\mathcal{M}^{-1}$ is the inverse Mehler-Fock transform, and $\mathcal{W}$ is the Whittaker transform as given in the definitions 5.3.2, 5.3.3, 5.3.4, 5.3.5.

Proof. Writing the Whittaker integral, it is straightforward to see that we have:

$$\int_0^\infty \frac{W_{\mu,i\nu}(2y)}{y} f(y)dy = \int_0^\infty \frac{W_{\mu,i\nu}(y)}{(y/2)} f(y/2)d(y/2) \tag{5.28}$$

$$= \int_0^\infty \frac{W_{\mu,i\nu}(y)}{y} f\left(\frac{y}{2}\right) dy = \int_0^\infty \frac{W_{\mu,i\nu}(y)}{y^2} yf\left(\frac{y}{2}\right) dy = \mathcal{W}\left[yf\left(\frac{y}{2}\right)\right] \tag{5.29}$$

Taking the integral formula 5.3.6, we then multiply both sides with respect to a function $f(y)$, and integrate with respect to y to obtain:

$$\int_0^\infty \frac{W_{\mu,i\nu}(2y)}{y} f(y)dy = \int_1^\infty \left(\frac{z+1}{z-1}\right)^{\mu/2} \mathcal{P}^{\mu}_{i\nu-1/2}(z)dz \int_0^\infty e^{-yz} f(y)dy \tag{5.30}$$

$$= \int_1^\infty \left(\frac{z+1}{z-1}\right)^{\mu/2} \mathcal{P}^{\mu}_{i\nu-1/2}(z)F(z)dz \tag{5.31}$$

where we have invoked the Fubini-Tonelli theorem to reverse the order of the integrals. Writing now $\bar{F}(z) = \left(\frac{z+1}{z-1}\right)^{\mu/2} F(z)$, we have:

$$\bar{F}(z) = \left(\frac{z+1}{z-1}\right)^{\mu/2} F(z) = \left(\frac{z+1}{z-1}\right)^{\mu/2} \int_0^\infty e^{-yz} f(y)dy = \left(\frac{z+1}{z-1}\right)^{\mu/2} \mathcal{L}f \tag{5.32}$$

$$= \int_1^\infty \mathcal{P}^{\mu}_{i\nu-1/2}(z)\bar{F}(z)dz = \mathcal{M}\bar{F}(z) = \mathcal{W}\left[yf\left(\frac{y}{2}\right)\right] \tag{5.33}$$

which can be rewritten on inversion of the Mehler-Fock transform to yield:

$$\int_0^\infty e^{-yz} f(y)dy = \mathcal{L}f = \left(\frac{z+1}{z-1}\right)^{-\mu/2} \mathcal{M}^{-1}[\mathcal{W}\left[yf\left(\frac{y}{2}\right)\right]] \tag{5.34}$$

as required. □

An equivalent formula exists for the Kontorovich-Lebedev transform, and is readily derived in the same fashion.

Theorem 5.3.9. *The Kontorovich-Lebedev form of the projection-slice theorem is given by:*

$$\mathcal{L}f(y) = \left(z^2-1\right)^{\mu/2} \mathcal{M}^{-1}\left(\mathcal{K}\left(\sqrt{\frac{2}{\pi}} y^{\mu+1/2} f(y)\right)\right) \tag{5.35}$$

where $\mathcal{K}$ is the Kontorovich-Lebedev transform as in 5.3.3, the Mehler-Fock and Laplace transform as in 5.3.2.

Proof. As before, we begin with the integral:

$$\int_1^\infty e^{-yz}\left(z^2-1\right)^{-\mu/2}\mathcal{P}^\mu_{i\nu-1/2}(z)dz=\sqrt{\frac{2}{\pi}}y^{\mu-1/2}K_{i\nu}(y) \tag{5.36}$$

Integrating both sides with respect to a function $f(y)$, and invoking the Fubini-Tonelli theorem to reverse the order of integration, we find:

$$\int_0^\infty \sqrt{\frac{2}{\pi}}y^{\mu-1/2}K_{i\nu}(y)f(y)dy=\int_0^\infty \left(z^2-1\right)^{-\mu/2}\mathcal{P}^\mu_{i\nu-1/2}(z)\int_0^\infty e^{-yz}f(y)dy \tag{5.37}$$

$$=\int_0^\infty \left(z^2-1\right)^{-\mu/2}\mathcal{P}^\mu_{i\nu-1/2}(z)F(z)dz \tag{5.38}$$

$$=\int_0^\infty \mathcal{P}^\mu_{i\nu-1/2}(z)\bar{F}(z)dz \tag{5.39}$$

Defining the function $\bar{F}(z)=\left(z^2-1\right)^{-\mu/2}F(z)$, we then may write:

$$\bar{F}(z)=\left(z^2-1\right)^{-\mu/2}F(z)=\mathcal{M}^{-1}\left(\int_0^\infty\sqrt{\frac{2}{\pi}}y^{\mu-1/2}K_{i\nu}(y)f(y)dy\right) \tag{5.40}$$

and inserting the definition of the Kontorovich-Lebedev transform 5.3.3 we obtain in a similar fashion to the previous proof, the formula:

$$F(z)=\mathcal{L}f(y)=\left(z^2-1\right)^{\mu/2}\mathcal{M}^{-1}\left(\mathcal{K}\left(\sqrt{\frac{2}{\pi}}y^{\mu+1/2}f(y)\right)\right) \tag{5.41}$$

□

We now employ these two formulae to derive the following corollary.

Corollary 5.3.10. *The Whittaker and Kontorovich-Lebedev transforms are related by the intertwining formula:*

$$\mathcal{W}\left[yf\left(\frac{y}{2}\right)\right]=\mathcal{M}\left[(z+1)^\mu\mathcal{M}^{-1}\left(\mathcal{K}\left(\sqrt{\frac{2}{\pi}}y^{\mu+1/2}f(y)\right)\right)\right] \tag{5.42}$$

where the transforms are as defined in 5.3.3, 5.3.2, 5.3.4.

Proof. We have the equality of the Laplace transform from 5.3.8, 5.3.9, writing this explicitly:

$$\mathcal{L}f(y)=\left(z^2-1\right)^{\mu/2}\mathcal{M}^{-1}\left(\mathcal{K}\left(\sqrt{\frac{2}{\pi}}y^{\mu+1/2}f(y)\right)\right) \tag{5.43}$$

$$=\left(\frac{z+1}{z-1}\right)^{-\mu/2}\mathcal{M}^{-1}[\mathcal{W}\left[yf\left(\frac{y}{2}\right)\right]] \tag{5.44}$$

Cancelling the numerical factors we obtain:

$$\left(\frac{z+1}{z-1}\right)^{-\mu/2}\mathcal{M}^{-1}[\mathcal{W}\left[yf\left(\frac{y}{2}\right)\right]]=\left(z^2-1\right)^{\mu/2}\mathcal{M}^{-1}\left(\mathcal{K}\left(\sqrt{\frac{2}{\pi}}y^{\mu+1/2}f(y)\right)\right) \tag{5.45}$$

$$=(z-1)^{\mu/2}(z+1)^{\mu/2}\mathcal{M}^{-1}\left(\mathcal{K}\left(\sqrt{\frac{2}{\pi}}y^{\mu+1/2}f(y)\right)\right) \tag{5.46}$$

Therefore we may write:

$$\mathcal{M}^{-1}[\mathcal{W}\left[yf\left(\frac{y}{2}\right)\right]] = \left(\frac{z+1}{z-1}\right)^{\mu/2}(z-1)^{\mu/2}(z+1)^{\mu/2}\mathcal{M}^{-1}\left(\mathcal{K}\left(\sqrt{\frac{2}{\pi}}y^{\mu+1/2}f(y)\right)\right) \tag{5.47}$$

$$= (z+1)^{\mu}\mathcal{M}^{-1}\left(\mathcal{K}\left(\sqrt{\frac{2}{\pi}}y^{\mu+1/2}f(y)\right)\right) \tag{5.48}$$

Inverting the Mehler-Fock transform, we may therefore write:

$$\mathcal{W}\left[yf\left(\frac{y}{2}\right)\right] = \mathcal{M}\left[(z+1)^{\mu}\mathcal{M}^{-1}\left(\mathcal{K}\left(\sqrt{\frac{2}{\pi}}y^{\mu+1/2}f(y)\right)\right)\right] \tag{5.49}$$

as required. □

Remark 86. Note the explicit integral formula for this may be written as:

$$\int_0^\infty \frac{W_{\mu,i\nu}(2y)}{y}f(y)dy = \mathcal{M}\left[(z+1)^{\mu}\mathcal{M}^{-1}\left(\mathcal{K}\left(\sqrt{\frac{2}{\pi}}y^{\mu+1/2}f(y)\right)\right)\right] \tag{5.50}$$

where we have obtained the left-hand side using the proof of 5.3.8.

We then obtain the following easy special case, implied by the argument $\mu = 0$ in the Whittaker function.

Theorem 5.3.11. *The Whittaker function and the Macdonald function are related by:*

$$W_{0,i\nu}(2y) = \sqrt{\frac{2y}{\pi}}K_{i\nu}(y) \tag{5.51}$$

Proof. We take the case $\mu = 0$ in 5.3.10, we have:

$$\int_0^\infty \frac{W_{\mu,i\nu}(2y)}{y}f(y)dy = \mathcal{M}\left[(z+1)^{\mu}\mathcal{M}^{-1}\left(\mathcal{K}\left(\sqrt{\frac{2}{\pi}}y^{\mu+1/2}f(y)\right)\right)\right] \tag{5.52}$$

and consequently, as the Mehler-Fock is an invertible transform we have $\mathcal{M}\mathcal{M}^{-1} = 1$ and hence:

$$\int_0^\infty \frac{W_{0,i\nu}(2y)}{y}f(y)dy = \mathcal{K}\left(\sqrt{\frac{2}{\pi}}y^{1/2}f(y)\right) \tag{5.53}$$

Using the formula for the Kontorovich-Lebedev and Whittaker transforms, we then have:

$$\int_0^\infty \frac{W_{0,i\nu}(2y)}{y}f(y)dy = \mathcal{K}\left(\sqrt{\frac{2}{\pi}}y^{1/2}f(y)\right) = \int_0^\infty \sqrt{\frac{2y}{\pi}}\frac{K_{i\nu}(y)}{y}f(y)dy \tag{5.54}$$

implying that $W_{0,i\nu}(2y) = \sqrt{\frac{2y}{\pi}}K_{i\nu}(y)$ as claimed. □

We shall now show a more direct application of the connection between these three systems of eigenfunctions as given through the action of the Laplace transform upon some simple diffusion problems.

5.4 Changes of Variables for Diffusion Generators

The following brief calculations shall show how the sets of eigenfunctions described by the Whittaker, Kontorovich-Lebedev and Mehler-Fock functions are related. This is achieved most directly through the differential representation of the Mehler-Fock function. We shall require the following lemma.

Lemma 5.4.1. *The two differential equations defined by:*

$$u'' + a_1(x)u' + a_2(x)u = 0 \tag{5.55}$$

and

$$U'' + U\left[\phi' + \phi^2 + \phi a_1 + a_2\right] = 0 \tag{5.56}$$

are equivalent, provided that:

$$u(x) = \exp\left(-\frac{1}{2}\int^x a_1(\tau)d\tau\right)U(x) \tag{5.57}$$

and $\phi' = -\frac{a_1}{2}$.

Proof. Take the transformation, known from the theory of differential equations, and integrating factor:

$$\phi(x) = -\frac{a_1(x)}{2} \tag{5.58}$$

$$u(x) = e^{\int \phi dx}U(x) \tag{5.59}$$

Substituting into the differential equation, one obtains the result. $\square$

Definition 5.4.2. We recall the DE satisfied by the associated Legendre functions [121] is 3.4.1:

$$\frac{\partial^2 f}{\partial \tau^2} + \coth\tau\frac{\partial f}{\partial \tau} + \left(-\frac{k^2}{\sinh^2\tau} + \left(\frac{1}{4} + \xi^2\right)\right)f = 0 \tag{5.60}$$

We now state the following two alternate forms of this differential equation.

Lemma 5.4.3. *The differential equation for the Mehler-Fock functions may be written:*

$$\frac{\partial^2 F}{\partial \tau^2} + \left(\frac{(1/4 - k^2)}{\sinh^2\tau} + \xi^2\right)F = 0 \tag{5.61}$$

or alternatively:

$$(z^2-1)\frac{\partial^2 f}{\partial z^2} + 2z\frac{\partial f}{\partial z} + \left(-k^2 + \frac{\left(\frac{1}{4} + \xi^2\right)}{(z^2-1)}\right)f = 0 \tag{5.62}$$

Proof. For the first part of the lemma, we take the transformation 5.4.1, applying this we have:

$$\left(\frac{\partial}{\partial \tau}\left[-\frac{\coth\tau}{2}\right] + \left[-\frac{\coth\tau}{2}\right]^2 - \frac{1}{2}(\coth\tau)^2 + \left(-\frac{k^2}{\sinh^2\tau} + \left(\frac{1}{4} + \xi^2\right)\right)\right)F = 0 \tag{5.63}$$

$$= -\frac{\partial^2 F}{\partial \tau^2} \tag{5.64}$$

or

$$\frac{\partial^2 F}{\partial \tau^2} + \left(-\frac{k^2}{\sinh^2 \tau} + \frac{1}{4}\left(\coth^2 \tau - 1\right) + \xi^2 \right) F \tag{5.65}$$

$$= \frac{\partial^2 F}{\partial \tau^2} + \left(\frac{(1/4 - k^2)}{\sinh^2 \tau} + \xi^2 \right) F = 0 \tag{5.66}$$

as required. To establish the claim in the second part of the lemma, we use a hyperbolic substitution. Removing any singular terms in the denominator of the differential equation, we have:

$$\frac{\partial^2 f}{\partial \tau^2} + \coth \tau \frac{\partial f}{\partial \tau} + \left(-\frac{k^2}{\sinh^2 \tau} + \left(\frac{1}{4} + \xi^2 \right) \right) f = 0 \tag{5.67}$$

and equivalently:

$$\sinh^2 \tau \frac{\partial^2 f}{\partial \tau^2} + \sinh \tau \cosh \tau \frac{\partial f}{\partial \tau} + \left(-k^2 + \left(\frac{1}{4} + \xi^2 \right) \sinh^2 \tau \right) f = 0 \tag{5.68}$$

Using $z = \cosh \tau$, $z^2 - \sinh^2 \tau = 1$, we can rewrite:

$$\frac{\partial f}{\partial \tau} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial \tau} = \sinh \tau \frac{\partial f}{\partial z} = \sqrt{z^2 - 1}\frac{\partial f}{\partial z} \tag{5.69}$$

$$\frac{\partial f}{\partial z} = \frac{1}{\sinh \tau}\frac{\partial f}{\partial \tau} \tag{5.70}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{\sinh \tau}\frac{\partial}{\partial \tau}\frac{1}{\sinh \tau}\frac{\partial f}{\partial \tau} = \frac{1}{\sinh^2 \tau}\frac{\partial^2 f}{\partial \tau^2} - \left(\frac{\cosh \tau}{\sinh^3 \tau} \right)\frac{\partial f}{\partial \tau} \tag{5.71}$$

$$= \frac{1}{\sinh^2 \tau}\left(\frac{\partial^2 f}{\partial \tau^2} - \coth \tau \frac{\partial f}{\partial \tau} \right) \tag{5.72}$$

We therefore have the derivatives as specified by the transformations:

$$(z^2 - 1)\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial \tau^2} - \coth \tau \frac{\partial f}{\partial \tau} \tag{5.73}$$

$$\frac{\partial f}{\partial z} = \frac{1}{\sinh \tau}\frac{\partial f}{\partial \tau} \tag{5.74}$$

$$z\frac{\partial f}{\partial z} = \coth \tau \frac{\partial f}{\partial \tau} \tag{5.75}$$

Substituting these into the differential equation, we obtain:

$$(z^2 - 1)\frac{\partial^2 f}{\partial z^2} + 2z\frac{\partial f}{\partial z} + \left(-k^2 + \frac{\left(\frac{1}{4} + \xi^2 \right)}{(z^2 - 1)} \right) f = 0 \tag{5.76}$$

as required. □

Remark 87. We can also solve the DE directly using a hyperbolic substitution. The Mehler-Fock equation can be written:

$$(z^2 - 1)\frac{\partial^2 f}{\partial z^2} + 2z\frac{\partial f}{\partial z} + \left(\frac{(\nu^2 + 1)}{4} - \frac{k^2}{(z^2 - 1)} \right) f = 0 \tag{5.77}$$

where we will have solution

$$f(z) = C_1 \mathcal{P}^k_{i\nu - 1/2}(z) + C_2 \mathcal{Q}^k_{i\nu - 1/2}(z) \tag{5.78}$$

We can see clearly from this how the different representations arise.

We shall now demonstrate how the different types of eigenfunctions defined by the generators of the Mehler-Fock differential equation are related. Each of these, as given by the Macdonald and Whittaker function equivalents to the Mehler-Fock type, is associated with an isospectral representation of the hyperbolic plane. We shall demonstrate this directly using the integral formulae 5.3.7, 5.3.6, as given in Gradshteyn and Ryzik 7.141.5:

$$\int_1^\infty e^{-ax}\left(\frac{x+1}{x-1}\right)^{\mu/2}\mathcal{P}^\mu_{\nu-1/2}(x)dx = \frac{W_{\mu,\nu}(2a)}{a} \tag{5.79}$$

We state the following series of lemmas which we shall require in order to determine the connection between these sets of eigenfunctions.

Lemma 5.4.4. *For the Mehler-Fock generator defined by the DE:*

$$\hat{\mathcal{H}} = (z^2-1)\partial_z^2 + 2z\partial_z + \left(\frac{\nu^2+1}{4} - \frac{k^2}{z^2-1}\right) \tag{5.80}$$

where $\hat{\mathcal{H}}f = 0$, there exists a function g such that:

$$[g(z)]^{-k/2}\hat{\mathcal{H}}\left([g(z)]^{k/2}f(z)\right) = \hat{\mathcal{H}}'[f(z)] \tag{5.81}$$

with

$$\hat{\mathcal{H}}' = (z^2-1)\partial_z^2 + 2(z-k)\partial_z + \frac{(\nu^2+1)}{4} \tag{5.82}$$

Proof. The transformation is defined through the integral relation 5.4.4. Substitution of $g(z) = \dfrac{z+1}{z-1}$ yields the desired result. We have:

$$(z^2-1)\frac{\partial^2 f}{\partial z^2} + 2z\frac{\partial f}{\partial z} + \left(\frac{(\nu^2+1)}{4} - \frac{k^2}{(z^2-1)}\right)f = 0 = \mathcal{H}[f(z)] \tag{5.83}$$

$$\mathcal{H}\left[\left(\frac{z+1}{z-1}\right)^{k/2}f(z)\right] = \left(\frac{z+1}{z-1}\right)^{k/2}\left[\frac{(\nu^2+1)}{4}f + 2(z-k)\frac{\partial f}{\partial z} + (z^2-1)\frac{\partial^2 f}{\partial z^2}\right] \tag{5.84}$$

$$= \mathcal{H}'[f(z)] \tag{5.85}$$

as required. □

Corollary 5.4.5. *Assume the boundary conditions $\dfrac{\partial f}{\partial z}|_{z=0^+} = f(0) = 0$. From 5.3.8, we know that the Laplace transform of the DE given by 5.4.4 is given by:*

$$\mathcal{L}[\mathcal{H}[f]] = \int_0^\infty e^{-px}\mathcal{H}'[f(x)]dx \tag{5.86}$$

$$= p^2\frac{\partial^2 F}{\partial p^2} + 2p\frac{\partial F}{\partial p} + \left(\frac{1}{4}(\nu^2+1) - 2pk - p^2\right)F + f'(0) + (2k+p)f(0) = 0 \tag{5.87}$$

$$= p^2\frac{\partial^2 F}{\partial p^2} + 2p\frac{\partial F}{\partial p} + \left(\frac{1}{4}(\nu^2+1) - 2pk - p^2\right)F = 0 \tag{5.88}$$

We immediately obtain the solution as a sum of Whittaker functions:

$$F(p) = C_1\frac{W_{-k,i\nu/2}(2p)}{p} + C_2\frac{M_{-k,i\nu/2}(2p)}{p} \tag{5.89}$$

Proof. An elementary application of Laplace transforms produces the formula. The boundary terms do not contribute to the eigenfunction representation of the solution, as they only contain inhomogenous terms which in our analysis are not required. In this situation, we take the case $1 < z < \infty$, such that the Laplace transform exists. As a result of the properties of the Whittaker function for large arguments, we have:

$$W_{\alpha,i\tau}(x) \sim e^{-x/2}x^{\alpha}\sum_{s=0}^{\infty}\frac{(1/2+i\tau-\alpha)_s\,(1/2-i\tau-\alpha)_s}{s!}(-x)^{-s} \tag{5.90}$$

and consequently one can see that the inverse Laplace transform will exist for:

$$\sum_{s=0}^{\infty}\frac{(1/2+i\tau-\alpha)_s\,(1/2-i\tau-\alpha)_s}{s!}(-x)^{-s} < \infty \tag{5.91}$$

and $\bar{f}(x) = \mathcal{O}\left(e^{x/2}x^{2-\alpha}\right)$. □

5.5 Construction of Index Transforms

As is known from classical analysis and transform theory, the Hankel transform is the radial Fourier transform. In the following section, we show a simple way that one can derive the Hankel, generalised Hankel and Kontorovich-Lebedev transforms through use of area integration in the complex plane. This is one entry point we shall use in order to define the basic forms of index integration. This method, employed in Chavel [11], differs from the standard kernel techniques we have employed to analyse the hyperbolic heat kernel. As is well known, the Hankel transform is given by a form of the two-dimensional Fourier transform, which in our context represents the projection-slice theorem. In this second formulation of index transforms, we outline the basic theory of transforms as it relates to the eigenfunction problems we have shown to be important on the various hyperbolic spaces. In particular, we shall consider the method of index transformation. Index transforms are more difficult in terms of tractability than the more familiar types of transforms from optics and wave theory. For this reason, we shall consider first a simplistic example related to the Hankel transform. Using similar methods, we shall then show how other types of transforms related to Bessel functions may be constructed, specifically, the Kontorovich-Lebedev transformation.

5.5.0.1 Hankel Transform

We shall now show how the Hankel transform may be derived using some simple concepts from complex integration theory. As the projection-slice theorem shows, there is a direct relationship between the two-dimensional Fourier transform and the Hankel transform, as given by the FAH cycle. What is less obvious is that this can be expressed in terms of an area integral in the complex plane. We define this area integral as:

Definition 5.5.1. The area integral in the complex plane is defined through the formula:

$$\iint_{\mathbb{C}} dA(z) = \frac{1}{2\pi}\iint_{\mathbb{C}} d^2\mu(z) \tag{5.92}$$

where $z = re^{i\theta} \in \mathbb{C}$, and the area element is given by $d^2\mu(z) = r dr d\theta$, $0 < r < \infty$, and $\theta \in [0, 2\pi)$.

In the following, we replicate the argument from Chavel [11] to derive the Hankel transform, and then propose some simple modifications that will allow us to generalise this simple example. We require the following definition from [11]:

Definition 5.5.2. There exists a transform $\Phi(w)$ associated with the complex area integral of a radial function, given by:

$$\Phi(w) = \frac{1}{2\pi} \int\int_{\mathbb{C}} \varphi(r) e^{i\mathrm{Re}(zw^*)} d^2\mu(z) = \mathcal{F}[\varphi(r)] \tag{5.93}$$

where $d^2\mu(z)$ is the area measure in the complex plane, and $z = re^{i\theta}$, and $w \in \mathbb{C}$. A function which is radial in the complex plane is a function solely of r.

We now apply this definition to derive the following two small lemmas, which yield the Hankel and generalised Hankel transforms.

Lemma 5.5.3. *Assume the cylindrical basis such $z = re^{i\theta}$, $w = \rho$, then the area overlap integral defined by 5.5.2 is given by the Hankel transform:*

$$\Phi(w) = \frac{1}{2\pi} \int_0^\infty r dr . \varphi(r) J_0(\rho r) \tag{5.94}$$

where $J_0(.)$ is the Bessel function of order zero.

Proof. Following the argument in [11], let us take a radial representation of $\mathbb{R}^2 \to \mathbb{C}$ which we shall write as $d^2\mu(z) = r dr d\theta$, $\mathrm{Re}\,(zw^*) = \frac{1}{2}(zw^* + z^*w) = r\rho\cos\theta$, where we have taken a polar-axial co-ordinate system such that $z = re^{i\theta}$, $w = \rho$. Writing out the area integral explicitly from 5.5.2, we have:

$$\Phi_0(\rho) \doteq \Phi(w) = \frac{1}{2\pi} \int\int_{\mathbb{C}} \varphi(r) e^{i\mathrm{Re}(zw^*)} d^2\mu(z) = \mathcal{F}[\varphi(r)] \tag{5.95}$$

In this case, we may write:

$$\Phi_0(\rho) = \frac{1}{2\pi} \int_0^\infty r dr \int_0^{2\pi} d\theta . \varphi(r) e^{ir\rho\cos\theta} \tag{5.96}$$

which, by computing the inner integral, we find:

$$\Phi_0(\rho) = \frac{1}{2\pi} \int_0^\infty r dr . \varphi(r) J_0(\rho r) \tag{5.97}$$

as claimed. □

We then are able to modify the area integral 5.5.2 to recover the generalised Hankel transform in the following corollary.

Corollary 5.5.4. *Assume the modification of 5.5.2 such that:*

$$\Phi_n(\rho) = \Phi_n(w) = \frac{1}{2\pi} \int\int_{\mathbb{C}} \varphi(r) e^{i\mathrm{Re}(zw^*) + in[\theta - \pi/2]} d^2\mu(z) \tag{5.98}$$

Then the area integral is given by the generalised Hankel transform:

$$\Phi_n(\rho) = \frac{1}{2\pi} \int_0^\infty r\varphi(r) J_n(\rho r) dr \tag{5.99}$$

Proof. Begin with the integral representation of the Bessel function, we have:

$$J_n(z) = \frac{1}{2\pi i^n} \int_0^{2\pi} e^{i(z\cos\theta + n\theta)} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{i(z\cos\theta + n[\theta - \pi/2])} d\theta \tag{5.100}$$

Substituting into the modified area integral, we find:

$$\Phi_n(w) = \frac{1}{2\pi} \int\int_{\mathbb{C}} \varphi(r) e^{i\mathrm{Re}(zw^*) + in[\theta - \pi/2]} d^2\mu(z) \tag{5.101}$$

and hence we recover the generalised Hankel transform:

$$\Phi_n(\rho) = \frac{1}{2\pi} \int_0^\infty r\varphi(r) J_n(\rho r) dr \tag{5.102}$$

as claimed. □

5.5.0.2 Kontorovich-Lebedev Transform

The Kontorovich-Lebedev transform can be understood as a similar area integral. Using the connection between the different hyperbolic transforms and the composition formula, we can write down the form of the kernel solution as an integral transform of the initial data. Further, this integral transform is given as a convolution product of the eigenfunction with the initial data. We state this in the following lemma.

Lemma 5.5.5. *In the hyperbolic plane $x^2 - y^2 = r^2$, given by the parametrisation:*

$$x = r\cosh\tau, y = r\sinh\tau \tag{5.103}$$

the overlap integral defined in an analogous way to 5.5.2:

$$\Phi(w) = \int\int_{\mathbb{H}} \varphi(\alpha) e^{-\mathrm{Re}(zw^*)} \cosh\alpha\tau . d^2\mu(z) \tag{5.104}$$

defines a Kontorovich-Lebedev integral:

$$\Phi_\alpha(w) = \int_0^\infty \frac{d\alpha}{\alpha^2} . \varphi(\alpha) K_\alpha(\rho r) \tag{5.105}$$

Proof. Let us consider the hyperbolic plane. In this case, we will have as in the definition $x^2 - y^2 = r^2$, given by the parametrisation:

$$x = r\cosh\tau, y = r\sinh\tau \tag{5.106}$$

. We may easily define the subsidiary variables $z = re^\tau = x + y$, $z^* = x - y = re^{-\tau}$, and $w = \rho$. Writing the overlap integral, we have:

$$\Phi(w) = \int\int_{\mathbb{H}} \varphi(\alpha) e^{-\mathrm{Re}(zw^*)} \cosh\alpha\tau . d^2\mu(z) \tag{5.107}$$

$$\Phi_\alpha(w) = \int_0^\infty \frac{d\alpha}{\alpha^2} \int_0^\infty d\tau . \varphi(\alpha) e^{-\rho r\cosh\tau} \cosh\alpha\tau \tag{5.108}$$

$$= \int\int_{\mathbb{H}} \varphi(\alpha) \exp\left(-\frac{\rho r}{2}\left[e^\tau + e^{-\tau}\right]\right) \cosh\alpha\tau . d^2\mu(z) \tag{5.109}$$

$$= \int\int_{\mathbb{H}} \varphi(\alpha) e^{-\rho r\cosh\tau} \cosh\alpha\tau . d^2\mu(z) \tag{5.110}$$

We then take our area increment in a way such that $d^2\mu(z) = \dfrac{d\tau d\alpha}{\alpha^2}$, which may be obtained from analysis of the hyperbolic geometry, and find:

$$\Phi_\alpha(\rho) = \int_0^\infty \frac{d\alpha}{\alpha^2} \int_0^\infty d\tau . \varphi(\alpha) e^{-\rho r\cosh\tau} \cosh\alpha\tau = \int_0^\infty \frac{d\alpha}{\alpha^2} . \varphi(\alpha) K_\alpha(\rho r) \tag{5.111}$$

thereby arriving at a form of the Kontorovich-Lebedev transformation as claimed. We have used the generating function for the modified Bessel function, which may be found in DLMF [121], formula 10.32.9. These are simple types of transforms that can be used to solve the heat equation for various different co-ordinate systems. □

5.6 Applications of the Hyperbolic Projection-Slice Theorem

We shall now outline the basic techniques one can use to evaluate index integrals directly. The following lays out the types of special functions and basic mathematical framework we require to determine these complicated integrals. We briefly discuss the different contexts in which index transforms can arise, selecting the physical theory of scattering and some problems in quantitative finance for this purpose.

5.6.1 Mathematical Apparatus

This section shall have recourse to the following sets of special functions; the modified Bessel functions of the first and second kind $\{K_\nu(z), I_\nu(z)\}$, which we refer to as the Macdonald and Basset functions for historical consistency. The Whittaker function of the second kind $W_{\mu,\nu}(z)$ shall be referred to as *the* Whittaker function, as we shall not need the other function in our calculations, so there will be no confusion. The function given by $\mathcal{P}^{\mu}_{i\nu-1/2}(z)$ is similarly to be referred to as the Mehler-Fock function for the duration of this work. We shall also require the Hermite polynomials $H_n(x)$, and the parabolic cylinder function $D_\alpha(z)$. For a description of these functions, their basic properties and other such information, references at [121,122] contain tabulated formulae, valuable descriptions of these functions and their derivation using representation theory in [69]. In this work, we have defined these functions in 3.4.2, 3.4.1, 3.4.3. This chapter shall not focus on the theory and structure of these special functions and we shall assume that the reader is familiar with their basic types and notations. It must also be made clear that the ranges of applicability of many of the formulae derived herein are restricted in domain. Special functions often have representations that are only valid in certain areas and any practical application must take this important property into account. Much of the work in this chapter revolves around the use of kernels, Green's functions and the like. The kernel of the system is denoted $K(x, x'; t)$, if a kernel has some need of specification we shall denote it with a subscript as in $K_A(x, x'; t)$. Similarly, the Green's function will be written $G(x, x'; E)$. Generally the dependent variable, which in this case is given by t, E will be written to the right hand side of the semicolon to distinguish it from the space variables. We assume that the reader is familiar with the definitions of the previous chapter, in particular the Green's function as the Fredholm resolvent, 4.7.18 and the spherical product composition law 3.3.45.

5.6.2 General Framework

Index integration differs from the standard types of integrals commonly encountered in analysis, where most formulae can be written in the form:

$$I_0(n; a, b) = \int_a^b f_n(x) d\tilde{\mathfrak{m}}(x) \tag{5.112}$$

By an index integral, we identify the generalised integral over the index of the function via:

$$I_1(x; a, b) = \int_a^b f_u(x) d\mathfrak{m}(u) \tag{5.113}$$

This differs markedly in behaviour from a standard integral. Indeed, by observation, it extends to functions which are continuously dependent on an index. In this case, rather than integrating over the variable of a function, we are integrating the input (a function) over a class of functions. A simple example might consist of the exponential function $f_u(x) = e^{-ux}$, in which case we are easily able to derive:

$$I_1(x; a, b) = \int_a^b e^{-ux} d\mathfrak{m}(u) \tag{5.114}$$

This example is confluent in that there is little difference between the index and standard integral, both give the Laplace transform of the measure distribution. In the same way that the Bessel functions form the natural extension of the Fourier series to systems of circular symmetry, index transforms over the Bessel functions take the form:

$$I_1(x; a, b) = \int_0^\infty K_{iu}(x) f(x) d\mathfrak{m}(u) \tag{5.115}$$

This is already a more difficult problem, as the index appears as a variable of integration. In general, this is a computationally difficult problem. However, as we shall show in this chapter, these difficulties can be resolved by a clever use of some integral transformation formulae which are available from the theory of the confluent hypergeometric function. Some results are available in tables of integrals, see e.g. [27,122,134]; however, many of the tabulated results suffer from the standard deficiency of having no known method of solution. In this chapter, we show how many of these results stem from similar observations in the theory of special functions, in particular those functions related to hyperbolic geometry.

5.6.3 Index Transforms from Scattering Theory

Index transforms can be understood as a description of certain scattered states in the continuous spectrum. In the calculations that follow, we shall show how a number of index transforms for the modified Bessel function may be derived using methods of Oberhettinger that hinge on the use of generating functions for Bessel functions more generally. Generating functions are a neat and concise way of describing families of continuous functions, and as is well-known from the study of classical analysis, many results in special function theory can be understood through the use of this technique. We shall show a simple way in which some basic index integrals arise in scattering theory, following the suggestions in the appendix of [72]. Note that this reference, whilst useful, does contain misprints, and the calculation method is not outlined in detail. For completeness we shall derive the correct formulae and demonstrate the techniques of index integration for some simple scattered waves. We shall state the following three simple types of index transforms, which we shall establish using techniques of scattering theory.

Lemma 5.6.1. *The following three index integrals may be established to be equivalent:*

$$K_{i\nu}(a)K_{i\nu}(b) = \int_0^\infty \cos(\nu t)K_0\left(\sqrt{a^2+b^2+2ab\cosh t}\right)dt \tag{5.116}$$

$$K_0\left(\sqrt{a^2+b^2+2ab\cosh t}\right) = \frac{2}{\pi}\int_0^\infty \cos(\nu t)K_{i\nu}(a)K_{i\nu}(b)d\nu \tag{5.117}$$

$$K_0\left(\sqrt{a^2+b^2-2ab\cos\theta}\right) = \frac{2}{\pi}\int_0^\infty \cosh(\nu(\pi-\theta))K_{i\nu}(a)K_{i\nu}(b)d\nu \tag{5.118}$$

where $K_{i\nu}(.)$ is the modified Bessel function with complex index, $K_0(.)$ is the modified Bessel function of order zero.

Proof. We require the following results, tabulated for special functions in DLMF [121]. The integral generating functions of the modified Bessel functions of the first and second kind may be written:

$$\pi I_\nu(z) = \int_0^\infty e^{z\cos t}\cos(\nu t)dt - \sin(\nu\pi)\int_0^\infty e^{-z\cosh t-\nu t}dt \tag{5.119}$$

$$K_\nu(z) = \frac{1}{2}\int_0^\infty x^{-\nu-1}\exp\left(-x-\frac{z^2}{4x}\right)dx \tag{5.120}$$

We also have the product formula given through:

$$\pi I_\nu(a)K_\nu(b) = \frac{1}{2}\int_0^\infty x^{-1}\exp\left(-x-\frac{a^2+b^2}{4x}\right)I_\nu\left(\frac{ab}{2x}\right)dx \tag{5.121}$$

See [121] formulae 10.32.4, 10.32.10, 10.32.15-18, also in Prudnikov [148], pp.321 2.15.20.8. Inserting the expression for $\pi I_\nu(z)$ into the product formula, we find:

$$\pi I_\nu(a)K_\nu(b) = \frac{1}{2}\int_0^\infty x^{-1}\exp\left(-x - \frac{a^2+b^2}{4x}\right)\left[\begin{array}{c}\int_0^\infty e^{(2x)^{-1}ab.\cos t}\cos(\nu t)dt \\ -\sin(\nu\pi)\int_0^\infty e^{-(2x)^{-1}ab.\cosh t - \nu t}dt\end{array}\right]dx \tag{5.122}$$

$$= \frac{1}{2}\int_0^\infty \cos(\nu t)dt\int_0^\infty x^{-1}\exp\left(-x - \frac{a^2+b^2-2ab\cos t}{4x}\right)dx \tag{5.123}$$

$$-\frac{1}{2}\sin(\nu\pi)\int_0^\infty e^{-\nu t}dt\int_0^\infty x^{-1}\exp\left(-x - \frac{a^2+b^2+2ab\cosh t}{4x}\right)dx \tag{5.124}$$

where we have swapped the order of integration under the standard conditions of Fubini-Tonelli. Using the integral representation for the Macdonald function, we find a product formula for the product of modified Bessel functions:

$$\pi I_\nu(a)K_\nu(b) = \int_0^\infty \cos(\nu t)K_0\left(\sqrt{a^2+b^2-2ab\cos t}\right)dt \tag{5.125}$$

$$-\sin(\nu\pi)\int_0^\infty e^{-\nu t}K_0\left(\sqrt{a^2+b^2+2ab\cosh t}\right)dt \tag{5.126}$$

Now, we have the formula connecting the Basset and Macdonald functions, viz.:

$$\frac{2\sin(\pi\nu)}{\pi}K_\nu(z) = I_{-\nu}(z) - I_\nu(z) \tag{5.127}$$

Then we may write by direct insertion into the product formula

$$\pi I_{-\nu}(z) = \int_0^\infty e^{z\cos t}\cos(\nu t)dt + \sin(\nu\pi)\int_0^\infty e^{-z\cosh t + \nu t}dt \tag{5.128}$$

$$\pi I_\nu(z) = \int_0^\infty e^{z\cos t}\cos(\nu t)dt - \sin(\nu\pi)\int_0^\infty e^{-z\cosh t - \nu t}dt \tag{5.129}$$

$$\pi(I_{-\nu}(z) - I_\nu(z)) = \sin(\nu\pi)\int_0^\infty e^{-z\cosh t}\left(e^{\nu t} + e^{-\nu t}\right)dt \tag{5.130}$$

$$= 2\sin(\nu\pi)\int_0^\infty e^{-z\cosh t}\cosh(\nu t)dt \tag{5.131}$$

implying the integral representation:

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh(\nu t)dt \tag{5.132}$$

In terms of the product formula, we have that:

$$\pi I_{-\nu}(a)K_{-\nu}(b) = \pi I_{-\nu}(a)K_\nu(b) = \int_0^\infty \cos(\nu t)K_0\left(\sqrt{a^2+b^2-2ab\cos t}\right)dt \tag{5.133}$$

$$+\sin(\nu\pi)\int_0^\infty e^{\nu t}K_0\left(\sqrt{a^2+b^2+2ab\cosh t}\right)dt \tag{5.134}$$

so we obtain e.g.

$$\pi\left(I_{-\nu}(a) - I_\nu(a)\right)K_\nu(b) = \sin(\nu\pi)\int_0^\infty e^{\nu t}K_0\left(\sqrt{a^2+b^2+2ab\cosh t}\right)dt \tag{5.135}$$

$$+\sin(\nu\pi)\int_0^\infty e^{-\nu t}K_0\left(\sqrt{a^2+b^2+2ab\cosh t}\right)dt \tag{5.136}$$

$$= 2\sin(\nu\pi) \int_0^\infty \cosh(\nu t) K_0 \left(\sqrt{a^2 + b^2 + 2ab\cosh t}\right) dt \tag{5.137}$$

$$\frac{\pi \left(I_{-\nu}(a) - I_\nu(a)\right)}{2\sin(\nu\pi)} K_\nu(b) = \int_0^\infty \cosh(\nu t) K_0 \left(\sqrt{a^2 + b^2 + 2ab\cosh t}\right) dt \tag{5.138}$$

or

$$K_\nu(a) K_\nu(b) = \int_0^\infty \cosh(\nu t) K_0 \left(\sqrt{a^2 + b^2 + 2ab\cosh t}\right) dt \tag{5.139}$$

Changing the index to a complex variable, we find the cosine transform:

$$K_{i\nu}(a) K_{i\nu}(b) = \int_0^\infty \cos(\nu t) K_0 \left(\sqrt{a^2 + b^2 + 2ab\cosh t}\right) dt \tag{5.140}$$

and by Fourier inversion, we derive the second index transformation in the lemma:

$$K_0 \left(\sqrt{a^2 + b^2 + 2ab\cosh t}\right) = \frac{2}{\pi} \int_0^\infty \cos(\nu t) K_{i\nu}(a) K_{i\nu}(b) d\nu \tag{5.141}$$

Using the substitution $t = i(\pi - \theta)$ we recover Oberhettinger's expression for the cylindrical wave [27]:

$$K_0 \left(\sqrt{a^2 + b^2 - 2ab\cos\theta}\right) = \frac{2}{\pi} \int_0^\infty \cosh(\nu(\pi - \theta)) K_{i\nu}(a) K_{i\nu}(b) d\nu \tag{5.142}$$

as required. □

Remark 88. This brief derivation is illustrative of many of the basic techniques in evaluation of index integrals. The use of integral representations, and interchanging of integration variables under the Fubini-Tonelli theorem is fundamental to many of the calculations contained in the following sections. Note that although the right hand side of the above index integral appears formidable, we are able to relate it to a fairly simple integral of Bessel functions, which is straightforward to evaluate.

5.6.4 Index Transforms from Mathematical Finance

The Yakubovich, Mehler-Fock and Whittaker kernels may be defined through eigenfunction decomposition formulae. Several types of connection formulae may be identified through tabulated integrals and results from the theory of special functions. We shall touch upon some basic results from the theory of mathematical finance and stochastic processes that are of interest in this work. We state the following three forms of the kernel solution to the heat equation associated with the modified Bessel, Mehler-Fock and Whittaker eigenfunctions. References for these systems of special functions may be found in Craddock [46] and Sousa and Yakubovich [56].

Definition 5.6.2. Craddock [46] defines the Yakubovich kernel via the integral:

$$K_Y(x, y; t) = \frac{2}{\pi^2} \int_0^\infty e^{-p^2 t} K_{ip}(x) K_{ip}(y) p \sinh(\pi p) dp \tag{5.143}$$

This is the solution to the PDE $u_t = x^2 u_{xx} + x u_x - x^2 u$.

Sousa and Yakubovich [56] define a number of other solutions to the heat equation, which we call the Mehler-Fock and Whittaker heat kernels, given in an analogous fashion by the formulae:

Definition 5.6.3. The Whittaker kernel is given by the index integral:

$$K_W(x,y;t) = \frac{2}{\pi^2\sqrt{xy}} \tag{5.144}$$

$$\times \int_0^\infty e^{-(p^2+\alpha^2)t} W_{\alpha,ip}(2x) W_{\alpha,ip}(2y) p \sinh(2\pi p) \left|\Gamma\left(1/2-\alpha+ip\right)\right|^2 dp \tag{5.145}$$

This is the fundamental solution to the heat-type equation $u_t = x^2 u_{xx} + x u_x - (x-\alpha)^2 u$.

Definition 5.6.4. The Mehler-Fock kernel is defined by the index integral [56]:

$$K_{MF}(x,y;t) = \frac{1}{\pi}\int_0^\infty e^{-(p^2+1/4)t} \mathcal{P}^{-\mu}_{ip-1/2}(x) \mathcal{P}^{-\mu}_{ip-1/2}(y) p \sinh(\pi p) \left|\Gamma\left(1/2+\mu+ip\right)\right|^2 dp \tag{5.146}$$

This is the fundamental solution to the heat equation $u_t = (x^2-1)u_{xx} + 2xu_x - \dfrac{\mu^2}{x^2-1}u$.

Remark 89. It is an interesting question in spectral theory to determine the exact relationship between these different kernels. As we shall show, use of the theory of diffusion has some interesting consequences for these seemingly different kernel formulae.

We note the following known integrals from tables [122] eqs. 7.141.5, 7.142.1:

Lemma 5.6.5. *The Laplace transforms connecting the Mehler-Fock, Whittaker and modified Bessel functions are:*

$$\int_1^\infty e^{-yz}\left(\frac{z+1}{z-1}\right)^{\mu/2} \mathcal{P}^{\mu}_{i\nu-1/2}(z)dz = \frac{W_{\mu,i\nu}(2y)}{y} \tag{5.147}$$

$$\int_1^\infty e^{-yz}(z^2-1)^{-\mu/2}\mathcal{P}^{\mu}_{i\nu-1/2}(z)dz = \sqrt{\frac{2}{\pi}} y^{\mu-1/2} K_{i\nu}(y) \tag{5.148}$$

Buchholz [134] (eq. 5.1, pp.60) reports the following relationship for the Whittaker function:

Lemma 5.6.6. *The Whittaker function has Laplace representation:*

$$W_{\kappa,\mu}(z) = \frac{z^{\mu+1/2}}{2^\kappa \Gamma\left(\dfrac{1}{2}+\mu-\kappa\right)} \int_1^\infty e^{-sz/2}\left(\frac{s+1}{s-1}\right)^\kappa (s^2-1)^{\mu-1/2} ds \tag{5.149}$$

Remark 90. There is obviously a direct relationship between these sets of different eigenfunctions which this work shall be primarily focused on through the application of index integrals. Note that these formulae have inversion integrals such as Mehler-Fock transform, Whittaker transform and Kontorovich-Lebedev transform for the different types of hyperbolic special functions.

5.6.5 Asian Option Pricing and Index Integration

We shall now calculate Asian options prices and show how these exotic derivatives are related to index integration and the projection-slice theorem in the hyperbolic plane. The papers of Linetsky [75], Geman and Yor [149], and Sousa and Yakubovich [56] are of use in developing the necessary stochastic calculus to specify the pricing formula. For completeness, we shall briefly review the basic equations, and refer the reader to [80], where the author of this thesis has employed techniques from classical numerical analysis to construct various approximations for pricing this option. The basic analysis begins with the exponential Brownian motion:

Definition 5.6.7. The exponential Brownian motion is defined by the time integral:

$$A_\tau^{(\nu)} = \int_0^\tau e^{2(B_u+\nu u)} du \tag{5.150}$$

where $\tau > 0$ is some constant, B_u a Brownian motion and ν is the drift.

In this regime, the put and call prices are computed using the time-invariant solution to the differential equation [75,149]. We have following Geman, Yor and Linetsky:

Theorem 5.6.8. *Put-call parity for Asian (arithmetic) options takes the form:*

$$P^{(\nu)}(k,\tau) = \mathbb{E}\left[(k - A_\tau^{(\nu)})^+\right] \tag{5.151}$$

$$V_p = e^{-rT}\mathbb{E}\left[(K - \mathcal{A}_T)^+\right] = e^{-rT}\left(\frac{4S_0}{\sigma^2 T}\right) P^{(\nu)}(k,\tau) \tag{5.152}$$

$$V_c = e^{-rT}\mathbb{E}\left[(\mathcal{A}_T - K)^+\right] = V_p + \frac{1-e^{-rT}}{rT} S_0 - e^{-rT} K \tag{5.153}$$

where the probability density functional is given by the Whittaker index integral:

$$P^{(\nu)}(k,\tau) = e^{-\nu^2\tau/2}\frac{(2k)^{-\kappa}e^{-1/(4k)}}{8\pi^2}\int_0^\infty e^{-p^2\tau/2} W_{\kappa,ip/2}\left(\frac{1}{2k}\right) d\mathfrak{m} + .. \tag{5.154}$$

and the measure is defined by:

$$d\mathfrak{m} = \left|\Gamma\left(\frac{\nu+ip}{2}\right)\right|^2 \sinh \pi p.pdp \tag{5.155}$$

and the scalar parameters are defined through:

$$\nu = \frac{2r}{\sigma^2} - 1, k = \frac{\tau K}{S_0}, \tau = \frac{T\sigma^2}{4}, \kappa = -\frac{(\nu+3)}{2} \tag{5.156}$$

Proof. See Linetsky, [75], eq. 16, Prop. 3.2.2, also Geman and Yor, Thm. 3.1 [149]. We shall not reiterate their existing proof, and direct the reader to these papers. □

We have the following known result from Geman and Yor, where the authors in [149] gave the canonical result for the fixed strike call option by exploiting the scaling property of Brownian motion.

Theorem 5.6.9. *The arithmetic Asian option is defined through the expectation value:*

$$\mathbb{E}_T\left[\left(\frac{1}{T-t_0}\int_{t_0}^T S_u du - K\right)^+\right] \tag{5.157}$$

$$= \frac{4S_t e^{-r(T-t)}}{(T-t_0)\sigma^2}\mathbb{E}^{\mathbb{Q}}_{\sigma^2 s/4}\left[\left(\int_0^{\sigma^2 s/4} \exp\left(2W_{s'} + \frac{4}{\sigma^2}\nu s'\right) ds' - \frac{\mathcal{K}\sigma^2}{4}\right)^+\right] \tag{5.158}$$

We may price call options for a fixed strike by using the transition probability density:

$$C(T,0;S_0,K) = \frac{4S_0 e^{-rT}}{T\sigma^2}\int_0^\infty (u - \frac{\mathcal{K}\sigma^2}{4})^+ f(u,0;\tau) du \tag{5.159}$$

where the fundamental diffusion equation is defined through:

$$2u^2\frac{\partial^2 f}{\partial u^2} + [(2\zeta+1)u + 1]\frac{\partial f}{\partial u} = -\frac{\partial f}{\partial t} \tag{5.160}$$

and the variable u is the exponential Brownian motion:

$$u = \int_0^{\sigma^2 s/4} \exp\left(2\left[W_{s'} + \zeta s'\right]\right) ds' \tag{5.161}$$

Proof. See Geman and Yor, [149]. □

Other known results for this type of contract include the floating strike call option formula, developed by Craddock in [46].

Theorem 5.6.10. *The price of a floating-strike arithmetic Asian option is defined through the payoff function:*

$$C_f(S_T, A_T, T) = \max\left(S_T - \frac{A_T}{T}, 0\right) \tag{5.162}$$

which satisfies the degenerate parabolic equation:

$$\frac{\partial C_f}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C_f}{\partial S^2} + S\frac{\partial C_f}{\partial \xi} + rS\frac{\partial C_f}{\partial S} - rC_f = 0 \tag{5.163}$$

This PDE is solved through the substitution, originally employed in Craddock [46], where the redundant variable is given through $C_f(S, A, t) = Sf(z, t)$ implying the PDE system:

$$-\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 f}{\partial z^2} - \left(\frac{1}{T} + rz\right)\frac{\partial f}{\partial z} = 0 \tag{5.164}$$

The redundant variable in this case is defined through the time average in a similar way to the Yor system, via:

$$\xi_t = \int_0^t S_u du \tag{5.165}$$

The change of variables in this case is given by the relationship:

$$z_t = \frac{1}{S}(k - \xi_t) = \frac{1}{S}\left(k - \frac{A_t}{T}\right) \tag{5.166}$$

Remark 91. It is well known from the theory of mathematical finance that the price of an Asian option can be related to the expectation value of an exponential Brownian motion as outlined in the theorems above. The diffusion equation given by Yor may be written as:

$$-\frac{\partial f}{\partial t} = 2x^2\frac{\partial^2 f}{\partial x^2} + (2\alpha x + 1)\frac{\partial f}{\partial x} \tag{5.167}$$

and various investigations have demonstrated that there is a close relationship between this PDE, the kernel solution and other sets of differential equations implied by the Laplace transforms above. This is of important practical significance in developing accurate pricing structures for these types of options. One formulation of the Asian option pricing algorithm [75] uses the index integral:

$$\int_0^\infty e^{-p^2\tau} p \sinh(\pi p) \left|\Gamma\left(1/2 - \mu + \frac{ip}{2}\right)\right|^2 W_{\mu,ip}(2x)dp \tag{5.168}$$

There may be relevance to other models of hyperbolic geometry such as in physics which we discuss in the concluding sections of this chapter.

5.7 Applications of Index Transforms to Special Function Theory

One simple technique that can be used to determine formulae for index transformations are the Whittaker-Bessel-Legendre integrals, which allow the reduction of more complicated integrals to known types involving Legendre equations. We shall briefly show several integrals in which this method of calculation is effective, and indicate some further types of problems that may arise for more complicated scenarios that may be of interest.

5.7.1 Composition Formula for Whittaker Functions

There is an integral formula which relates the Whittaker and Mehler-Fock functions. The Mehler-Fock, otherwise known by toroidal, ring or conical functions, are a special case of the associated Legendre functions, which we have defined in 3.4.1, 3.4.3.

Definition 5.7.1. The basic integral [122] given by eq. 7.141.5, which expresses the hyperbolic form of the projection-slice theorem reads as:

$$\int_1^\infty e^{-yz}\left(\frac{z+1}{z-1}\right)^{k/2}\mathcal{P}^k_{i\nu-1/2}(x)dx=\frac{W_{k,i\nu}(2y)}{y} \tag{5.169}$$

where $\mathcal{P}^k_{i\nu-1/2}(x)$ is the Mehler-Fock function, and $W_{k,i\nu}(2y)$ is the Whittaker function.

This is suggestive of the type of transformation we can expect to make to recover a Legendre-type equation. Consider now the index transform derived in 4.9.2:

$$\int_0^\infty \nu\sinh(\pi\nu)\Gamma(\lambda+i\nu)\Gamma(\lambda-i\nu)K_{i\nu}(a)K_{i\nu}(b)d\nu=\frac{\pi^{3/2}\Gamma(\lambda+1/2)}{2}\left(\frac{a+b}{2ab}\right)^{-\lambda}K_\lambda(a+b) \tag{5.170}$$

We wish to find a generalised form of this, valid for the Whittaker function. To begin, we shall derive the completeness relationship for the Mehler-Fock functions, in the spirit of the initial investigations of Feynman as discussed in 0.1.4. We shall show the following, which is covered using different methodologies in Sousa and Yakubovich [56], see also [13] for discussion:

Lemma 5.7.2. *The completeness relation for the Mehler-Fock functions satisfies the decomposition formula:*

$$\pi\delta(x-y)=\int_0^\infty dp.p\sinh(\pi p)|\Gamma\left(ip-\mu+1/2\right)|^2\mathcal{P}^\mu_{ip-1/2}(y)\mathcal{P}^\mu_{ip-1/2}(x) \tag{5.171}$$

where $\delta(x-y)$ is the Dirac delta function.

Proof. The forward and inverse Mehler-Fock transforms satisfy:

$$\mathcal{F}^{-1}[f(p)]=\int_1^\infty f(p)\mathcal{P}^m_{i\nu-1/2}(p)dp=F(\nu) \tag{5.172}$$

as defined in [56], [147]. Application of the forward and then the inverse transform results in no net change, i.e. the identity transformation, we therefore have:

$$f(\tau)=\mathcal{F}\mathcal{F}^{-1}[f(\tau)]=\mathcal{F}^{-1}\mathcal{F}[f(\tau)] \tag{5.173}$$

Writing this out in full, we find:

$$\mathcal{F}^{-1}\left[F(\nu)\right]=f(p)=\frac{1}{\pi}\int_0^\infty \nu\sinh(\pi\nu)\left|\Gamma\left(\frac{1}{2}-m+i\nu\right)\right|^2\mathcal{P}^m_{i\nu-1/2}(p)F(\nu)d\nu \tag{5.174}$$

or

$$\int_1^\infty \frac{1}{\pi}|\Gamma\left(ip-\mu+1/2\right)|^2 p\sinh(\pi p)dy.\mathcal{P}^\mu_{ip-1/2}(y)\int_0^\infty \mathcal{P}^\mu_{ip-1/2}(x)f(p)dp=f(p) \tag{5.175}$$

$$=\frac{1}{\pi}\int_1^\infty dy\int_0^\infty dp.p\sinh(\pi p)|\Gamma\left(ip-\mu+1/2\right)|^2\mathcal{P}^\mu_{ip-1/2}(y)\mathcal{P}^\mu_{ip-1/2}(x)f(p) \tag{5.176}$$

where we have invoked Fubini-Tonelli to interchange the order of integration. So, we can see that the kernel, represented by the inner integral gives the completeness relation.

$$\pi\delta(x-y)=\int_0^\infty dp.p\sinh(\pi p)|\Gamma\left(ip-\mu+1/2\right)|^2\mathcal{P}^\mu_{ip-1/2}(y)\mathcal{P}^\mu_{ip-1/2}(x) \tag{5.177}$$

which establishes the claim in the lemma. □

We now derive a generalisation of this index transform, which relates to the Whittaker function.

Lemma 5.7.3. *There exists a composition formula for the Whittaker function, given by the index Whittaker transform:*

$$\frac{2\Gamma(1-\mu)}{\pi(a+b)}W_{\mu,1/2}(2(a+b)) = \frac{2}{\pi^2}\int_0^\infty p\sinh(\pi p)|\Gamma(-\mu+1/2+ip)|^2\frac{W_{\mu,ip}(2a)}{a}\frac{W_{\mu,ip}(2b)}{b}dp \tag{5.178}$$

Proof. If we take the index Whittaker kernel 5.6.3, and set the time component equal to zero, we have: the Whittaker index kernel $K_W(a,b;0) = K(a,b)$, defined by:

$$K(a,b) = \frac{2}{\pi^2}\int_0^\infty p\sinh(\pi p)|\Gamma(-\mu+1/2+ip)|^2\frac{W_{\mu,ip}(2a)}{a}\frac{W_{\mu,ip}(2b)}{b}dp \tag{5.179}$$

$$= \frac{2}{\pi^2}\int_0^\infty dp.p\sinh(\pi p)|\Gamma(-\mu+1/2+ip)|^2$$

$$\times\int_1^\infty e^{-ax}\left(\frac{x+1}{x-1}\right)^{\mu/2}\mathcal{P}^\mu_{ip-1/2}(x)dx\int_1^\infty e^{-by}\left(\frac{y+1}{y-1}\right)^{\mu/2}\mathcal{P}^\mu_{ip-1/2}(y)dy \tag{5.180}$$

$$= \frac{2}{\pi^2}\int_1^\infty dx.e^{-ax}\left(\frac{x+1}{x-1}\right)^{\mu/2}\int_1^\infty dy.e^{-by}\left(\frac{y+1}{y-1}\right)^{\mu/2}$$

$$\times\int_0^\infty p\sinh(\pi p)|\Gamma(-\mu+1/2+ip)|^2\mathcal{P}^\mu_{ip-1/2}(x)\mathcal{P}^\mu_{ip-1/2}(y)dp \tag{5.181}$$

$$= \frac{2}{\pi}\int_1^\infty dx.e^{-ax}\left(\frac{x+1}{x-1}\right)^{\mu/2}\int_1^\infty dy.e^{-by}\left(\frac{y+1}{y-1}\right)^{\mu/2}\delta(x-y) \tag{5.182}$$

$$= \frac{2}{\pi}\int_1^\infty dx.e^{-(a+b)x}\left(\frac{x+1}{x-1}\right)^{\mu} \tag{5.183}$$

where in the second step we have invoked Fubini-Tonelli to interchange the order of integration, and then inserted the completeness relation for the Mehler-Fock functions 5.7.2. To solve this integral, note the known result from [122], we have formula 3.385.3 which gives the integral representation for the Whittaker function:

$$\int_u^\infty (x+\beta)^{2\nu-1}(x-u)^{2\rho-1}e^{-\mu x}dx = \frac{(u+\beta)^{\nu+\rho-1}}{\mu^{\nu+\rho}}\exp\left(\frac{(\beta-u)\mu}{2}\right)\Gamma(2\rho)W_{\nu-\rho,\nu+\rho-1/2}\left([u+\beta]\mu\right) \tag{5.184}$$

If we substitute $\beta = u = 1$, $\nu = \frac{\mu}{2}+\frac{1}{2}$, $\rho = -\frac{\mu}{2}+\frac{1}{2}$, $\mu = (a+b)$, we find:

$$\frac{2}{\pi}\int_1^\infty dx.e^{-(a+b)x}\left(\frac{x+1}{x-1}\right)^{\mu} = \frac{2\Gamma(1-\mu)}{\pi(a+b)}W_{\mu,1/2}(2(a+b)) \tag{5.185}$$

which implies:

$$K(a,b) = \frac{2\Gamma(1-\mu)}{\pi(a+b)}W_{\mu,1/2}(2(a+b)) \tag{5.186}$$

$$= \frac{2}{\pi^2}\int_0^\infty p\sinh(\pi p)|\Gamma(-\mu+1/2+ip)|^2\frac{W_{\mu,ip}(2a)}{a}\frac{W_{\mu,ip}(2b)}{b}dp \tag{5.187}$$

and we have proved the result in the lemma. □

We immediately obtain the following corollaries as special cases of the Whittaker composition formula.

Corollary 5.7.4. *The composition formula for the modified Bessel function as derived in 4.9.1, 4.9.2 is a special case of the Whittaker composition formula 5.7.3. In particular, we have the following:*

$$\int_0^\infty p\sinh(\pi p)|\Gamma(1/2+ip)|^2 K_{ip}(a)K_{ip}(b)dp = \frac{\pi^{3/2}}{2}\left(\frac{a+b}{2ab}\right)^{-1/2} K_{1/2}(a+b) \tag{5.188}$$

and

$$\int_0^\infty p\sinh(\pi p)\Gamma(\lambda+ip)\Gamma(\lambda-ip)K_{ip}(a)K_{ip}(b)dp = \frac{\pi^{3/2}\Gamma(\lambda+1/2)}{2}\left(\frac{a+b}{2ab}\right)^{-\lambda} K_\lambda(a+b) \tag{5.189}$$

Proof. If we take the formula for the index Whittaker kernel, the special case $\lambda = 0$ yields

$$\frac{2\Gamma(1-\mu)}{\pi(a+b)}W_{0,1/2}(2(a+b)) = \frac{2}{\pi^2}\int_0^\infty p\sinh(\pi p)|\Gamma(1/2+ip)|^2\frac{W_{0,ip}(2a)}{a}\frac{W_{0,ip}(2b)}{b}dp \tag{5.190}$$

Using the conversion formula for the modified Bessel function, we can write:

$$W_{0,\nu}(2z) = \sqrt{\frac{2z}{\pi}}K_\nu(z) \tag{5.191}$$

so we find:

$$\frac{2}{\pi(a+b)}\sqrt{\frac{2(a+b)}{\pi}}K_{1/2}(a+b) = \frac{2}{\pi^2}\int_0^\infty p\sinh(\pi p)|\Gamma(1/2+ip)|^2\frac{1}{ab}\sqrt{\frac{2a}{\pi}}K_{ip}(a)\sqrt{\frac{2b}{\pi}}K_{ip}(b)dp \tag{5.192}$$

$$= \frac{4}{\pi^3\sqrt{ab}}\int_0^\infty p\sinh(\pi p)|\Gamma(1/2+ip)|^2K_{ip}(a)K_{ip}(b)dp \tag{5.193}$$

Simplifying, we find the formula:

$$\frac{1}{\sqrt{2}}\frac{\pi^2}{\sqrt{\pi}}\left(\frac{ab}{a+b}\right)^{1/2}K_{1/2}(a+b) = \int_0^\infty p\sinh(\pi p)|\Gamma(1/2+ip)|^2K_{ip}(a)K_{ip}(b)dp \tag{5.194}$$

which can be rewritten as:

$$\int_0^\infty p\sinh(\pi p)|\Gamma(1/2+ip)|^2 K_{ip}(a)K_{ip}(b)dp = \frac{\pi^{3/2}}{2}\left(\frac{a+b}{2ab}\right)^{-1/2} K_{1/2}(a+b) \tag{5.195}$$

which establishes the first leg of the corollary. For the second, from the integral representation for the Whittaker functions defined by:

$$\int_u^\infty (x+\beta)^{2\nu-1}(x-u)^{2\rho-1}e^{-\mu x}dx = \frac{(u+\beta)^{\nu+\rho-1}}{\mu^{\nu+\rho}}\exp\left(\frac{(\beta-u)\mu}{2}\right)\Gamma(2\rho)W_{\nu-\rho,\nu+\rho-1/2}(u\mu+\beta\mu) \tag{5.196}$$

we have:

$$\int_u^\infty (x+\beta)^{2\nu-1}(x-u)^{2\rho-1}e^{-\mu x}dx = \frac{(u+\beta)^{\nu+\rho-1}}{\mu^{\nu+\rho}}\exp\left(\frac{(\beta-u)\mu}{2}\right)\Gamma(2\rho)W_{\nu-\rho,\nu+\rho-1/2}([u+\beta]\mu) \tag{5.197}$$

which in turn gives, for $\beta = u = 1$, $\nu = \frac{k}{2} + \frac{1}{2}$, $\rho = \frac{k}{2} + \frac{1}{2}$, $\mu = (a+b)$, the following:

$$\int_1^\infty (x+1)^k (x-1)^k e^{-(a+b)x} dx = \int_1^\infty (x^2-1)^k e^{-(a+b)x} dx \tag{5.198}$$

$$= \frac{2^k}{(a+b)^{k+1}} \Gamma(k+1) W_{0,k+1/2}\left(2(a+b)\right) \tag{5.199}$$

Using the representation of the modified Bessel function 3.4.2, we find:

$$W_{0,\nu}(2z) = \sqrt{\frac{2z}{\pi}} K_\nu(z) \tag{5.200}$$

$$W_{0,k+1/2}\left(2(a+b)\right) = \sqrt{\frac{2(a+b)}{\pi}} K_{k+1/2}(a+b) \tag{5.201}$$

$$\int_1^\infty (x^2-1)^k e^{-(a+b)x} dx = \frac{2^k}{(a+b)^{k+1}} \Gamma(k+1) \sqrt{\frac{2(a+b)}{\pi}} K_{k+1/2}(a+b) \tag{5.202}$$

$$= \left(\frac{a+b}{2}\right)^{-k-1/2} \frac{\Gamma(k+1)}{\sqrt{\pi}} K_{k+1/2}(a+b) \tag{5.203}$$

In terms of the kernel formula implied by this relationship, if we take e.g.

$$\int_0^\infty d\mathfrak{m}(p;k) K_{ip}(a) K_{ip}(b) (ab)^{k+1/2}$$

$$= (ab)^{k+1/2} \int_0^\infty d\mathfrak{m}(p) \times \tag{5.204}$$

$$\left[\sqrt{\frac{\pi}{2}} \int_1^\infty e^{-ax} (x^2-1)^{-k/2} \mathcal{P}^k_{ip-1/2}(x) dx\right] \left[\sqrt{\frac{\pi}{2}} \int_1^\infty e^{-by} (y^2-1)^{-k/2} \mathcal{P}^k_{ip-1/2}(y) dy\right] \tag{5.205}$$

$$= (ab)^{k+1/2} \frac{\pi}{2} \int_1^\infty dx. e^{-ax} (x^2-1)^{-k/2} \int_1^\infty dy. e^{-by} (y^2-1)^{-k/2} \int_0^\infty d\mathfrak{m}(p) \mathcal{P}^k_{ip-1/2}(x) \mathcal{P}^k_{ip-1/2}(y) \tag{5.206}$$

$$= (ab)^{k+1/2} \frac{\pi^2}{2} \int_1^\infty dx. e^{-(a+b)x} (x^2-1)^{-k} \tag{5.207}$$

$$= \left(\frac{a+b}{2ab}\right)^{-k-1/2} \frac{\pi^{3/2}\Gamma(k+1)}{2} K_{k+1/2}(a+b) \tag{5.208}$$

where we used the completeness relationship implied by:

$$\pi\delta(x-y) = \int_0^\infty d\mathfrak{m}(p) \mathcal{P}^k_{ip-1/2}(x) \mathcal{P}^k_{ip-1/2}(y) \tag{5.209}$$

which, in turn, implies the following identity:

$$\int_0^\infty p \sinh(\pi p) \Gamma(\lambda + ip) \Gamma(\lambda - ip) K_{ip}(a) K_{ip}(b) dp = \frac{\pi^{3/2}\Gamma(\lambda+1/2)}{2} \left(\frac{a+b}{2ab}\right)^{-\lambda} K_\lambda(a+b) \tag{5.210}$$

for non-integer values, thereby proving the lemma. Obviously, the measure must be given by:

$$d\mathfrak{m}(p;k) = p \sinh(\pi p) \Gamma(k+ip) \Gamma(k-ip) \tag{5.211}$$

□

These examples of index integration serve as our base case for deriving some more complicated formulae in the following subsections. We note that the technique of introducing the projection-slice theorem for the hyperbolic plane results in a change of Whittaker functions down to more manageable special functions, in this case the Mehler-Fock functions. Using the completeness relationship then allows rapid solution of the integral in this case. As we shall see, this is a more generally applicable technique that is useful in evaluation of these difficult integrals.

5.7.2 A Mixed Index Integral

Using completeness relations, it is possible to derive a mixed index integral involving the modified Bessel and Mehler-Fock functions. This brief calculation is descriptive of a more general method which we shall explore in the remainder of this chapter.

Theorem 5.7.5. *The mixed index integral defined by the product of Mehler-Fock and Macdonald functions has solution:*

$$\int_0^\infty p\sinh(\pi p)\left|\Gamma\left(-\frac{\mu}{2}+1/2+ip\right)\right|^2 K_{ip}(a)\mathcal{P}^{\mu}_{ip-1/2}(y)dp \tag{5.212}$$

$$=\sqrt{\frac{\pi^3}{2}}a^{-\mu+1/2}e^{-ay}(y^2-1)^{-\mu/2} \tag{5.213}$$

where the Mehler-Fock and Macdonald (modified Bessel) functions are defined in 3.4.1, 3.4.2.

Proof. We begin our calculation with the completeness relationship for Mehler-Fock functions (see e.g. [56]), and in this work 5.7.2:

$$\pi\delta(x-y)=\int_0^\infty p\sinh(\pi p)\left|\Gamma\left(-\mu+1/2+ip\right)\right|^2 \mathcal{P}^{\mu}_{ip-1/2}(x)\mathcal{P}^{\mu}_{ip-1/2}(y)dp \tag{5.214}$$

The integral in question may be written:

$$I=\int_0^\infty p\sinh(\pi p)\left|\Gamma\left(-\mu+1/2+ip\right)\right|^2 K_{ip}(a)\mathcal{P}^{\mu}_{ip-1/2}(y)dp \tag{5.215}$$

Using known integral relations between the Macdonald and Mehler-Fock functions 5.3.7, [122] (eqs. 7.141.5, 7.142.1) we may write:

$$\int_1^\infty e^{-ax}(x^2-1)^{-\mu/2}\mathcal{P}^{\mu}_{ip-1/2}(x)dx=\sqrt{\frac{2}{\pi}}a^{\mu-1/2}K_{ip}(a) \tag{5.216}$$

$$K_{ip}(a)=\sqrt{\frac{\pi}{2}}a^{-\mu+1/2}\int_1^\infty e^{-ax}(x^2-1)^{-\mu/2}\mathcal{P}^{\mu}_{ip-1/2}(x)dx \tag{5.217}$$

Rewriting the integral, and inserting the relationship for the Mehler-Fock function, we obtain the following integral:

$$I=\sqrt{\frac{\pi}{2}}a^{-\mu+1/2}\int_0^\infty dp.p\sinh(\pi p)\left|\Gamma\left(-\mu+1/2+ip\right)\right|^2\mathcal{P}^{\mu}_{ip-1/2}(y) \tag{5.218}$$

$$\times\int_1^\infty dx.e^{-ax}(x^2-1)^{-\mu/2}\mathcal{P}^{\mu}_{ip-1/2}(x) \tag{5.219}$$

$$=\sqrt{\frac{\pi}{2}}a^{-\mu+1/2}\int_1^\infty dx.e^{-ax}(x^2-1)^{-\mu/2} \tag{5.220}$$

$$\times \int_0^\infty dp.p \sinh(\pi p) \left|\Gamma\left(-\mu + 1/2 + ip\right)\right|^2 \mathcal{P}^\mu_{ip-1/2}(x)\mathcal{P}^\mu_{ip-1/2}(y) \tag{5.221}$$

$$= \sqrt{\frac{\pi}{2}} a^{-\mu+1/2} \int_1^\infty dx.e^{-ax}(x^2-1)^{-\mu/2}\pi\delta(x-y) \tag{5.222}$$

$$= \sqrt{\frac{\pi^3}{2}} a^{-\mu+1/2} e^{-ay}(y^2-1)^{-\mu/2} \tag{5.223}$$

as required, after substitution of the completeness relation. Note also the integral formula, from Prudnikov, Vol. III, 2.17.27.20 [126]:

$$\int_0^\infty x \sinh(n\pi x)\Gamma\left(\frac{1}{2} - \mu + ix\right)\Gamma\left(\frac{1}{2} - \mu - ix\right) K_{ilx}(b)\mathcal{P}^\mu_{ix-1/2}(c)dx = J_{n,l} \tag{5.224}$$

$$J_{1,1} = 2^{-1/2}\pi^{3/2}b^{1/2-\mu}(c^2-1)^{-\mu/2}e^{-ac} \tag{5.225}$$

which confirms that this is indeed the correct formula. We have established the theorem. □

Remark 92. This simple technique of expansion, interchange and resolution using the completeness identity may be applied to a large class of index transforms. As we shall show, many different integrals can be evaluated by this method.

5.7.3 Second Mixed Integral

A similar integral between the modified Bessel and Whittaker functions may be found using the hyperbolic extension of the projection-slice theorem. We state this as the following theorem:

Theorem 5.7.6. *The second index integral of mixed type is:*

$$\int_0^\infty p \sinh(\pi p) \left|\Gamma\left(\frac{1}{2} - \mu + ip\right)\right|^2 W_{\mu,ip}(2x)K_{ip}(y)dp \tag{5.226}$$

$$= \sqrt{\frac{\pi^3}{2}} xy^{-\mu+1/2}e^{x+y}(x+y)^{\mu-1}\Gamma(1-\mu) \tag{5.227}$$

Proof. Writing the integral, and applying the transformation formula given by the integrals 5.3.6, 5.3.7, then invoking the Fubini-Tonelli theorem to interchange the order of integrals, we have:

$$I = \int_0^\infty p \sinh(\pi p) \left|\Gamma\left(\frac{1}{2} - \mu + ip\right)\right|^2 W_{\mu,ip}(2x)K_{ip}(y)dp \tag{5.228}$$

$$= \sqrt{\frac{\pi}{2}} xy^{-\mu+1/2} \int_0^\infty p \sinh(\pi p) \left|\Gamma\left(\frac{1}{2} - \mu + ip\right)\right|^2 \sqrt{\frac{2}{\pi}} y^{\mu-1/2} \frac{W_{\mu,ip}(2x)}{x} K_{ip}(y)dp \tag{5.229}$$

$$= \sqrt{\frac{\pi}{2}} xy^{-\mu+1/2} \int_0^\infty dp.p \sinh(\pi p) \left|\Gamma\left(\frac{1}{2} - \mu + ip\right)\right|^2 \tag{5.230}$$

$$\times \int_1^\infty e^{-xu}\left(\frac{u+1}{u-1}\right)^{-\mu/2} \mathcal{P}^\mu_{ip-1/2}(u)du \int_1^\infty e^{-yv}(v^2-1)^{-\mu/2}\mathcal{P}^\mu_{ip-1/2}(v)dv \tag{5.231}$$

Rearrangement via the Fubini-Tonelli theorem yields:

$$I = \sqrt{\frac{\pi}{2}}\pi.xy^{-\mu+1/2} \int_1^\infty \int_1^\infty dudv.e^{-(xu+yv)}\left(\frac{u+1}{u-1}\right)^{-\mu/2}(v^2-1)^{-\mu/2} \tag{5.232}$$

$$\times \int_0^\infty dp.p\sinh(\pi p) \left|\Gamma\left(\frac{1}{2} - \mu + ip\right)\right|^2 \mathcal{P}^\mu_{ip-1/2}(u)\mathcal{P}^\mu_{ip-1/2}(v) \tag{5.233}$$

$$= \sqrt{\frac{\pi^3}{2}} xy^{-\mu+1/2} \int_1^\infty \int_1^\infty dudv.e^{-(xu+yv)} \left(\frac{u+1}{u-1}\right)^{-\mu/2} (v^2-1)^{-\mu/2}\delta(u-v) \tag{5.234}$$

$$= \sqrt{\frac{\pi^3}{2}} xy^{-\mu+1/2} \int_1^\infty du.e^{-(x+y)u} \left(\frac{u+1}{u-1}\right)^{-\mu/2} (u^2-1)^{-\mu/2} \tag{5.235}$$

$$= \sqrt{\frac{\pi^3}{2}} xy^{-\mu+1/2} \int_1^\infty du.e^{-(x+y)u} \left(\frac{u+1}{u-1}\right)^{-\mu/2} (u-1)^{-\mu/2}(u+1)^{-\mu/2} \tag{5.236}$$

$$= \sqrt{\frac{\pi^3}{2}} xy^{-\mu+1/2} \int_1^\infty du.e^{-(x+y)u}(u+1)^{-\mu} \tag{5.237}$$

Using Prudnikov, Vol. I, 2.3.4.2 . Formula 4.239 [126], we have the gamma function integral:

$$\int_0^\infty \frac{e^{-px}}{(x+z)^\rho} dx = p^{\rho-1} e^{pz}\Gamma(1-\rho, pz) \tag{5.238}$$

Alternatively, we may write the known integral of the gamma distribution:

$$\int_1^\infty \frac{e^{-(x+y)u}}{(u+1)^\mu} = (x+y)^{\mu-1} e^{-(x+y)}\Gamma(1-\mu, x+y) \tag{5.239}$$

Substituting the change of variables:

$$u' = u+1, u = u'-1 \tag{5.240}$$

we have

$$u = 1, u' = 0 \tag{5.241}$$

and

$$u = \infty, u' = \infty \tag{5.242}$$

$$du' = du \tag{5.243}$$

and hence the integral transforms to:

$$\int_1^\infty \frac{e^{-(x+y)u}}{(u+1)^\mu} du = \int_0^\infty \frac{e^{x+y}e^{-u'(x+y)}}{(u')^\mu} du' \tag{5.244}$$

$$= e^{x+y} \int_0^\infty \frac{e^{-p(x+y)}}{p^\mu} dp \tag{5.245}$$

$$= e^{x+y}(x+y)^{\mu-1}\Gamma(1-\mu) \tag{5.246}$$

We therefore obtain the final result:

$$I = \sqrt{\frac{\pi^3}{2}} xy^{-\mu+1/2} e^{x+y}(x+y)^{\mu-1}\Gamma(1-\mu) \tag{5.247}$$

□

Remark 93. By inversion of either the Whittaker or Kontorovich-Lebedev transform, we can find the formula for an eigenfunction source term. As we shall see in later sections of this work, methods of complex analysis and residue calculus can be applied to solve index integrals of similar types to these. Note that Oberhettinger, [27], pp.17 contains an identical formula, but with a sign error in the exponent.

5.7.4 Cosine Transform

As we have seen in 5.6.8, the Asian option pricing formula is related to exponential Brownian motion, which can be understood through the arithmetic weighting of geometric Brownian motion increments. As discussed by Linetsky [75], Geman and Yor [149], also in Pintoux and Privault [136], [137], the dynamics which governs the option price for this exotic contract can be related to certain PDEs associated with the Whittaker function.

The index integral related to the Asian option pricing problem as discussed in 5.6.8, 5.6.9, 5.6.10 may be written as a Laplace transform of a Mehler-Fock kernel of similar type. The cosine transform of the Whittaker function may be found. The solution is given by a parabolic cylinder function consistent with known results. The Asian option pricing problem may be solved through use of this cosine transform. This requires evaluation of a known integral that was analysed previously by Ramanujan [150] and Bateman [151].

Theorem 5.7.7. *The cosine transform of the Whittaker function is given by the parabolic cylinder function:*

$$\int_0^\infty W_{\mu,ip}(2u)\cos(py)dp = \sqrt{\frac{\pi u}{2}}2^{-\mu}\exp\left(-u\sinh^2\frac{y}{2}\right)D_{2\mu}\left(\cosh\frac{y}{2}\sqrt{2u}\right) \tag{5.248}$$

Corollary 5.7.8. *The Asian option price as defined in 5.6.8 may be given in terms of the parabolic cylinder functions, following the result in 5.7.7 as:*

$$I(x;\tau) = \int_0^\infty e^{-p^2\tau}p\sinh(\pi p)\left|\Gamma\left(1/2-\mu+\frac{ip}{2}\right)\right|^2 W_{\mu,ip}(2x)dp \tag{5.249}$$

which can be written:

$$I(x;\tau) = \frac{2}{\pi}\sqrt{\frac{\pi u}{2}}2^{-\mu}\int_0^\infty dy.D_{2\mu}\left(\cosh\frac{y}{2}\sqrt{2x}\right)\times \tag{5.250}$$

$$\int_0^\infty e^{-p^2\tau}p\sinh(\pi p)\left|\Gamma\left(1/2-\mu+\frac{ip}{2}\right)\right|^2\cos(py)dp \tag{5.251}$$

Proof. The formula for the Asian option pricing problem may be found in Linetsky [75], also Yor [77], in either case the basic integral is given by the index transform kernel:

$$\int_0^\infty e^{-p^2\tau}p\sinh(\pi p)\left|\Gamma\left(\frac{1}{2}-\mu+\frac{ip}{2}\right)\right|^2 W_{\mu,ip}(2x)dp \tag{5.252}$$

$$= x\int_0^\infty e^{-p^2\tau}p\sinh(\pi p)\left|\Gamma\left(\frac{1}{2}-\mu+\frac{ip}{2}\right)\right|^2\frac{W_{\mu,ip}(2x)}{x}dp \tag{5.253}$$

$$= x\int_0^\infty dp.e^{-p^2\tau}p\sinh(\pi p)\left|\Gamma\left(\frac{1}{2}-\mu+\frac{ip}{2}\right)\right|^2\int_1^\infty e^{-xu}\left(\frac{u+1}{u-1}\right)^{-\mu/2}\mathcal{P}^{\mu}_{ip-1/2}(u)du \tag{5.254}$$

$$= x\int_1^\infty du.e^{-xu}\left(\frac{u+1}{u-1}\right)^{-\mu/2}\int_0^\infty dp.e^{-p^2\tau}p\sinh(\pi p)\left|\Gamma\left(\frac{1}{2}-\mu+\frac{ip}{2}\right)\right|^2\mathcal{P}^{\mu}_{ip-1/2}(u) \tag{5.255}$$

The inner integral here is of primary interest. We note the connection between integrals of this type and the so-called Yor integrals. We shall now show some methods that can be used to evaluate this integral by relying on various transforms between groups of special functions. Calculating the cosine transform of the Whittaker function:

$$g_c(y) = \int_0^\infty f(x)\cos(xy)dx \tag{5.256}$$

$$g_c(y) = \int_0^\infty W_{\mu,ip}(2u)\cos(py)dp \tag{5.257}$$

$$= \int_0^\infty u \int_1^\infty e^{-ux}\left(\frac{x+1}{x-1}\right)^{\mu/2} \mathcal{P}^\mu_{ip-1/2}(x)\cos(py).dxdp \tag{5.258}$$

$$= u\int_1^\infty dx.e^{-ux}\left(\frac{x+1}{x-1}\right)^{\mu/2}\int_0^\infty dp.\mathcal{P}^\mu_{ip-1/2}(x)\cos(py) \tag{5.259}$$

Oberhettinger eq. 12.85 [27] gives the following cosine transform of the Mehler-Fock function:

$$\int_0^\infty dp.\mathcal{P}^\mu_{ip-1/2}(x)\cos(py) = \sqrt{\frac{\pi}{2}}\frac{\left(x^2-1\right)^{\mu/2}(x-\cosh y)^{-\mu-1/2}}{\Gamma\left(1/2-\mu\right)} \tag{5.260}$$

$$\cosh y < x \tag{5.261}$$

so we obtain the cosine transform of the Whittaker function via:

$$g_c(y) = \sqrt{\frac{\pi}{2}}\frac{u}{\Gamma\left(1/2-\mu\right)}\int_1^\infty dx.e^{-ux}\left(x+1\right)^\mu (x-\cosh y)^{-\mu-1/2} \tag{5.262}$$

Consider now the function:

$$f(x) = \begin{cases} 0 & x < \beta \\ (x+\alpha)^{2\mu'-1}(x-\beta)^{2\nu'-1} & x > \beta \end{cases} \tag{5.263}$$

We have $\alpha < \beta < x$ by assumption, and therefore $f(x) = 0 \forall x \in [0,\alpha],[0,\beta]$, and we may write:

$$\int_0^\infty e^{-ax}f(x)dx = \int_\alpha^\infty e^{-ax}f(x)dx = \int_\beta^\infty e^{-ax}f(x)dx \tag{5.264}$$

Using the Laplace transform that may be found in Oberhettinger [27], eq. 3.24, pp. 24, we find the confluent hypergeometric function formula:

$$\int_0^\infty e^{-ax}f(x)dx = \frac{(\alpha+\beta)^{\mu'+\nu'-1}}{a^{\mu'+\nu'}}\exp\left(\frac{1}{2}(\alpha-\beta)a\right)\Gamma(2\nu')W_{\mu'-\nu',\mu'+\nu'-1/2}([\alpha+\beta]a) \tag{5.265}$$

where $\Re\nu > 0, |\arg a| < \pi, \Re a > 0$. Writing the cosine transform, we then have:

$$g_c(y) = \sqrt{\frac{\pi}{2}}\frac{u}{\Gamma\left(1/2-\mu\right)}\int_1^\infty dx.e^{-ux}\left(x+1\right)^\mu (x-\cosh y)^{-\mu-1/2} \tag{5.266}$$

and $2\mu'-1=\mu$, $2\nu'-1=-\mu-1/2$ with $\alpha=1$, $\beta=\cosh y$ and $a=u$. Writing this out in full, we then have the Whittaker function:

$$g_c(y) = \sqrt{\frac{\pi}{2}}\frac{u}{\Gamma\left(1/2-\mu\right)}\frac{(\alpha+\beta)^{\mu'+\nu'-1}}{a^{\mu'+\nu'}}\exp\left(\frac{1}{2}(\alpha-\beta)a\right)\Gamma(2\nu')W_{\mu'-\nu',\mu'+\nu'-1/2}([\alpha+\beta]a) \tag{5.267}$$

$$= \sqrt{\frac{\pi}{2}}\frac{u}{\Gamma\left(1/2-\mu\right)}\frac{(1+\cosh y)^{\mu'+\nu'-1}}{u^{\mu'+\nu'}}\exp\left(\frac{1}{2}(1-\cosh y)u\right)\Gamma(-\mu+1/2)W_{\mu'-\nu',\mu'+\nu'-1/2}([1+\cosh y]u) \tag{5.268}$$

Inserting the expressions for the parameters, we obtain $\mu' = \frac{\mu}{2}+\frac{1}{2}$ and $\nu' = -\frac{\mu}{2}+\frac{1}{4}$, hence:

$$g_c(y) = \sqrt{\frac{\pi}{2}}\frac{u}{\Gamma\left(1/2-\mu\right)}\frac{(1+\cosh y)^{-1/4}}{u^{3/4}}\exp\left(\frac{1}{2}(1-\cosh y)u\right)\Gamma(-\mu+1/2)W_{\mu+1/4,1/4}([1+\cosh y]u) \tag{5.269}$$

or

$$g_c(y) = \sqrt{\frac{\pi}{2}} u^{3/4} \left(2\cosh^2\frac{y}{2}\right)^{-1/4} \exp\left(-u\sinh^2\frac{y}{2}\right) W_{\mu+1/4,1/4}(2u\cosh^2\frac{y}{2}) \tag{5.270}$$

Using the expression for the parabolic cylinder function, we have the Whittaker function relation:

$$W_{\mu+1/4,1/4}(z) = D_{2\mu}\left(\sqrt{2z}\right)\frac{z^{1/4}}{2^{\mu}} \tag{5.271}$$

$$W_{\mu+1/4,1/4}(2u\cosh^2\frac{y}{2}) = D_{2\mu}\left(\sqrt{4u\cosh^2\frac{y}{2}}\right)\frac{(2u\cosh^2\frac{y}{2})^{1/4}}{2^{\mu}} \tag{5.272}$$

$$= D_{2\mu}\left(2\sqrt{u}\cosh\frac{y}{2}\right)\frac{(2u\cosh^2\frac{y}{2})^{1/4}}{2^{\mu}} \tag{5.273}$$

Simplifying the cosine transform of the Whittaker function, we then may write:

$$g_c(y) = \sqrt{\frac{\pi}{2}} u^{-3/4} \left(2\cosh^2\frac{y}{2}\right)^{-1/4} \exp\left(-u\sinh^2\frac{y}{2}\right) D_{2\mu}\left(2\sqrt{u}\cosh\frac{y}{2}\right)\frac{(2u\cosh^2\frac{y}{2})^{1/4}}{2^{\mu}} \tag{5.274}$$

$$= \sqrt{\frac{\pi}{2}} u^{1/4-3/4} \left(2\cosh^2\frac{y}{2}\right)^{-1/4+1/4} \frac{1}{2^{\mu}} \exp\left(-u\sinh^2\frac{y}{2}\right) D_{2\mu}\left(2\sqrt{u}\cosh\frac{y}{2}\right) \tag{5.275}$$

$$= \sqrt{\frac{\pi u}{2}}\frac{1}{2^{\mu}} \exp\left(-u\sinh^2\frac{y}{2}\right) D_{2\mu}\left(2\sqrt{u}\cosh\frac{y}{2}\right) \tag{5.276}$$

$$= \int_0^\infty W_{\mu,ip}(2u)\cos(py)dp \tag{5.277}$$

This is consistent with the cosine transform pair that may be found in Oberhettinger [28], eq. 22.28, pp. 109, as can be seen through the following:

$$\int_0^\infty W_{\mu,ip}(a)\cos(py)dp = \sqrt{\frac{\pi a}{4}}\frac{1}{2^{\mu}} \exp\left(-\frac{a}{2}\sinh^2\frac{y}{2}\right) D_{2\mu}\left(2\sqrt{\frac{a}{2}}\cosh\frac{y}{2}\right) \tag{5.278}$$

$$= \frac{1}{2}\sqrt{\pi a}.2^{-\mu} \exp\left(-\frac{a}{2}\sinh^2\frac{y}{2}\right) D_{2\mu}\left(\sqrt{2a}\cosh\frac{y}{2}\right) \tag{5.279}$$

We have thus established the following cosine transform:

$$\int_0^\infty W_{\mu,ip}(2u)\cos(py)dp = \sqrt{\frac{\pi u}{2}} 2^{-\mu} \exp\left(-u\sinh^2\frac{y}{2}\right) D_{2\mu}\left(\cosh\frac{y}{2}\sqrt{2u}\right) = \mathscr{I}_{\mu}(y,u) \tag{5.280}$$

Given that the original equation for the Asian option price is defined through:

$$I(x;\tau) = \int_0^\infty e^{-p^2\tau} p\sinh(\pi p)\left|\Gamma\left(1/2-\mu+\frac{ip}{2}\right)\right|^2 W_{\mu,ip}(2x)dp \tag{5.281}$$

we may invert the cosine transform for the Whittaker function via:

$$W_{\mu,ip}(2u) = \frac{2}{\pi}\sqrt{\frac{\pi u}{2}} 2^{-\mu}\int_0^\infty \cos(py)\exp\left(-u\sinh^2\frac{y}{2}\right) D_{2\mu}\left(\cosh\frac{y}{2}\sqrt{2u}\right)dy \tag{5.282}$$

We may then rearrange to obtain the compact form for the option price:

$$I(x;\tau) = \frac{2}{\pi}\sqrt{\frac{\pi u}{2}} 2^{-\mu}\int_0^\infty dy.D_{2\mu}\left(\cosh\frac{y}{2}\sqrt{2x}\right)\times \tag{5.283}$$

$$\int_0^\infty e^{-p^2\tau} p\sinh(\pi p)\left|\Gamma\left(1/2-\mu+\frac{ip}{2}\right)\right|^2 \cos(py)dp \tag{5.284}$$

thereby establishing the claims in the theorem and corollary. □

Remark 94. The difficulty here lies in evaluating the inner integral. There are some results from a certain mathematical journal of Ramanujan [150] that are of use here.

5.7.5 Convolution Theory of Asian Option

The Asian option pricing formula 5.6.8, 5.6.9, 5.6.10 can be understood as a cosine transform as shown in 5.7.7, 5.7.10. The solution to the index transform may be written as a convolution known to Ramanujan and Bateman. This formula may be much reduced using the Fubini-Tonelli theorem to give a nested integral related to the Mehler-Fock transform. We shall now show how the results from the cosine transform give useful formulae for the Asian option pricing schemata.

Lemma 5.7.9. *Taking the definition of the cosine transform of the Whittaker function as in 5.7.7, the Asian option pricing problem as given in [75,77,152] is defined by the integral:*

$$I = \int_0^\infty e^{-p^2\tau} p \sinh(\pi p) \left| \Gamma\left(\frac{1}{2} - \mu + \frac{ip}{2}\right) \right|^2 W_{\mu,ip}(2u) dp \tag{5.285}$$

This integral can be written in the form:

$$I = \frac{2}{\pi} \int_0^\infty dy. \mathscr{J}_\mu(y,u) \mathscr{S}_\mu(y,\tau) \tag{5.286}$$

where the respective integrands are given by the inverse cosine transform (see 5.7.7):

$$W_{\mu,ip}(2u) = \mathcal{F}_c^{-1}\left[\mathscr{J}_\mu(y,u)\right] \tag{5.287}$$

and the index integral:

$$\mathscr{S}_\mu(y,\tau) = \int_0^\infty dp. e^{-p^2\tau} p \sinh(\pi p) \cos(py) \left| \Gamma\left(\frac{1}{2} - \mu + \frac{ip}{2}\right) \right|^2 \tag{5.288}$$

Theorem 5.7.10. *The index integral as given in 5.7.9 defines the solution to the Asian option pricing problem.*

$$\mathscr{S}_\mu(y,\tau) = \int_0^\infty dp. e^{-p^2\tau} p \sinh(\pi p) \cos(py) \left| \Gamma\left(\frac{1}{2} - \mu + \frac{ip}{2}\right) \right|^2 \tag{5.289}$$

This integral may be written as the convolution product

$$f_1(w,\tau) \star f_2(w,\mu) = \int_{-\infty}^{+\infty} f_1(w,\tau) f_2(y-w,\mu) dw \tag{5.290}$$

$$= \frac{\pi}{4\tau^{3/2}} \Gamma\left(\frac{1}{2} - \mu\right) \Gamma(1-\mu) \int_{-\infty}^{+\infty} dw. \exp\left(\frac{\pi^2 - w^2}{4\tau}\right) \frac{\left(\pi \cos\left(\frac{\pi w}{2\tau}\right) - y \sin\left(\frac{\pi w}{2\tau}\right)\right)}{(\cosh y \cosh w - \sinh y \sinh w)^{1-2\mu}} \tag{5.291}$$

Proof. We begin by proving the lemma. Taking the definition of the Whittaker cosine transform as in 5.7.7, we may write:

$$W_{\mu,ip}(2u) = \mathcal{F}_c^{-1}\left[\mathscr{J}_\mu(y,u)\right] \tag{5.292}$$

$$I = \int_0^\infty e^{-p^2\tau} p \sinh(\pi p) \left| \Gamma\left(\frac{1}{2} - \mu + \frac{ip}{2}\right) \right|^2 \mathcal{F}_c^{-1}\left[\mathscr{J}_\mu(y,u)\right] dp \tag{5.293}$$

$$= \frac{2}{\pi} \int_0^\infty dy. \mathscr{J}_\mu(y,u) \int_0^\infty dp. e^{-p^2\tau} p \sinh(\pi p) \cos(py) \left| \Gamma\left(\frac{1}{2} - \mu + \frac{ip}{2}\right) \right|^2 \tag{5.294}$$

The inner integral in this case is given by the cosine transform:

$$I = \frac{2}{\pi} \int_0^\infty dy. \mathscr{J}_\mu(y,u) \mathscr{S}_\mu(y,\tau) \tag{5.295}$$

$$\mathscr{S}_\mu(y,\tau) = \int_0^\infty dp. e^{-p^2\tau} p \sinh(\pi p) \cos(py) \left| \Gamma\left(\frac{1}{2} - \mu + \frac{ip}{2}\right) \right|^2 \tag{5.296}$$

which proves the lemma. To prove the theorem, we have by inspection that the functions $e^{-p^2\tau} p \sinh(\pi p)$ and $\left| \Gamma\left(\frac{1}{2} - \mu + \frac{ip}{2}\right) \right|^2$ are even in p. It is known that the Fourier transform is given by the sum of the sine and cosine transforms; as both of these functions are even in p, their product is even in p, and we are free to use the complex exponential Fourier convolution theorem in lieu of the convolution transform for the cosine transform. With abuse of notation, we have that the Fourier cosine transform is self inverse, which follows from the definition. Writing the inner integral as above, we have:

$$\mathscr{S}_\mu(y,\tau) = \int_0^\infty dp. \cos(py) F_1(p) F_2(p) \tag{5.297}$$

with

$$F_1(p) = e^{-p^2\tau} p \sinh(\pi p) \tag{5.298}$$

$$F_2(p) = \left| \Gamma\left(\frac{1}{2} - \mu + \frac{ip}{2}\right) \right|^2 \tag{5.299}$$

In transform notation, we have $\mathcal{F}_c = \mathcal{F}_c^{-1}$, and also:

$$\mathcal{F}_c \left[F_1(p) F_2(p) \right] = f_1 \star f_2 = \mathcal{F}_c^{-1}[F_1] \star \mathcal{F}_c^{-1}[F_2] \tag{5.300}$$

where the first follows from the known fact that the cosine transform is self inverse. We have then the transform pairs:

$$f_1 = \mathcal{F}_c \left[e^{-p^2\tau} p \sinh(\pi p) \right] = \mathcal{F} \left[e^{-p^2\tau} p \sinh(\pi p) \right] \tag{5.301}$$

and

$$f_2 = \mathcal{F}_c \left[\left| \Gamma\left(\frac{1}{2} - \mu + \frac{ip}{2}\right) \right|^2 \right] = \mathcal{F} \left[\left| \Gamma\left(\frac{1}{2} - \mu + \frac{ip}{2}\right) \right|^2 \right] \tag{5.302}$$

Evaluating the first component, we have the cosine transform:

$$f_1 = \mathcal{F}_c \left[e^{-p^2\tau} p \sinh(\pi p) \right] = \int_0^\infty dp. \cos(py) e^{-p^2\tau} p \sinh(\pi p) \tag{5.303}$$

$$= \frac{\sqrt{\pi}}{4} \exp\left(\frac{\pi^2 - y^2}{4\tau} \right) \times \left(\pi \cos\left(\frac{\pi y}{2\tau}\right) - y \sin\left(\frac{\pi y}{2\tau}\right) \right) \tag{5.304}$$

For the second transform, by substitution it is simple to show that:

$$\int_0^\infty \left| \Gamma\left(\frac{1}{2} - \mu + \frac{ip}{2}\right) \right|^2 \cos(px) dp = 2 \int_0^\infty \left| \Gamma\left(\frac{1}{2} - \mu + ip\right) \right|^2 \cos(2px) dp \tag{5.305}$$

In particular, as commented previously, the integral given in the second part of the convolution was analysed both by Ramanujan and Bateman in different contexts, see [150,151]. We may use the following formula from Ramanujan (see e.g. [150], eq. 11.1.2) to evaluate the second integral:

$$\int_0^\infty \left| \Gamma(a + ix) \right|^2 \cos(2mx) dx = \frac{\sqrt{\pi}}{2} \Gamma(a) \Gamma(a + 1/2) \left(\cosh m\right)^{-2a} \tag{5.306}$$

and using $a = \frac{1}{2} - \mu, m = x$ we obtain:

$$\int_0^\infty \left|\Gamma\left(\frac{1}{2} - \mu + \frac{ip}{2}\right)\right|^2 \cos(px)dp = 2\frac{\sqrt{\pi}}{2}\Gamma\left(\frac{1}{2} - \mu\right)\Gamma\left(1 - \mu\right)\left(\cosh x\right)^{-1+2\mu} \tag{5.307}$$

$$= \sqrt{\pi}\Gamma\left(\frac{1}{2} - \mu\right)\Gamma\left(1 - \mu\right)\left(\cosh x\right)^{-1+2\mu} \tag{5.308}$$

which implies we have $\mu < 1/2$ for reasonable growth conditions. Computing the convolution, and using the fact that the functions are even, their product is even, and one may replace the cosine transforms by the complex exponential counterparts and vice versa, one finds:

$$\mathcal{F}_c\left[F_1(p)F_2(p)\right] = f_1 \star f_2 \tag{5.309}$$

$$= \int_{-\infty}^{+\infty} f_1(w)f_2(y - w)dw \tag{5.310}$$

$$\frac{\sqrt{\pi}}{4\tau^{3/2}}\sqrt{\pi}\Gamma\left(\frac{1}{2} - \mu\right)\Gamma\left(1 - \mu\right)\int_{-\infty}^{+\infty} dw.\exp\left(\frac{\pi^2 - w^2}{4\tau}\right) \tag{5.311}$$

$$\times\left(\pi\cos\left(\frac{\pi w}{2\tau}\right) - y\sin\left(\frac{\pi w}{2\tau}\right)\right)\left(\cosh(y - w)\right)^{-1+2\mu} \tag{5.312}$$

$$= \frac{\pi}{4\tau^{3/2}}\Gamma\left(\frac{1}{2} - \mu\right)\Gamma\left(1 - \mu\right)\int_{-\infty}^{+\infty} dw.\exp\left(\frac{\pi^2 - w^2}{4\tau}\right)\frac{\left(\pi\cos\left(\frac{\pi w}{2\tau}\right) - y\sin\left(\frac{\pi w}{2\tau}\right)\right)}{\left(\cosh(y - w)\right)^{1-2\mu}} \tag{5.313}$$

$$= \frac{\pi}{4\tau^{3/2}}\Gamma\left(\frac{1}{2} - \mu\right)\Gamma\left(1 - \mu\right)\int_{-\infty}^{+\infty} dw.\exp\left(\frac{\pi^2 - w^2}{4\tau}\right)\frac{\left(\pi\cos\left(\frac{\pi w}{2\tau}\right) - y\sin\left(\frac{\pi w}{2\tau}\right)\right)}{\left(\cosh y\cosh w - \sinh y\sinh w\right)^{1-2\mu}} \tag{5.314}$$

where we have used the hyperbolic subtraction formula:

$$\cosh(y - w) = \cosh y\cosh w - \sinh y\sinh w \tag{5.315}$$

as required to establish the theorem. We note that this function is real valued, this is to be expected under the conditions of the cosine transform, as any Fourier transform that is real is even, and any even function has the cosine transform equal to the complex exponential Fourier transform, which is consistent with the assumptions we have used to derive this result. This is a generalised form of the type of integral encountered in deriving the second representation of the hyperbolic heat kernel, as in 4.7.17. With appropriate substitutions, one can recover this as a special case of this formula. This is one way in which the connection between Asian options (exotic options with arithmetic averaging features) and hyperbolic heat kernels can be understood. □

Remark 95. Bateman [151] gives an alternative formula for the integral:

$$\int_0^\infty |\Gamma(a + ix)|^2\cos(xy)dx = \pi\Gamma(2a)2^{-2a}\cosh^{-2a}\left(\frac{y}{2}\right) \tag{5.316}$$

which may be used to establish a similar theory of the convolution and cosine transform in this context. Another gamma function integral formula of the same type may be found in Oberhettinger [28], see eq. 9.1, pp. 47.

We shall now establish a closely related representation to this integral by using a certain Mellin transform, available in Bateman's reference.

Theorem 5.7.11. *Given the Mellin transform (Bateman eq. 7.5.20 [151]):*

$$\int_0^\infty x^{s-1} 2\sqrt{a} e^{-x+a/2} K_{2\nu}(2\sqrt{ax}) dx = |\Gamma(s+\nu+1/2)|^2 W_{-s,\nu}(a) \tag{5.317}$$

$$= |\Gamma(s+\nu+1/2)|^2 W_{s,\nu}(a) \tag{5.318}$$

the Asian option pricing problem as in 5.7.7, 5.7.9, 5.7.10:

$$I = \int_0^\infty e^{-p^2\tau} p \sinh(\pi p) \left| \Gamma\left(\frac{1}{2} - \mu + ip\right)\right|^2 W_{\mu,ip}(2u) dp \tag{5.319}$$

can be written as the Mellin transform:

$$I = 2\sqrt{a} e^{a/2} \int_0^\infty dx. x^{-\mu-1} e^{-x} \mathscr{S}(a, x, \tau) \tag{5.320}$$

where the inner integral is given by:

$$\mathscr{S}(a, x, \tau) = \int_0^\infty dp. e^{-p^2\tau} p \sinh(\pi p) K_{2ip}(2\sqrt{ax}) \tag{5.321}$$

Proof. We take the definition of the Asian option pricing integral as in 5.7.10, we have:

$$I = \int_0^\infty e^{-p^2\tau} p \sinh(\pi p) \left| \Gamma\left(\frac{1}{2} - \mu + ip\right)\right|^2 W_{\mu,ip}(2u) dp \tag{5.322}$$

Simple rearrangement of Bateman's relationship for the Mellin transform yields:

$$|\Gamma(-\mu + ip + 1/2)|^2 W_{\mu,ip}(a) = \int_0^\infty x^{-\mu-1} 2\sqrt{a} e^{-x+a/2} K_{2ip}(2\sqrt{ax}) dx \tag{5.323}$$

Inserting this into the integral, and invoking Fubini-Tonelli we obtain:

$$I = \int_0^\infty dp. e^{-p^2\tau} p \sinh(\pi p) \int_0^\infty dx. x^{-\mu-1} 2\sqrt{a} e^{-x+a/2} K_{2ip}(2\sqrt{ax}) \tag{5.324}$$

$$= 2\sqrt{a} e^{a/2} \int_0^\infty dx. x^{-\mu-1} e^{-x} \int_0^\infty dp. e^{-p^2\tau} p \sinh(\pi p) K_{2ip}(2\sqrt{ax}) \tag{5.325}$$

which reduces to the Mellin integral:

$$\mathscr{S}(a, x, \tau) = \int_0^\infty dp. e^{-p^2\tau} p \sinh(\pi p) K_{2ip}(2\sqrt{ax}) \tag{5.326}$$

$$I = 2\sqrt{a} e^{a/2} \int_0^\infty dx. x^{-\mu-1} e^{-x} \mathscr{S}(a, x, \tau) \tag{5.327}$$

as stated in the theorem. □

Remark 96. The index transform given through

$$\int_0^\infty dp. e^{-p^2\tau} p \sinh(\pi p) K_{2ip}(u) \tag{5.328}$$

is therefore of primary importance. This is, up to scaling, equivalent to the Yor integral for the modified Bessel function.

We shall now briefly comment on how this integral is related to a similar integral for the Mehler-Fock transform, using the projection-slice theorem for the hyperbolic functions. Writing the Yor-type integral, we have:

$$I = \int_0^\infty dp.e^{-p^2\tau} p \sinh(\pi p) K_{ip}(u) \tag{5.329}$$

$$= \sqrt{\frac{\pi}{2}} y^{-\mu+1/2} \int_1^\infty dv.e^{-yv}(v^2-1)^{-\mu/2} \int_0^\infty dp.e^{-p^2\tau} p \sinh(\pi p) \mathcal{P}^{\mu}_{ip-1/2}(v) \tag{5.330}$$

which shows that, in this representation, what is important is the Yor integral for the generalised Mehler-Fock function via:

$$\int_0^\infty dp.e^{-p^2\tau} p \sinh(\pi p) \mathcal{P}^{\mu}_{ip-1/2}(v) \tag{5.331}$$

Evaluation of this integral is related to certain problems of eigenvalue decomposition in the hyperbolic plane as analysed in the previous chapters of this work.

5.7.6 Yor Integral for the Mehler-Fock Transform

The Yor integral is the restriction of the eigenfunction decomposition of the kernel to a zero initial condition. The Yor integral may be found using the same process of interchange of the integrals and application of the Fubini-Tonelli theorem. In particular, one may recover a complementary formula for the Mehler-Fock kernel that is similar to that obtained from the convolution theorem. We shall utilise the results established for the hyperbolic heat kernel as in the previous chapters, see e.g. 4.7.16. Some results are known, for example [11,124] give a formula we may use for the Mehler-Fock kernel.

Theorem 5.7.12. *The hyperbolic heat kernel is given by the eigenfunction decomposition in the continuous spectrum:*

$$\mathcal{K}(v,z;\tau) = \int_0^\infty dp.e^{-p^2\tau} p \sinh(\pi p) \mathcal{P}^{\mu}_{ip-1/2}(z) \mathcal{P}^{\mu}_{ip-1/2}(v) \tag{5.332}$$

The Yor integral is the special case of this kernel with initial value zero.

$$I = \mathcal{K}(v,0;\tau) \tag{5.333}$$

There exists a Weierstrass integral representation of this integral as given in this work (see 4.7.16, also original discussion in Chavel [11]):

$$\mathcal{K}(v,z;\tau) = \mathcal{K}(z,v;\tau) = \frac{1}{\sqrt{2}\pi^2} \int_{d(z,v)}^\infty \frac{\beta \exp\left(-\frac{\beta^2}{4\tau}\right)}{\sqrt{\cosh\beta - \cosh d(z,v)}} d\beta \tag{5.334}$$

$$\mathcal{K}(v,0;\tau) = \frac{1}{\sqrt{2}\pi^2} \int_{d(z,0)}^\infty \frac{\beta \exp\left(-\frac{\beta^2}{4\tau}\right)}{\sqrt{\cosh\beta - \cosh d(z,0)}} d\beta \tag{5.335}$$

Corollary 5.7.13. *Under the conditions of hyperbolic space, the distance function as in 5.7.12 may be written as derived in 3.3.12:*

$$\cosh d(v,0) = \cosh R \cosh r + \sinh R \sinh r \cos\theta \tag{5.336}$$

This defines a Gaussian integral:

$$\mathcal{K}(v,0;\tau) = \frac{1}{2\pi^2} \int_1^\infty \frac{\beta \exp\left(-\frac{\beta^2}{4\tau}\right)}{\sinh\frac{\beta}{2}} d\beta \tag{5.337}$$

Proof. We direct the reader to the earlier proof of the hyperbolic heat kernel in this work, see e.g. the arguments contained in 4.7.16. If we examine the distance formula in this situation

$$\cosh d(v,0) = \cosh R \cosh r + \sinh R \sinh r \cos\theta \tag{5.338}$$

$$\mathcal{K}(v,0;\tau) = \frac{1}{\sqrt{2}\pi^2}\int_1^\infty \frac{\beta \exp\left(-\frac{\beta^2}{4\tau}\right)}{\sqrt{\cosh\beta - 1}} d\beta \tag{5.339}$$

Using the double angle formula as before, we arrive at the integral representation:

$$\mathcal{K}(v,0;\tau) = \frac{1}{2\pi^2}\int_1^\infty \frac{\beta \exp\left(-\frac{\beta^2}{4\tau}\right)}{\sinh\frac{\beta}{2}} d\beta \tag{5.340}$$

$$\cosh\beta - 1 = 2\sinh^2\frac{\beta}{2} \tag{5.341}$$

as required. □

We shall now state a similar theorem related to an analogous representation of the Mehler-Fock functions. This way to directly evaluate the Yor integral stems from the use of the Dirichlet-Murphy integral representation for the Legendre function:

Theorem 5.7.14. *Given the Dirichlet-Murphy integral representation (see [121], 14.12.4):*

$$\mathcal{P}_\nu^{-\mu}(\cosh x) = \sqrt{\frac{2}{\pi}}\frac{\Gamma(\mu+1/2)\sinh^\mu x}{\Gamma(\mu+\nu+1)\Gamma(\mu-\nu)}\int_0^\infty \frac{\cosh\left((\nu+1/2)\,u\right)}{(\cosh x + \cosh u)^{\mu+1/2}}du \tag{5.342}$$

then the generalised Yor integral is given by the integral:

$$I = \int_0^\infty e^{-p^2\tau}|\Gamma(-\mu+ip+1/2)|^2\mathcal{P}^\mu_{ip-1/2}(\cosh x)p\sinh(\pi p)dp \tag{5.343}$$

$$= \frac{1}{8}\frac{\sqrt{2}}{\tau^{3/2}}\Gamma(-\mu+1/2)\sinh^{-\mu}x \tag{5.344}$$

$$\times\int_0^\infty du.\frac{1}{(\cosh x+\cosh u)^{-\mu+1/2}}\exp\left(\frac{\pi^2-u^2}{4\tau}\right)\left(\pi\cos\left(\frac{\pi u}{2\tau}\right) - u\sin\left(\frac{\pi u}{2\tau}\right)\right) \tag{5.345}$$

Proof. We take the formula for the Dirichlet-Murphy integral representation as in [121], 14.12.4, using $\nu = ip - 1/2$ we find

$$\mathcal{P}^{-\mu}_{ip-1/2}(\cosh x) = \sqrt{\frac{2}{\pi}}\frac{\Gamma(\mu+1/2)\sinh^\mu x}{\Gamma(\mu+ip+1/2)\Gamma(\mu-ip+1/2)}\int_0^\infty \frac{\cosh(ipu)}{(\cosh x+\cosh u)^{\mu+1/2}}du \tag{5.346}$$

We write the associated Legendre function using this integral formula:

$$|\Gamma(-\mu+ip+1/2)|^2\mathcal{P}^\mu_{ip-1/2}(\cosh x) \tag{5.347}$$

$$= \sqrt{\frac{2}{\pi}}\Gamma(-\mu+1/2)\sinh^{-\mu}x\int_0^\infty \frac{\cos(pu)}{(\cosh x+\cosh u)^{-\mu+1/2}}du \tag{5.348}$$

and further, inserting this into the expression for the Yor integral we find:

$$\int_0^\infty e^{-p^2\tau}|\Gamma(-\mu+ip+1/2)|^2\mathcal{P}^\mu_{ip-1/2}(\cosh x)p\sinh(\pi p)dp \tag{5.349}$$

$$= \int_0^\infty dp.e^{-p^2\tau}\sqrt{\frac{2}{\pi}}\Gamma(-\mu+1/2)\sinh^{-\mu}x.p\sinh(\pi p)\int_0^\infty du.\frac{\cos(pu)}{(\cosh x+\cosh u)^{-\mu+1/2}} \tag{5.350}$$

Interchanging the integrals under Fubini-Tonelli, we find the following:

$$I = \sqrt{\frac{2}{\pi}}\Gamma(-\mu+1/2)\sinh^{-\mu}x \tag{5.351}$$

$$\times\int_0^\infty du.\frac{1}{(\cosh x+\cosh u)^{-\mu+1/2}}\times\int_0^\infty dp.e^{-p^2\tau}p\sinh(\pi p)\cos(pu) \tag{5.352}$$

We may readily evaluate the inner integral, finding:

$$\int_0^\infty dp.e^{-p^2\tau}p\sinh(\pi p)\cos(pu) = \frac{1}{8}\frac{\sqrt{\pi}}{\tau^{3/2}}\exp\left(\frac{\pi^2-u^2}{4\tau}\right)\left(\pi\cos\left(\frac{\pi u}{2\tau}\right)-\sin\left(\frac{\pi u}{2\tau}\right)\right) \tag{5.353}$$

$$I = \int_0^\infty e^{-p^2\tau}|\Gamma(-\mu+ip+1/2)|^2\mathcal{P}^\mu_{ip-1/2}(\cosh x)p\sinh(\pi p)dp \tag{5.354}$$

$$= \frac{1}{8}\frac{\sqrt{2}}{\tau^{3/2}}\Gamma(-\mu+1/2)\sinh^{-\mu}x \tag{5.355}$$

$$\times\int_0^\infty du.\frac{1}{(\cosh x+\cosh u)^{-\mu+1/2}}\exp\left(\frac{\pi^2-u^2}{4\tau}\right)\left(\pi\cos\left(\frac{\pi u}{2\tau}\right)-u\sin\left(\frac{\pi u}{2\tau}\right)\right) \tag{5.356}$$

as required. □

Remark 97. This formula is extremely similar to that obtained using the convolution method for the Asian option in 5.7.10, where we obtained a result that was proportional to:

$$\tau^{-3/2}\int_{-\infty}^{+\infty} e^{(\pi^2-w^2)/4\tau}\left(\pi\cos\left(\frac{\pi w}{2\tau}\right)-w\sin\left(\frac{\pi w}{2\tau}\right)\right)\cosh^{\mu-1}(y-w)dw \tag{5.357}$$

As discussed prior, we have shown that this relates to various special cases of the hyperbolic heat kernel. As the previous calculations have shown, it is possible that by exploring the link between these sets of special functions we may find new representations for the solutions of diffusion equation in hyperbolic spaces. It is clear from the computational results that a result in any one of the domains of Whittaker, modified Bessel or Mehler-Fock functions implies a complementary formula should exist for the other groups of functions in the set. This is the informational content of the projection-slice theorem in this context; as the sets of eigenfunctions are isospectral, one can map cleanly from one set to another.

5.7.7 Special Cases of Yor Integrals

The standard and associated form of the Mehler-Fock kernel and transforms may be found using the Yor integral. In the special case of an associated Legendre (Mehler-Fock) function of integer order, one may recover a cosine transform related to the error function. In the following, we loosen the strict definition of the hyperbolic heat kernel as given by the eigenfunction decomposition, and shall consider various other situations.

Lemma 5.7.15. *The generalised form of the Yor integral in 5.7.14 is given by the formula:*

$$\mathcal{K}(w,0;\tau) = \int_0^\infty e^{-p^2\tau}|\Gamma(ip+a+\mu)|^2\mathcal{P}^\mu_{ip-1/2}(\cosh x)p\sinh(\pi p)dp \tag{5.358}$$

The special case $a = 1/2, \mu = 0$ defines the Mehler-Fock transform of $e^{-p^2\tau}$:

$$\mathcal{K}(w, 0; \tau) = \pi \int_0^\infty e^{-p^2\tau} \mathcal{P}_{ip-1/2}(\cosh x) p \tanh(\pi p) dp \tag{5.359}$$

Proof. Writing the integral, we have as in the lemma

$$\mathcal{K}(w, 0; \tau) = \int_0^\infty e^{-p^2\tau} |\Gamma(ip + 1/2)|^2 \mathcal{P}_{ip-1/2}(\cosh x) p \sinh(\pi p) dp \tag{5.360}$$

using gamma function identities we have further that

$$|\Gamma(ip + 1/2)|^2 = \frac{\pi}{\cosh(\pi p)} \tag{5.361}$$

and we obtain:

$$\mathcal{K}(w, 0; \tau) = \pi \int_0^\infty e^{-p^2\tau} \mathcal{P}_{ip-1/2}(\cosh x) p \tanh(\pi p) dp \tag{5.362}$$

which is the standard expression for the Mehler-Fock transform, as required in the establishment of the lemma. □

Lemma 5.7.16. *If, contrary to the previous lemma, we have $a = -1/2, \mu = +1/2$, then the kernel is given by the Gaussian fundamental solution:*

$$\mathcal{K}(w, 0; \tau) = \left(\frac{2\pi}{\sinh x}\right)^{1/2} \int_0^\infty e^{-p^2\tau} \cos(px)\, dp \tag{5.363}$$

$$= \left(\frac{2\pi}{\sinh x}\right)^{1/2} \sqrt{\frac{\pi}{\tau}} \exp\left(-\frac{x^2}{4\tau}\right) \tag{5.364}$$

Proof. Substituting the values in the generalised Yor integral, we have the gamma function identity:

$$|\Gamma(ip)|^2 = \frac{\pi}{p \sinh(\pi p)} \tag{5.365}$$

We find a different kernel, which defines a different type of transform via:

$$\mathcal{K}(w, 0; \tau) = \pi \int_0^\infty e^{-p^2\tau} \mathcal{P}^{1/2}_{ip-1/2}(\cosh x) dp \tag{5.366}$$

Using DLMF formula 14.5.6 [121] we obtain:

$$\mathcal{P}^{1/2}_{ip-1/2}(\cosh x) = \left(\frac{2}{\pi \sinh x}\right)^{1/2} \cosh(ipx) = \left(\frac{2}{\pi \sinh x}\right)^{1/2} \cos(px) \tag{5.367}$$

and the kernel simplifies to:

$$\mathcal{K}(w, 0; \tau) = \left(\frac{2\pi}{\sinh x}\right)^{1/2} \int_0^\infty e^{-p^2\tau} \cos(px)\, dp \tag{5.368}$$

$$= \left(\frac{2\pi}{\sinh x}\right)^{1/2} \sqrt{\frac{\pi}{\tau}} \exp\left(-\frac{x^2}{4\tau}\right) \tag{5.369}$$

which is a Gaussian kernel, up to scaling. □

There exists another special case, given by the Legendre polynomials. We may find this representation by using the associated Legendre functions of integer order.

Lemma 5.7.17. The special case of the generalised Yor integral $a = -1/2, \mu = -n + 1/2$ is given by the kernel:

$$\mathcal{K}(w, 0; \tau) = \int_0^\infty e^{-p^2\tau} |\Gamma(-n + ip)|^2 \mathcal{P}_{ip-1/2}^{-n+1/2}(\cosh x) p \sinh(\pi p) dp \tag{5.370}$$

$$= \int_0^\infty \prod_{k=1}^n \frac{1}{(k^2 + p^2)} e^{-p^2\tau} \mathcal{P}_{ip-1/2}^{-n+1/2}(\cosh x) dp \tag{5.371}$$

The restriction $n = 1$ gives the Gaussian error function integral:

$$\int_0^\infty \frac{1}{(1 + p^2)} e^{-p^2\tau} \mathcal{P}_{ip-1/2}^{-1/2}(\cosh x) dp \tag{5.372}$$

$$= \left(\frac{2}{\pi \sinh x}\right)^{1/2} \int_0^\infty \frac{1}{(1 + p^2)} e^{-p^2\tau} \cos(px) dp \tag{5.373}$$

$$= \frac{\pi e^\tau}{4} \left(e^x \Phi_c \left(\sqrt{\tau} + \frac{x}{2\sqrt{\tau}}\right) + e^{-x} \Phi_c \left(\sqrt{\tau} - \frac{x}{2\sqrt{\tau}}\right)\right) \tag{5.374}$$

Proof. Using as before the gamma function identities, we have:

$$\mathcal{K}(w, 0; \tau) = \int_0^\infty e^{-p^2\tau} |\Gamma(-n + ip)|^2 \mathcal{P}_{ip-1/2}^{-n+1/2}(\cosh x) p \sinh(\pi p) dp \tag{5.375}$$

$$= \int_0^\infty \prod_{k=1}^n \frac{1}{(k^2 + p^2)} e^{-p^2\tau} \mathcal{P}_{ip-1/2}^{-n+1/2}(\cosh x) dp \tag{5.376}$$

$$\int_0^\infty \frac{1}{(1 + p^2)} e^{-p^2\tau} \mathcal{P}_{ip-1/2}^{-1/2}(\cosh x) dp \tag{5.377}$$

$$= \left(\frac{2}{\pi \sinh x}\right)^{1/2} \int_0^\infty \frac{1}{(1 + p^2)} e^{-p^2\tau} \cos(px) dp \tag{5.378}$$

$$= \frac{\pi e^\tau}{4} \left(e^x \Phi_c \left(\sqrt{\tau} + \frac{x}{2\sqrt{\tau}}\right) + e^{-x} \Phi_c \left(\sqrt{\tau} - \frac{x}{2\sqrt{\tau}}\right)\right) \tag{5.379}$$

as required. □

5.8 Comments

Many of the formulae in this chapter have depended on the clever use of integral generating functions, for example we used the expressions of Oberhettinger [72,74]

$$\pi I_\nu(z) = \int_0^\infty e^{z \cos t} \cos(\nu t) dt - \sin(\nu\pi) \int_0^\infty e^{-z \cosh t - \nu t} dt \tag{5.380}$$

$$K_\nu(z) = \frac{1}{2} \int_0^\infty x^{-\nu-1} \exp\left(-x - \frac{z^2}{4x}\right) dx \tag{5.381}$$

and the product formula:

$$\pi I_\nu(a) K_\nu(b) = \frac{1}{2} \int_0^\infty x^{-1} \exp\left(-x - \frac{a^2 + b^2}{4x}\right) I_\nu\left(\frac{ab}{2x}\right) dx \tag{5.382}$$

to derive an expression for a basic index integral of surprising generality. The last of these expressions represents the Green's function in a sense, and allows the question. Given that there exist such formulae for the modified Bessel functions as stated, what counterpart might we expect for the Whittaker function pair? In this case, some similar formulae might give readily computable expressions for the Green's function. Some clues are given in the paper of Becker [153], see also Otunuga [154] for an excellent and topical discussion of the use of Whittaker functions to model population infection dynamics, the equivalent formula being given for the Green's function as:

$$G(x, y; E) = \frac{\pi^2 \Gamma(1/2 - \kappa + \mu)}{\Gamma(1 + 2\mu)} W_{\kappa,\mu}(x) M_{\kappa,\mu}(y) \tag{5.383}$$

$$= \int_0^\infty \frac{u \sinh(2\pi u)}{\mu^2 + u^2} \left|\Gamma\left(1/2 - \kappa - iu\right)\right|^2 W_{\kappa,iu}(x) W_{\kappa,iu}(y) du + \text{finite terms} \tag{5.384}$$

the finite terms coming from a close analysis of the residue calculus of the Whittaker functions. One way in which these types of formulae could be approached from a new direction might be to use the expressions for the Whittaker functions of the first and second kinds as given through the modified Bessel integrals [122] eq. 7.629.2:

$$\int_0^\infty t^{-k} e^{-a/2t} e^{-st} W_{k,\mu}\left(\frac{a}{t}\right) dt = 2\sqrt{a} s^{k-1/2} K_{2\mu}(2\sqrt{as}) \tag{5.385}$$

Inversion of any of the possible transforms on the left hand side of the integral allows description of the Whittaker function in the positive plane. It is straightforward to see that interesting equivalencies will result depending on whether one chooses the Mellin, Weierstrass or Laplace transform to invert. Similar relationships exist for the Basset function and the Whittaker equation of the first kind and it is likely that such a theory might be extremely useful in the field of differential equations, representation theory and special functions. The right hand side, which gives solution of the integral is also the eigenfunction defined through the Mehler-Fock conversion formula used throughout this chapter, viz.:

$$\int_1^\infty e^{-ax} (x^2 - 1)^{-\mu/2} \mathcal{P}^{\mu}_{ip-1/2}(x) dx = \sqrt{\frac{2}{\pi}} a^{\mu - 1/2} K_{ip}(a) \tag{5.386}$$

Grosche and Steiner eq. 6.4.18 [14,17] give an analogous result using a path integral methodology, finding the representation for the Green's function defined by:

$$G(z, w; E) = C W_{k,\mu}(z) M_{k,\mu}(w) \tag{5.387}$$

$$= \frac{i}{\hbar} \int_0^\infty dT. e^{iTE/\hbar} \oiint \mathscr{D}[x(t)] e^{i/\hbar \int \mathcal{L}(x,\dot{x}) dt} \tag{5.388}$$

and the functional inside the path integral is given by the Morse oscillator Lagrangian:

$$\mathcal{L}(x, \dot{x}) = \frac{m}{2} \dot{x}^2 - \frac{\hbar^2 V_0^2}{2m} (e^{2x} - 2\alpha e^x) \tag{5.389}$$

$$z = 2V_0 e^x, w = 2V_0 e^y \tag{5.390}$$

Although it is beyond the scope of this work, it is interesting to observe that the formula developed using the path integral method is far more general. Even the continuous index transform component has a wide sense validity. It is hoped that future investigations will uncover further aspects of these types of useful relationships between different realms of modern mathematics. The product formula of Buchholz [134] also gives insight into other representations of the Whittaker function that might be obtained. The integrand here is of primary interest, as it again can be used to formulate the Whittaker function via the previous integrals used in this section. If we take the formula from Prudnikov [126], Vol. II, 2.16.9.8, it is known:

$$\int_0^\infty \frac{\exp\left(-p\sqrt{x^2+z^2}\right)}{\sqrt{x^2+z^2}} K_\nu(cx)dx = \frac{1}{2}\sec\left(\frac{\nu\pi}{2}\right) K_{\nu/2}\left(\frac{z_+}{2}\right) K_{\nu/2}\left(\frac{z_-}{2}\right) \tag{5.391}$$

for the variables:

$$\Re(z), \Re(p+c) > 0; |\Re(\nu)| < 1 \tag{5.392}$$

$$z_\pm = z(p \pm \sqrt{p^2-c^2}) \tag{5.393}$$

The Buchholz formula 6.1.4α [134] contains similar information, for the Whittaker function:

$$W_{\chi,\mu/2}(a_1 t) W_{\chi,\mu/2}(a_2 t) = \frac{(a_1 a_2)^{1/2}\cdot 2t}{\Gamma\left(\frac{1+\mu}{2}-\chi\right)\Gamma\left(\frac{1-\mu}{2}-\chi\right)} \tag{5.394}$$

$$\times \int_0^\infty e^{-\frac{1}{2}(a_1+a_2)t\cosh v} K_\mu\left(t\sqrt{a_1 a_2}\sinh v\right)\left(\coth\frac{v}{2}\right)^{2\chi} dv \tag{5.395}$$

For the special case of the variables given by:

$$\chi = 0, t = 1, a_1 = x, a_2 = y \tag{5.396}$$

then we may write:

$$W_{0,\mu/2}(x) W_{0,\mu/2}(y) = \frac{2(xy)^{1/2}}{\Gamma\left(\frac{1+\mu}{2}\right)\Gamma\left(\frac{1-\mu}{2}\right)} \int_0^\infty e^{-\frac{1}{2}(x+y)\cosh v} K_\mu\left(\sqrt{xy}\sinh v\right) dv \tag{5.397}$$

which simplifies upon substitution of the modified Bessel function for the Whittaker function with first argument zero:

$$\frac{\sqrt{xy}}{\pi} K_\mu\left(\frac{x}{2}\right) K_\mu\left(\frac{y}{2}\right) = \frac{2\sqrt{xy}}{\Gamma\left(\frac{1+\mu}{2}\right)\Gamma\left(\frac{1-\mu}{2}\right)} \int_0^\infty e^{-\frac{1}{2}(x+y)\cosh v} K_\mu\left(\sqrt{xy}\sinh v\right) dv \tag{5.398}$$

hence, finally obtaining:

$$K_\mu\left(\frac{x}{2}\right) K_\mu\left(\frac{y}{2}\right) = \frac{2\pi}{\Gamma\left(\frac{1+\mu}{2}\right)\Gamma\left(\frac{1-\mu}{2}\right)} \int_0^\infty e^{-\frac{1}{2}(x+y)\cosh v} K_\mu\left(\sqrt{xy}\sinh v\right) dv \tag{5.399}$$

$$\Gamma\left(\frac{1+\mu}{2}\right)\Gamma\left(\frac{1-\mu}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi}{2}(\mu+1)\right)} = \frac{\pi}{\cos\left(\frac{\pi\mu}{2}\right)} \tag{5.400}$$

$$\frac{1}{2} K_\mu\left(\frac{x}{2}\right) K_\mu\left(\frac{y}{2}\right) = \cos\left(\frac{\pi\mu}{2}\right) \int_0^\infty e^{-\frac{1}{2}(x+y)\cosh v} K_\mu\left(\sqrt{xy}\sinh v\right) dv \tag{5.401}$$

Of course, other similar formulae exist for the Basset and mixed modified Bessel products, so this is likely to be a fruitful area of analysis. The integral above may be very simply inverted using the appropriate form of Laplace transformation; this process leads to a plethora of other equivalences between the different classes of special functions.

CHAPTER 6

Radon Transforms & the EPD Equation

6.1 Introduction and Review

This chapter details some results regarding the Euler-Poisson-Darboux equation related to spherical transformation laws of special functions. Taking our inspiration from Helgason [81], we derive the Radon transform and show that the solution to the EPD (Euler-Poisson-Darboux abbr.) equations is given through this integral transformation. We then use some insights from projective geometry and differential geometry of spherical spaces to derive the Laplacian operator. In doing so, we encounter a number of generalised EPD equations which can be seen as different types of singular hyperbolic PDEs. In particular, we cover a range of time dispersive systems where the time variable appears as a stochastic quantity. Using some relationships from Sturm-Liouville analysis as given through the Bose invariant as in 3.5.11, we are able to derive integral kernels and different fundamental solutions for these systems. We discuss the Weierstrass transform, essentially solvable operators, Bessel potentials and the classification of such hyperbolic singular PDEs. Although there are many known results for systems that are dispersive in space, the situation where dispersion occurs in a complementary fashion in the time co-ordinate is somewhat of a mystery, obscured by its novelty. In this chapter we discuss some modifications of linear-in-time operators, such as is standard in Sturm-Liouville theory, and allow the time variable to be of a diffusive nature. This simple change is sufficient to generate a large number of different relationships between various groups of special functions, including those of Whittaker, Mehler-Fock and Macdonald type. Using the methods explored in the series of papers by Craddock and Lennox [40,42], a further generalisation allows the use of essentially exact kernels to be written down. The solutions developed in this chapter differ from the normal situation most will be familiar with from Sturm-Liouville theory, in that the kernel elements and transition probability densities occur in time, not in space. We develop a theory herein whereby by using a complementarity principle we are able to directly evaluate these matrix elements from known formulae in spectral theory. Using the techniques of integral transform theory, these results are then mapped to a number of different classes of PDEs, giving some simple ways to represent the various special functions as aspects of the same underlying PDE structure.

This chapter shall focus on the topic of spherical functions, with applications to some modified forms of the Euler-Poisson-Darboux equation. We shall be primarily interested in the symmetries of these differential equations. Our synthesis shall cover a wide range of modern mathematics, let us now discuss the relevant literature as related to the calculations contained herein.

One fundamental idea within the modern theory of partial differential equations is the concept of a transformation of an equation. Many mathematicians have approached this in different ways, some offering classification schemes, others presenting formal algorithms for generating these symmetries. In many respects, the modern stochastic theory of PDEs, in the papers of Feynman [18] and Kac [21] defined the basic equivalence between stochastic processes and path integrals given through expectations of functionals. Feynman [18–20] was able to recover a number of different formulae related to kernel solutions of these PDEs, this chapter hopes to capture the spirit of elegance that these succinct formulae give as solutions for the harmonic oscillator and other quadratic systems.

Craddock and Lennox, also appearing in Craddock [40–42,46] have analysed the Lie symmetry groups of a number of systems closely related to the problems encountered in this chapter. The authors used the method originally explored in [155] to define the Lie symmetries via construction of continuous differential operators to represent the symmetry groups. In a contrasting fashion, in the papers of Albanese et. al [61,62], also Kuznetsov [156], the authors were able to derive various symmetries by using a Bose invariant potential. This method, which is equivalent to using a Schwartz derivative and associated bracket, is interesting in that it reveals the existence of scale factors. These scale factors again go back to e.g. the Feller decomposition [117,157,158] into so-called mass and scale factors within the representation of the second order PDE. The Bose invariant potential form is of great utility in that any second order PDE may be reduced to a Gaussian diffusion plus a potential term, a situation much simplified and easier to solve.

Another foundation in the development of the analysis of PDEs is covered by Mercer's theorem [67]. Mercer was able to find an equivalence between the existence of positive definite kernels, distance functions and projective homomorphisms. We take this as one of our central principles; many of the calculations enclosed in this chapter rely on the existence of kernels with properties given through these sort of group laws.

We note the advanced development and progress in the understanding of the link between kernels and diffusion processes given in the paper of Sousa and Yakubovich [56]. This work outlines some ways in which the link between kernels and integral transforms can be understood for a number of hyperbolic systems, specifically those of Mehler-Fock, Macdonald/Kontorovich-Lebedev, and Whittaker. These transform pairs are associated with their underlying differential equation by a clever use of Titchmarsh's spectral theory of differential equations. Using this method, the authors were able to uncover a number of closely related examples to those given in this work. We depart from their methods in that we shall pursue a different type of kernel, whereas in the systems given through these special functions have continuous degrees of freedom, in our case we shall be dealing with periodic, exponentially decaying Bessel functions associated with wave kernels and the equivalent operators for the EPD equation.

Classification schemes of similar types of differential equations have been applied in Avram et. al [159], see also [160–162] for a description of the Pearson classification system. Our problems shall differ from these other examples, as we shall be dealing with singular hyperbolic PDEs instead of the more familiar situation of parabolic equations. For historical interest, the classification system originally proposed by Pearson appeared in [93]. This is an important result that shows that many different probability distributions familiar to classical analysis are related to a common representation. This can be thought of as a type of potential theory; in fact, as it is possible to define large classes of probability density functions as the solution of second order PDEs, so the classification scheme of Pearson reduces to an approximation of the potential function. We shall touch upon this topic, and the related issue of potential theory as it relates to the Bose invariant distribution and scale transformations.

Let us now briefly cover the relevant results from the theory of quantum systems we shall require. Grosche, also Grosche and Steiner [14,16,17] is the canonical reference for path integrals for various potentials. The path integral is equivalent to the Green's function of the PDE, which in our situation can be understood as essentially the Laplace transform of this functional. Grosche and Steiner used a path integral formalism to solve for the kernel of the quantum equations. These equations can be seen as the heat equation in imaginary time, via the Wick rotation, or using properties of Wiener integrals. We shall avoid the difficult questions of measure in this chapter and focus primarily on the development of kernel formulae via some clever integral tricks. The calculations contained in [14,16] lead the way to our primary results in this chapter. In a sense, we are pursuing a Bessel-wave in complex time version of their system, which is a quantum wave function. Our approach is very similar, in that we also use decompositions over eigenfunctions to extract the kernel. These papers and results are very useful in gaining insight into the properties of wave kernels, heat kernels and the like. The analysis of the hyperbolic plane in our case shall be modified to deal with a hyperbola in time, which will result in modification of the underlying differential equations.

Importantly, in [14,16], the authors have found a number of different formulae related to special functions, including Macdonald, Mehler-Fock, Whittaker and hypergeometric types. These are all associated to the geometry of various representations of the hyperbolic plane, pseudosphere, etc. This connection is dependent on the existence of a metric tensor, which is given through the differential geometry. Kuzmak, in [66], derived a simple relationship for the Fubini-Study metric on projective state space that can be applied to find this useful tensor. In this chapter, we use this approach to find the spherical metric tensor on the basis of some simple unitary transformations applied to the state. The results in [66] are surprisingly general, extending to spherical and hyperbolic groups, the Heisenberg group and other higher dimensional objects. In our view, the use of the Fubini-Study metric is a highly direct way in which to engage the connection between matrix groups, geometry and the continuous differential operators. Other research has shown that this is an insightful way in which to generate special functions and understand the geometry of states, and this shall function as an important part of our hypothesis in solving the EPD systems. Development of the eigenfunctions is dependent on the existence of the Laplace operator, and its solutions. By using this method we can very easily derive the Laplace operator that corresponds to a particular unitary operator without having to resort to complications of representation theory in the main.

In a more group-theoretic setting, we consider the works of Godements [71], Dieudonne [70], also Helgason [114], as our standard axiomatic basis in the theory of spherical functions. These types of functions and their relationship with transform pairs is well-known, in particular we thread the needle between spherical groups, product formulae, positive definite kernels and unitary matrices. Alternative perspectives on such systems of product formulae include the use of group characters to derive related product formulae in [163], hypergroups and multiplication laws for the Jacobi polynomials in Connett [164,165]. Extensions of the idea of product multiplication kernels to index transforms and the index Whittaker convolution has been carried out by Sousa [37]. This is an important result, as the Whittaker function is descriptive of a large class of special functions as limiting cases in one way or another.

In this chapter we shall be interested in the intersection between these theories; on the one hand, the theory of groups and zonal decompositions, and on the other, the theory of special functions and diffusion processes. The link is given through representation theory, the extensive reference text of Vilenkin [69] is of use in this matter. Our approach in this chapter complements the strict application of representation theory to our problem; we adopt a more nuanced tactical regime and rely solely on geometric constructs that arise as a consequence of the projective state. This simple set of axioms is sufficient to generate many of the results that may be obtained. However, in one respect the approach given in [69] is more general in that this work considers kernels on groups as a fundamental part of the generating process for constructing integral relations. This is carried out by a sophisticated use of Mellin transforms applied to group representations, given through the kernel. The product laws of the group then go over into convolution kernels. Although we shall not touch upon the concept of a kernel of a group, at points our results will point towards similar aspects of analysis and we shall comment on this when appropriate.

This chapter shall necessitate understanding of transform theory in order to engage with the concepts. Some aspects of the following proof require familiarisation with the concepts of heat kernels, EPD equations, transmutation operators and so on. These can all be understood in terms of the Weierstrass transform of the heat kernel, or generalised EPD transform given through the modified Bessel solution operator. We direct the interested reader to the excellent papers of Bragg and Dettman [98,99,166,167] who have derived a number of interesting integral relationships between fundamental solutions of the heat, wave and EPD equations. This idiom is central to the calculations explored in this work, in particular the concept of using integral transforms to move from one solution space to another is of critical importance to our proof.

Another important transformation we shall encounter in this chapter is the Radon transform. This transform is of much utility in the field of computer tomography and photoacoustics. We are interested primarily in the following fact, established in Helgason [81] (and potentially much earlier): that one solution to the EPD equations in space is given through the Radon transform. This transform, which amounts to integrating a delta function over some hyperplane, and Fourier transforming, is of considerable interest and practical relevance especially as imaging and scattering resolutions improve. The understanding of such techniques of tomography is of relevance beyond their purpose as serving as solutions to EPD equations; by linking the geometry, scattering processes, transforms and filters, we are able to generate some very interesting results complementary to our calculations regarding spherical functions. The link is, of course, given through the kernel or modifications thereof. We direct the reader to the works of Deans [82,83], who developed a number of techniques of understanding and deriving Radon transforms that we shall require in this proof.

Other interesting proofs on similar systems are given by Rubin, who covered vertical slice tomography in [168], Agranovsky et. al who looked at spherical mean transforms as descriptive methods in twisted spherical means for the Heisenberg group, Antipov, Estrada and Rubin in [168] who looked at inverse problems for spherical mean transforms in constant curvature spaces. Although many of these works are allied to the EPD equations, few authors appear to have worked out the kernel formulae. Our work shall develop some ways in which we can compute these important functionals of the probability space. Radon transforms and other associated integral transforms have applicability far beyond the simple techniques considered in this chapter, and the development of new techniques of analysis for imaging science is an exciting prospect.

Further works on the Radon transform relevant to this chapter include those in [169, 170], who looked at some applications of Gegenbauer and Hankel type representations on the unit ball to calculate the spherical mean transform. These important papers establish some results related to the Gegenbauer transform that we shall require in this chapter. The authors encountered some interesting representations of the hyperbolic plane which were given by the associated Legendre functions. Our calculation shall shed further light on this result and present it in a context which reveals the link in a concerted fashion for wave, heat and EPD kernels. Nguyen, in [171] applied similar techniques of spherical mean transforms to solve some interesting questions in thermoacoustic tomography. It is interesting to think about the potential applications, in principle all one needs is a source, a scatterer, and an array of detectors to measure any number of things via similar methods, such as geology, medical imaging, sonar, fracture testing and other areas where the internal details of an object contain information of value. Moon in [172] has defined a number of properties related to inversion of the Radon transform that relate to the EPD systems in this chapter. These turn out to be given through spherical Bessel functions. Our work complements this approach in many respects although we depart from the strict methodology of Radon transformations the underlying results share much in common due to the relationship between the Radon transform and EPD problems.

Finally, we note that this proof references a number of integral formulae related to Bessel and Legendre functions. These integrals may be found in tables [29,122] where appending references may be found to support the formulae. We note that Vilenkin [69] contains a large number of different integral formulae related to some topics we shall discuss in this proof.

6.2 Outline of Chapter

In order of appearance, this chapter shall address as follows: the development of composition formulae descriptive of a group using some analysis from the theory of spherical transforms; the derivation of the Radon transform; derivation of the Euler-Poisson-Darboux solution via spherical transforms using insights from projective geometry and differential analysis; analysis of characters associated to spherical functions, specifically the Bessel functions used in this chapter. We then go on to apply these concepts to a series of time dispersive systems which generalise the EPD equation; analyse the special case of a Bessel time-potential using Lipschitz continuity and derive the Jacobi transform. The final part of the chapter deals with some more advanced topics including the kernels associated with generalised EPD equations, the classification of singular PDEs, and the Weierstrass transform as a transmutation operator between the wave, heat and EPD kernels. We close with a discussion of some future directions that are opened up via the analytical methods explored in this chapter.

6.3 Spherical Systems

We shall begin this work with a brief calculation related to spherical decompositions and induced representations. The basic form of a spherical function is given through the Haar integral formula [70,71,113]:

Definition 6.3.1. A spherical zonal function satisfies the zonal formula 7.3.15:

$$\int_K f(xky)dk = f(x)f(y) \tag{6.1}$$

where $k \in K$ is the group element, integration indicated appropriately through the Haar measure dk, where $f(e) = 1$ and f is positive definite in the sense of 3.3.1.

Remark 98. As the following calculations show, the group theory of the affine projection is given by the two dimensional hyperbolic space. This space has a spherical decomposition given by the modified Bessel function composition formula.

Theorem 6.3.2. *The spherical product formula for the Bessel function is given by one of the following three forms, involving the standard Bessel functions or the modified Bessel functions of the first and second kinds.*

$$K_0(a)K_0(b) = \int_0^\infty K_0\left(\sqrt{a^2+b^2+2ab\cosh t}\right) dt \tag{6.2}$$

$$\int_0^\pi J_0\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right) dx = \pi J_0(\alpha)J_0(\beta) \tag{6.3}$$

$$\int_{-\pi/2}^{+\pi/2} I_0\left(\sqrt{\alpha^2+\beta^2+2\alpha\beta\cos\theta}\right)d\theta = \pi I_0(\alpha)I_0(\beta) \tag{6.4}$$

Proof. Using stereographic projection and representation theory, we have the group equivalent given through the spherical zonal formula 7.3.15:

$$\int_K f(gkh)dk = f(g)f(h) \tag{6.5}$$

In terms of the kernel structure, we know from Mercer's theorem, 3.3.11 [67] that the kernel solution will be given by some functional of the distance 3.3.13. In such case, we find the expression

$$\int_K f(d_K(x,y))dk = f(x)f(y) \tag{6.6}$$

We take the group representation induced by the projection:

$$g \to g\cdot\mathcal{O}_1 \tag{6.7}$$

As before, we can characterise this using the group homomorphism 3.3.8:

$$\int f(gkh\cdot\mathcal{O}_1)dk = \int f(d(x,\mathcal{O}_1))dk = f(d(g,\mathcal{O}_1))f(d(h,\mathcal{O}_1)) \tag{6.8}$$

The distance between two points $(r,0)$ and (s,τ) in two dimensional hyperbolic coordinates is given by:

$$d(x,\mathcal{O}_1) = (r^2+s^2+2rs\cosh\tau)^{1/2} \tag{6.9}$$

$$d(g,\mathcal{O}_1) = r \tag{6.10}$$

$$d(h,\mathcal{O}_1) = s \tag{6.11}$$

and we obtain:

$$\int f(\sqrt{r^2+s^2+2rs\cosh\tau})d\tau = f(r)f(s) \tag{6.12}$$

We have already derived the cosine transform of the modified Bessel function in 5.6.1, where we have shown:

$$K_\nu(a)K_\nu(b) = \int_0^\infty \cosh(\nu t)K_0\left(\sqrt{a^2+b^2+2ab\cosh t}\right)dt \tag{6.13}$$

Analytically continuing the index, we may write:

$$K_{i\nu}(a)K_{i\nu}(b) = \int_0^\infty \cosh(i\nu t)K_0\left(\sqrt{a^2+b^2+2ab\cosh t}\right)dt \tag{6.14}$$

$$= \int_0^\infty \cos(\nu t)K_0\left(\sqrt{a^2+b^2+2ab\cosh t}\right)dt \tag{6.15}$$

and taking ν approaching zero, we find:

$$K_0(a)K_0(b) = \int_0^\infty K_0\left(\sqrt{a^2+b^2+2ab\cosh t}\right)dt \tag{6.16}$$

which yields the first part of the proof. To find the second formula, we use integral formula 6.684 from [122], finding:

$$\int_0^\pi \sin^{2\nu} x \frac{J_\nu\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)}{\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)^\nu}dx = 2^\nu\sqrt{\pi}\Gamma(\nu+1/2)\frac{J_\nu(\alpha)}{\alpha^\nu}\frac{J_\nu(\beta)}{\beta^\nu} \tag{6.17}$$

so we may write the special case as:

$$\int_0^\pi J_0\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)dx = \pi J_0(\alpha)J_0(\beta) \tag{6.18}$$

which yields the first part of the claim. To establish the remaining parts, we apply the relationship between modified and standard Bessel functions. We have $I_\nu(z) = i^{-\nu}J_{i\nu}(iz)$, and therefore find:

$$\int I_0\left(\sqrt{\alpha^2+\beta^2+2\alpha\beta\cos x}\right)dx = \pi I_0(\alpha)I_0(\beta) \tag{6.19}$$

To evaluate this integral, note that from [173], we may write:

$$I_\nu(ax)I_\nu(bx) = \frac{(abx)^\nu}{2^\nu\sqrt{\pi}\Gamma(\nu+1/2)}\int_{-1}^{+1}\frac{(1-u^2)^{\nu-1/2}}{(a^2+b^2+2ab|u|)^{\nu/2}}I_\nu\left(x\sqrt{a^2+b^2+2|ab|u}\right)du \tag{6.20}$$

Taking the special case $\nu = 0$, we have:

$$I_0(ax)I_0(bx) = \frac{1}{\pi}\int_{-1}^{+1}\frac{1}{\sqrt{1-u^2}}I_0\left(x\sqrt{a^2+b^2+2|ab|u}\right)du \tag{6.21}$$

which specialises to:

$$I_0(a)I_0(b) = \frac{1}{\pi}\int_{-1}^{+1}\frac{1}{\sqrt{1-u^2}}I_0\left(\sqrt{a^2+b^2+2|ab|u}\right)du \tag{6.22}$$

for $x = 1$. We then have:

$$I_0(a)I_0(b) = \frac{1}{\pi}\int_{-1}^{0}\frac{1}{\sqrt{1-u^2}}I_0\left(\sqrt{a^2+b^2+2|ab|u}\right)du \tag{6.23}$$

$$+\frac{1}{\pi}\int_{0}^{1}\frac{1}{\sqrt{1-u^2}}I_0\left(\sqrt{a^2+b^2+2|ab|u}\right)du \tag{6.24}$$

Using the substitutions $u = \cos\theta, du = -\sin\theta d\theta$, $\sin\theta = \sqrt{1-u^2}$, we then obtain:

$$I_0(a)I_0(b) = \frac{2}{\pi}\int_0^{\pi/2} I_0\left(x\sqrt{a^2+b^2+2|ab|\cos\theta}\right)d\theta \tag{6.25}$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} I_0\left(x\sqrt{a^2+b^2+2|ab|\cos\theta}\right)d\theta \tag{6.26}$$

as required. □

Remark 99. These are the basic types of identities that spherical functions are described by. This is a powerful technique in analysis which is useful in determining many integral kernels. One may develop a similar type of decomposition to define a number of other different types of hyperbolic functions, including those of Whittaker, Mehler-Fock and other Bessel functions. It is clear that such integral kernels depend on the nature of the distance function, as expressed in this case through the variable $d(a,b) = \sqrt{a^2+b^2+2|ab|\cosh\tau}$.

6.4 Radon Measure and Spherical Transform

6.4.1 Derivation of Radon Transform

The Radon transform of a function can be associated with its two dimensional Fourier transform. This may be derived by looking at the rotational part of the transform. We shall now derive some simple formulae that we shall require in the following calculation. Briefly, the Laplacian operator on a hyperbolic space has a form which may be derived in a simple way. In this context, we may write a rotation in the plane as matrix operations, and evaluate the Radon transform as arising from the action on the group. The following derivation follows closely proofs which may be found in [81,83], which may be readily understood in terms of the group theory and harmonic analysis of the affine displacement operator. We shall require the following theorem of group integration and Fourier transformation, which may be related to group invariant action and integration, see e.g. [69], (section I.2.3, pp. 27).

Definition 6.4.1. Let G be a locally compact group with left Haar measure $\mu(g)$, $g \in G$. Let the irreducible inner representation of G be given by $(\pi_\lambda, \mathscr{H})$, where $\mathscr{H}$ is a Hilbert space. Then the Fourier transform of f on G is the group integral:

$$\hat{f}(\pi_\lambda)(h) \doteq \hat{f}(h) = \int_G f(g)\pi_\lambda\,(gh)\,d\mu(g) \tag{6.27}$$

On application of the theory of the Fourier integral, we have the following corollary:

Corollary 6.4.2. *The Fourier integral on the Euclidean group $G = \mathbb{R}^n$ as given by 6.4.1 is defined by the displacement integral:*

$$\hat{f}(\mathbf{a}) = \int_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})e^{-i\mathbf{a}\cdot\mathbf{x}}d\mathbf{x} \tag{6.28}$$

Proof. Using 6.4.1, we have:

$$\hat{f}(\pi_\lambda)(h) \doteq \hat{f}(h) = \int_G f(g)\pi_\lambda\,(gh)\,d\mu(g) \tag{6.29}$$

Substituting $\pi_\lambda\,(\mathbf{xa}) = e^{-i\mathbf{a}\cdot\mathbf{x}}$, and using the measure with abuse of notation $d\mu(\mathbf{x}) = d\mathbf{x}$ we have immediately:

$$\hat{f}(\mathbf{a}) = \int_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})e^{-i\mathbf{a}\cdot\mathbf{x}}d\mathbf{x} \tag{6.30}$$

as required. Note that we must take into account an appropriate volume scale in this formula for it to be numerically correct. □

We shall now state a simple lemma which we may use to obtain the two-dimensional Radon transform through use of unitary invariance.

Lemma 6.4.3. *Take the Fourier integral on the Euclidean space $\mathbf{x} \in \mathbb{R}^n$ as given by 6.4.2, we have:*

$$\left(\hat{\mathcal{F}}_{\mathbf{a}}\right) f = \int e^{-i\mathbf{x}\cdot\mathbf{a}} f(\mathbf{x})d\mathbf{x} \tag{6.31}$$

Assume further invariance under the unitary action as specified by the isomorphism $\hat{U}\mathbf{x}\hat{U}^{-1} = \mathbf{x}$, likewise for $\hat{U}\mathbf{a}\hat{U}^{-1} = \mathbf{a}$. Then the Fourier integral under the isomorphism is invariant, i.e.:

$$\left(\hat{\mathcal{F}}_{\mathbf{a}}\right) f = \left(\hat{\mathcal{F}}_{\hat{U}\mathbf{a}\hat{U}^{-1}}\right) f \tag{6.32}$$

We shall now use the expression for the Fourier transform in Euclidean space to derive a type of Radon transform. The most basic example follows from derivation of the formula for the two-dimensional Fourier transform. We do this in order to illustrate the fundamental concepts in tomography, as later we shall extend this derivation to higher dimensional Euclidean space to derive more general forms of Radon transforms and relate them to the Euler-Poisson-Darboux equation.

Corollary 6.4.4. *The Fourier transform is given by a continuous integral over the group defined by the group representation $\pi_\lambda(\mathbf{xa}) = e^{-i\mathbf{a}\cdot\mathbf{x}}$, as defined in 6.4.2, 6.4.1. In two dimensions, this can be computed as the Fourier integral:*

$$\left(\hat{\mathcal{F}}_{\mathbf{a}}\right) f = \int e^{-i\mathbf{x}\cdot\mathbf{a}} f(\mathbf{x}) d\mathbf{x} \tag{6.33}$$

where the vector of transformation is given by the rotation:

$$\mathbf{a} = a \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} \tag{6.34}$$

Proof. Writing the 2-D Fourier transform explicitly, we obtain:

$$\left(\hat{\mathcal{F}}_{\mathbf{a}}\right) f = \int e^{-i\mathbf{x}\cdot\mathbf{a}} f(\mathbf{x}) d\mathbf{x} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i(a_x x + a_y y)} f(x, y) dx dy \tag{6.35}$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-ia(x\cos\theta + y\sin\theta)} f(x, y) dx dy \tag{6.36}$$

We shall use this formula in the following calculations to derive a simple form of the Radon transform. □

For the following brief calculation we shall require the definition of the group operation in this space.

Definition 6.4.5. The group action in the Fourier space is given by the rotation:

$$\hat{g}\mathbf{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{6.37}$$

The following lemmas may be readily obtained from the group action.

Lemma 6.4.6. *For the group action specified through 6.4.5, the rotated frame is given by:*

$$\hat{g}\mathbf{x} = \begin{bmatrix} x(u, s) \\ y(u, s) \end{bmatrix} \tag{6.38}$$

where the coordinate transform in u is:

$$x(u, s)\cos\theta + y(u, s)\sin\theta = u \tag{6.39}$$

Proof. The proof follows from substitution of the constituent parts, we have:

$$\hat{g}\mathbf{x} = \begin{bmatrix} x(u, s) \\ y(u, s) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ s \end{bmatrix} = \begin{bmatrix} u\cos\theta - s\sin\theta \\ s\cos\theta + u\sin\theta \end{bmatrix} \tag{6.40}$$

and hence the result follows:

$$x(u, s)\cos\theta + y(u, s)\sin\theta = u \tag{6.41}$$

The second coordinate s is solved in an analogous fashion, we shall not repeat the proof. □

We may derive a simple lemma as a consequence, as is well known in classical symplectic geometry and probability/measure theory, the integral on the group is given by an integral over the Jacobian measure. The Jacobian matrix gives the necessary conversion factor for rotation from one reference frame to another, or transformation of coordinates in a more general setting.

Lemma 6.4.7. *The Jacobian matrix associated with the coordinate transformation and group action of the rotation matrix 6.4.6 is:*

$$\hat{J} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{6.42}$$

The determinant satisfies $\det \hat{J} = 1$.

Proof. This is elementary, and proceeds from application of the Jacobian via:

$$\hat{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \tag{6.43}$$

Use of the coordinate transformation from 6.4.6 yields the result. The determinant follows as a consequence. □

The following theorem from [81] results from an application of the preceding lemmas.

Theorem 6.4.8. *The operator defined through:*

$$(\mathcal{R}_{\mathbf{a}})f = \int_{-\infty}^{+\infty} f(u\cos\theta - s\sin\theta, s\cos\theta + u\sin\theta)ds \tag{6.44}$$

is known as the two-dimensional Radon transform. The group integral given by the Fourier transform as in 6.4.2, 6.4.4 satisfies:

$$\left(\hat{\mathcal{F}}_{\mathbf{a}}\right) f = (\mathcal{F}_u \mathcal{R}_{\mathbf{a}})f \tag{6.45}$$

where the Fourier transform in radial coordinates is defined through:

$$(\mathcal{F}_u)g(u) = \int_{-\infty}^{+\infty} e^{-iau} g(u)du \tag{6.46}$$

Proof. Writing the group integral, we have:

$$\left(\hat{\mathcal{F}}_{\mathbf{a}}\right) f = \int e^{-i\mathbf{a}\cdot\mathbf{x}} f(\mathbf{x})d\mathbf{x} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i(a_x x + a_y y)} f(x,y)dxdy \tag{6.47}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i\mathbf{a}\cdot\mathbf{x}} f(\hat{J}^{-1}\mathbf{u})|\det \hat{J}|duds \tag{6.48}$$

Projecting down with the vector $\mathbf{a} = [1,0]^T$ we obtain the homomorphism via the Jacobian transformation:

$$\hat{J}^{-1}\mathbf{u} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ s \end{bmatrix} = \begin{bmatrix} u \\ s \end{bmatrix} = \begin{bmatrix} u\cos\theta - s\sin\theta \\ u\sin\theta + s\cos\theta \end{bmatrix} \tag{6.49}$$

and the dot product as:

$$\mathbf{a}\cdot\mathbf{x} = a[1,0]\cdot\begin{bmatrix} x(s)\cos\theta + y(s)\sin\theta \\ -x(s)\sin\theta + y(s)\cos\theta \end{bmatrix} = a[x(s)\cos\theta + y(s)\sin\theta] = au \tag{6.50}$$

Hence we find the Radon transform in the form:

$$\left(\hat{\mathcal{F}}_{\mathbf{a}}\right) f=\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-iau} f(u\cos\theta - s\sin\theta, s\cos\theta + u\sin\theta) du ds \tag{6.51}$$

as specified in the theorem. The operator defined through

$$(\mathcal{R}_{\mathbf{a}})f=\int_{-\infty}^{+\infty} f(u\cos\theta - s\sin\theta, s\cos\theta + u\sin\theta) ds \tag{6.52}$$

is known as the two-dimensional Radon transform, and can be found in the works of Deans at [82,83]. □

Remark 100. Note that we can write this in terms of the delta function:

$$(\mathcal{R}_s)f=\int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} du. f(x,y)\delta(x-u\cos\theta+s\sin\theta)\delta(y-s\cos\theta-u\sin\theta) \tag{6.53}$$

6.5 Geometry of Euclidean Space and the EPD Equation

We shall outline two main results in differential analysis that we shall require in the following sections. We would like a simple way to derive the metric tensor and relate this to the Laplacian. As is well known, results from differential geometry imply that knowledge of the metric is sufficient to define a Laplacian. We shall outline such a scheme, and show how the differential geometry from a known Laplacian defines a spherical metric restricted to the surface of a sphere in n-dimensional projective space. To complement this approach, using the Fubini-Study metric and a projective geometric method, we are able to define this metric as being essentially equivalent to that defined on the n-dimensional spherical surface.

6.5.1 Metric Tensor via Laplacian

The reduced form of the Laplacian on the sphere can be identified with a metric tensor via basic differential geometry. We shall require the following known result in differential calculus:

Definition 6.5.1. The Laplacian in n-dimensional spherical polar co-ordinates is defined by:

$$\nabla^2 h = r^{-(n-1)}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial h}{\partial r}\right)+\frac{1}{r^2}\nabla^2_{S^{n-1}} h \tag{6.54}$$

The reduced Laplacian (Laplace-Beltrami operator) on the sphere S^{n-1} is identified with the component:

$$\nabla^2_{S^{n-1}} h = \nabla^2 h\left(\frac{x}{|x|}\right) \tag{6.55}$$

Remark 101. Note that the reduced Laplacian operates on the angular part of the function, if we write $x = r\omega(\theta,\phi)$, then obviously $x/|x| = \omega(\theta,\phi)$.

The following corollary results from an easy application of 6.5.1.

Corollary 6.5.2. *The reduced Laplacian in 3-dimensional space is given by the operator:*

$$\nabla^2_{S^2} h = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial h}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 h}{\partial\phi^2} \tag{6.56}$$

This is the Laplace-Beltrami operator.

Proof. We have the definition for the Laplace operator from 6.5.1, substituting $n = 3$, we have:

$$\nabla^2 h = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial h}{\partial r}\right) + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial h}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 h}{\partial\phi^2}\right) \tag{6.57}$$

the reduced Laplacian then being given through the angular part of the Laplacian:

$$\nabla^2_{S^2} h = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial h}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 h}{\partial\phi^2} \tag{6.58}$$

□

We shall now apply some simple methods from differential geometry as in 6.5.4, 6.5.8 to derive the metric we can associate with the reduced sphere. In particular, we have the following result, which is congruent with the results we derived earlier relating the hyperbolic brachistochrone equation and various metrics in hyperbolic and spherical spaces. We review this briefly, as it allows for this chapter to be self-contained for those who have already covered the material in preceding chapters.

Theorem 6.5.3. *For the reduced sphere S^2 which is defined by the 3-dimensional Laplacian 6.5.1, the metric tensor $g_{\alpha\beta}$ is given by:*

$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix} \tag{6.59}$$

Proof. The Laplacian in a curvilinear space is given by 3.2.34:

$$\nabla^2 f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\alpha}\left(\sqrt{g}g^{\alpha\beta}\frac{\partial f}{\partial x_\beta}\right) \tag{6.60}$$

$$= \frac{1}{\sqrt{g}}\frac{\partial}{\partial\theta}\left(\sqrt{g}g^{\theta\theta}\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial\phi}\left(\sqrt{g}g^{\phi\phi}\frac{\partial f}{\partial x_\phi}\right) \tag{6.61}$$

where the second step follows by application to the reduced sphere, as commented on in the preceding remark. The coefficients of the metric are readily identified thus:

$$\frac{1}{\sqrt{g}} = \frac{1}{\sin\theta} \Rightarrow \sqrt{g} = \sin\theta \tag{6.62}$$

$$\sqrt{g}g^{\theta\theta} = \sin\theta \Rightarrow g^{\theta\theta} = 1 \tag{6.63}$$

$$\sqrt{g}g^{\phi\phi} = \frac{1}{\sin\theta} \Rightarrow g^{\phi\phi} = \frac{1}{\sin^2\theta} \tag{6.64}$$

We therefore extract the covariant metric tensor:

$$g^{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{\sin^2\theta} \end{bmatrix} \tag{6.65}$$

Lowering both indices yields the result. □

Remark 102. In a way this result is begging the question, given that the Laplacian obviously depends on the metric. The natural question that one may ask is how such metric tensors may arise. One answer to this conundrum is to use the Fubini-Study projective metric on the space of states. This space is given by a spherical surface, which we recognise as being equivalent to the restricted sphere S^{n-1}.

6.5.2 Metric Tensor via Fubini-Study Projection

The metric tensor associated with the reduced Laplacian may be derived using a unitary transformation. This unitary transform defines a projective operator which can be associated with the Fubini-Study metric. As the following shall show, We recover an identical metric on the spherical surface using the projective state space methodology. In particular, using insights from projective geometric and the representation theory of unitary transformations, these types of Laplacians may be found using the Fubini-Study metric. The geometry associated with metrics as found through the Laplacian is indeed implied through the equivalence, shown initially in Kuzmak et. al [66]. We shall use the following theorem outlined in 3.2.33, and originally outlined in [66].

Theorem 6.5.4. *The Fubini-Study metric is given by the tensor:*

$$F_{\alpha\beta} = \left\langle \bar{\Psi}_\alpha \right| \Psi_\beta\rangle - \left\langle \bar{\Psi}_\alpha \right| \Psi\rangle \left\langle \Psi \right| \Psi_\beta\rangle = \left\langle \bar{\Psi}_\alpha \right| \Big(\mathbf{1} - \hat{P}(\tau, \varphi, \psi)\Big) \Psi_\beta\rangle \tag{6.66}$$

where $\hat{P}$ is the projection operator, as given in 3.2.19, the state being defined as in 3.2.19, and the derivatives taken term by term such that:

$$|\psi_\beta\rangle = \frac{\partial}{\partial x_\beta} |\Psi\rangle \tag{6.67}$$

The Fubini-Study metric is realised as the real part of this matrix tensor:

$$g_{\alpha\beta} = \mathfrak{Re}\left[F_{\alpha\beta}\right] \tag{6.68}$$

It has been shown in previous chapters 3.2.42 that this metric may be used in order to derive a number of special functions, including the Mehler-Fock functions in hyperbolic space. We shall now briefly calculate the geometry of states and how one may go about constructing these types of projective operators, descriptive of the metric tensor as above.

Definition 6.5.5. Following the arguments in 4.3.17, the generator of the Cartan decomposition of SU(2) is the matrix:

$$\hat{S}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6.69}$$

The following lemmas establish the unitary transformations necessary to construct the Fubini-Study metric.

Lemma 6.5.6. *There exists a unitary transformation, such that:*

$$\hat{U}_z(\phi) = \exp\left(-i\phi\hat{S}_z\right) = \begin{bmatrix} e^{-i\phi} & 0 \\ 0 & e^{+i\phi} \end{bmatrix} \tag{6.70}$$

Proof. This follows from exponentiation of the operator $\hat{S}_z$. □

Lemma 6.5.7. *There exists a unitary transformation $Y\hat{(\alpha)}$ such that:*

$$\hat{U}(\phi; \alpha) = \hat{Y}(\alpha)\hat{U}_z(\phi)\hat{Y}^\dagger(\alpha) = \begin{bmatrix} \cos\phi & -ie^{-i\alpha}\sin\phi \\ -ie^{i\alpha}\sin\phi & \cos\phi \end{bmatrix} \tag{6.71}$$

Proof. This is the Euler decomposition, see 4.3.17 for proof. The matrix is given in the calculation of the decomposition immediately following 4.3.17. □

We shall now state and prove the major result of this section.

Theorem 6.5.8. *The Fubini-Study metric associated to the unitary transformation $\hat{U}$ satisfies:*

$$g_{jk} = \frac{1}{4}\begin{bmatrix} \sin^2\alpha & 0 \\ 0 & 1 \end{bmatrix} \tag{6.72}$$

Up to permutation of indices, this is identical to the matrix derived in 4.4.3 using the Laplacian on the reduced sphere instead of the hyperboloid.

Proof. This follows from a direct application of 6.5.4, and the unitary transformation given through 6.5.7. We have, following Kuzmak et. al [66]:

$$g_{jk} = \mathfrak{Re}\left[\langle\bar{\psi_j}|\, \psi_k\rangle - \langle\bar{\psi_j}|\, \Psi\rangle \langle\Psi|\, \psi_k\rangle\right] \tag{6.73}$$

$$|\psi_j\rangle = \frac{\partial}{\partial x^j}|\Psi\rangle \tag{6.74}$$

The action of the unitary operator on an initial state is given through:

$$|\Psi\rangle = \begin{bmatrix} \cos\phi & -ie^{-i\alpha}\sin\phi \\ -ie^{i\alpha}\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\phi \\ -ie^{i\alpha}\sin\phi \end{bmatrix} \tag{6.75}$$

Evaluating the derivatives, we find:

$$|\psi_\phi\rangle = \frac{\partial}{\partial\phi}|\Psi\rangle = \begin{bmatrix} -\sin\phi \\ -ie^{i\alpha}\cos\phi \end{bmatrix} \tag{6.76}$$

$$|\psi_\alpha\rangle = \frac{\partial}{\partial\alpha}|\Psi\rangle = \begin{bmatrix} 0 \\ e^{i\alpha}\sin\phi \end{bmatrix} \tag{6.77}$$

The projection operator is given by:

$$|\Psi\rangle\langle\Psi| = \begin{bmatrix} \cos\phi \\ -ie^{i\alpha}\sin\phi \end{bmatrix}\begin{bmatrix} \cos\phi & ie^{-i\alpha}\sin\phi \end{bmatrix} \tag{6.78}$$

$$= \begin{bmatrix} \cos^2\phi & ie^{-i\alpha}\sin\phi\cos\phi \\ -ie^{i\alpha}\sin\phi\cos\phi & \sin^2\phi \end{bmatrix} \tag{6.79}$$

We can use the basic formula for the Fubini-Study metric and the adjoint to find:

$$F_{jk} = \langle\bar{\psi_j}|\, \psi_k\rangle - \langle\bar{\psi_j}|\, \Psi\rangle\langle\Psi|\, \psi_k\rangle \tag{6.80}$$

$$= \langle\bar{\psi_j}|\left(\mathbf{1} - \hat{P}\right)|\psi_k\rangle \tag{6.81}$$

Evaluating, we obtain the tensor:

$$F_{jk} = \begin{bmatrix} (\sin\phi\cos\phi)^2 & -i\sin\phi\cos\phi \\ i\sin\phi\cos\phi & 1 \end{bmatrix} \tag{6.82}$$

$$g_{jk} = \begin{bmatrix} \frac{1}{4}\sin^2 2\phi & 0 \\ 0 & 1 \end{bmatrix} \tag{6.83}$$

$$ds^2 = \frac{1}{4}\sin^2 2\phi . d\alpha^2 + d\phi^2 \tag{6.84}$$

We can now associate this with a sub-Riemannian metric defined by

$$g_{jk} = \frac{1}{4}\begin{bmatrix} \sin^2\alpha & 0 \\ 0 & 1 \end{bmatrix} \tag{6.85}$$

with $\alpha = 2\phi$ as required to establish the claim.

$\square$

Remark 103. Note that we have derived the Panaratcham-Berry phase in the form:

$$\hat{b} = \frac{1}{2}\sin 2\phi \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{6.86}$$

which may be of some physical interest. In a sense, the theory of the Fubini-Study metric is "primal", or at least primitive to the theory of the Laplacian. With the hypothesis that the correct metric to describe the geometry is that of projective state space, the Laplacian is then a derivate operator, defined in all respects through the properties of this metric. Analysis of the unitary operators associated to a particular state space is then of crucial importance in determining the associated differential operators that describe any particular group.

6.5.3 Radial Component- EPD Equation

The radial part of the Laplacian may be used to define a generalised Euler-Poisson-Darboux (EPD) equation. We consider the time and space counterparts. If we examine the radial part of the Laplacian in n-dimensional space, as the following small lemmas show, these result in various types of partial differential equations related to the Radon transform.

Lemma 6.5.9. *The radial form of the Laplace operator is:*

$$\nabla_r^2 h = \frac{\partial^2 h}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial h}{\partial r} \tag{6.87}$$

Proof. Elementary. Use the expression for the Laplacian:

$$\nabla_r^2 h = r^{-(n-1)}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial h}{\partial r}\right) \tag{6.88}$$

$$= \frac{\partial^2 h}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial h}{\partial r} \tag{6.89}$$

□

Lemma 6.5.10. *The generalised EPD equation analogous to either the wave or quantum equation with potential is specified through the operator identity:*

$$\hat{A}h = \frac{\partial^2 h}{\partial r^2} + \frac{(1+2\eta)}{r}\frac{\partial h}{\partial r} + \hat{B}h \tag{6.90}$$

where $\hat{A}, \hat{B}$ are linear operators acting on the function $h(.)$.

Proof. If we take the formula for the Laplacian, we may generalise the problem in the following way. The dynamic equation may be given a Helmholtz form, following from the expression given by the equivalent to Laplace's equation:

$$\hat{A}h = \nabla^2 h + \hat{B}h \tag{6.91}$$

We have the Laplacian operator directly:

$$\hat{A}h = \frac{\partial^2 h}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial h}{\partial r} + \hat{B}h \tag{6.92}$$

whence, using $(n-1) = 1+2\eta$ we find the generalised EPD equation:

$$\hat{A}h = \frac{\partial^2 h}{\partial r^2} + \frac{(1+2\eta)}{r}\frac{\partial h}{\partial r} + \hat{B}h \tag{6.93}$$

and establish the claim. □

Theorem 6.5.11. *The generalised space-EPD equation may be written:*

$$\hat{A}h = \hat{\mathscr{H}}(r)h + \hat{B}h \tag{6.94}$$

where the space-EPD operator is given by:

$$\frac{\partial^2}{\partial r^2} + \frac{(1+2\eta)}{r}\frac{\partial}{\partial r} = \hat{\mathscr{H}}(r, \partial_r) \doteq \hat{\mathscr{H}}(r) \tag{6.95}$$

as in 6.5.10. The generalised time-EPD equation is given by the time-space inversion:

$$\hat{A}h = \hat{\mathscr{H}}(t)h + \hat{B}h \tag{6.96}$$

Proof. This follows as a result of the generalisation of 6.5.10 to non-integer dimensions. The time equivalent for the space-EPD equation follows by replacement of r with t. □

Remark 104. Note that in some ways this equation may be seen as generalising the dimension to non-integer values in a similar way to other types of stochastic processes have a dimensionality associated with them, as with the square Bessel process, Platen process etc. The example with the Laplacian obviously has $\hat{A} = \nabla^2$, however in this chapter we depart from this simplistic analogy and give full solutions for new types of EPD equation.

By modification of the operators $\hat{A}, \hat{B}$ and $\hat{\mathscr{H}}(.)$ one may generate a large class of differential equations with some very interesting properties. As we shall show, these types of hyperbolic equations have deep links to special function theory and the theory of kernels. However, there is one key difference. Systems with dispersion in space are relatively easy to handle conceptually. However, if we take a system which has dispersion in time, the time will then be described through a probability distribution instead of being a deterministic parameter. The following calculations intend to render the EPD equations, heat equations and wave equations under the same underlying umbrella, being the kernel method. The twist in this case is that we shall be examining what amounts to a transition probability for the system to jump from one time to another, at a particular space point.

6.5.4 Solution to EPD Equation via Radon Transform

We shall now discuss some simple principles related to the EPD equation. Resources regarding the Radon transform may be found in the standard reference texts [82,83], see also [171,172,174–176] for an in-depth discussion of the relationship between integral measures on spherical spaces, the Radon transform, tomography and other related topics. Our principal interest in these sorts of calculations is the following known fact [81]; one solution to the EPD equation is given through the Radon transform on the space. This fact is not immediately apparent, so we shall briefly outline how this may be achieved using some simple insights from the theory of spherical groups. Helgason, in the seminal work [114] has developed a generalised theory of such transforms. The interested reader is directed to a very thorough and useful discussion of Radon transforms and all their applications contained in [82,83]. Many of the PDE systems encountered in this chapter may be taken to be variations on the theme of the EPD equation; the question we shall discuss in the concluding remarks is the existence of equivalent Radon transformations to solve these PDEs in the following way. The following calculation shows, following the arguments in [81], that the Radon transform solves the two dimensional Fourier transform. The radial part is given by a Gegenbauer transform in the radial co-ordinate, and further, the spherical mean value can be defined through the Radon transform. This is given by the integral over all points affine transported around some origin. We shall demonstrate this through a direct application of commutation relations with the radial Laplace operator, which defines the EPD equation as in 6.5.9. One basic route [81,83] to develop the generalised Radon transform is to use the Fourier transform on a sphere. We shall prove the following lemmas to illustrate the link between this method, and the prior calculations where we found the Radon transform using the two dimensional Fourier transform, and the Laplacian on the reduced sphere. We shall use these lemmas to derive a basic concept in the theory of Radon transforms, which is known as the central slice theorem.

Notation 5. In the calculation that follows, we shall use the following summary notation to simplify the mathematics. We shall write the measure for the space $\mathbf{x} \in G$ as:

$$d\mathbf{x} = d^n x = dx_1 dx_2 ... dx_n \tag{6.97}$$

and in the Fourier transformed space (conjugate space):

$$d\mathbf{a} = \frac{1}{(2\pi)^n} d^n a = \frac{1}{(2\pi)^n} da_1 da_2 ... da_n \tag{6.98}$$

Definition 6.5.12. Let $f(\mathbf{x}) \in \mathrm{L}^1(\mathbb{R}^n)$. The Fourier transform given by the group integral as in 6.4.4, 6.4.2:

$$\hat{f}(\mathbf{a}) = \int_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{a}\cdot\mathbf{x}} d\mathbf{x} \tag{6.99}$$

If $\hat{f} \in \mathrm{L}^1(\mathbb{R}^n)$, then we have the inversion:

$$f(\mathbf{x}) = \int_{\mathbf{a} \in \mathbb{R}^{n*}} \hat{f}(\mathbf{a}) e^{i\mathbf{a}\cdot\mathbf{x}} d\mathbf{a} \tag{6.100}$$

The conditions of the Riemann-Lesbesgue lemma require that for the transform and inversion to exist, we must have:

$$\|f\|_{L^1} = \int_{\mathbb{R}^n} |f(\mathbf{x})| d\mathbf{x} < \infty \tag{6.101}$$

i.e. the inverse Fourier transform converges to zero at infinity:

$$\lim_{|\mathbf{a}| \to \infty} |\hat{f}(\mathbf{a})| = 0 \tag{6.102}$$

Given that the n-dimensional sphere grows as:

$$\int_{\mathbb{R}^n} d\mathbf{x} = V_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} |\mathbf{x}|^n \tag{6.103}$$

we have the asymptotic limit $|f(\mathbf{x})| = \mathcal{O}\left(|\mathbf{x}|^{-N}\right)$ for some $N > n$ as $|\mathbf{x}| \to \infty$.

Lemma 6.5.13. *The inversion formula for the Fourier transform:*

$$\hat{f}(\mathbf{a}) = \int_{-\infty}^{+\infty} e^{-it} dt \int_{\mathbf{x} \in \mathbb{R}^n} \delta(t - \mathbf{a}\cdot\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \tag{6.104}$$

where $\delta(.)$ is the Dirac delta function.

Proof. We have the exponential function in terms of the Dirac delta function:

$$\int_{-\infty}^{+\infty} \delta(t - \mathbf{a}\cdot\mathbf{x}) e^{-it} dt = e^{-i\mathbf{a}\cdot\mathbf{x}} \tag{6.105}$$

Inserting this into the inversion formula from 6.5.12, we have:

$$\hat{f}(\mathbf{a}) = \int_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{a}\cdot\mathbf{x}} d\mathbf{x} \tag{6.106}$$

$$= \int_{-\infty}^{+\infty} e^{-it} dt \int_{\mathbf{x} \in \mathbb{R}^n} \delta(t - \mathbf{a}\cdot\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \tag{6.107}$$

as required. We have invoked the Fubini-Tonelli theorem at the second step to interchange the order of integration. □

We immediately obtain the following corollary, which derives the Radon transform in terms of the Fourier inversion problem.

Corollary 6.5.14. *The inner integral of 6.5.13 is defined by the integral transform:*

$$[\hat{\mathcal{R}}_{\mathbf{a},\mathbf{x}}(t)]f = \int \delta(t - \mathbf{a}\cdot\mathbf{x}) f(\mathbf{x}) d\mathbf{x} = g(\mathbf{a}, t) \tag{6.108}$$

This is the n-dimensional Radon transform.

We now take the following results from [81–83]. This series of brief theorems establish the relationship between Gegenbauer functions, spherical transform theory and the Radon transform as formulated alternatively using the methods explored in the previous sections.

Theorem 6.5.15. *The orthonormality conditions for the Radon transform eigenfunctions are given by:*

$$\int_{|\boldsymbol{\omega}|=1} S_{l,k}(\boldsymbol{\omega}) S_{l',k'}(\boldsymbol{\omega}) d\boldsymbol{\omega} = \delta_{kk'}\delta_{ll'} \tag{6.109}$$

where the integral measure is defined on the surface of the sphere $|\boldsymbol{\omega}| = 1$.

Proof. See [82,83] for an exhaustive proof, we merely point out the major points in the decomposition. A radial decomposition is given through the radial parametrisation formula $\mathbf{x} = r\boldsymbol{\omega}$. In such case, the eigenfunctions will decouple to give $f(\mathbf{x}) = \phi_k(r) S_{l,k}(\boldsymbol{\omega})$, which we call the spherical decomposition formula of the eigenfunction. The orthonormality relationship for the eigenfunctions follows as a consequence. For example, one can choose the basis set given by the real spherical harmonics, which have values:

$$S_{lm}(\hat{\mathbf{x}}) = Y_{lm} = \begin{cases} \dfrac{i}{\sqrt{2}}\left(Y_l^m - (-1)Y_l^{-m}\right) & m < 0 \\ Y_l^0 & m = 0 \\ \dfrac{1}{\sqrt{2}}\left(Y_l^m + Y_l^{-m}\right) & m > 0 \end{cases} \tag{6.110}$$

where the Y_l^m are the standardised form of the spherical harmonics, normalised to the unit sphere. The properties of these functions have been known since the times of Laplace, and we shall not dwell on the choice of basis function, all we shall require is their existence, which is basic to any practice of calculus in spherical geometry. □

The following result is known which we quote from [82,83]:

Theorem 6.5.16. *The convolution formula equivalent for the radial transform is given by the equation:*

$$[\hat{\mathcal{R}}_{\boldsymbol{\xi},\boldsymbol{\omega}}(p)]\left[\phi_k(r) S_{l,k}(\boldsymbol{\omega})\right] = \left[\hat{\mathcal{G}}\phi_k(r)\right] S_{l,k}(\boldsymbol{\xi}) = \bar{\phi}_k(p) S_{l,k}(\boldsymbol{\xi}) \tag{6.111}$$

where the transform is given by a Gegenbauer transform in the radial co-ordinate:

$$\bar{\phi}_k(p) = \hat{\mathcal{G}}\phi_k(r) = \frac{2\pi^{n/2}}{\Gamma(n/2) C_k^{\nu}(1)} \int_{|p|}^{\infty} r^{2\nu} \phi_k(r) C_k^{\nu}\left(\frac{p}{r}\right)\left(1 - \frac{p^2}{r^2}\right)^{\nu - 1/2} dr \tag{6.112}$$

and the Radon transform $\hat{\mathcal{R}}_{\mathbf{a},\mathbf{x}}(t)$ is as derived in 6.5.13, 6.4.4, $C_k^{\nu}(.)$ being the Gegenbauer function [8].

We have, also quoting a result, this time from Helgason [81] (see I.3.3.1, pp. 15), the following inversion theorem:

Theorem 6.5.17. *For the Radon transform derived in 6.4.4, 6.5.13, the transform is defined through the Dirac delta integral on the sphere:*

$$[\hat{\mathcal{R}}_{\mathbf{a},\mathbf{x}}(t)]f = \int d\mathbf{x}\delta(t - \mathbf{a}\cdot\mathbf{x})f(\mathbf{x}) = g(\mathbf{a}, t) \tag{6.113}$$

This has functional inversion in the same sense as Fourier that can be written:

$$cf(\mathbf{x}) = \left(-\hat{L}\right)^{(n-1)/2}[g(\mathbf{a}, t)] \tag{6.114}$$

where the constant is explicitly written as:

$$c = \frac{(4\pi)^{(n-1)/2}\Gamma(n/2)}{\Gamma(1/2)} \tag{6.115}$$

and the differential operator for radial functions $f(r)$ is given by the Euler-Poisson-Darboux operator:

$$\hat{L}f(r) = \frac{\partial^2 f}{\partial r^2} + \frac{n-1}{r}\frac{\partial f}{\partial r} \tag{6.116}$$

Proof. See e.g. the arguments in [81], section I.3.3.1. Another calculation using a different method involving the projective state space of the reduced sphere may be found in Hafoud, [84–86], in particular Thm. 1.1 in [84]. □

We have also the following definition of the mean value on a sphere, given by the affine transform via:

Definition 6.5.18. The mean value on the sphere [81] is defined through the transform:

$$(\hat{M}f)(x) = \int_{\mathbf{k}\in K} f(x + \mathbf{k}\cdot\mathbf{y})d\mathbf{k} \tag{6.117}$$

where we drop the explicit dependence of $\hat{M}$ on $\mathbf{y}$ for notational convenience. Generally we shall take the integration such that $K = \mathbb{R}^n$, and $|\mathbf{k}| = 1$, but this transformation exists on other non-Euclidean spaces as well, see e.g. [81] for work on symmetric spaces, which in this thesis are discussed in the context of Minkowski space and SO(p,q).

This may be understood in a similar way to our previous discussion where we derived the Radon transform from the spherical Fourier transformation. We now prove the following simple lemma, which is a basic property of the generalised EPD systems we have discussed in the previous sections, see e.g. 6.5.9, 6.5.10, 6.5.11.

Lemma 6.5.19. *The Euler-Poisson-Darboux equation is specified by the radial part of the Laplacian and obeys the wave-type PDE $\nabla^2_{rad}f = f_{tt}$. In 3 dimensional space this is given by the PDE:*

$$\nabla^2_{rad}f(r,t) = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} = \frac{\partial^2 f}{\partial t^2} \tag{6.118}$$

The n-dimensional generalisation is given by the related formula:

$$\nabla^2_{rad.}f(r,t) = \frac{\partial^2 f}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial f}{\partial r} = \frac{\partial^2 f}{\partial t^2} \tag{6.119}$$

where n is the dimension of the system.

Proof. Using the expression for the radius, we have $r = \sqrt{\sum x_i^2}$. The radial part of the Laplacian is given through the expression:

$$\nabla^2 f(r) = \sum_i \frac{\partial}{\partial x_i} \frac{\partial f(r)}{\partial x_i} = \sum_i \frac{\partial r}{\partial x_i} \frac{\partial}{\partial r} \left[\frac{\partial r}{\partial x_i} \frac{\partial f(r)}{\partial r} \right] \tag{6.120}$$

which in 3-dimensional space yields:

$$\nabla^2_{rad} f(r) = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} \tag{6.121}$$

Obviously, we have the n-dimensional generalisation:

$$\nabla^2_{rad.} f(r) = \frac{\partial^2 f}{\partial r^2} + \frac{(n-1)}{r} \frac{\partial f}{\partial r} \tag{6.122}$$

as required to establish the claim. □

We shall now demonstrate a number of simple results that follow from the construction of the spherical mean value transform, which henceforth we shall refer to as the Radon transform or forward Radon transform. In particular, we follow the proofs contained in [81,83] to show that the radial part of the Laplacian as given by the EPD operator in 6.5.9 commutes with the Radon transform.

Lemma 6.5.20. *Assume the mean value transform as given by the Radon transform:*

$$(\hat{M} f)(x) = g(x) = \int_K f(x + \mathbf{k} \cdot \mathbf{y}) d\mathbf{k} \tag{6.123}$$

and also the radial Laplacian as derived respectively in 6.5.9:

$$\nabla^2_{rad.} f(r) = \frac{\partial^2 f}{\partial r^2} + \frac{(n-1)}{r} \frac{\partial f}{\partial r} \tag{6.124}$$

Then the following statements are true:

1) The Laplacian and the mean value transform commute:

$$(\nabla^2_{rad.} \hat{M} f)(x) = (\hat{M} \nabla^2_{rad.} f)(x) \tag{6.125}$$

This may be found in Helgason's 1959 treatise, Thm 16, pp. 276.

2) The Laplacian on the reduced sphere (radial component) of a radial function is given by:

$$\nabla^2_{rad.} \varphi(r) = \Omega_{n-1} \int r^{n-2} \nabla^2_{rad.} F(r) dr \tag{6.126}$$

where the radial function is given by the Radon or mean value transform via:

$$F(r) = (\hat{M} f)(\mathbf{x}) \tag{6.127}$$

Remark 105. This establishes that the mean value transform is a solution of the Darboux formula, following the arguments in Helgason [81]. See also [83] for application to other differential equations similar to those encountered in this work.

Proof. We shall address each part of the proof in turn. For the first statement, we may take the mean value transform as in 6.5.20. Writing the mean value transform in Radon form:

$$(\hat{M} f)(x) = g(x) = \int_K f(x + \mathbf{k} \cdot \mathbf{y}) d\mathbf{k} \tag{6.128}$$

We shall now show that this transformation commutes with the radial Laplacian. If we evaluate the composition, we have:

$$f(x+\mathbf{k}\cdot\mathbf{y}) = f(\hat{T}_x\hat{R}_{\mathbf{k}}\mathbf{y}) \tag{6.129}$$

where the radial and rotational operators are given by:

$$f(\hat{R}_{\mathbf{k}}\mathbf{y}) = f(\mathbf{k}\cdot\mathbf{y}) \tag{6.130}$$

$$f(\hat{T}_x a) = f(x+a) \tag{6.131}$$

We assume, of course, that the function has support on the set $f(x+\mathbf{k}\cdot\mathbf{y})$, which can be reduced if necessary by homomorphism. Then we have:

$$\nabla^2_{rad}\hat{M}f = \nabla^2_{rad}\int_{\mathbf{k}\in K} f(x+\mathbf{k}\cdot\mathbf{y})d\mathbf{k} \tag{6.132}$$

$$= \int_{\mathbf{k}\in K} \nabla^2_{rad} f\,|_{x+\mathbf{k}\cdot\mathbf{y}}\, d\mathbf{k} \tag{6.133}$$

$$= \hat{M}\nabla^2_{rad} f \tag{6.134}$$

where we have differentiated under the integral sign, and used the assumed fact that the function is radial as per the assumptions of this section. This establishes, in a simplistic sense, the commutative property between the radial part of the Laplacian (Laplace-Beltrami operator) and the mean value transform, which is equivalent to the Radon transform in a different reference frame. This commutative property is generally used to identify symmetry groups of transforms, we have discussed this at length and shall not continue with analysis of intertwining relations and so on. However, the concluding remarks of this chapter discuss the back-projection formula for the Radon transform, which has a similar form to the type of formula encountered when using the projection-slice theorem in the hyperbolic plane. To prove the second part of the statement, we have following Helgason [81], the group homomorphism, where we identify only radial functions, and hence the Radon transform is radially invariant. In such a case, all points along the same ray are equivalent, and we may employ the first part of the statement to derive the form of the EPD Laplacian.

$$\bar{\phi}(\mathbf{g}\cdot 0) = \int_K \phi\left([\mathbf{g}\mathbf{k}]\cdot\boldsymbol{\xi}_0\right)d\mathbf{k} \tag{6.135}$$

$$= \int_K \left[\int_{\boldsymbol{\xi}_0} \phi\left([\mathbf{g}\mathbf{k}]\cdot\boldsymbol{\xi}_0\right)d\mathfrak{m}(\mathbf{y})\right]d\mathbf{k} \tag{6.136}$$

$$= \int_{\boldsymbol{\xi}_0} d\mathfrak{m}(\mathbf{y})\int_K \phi\left([\mathbf{g}\mathbf{k}]\cdot\mathbf{y}\right)d\mathbf{k} \tag{6.137}$$

We identify via the group homomorphism the Radon transform, and consider only radial functions, in which case

$$\bar{\phi}(\mathbf{g}\cdot 0) = (\hat{M}\phi)(\mathbf{g}\cdot 0) = \int_K \phi\left([\mathbf{g}\mathbf{k}]\cdot\mathbf{y}\right)d\mathbf{k} \tag{6.138}$$

where the volume of the Euclidean space is defined through

$$\int_0^\infty dr.r^{n-2}\int d\Omega_{n-1} = \int dV = \int d\mathbf{k} = V \tag{6.139}$$

and Ω^{n-1} is the area of a unit sphere in $\mathbb{R}^{n-1}$. Separating out the radial component, we therefore have:

$$\bar{\phi}(\mathbf{g}\cdot 0) = \int_0^\infty dr.r^{n-2}\int d\Omega_{n-1}\hat{M}\phi \tag{6.140}$$

$$= \Omega_{n-1} \int_0^\infty dr.r^{n-2} \hat{M}\phi \tag{6.141}$$

$$= \Omega_{n-1} \int r^{n-2} dr.(\hat{M}f)(\mathbf{x}) = \varphi(r) \tag{6.142}$$

Evaluating the radial Laplace operator, we find:

$$\nabla^2_{rad.}\varphi(r) = \Omega_{n-1} \int r^{n-2} \nabla^2_{rad.} F(r) dr \tag{6.143}$$

$$F(r) = (\hat{M}f)(\mathbf{x}) \tag{6.144}$$

which establishes the second leg of the lemma. We have assumed that the function vanishes sufficiently quickly on the boundary of r to allow the Laplacian to pass through the integral. □

Remark 106. This is sufficient for our purposes in this chapter, to notice that such transforms exist and that special functions are related to the different geometries implied through the Laplace operator. We shall now discuss some more involved aspects in the differential analysis of the types of systems implied through the EPD operator. In particular, we shall be generalising such results to a situation which essentially has a non-integer dimension. As a result, instead of being purely spherical as in this example, the situation is changed to hyperbolic in some sense. There are many established results in the field of computer tomography, see for example [83,171,172] for an in-depth analysis of the topic. Our focus shall differ from the standard methods in spherical functions in that we shall be using some clever tricks from kernel theory and analysis to find solutions to the EPD equations. This circumvents this sometimes messy process involving group algebraic integrals and the like. However, in order that our solutions be correct and solve the EPD equations in the appropriate way, there is a necessary equivalence relationship between the results that may be written down using the methods of spherical operators and those of Mercer kernels. This is a highly complex question far beyond the scope of this chapter; we seek only to show some few examples of the phenomena involving EPD, heat and wave kernel solutions to their respective domains, and some simple generalisations.

6.6 Characters and Product Formulae

The product formulae can be seen as Wigner transforms using the eigenfunctions. This can be used to define a Wigner index transform, and a momentum Wigner function over the different eigenfunction spectra. A similar product formula may be found using the theory of the kernel and group character representations. The spherical formula for the modified Bessel functions may be found using an integral theorem. This implies an eigenfunction decomposition of the kernel. Further, there is a sequence of polynomials that can be associated with the Laplace and sine transforms of the modified Bessel functions defined through the kernel. These polynomials share many properties with Chebyshev polynomials but are defined in terms of hyperbolic functions. From group representation theory of the kernel and the distribution of states [177], we know that the Wigner transform gives the following product decomposition of the eigenfunctions:

Definition 6.6.1. The Wigner transform of an eigenfunction in some Hilbert space $\psi \in \mathcal{H}$

$$\mathcal{W}_u(z,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{-ipy/\hbar} \psi_u^* \left(z + \frac{y}{2}\right) \psi_u \left(z - \frac{y}{2}\right) dy \tag{6.145}$$

This is the correct transform to use for a system with a discrete system, with a set of discrete eigenfunctions.

We can also form the index Wigner transform, which is the counterpart of the Wigner transform for a system with a single eigenfunction, with a single continuous eigenvalue, i.e. a continuous spectrum.

Definition 6.6.2. The index Whittaker transform of a suitable function ψ_u, where we have u some continuous parameter, is defined by:

$$\mathscr{W}_y(z,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{-ipy/\hbar}\psi^*_{u+y/2}(z)\,\psi_{u-y/2}(z)\,dy \tag{6.146}$$

by extension of the ordinary Wigner transform to a system with a single eigenfunction, depending continuously on u and the argument z.

This immediately results in product composition formulae, given by the corollary below.

Corollary 6.6.3. *The product formulae in terms of eigenfunctions may be written in terms of the Fourier inversion formulae for the Wigner and index Wigner distributions respectively, defined in 6.6.1.*

$$\psi^*_\mu(z)\,\psi_\nu(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\mathscr{W}_{(\mu+\nu)/2}(z,p)e^{ip(\mu-\nu)/\hbar}dp \tag{6.147}$$

$$\psi^*_\mu(x)\,\psi_\mu(y) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\mathcal{W}_\mu\left(\frac{x+y}{2},p\right)e^{ip(x-y)/\hbar}dp \tag{6.148}$$

These formulae apply in the discrete and continuous domains.

Proof. Simply invert the Fourier transforms as written. See for example [178] for a description of the application of a similar type of transform. □

We shall now show how one may develop a theory of characters using product formulae over groups. This is the natural generalisation of the product formulae given through the Wigner quasidistribution 6.6.1. In particular, we quote the following results from the literature, see e.g. [69,113,163].

Theorem 6.6.4. *The spherical decomposition of a function over a locally compact group K is given by the group integral:*

$$\int_K f(\mathbf{gkh})d\mathbf{k} = f(\mathbf{g})f(\mathbf{h}) \tag{6.149}$$

where integration is over the variable $\mathbf{k}$ which gives the parametrisation. The Wigner decomposition formula is given by the extension of this decomposition to the case $\int_K f(\mathbf{gkh})d\mathbf{k} = f(\mathbf{g}^\dagger)f(\mathbf{h})$, where the group homomorphism is through $f(\mathbf{g}^\dagger) = f^*(\mathbf{g})$.

Proof. See Dieudonne [70,71] for a thorough discussion of spherical decomposition theorems, convolution theorems and the like. We take this representation of the composition formulae as axiomatic in this work. We have discussed this in this thesis, see for example 3.3.45. □

More recently, Vainerman in [163] has established the following relationship which gives a product law using characters and an integral kernel. In many ways, this blends together the concepts that we have employed to derive the hyperbolic heat kernel, using integrals and kernel decompositions, and the more advanced techniques that stem from the use of the abstract axioms of group representation theory and spherical functions.

Theorem 6.6.5. *The invariant product on a locally compact group K [163] is given by the group integral of the kernel $K(\mathbf{g}, \mathbf{h}; \mathbf{s}) = f(\mathbf{gkh})$, which is the unique positive definite function that carries the group structure via the representation theory, as constructed in 3.3.1.*

$$\Delta(f)[\mathbf{g}, \mathbf{h}] = \int_K f(\mathbf{gkh})d\mathbf{k} = \int_Q K(\mathbf{g}, \mathbf{h}; \mathbf{s}) f(\mathbf{s}) d\mathfrak{m}(\mathbf{s}) \tag{6.150}$$

where the group G has subgroup K, and we take the group integration over the quotient group:

$$Q = G/K \tag{6.151}$$

Further, the group character is defined through the special case of this decomposition where we identify $\Delta(f)[\mathbf{g}, \mathbf{h}] = f(\mathbf{g})f(\mathbf{h})$.

$$\int_Q K(\mathbf{g}, \mathbf{h}; \mathbf{s})\chi_\sigma(\mathbf{s})d\mathfrak{m}(\mathbf{s}) = \chi_\sigma(\mathbf{g})\chi_\sigma(\mathbf{h}) \tag{6.152}$$

where the group character is expressed through the function $f(\mathbf{g}) = \chi_\sigma(\mathbf{g})$.

The final result we shall require for the following analysis gives the group character in terms of the invariant orbits of the group. A thorough analysis of this may be found in [69]. In particular, we take the following observation.

Theorem 6.6.6. *Vilenkin [69] gives the following formula for the character on a locally compact group:*

$$\chi_\sigma(\mathbf{g}) = \chi_\sigma(\mathbf{g}_1^{-1}\mathbf{g}\mathbf{g}_1) \tag{6.153}$$

where $\chi_\sigma(\mathbf{g})$ is the group character, and the orbit is given through the identification of the points where $\mathbf{g} = \mathbf{g}_1^{-1}\mathbf{g}\mathbf{g}_1$.

We also have the following corollary, immediately derivable from 6.6.6:

Corollary 6.6.7. *The character must also respect*

$$\chi_\sigma(\mathbf{g}_1^{-1}\mathbf{g}_1) = \chi_\sigma(\mathbf{g}_1^{-1})\chi_\sigma(\mathbf{g}_1) = \chi_\sigma(\mathbf{1}) = \mathbf{e} \tag{6.154}$$

where $\mathbf{1}, \mathbf{e}$ are the respective identity elements for the group and the group character.

Proof. This is the reduction of 6.6.6, with the special case $\mathbf{g} = \mathbf{1}$. As such, the character formula reduces to:

$$\chi_\sigma(\mathbf{1}) = \chi_\sigma(\mathbf{g}_1^{-1}\mathbf{1}\mathbf{g}_1) \tag{6.155}$$

and using the decomposition formula 6.6.5, we have:

$$\chi_\sigma(\mathbf{g}_1^{-1}\mathbf{g}_1) = \chi_\sigma(\mathbf{g}_1^{-1})\chi_\sigma(\mathbf{g}_1) \tag{6.156}$$

To close the proof, note that any group representation must map the identity element to the corresponding element of the group. If such a representation is unique, i.e. exists for only one element, then the group representation is faithful.

$$\chi_\sigma(\mathbf{1}) = \mathbf{e} \tag{6.157}$$

Together, these statements prove the system of equalities in the corollary. □

The following brief calculation shows how one may derive an associated form of functions that give the group character for a spherical decomposition related to the Bessel functions. We include this example to showcase the methodology of spherical functions, and to generalise the results we have analysed at the beginning of this chapter, as in 5.6.1.

Lemma 6.6.8. *Assume the distance function is given by the law of cosines, then for any two points, which we represent by the members of the group* $\mathbf{g}, \mathbf{h} \in K$*, we then have the relation:*

$$d_K(\mathbf{g}, \mathbf{h}) = \sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos x} \tag{6.158}$$

and the distance for the two points from the origin is given by $d_K(\mathbf{g}, \mathbf{0}) = \alpha$, $d_K(\mathbf{h}, \mathbf{0}) = \beta$. *If we identify the group homomorphism through the mapping:*

$$f(\mathbf{gkh}) \sim f(d_K(\mathbf{g}, \mathbf{h})) \tag{6.159}$$

Then there exists a spherical decomposition formula for the Bessel functions given by:

$$\int_K f(\mathbf{gkh}) d\mathbf{k} = f(\mathbf{g}) f(\mathbf{h}) \tag{6.160}$$

where the mapping is defined through the group action:

$$f(d_K(\mathbf{g}, \mathbf{h})) = \frac{u\left(d_K(\mathbf{g}, \mathbf{h})\right)}{\left(d_K(\mathbf{g}, \mathbf{h})\right)^{\nu}} \tag{6.161}$$

This function satisfies:

$$u(z) = J_{\nu}(z) \tag{6.162}$$

where $J_{nu}(\alpha)$ *is the Bessel function of order* ν.

Proof. We have the results implied from integral tables [122], also [173], where the authors showed an equivalent result:

$$\int_0^{\pi} \sin^{2\nu} x \frac{J_{\nu}\left(\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos x}\right)}{\left(\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos x}\right)^{\nu}} dx = 2^{\nu} \sqrt{\pi} \Gamma(\nu + 1/2) \frac{J_{\nu}(\alpha)}{\alpha^{\nu}} \frac{J_{\nu}(\beta)}{\beta^{\nu}} \tag{6.163}$$

If we take the group integral measure to be given by:

$$d\mathbf{k} = d\mathfrak{m}(x) = \frac{1}{2^{\nu}\sqrt{\pi}\Gamma(\nu + 1/2)} \sin^{2\nu} x dx \tag{6.164}$$

and identify group integration via:

$$\int_0^{\pi} d\mathfrak{m}(x) = \int_K d\mathbf{k} \tag{6.165}$$

i.e. take only radial functions, as is appropriate for this type of scenario, we then have:

$$f(\mathbf{gkh}) \sim f(d_K(\mathbf{g}, \mathbf{h})) = \frac{u\left(d_K(\mathbf{g}, \mathbf{h})\right)}{\left(d_K(\mathbf{g}, \mathbf{h})\right)^{\nu}} = \frac{J_{\nu}\left(d_K(\mathbf{g}, \mathbf{h})\right)}{\left(d_K(\mathbf{g}, \mathbf{h})\right)^{\nu}} \tag{6.166}$$

as required. Observing that this can be written in the form of the spherical decomposition formula, with appropriate choice of kernel as specified in the lemma we establish the result. □

Remark 107. This is very much in the spirit and formulation of the earlier results related to the hyperbolic heat kernel. We have already shown how different distance or pseudodistance functions give different variants of the metric and Laplacian. This type of formula will have other equivalent relations for the modified Bessel functions of both types.

Corollary 6.6.9. *The spherical decomposition formula for the modified Bessel function is given by:*

$$K_0(a)K_0(b) = \int_0^\infty K_0\left(\sqrt{a^2+b^2+2ab\cosh t}\right)dt \tag{6.167}$$

as derived previously in 6.6.9. This is a special case of 6.6.8, where the distance function is over hyperbolic space. Alternatively, the Basset function (modified Bessel function of the first kind $I_v(x)$) satisfies a similar relation:

$$\int_K f(\mathbf{gkh})d\mathbf{k} = f(\mathbf{g})f(\mathbf{h}) \tag{6.168}$$

where the group action is:

$$f(\mathbf{gkh}) \sim f(d_K(\mathbf{g},\mathbf{h})) = \frac{u\left(d_K(\mathbf{g},\mathbf{h})\right)}{\left(d_K(\mathbf{g},\mathbf{h})\right)^\nu} = \frac{I_\nu\left(d_K(\mathbf{g},\mathbf{h})\right)}{\left(d_K(\mathbf{g},\mathbf{h})\right)^\nu} \tag{6.169}$$

and the distance function occurs with the change of sign:

$$d_K(\mathbf{g},\mathbf{h}) = \sqrt{\alpha^2+\beta^2+2\alpha\beta\cos x} \tag{6.170}$$

Proof. We have already proven the first part in 6.6.9. To prove the second, note the integral from [173]:

$$I_v(a)I_v(b) = \frac{(ab)^\nu}{2^\nu\sqrt{\pi}\Gamma\left(\nu+1/2\right)}\int_{-1}^{+1}\frac{(1-u^2)^{\nu-1/2}}{(a^2+b^2+2|ab|u)^{\nu/2}}I_\nu\left(\sqrt{a^2+b^2+2|ab|u}\right)du \tag{6.171}$$

If we take the group integral measure as:

$$\int_0^\pi d\mathfrak{m}(x) = \int_K d\mathbf{k} = -\frac{(\alpha\beta)^\nu}{2^\nu\sqrt{\pi}\Gamma\left(\nu+1/2\right)}\int_0^\pi \sin^{2\nu}xdx \tag{6.172}$$

we obtain a similar relationship, with an extra sign from the integral change:

$$f(\mathbf{gkh}) \sim f(d_K(\mathbf{g},\mathbf{h})) = \frac{u\left(d_K(\mathbf{g},\mathbf{h})\right)}{\left(d_K(\mathbf{g},\mathbf{h})\right)^\nu} = \frac{I_\nu\left(d_K(\mathbf{g},\mathbf{h})\right)}{\left(d_K(\mathbf{g},\mathbf{h})\right)^\nu} \tag{6.173}$$

We note that this type of function appears in our calculation of the hyperbolic heat kernel, and our identification of the solutions to various index integrals. □

Remark 108. The association of Chebyshev polynomials and character formulae is well-known, what is less clear is their connection to e.g. the heat equation. One can show that the homogenous polynomial solutions to Laplace's equation $\nabla^2 f(x,y) = 0$ are the Chebyshev polynomials. In a sense one can see them as arising as special solutions and limiting cases of many of the new types of equations we shall now calculate, and as we shall show there are many formulae relating the Bessel functions and Chebyshev polynomials that appear in the context of the Euler-Poisson-Darboux equation. The interested reader who is interested in other results regarding product formulae is directed to Sousa [179], also Connett [164,165], the latter having computed recently a number of identities related to the index Whittaker transform. Connett's work focused in a different direction, arriving at an analogous result for the Jacobi polynomials. Both of these works have developed theories of the kernel solutions for various PDE systems and groups, in the case of Sousa and Yakubovich they have analysed heat kernels for diffusion processes that may be associated with hyperbolic space, and in many regards the analysis of the connection between groups of functions and the groups themselves forms the basis of our understanding throughout this thesis. In many respects, the idea and foundational premise of representation theory, that representations preserve group relations and identity, allows for a certain fungibility between the kernels that may be developed from the perspective of differential analysis, transformations of PDEs and so on, and the kernels that arise from group theory and positive definite functions. The Radon transform is no different in this regard, and as has been shown in Helgason, there is a close link between the problems we have covered in deriving the hyperbolic heat kernel, and the PDE systems associated with the Radon transform. In particular, Helgason states for the unit sphere in $n+1$-dimensional Euclidean space $X = S^n(0) \subset \mathbb{R}^{n+1}$, the k-sphere which is given by the projection $\xi \in \Xi$, then the Radon transform can be written as:

$$\hat{f}(\mathbf{x}) = \Omega_k \int_0^{\pi} (\hat{M}f)(\mathbf{x}) \sin^{k-1} r dr \tag{6.174}$$

where Ω_k is the standard area measure associated with the metric of the k-dimensional reduced sphere:

$$ds^2 = dr^2 + \sin^2 r d\Omega^2 \tag{6.175}$$

on $\mathbb{R}^{n+1}$. The irreducible decompositions are given by the homogenous polynomials on the spaces $\mathcal{H}_s$, and satisfy:

$$\hat{L}h_s = -s(s+n-1)h_s \tag{6.176}$$

for $s \geq 0$ and $h_s \in \mathcal{H}_s$. The decomposition of the Laplacian into radial and angular components satisfies:

$$\hat{L}_{tot} = \hat{L}_{rad} + \frac{\hat{L}}{r^2} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{\hat{L}}{r^2} \tag{6.177}$$

on this space, and the relationship with the Euler-Poisson-Darboux equation is apparent. Further, there exists a special antipodal solution, given by the spherical harmonics, which satisfy the boundary conditions $\varphi_s(o) = 1$, or $\varphi_s'(o) = 0$ with PDE given by the Gegenbauer/associated Legendre equation:

$$\frac{\partial^2 \varphi_s}{\partial r^2} + (n-1)\cot r \frac{\partial \varphi_s}{\partial r} = -s(s+n-1)\varphi_s \tag{6.178}$$

This differential equation is obviously related to the equation we analysed in solving the hyperbolic heat kernel, where we obtained $u_t = u_{rr} + u_r \coth r$, which obviously extends to the reduced sphere in this context. We can see from this perspective how the theories of measures, transforms, homogenous functions, kernels and a plethora of other concepts can be understood using very simple relations from the theory of group representations. Indeed, one might even start from the point of assuming a measure density of the form $\int_0^{\pi} \sin^{k-1} r dr = \int_0^{\pi} d\mathfrak{m}(r)$, and the identification between the theories of spherical functions as developed in the previous section, the method of Helgason, and the theory of the hyperbolic heat kernel and spherical extension is complete. In essence, these analytical regimes all use similar basis sets of eigenfunctions that have spectral characteristics that can be transformed in one another. This can be seen in the known connections between Gegenbauer, associated Legendre and Jacobi functions. For example, the orthogonality relationship for the Gegenbauer polynomials may be written:

$$\int_{-1}^{+1} C_n^{\alpha}(x) C_{n'}^{\alpha}(x)(1-x^2)^{\alpha-1/2} dx = \delta_{nn'} \frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)|\Gamma(\alpha)|^2} \tag{6.179}$$

and it is simple to see that the hyperbolic equivalent for $|x| > 1$ will have a similar set of orthonormality formulae. We can see that the integral measure of these functions is essentially equivalent to that we obtained using the logic of the Radon transform. The connection is of course provided through conversion formulae between associated Legendre, Gegenbauer and Jacobi functions. The PDE satisfied by Gegenbauer functions on $|x| < 1$ is:

$$(1 - x^2)f_{xx} - (2\alpha + 1)xf_x + n(n + 2\alpha)f = 0 \tag{6.180}$$

which we can relate to the various groups of special functions we have encountered when analysing the Radon transform. Note that this function may be modified such that it has validity in the hyperbolic domain equivalent to the spherical scenario we are dealing with in this chapter, for instance one can take $|x| > 1$ to generate these types of groups. In this case, instead of the special orthogonal group $SO(n, 1)$, which is associated with the Gegenbauer polynomials, one encounters the hyperbolic equivalent $SH(n, 1)$, which can be associated with Gegenbauer functions which may be employed to compute a theory of the heat kernel in the hyperbolic space given by the reduced pseudosphere. Obviously there is an equivalent basis set in terms of the associated Legendre functions

We wish to emphasise the deeper connection here between the various sets of Bessel functions (Macdonald, Basset, etc.) and the representation theory of groups. Although we have used a simplistic form of analysis in order to associate the Bessel functions with distance functions that are obviously related to a spherical or hyperbolic space, this is actually only an aspect of a greater theory which we will cover in depth in the remaining parts of this chapter. We can see that the Euler-Poisson-Darboux equation relates to a metric on a reduced sphere, and this necessarily implies the existence of distance metrics that are related to the Bessel functions. Analysis of this connection, and its generalisation will be sufficient to generate a large class of special functions using this simple apparatus.

By using the concepts of representation theory and spherical functions together with differential geometry we are able to gain insight into the structure of special functions. The understanding we have developed in using these types of methods allows us to define products, composition lemmas and convolution laws that are extremely important in determining many different mathematical and physical phenomena. We can see the structures we have observed as arising as a result of the spherical decomposition law combined with the distance formula and metric. Both of these complementary viewpoints combine to give us meaningful, succinct formulae for this type of zonal decomposition.

6.7 Time Dispersive Systems

We shall now discuss some more general principles related to time dependence in these types of PDEs. The following simple examples shall illustrate the central principles. If we think about a system which is continuous in time, but yet has the same properties of the spherical systems we have discussed, the natural consideration is to take a system with basic operator structure as below.

Definition 6.7.1. Any linear PDE that is solely a function of derivatives in time and space may be written in the form:

$$\hat{T}(t, \partial_t, \partial_t^2, ..)f(x, t) = \hat{H}(x, \partial_x, \partial_x^2, ..)f(x, t) \tag{6.181}$$

We call this a time dispersive system if the operator in time is a function of higher order derivatives greater than one. For example, the Schrödinger equation or heat equation are both first order in time. They are not time dispersive, the time change is additive. The wave equation and D'Alambert equations are second order in time, they are therefore time dispersive.

Under this formalism, we have immediately the following:

Corollary 6.7.2. *If we take only operators which are of second order in space and time, the generalised dispersion equations are of the form:*

$$\hat{T}(t, \partial_t, \partial_t^2)f(x, t) = \hat{H}(x, \partial_x, \partial_x^2)f(x, t) \tag{6.182}$$

Proof. This follows as a special case of 6.7.1. □

Remark 109. It is known that the class of second order differential operators in space and first order in time describes a large class of special functions. However, in this chapter we shall be concerned with the situation where the operator in time can have multiple degrees of freedom.

Corollary 6.7.3. *The wave equation operator under the time dispersion formalism is given by the special case:*

$$\hat{T}(\partial_t^2)f(x,t) = \hat{H}(\partial_x^2)f(x,t) \Rightarrow \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} \tag{6.183}$$

Proof. This follows as a special case of 6.7.1. □

We shall now consider some general statements regarding the time kernel of the system. We take the following definition as axiomatic to our discussion, viz.:

Definition 6.7.4. The heat kernel defines a transition probability density that may be written as:

$$K(x,t;y,t-t') \tag{6.184}$$

which gives the the probability that the state travels from y at time $t-t'$ to x at t.

We now point out that the standard ansatz as applied to the quantum dynamic equations, Liouville equations and other equations which are first order in time do not apply. Directly, we have:

Lemma 6.7.5. *For an equation which is first order in time, such as the heat equation or Schrödinger equation, the kernel function is time-translation invariant, in which case one may write:*

$$K(x,t;y,t') = K(x,t-t';y,0) = K(x,y;\tau) \tag{6.185}$$

i.e. the transition probability density defines a time translation invariant restriction of the kernel. For a time-dispersive situation, such as the wave or Euler-Poisson-Darboux equations this is no longer necessarily the case.

Proof. Using the kernel solution to the heat equation, we have:

$$K(x,y;t) = e^{t\partial_x^2} \tag{6.186}$$

where $u_t = u_{xx}$, and the integral kernel satisfies:

$$u(x,t) = \int K(x,y;t)u(y,0)d\mathfrak{m}(y) \tag{6.187}$$

By inspection, we have $K(x,y;t)K(y,z;s) = K(x,z;t+s)$, which implies time translation invariance. □

In the following sections, we depart from the formalism of the heat kernel and hyperbolic spaces and consider some more advanced types of kernel structures that result from eigenfunction decompositions for the wave and EPD equation. As these equations are no longer first order in time, they will not be time translation invariant. Indeed, in systems where we have time evolving according to second order and lower derivatives, the time component will be diffusive, in the same way that a second order differential equation in space and first order in space, i.e. the heat equation implies the existence of martingale measures, systems with this sort of dispersion characteristic inherent in the time parameter imply the existence of kernel functions where the roles of space and time are interchanged.

In the same way that a system obeying the heat equation can be characterised through a measurement of the distribution of x at two points at any one time, expressed through the kernel density $K(x, x'; t)$, many of these properties are preserved when one interchanges the time and space co-ordinate. One is naturally led to consider the question of the time a wave crosses a point and the time it returns from a point of reflection. In such a case, one takes the solution to e.g. the wave equation through a kernel of the type $K(t, t'; x)$, i.e. a wave kernel. We can understand this in pure physical terms, in that specification of the time a wave crosses a point, and when it crosses upon reflection, is sufficient to determine the speed of the wave, which determines the state. In the calculations contained in the remainder of this chapter we shall be discussing problems with these more general characteristics. By examining the different types of eigenfunction decompositions that apply to systems such as the wave and Euler-Poisson-Darboux equation in the time and space domain, we point to a more general theory of equivalence between these wave, heat and EPD equations which we shall calculate in the final part of this thesis using the Weierstrass transform.

Remark 110. We shall now determine the necessary principles for dealing with this special case of hyperbolic equations. In particular, we shall describe an eigenfunction-related method that gives a construction analogous to the Mehler kernel for the wave and EPD equations.

6.7.1 Lipschitz Continuity

Systems that are second order in time can be understood in a similar way to systems that are second order in space. In particular, the Lipschitz conditions define regions of uniqueness, stability and explosion in terms of time. We shall briefly set some rough guidelines on the continuity required in the functions which appear in our time dispersion equations. We take the following basic definition which extends from the Lipschitz conditions in space.

Definition 6.7.6. If we have an operator which may be written as:

$$\hat{T}f(t) = \frac{\sigma^2(t)}{2}\frac{\partial^2 f}{\partial t^2} + \mu(t)\frac{\partial f}{\partial t} + c(t)f(t) \tag{6.188}$$

we can state the following Lipschitz conditions:

$$|\mu(t)|^2 + |\sigma(t)|^2 \le L(1 + |t|^2) \tag{6.189}$$

$$|\mu(t) - \mu(s)| + |\sigma(t) - \sigma(s)| \le C|t - s| \tag{6.190}$$

and in generalised form:

$$|\mu(t)|^2 + |\sigma(t)|^2 \le L(1 + |t|^\beta) \tag{6.191}$$

$$|\mu(t) - \mu(s)| + |\sigma(t) - \sigma(s)| \le C|t - s|^\alpha \tag{6.192}$$

The Lipschitz conditions define regions of uniqueness and continuity. The second group of conditions can be understood in terms of Hölder continuity. These conditions imply moderate rates of growth in the functions multiplying the first and second order derivatives as time goes on. In the context of stochastic equations, we would refer to these as the diffusion and drift, and we would be examining the long time and long space behaviour of these coefficients. In our situation, we will have functions of time which play a similar role to these parameters, although the context is different many properties have analogous counterparts.

Remark 111. As we shall be discussing the case where the right-hand side of the equation 6.7.1 can be made to vanish through some type of transform, the dispersion remaining is principally in the time co-ordinate. The Lipschitz conditions will then come in use in order to determine the necessary continuity properties of the systems we shall encounter.

6.7.2 Time Kernel Equations

Time kernels are the solutions to time dispersive equations. In the same way that one may use the kernel for the heat equation, and derive it using an eigenfunction decomposition, the wave equation has a kernel solution which may be found using a related method. We exhibit several examples of kernel solutions for the wave equation using eigenfunctions, recovering the standard expression for a standing wave. We shall begin our calculation in this subsection with a simplification of the more general case of time dispersive equations. This simplification takes the form whereby we have some sort of dispersion occurring such that we can transform the space co-ordinate to remove the spatial dependence. If we take a generalisation of the wave-equation, we might write the following special case of the time dispersion equation:

Example 6.7.7. The generalised time dispersion equation which is second order in space and of any order in time is given by the expression:

$$\hat{T}(t, t^2, .., \partial_t, \partial_t^2, ..)f(x,t) = \hat{T}(t^i, \partial_t^i)f(x,t) = \frac{\partial^2 f}{\partial x^2} \tag{6.193}$$

If an equation of this type is of higher order than first in either the time or space co-ordinate, i.e. depends on derivatives that are higher than first order, then we say that the equation is dispersive in this co-ordinate.

We shall now show how this simple example leads to broader conclusions in analysis of this type of conjoined system. We have the following lemma:

Lemma 6.7.8. *For the simplest case of a system which is second order in space and time dispersive, we have:*

$$\mathcal{F}\left[\hat{T}(t^i, \partial_t^i)f(x,t)\right] = \hat{T}(t^i, \partial_t^i)\mathcal{F}\left[f(x,t)\right] = \mathcal{F}\left[\frac{\partial^2 f}{\partial x^2}\right] \tag{6.194}$$

The Fourier transform in space commutes with the time dispersion operator.

$$\mathcal{F}\left[\hat{T}(t^i, \partial_t^i)f(x,t)\right] = \hat{T}(t^i, \partial_t^i)\mathcal{F}\left[f(x,t)\right] \tag{6.195}$$

Proof. Following the arguments contained in the Radon transform 6.5.20, we have:

$$\mathcal{F}\left[\hat{T}(t^i, \partial_t^i)f(x,t)\right] = \hat{T}(t^i, \partial_t^i)\mathcal{F}\left[f(x,t)\right] = \mathcal{F}\left[\frac{\partial^2 f}{\partial x^2}\right] \tag{6.196}$$

which can be rewritten using standard results from Fourier theory as:

$$\hat{T}(t^i, \partial_t^i)g(s,t) = -s^2 g(s,t) \tag{6.197}$$

$$g(s,t) = \mathcal{F}\left[f(x,t)\right] \tag{6.198}$$

□

We now state a more general form of this expression in order to identify the particular relationship between the eigenfunctions and the symmetries under the Fourier transform.

Definition 6.7.9. If $\hat{H}(x^i, \partial_x^i)$ is any differential operator of at most second order in space, and containing a potential which falls between second order and inverse square so as to contain closed orbits, then there exists some operator such that we can write:

$$\mathcal{F}\left[\hat{H}(x^i, \partial_x^i)f(x,t)\right] = -s^2 g(s,t) \tag{6.199}$$

which we term the diagonalising transform for the operator.

Remark 112. For example, the Laplace operator is diagonalised by the Fourier transform as above, and one can show that the diagonalising operator for e.g. the modified Bessel equation is the Kontorovich-Lebedev transform, as in [41,46,147].

We obtain the following corollary as a result.

Corollary 6.7.10. *Any operator in space that possesses a diagonalising transform may therefore have a time dispersion equation which is of the simple form:*

$$\hat{T}(t^i, \partial_t^i)g(s,t) = -s^2 g(s,t) \tag{6.200}$$

The solution to the original equation is given by the inverse transformation:

$$f(x,t) = \mathcal{F}^{-1}[g(s,t)] \tag{6.201}$$

Proof. Apply the results from the lemma and definition 6.7.7, and the example as given in 6.7.9, we obtain the result from the Fourier transform. □

Remark 113. This is the generalised form of the time-EPD equation; in particular, we note that this equation may be solved by effectively treating s as a constant, and solving for the time variation in the function.

We shall now consider some basic examples of time dispersive kernels to illustrate the concepts. It is clear therefore that the role of the kernel in this situation is played through the function $g(s,t)$.

Example 6.7.11. The wave equation is defined by the time dispersive equation and eigenfunction problem:

$$\hat{T}(t^i, \partial_t^i)g(s,t) = \frac{\partial^2 g}{\partial t^2} = -s^2 g(s,t) \tag{6.202}$$

Assume boundary condition specified by $\frac{\partial g}{\partial t}|_{t=0} = 1$. Although this is second order in time, for now we leave the second boundary condition unspecified for the purpose of argument. Then the solution to this differential equation is the sinusoid:

$$g(s,t) = C_1(s)\sin(st) \tag{6.203}$$

where $C_1(s)$ is a constant in time, but possibly dependent on s.

Remark 114. We can equally choose the complimentary solution given by the boundary condition:

$$\frac{\partial g}{\partial t}|_{t=0} = 0 \tag{6.204}$$

to obtain the cosine instead of the sine solution.

We also have the following definition of the Yor integral which we shall use to derive various forms of the wave kernel.

Definition 6.7.12. The Yor integral for the wave equation is the one-sided kernel solution specified through the boundary condition $t' = 0$, i.e.:

$$K(t;x) = K(t,0;x) = \int_{-\infty}^{+\infty} e^{-isx}\psi_s(x)d\mathfrak{m}(x) \tag{6.205}$$

where $K(t,t';x)$ is the wave kernel solution, the system eigenfunction is given by $\psi_s(x)$, and the measure of the space is $d\mathfrak{m}(x)$.

We emphasise that, although the solutions share many properties in common with the heat equation, the decompositions of the kernel are not Laplace transforms, but are given by Fourier transforms over the eigenfunctions.

Lemma 6.7.13. *A weak solution to the wave equation 6.7.3, 6.7.13:*

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2} \tag{6.206}$$

is given by the eigenfunction decomposition:

$$K(t, t'; x) = \int_{-\infty}^{+\infty} e^{-isx} \sin(st)\sin(st')ds \tag{6.207}$$

where $K(t, t'; x)$ is the time kernel of the wave equation. This kernel is given by the Dirac delta function sum:

$$K(t, \tau; x) = \frac{\pi}{2}\left(\delta(x + \tau - t) - \delta(x - \tau - t) - \delta(x + \tau + t) + \delta(x - \tau + t)\right) \tag{6.208}$$

Proof. A weak solution satisfies the PDE in distribution, i.e. for any suitable test function on the space we have:

$$\int dx.\eta(x)\frac{\partial^2}{\partial x^2}K(t, \tau; x) = \int dx.\eta(x)\frac{\partial^2}{\partial t^2}K(t, \tau; x) \tag{6.209}$$

Applying this to $K(t, \tau; x) = \int_0^\infty e^{-isx}\sin(st)\sin(s\tau)ds$, we have:

$$\int dx.\eta(x)\frac{\partial^2}{\partial x^2}K(t, \tau; x) = \int dx.\eta(x)\frac{\partial^2}{\partial x^2}\int_{-\infty}^{+\infty} e^{-isx}\sin(st)\sin(s\tau)ds \tag{6.210}$$

$$= \int dx.\eta(x)\int_{-\infty}^{+\infty}(-s^2)e^{-isx}\sin(st)\sin(s\tau)ds \tag{6.211}$$

On the other side of the equality, we have:

$$\int dx.\eta(x)\frac{\partial^2}{\partial t^2}K(t, \tau; x) = \int dx.\eta(x)\frac{\partial^2}{\partial t^2}\int_{-\infty}^{+\infty} e^{-isx}\sin(st)\sin(s\tau)ds \tag{6.212}$$

$$= \int dx.\eta(x)\int_{-\infty}^{+\infty}(-s^2)e^{-isx}\sin(st)\sin(s\tau)ds \tag{6.213}$$

where we have differentiated under the integral sign. To prove the delta function sum, simply take the formula for the product of two sine waves as the sum and the difference of frequencies as in physics via $\frac{1}{2}\left(\cos(\alpha - \beta) - \cos(\alpha + \beta)\right) = \sin\alpha\sin\beta$ and evaluate the Fourier transform directly. These formulae are only true in distribution, as in they need to be integrated against a function to have mathematical meaning. □

We now proceed to identify a number of other examples of these types of kernels which follow from the sine eigenfunction. Our first example of this type is the Yor integral, given by the special solution where one boundary condition is held fixed at unity. In this case, the eigenfunction collapses, and the kernel formula takes a simplified form.

Example 6.7.14. The Yor integral for the wave equation is the one-sided Fourier transform of the eigenfunction defined by the transform:

$$K(\tau; x) = \int e^{-isx}\phi_s(\tau)ds \tag{6.214}$$

where $\phi_s(t)$ is the eigenfunction for the wave-type PDE $u_{tt} = u_{xx}$. For the wave equation with boundary condition $u(\tau, 0) = 0$ the Yor integral may be written in the form of a system of advanced and retarded waves from a point source:

$$K(\tau; x) = -i\pi[\delta(x - \tau) - \delta(x + \tau)] \tag{6.215}$$

where $\delta(.)$ is the Dirac delta function.

Proof. The Yor integral is the special case of the kernel such that:

$$K(\tau; x) = \int e^{-isx} \phi_s(\tau) ds \tag{6.216}$$

where $\phi_s(\tau)$ is the eigenfunction of the wave equation. Evaluating the Yor integral, we find the wave equation kernel:

$$K(\tau; x) = \int_{-\infty}^{+\infty} e^{-isx} \sin(s\tau) ds \tag{6.217}$$

$$= \int_{-\infty}^{+\infty} e^{-isx} \sin(s\tau) ds \tag{6.218}$$

$$= -i\pi[\delta(x - \tau) - \delta(x + \tau)] \tag{6.219}$$

as required. To show that it satisfies the boundary condition, we use the evenness property of the delta function, we have $\delta(-\tau) = \delta(\tau)$, taking $x = 0$ in the kernel equation, and assuming that the fundamental solution can be written via the kernel, we have $K(\tau; 0) = -i\pi\,[\delta(-\tau) - \delta(\tau)] = 0$. Note that the wave equation is linear, and hence obeys the superposition principle, i.e. waves add up, and the solutions to the equation (if any) that obey the boundary conditions will be invariant under multiplication by a scalar, and addition of solutions. □

Remark 115. Knowing that the wave solution is the superposition of waves originating from different sources in this way, we can identify this type of kernel as an elementary type of solution that arises in the study of wave equations.

We shall now use this form of kernel to construct a familiar type of wave solution. We shall require the following lemma:

Lemma 6.7.15. *For the generalised time dispersive equation of the form:*

$$\hat{T}(t^i, \partial_t^i) f(x, t) = \hat{H}(x^i, \partial_x^i) f(x, t) \tag{6.220}$$

we can find a solution through eigenfunction decomposition, as in the Yor integral. We may write this special solution in terms of the initial value problem, where the definitions of the operators are as in 6.7.1, a kernel solution exists and is defined through the translation property:

$$f(x, t) = \int K(t, 0; y) f(y, 0) dy \tag{6.221}$$

Proof. See e.g. this reference 6.7.4, 6.7.11, also the derivation in 3.3.20. This is not a proof, per se. We merely wish to illustrate consistency with our previous methods using eigenfunction decomposition, as was employed to find the solution for the hyperbolic heat kernel. □

On application of the preceding lemma for the Yor integral, we have immediately the following result.

Corollary 6.7.16. *Assume the wave equation:*

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2} \tag{6.222}$$

and the kernel solution given by the shifted Yor integral, analogous to 6.7.14:

$$K(t,\tau;x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{-isx}\sin\left(s(t-\tau)\right)ds \tag{6.223}$$

For initial conditions given by the sinusoid $f(x,0)=\sin(x)$, $f_t(x,0)=0$, the time dependent solution is given by a standing wave:

$$f(x,t) = \sin x\cos t \tag{6.224}$$

Proof. In this simple case we have e.g. the wave equation:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2} \tag{6.225}$$

Apply the formula for the shifted Yor integral, we have:

$$K(t,\tau;x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{-isx}\sin\left(s(t-\tau)\right)ds \tag{6.226}$$

$$= \frac{1}{2}[\delta(x+t-\tau)+\delta(x-(t-\tau))] \tag{6.227}$$

We use the factor of $\frac{1}{2\pi i}$ for numerical convenience. As is known from Fourier transform theory, this is a matter of convention. As the equations are linear, we are free to make this choice. Assuming a sinusoidal boundary condition initially, we have:

$$f(x,t) = \int \frac{1}{2}[\delta(x+t)+\delta(x-t)]\sin(x)dx = \frac{1}{2}\left(\sin(x+t)+\sin(x-t)\right) \tag{6.228}$$

$$= \sin x\cos t \tag{6.229}$$

which is a standing wave solution, obviously satisfying the boundary conditions and the wave equation. □

Remark 116. We can see how knowledge of the kernel comes about via solution of the eigenfunction equation. Once we can write down the decomposition and evaluate the kernel, it is just a matter of solving a few integrals to find the solution at any position and time given appropriate boundary conditions.

6.7.3 Principles of Time Dispersive Kernels

Time kernels have a similar set of axioms to those associated with systems that are dispersive in space. We shall summarise the results in the following principle for the generalised wave equation.

Theorem 6.7.17. *Assume the separable time dispersive equation given by:*

$$\hat{T}(t^i,\partial_t^i)f(x,t) = \hat{H}(x^i,\partial_x^i)f(x,t) \tag{6.230}$$

Assume further that this system has eigenvalues and eigenfunctions in the time domain that are specified through:

$$\hat{T}(t^i,\partial_t^i)\phi_s(\mathbf{x};t) = -\lambda^2(s)\phi_s(\mathbf{x};t) \tag{6.231}$$

We say that a diagonalising or eigenfunction transformation exists if we can transform the operator $\hat{H}(x^i, \partial_x^i)$ into the eigenvalue spectrum:

$$\mathcal{F}_{\mathbf{x}}[\hat{H}(x^i, \partial_x^i) f(\mathbf{x}, t)] = \mathcal{F}_{\mathbf{x}}\left[\hat{T}(t^i, \partial_t^i) f(\mathbf{x}, t)\right] = \hat{T}(t^i, \partial_t^i)\left[\mathcal{F}_{\mathbf{x}} f(\mathbf{x}, t)\right] \tag{6.232}$$

via $\hat{T}(t^i, \partial_t^i) g(\mathbf{s}, t) = -\lambda^2(s) g(\mathbf{s}, t)$. Assume such a transformation exists for the second order system. Then the kernel solution to the second order system: $\hat{H}(x^i, \partial_x^i) = \partial_x^2$ is given by:

$$K(t, \tau; x) = \int \exp(-i\lambda_u x)\phi_u^*(t)\phi_u(\tau) d\mathfrak{m}(u) \tag{6.233}$$

where the integral is to be carried out over the eigenvalue spectrum u, which may be continuous or discrete depending on the spectral characteristics of the system.

Proof. We assume the precepts of the theorem, that there exists a diagonalising transformation:

$$\hat{T}(t^i, \partial_t^i) g(\mathbf{s}, t) = -\lambda^2(s) g(\mathbf{s}, t) \tag{6.234}$$

the eigenfunctions of the time dispersive operator:

$$\hat{T}(t^i, \partial_t^i)\phi(\mathbf{u}; t) = -\lambda^2(\mathbf{u})\phi(\mathbf{u}; t) = -\lambda_u^2 \phi_{\mathbf{u}}(t) \tag{6.235}$$

and that the space operator is given through the second derivative, $\hat{H}(x^i, \partial_x^i) = \partial_x^2$. Then the kernel is given through the eigenvalue combination:

$$K(t, \tau; x) = \int \exp(-i\lambda_u x)\phi_u^*(t)\phi_u(\tau) d\mathfrak{m}(u) \tag{6.236}$$

according to an identical argument to 6.7.15. The integral in this case is over the index of the eigenfunctions. This is for a system with a single continuous eigenvalue, i.e. a continuous spectrum. Depending on the example, the integral may be replaced by a sum with appropriate weighting for a system with a discrete spectrum. □

We now demonstrate a theorem which is pervasive in the continuous representation of kernels. We shall require the following:

Lemma 6.7.18. *The initial condition of the time dispersive kernel $K(z, \tau; 0)$ is given by the delta function solution:*

$$\delta(z - \tau) = \int \phi_u^*(z)\phi_u(\tau) d\mathfrak{m}(u) = K(z, \tau; 0) \tag{6.237}$$

where $\phi_u(\tau)$ is the eigenfunction of the time dispersive operator, $\mathfrak{m}(u)$ is the measure in the continuous state space and $\delta(.)$ is the Dirac delta function.

Proof. Applying 6.7.17, we take the special case $x = 0$, and obtain the lemma.

$$\delta(z - \tau) = \int \phi_u^*(z)\phi_u(\tau) d\mathfrak{m}(u) = K(z, \tau; 0) \tag{6.238}$$

□

This closes our introductory investigations of these types of integral operators. We shall now move on to discuss more advanced topics in the analysis of special functions after some brief remarks.

Remark 117. This class of equations is more general than one might think at face value. For example, many second order differential equations may be transformed into a similar form with the addition of a potential term. This means that we will be able to recover a similar schemata for many other systems other than the illustrated Fourier system with the wave equation. Using the kernel, we can recover the solution at any time given some initial boundary conditions:

$$f(x,t) = \int\int K(t,\tau;y) f(y,\tau) dy d\tau \tag{6.239}$$

Our basic algorithm will proceed in the following way. We shall determine the diagonalising operators for the time-dispersive EPD equations using the Bose invariant technique, and use an inverse transform to recover the kernel of the implied wave equation. The kernel in this case is a function of two times, evaluated at a space point. We expect this type of system to have some interesting behaviour when compared to e.g. the standard examples known through the wave, heat and quantum diffusion equations.

6.7.4 Time Potentials

Time dispersive equations have a potential theory that can be understood using the concepts of Sturm-Liouville theory and the Bose invariant. We demonstrate the basic principles for application in the following calculations. Let us now consider the relationship between the basic structure we have described for the wave equation, and more generalised forms given through EPD-type time equations. In general, we will have the time dependent eigenfunction equation specified through the PDE system:

Definition 6.7.19. The eigenvalue problem for a second order time dispersive PDE is defined by:

$$\hat{T}u(t) = -\lambda^2 u(t) \tag{6.240}$$

where the diffusive time operator is given through:

$$\hat{T} = a(t)\partial_t^2 + b(t)\partial_t + c(t) \tag{6.241}$$

This is defined in direct extension to the quantum equation (first order in time) and the wave equation (second order in time).

We quote the following results for the Bose invariant, which extends easily from the case of space dependence to that of time. This important result from Sturm-Liouville theory enables us to rearrange the fundamental differential equation defined in 6.7.19 into a diagonalised form.

Theorem 6.7.20. *The second order time dispersive equation:*

$$a(t)\partial_t^2 u + b(t)\partial_t u + c(t)u = (\hat{T} + \lambda^2)u = 0 \tag{6.242}$$

is equivalent to the wave equation with time potential:

$$\frac{\partial^2 f}{\partial t^2} + (V(t) - E)f(x,t) = 0 \tag{6.243}$$

where the Bose invariant potential is given by:

$$V(t) - E = \frac{4ac - b^2 + 2(ba' - ab')}{4a^2} \tag{6.244}$$

Proof. We have calculated the space equivalent (see e.g. 3.5.11) in this work. An extensive discussion of the topic may be found in the works of Albanese, Kuznetsov and Lawi [61,62,156]. □

The following results carry through automatically into the time domain. We have already proved these theorems in the context of kernel diffusions in space and the heat equation 3.5.11, so will not prove them again, as all that is required is the substitution $x = t$ into the expressions.

Theorem 6.7.21. *The operator equation for the second order time dispersion 6.7.7:*

$$\hat{\mathcal{A}}(t, \partial_t) = a(t)\partial_t^2 + b(t)\partial_t + c(t) \tag{6.245}$$

has a scale factor that transforms the equation via:

$$([a(t)]^{-1}h(t)\hat{\mathcal{A}}(t, \partial_t)[h(t)]^{-1})f = \frac{\partial^2 f}{\partial t^2} + (V(t) - E)f(x, t) \tag{6.246}$$

where the scale factor is given by the formula:

$$h(t) = \exp\left(\int^t \frac{b(t')}{2a(t')}dt'\right) \tag{6.247}$$

Remark 118. Calculating the scale factor and the Bose invariant normal form of any time dispersive equation is therefore of paramount importance in calculating the potential function we can associate to time diffusion. Note that we are free to move the scalar parameter through this expression, either by absorbing it into $c(t)$ or into the potential function. We term the generalised operator defined through

$$V(t) - E = \hat{\mathcal{V}}(t) \tag{6.248}$$

to be the time potential of the system. As we shall show, the use of this transformation is sufficient to bring the system into a diagonalised form. In this situation, we are left with a quantum potential equation which we are able to solve in terms of eigenfunction expansions. It is then a matter of inverting the transformation to determine the solution to the hyperbolic PDE.

6.8 Theory of the Time Bessel Potential

There exists a time form of the EPD equation, derived in this work for the space equivalent using properties of hyperbolic diffusion and the Laplacian 6.5.19, 6.5.9, 6.5.10. The time-EPD equation can be solved using the concatenation of the Fourier and Laplace transforms. The solution is given by a Bessel function in time. The inverse Fourier transform of the solution is given by a Chebyshev polynomial multiplied by Heaviside step functions, with the Green's function solved in a similar way. As shall be a common theme in the following sections, the process of Fourier inversion when combined with Laplace transformation has some interesting outcomes for the development of exact or essentially exact solutions for systems of this type.

Definition 6.8.1. Assume a separable system defined by a PDE of the form:

$$\hat{H}(x^i, \partial_x^i)f(\mathbf{x}, t) = \hat{T}(t^i, \partial_t^i)f(\mathbf{x}, t) \tag{6.249}$$

where $f(\mathbf{x}, t)$ is a function representing the solution to the PDE. Further, assume that $E, t > 0$ and $\boldsymbol{k}, \mathbf{x} \in \mathbb{R}^n$, with $f(\mathbf{x}, t) = \mathcal{O}\left(|\mathbf{x}|^{-N}e^{ct}\right)$ for some $N > n$ and $c < E$. The measure of the integral is $d\mathbf{x} = d^n x = dx_1 dx_2 ... dx_n$, and the Green's function is given by the Fourier-Laplace integral:

$$G(\boldsymbol{k}, E) = \int_0^\infty e^{-Et} \int_{\mathbb{R}^n} e^{-i\boldsymbol{k}\cdot\mathbf{x}} f(\mathbf{x}, t) d\mathbf{x} \tag{6.250}$$

The Green's function is the solution to the Fourier-Laplace transformed kernel equations and satisfies:

$$G(\boldsymbol{k}, E) = \mathcal{F}_{\boldsymbol{x}}\left[\mathcal{L}_t\left[f(\boldsymbol{x}, t)\right]\right] = \mathcal{L}_t\left[\mathcal{F}_{\boldsymbol{x}}\left[f(\boldsymbol{x}, t)\right]\right] \tag{6.251}$$

Remark 119. We choose specifically the limiting behaviour $f(\mathbf{x}, t) = \mathcal{O}\left(|\mathbf{x}|^{-N} e^{ct}\right)$ such that the Laplace and Fourier transformations exist and be well-defined. As can be seen from the definition, the time component must grow at a rate less than e^{Et}, and the sphere in Euclidean n-dimensional space expands at a rate $|\mathbf{x}|^N$ as we change the dimension, hence we may only take functions with this type of growth rate for our formulae for the Green's function and kernel to exist.

We shall illustrate the utility of such transformation methods with a brief series of lemmas involving the time-EPD diffusion problem.

Definition 6.8.2. Assume that $t > 0$, and $\mathbf{x} \in \Omega_k \subset \mathbb{R}^n$. The basic form of the time-EPD equation is given by the special case of the operator EPD system 6.5.11.

$$\frac{\partial^2 f}{\partial t^2} + \frac{\alpha}{t}\frac{\partial f}{\partial t} = \nabla^2 f \tag{6.252}$$

where ∇^2 is the Laplacian.

We shall now prove the following lemma, where we establish the solution to the time-EPD equation.

Lemma 6.8.3. *For the Euler-Poisson-Darboux equation in time:*

$$\frac{\partial^2 u}{\partial t^2} + \frac{\alpha}{t}\frac{\partial u}{\partial t} = \nabla^2 u \tag{6.253}$$

where the co-ordinates are Cartesian $\mathbf{x} = \sum x_i \mathbf{e}_i$, *the Laplacian given as* $\nabla^2 = \nabla \cdot \nabla$, *the fundamental solution may be written using the Fourier transform on Euclidean space:*

$$u(\mathbf{x}, t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} U(\xi, t) e^{i\mathbf{x}\cdot\xi} d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} C_1(|\xi|) t^{(1-\alpha)/2} J_{(\alpha-1)/2}(|\xi| t) e^{i\mathbf{x}\cdot\xi} d\xi \tag{6.254}$$

where $J_\mu(.)$ *is the Bessel function,* $|\xi| = \sqrt{\sum_j \xi_j^2}$ *and* $C_1(|\xi|)$ *is the Fourier transform of* $u(\mathbf{x}, 0) = f(\mathbf{x})$. *This function is finite and differentiable at the origin in time:*

$$\lim_{t\to 0^+} \frac{1}{t}\frac{\partial}{\partial t}\left(u(\mathbf{x}, t)\right) < \infty \tag{6.255}$$

as is its Fourier transform, and satisfies the initial value problem:

$$u(\mathbf{x}, 0) = f(\mathbf{x}), u_t(\mathbf{x}, 0) = 0 \tag{6.256}$$

Proof. We begin with the EPD equation:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\alpha}{t}\frac{\partial u}{\partial t} = \nabla^2 u \tag{6.257}$$

which, for $\alpha = 0$ reduces to the wave equation. To establish that this is a solution to the initial value problem, we write the Fourier transform:

$$U(\xi, t) = \int_{\mathbb{R}^n} u(\mathbf{x}, t) e^{-i\mathbf{x}\cdot\xi} d^n\mathbf{x} \tag{6.258}$$

The transformed PDE is then given by:

$$\frac{\partial^2 U}{\partial t^2} + \frac{\alpha}{t}\frac{\partial U}{\partial t} + |\xi|^2 U = 0 \tag{6.259}$$

The solution may be written:

$$U(\xi, t) = \int_{\mathbb{R}^n} u(\mathbf{x}, t) e^{-i\mathbf{x}\cdot\xi} d\mathbf{x} \tag{6.260}$$

and we obtain the PDE system in the Fourier transformed variable as:

$$\frac{\partial^2 U}{\partial t^2} + \frac{\alpha}{t}\frac{\partial U}{\partial t} + |\xi|^2 U = 0 \tag{6.261}$$

which is solved by the Bessel functions:

$$U(\xi, t) = C_1(|\xi|) t^{(1-\alpha)/2} J_{(\alpha-1)/2}(|\xi|t) + C_1(|\xi|) t^{(1-\alpha)/2} Y_{(\alpha-1)/2}(|\xi|t) \tag{6.262}$$

We have the limits as we approach the origin from the positive direction:

$$\lim_{t\to 0^+} \frac{1}{t}\frac{\partial}{\partial t}\left(t^{(1-\alpha)/2} Y_{(\alpha-1)/2}(|\xi|t)\right) = \infty \tag{6.263}$$

$$\lim_{t\to 0^+} \frac{1}{t}\frac{\partial}{\partial t}\left(t^{(1-\alpha)/2} J_{(\alpha-1)/2}(|\xi|t)\right) < \infty \tag{6.264}$$

and hence only the solution involving Bessel-J type functions has validity in our domain, $C_2 = 0$. To see this explicitly, we take the series representation:

$$t^{(1-\alpha)/2} J_{(\alpha-1)/2}(|\xi|t) = t^{(1-\alpha)/2}\left(\frac{|\xi|t}{2}\right)^{(\alpha-1)/2} \sum_{n=0}^{\infty} (-1)^n \frac{(|\xi|t)^{2n}}{2^{2n}\Gamma\left(n+1+\frac{\alpha-1}{2}\right) n!} \tag{6.265}$$

$$= \left(\frac{|\xi|}{2}\right)^{\nu} \sum_{n=0}^{\infty} (-1)^n \frac{(|\xi|t)^{2n}}{2^{2n}\Gamma\left(n+1+\nu\right) n!} \tag{6.266}$$

where $\nu = \dfrac{\alpha-1}{2}$. Note that we have, at the origin:

$$\lim_{t\to 0^+}\left(t^{(1-\alpha)/2} J_{(\alpha-1)/2}(|\xi|t)\right) < \infty \tag{6.267}$$

and therefore we have:

$$\lim_{t\to 0^+} \frac{1}{t}\frac{\partial}{\partial t}\left(t^{(1-\alpha)/2} J_{(\alpha-1)/2}(|\xi|t)\right) = \left(\frac{|\xi|}{2}\right)^{\nu} \sum_{n=0}^{\infty} (-1)^n \frac{\partial}{\partial t}\left(\frac{(|\xi|t)^{2n}}{2^{2n}\Gamma\left(n+1+\nu\right) n!}\right) \tag{6.268}$$

$$= \left(\frac{|\xi|}{2}\right)^{\nu}\left(-\frac{2|\xi|t}{2^2 2!\Gamma(\nu+2)} + \text{H.O.T.}\right)_{t=0} \tag{6.269}$$

$$= 0 \tag{6.270}$$

$$\lim_{t\to 0^+} \frac{1}{t}\frac{\partial U(\xi,t)}{\partial t} \to -\left(\frac{|\xi|}{2}\right)^{\nu} \cdot \frac{2|\xi|t}{2^2 2!\Gamma(\nu+2)}\Big|_{t\downarrow 0^+} < \infty \tag{6.271}$$

i.e. the Fourier transform of the solution is finite as we approach the origin from the positive direction. In terms of the fundamental solution, we may use the Fourier inversion theorem to obtain:

$$U(\xi,t) = \int_{\mathbb{R}^n} u(\mathbf{x},t)e^{-i\mathbf{x}\cdot\xi}d\xi \tag{6.272}$$

$$u(\mathbf{x},t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} U(\xi,t)e^{i\mathbf{x}\cdot\xi}d\xi \tag{6.273}$$

and our solution is then:

$$u(\mathbf{x},t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} U(\xi,t)e^{i\mathbf{x}\cdot\xi}d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} C_1(|\xi|)t^{(1-\alpha)/2}J_{(\alpha-1)/2}(|\xi|t)e^{i\mathbf{x}\cdot\xi}d\xi \tag{6.274}$$

Now we have the initial value problem, with solution expressed as:

$$u(\mathbf{x},0) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} C_1(|\xi|)J_{(\alpha-1)/2}(|\xi|t)e^{i\mathbf{x}\cdot\xi}d\xi = f(\mathbf{x}) \tag{6.275}$$

The limit on the fundamental solution yields:

$$\lim_{t\to 0^+}\frac{1}{t}\frac{\partial}{\partial t}\left(u(\mathbf{x},t)\right) = \frac{1}{(2\pi)^n}\lim_{t\to 0^+}\frac{1}{t}\frac{\partial}{\partial t}\int_{\mathbb{R}^n} U(\xi,t)e^{i\mathbf{x}\cdot\xi}d\xi \tag{6.276}$$

$$= \frac{1}{(2\pi)^n}\lim_{t\to 0^+}\int_{\mathbb{R}^n}\frac{1}{t}\frac{\partial U(\xi,t)}{\partial t}e^{i\mathbf{x}\cdot\xi}d\xi \tag{6.277}$$

$$= -\frac{1}{(2\pi)^n}\lim_{t\to 0^+}\int_{\mathbb{R}^n}\frac{1}{t}\left(\frac{|\xi|}{2}\right)^{\nu}\cdot\frac{2|\xi|t}{2^2 2!\Gamma(\nu+2)}e^{i\mathbf{x}\cdot\xi}d\xi \tag{6.278}$$

$$< \infty \tag{6.279}$$

as required. □

Following this result, we have the particular solution given by a special case of the preceding lemma. This involves an integral which may be solved using results from tables.

Lemma 6.8.4. *A particular solution of the time-EPD equation problem given by 6.8.2 is given by the inverse Fourier transform of a Bessel function:*

$$f(\boldsymbol{x},t) = C_1 t^{1/2-\alpha/2}\mathcal{F}^{-1}\left[J_{\alpha/2-1/2}(|\boldsymbol{k}|t)\right] \tag{6.280}$$

The solution to this inversion problem may be written as a linear combination of Heaviside functions, multiplied by a Chebyshev polynomial.

$$f(\boldsymbol{x},t) = C\frac{t^{1/2-\alpha/2}}{\sqrt{t^2-|\boldsymbol{x}|^2}}T_{\alpha/2-1/2}\left(\frac{|\boldsymbol{x}|}{t}\right)\left[H(|\boldsymbol{x}|+|t|)-H(|\boldsymbol{x}|-|t|)\right] \tag{6.281}$$

Proof. The basic time-EPD equation from the definition is given through:

$$\frac{\partial^2 f}{\partial t^2}+\frac{\alpha}{t}\frac{\partial f}{\partial t} = \nabla^2 f \tag{6.282}$$

If we Fourier transform, it is straightforward to see that the equation may also be written as:

$$\frac{\partial^2 R}{\partial t^2}+\frac{\alpha}{t}\frac{\partial R}{\partial t} = -|\boldsymbol{k}|^2 R = -k^2 R \tag{6.283}$$

which is readily solved to give:

$$R(\boldsymbol{k},t) = t^{1/2-\alpha/2}\left[C_1 J_{\alpha/2-1/2}(|\boldsymbol{k}|t)+C_2 Y_{\alpha/2-1/2}(|\boldsymbol{k}|t)\right] \tag{6.284}$$

Inverting the Fourier transform, we may write the solution as:

$$f(\boldsymbol{x}, t) = C_1 t^{1/2-\alpha/2} \mathcal{F}^{-1}\left[J_{\alpha/2-1/2}(|\boldsymbol{k}|t)\right] \tag{6.285}$$

which is the first part of the lemma. Evaluating the integral using [31] eq. 2.13.2, pp. 164, alternatively GR 6.671.9 [122] we gain the result, via:

$$\mathcal{F}^{-1}\left[J_{\alpha/2-1/2}(|\boldsymbol{k}|t)\right] = \frac{2i(-1)^{(\alpha-3)/4}}{\sqrt{t^2-|\boldsymbol{x}|^2}} T_{\alpha/2-1/2}\left(\frac{|\boldsymbol{x}|}{t}\right) \tag{6.286}$$

$$\times\left[H(|\boldsymbol{x}|+|t|) - H(|\boldsymbol{x}|-|t|)\right] \tag{6.287}$$

This integral may alternatively be performed using computer algebra packages such as Mathematica or Maple. In the tables, the cases $|\mathbf{x}| < t$ and $|\mathbf{x}| > t$ have not both been taken into account, which follows from the residue calculus. □

6.8.1 Kernel Solution to Time-Bessel EPD Equation

The Bose invariant principle and the eigenfunction decomposition allow solution of the time-Bessel EPD equation. The solution to the time-Bessel EPD equation is given by the Mehler-Fock function of the distance. This distance can be understood in terms of the hyperbolic plane, and the hyperbolic heat kernel equivalent. The solution to the time-Bessel EPD equation may be extracted using either the sine, cosine, or exponential form of the Fourier transform. As the following shows, by differentiation under the integral sign, the kernels defined through the eigenfunction decomposition integrals satisfy the time-Bessel EPD equations, which we shall verify directly.

Theorem 6.8.5. *For the time-EPD equation, we have PDE given by the second order system:*

$$\partial_t^2 u + \frac{(1+2\eta)}{t}\partial_t u = \nabla^2 u \tag{6.288}$$

The eigenfunctions for this PDE are given by the Bessel functions:

$$u_k(t) = \sqrt{t}\left[C_1 J_\eta(kt) + C_2 Y_\eta(kt)\right] \tag{6.289}$$

Proof. We take the n-dimensional Fourier $f = \mathcal{F}_{\mathbf{x}} u$, then we have:

$$\mathcal{F}_{\mathbf{x}} \nabla^2 u = \nabla^2 \mathcal{F}_{\mathbf{x}} u \tag{6.290}$$

for

$$\partial_t^2 u + \frac{(1+2\eta)}{t}\partial_t u = \nabla^2 u \tag{6.291}$$

which transforms to:

$$\nabla^2 f = -|\mathbf{k}|^2 f = -k^2 f \tag{6.292}$$

where we used $\mathcal{F}_{\mathbf{x}} \nabla^2 u = \nabla^2 \mathcal{F}_{\mathbf{x}} u$. The solution follows as the Bessel functions:

$$u_k(t) = \sqrt{t}\left[C_1 J_\eta(kt) + C_2 Y_\eta(kt)\right] \tag{6.293}$$

□

We therefore gain the following immediate formula for the scale factor of the system and eigenfunction solution, as defined in 3.5.8, 6.7.20.

Corollary 6.8.6. *The scale factor is given through:*

$$h(t) = \exp\left(\int^t \frac{b(t')}{2a(t')}dt'\right) = \exp\left(\int^t \frac{(1+2\eta)}{2t'}dt'\right) = t^{\eta+1/2} \tag{6.294}$$

The solution to the eigenfunction time-EPD problem satisfies:

$$f(k,t) = [h(t)]^{-1}u_k(t) = t^{-\eta}\left[C_1 J_\eta(kt) + C_2 Y_\eta(kt)\right] \tag{6.295}$$

Proof. An application of the formula from 3.5.8, 6.7.20 yields for the time-EPD equation:

$$h(t) = \exp\left(\int^t \frac{b(t')}{2a(t')}dt'\right) = \exp\left(\int^t \frac{(1+2\eta)}{2t'}dt'\right) = t^{\eta+1/2} \tag{6.296}$$

Solving for the solution of the original system in terms of the Bose solution, we have the expression $f(k,t) = [h(t)]^{-1}u_k(t)$, hence we obtain the result. □

We now have the following result, which follows as a simple application of the eigenfunction representation of the kernel.

Theorem 6.8.7. *Assume the space part of the EPD equation has eigenfunction $\phi_\eta(\tau;k)$, with:*

$$\nabla^2\phi_\eta(\tau;k) = -k^2\phi_\eta(\tau;k) \tag{6.297}$$

The EPD equation:

$$\partial_t^2 u + \frac{(1+2\eta)}{t}\partial_t u = \nabla^2 u \tag{6.298}$$

satisfies a kernel decomposition formula:

$$K(t,\tau;x) = \int_{-\infty}^{+\infty} e^{-ikx}\phi_\eta(t;k)\phi_\eta(\tau;k)d\mathfrak{m}(k) \tag{6.299}$$

In the case of the time-EPD equation, we may employ the results from 6.7.13, 6.7.17 to derive this kernel for the system of eigenfunctions given by Bessel functions with replacement of the Fourier exponential transform by either the cosine or sine Fourier transform. The kernel solutions resulting from the the sine and cosine transform, which satisfy the EPD equation with different boundary conditions on the space component are respectively:

$$K(t,\tau;x) = \frac{A}{2(t\tau)^{\eta+1/2}}\mathcal{P}_{\eta-1/2}(\cosh d) \tag{6.300}$$

and

$$K(t,\tau;x) = -\frac{C}{2(t\tau)^{\eta+1/2}}\mathcal{Q}_{\eta-1/2}(\cosh d) \tag{6.301}$$

where the hyperbolic distance is given by

$$\cosh d = \frac{t^2+\tau^2-x^2}{2\tau t} \tag{6.302}$$

the functions $\mathcal{P}_{\eta-1/2}(.)$, $\mathcal{Q}_{\eta-1/2}(.)$ are the associated Legendre functions of Mehler-Fock type, and A, C are constants.

Proof. The kernel for the equation is constructed from the eigenfunctions using the integral expansion 6.7.17. We have:

$$K(t,\tau;x) = \frac{C}{(t\tau)^\eta}\int_{-\infty}^{\infty} e^{-ikx}J_\eta(kt)J_\eta(k\tau)dk \tag{6.303}$$

$$= \frac{2iC}{2(t\tau)^\eta}\int_0^\infty \sin(kx)J_\eta(kt)J_\eta(k\tau)dk \tag{6.304}$$

$$= \frac{A}{2(t\tau)^{\eta+1/2}}\mathcal{P}_{\eta-1/2}(\cosh d) \tag{6.305}$$

where $A = 4iC$ and the distance function in the argument is the hyperbolic distance formula:

$$\cosh d = \frac{t^2+\tau^2-x^2}{2\tau t} \tag{6.306}$$

where we have used the sine integral formula from Oberhettinger [29] eq. II.13.38. This gives the first part of the kernel solution. The second part is readily obtained in a similar fashion, we have the corresponding integral formula from the Hankel-type transform:

$$K(t,\tau;x) = \frac{C}{2(t\tau)^\eta}\int_0^\infty \cos(kx)J_\eta(kt)J_\eta(k\tau)dk \tag{6.307}$$

$$= -\frac{C}{2(t\tau)^{\eta+1/2}}\mathcal{Q}_{\eta-1/2}(\cosh d) \tag{6.308}$$

as required, where we have used integral formula eq. I.13.37 [29]. By inspection, both of these satisfy the time-EPD equation:

$$\partial_t^2 f + \frac{(1+2\eta)}{t}\partial_t f = \nabla^2 f \tag{6.309}$$

□

We find the following corollary, which is a specific example of a more general theory according to the precepts of kernel theory and distance functions.

Corollary 6.8.8. *The kernel solution to the time-EPD equations may be written as:*

$$K(t,\tau;x) = A_{\tau,t}f(d(t,\tau,x)) \tag{6.310}$$

where $d(t,\tau,x)$ is the hyperbolic distance function, $f()$ is the Mehler-Fock function of the first or second kind, $Q_{\eta-1/2}(.)$ or $P_{\eta-1/2}(.)$, and $A_{\tau,t} = -\frac{C}{2(t\tau)^{\eta+1/2}}$, C a constant, depending on the boundary conditions in x.

Proof. This follows as a direct consequence of the preceding theorem, with distance function as defined therein. One can show directly through differentiation under the integral sign that the eigenfunction expansions satisfy the time-EPD equations. □

Remark 120. We may change our position, whether inside or outside the light cone $t^2+\tau^2-x^2$ as implied through the hyperbolic distance, consequently we will alter the solution which depends on the boundary conditions. The relationship that the kernel be a function of distance is a natural consequence of group homomorphism identities on the hyperbolic plane, see [67] for a historical reference, also [14,16] for application in quantum mechanics. These formulae are interesting in that the kernel formulae defined through these expressions are similar in nature to the Green's function, as developed in the path integral formalism in [14,16], see also [56] for a discussion of similar techniques when related to index transforms and diffusion problems, and in this thesis, our work on the hyperbolic heat kernel 7.4.36. Depending on the nature of the boundary conditions and so on we can expect that there exist other complementary solutions defined through the various types of associated Legendre functions. This is quite obvious after consulting tables of Bessel integrals that these types of equations have solutions of this succinct format. It is interesting to think about the nature of transformation in this context. Obviously these results carry over into the (2+1) spacetime as opposed to (1+2) spacetime we have calculated in this chapter.

6.8.2 Lipschitz Conditions and Continuity

The Lipschitz conditions for time dispersive systems are well-defined and give meaningful information. In particular, it is possible to define the parameters which are non-exploding through the first condition. We have the Lipschitz conditions as in 6.7.6:

Lemma 6.8.9. *For the system given by the second order differential equation:*

$$\hat{T}f(t) = \frac{\sigma^2(t)}{2}\frac{\partial^2 f}{\partial t^2} + \mu(t)\frac{\partial f}{\partial t} + c(t)f(t) \tag{6.311}$$

if we have $\sigma(t) = \sqrt{2}$ *and* $\mu(t) = \dfrac{1+2\eta}{t}$ *for the variance and drift, then we have:*

$$\hat{T}f(t) = \frac{\partial^2 f}{\partial t^2} + \frac{(1+2\eta)}{t}\frac{\partial f}{\partial t} + c(t)f(t) \tag{6.312}$$

The Lipschitz conditions satisfy:

$$|\mu(t)|^2 + |\sigma(t)|^2 \leq L(1+|t|^{-2}) \tag{6.313}$$

$$|\mu(t)-\mu(s)| + |\sigma(t)-\sigma(s)| \leq C|t-s| \tag{6.314}$$

i.e. the first of these conditions is singular at $t = 0$*, implying that the equation is not continuous at this point. This can be seen from the drift coefficient which is not defined at zero, in stochastic terms, this point is strongly excluding.*

Proof. This is amenable to direct analysis. We have drift and diffusion in time:

$$\frac{\sigma^2(t)}{2} = 1, \mu(t) = \frac{(1+2\eta)}{t} \tag{6.315}$$

Calculating the Lipschitz conditions directly, we have:

$$|\mu(t)|^2 + |\sigma(t)|^2 = 4 + (1+2\eta)^2 t^{-2} = 4(1 + \frac{(1+2\eta)^2}{4}t^2) \tag{6.316}$$

$$= 4(1+(t\lambda)^{-2}) \leq L(1+|t|^{-2}) \tag{6.317}$$

and likewise for the second condition:

$$|\mu(t)-\mu(s)| + |\sigma(t)-\sigma(s)| = |\mu(t)-\mu(s)| = \left|\frac{(1+2\eta)}{t} - \frac{(1+2\eta)}{s}\right| \tag{6.318}$$

$$= |(1+2\eta)|\left|\frac{1}{t} - \frac{1}{s}\right| = |(1+2\eta)|\left|\frac{(s-t)}{ts}\right| = |(1+2\eta)|(ts)^{-1}|t-s| \tag{6.319}$$

$$\leq C|t-s| \tag{6.320}$$

We also gain the parameter range before blowup, we require λ less than unity, hence:

$$\frac{(1+2\eta)^2}{4} \leq 1 \tag{6.321}$$

$$-2 \leq (1+2\eta) \leq 2 \tag{6.322}$$

□

Remark 121. As we shall see in later sections of this chapter, the systems implied by the EPD equations and their confluent hypergeometric counterparts are in a sense defined by this property. This is discussed further in the conclusions, where we analyse the different continuity conditions that can be associated with the various classes of special functions.

6.8.3 Jacobi Transform

We shall now derive the Green's function, which can be computed from successive Fourier and Laplace transforms of the solution to the EPD equation. As we have already derived the space part of this, in that we have established the Fourier transform as given through the Bessel functions, we only require the further application of a Laplace transform to derive this result. This special solution is given by a formula that is in terms of the hypergeometric function or a generalised Jacobi function.

Lemma 6.8.10. *The Green's function for the time-Bessel EPD equation 6.7.21 is given by the formula:*

$$G(\boldsymbol{k}, E) = \frac{\left(\frac{|\boldsymbol{k}|}{2}\right)^{(1-\alpha)/2}}{E\Gamma\left(\frac{\alpha+1}{2}\right)} {}_2F_1\left[\ [1/2, 1], \quad [(\alpha+1)/2], \quad -\frac{|\boldsymbol{k}|^2}{E^2}\ \right] \tag{6.323}$$

where ${}_2F_1(.)$ *is the hypergeometric function. The Jacobi representation of the solution is:*

$$G(\boldsymbol{k}, E) = \frac{\sqrt{\pi}}{E\Gamma\left(\frac{\alpha}{2}\right)}\left(\frac{2}{|\boldsymbol{k}|}\right)^{(1-\alpha)/2} P_{-1/2}^{[(\alpha-1)/2, 1-\alpha/2]}\left(1+\frac{2|\boldsymbol{k}|^2}{E^2}\right) \tag{6.324}$$

Proof. If we take the solution to the time-EPD equation of Bessel type, we have already shown in this work in 6.8.2 that the solution in momentum space can be written:

$$f(\boldsymbol{k}, t) = Ct^{(1-\alpha)/2} J_{(\alpha-1)/2}(|\boldsymbol{k}|t) \tag{6.325}$$

for the space Hamiltonian operator $\mathscr{H} = \nabla^2$. Writing the Green's function as an integral, we have:

$$G(\boldsymbol{k}, E) = \int e^{-Et} f(\boldsymbol{k}, t) dt \tag{6.326}$$

$$G(\boldsymbol{k}, E) = \frac{\left(\frac{|\boldsymbol{k}|}{2}\right)^{(1-\alpha)/2}}{E\Gamma\left(\frac{\alpha+1}{2}\right)} {}_2F_1\left[\ [1/2, 1], \quad [(\alpha+1)/2], \quad -\frac{|\boldsymbol{k}|^2}{E^2}\ \right] \tag{6.327}$$

Using standard tabulated results (see e.g. [122], eq. 6.621.1 for the hypergeometric function we obtain the second formula. □

Remark 122. Note the useful Laplace transform:

$$\int_0^\infty e^{-Et} J_{k/2-1/2}(\lambda t) dt = \frac{\lambda^{k/2-1/2}}{\sqrt{E^2+\lambda^2}}\left(\sqrt{E^2+\lambda^2}+E\right)^{1/2-k/2} \tag{6.328}$$

which again can be seen as either a Laplace transform, as written, or a Hankel transform via $t = \sqrt{z}, \lambda = \sqrt{w}$. Either of these transforms may be inverted to give representations for the Chebyshev polynomials. These polynomials naturally arise as the characters of spherical groups.

6.9 Generalised EPD Equations

In the same way that the heat kernel has Mehler-Fock, Macdonald and Whittaker eigenfunction solutions in the hyperbolic domain, the time-Bessel EPD equation has a number of generalisations. We present 8 other PDEs of similar type related to the Whittaker, modified Bessel, Laguerre and Legendre functions. We demonstrate through these examples that an alternative schemata of special function theory may be developed in this way. Many generalisations of the EPD equations are easily reached by some simple modifications. In particular, the examples of the Whittaker and Mehler-Fock wave kernels are closely allied with those of the Bessel function. As shown through the projection-slice theorem (see 5.3.8, 5.3.9), these functions arise through integral transforms of Bessel-type equations. We shall consider modifications in the time operator, and save the analysis of a similar change to the Laplacian in the concluding sections of this chapter. We shall list the equations and their associated eigenfunctions. We take the following basic definition for the EPD system:

Definition 6.9.1.

$$\hat{T}(t^i, \partial_t^i)\phi(t;u) = -\lambda^2(u)\phi(t;u) \tag{6.329}$$

Various types of modified EPD systems are given by the following:

6.9.1 Whittaker Type I

Example 6.9.2.

$$-\frac{\partial^2 f}{\partial t^2} - \frac{(2+\alpha)}{t}\frac{\partial f}{\partial t} + (\frac{2}{t} - \frac{1}{q^2})f = 0 \tag{6.330}$$

$$f(t) \sim C\left(\frac{u}{q}\right)^{-\alpha/2-1} W_{iq,(\alpha+1)/2}\left(\frac{2it}{q}\right) \tag{6.331}$$

Depending on the nature of the spectrum, the corresponding Whittaker function is replaced by its equivalent counterpart.

6.9.2 Whittaker Type II

Example 6.9.3.

$$-t\frac{\partial^2 f}{\partial t^2} - \xi^2(\alpha+1)\frac{\partial f}{\partial t} + \xi^2(k^2 t + 2)f = 0 \tag{6.332}$$

$$f(t) = C_1 2^{-(\alpha+1)/2}(\xi k t)^{-(\alpha+1)/2} W_{-\xi/k,\alpha/2}(2\xi k t) \tag{6.333}$$

Remark 123. This type of equation is symmetrical in the variables which occur in the equation, modulo the factors occurring in the Whittaker indices. We shall discuss the reasons for this in the following sections, for our needs it suffices to note that if we transform another differential operator in a similar way to how we transformed the Laplacian, we might expect to end up with equations such as these. This necessarily implies certain relationships for so-called biharmonic equations and similar differential operators. The solutions to these higher-order equations will naturally be given through Fourier inversion of the kernels composed of the eigenfunction solutions, as demonstrated with the Bessel-type example.

6.9.3 Whittaker Type III

Example 6.9.4.

$$-t^2\frac{\partial^2 f}{\partial t^2} + \beta^2\frac{\partial f}{\partial t} + \left(\frac{k^2}{t} + \mu^2 - \frac{1}{4}\right)f = 0 \tag{6.334}$$

$$f(t) = C_1 t e^{-\beta^2/(2t)}(\beta^2)^{-\mu-1/2} W_{-1-(k/\beta)^2,\mu}\left(\frac{\beta^2}{t}\right) \tag{6.335}$$

Remark 124. Note that this special form contains a Gaussian prefactor. This is associated to the Weierstrass transform, which we saw appear in this work in 4.7.16. The relationship between these types of Gaussian factors is related to the transform theory between special functions.

6.9.4 Bessel Type I

This type of modified EPD equation is one way in which we can see the modified Bessel functions arise. These functions are important in the mathematics of index transforms and spectral theory, see e.g. [37,56].

Example 6.9.5.

$$t\frac{\partial^2 f}{\partial t^2} - \alpha\frac{\partial f}{\partial t} - \left(\frac{L^2}{t} + \lambda^2\right) f = 0 \tag{6.336}$$

$$L^2 = -\frac{1}{2}\left(\alpha\left(\frac{\alpha}{2}+1\right) + \frac{\nu^2}{2} + \frac{1}{2}\right) \tag{6.337}$$

$$f(t) = C_1 t^{(\alpha+1)/2} K_{i\nu}(2\lambda\sqrt{t}) \tag{6.338}$$

Remark 125. This may be evaluated using the methods of index transforms, if we take the EPD Laplacian variable to be the index of the function. Craddock [46] has analysed similar types of expressions with regards to mathematical finance and the kernels associated with Asian options. However, if we instead take the variable inside the function to the Fourier variable, we arrive at a much more soluble example of a kernel function we can associate to an EPD equation. We discuss this at the conclusion of this list.

6.9.5 Whittaker Type IV

Example 6.9.6.

$$\frac{\partial^2 f}{\partial t^2} - \alpha\frac{\partial f}{\partial t} - \frac{k}{t} f = 0 \tag{6.339}$$

$$f(t) = C_1 U\left(1 + \frac{k}{\alpha}, 2, \alpha t\right) = C_1 W_{-k/\alpha, 1/2}(t\alpha)e^{t\alpha/2} \tag{6.340}$$

6.9.6 Laguerre Type I

Example 6.9.7.

$$\frac{\partial^2 f}{\partial t^2} - \alpha\frac{\partial f}{\partial t} + \frac{n\alpha}{t} f = 0 \tag{6.341}$$

$$f(t) = C_1 \frac{t}{n} L_1^{(n-1)}(\alpha t) \tag{6.342}$$

Remark 126. The conjugate function in this case is taken by the Kummer-U function. This is interesting, in that for particular values of the constants, we obtain a polynomial solution. Using the theory of the associated Laguerre function, it is possible to see how one may use this type of expression as a starting point for further investigation of higher dimensional waves. In this case, the eigenvalue does not enter as a squared term, only linearly, indicating that the Fourier transform is acting on a derivative of first order.

6.9.7 Legendre Type I

Example 6.9.8.

$$(1-t^2)\frac{\partial^2 f}{\partial t^2} - 2t\frac{\partial f}{\partial t} - \left(\frac{\lambda^2}{1-t^2} + k^2 + \frac{1}{4}\right) f = 0 \tag{6.343}$$

$$f(t) = C\mathcal{P}^{\lambda}_{ik-1/2}(t) \tag{6.344}$$

6.9.8 Legendre Type II

Example 6.9.9.

$$(1-t^2)\frac{\partial^2 f}{\partial t^2} - 2\beta t\frac{\partial f}{\partial t} - \left(\frac{\lambda^2}{1-t^2} + k^2 + \frac{1}{4}\right) f = 0 \tag{6.345}$$

$$f(t) = C(t^2-1)^{1/2-\beta/2}\mathcal{P}^{i\nu}_{ik-1/2}(t) \tag{6.346}$$

$$\beta^2 + \lambda^2 - 2\beta + 1 = -\nu^2 \tag{6.347}$$

$$\beta = 1 \pm i\sqrt{\lambda^2+\nu^2} \tag{6.348}$$

Remark 127. This gives a spherical transform, related to the Gegenbauer functions and projective space (see Hafoud, [84–86]).

Remark 128. One can see from this perspective how the EPD class of equations generates most special functions we can expect to encounter. There are several reasons for this. Firstly, we must point out that there are a limited number of objects with the necessary properties. Given a problem with large enough scope, we can expect to encounter each of the different objects that span the various solution spaces, with each playing a role in turn. It is for this reason that we can expect similar types of equations to define kernels that are important in other contexts, such as in physics or financial engineering. This chapter hopefully shows how many different problems can be approached in a similar fashion; by using kernels, some symmetry properties and integral formulae we are able to derive some very interesting results. This list is by no means exhaustive. We merely state these results as interesting examples of singular differential equations from which one may construct different kernel operators to satisfy the EPD-type relations. Each of these expressions will have an associated time potential, which we can use to read off the eigenvalue equation.

6.9.9 Bessel Kernel

The time-Bessel EPD equation 6.8.2 defines a PDE system, with a kernel solution. This kernel is given by an eigenfunction decomposition. Using known integrals, the kernel is equivalent to the Mehler-Fock solution. We state the following simple lemma which is derived using the eigenfunction expansion of the kernel.

Lemma 6.9.10. *For the case of the Bessel Type I EPD equation, with PDE system defined through:*

$$t\frac{\partial^2 f}{\partial t^2} - \alpha\frac{\partial f}{\partial t} - \left(\frac{L^2}{t}\right) f = -\nabla^2 f \tag{6.349}$$

The solution to this equation is given by the kernel:

$$K(t,\tau;x) = A\frac{\pi^2(t\tau)^{\alpha/2+1/4}}{8\cosh(\pi\nu)}\mathcal{P}_{i\nu-1/2}\left(\frac{t+\tau+x^2/4}{2\sqrt{t\tau}}\right) \tag{6.350}$$

Proof. Taking the eigenfunction decomposition of the solution, we have ODE system:

$$t\frac{\partial^2 f}{\partial t^2} - \alpha\frac{\partial f}{\partial t} - \left(\frac{L^2}{t} + \lambda^2\right) f = 0 \tag{6.351}$$

with eigenfunction solution:

$$f(t) = C_1 t^{(\alpha+1)/2} K_{i\nu}(2\lambda\sqrt{t}) \tag{6.352}$$

We may then take the Fourier-transformed variable to be λ^2 for simplicity, which leads to the PDE system:

$$t\frac{\partial^2 f}{\partial t^2} - \alpha\frac{\partial f}{\partial t} - \left(\frac{L^2}{t}\right) f = -\nabla^2 f \tag{6.353}$$

In an analogous fashion through using the eigenfunction representation, we may write down one sort of kernel solution to this equation in the form:

$$K(t,\tau;x) = A\int_0^\infty \cos(\lambda x)(t\tau)^{(\alpha+1)/2}K_{i\nu}(2\lambda\sqrt{t})K_{i\nu}(2\lambda\sqrt{\tau})d\lambda \tag{6.354}$$

$$= A(t\tau)^{(\alpha+1)/2}\int_0^\infty \cos(\lambda x)K_{i\nu}(2\lambda\sqrt{t})K_{i\nu}(2\lambda\sqrt{\tau})d\lambda \tag{6.355}$$

The following formula for the cosine transform is available, [29] eq 17.23:

$$\int_0^\infty \cos(\lambda y)K_{i\nu}(\lambda a)K_{i\nu}(\lambda b)d\lambda = \frac{\pi^2}{4\cos(\pi i\nu)\sqrt{ab}}\mathcal{P}_{i\nu-1/2}\left(\frac{a^2+b^2+x^2}{2ab}\right) \tag{6.356}$$

$$\int_0^\infty \cos(\lambda x)K_{i\nu}(2\lambda\sqrt{t})K_{i\nu}(2\lambda\sqrt{\tau})d\lambda = \frac{\pi^2}{4\cos(\pi i\nu)\sqrt{4\sqrt{t\tau}}}\mathcal{P}_{i\nu-1/2}\left(\frac{4t+4\tau+x^2}{2.4.\sqrt{t\tau}}\right) \tag{6.357}$$

$$= \frac{\pi^2}{8\cosh(\pi\nu)(t\tau)^{1/4}}\mathcal{P}_{i\nu-1/2}\left(\frac{t+\tau+x^2/4}{2\sqrt{t\tau}}\right) \tag{6.358}$$

We then obtain the following kernel formula:

$$K(t,\tau;x) = A\frac{\pi^2(t\tau)^{\alpha/2+1/4}}{8\cosh(\pi\nu)}\mathcal{P}_{i\nu-1/2}\left(\frac{t+\tau+x^2/4}{2\sqrt{t\tau}}\right) \tag{6.359}$$

which is readily shown to satisfy the PDE system in the lemma. □

Remark 129. In a directly analogous fashion, one is able to find various other representations of kernel solutions, whether by changing the boundary conditions through the eigenfunctions, or modification of the cosine transform to a sine, exponential or otherwise. Such kernels are familiar in another setting, in Grosche [14,16] the authors have discussed a very similar problem related to the path integral on the hyperbolic plane. These types of formulae appear as Green's functions, whereas here they appear as kernels, or as we term them, "wave kernels". This is due to the essential symmetry between (2+1)-spacetime and (1+2)-spacetime.

6.10 Essentially Solvable Operators

We shall now show how this class of equations can be generalised in a different way, and various other kernel formulae can be obtained. The method we shall use is referred to in the literature as that of essentially solvable equations, utilised first in Craddock and Lennox [40–42]. The following calculation demonstrates that the time-Bessel EPD equation further generalises to give a class of essentially solvable operators. We exhibit three of this class, given by the Mehler-Fock, Kontorovich-Lebedev and Whittaker transforms.

For the following, let us assume that we have an operator equation that separates into time and space components as before:

Definition 6.10.1.

$$\mathcal{F}\left[\hat{H}(x^i,\partial_x^i)f(x,t)\right] = -s^2g(s,t) \tag{6.360}$$

$$\hat{T}(t^i,\partial_t^i)g(s,t) = -s^2g(s,t) \tag{6.361}$$

Remark 130. This represents the expression

$$\hat{T}(t^i,\partial_t^i)f(x,t) = \hat{H}(x^i,\partial_x^i)f(x,t) \tag{6.362}$$

in the diagonalised eigenspace.

We shall now show how one may apply this to more complicated examples than the Laplacian as calculated earlier. As a simple demonstration, if we take the modified Bessel equation in the variable x, we will have the index transform satisfying the formula as given.

Theorem 6.10.2. *Assuming an EPD type relation, we have a kernel for the "free" EPD system defined as:*

$$K(t,\tau;s) = \frac{A}{2(t\tau)^\eta}\int_0^\infty \sin(kx)J_\eta(kt)J_\eta(k\tau)dk = \frac{A}{2(t\tau)^{\eta+1/2}}\mathcal{P}_{\eta-1/2}(\cosh d) \tag{6.363}$$

$$\cosh d = \frac{t^2+\tau^2-s^2}{2\tau t} \tag{6.364}$$

which solves:

$$\frac{\partial^2 f}{\partial t^2} + \frac{\alpha}{t}\frac{\partial f}{\partial t} = \nabla^2 f \tag{6.365}$$

where $\alpha = 1+2\eta$.

Proof. This is merely a replication of the results in 6.8.4, 6.8.2. We begin with Fourier transformation of the time-EPD equation:

$$\frac{\partial^2 f}{\partial t^2} + \frac{\alpha}{t}\frac{\partial f}{\partial t} = \nabla^2 f \tag{6.366}$$

which is equal to $-s^2 f$. The result follows automatically as a consequence of the eigenfunction decomposition of the kernel for the EPD equation. □

Remark 131. We can see, therefore, that the nature of the diagonalising transformation and operator on the right-hand side (space dependent) of the EPD equation is important. We have exploited the relationship that effectively the Fourier transform diagonalises the Laplacian. As the following shall show, we can easily generalise this property to gain other types of solutions.

The general theorem is as below.

Theorem 6.10.3. *The essentially exact solution is given through the integral formula:*

$$K(t,\tau;x) = \frac{A}{\pi^2(t\tau)^{\eta+1/2}}\int_0^\infty \phi_s(x)\mathcal{P}_{\eta-1/2}(\cosh d)d\mathfrak{m}(s) \tag{6.367}$$

where $\phi_s(x)$ *is some eigenfunction,* $d\mathfrak{m}(s)$ *the associated measure and* A *is a constant.*

We shall now apply this to some generalisations of the time-Bessel EPD equation. The basic process may be summarised in the following theorem:

Theorem 6.10.4. *Assume the generalised EPD system:*

$$\frac{\partial^2 f}{\partial t^2} + \frac{\alpha}{t}\frac{\partial f}{\partial t} = \hat{\mathcal{L}} f \tag{6.368}$$

where $\mathcal{L}$ *is some operator that acts in the space* x. *Then the solution of this problem is equivalent to the eigenfunction ODE:*

$$\frac{\partial^2 g}{\partial t^2} + \frac{\alpha}{t}\frac{\partial g}{\partial t} = -s^2 g \tag{6.369}$$

where $\mathcal{F}f = g(s,t)$ *and* $f(x,t) = \mathcal{F}^{-1}g$.

Proof. If we take the transform and operator such that

$$\mathcal{F}[\hat{\mathcal{L}}f] = -s^2 g \tag{6.370}$$

we readily obtain the second form of the solution, as the time derivatives commute with the transformation in space. Inverting the transform, we obtain the result. Note that a shifted or generalised eigenvalue of the form $\lambda^2(s)$ (as opposed to strictly s^2, as above), does not change the general methodology. □

We now apply this theorem to three major examples in index transform theory.

6.10.1 Kontorovich-Lebedev Transform

If we consider the replacement of the Fourier transform by its counterparts in hyperbolic space, one simple example is offered by the Kontorovich-Lebedev transform. We have the following result as a simple application of 6.10.4:

Lemma 6.10.5. *The kernel solution to the extended EPD system given by:*

$$\frac{\partial^2 f}{\partial t^2} + \frac{\alpha}{t}\frac{\partial f}{\partial t} = \hat{\mathcal{L}}f \tag{6.371}$$

$$\frac{\partial^2 g}{\partial t^2} + \frac{\alpha}{t}\frac{\partial g}{\partial t} = -s^2 g \tag{6.372}$$

is given by the Kontorovich-Lebedev transformation:

$$K(t,\tau;x) = \frac{A}{\pi^2(t\tau)^{\eta+1/2}}\int_0^\infty K_{is}(x)\mathcal{P}_{\eta-1/2}(\cosh d)\sinh(\pi s)s ds \tag{6.373}$$

Proof. We know that the Kontorovich-Lebedev transform pair is given by:

$$\mathcal{F}f = g(s,t) = \int_0^\infty K_{is}(x)f(x,t)\frac{dx}{x} \tag{6.374}$$

$$f(x,t) = \mathcal{F}^{-1}g = \frac{2}{\pi^2}\int_0^\infty K_{is}(x)g(s,t)\sinh(\pi s)s ds \tag{6.375}$$

It is known (see e.g. [46,147]) that this diagonalises the modified Bessel equation operator.

$$\hat{\mathcal{L}}f = x^2\frac{\partial^2 f}{\partial x^2} + x\frac{\partial f}{\partial x} - x^2 f \tag{6.376}$$

Then we have

$$\mathcal{F}[\hat{\mathcal{L}}f] = -s^2 g \tag{6.377}$$

Inverting the Kontorovich-Lebedev transformation, we obtain the essentially exact solution. □

Remark 132. This technique is immediately applicable to any operator for which we can find an invertible transform pair. This is a doubly difficult integral to evaluate, in that the parameter of integration appears both within the argument of the Legendre function and the index of the Macdonald function. This example leads itself to ready generalisation, and we shall show how other complicated differential equations have solutions that can be written down in a similar fashion.

6.10.2 Mehler-Fock Transform

Given that the Mehler-Fock function appears in multiple contexts throughout this work, the natural generalisation of the essentially exact solution is given through the Mehler-Fock index transform. We take the following definition for the operator $\mathcal{L}$:

Definition 6.10.6. The Mehler-Fock space operator is defined by the differential operator:

$$\hat{\mathcal{L}}f = (x^2-1)\frac{\partial^2 f}{\partial x^2} + 2x\frac{\partial f}{\partial x} - \frac{\mu^2}{(x^2-1)}f \tag{6.378}$$

with eigenfunction equation:

$$\hat{\mathcal{L}}f = -\left(\lambda^2 + 1/4\right) f \tag{6.379}$$

Lemma 6.10.7. *The Mehler-Fock equivalent to 6.10.4, 6.10.5 is given by:*

$$K(t,\tau;x) = C\int_0^\infty \mathcal{P}^{-\mu}_{is-1/2}(x)\mathcal{P}_{\eta-1/2}(\cosh d)\left|\Gamma\left(1/2+\mu+is\right)\right|^2 s\sinh(\pi s)ds \tag{6.380}$$

where $C(t,\tau) = \dfrac{A}{\pi^2(t\tau)^{\eta+1/2}}$ *and* A *is a constant.*

Proof. We take the space part of the EPD equation to be given through the differential operator of the Mehler-Fock eigenfunction:

$$\hat{\mathcal{L}}f = (x^2-1)\frac{\partial^2 f}{\partial x^2} + 2x\frac{\partial f}{\partial x} - \frac{\mu^2}{(x^2-1)}f \tag{6.381}$$

$$\hat{\mathcal{L}}f = -\left(\lambda^2 + 1/4\right) f \tag{6.382}$$

We then employ the index transform pair defined in Sousa and Yakubovich [56], given by:

$$\mathcal{F}f = g(s,t) = \int_0^\infty \mathcal{P}^{\mu}_{is-1/2}(x)f(x,t)dx \tag{6.383}$$

$$f(x,t) = \mathcal{F}^{-1}g = \frac{1}{\pi}\int_0^\infty \mathcal{P}^{\mu}_{is-1/2}(x)g(s,t)\left|\Gamma\left(1/2+\mu+is\right)\right|^2 s\sinh(\pi s)ds \tag{6.384}$$

The equivalent transform in this case is readily evaluated through the application of 6.10.4:

$$K(t,\tau;x) = C\int_0^\infty \mathcal{P}^{-\mu}_{is-1/2}(x)\mathcal{P}_{\eta-1/2}(\cosh d)\left|\Gamma\left(1/2+\mu+is\right)\right|^2 s\sinh(\pi s)ds \tag{6.385}$$

where $C = \dfrac{A}{\pi^2(t\tau)^{\eta+1/2}}$, as required. □

We demonstrate in the following two lemmas a special case of this expression that is amenable to more exact analysis beyond the transform.

Lemma 6.10.8. *The special case of 6.10.7 given by* $\mu = 1/2$ *defines a time kernel of the form:*

$$K(t,\tau;x) = \frac{A}{\pi^2(t\tau)^{\eta+1/2}}\left(\frac{2}{\pi\sinh\xi}\right)^{1/2}\int_0^\infty s\sin(s\xi)\mathcal{P}_{\eta-1/2}(\cosh d)ds \tag{6.386}$$

where the parameters are given through:

$$x = \cosh\xi \tag{6.387}$$

$$\cosh d = \frac{t^2+\tau^2-x^2}{2\tau t} \tag{6.388}$$

Proof. Substitution of $\mu = 1/2$ into the previous lemma yields:

$$K(t,\tau;x) = \frac{A}{\pi^2(t\tau)^{\eta+1/2}} \int_0^\infty \mathcal{P}^{-1/2}_{is-1/2}(x)\mathcal{P}_{\eta-1/2}(\cosh d) \left|\Gamma(1+is)\right|^2 s\sinh(\pi s)ds \quad (6.389)$$

$$= \frac{A}{\pi^2(t\tau)^{\eta+1/2}} \int_0^\infty \mathcal{P}^{-1/2}_{is-1/2}(\cosh\xi)\mathcal{P}_{\eta-1/2}(\cosh d) \left|\Gamma(1+is)\right|^2 s\sinh(\pi s)ds \quad (6.390)$$

$$= \frac{A}{\pi^2(t\tau)^{\eta+1/2}} \int_0^\infty \left(\frac{2}{\pi\sinh\xi}\right)^{1/2} \frac{\sinh(is\xi)}{is}\mathcal{P}_{\eta-1/2}(\cosh d)\frac{\pi s}{\sinh(\pi s)} s\sinh(\pi s)ds \quad (6.391)$$

$$= \frac{A}{\pi^2(t\tau)^{\eta+1/2}} \left(\frac{2}{\pi\sinh\xi}\right)^{1/2} \int_0^\infty s\sin(s\xi)\mathcal{P}_{\eta-1/2}(\cosh d)ds \quad (6.392)$$

as required, the parameters as given above. We have used formulae 14.5.16 from DLMF [8] to evaluate the associated Legendre function of order $-1/2$. □

Remark 133. Our formulae differ from those of Sousa and Yakubovich [56] insofar as the wave equation can be considered different from the heat equation. As we can see, by using fairly rudimentary tools in kernel theory, we are able to cover a swathe of different classes of special functions.

We note the following simple formula for the index transform of the product of Mehler-Fock and modified Bessel functions is easily derived using the Hankel function.

Lemma 6.10.9. *The index transform of the Mehler-Fock function and modified Bessel function of order $2is$ satisfies the integral relation:*

$$\int_0^\infty \mathcal{P}_{is-1/2}(1+2x^2)K_{2is}(y)s\tanh(\pi s)ds = \frac{\pi xyJ_0(xy)}{2\cosh(2\pi y)} \quad (6.393)$$

where $J_0(.)$ is the Bessel function of order zero.

Proof. We have access to the following Hankel transform [28], eq. 1.9.17:

$$\int_0^\infty xJ_0(xy)\mathcal{P}_{is-1/2}\left(1+2x^2\right)dx = \frac{2}{\pi y}\cosh(2\pi y)K_{2is}(y) \quad (6.394)$$

However, instead of assuming this formula as axiomatic, we can derive it in the following way using some simple steps. Taking the general integral formula from Prudnikov [126], Vol. III, 2.17.12.8-9:

$$\int_a^\infty (x+a)^{-\mu/2}J_{-\mu}(b\sqrt{x-a})\mathcal{P}^\mu_\nu\left(\frac{x}{a}\right)dx = -\frac{2^{5/2-\mu}}{\pi}\sqrt{a}b^{\mu-1}\sin(\nu\pi)K_{2\nu+1}\left(b\sqrt{2a}\right) \quad (6.395)$$

which for special case $\mu = 0$ reduces to:

$$\int_a^\infty J_0(b\sqrt{x-a})\mathcal{P}_\nu\left(\frac{x}{a}\right)dx = -\frac{2^{5/2}}{\pi}\frac{\sqrt{a}}{b}\sin(\nu\pi)K_{2\nu+1}\left(b\sqrt{2a}\right) \quad (6.396)$$

and furthermore, if we have $a = 1, b = \dfrac{y}{\sqrt{2}}, \nu = is - 1/2$, we find:

$$\int_1^\infty J_0\left(y\sqrt{\frac{x-1}{2}}\right)\mathcal{P}_{is-1/2}(x)dx = -\frac{2^{5/2}}{\pi}\frac{\sqrt{2}}{y}\sin((is-1/2)\pi)K_{2is}(y) \quad (6.397)$$

Using the complex trigonometric identity which is easily proven:

$$\sin((is-1/2)\pi) = -\cosh(\pi s) \quad (6.398)$$

we find

$$\int_1^\infty J_0\left(y\sqrt{\frac{x-1}{2}}\right)\mathcal{P}_{is-1/2}(x)dx = \frac{8}{\pi y}\cosh(\pi s)K_{2is}(y) \tag{6.399}$$

which can be written in the form:

$$J_0\left(y\sqrt{\frac{x-1}{2}}\right) = \frac{8}{\pi y}\int_0^\infty \mathcal{P}_{is-1/2}(x)s\tanh(\pi s)\cosh(\pi s)K_{2is}(y)ds \tag{6.400}$$

$$= \frac{8}{\pi y}\int_0^\infty \mathcal{P}_{is-1/2}(x)s\sinh(\pi s)K_{2is}(y)ds \tag{6.401}$$

Applying the substitution $x' = \sqrt{\frac{x-1}{2}}$, we have the inversion relations:

$$x'^2 = \frac{1}{2}(x-1) \tag{6.402}$$

$$2x'^2 + 1 = x \tag{6.403}$$

and hence:

$$J_0\left(yx'\right) = \frac{8}{\pi y}\int_0^\infty \mathcal{P}_{is-1/2}(2x'^2+1)s\sinh(\pi s)K_{2is}(y)ds \tag{6.404}$$

Alternatively, using the integral formula that may be found in Prudnikov [126], Vol. III, eq. 2.17.27.16:

$$\int_0^\infty x\sinh(\pi x)K_{2ix}(b)\mathcal{P}^\mu_{ix-1/2}(c)dx = 2^{-3\mu/2-3}\pi(c+1)^{\mu/2}b^{\mu+1}J_{-\mu}(z_-) \tag{6.405}$$

where

$$c > 1, z_- = b\sqrt{\frac{c-1}{2}} \tag{6.406}$$

one finds for the special case $\mu = 0$

$$\int x\sinh(\pi x)K_{2ix}(b)\mathcal{P}_{ix-1/2}(c)dx = \frac{\pi}{8}bJ_0\left(b\sqrt{\frac{c-1}{2}}\right) \tag{6.407}$$

Applying the inverse transform via the Mehler-Fock function, we find immediately:

$$xJ_0(xy) = \frac{2\cosh(2\pi y)}{\pi^2 y}\int_0^\infty \mathcal{P}_{is-1/2}(1+2x^2)K_{2is}(y)\left|\Gamma\left(1/2+is\right)\right|^2 s\sinh(\pi s)ds \tag{6.408}$$

$$= \frac{2\cosh(2\pi y)}{\pi y}\int_0^\infty \mathcal{P}_{is-1/2}(1+2x^2)K_{2is}(y)s\tanh(\pi s)ds \tag{6.409}$$

and hence:

$$\int_0^\infty \mathcal{P}_{is-1/2}(1+2x^2)K_{2is}(y)s\tanh(\pi s)ds = \frac{\pi xyJ_0(xy)}{2\cosh(2\pi y)} \tag{6.410}$$

where we have used the trigonometric substitution for the gamma function, see DLMF 5.4.1 [8]. □

Remark 134. Understanding how these types of kernel formulae come about is of principal importance in laying out the best theory of these objects. We can see the interaction between the various groups of special functions, reflected in the appearance of the different index transforms.

We can demonstrate several other similar formulae of this type, using related integrals from Prudnikov [126]. One formula is supplied, that is similar to that above may be found at Vol. II, 2.17.26.15, where the table entry reads as:

$$\int_0^\infty x\tanh(\pi nx)K_{ilx}(b)\mathcal{P}_{ix-1/2}(c)dx = I_{n,l} \tag{6.411}$$

However, this result is doubtful and does not necessarily lead to new formulae. The following uses the Lommel function to derive a new integral relation of the same type. As this is the only use of the Lommel function in this work, we direct the interested reader to DLMF, 11.9 [8], and also Watson, [180], 10.7–10.75.

Lemma 6.10.10. *The index transform of the Mehler-Fock function and Lommel function of order* $(0, 2ip)$ *satisfies the integral relation:*

$$K_0\left(yx\right) = \frac{4}{y}\int_0^\infty p\tanh(\pi p)\mathcal{P}_{ip-1/2}(2x^2+1)S_{0,2ip}(y)dp \tag{6.412}$$

where $K_0\left(.\right)$ *is the modified Bessel function of order zero,* $S_{0,2ip}\left(.\right)$ *the Lommel function of first argument zero, second argument* $2ip$*.*

Proof. If we take the integral formula from Prudnikov [126], Vol. III, 2.17.15.5:

$$\int_a^\infty (x+a)^{\mu/2}K_\mu(b\sqrt{x-a})\mathcal{P}_\nu^\mu\left(\frac{x}{a}\right)dx = \frac{2^{3/2-\mu}\sqrt{a}}{b^{\mu+1}}S_{2\mu,2\nu+1}(b\sqrt{2a}) \tag{6.413}$$

Substituting the special case $\mu = 0, a = 1, b = \dfrac{y}{\sqrt{2}}$ we find:

$$\int_1^\infty K_0\left(y\sqrt{\frac{y-1}{2}}\right)\mathcal{P}_\nu(x)dx = \frac{2^{3/2}}{\left[y/\sqrt{2}\right]^{-1}}S_{0,2\nu+1}\left(\frac{y}{\sqrt{2}}\cdot\sqrt{2}\right) \tag{6.414}$$

$$= \frac{4}{y}S_{0,2\nu+1}(y) \tag{6.415}$$

and hence we may write:

$$\int_1^\infty K_0\left(y\sqrt{\frac{y-1}{2}}\right)\mathcal{P}_\nu(x)dx = \frac{4}{y}S_{0,2\nu+1}(y) \tag{6.416}$$

implying that we have:

$$\int_1^\infty K_0\left(y\sqrt{\frac{y-1}{2}}\right)\mathcal{P}_{ip-1/2}(x)dx = \frac{4}{y}S_{0,2ip}(y) \tag{6.417}$$

Inversion of the Mehler-Fock transform then gives:

$$K_0\left(y\sqrt{\frac{y-1}{2}}\right) = \frac{4}{y}\int_0^\infty p\tanh(\pi p)\mathcal{P}_{ip-1/2}(x)S_{0,2ip}(y)dp \tag{6.418}$$

and once more, using the substitution into the integral, we find:

$$2x'^2+1 = x \tag{6.419}$$

$$K_0\left(yx'\right) = \frac{4}{y}\int_0^\infty p\tanh(\pi p)\mathcal{P}_{ip-1/2}(2x'^2+1)S_{0,2ip}(y)dp \tag{6.420}$$

as required. □

Finally, we shall comment on the connection between these types of integral formulae and the convolution theory of the modified Bessel function, as discussed in Yakubovich's collated reference [33]. The author in this text has shown the following result (see e.g. formula 3.2 in [33], also Chapter 15, pp 213 in [36]).

Lemma 6.10.11. *The convolution product of any two L^2 functions f, g may be written:*

$$(f \star g)(y) = \frac{1}{2\sqrt{y}} \int_0^\infty \int_0^\infty \exp\left(-y\left(\frac{u^2+v^2}{2uv}\right) - \frac{uv}{2y}\right) f(u)g(v)dudv \tag{6.421}$$

For the special case $f(u) = \sqrt{u}J_0(xu)$ and $g(v) = \frac{1}{\sqrt{v}}$ this reduces to the expression:

$$(f \star g)(y) = \frac{1}{2\sqrt{y}} \int_0^\infty \int_0^\infty \exp\left(-y\left(\frac{u^2+v^2}{2uv}\right) - \frac{uv}{2y}\right) \frac{J_0(xu)}{v}dudv \tag{6.422}$$

$$= \frac{1}{\sqrt{y}} \int_0^\infty K_0(\sqrt{u^2+y^2})J_0(xu)du \tag{6.423}$$

An alternative integral representation may be found:

$$(f \star g)(y) = \frac{1}{2\pi\sqrt{y}} \int_0^\infty \frac{dv}{v} \int_0^{2\pi} d\theta . \frac{K_1\left(\sqrt{y^2+v^2-2ixyv\cos\theta}\right)}{\sqrt{y^2+v^2-2ixyv\cos\theta}} \tag{6.424}$$

where $K_1(.)$ is the modified Bessel function of order 1.

Proof. This is found through direct substitution. The reader is directed to Yakubovich for further details, see e.g. 3.2-3.4, [33].

$$(f \star g)(y) = \frac{1}{2\sqrt{y}} \int_0^\infty \int_0^\infty \exp\left(-y\left(\frac{u^2+v^2}{2uv}\right) - \frac{uv}{2y}\right) f(u)g(v)dudv \tag{6.425}$$

Using the substitutions for the functions to be convolved, we have following Yakubovich [33]:

$$f(u) = \sqrt{u}J_0(xu) \tag{6.426}$$

$$g(v) = \frac{1}{\sqrt{v}} \tag{6.427}$$

and obtain the following:

$$(f \star g)(y) = \frac{1}{2\sqrt{y}} \int_0^\infty \int_0^\infty \exp\left(-y\left(\frac{u^2+v^2}{2uv}\right) - \frac{uv}{2y}\right) \frac{J_0(xu)}{v}dudv \tag{6.428}$$

$$= \frac{1}{\sqrt{y}} \int_0^\infty K_0(\sqrt{u^2+y^2})J_0(xu)du \tag{6.429}$$

The second form of the integral to be obtained may be reached through application of a different integral formula, by rearranging the integrals under Fubini-Tonelli. Then we have:

$$(f \star g)(y) = \frac{1}{2\sqrt{y}} \int_0^\infty \int_0^\infty \exp\left(-y\left(\frac{u^2+v^2}{2uv}\right) - \frac{uv}{2y}\right) \frac{J_0(xu)}{v}dudv \tag{6.430}$$

$$= \frac{1}{2\sqrt{y}} \int_0^\infty \frac{dv}{v} \int_0^\infty du \exp\left(-y\left(\frac{u^2+v^2}{2uv}\right) - \frac{uv}{2y}\right) J_0(xu) \tag{6.431}$$

$$= \frac{1}{2\sqrt{y}} \int_0^\infty \frac{dv}{v} \int_0^\infty du . \exp\left(-\alpha u - \frac{\beta}{u}\right) J_0(xu) \tag{6.432}$$

Using the integral representation of the Bessel function, we find:

$$J_0(xu) = \frac{1}{2\pi}\int_0^{2\pi} e^{ixu\cos\theta} d\theta \tag{6.433}$$

and hence the convolution goes over into:

$$(f \star g)(y) = \frac{1}{2\sqrt{y}}\int_0^\infty \frac{dv}{v}\int_0^\infty du. \frac{1}{2\pi}\int_0^{2\pi} d\theta. \exp\left(-(\alpha - ix\cos\theta)u - \frac{\beta}{u}\right) \tag{6.434}$$

$$= \frac{1}{4\pi\sqrt{y}}\int_0^\infty \frac{dv}{v}\int_0^\infty du \int_0^{2\pi} d\theta. \exp\left(-\alpha' u - \frac{\beta}{u}\right) \tag{6.435}$$

$$\alpha' = \alpha - ix\cos\theta \tag{6.436}$$

Simplifying, we find the result:

$$(f \star g)(y) = \frac{1}{4\pi\sqrt{y}}\int_0^\infty \frac{dv}{v}\int_0^{2\pi} d\theta \int_0^\infty du. \exp\left(-\alpha' u - \frac{\beta}{u}\right) \tag{6.437}$$

$$= \frac{1}{4\pi\sqrt{y}}\int_0^\infty \frac{dv}{v}\int_0^{2\pi} d\theta. 2\sqrt{\frac{\beta}{\alpha'}} K_1(2\sqrt{\alpha'\beta}) \tag{6.438}$$

$$= \frac{1}{2\pi\sqrt{y}}\int_0^\infty \frac{dv}{v}\int_0^{2\pi} d\theta. \sqrt{\frac{\beta}{\alpha'}} K_1(2\sqrt{\alpha'\beta}) \tag{6.439}$$

Resolving the various coefficients, we find:

$$\sqrt{\frac{\beta}{\alpha'}} = \frac{1}{\sqrt{y^2 + v^2 - 2ixyv\cos\theta}} \tag{6.440}$$

and

$$\sqrt{\alpha'\beta} = \sqrt{\frac{y^2}{4} + \frac{v^2}{4} - \frac{ixyv}{2}\cos\theta} \tag{6.441}$$

$$2\sqrt{\alpha'\beta} = \sqrt{y^2 + v^2 - 2ixyv\cos\theta} \tag{6.442}$$

hence obtaining the result:

$$(f \star g)(y) = \frac{1}{2\pi\sqrt{y}}\int_0^\infty \frac{dv}{v}\int_0^{2\pi} d\theta. \frac{K_1\left(\sqrt{y^2 + v^2 - 2ixyv\cos\theta}\right)}{\sqrt{y^2 + v^2 - 2ixyv\cos\theta}} \tag{6.443}$$

as required. Alternative integral forms exist for the Bessel and modified Bessel functions; also, using the derivative formula for the modified Bessel function specified through:

$$f = K_0\left(\sqrt{y^2 + v^2 - 2ixyv\cos\theta}\right) \tag{6.444}$$

and

$$\frac{\partial f}{\partial \theta} = -ivxy\sin\theta \frac{K_1\left(\sqrt{y^2 + v^2 - 2ixyv\cos\theta}\right)}{\sqrt{v^2 + y^2 - 2ixyv\cos\theta}} \tag{6.445}$$

allows simplification of the right-hand side of the index integral. The proof is complete. □

6.10.3 Whittaker Transform

Although at this point it seems possible to keep generating endless classes of different types of different essentially exact solutions, we shall not be carrying out further analysis of this type further than the Whittaker transform [37,56,112,179]. This valuable transform pair is given by the formulae:

Definition 6.10.12. The forward and inverse Whittaker transforms are defined by 5.3.4:

$$\mathcal{F}f = g(s,t) = \int_0^\infty \frac{dx}{x^2} W_{\alpha,is}(x) f(x,t) \tag{6.446}$$

$$f(x,t) = \mathcal{F}^{-1}g = \frac{1}{\pi}\int_0^\infty W_{\alpha,is}(x) g(s,t) \left|\Gamma\left(1/2-\alpha+is\right)\right|^2 s\sinh(2\pi s)ds \tag{6.447}$$

Following the arguments as for the Mehler-Fock and Kontorovich-Lebedev transforms, we have immediately the following kernel solution of essentially exact form:

Lemma 6.10.13. *The essentially exact solution associated to the Whittaker (in space) and time-Bessel EPD equation is given by the kernel:*

$$K(t,\tau;x) = C\int_0^\infty W_{\alpha,is}(x)\mathcal{P}_{\eta-1/2}(\cosh d) \left|\Gamma\left(1/2-\alpha+is\right)\right|^2 s\sinh(2\pi s)ds \tag{6.448}$$

where the distance function is:

$$\cosh d = \frac{t^2+\tau^2-s^2}{2\tau t} \tag{6.449}$$

and the constant $C = \dfrac{A}{\pi(t\tau)^{\eta+1/2}}$.

Proof. This follows an identical method to 6.10.5, 6.10.7 with replacement of the transform in 6.10.4 by the Whittaker index transform. □

Remark 135. We note the relative paucity of cosine transforms and the like related to the Whittaker function, which causes many essential difficulties in computing these formulae exactly (hence 'essentially' exact). For that reason we discuss it briefly as it can be related to one of the EPD equations and various other problems in pricing Asian options. Given the cosine transform, [29]:

$$\int_0^\infty W_{\mu,ip}(2u)\cos(py)dp = \sqrt{\frac{\pi u}{2}}2^{-\mu}\exp\left(-u\sinh^2\frac{y}{2}\right)D_{2\mu}\left(\cosh\frac{y}{2}\sqrt{2u}\right) \tag{6.450}$$

we can use the cosine convolution theorem to derive the following product formula:

$$\int_0^\infty W_{\mu,ip}(2a)W_{\mu,ip}(2b)\cos(py)dp = \mathcal{F}[A.B] = \mathcal{F}[A]\star\mathcal{F}[B] \tag{6.451}$$

As an area of future investigation this seems to be ripe for numerical testing, as the left-hand side of the cosine transform is computationally intensive to calculate. Parabolic cylinder functions are much more straightforward to evaluate than index Whittaker integrals, so this may prove to be a useful method for pricing of Asian options and other related instruments.

This concludes our discussion of essentially exact solutions. We can see how for systems of the particular configuration we have chosen, it is possible to arrange the variables in the space and time dimensions so they do not interfere. This property is crucial to the successful operation of this calculation.

6.11 Classification of Singular Hyperbolic Equations

We shall now discuss some simple ways in which we can classify the various different differential systems considered in this chapter. In the same way that the Pearson classification scheme may be applied to define classes of exponential families related to probability distributions, the Bose invariant may be used to define a classification scheme for the time-EPD equation and its generalisations. This may be achieved using the discriminant and the commutator that emerge from the invariant potential. We shall derive the Pearson classification system using the following:

Lemma 6.11.1. *Assume the equivalent time dispersive systems, given by either the wave equation with potential or the second order in time operator equation:*

$$\frac{\partial^2 f}{\partial t^2} + (V(t) - E)f(x,t) = 0 \tag{6.452}$$

$$a(t)\partial_t^2 u + b(t)\partial_t u + c(t)u = 0 \tag{6.453}$$

$$V(t) - E = \frac{4ac - b^2 + 2(ba' - ab')}{4a^2} \tag{6.454}$$

where the equivalence is given through the Bose invariant potential. In the regime where $f(x,t)$ does not change too quickly:

$$\frac{f''}{f} << \left(\frac{f'}{f}\right)^2 \tag{6.455}$$

and assuming there exists a suitable linearisation of the potential:

$$\sqrt{V(t) - E} = \frac{\sum a_n (t-\mu)^n}{\sum b_n (t-\mu)^n} \tag{6.456}$$

the values of a_n, μ define a classification system for the PDE system.

Corollary 6.11.2. *The Pearson classification system is defined by the special case of 6.11.1 with the linear-quadratic approximation:*

$$\sqrt{V(t) - E} \sim -\frac{a + (t-\mu)}{b_0 + b_1(t-\mu) + b_2(t-\mu)^2} \tag{6.457}$$

Proof. Observe that the Bose invariant potential may be written 3.5.11:

$$V(t) - E = \frac{4ac - b^2 + 2(ba' - ab')}{4a^2} \tag{6.458}$$

If we examine the potential equation, using what amounts to a Schwartzian derivative, or in quantum mechanics the so-called "quantum potential", or Hamilton-Jacobi equations, we may write out the expression for the diagonalised solution in the form:

$$\frac{f''}{f} = (V(t) - E) \tag{6.459}$$

$$\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 = \frac{\partial^2}{\partial t^2}(\ln f) \tag{6.460}$$

$$\frac{f''}{f} = \frac{\partial^2}{\partial t^2}(\ln f) + \left(\frac{f'}{f}\right)^2 \tag{6.461}$$

$$\frac{\partial^2}{\partial t^2}(\ln f) + \left(\frac{f'}{f}\right)^2 = (V(t) - E) \tag{6.462}$$

Applying the axiom from the lemma, we have the approximation in the regime where $f(t)$ does not change too quickly:

$$\left(\frac{f'}{f}\right)^2 \sim (V(t) - E) \tag{6.463}$$

which we may rewrite as:

$$\frac{f'}{f} \sim \sqrt{V(t) - E} \tag{6.464}$$

Linearisation of the right-hand side results in the following series approximation:

$$\sqrt{V(t) - E} = \frac{\sum a_n (t-\mu)^n}{\sum b_n (t-\mu)^n} \tag{6.465}$$

and taking the linear-quadratic approximant, we find:

$$\sqrt{V(t) - E} \sim -\frac{a + (t-\mu)}{b_0 + b_1(t-\mu) + b_2(t-\mu)^2} \tag{6.466}$$

which is Pearson's classification system. □

Pearson's system of classification is known historically, and many useful distributions familiar to probability theory fall under this schemata. However, we shall pursue a more direct way in which we may approach the question of classification. Essentially, as the calculation in defining the change between the second order system and the wave equation with potential is given by the Bose invariant, it makes sense to do a basic classification in terms of the discrimination and commutative factors which appear in this potential. The following is known from our prior calculations 3.5.11:

Theorem 6.11.3. *The Bose invariant potential for the system defined by the second order ODE is given by:*

$$V(t) - E = \frac{4ac - b^2 + 2(ba' - ab')}{4a^2} \tag{6.467}$$

where the ODE system is defined through either the diagonalised form:

$$\frac{\partial^2 f}{\partial t^2} + (V(t) - E) f(x,t) = 0 \tag{6.468}$$

or the second order ODE in time:

$$a(t)\partial_t^2 u + b(t)\partial_t u + c(t) u = 0 \tag{6.469}$$

The component parts of this potential are given by the parabolic discriminant:

$$\Delta = b^2 - 4ac \tag{6.470}$$

and the commutator:

$$[a, b] = ba' - ab' \tag{6.471}$$

where [,] is the differential commutator. The discriminant and the commutator can be used to form a classification system for the different PDE systems given by the various types of generalised EPD equations.

Remark 136. These two factors (discriminant and commutator) determine the form of the time potential we can associate with each of these hyperbolic equations. Leonenko et. al in [159–162] have utilised similar classification systems for the analysis of some probability density functions.

We now apply this classification scheme with appropriate modifications for our family of PDEs generated by the Euler-Poisson-Darboux equation.

Theorem 6.11.4. *Classification of differential systems that generalise from the EPD equation in time.*

Function	Bose Potential	Discriminant	Commutator
Whittaker Type I	$\dfrac{1}{q^2} - \dfrac{2}{t} + \dfrac{C_1(\alpha)}{t^2}$	$\dfrac{a_1}{t^2} + \dfrac{a_2}{t} + a_3$	$-\dfrac{(2+\alpha)}{t^2}$
Whittaker Type II	$-k^2\xi^2 - \dfrac{2\xi^2}{t} + \dfrac{C_2(\alpha)}{t^2}$	$a_1t^2 + a_2t + a_3$	$\xi^2(\alpha+1)$
Whittaker Type III	$\dfrac{-\mu^2 + 1/4}{t^2} - \dfrac{(\beta^2+k^2)}{t^3} - \dfrac{\beta^4}{4t^4}$	$a_1t^2 + a_2t + a_3$	$-2t\beta^2$
Bessel Type I	$\dfrac{\lambda^2}{t} + \dfrac{\left(L^2 - \dfrac{\alpha^2}{4} - \dfrac{\alpha}{2}\right)}{t^2}$	$a_1t + a_2$	$-\alpha$
Whittaker Type IV	$-\dfrac{k}{t} - \dfrac{\alpha^2}{4}$	$\alpha^2 + \dfrac{4k}{t}$	0
Laguerre Type I	$\dfrac{n\alpha}{t} - \dfrac{\alpha^2}{4}$	$\alpha^2 - \dfrac{4\alpha n}{t}$	0
Legendre Type I	$\dfrac{1+C_3}{u^2} + \dfrac{C_4}{u}$	$b_1t^2 + b_2$	$2(3t^2-1)$
Legendre Type II	$\dfrac{C_5}{u^2} + \dfrac{C_6}{u}$	$q_1t^2 + q_2$	$2\beta(3t^2-1)$

Remark 137. Note that $u = 1-t^2$ in this table for neatness; we have also collected constant factors together to see the functional form. We can see that by classification of the basic parts of the potential into linear, quadratic, inverse quadratic and generalised inverse quadratic we can recover most of these different systems. It is interesting to contrast this with Pearson's classification system; indeed, one can hope to recover a similar set of expressions for the probability density functions that are derivable from the kernels of these differential equations that are comparable. This table is by no means exhaustive, we merely include it to illustrate the families of PDEs encountered in this chapter.

6.12 Comments

It is well known [18,21], that the solution to the Cauchy problem specified through

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2} - V(x)f(x,t) = -\frac{\partial f}{\partial t} \tag{6.472}$$

where the solution is given through the expectation value:

$$f(x,t) = \mathbb{E}\left[\exp\left(-\int_t^T V(X_s,s)ds\right)\phi(X_T)\,|X_t = x\right] \tag{6.473}$$

$$f(x,T) = \phi(X_T) \tag{6.474}$$

However, in this chapter we have mainly dealt with systems of the type:

$$\frac{\partial^2 f}{\partial t^2} + V(t)f(t;x) = 0 \tag{6.475}$$

It is interesting to think about the differences and similarities between such systems and those of Feynman-Kac. In a system which is dispersive, such as the heat equation, any disturbance propagates instantaneously throughout the system. We can think of this as a reflection of the entropy principle; in a system where heat is being measured, we don't necessarily care where heat "goes", other than it left the object by some means. This is an example of a parabolic differential equation. However, in a wave-equation scenario, and assuming we are talking of the classical wave equation, then all disturbances propagate at a finite speed. Contrasting this with equations such as the time-dispersive equation above, which can be thought of as essentially a generalised EPD equation. We know that this equation has a solution that, up to some integral transform, is equivalent to the heat equation solution by the results of [98]. One would therefore think that the time dispersive systems behave more like "heat" than like "waves". We can see this reflected through the difference in solutions between the Bessel-type kernels and the toroidal harmonics.

CHAPTER 7

The Weierstrass Transform

7.1 Introduction and Review

The Weierstrass transform provides another perspective related to the hyperbolic heat kernel, as explored in the preceding chapters. As we have shown in these calculations, there exists a formulation of the heat kernel on these hyperbolic spaces that may be described through the use of a convolution operator and integral transform based on the Gaussian function. In the following chapter, we analyse an alternative description of the hyperbolic heat kernel using the method of the solution operator.

Heaviside, in [100,101] (c. 1893-94) was the original proponent of the method of the solution operator; in developing the theory of the Laplace transform, he was able to derive the now well-known relationship between differential and algebraic equations. At the time, this somewhat novel concept was a source of much controversy. In this chapter, we show how the link between various representations of the hyperbolic plane and heat kernel can be derived using some techniques from operator calculus. The basis of these calculations is the simple observation that the solution space of differential operators may be found by extending the argument of functions from numbers to operators. In many ways this is similar to the exponentiation of matrices; indeed, by treating the operator as an effective constant, we are able to recover much of the structure of the solution.

Hirschmann and Widder [95,96] (Chapter VIII, sections inclusive) outline a theory of the Weierstrass transform based on the infinite product methodology. The authors in this text derive the major results required for the definition of the Weierstrass transform theory, including existence, uniqueness, inversion and convolution theorems. In this chapter, we show how these results can be understood from an alternative perspective, and derive the major results using an eigenfunction theory of the hyperbolic wave equation equivalent to the heat kernel. Further, we analyse the relationship between the heat, wave and Euler-Poisson-Darboux equations and show how the transformation theory of eigenfunctions can be applied in this context. In doing so, we show how the eigenfunction decompositions of the kernel explored in Chapters 1-3 in this text may be generalised to give a number of different analogous eigenfunction systems related to special function theory.

We note that analysis of the Weierstrass transform may be found in both Karunakaran [94] and Ditzian [97], in particular relating to the convolution theorem and development of the inversion theorem of this transform. Ditzian [97] has analysed the convergence and conditional convergence of this transform, and we shall take many of the basic structures as given in order to derive our results in this chapter.

7.2 Outline of Chapter

In the following chapter, we analyse some complimentary problems related to the hyperbolic heat kernel, which are given by the wave kernel equivalent and the Euler-Poisson-Darboux equation. We first derive the solution operator form of the wave kernel, and demonstrate how this extends to a convolution theorem for the Weierstrass transform. We give a number of simple applications that allow the derivation of the Yakubovich heat kernel, sine/cosine transforms of the hyperbolic heat kernel and the basic theory of the Green's function in the context of the hyperbolic heat kernel and wave equation. A brief discussion of some orthogonal functions related to the Hermite polynomials is given, and we show the relationship between these polynomials and the Gaussian heat kernel as is well known from the results of Mehler and Feynman.

Following these brief introductory calculations, we show how the solution operator method may be applied within the context of the hyperbolic wave kernel. Using advanced methods based on the theory of eigenfunction decomposition and special functions, we demonstrate how the Weierstrass transform gives the connection between the wave, heat and Euler-Poisson-Darboux (EPD) equation. This gives rise to a natural generalisation of the EPD equation, related to a two-time transition probability density that gives the kernel solution. We show how this is related to a special decomposition of the associated Legendre functions over the Bessel functions, and comment on the relationship between these systems and the heat and wave equations. We conclude with a close analysis of the Euler-Poisson-Darboux equation, and prove that the solution to this equation can be recast in terms of a hyperbolic PDE by use of a combined Fourier-Laplace transform of the original system.

As we have shown in the preceding chapters, the theory of the hyperbolic heat kernel implies the existence of a general integral transform that is based on the Gaussian function. Gaussian integrals have a deep relation to the structure of the heat kernel. As the following calculations show, the observations given by McKean in [2] may be derived using an alternative methodology based upon the solution operator. The solution operator extends the argument of a function to operators, and is consistent with the philosophy of Heaviside [100,101], who originated this method to describe the solution via Laplace transforms. In using a Laplace transform, it is well known that the Laplace transform converts differential equations into algebraic equations. Heaviside's observation that one may treat the operator as an effective constant allows this transition, which follows from the solution of these algebraic equations.

7.3 Weierstrass Transform and the Heat Kernel

We shall begin with some simple definitions that follow from [98,99], see also [94] for an advanced discussion on the topic. Briefly, our need stems from the necessity of finding solutions to Gaussian integrals, as in 4.7.16. We take in the following our fundamental definition:

Definition 7.3.1. The kernel equation in x, y and t can be written:

$$u(x,t) = \int k(x,y;t) f(y) dy \tag{7.1}$$

where $u(x,t)$ is the solution to the system, $k(x,y;t)$ is the kernel of the heat equation (or PDE system more generally) and $f(y)$ is the initial condition. For the following consider the specific example of a one dimensional heat equation $u_t = u_{xx}$.

We have the following simple application of this definition:

Lemma 7.3.2. *The one-dimensional heat kernel is:*

$$k(x,y;t) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} \tag{7.2}$$

and is the solution to the heat equation $u_{xx} = u_t$ with initial condition $u(x,0) = \delta(x)$. Then the solution to the initial value problem $u(x,0) = f(x)$ may be defined in terms of the integral transform:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4t} f(y)dy \tag{7.3}$$

where we call the action taking $f(y)$ into $u(x,t)$ the Weierstrass transform.

We have the following result for the distribution $k(x,y;t)$, which gives a vector law for the mean and covariance matrix.

Lemma 7.3.3. *The kernel of the Weierstrass transform is normalised in the variables x and y:*

$$\int_{-\infty}^{+\infty} k(x,y;t)dx = \int_{-\infty}^{+\infty} k(x,y;t)dy = 1 \tag{7.4}$$

The mean vector of this quasidistribution function is given by $z' = \hat{X}z$

$$\hat{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{7.5}$$

where the vector is given by:

$$z = \begin{bmatrix} x \\ y \end{bmatrix} \tag{7.6}$$

The variances are given by:

$$\int_{-\infty}^{+\infty} x^2 k(x,y;t)dx - \left(\int_{-\infty}^{+\infty} xk(x,y;t)dx\right)^2 = \langle x^2 \rangle - \langle x \rangle^2 = 2t \tag{7.7}$$

and similarly for y.

Proof. The normalisation of the quasidistribution follows from $k(x,y;t) = k(y,x;t)$, and the standard integral of the Gaussian distribution. Similarly, for the mean, we can extract the values using standard integrals. The variance follows as an application of the basic results for variance of a distribution.

□

7.3.1 Kernel Inversion Formula

Let us now examine the question of the kernel inversion formula. A known result [94], gives the following transform pair:

Lemma 7.3.4. *The forward and inverse Weierstrass transforms are defined by the integral formulae:*

$$\hat{\phi}(w) = \int_{-\infty}^{+\infty} k(w-z;1)\phi(z)dz \tag{7.8}$$

$$\phi(z) = \lim_{t\to 1^-} \int_{-\infty}^{+\infty} k(\xi + iz, t)\hat{\phi}(i\xi)d\xi \tag{7.9}$$

$$= \lim_{t\to 1^-} \frac{1}{i} \int_{-i\infty}^{+i\infty} k(-i\xi + iz, t)\hat{\phi}(\xi)d\xi \tag{7.10}$$

where $k(x,y;t)$ is the Gaussian kernel as defined in 7.3.1.

We shall sketch a simple proof of this fact below.

Proof. In either case, we need to show that the function

$$u(z,t) = \lim_{t\to 1^-} \frac{1}{i}\int_{-i\infty}^{+i\infty} k(-i\xi + iz, t)\hat{\phi}(\xi)d\xi \tag{7.11}$$

has the property

$$\phi(z) = \lim_{t\to 1^-} u(z,t) \tag{7.12}$$

$$\hat{\phi}(i\xi) = \int_{-\infty}^{+\infty} k(i\xi - s; 1)\phi(s)ds \tag{7.13}$$

$$= \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{+\infty} \exp\left(-\frac{(i\xi - s)^2}{4}\right)\phi(s)ds \tag{7.14}$$

This is readily reduced to the following two dimensional integral:

$$u(z,t) = \frac{1}{\sqrt{4\pi}}\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left(-\frac{(\xi + iz)^2}{4t}\right)\exp\left(-\frac{(i\xi - s)^2}{4}\right)\phi(s)dsd\xi \tag{7.15}$$

$$= \frac{1}{4\pi\sqrt{t}}\int_{\mathbb{R}^2} \exp\left(-\frac{\xi'^2}{4t}\right)\exp\left(-\frac{(i\xi' - s')^2}{4}\right)\phi(s' + z)ds'd\xi' \tag{7.16}$$

where the last step has used time translation in the integration variables to simplify the expression, thereby moving the convolution inside the boundary condition. Switching the order of integration, we recover:

$$u(z,t) = \frac{1}{4\pi\sqrt{t}}\int_{-\infty}^{+\infty} ds'.\phi(s' + z)\int_{-\infty}^{+\infty} d\xi'.\exp\left(-\frac{\xi'^2}{4t}\right)\exp\left(-\frac{(i\xi' - s')^2}{4}\right) \tag{7.17}$$

The inner integral may be rearranged to find the Fourier transform of a Gaussian viz.:

$$\int_{-\infty}^{+\infty} d\xi'.\exp\left(-\frac{\xi'^2}{4t}\right)\exp\left(-\frac{(i\xi' - s')^2}{4}\right) \tag{7.18}$$

$$= \int_{-\infty}^{+\infty} d\xi'.\exp\left(\frac{is'\xi'}{2} - \frac{s'^2}{4} + \frac{1}{4}\left(\frac{t-1}{t}\right)\xi'^2\right) \tag{7.19}$$

$$= \exp\left(-\frac{s'^2}{4}\right)\int_{-\infty}^{+\infty} d\xi'.\exp\left(\frac{is'\xi'}{2}\right)\exp\left(+\frac{1}{4}\left(\frac{t-1}{t}\right)\xi'^2\right) \tag{7.20}$$

$$= 2\exp\left(-\frac{s'^2}{4}\right)\int_{-\infty}^{+\infty} d\eta.\exp\left(is'\eta\right)\exp\left(-\left(\frac{1-t}{t}\right)\eta^2\right) \tag{7.21}$$

Collating constants and factors, we find the simplified expression:

$$u(z,t) = C\int_{-\infty}^{+\infty} ds'.\phi(s' + z)\exp\left(-\frac{s'^2}{4}\right)\int_{-\infty}^{+\infty} d\eta.\exp\left(is'\eta\right)\exp\left(-\left(\frac{1-t}{t}\right)\eta^2\right) \tag{7.22}$$

where $C = \dfrac{1}{2\pi\sqrt{t}}$. Evaluating the inner integral on the interval where $t < 1$ strictly, we find:

$$\int_{-\infty}^{+\infty} d\eta.\exp\left(is'\eta\right)\exp\left(-\left(\frac{1-t}{t}\right)\eta^2\right) = \int_{-\infty}^{+\infty} d\eta.\exp\left(is'\eta\right)\exp\left(-\lambda\eta^2\right) = f(s') \tag{7.23}$$

Using results from standard integral theory, we write:

$$\int_{-\infty}^{+\infty} d\eta . \exp\left(is'\eta\right)\exp\left(-\lambda\eta^2\right) = \sqrt{\frac{\pi}{\lambda}} e^{-s'^2/4\lambda} \tag{7.24}$$

and hence we obtain, after replacing the dummy variables:

$$u(z,t) = \frac{1}{2\pi\sqrt{t}}\sqrt{\frac{\pi}{\lambda}} \int_{-\infty}^{+\infty} \phi(s+z)\exp\left(-\frac{s^2}{4} - \frac{s^2}{4\lambda}\right) ds \tag{7.25}$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \phi(s+z) e^{-s^2/4}\omega_\lambda(s) ds \tag{7.26}$$

where the function inside the integral is defined through

$$\omega_\lambda(s) = \frac{1}{\sqrt{2\lambda}} e^{-s^2/4\lambda} \tag{7.27}$$

Taking the limit, we have

$$\lim_{t\to 1^-}\left(\frac{1-t}{t}\right) = \lim_{t\to 1^-} \lambda = 0^+ \tag{7.28}$$

$$u(z,t) = \int_{-\infty}^{+\infty} \phi(s+z) e^{-s^2/4}\omega_\lambda(s) ds \tag{7.29}$$

Using the known initial condition for a point source, we may write:

$$\omega_1(s) = \delta(s-0) \tag{7.30}$$

hence obtaining easily the property:

$$u(z,1) = \int_{-\infty}^{+\infty} \phi(s+z) e^{-s^2/4}\delta(s-0) ds \tag{7.31}$$

$$= \phi(0+z) e^{-0^2/4} = \phi(z) \tag{7.32}$$

□

This argument, adapted from [94,98] is sufficient to establish the existence of the forward and inverse Weierstrass transform.

7.3.2 Generalised Weierstrass Transform

We shall now make some more general statements regarding the kernel and kernel inverse, as derived in the previous section. Following the results derived in 7.3.4, 7.3.2, we have the expressions for the transform and inverse as:

Definition 7.3.5. The inverse Weierstrass transform can be written as the integral kernel:

$$\lim_{t\to 1^-} \frac{1}{i}\int_{-i\infty}^{+i\infty} \mathcal{G}(w-z;1)\hat{\phi}(w) dz = \phi(z) \tag{7.33}$$

where the forward transform is given by:

$$\hat{\phi}(w) = \int_{-\infty}^{+\infty} k(w-z;1)\phi(z) dz \tag{7.34}$$

and the Weierstrass kernel is as defined in 7.3.1.

Lemma 7.3.6. *The kernel of the inverse Weierstrass transform as in 7.3.5 is given by the formula:*

$$\mathcal{G}(w, z; t) = \frac{1}{i\sqrt{4\pi t}} e^{(z-w)^2/4t} \tag{7.35}$$

Proof. Using the results of the calculated kernels, we also have demonstrated that the transform pair may explicitly be written in the form:

$$\hat{\phi}(w) = \int_{-\infty}^{+\infty} k(w - z; 1)\phi(z)dz \tag{7.36}$$

$$\phi(z) = \lim_{t \to 1^-} \frac{1}{i} \int_{-i\infty}^{+i\infty} k(-iw + iz; t)\hat{\phi}(w)dw \tag{7.37}$$

We therefore can conclude that

$$\mathcal{G}(w - z; 1) = \frac{1}{i} \lim_{t \to 1^-} k(-iw + iz; t) = \frac{1}{i} k(-iw + iz; 1) \tag{7.38}$$

$$= \frac{1}{i} k(i[z - w]; 1) = \frac{1}{i\sqrt{4\pi}} e^{(z-w)^2/4} \tag{7.39}$$

in line with known results for the Weierstrass transform pair. Depending on the nature of the function, we may move between these two different representations by a rotation in the complex plane. The kernel inverse is simply found via substitution, giving:

$$\mathcal{G}(w, z; t) = \frac{1}{i\sqrt{4\pi t}} e^{(z-w)^2/4t} \tag{7.40}$$

as required. □

We have the following simple corollary as an application of the inverse kernel formula.

Corollary 7.3.7. *The symmetry relations between the forward and inverse kernels take the form of time translation invariance:*

$$\mathcal{G}(x, y; t) = \mathcal{G}(x - y; t) \tag{7.41}$$

$$k(x, y; t) = k(x - y; t) \tag{7.42}$$

Inversion symmetry:

$$k(z; t) = k(-z; t) \tag{7.43}$$

and complex inversion:

$$k(iz; t) = \frac{1}{\sqrt{4\pi t}} e^{z^2/4t} = i\mathcal{G}(z; t) \tag{7.44}$$

Proof. These all follow as an application of 7.3.5, with the results in 7.3.6. Writing the complex variable:

$$k(z; t) = \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} \tag{7.45}$$

it is simple to see that we have e.g.

$$k(iz; t) = \frac{1}{\sqrt{4\pi t}} e^{z^2/4t} = i\mathcal{G}(z; t) \tag{7.46}$$

from which the results in the corollary follow. □

We now state some simple definitions of the convolution theory for the Gaussian type integrals as given by the forward and inverse Weierstrass transforms.

Definition 7.3.8. The convolution integral for the Weierstrass transform is defined by the following star-product formula:

$$\int_{-\infty}^{+\infty} e^{-w^2/4\tau} f(w)g(y-w)dw = \int_{-\infty}^{+\infty} a(w)g(y-w)dw = (a \star g)(y) \tag{7.47}$$

where we take $a(w) = e^{-w^2/4\tau} f(w)$.

We also have the following theorem from [94], where the authors identify the product relationships for the Weierstrass convolution.

Theorem 7.3.9. *The product formula for convolutions [94] reads as:*

$$(a \star b^{(n)})(y) = (a^{(n)} \star b)(y) = (a \star b)^{(n)}(y) \tag{7.48}$$

where the convolution product is defined through:

$$(a \star b)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(s)b(y-s)ds \tag{7.49}$$

The convolution product is associative and commutative by nature of the shift invariance of the real line. Standard Fourier theorems apply regarding products of transforms via the homomorphism:

$$\mathcal{F}[a \star b] = \mathcal{F}[a]\,\mathcal{F}[b] \tag{7.50}$$

$$\mathcal{F}[ab] = \mathcal{F}[a] \star \mathcal{F}[b] \tag{7.51}$$

We shall now demonstrate some general applications of convolution theory to the analysis of exact solutions for various hyperbolic kernels and Green's functions. In doing so, we shall apply the methodology of projection/slice type decompositions of integrals, a clever implementation of product formulae, and the repeated use of the Fubini-Tonelli theorem. This will reduce many of the integrals we found before when analysing the hyperbolic heat kernel into a Weierstrass transform of various types.

7.3.3 Sine and Cosine Transform of Yakubovich Kernel

We shall state a new result that may be developed using a form of the convolution theorem for the Macdonald functions, which may be reduced to a certain type of Gaussian integral related to the Weierstrass transform. We shall prove the following lemma.

Lemma 7.3.10. *The Yakubovich kernel [46] is given by the integral:*

$$K_Y(x,y;t) = \frac{2}{\pi^2} \int_0^{\infty} e^{-p^2 t} K_{ip}(x) K_{ip}(y) p \sinh(\pi p) dp \tag{7.52}$$

which, following the arguments in 5.6.2, satisfies the hyperbolic heat equation in the Bessel form $u_t = x^2 u_{xx} + x u_x - x^2 u$. This kernel has Weierstrass representation:

$$K_Y(x,y;t) = \frac{e^{\pi^2/4t}}{4(\pi t)^{3/2}} \int_0^{\infty} dz. e^{-z^2/4t} K_0\left(\sqrt{x^2+y^2+2xy\cosh z}\right) g(z,t) \tag{7.53}$$

where $K_0(x)$ is the modified Bessel function of zeroth order and

$$g(z,t) = \left[\pi \cos\left(\frac{z\pi}{2t}\right) - z \sin\left(\frac{z\pi}{2t}\right)\right] \tag{7.54}$$

Proof. We begin with the formula from the lemma for the Yakubovich kernel.

$$K_Y(x,y;t) = \frac{2}{\pi^2}\int_0^\infty e^{-p^2t}K_{ip}(x)K_{ip}(y)p\sinh(\pi p)dp \tag{7.55}$$

Consulting tables of integrals [31](eq. 1.19.14, pp99), also see discussion of the scattering theory of Oberhettinger (this work 5.6.1, original text at [72]), we find the transform pair:

$$K_{ip}(x)K_{ip}(y) = \int_0^\infty \cos(pz)K_0\left(\sqrt{x^2+y^2+2xy\cosh z}\right)dz \tag{7.56}$$

Applying the inverse cosine transform, we evaluate the product inside the integral:

$$K_Y(x,y;t) = \frac{2}{\pi^2}\int_0^\infty dp.p\sinh(\pi p)e^{-p^2t}\int_0^\infty dz.\cos(pz)K_0\left(\sqrt{x^2+y^2+2xy\cosh z}\right) \tag{7.57}$$

$$= \frac{2}{\pi^2}\int_0^\infty dz.K_0\left(\sqrt{x^2+y^2+2xy\cosh z}\right)\int_0^\infty dp.p\sinh(\pi p)\cos(pz)e^{-p^2t} \tag{7.58}$$

under the conditions of the Fubini-Tonelli theorem. Using the known integral:

$$\int_0^\infty dp.p\sinh(\pi p)\cos(pz)e^{-p^2t} = \frac{\sqrt{\pi}}{4t^{3/2}}\exp\left(\frac{\pi^2-z^2}{4t}\right)\left[\pi\cos\left(\frac{z\pi}{2t}\right) - z\sin\left(\frac{z\pi}{2t}\right)\right] \tag{7.59}$$

we arrive at

$$K_Y(x,y;t) = \frac{e^{\pi^2/4t}}{4(\pi t)^{3/2}}\int_0^\infty dz.e^{-z^2/4t}K_0\left(\sqrt{x^2+y^2+2xy\cosh z}\right)g(z,t) \tag{7.60}$$

where $g(z,t) = \left[\pi\cos\left(\frac{z\pi}{2t}\right) - z\sin\left(\frac{z\pi}{2t}\right)\right]$ thereby establishing the claim in the lemma. □

We shall now review some established results from the literature that give other forms of Weierstrass transformations that are related to the hyperbolic heat kernel. Craddock, in [46] obtained the following:

Theorem 7.3.11. *The Yakubovich heat kernel as given by:*

$$K_Y(x,y;t) = \frac{2}{\pi^2}\int_0^\infty e^{-p^2t}K_{ip}(x)K_{ip}(y)p\sinh(\pi p)dp \tag{7.61}$$

has second Weierstrass form:

$$K_Y(x,y;t) = \frac{1}{4\sqrt{\pi}t^{3/2}}\int_{d(z,0)}^\infty J_0\left(2xy\cosh z - x^2 - y^2\right)ze^{-z^2/4t}dz \tag{7.62}$$

where $J_0(x)$ is the Bessel function of the first kind of zeroth order.

Proof. We review the basic calculation as outlined in [46] for comparison with the preceding lemma. Begin with the following integral:

$$\int_0^\infty K_{ip}(x)K_{ip}(y)\sinh(\pi p)\sin(pz)dp = \mathscr{J}(x,y,z) \tag{7.63}$$

$$= \begin{cases} \frac{\pi^2}{4}J_0\left(2xy\cosh z - x^2 - y^2\right) & u > 0 \\ 0 & u < 0 \\ u = 2xy\cosh z - x^2 - y^2 \end{cases} \tag{7.64}$$

obtained from [31] (pp. 189). Substitution into the Yakubovich kernel yields:

$$K_Y(x,y;t) = \frac{2}{\pi^2}\int_0^\infty pe^{-p^2 t}\mathcal{F}_s^{-1}\left[\mathscr{J}(x,y,z)\right]dp \tag{7.65}$$

$$d(z,0) = \cosh^{-1}\left(\frac{x^2+y^2}{2xy}\right) \Leftrightarrow u = 0 \tag{7.66}$$

$$K_Y(x,y;t) = \frac{4}{\pi^3}\int_0^\infty dp.pe^{-p^2 t} \times \int_0^\infty dz.\sin(pz)\mathscr{J}(x,y,z) \tag{7.67}$$

$$= \frac{4}{\pi^3}\int_0^\infty dz.\mathscr{J}(x,y,z)\int_0^\infty dp.pe^{-p^2 t}\sin(pz) \tag{7.68}$$

$$= \frac{1}{4\sqrt{\pi}t^{3/2}}\int_{d(z,0)}^\infty J_0\left(2xy\cosh z - x^2 - y^2\right)ze^{-z^2/4t}dz \tag{7.69}$$

i.e. the result as claimed, where we used the known integral:

$$\int_0^\infty dp.pe^{-p^2 t}\sin(pz) = \frac{z\sqrt{\pi}}{4t^{3/2}}e^{-z^2/4t} \tag{7.70}$$

□

Remark 138. The result of Craddock [46] gives an incomplete Weierstrass transform. We can see how this is a generalisation from the standard transform pair, where the standard Laplace transform is taken into an incomplete transform. It is possible that a duplication formula exists which allows translation between these two kernel formulae, as they both represent the same fundamental object.

We shall now briefly discuss some other applications of the Weierstrass transformation. These arise in the context of various other heat kernel representations in the hyperbolic plane. In particular, we have analysed these in detail in 7.3.12. The essential role of the Weierstrass transform is further emphasised by the formula for the Mehler-Fock representation of the hyperbolic heat kernel 7.3.12, also [11,124], where what amounts to either the Yor integral or a space translation invariant kernel is given by the formula:

Theorem 7.3.12. *The hyperbolic heat kernel may be written as the Weierstrass transform:*

$$\mathcal{K}(x,0;\tau) = \frac{1}{\sqrt{2}\pi^2}\int_{d(x,0)}^\infty \frac{z\exp\left(-\frac{z^2}{4\tau}\right)}{\sqrt{\cosh z - \cosh d(x,0)}}dz \tag{7.71}$$

where $d(x,0)$ *is the hyperbolic distance between the point* x *and the origin.*

Remark 139. See e.g. 4.7.16 for the proof of this using the results contained in this research, based on earlier work by Chavel [11] and comments by McKean.

Further, some calculations of Sousa et. al in [56] have demonstrated the utility of Bougerol's identity, in particular they obtained the following:

Theorem 7.3.13. *There exists a second representation for the hyperbolic heat kernel given by the Weierstrass transform over the hyperbolic generating function:*

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty} e^{ix\sinh z}e^{-z^2/4t}dz = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty} w(x;z)e^{-z^2/4t}dz \tag{7.72}$$

Proof. See [56] for the detailed calculation. We comment that the following Fourier transform and generating function exists for the modified Bessel function:

$$K_0(u) = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{iut}}{\sqrt{t^2+1}}dt = \int_0^{\infty} e^{-u\cosh t}dt \tag{7.73}$$

which is similar in many ways to the function appearing inside the Weierstrass transform in Sousa's result. It is likely that this is a form of Bessel function of one kind or another. □

We can see through all these different examples that the common thread of analysis across these different types of systems is one of Gaussian convolution. This is a powerful concept readily adaptable to many applications. It may turn out that the process of deconvolution is of primary interest in the field of image processing, as a similar technique may be used to deblur an otherwise unclear image. These types of equations are quite similar to those of chirped wavelets. We turn now to some more general considerations related to the Laplacian, Green's function and the relation to the kernel.

7.3.4 Laplacian and Green's Function

As is well known from spectral theory, the kernel and the Green's function are Laplace transformations of one another, and hence knowledge of one is often sufficient to develop the solution for the other. We shall now show how one may derive the Green's function on the hyperbolic plane using the modified Bessel representation for the heat kernel. This method differs from the use of the Mehler-Fock functions directly, as we shall see there exists a technique whereby we can find the resolvent function, which is essentially a shifted inverse of the Hamiltonian operator of the system. A brief description of similar techniques when applied to the problem of Asian option pricing may be found in [136,137].

Lemma 7.3.14. *The Green's function for the hyperbolic heat kernel $\nabla^2 u = -\partial_t u$ is given by the Fourier transform:*

$$\int_{-\infty}^{+\infty} e^{izt} G(x, y; \sqrt{iz})dz = K(x, y; t) \tag{7.74}$$

where the Green's function can be written as the eigenvalue decomposition of the resolvent operator:

$$G(x, y; \sqrt{iz}) = \frac{2}{\pi^2 x}\int_0^{\infty} \frac{1}{\nu^2 + iz} K_{i\nu}(x)K_{i\nu}(y)\nu\sinh(\pi\nu)d\nu \tag{7.75}$$

and $K_{i\nu}(x)$ is the modified Bessel function.

Proof. The heat kernel diffusion equation may be written using the hyperbolic Laplacian (see e.g. 4.5.1 in this work, also [2]) in standard form as in the lemma:

$$-\frac{\partial u}{\partial t} = \nabla^2 u \tag{7.76}$$

The kernel of this is given by the solution operator, where we formally take our definition of the exponentiated Laplacian through the integral transform:

$$e^{-t\nabla^2}u(y, 0) = u(x, t) \tag{7.77}$$

$$e^{-t\nabla^2}u(y, 0) = \int K(x, y; t)u(y, 0)dz \tag{7.78}$$

The Green's function for the kernel may be simply written as the Fourier transform via:

$$G(x, y; p) = -\int_0^{\infty} e^{-p^2 t}K(x, y; t)dt \tag{7.79}$$

where we take our definition of the Green's function via the Fredholm resolvent as in 4.7.18, this necessitates a choice of exponent so that it is negative definite in the Laplace transform, which is satisfied by $-p^2$.

$$K(x,y;t) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{p^2 t} G(x,y;p)d(p^2) \tag{7.80}$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{izt} G(x,y;\sqrt{iz})dz \tag{7.81}$$

Note that the constant factor is not essential to the argument and may be absorbed into the definition of either the Fourier transform or the normalisation of the kernel/Green's function. In our system, the Hamiltonian is given through the hyperbolic Laplace operator, and as we have shown this defines a large class of similar systems, all of whom are related through similar sets of PDEs. Using the eigenvalue decomposition of the resolvent, which amounts to evaluating the inverse Hamiltonian operator for the system, we may find a representation for the Green's function.

In terms of the Green's function, for the Yakubovich kernel, we have:

$$G(x,y;p) = \frac{2}{\pi^2 x}\int_0^{\infty} \frac{1}{\nu^2+p^2} K_{i\nu}(x)K_{i\nu}(y)\nu\sinh(\pi\nu)d\nu = (\hat{H}-p^2)^{-1} \tag{7.82}$$

$$G(x,y;\sqrt{iz}) = \frac{2}{\pi^2 x}\int_0^{\infty} \frac{1}{\nu^2+iz} K_{i\nu}(x)K_{i\nu}(y)\nu\sinh(\pi\nu)d\nu \tag{7.83}$$

$$\int_{-\infty}^{+\infty} e^{izt} G(x,y;\sqrt{iz})dz = \int_{-\infty}^{+\infty} dz.e^{izt}\frac{2}{\pi^2 x}\int_0^{\infty} \frac{1}{\nu^2+iz} K_{i\nu}(x)K_{i\nu}(y)\nu\sinh(\pi\nu)d\nu \tag{7.84}$$

$$= \frac{2}{\pi^2 x}\int_0^{\infty} d\nu.K_{i\nu}(x)K_{i\nu}(y)\nu\sinh(\pi\nu)\times\int_{-\infty}^{+\infty} dz.\frac{e^{izt}}{\nu^2+iz} \tag{7.85}$$

$$= \frac{4}{\pi x}\int_0^{\infty} e^{-\nu^2 t} K_{i\nu}(x)K_{i\nu}(y)\nu\sinh(\pi\nu)d\nu \tag{7.86}$$

$$= K(x,y;t) \tag{7.87}$$

as required to establish the lemma. A second path proceeds using the following known result, see Prudnikov [126], Vol. II, 2.16.52.11-12, where the integral formula for the Green's function may be found:

$$G(x,y;s) = \int_0^{\infty} dt.e^{-st}\int_0^{\infty} e^{-\rho^2 t} K_{i\rho}(x)K_{i\rho}(y)\rho\sinh(\pi\rho)d\rho \tag{7.88}$$

$$= \int_0^{\infty} K_{i\rho}(x)K_{i\rho}(y)\rho\sinh(\pi\rho)d\rho\int_0^{\infty} e^{-st}e^{-\rho^2 t}dt \tag{7.89}$$

$$= \int_0^{\infty} \frac{\rho\sinh(\pi\rho)}{\rho^2+s} K_{i\rho}(x)K_{i\rho}(y)d\rho \tag{7.90}$$

We therefore conclude that the Green's function may be written identically as before:

$$\int_0^{\infty} \frac{x\sinh(\pi x)}{x^2+z^2} K_{ix}(b)K_{ix}(c)dx = \begin{cases} \frac{\pi^2}{2} I_z(c)K_z(b) & 0<c<b \\ \frac{\pi^2}{2} I_z(b)K_z(c) & 0<b<c \end{cases} \tag{7.91}$$

$$G(x,y;s) = \begin{cases} \frac{\pi^2}{2} I_{\sqrt{s}}(x)K_{\sqrt{s}}(y) & 0<x<y \\ \frac{\pi^2}{2} I_{\sqrt{s}}(y)K_{\sqrt{s}}(x) & 0<y<x \end{cases} \tag{7.92}$$

□

We can see how this virtuous circle of operators depends entirely on the structure of the eigenfunctions. It is expected that similar relationships exist for the Whittaker and Mehler-Fock kernels as a consequence of the projection-slice theorem, as in 5.3.8, 5.3.9. We shall now comment on the relationship of the product formula to the various different representations for the kernel we have uncovered. As is apparent, this is a system in which specification of the functional dependence of the product $K_{i\nu}(x)K_{i\nu}(y)$ is of paramount importance.

7.3.5 Product Formulae and Parabolic Cylinder Functions

Sousa and Yakubovich [112] have analysed the nature of the product of two Whittaker functions, defining the convolution theorem for the Whittaker transform. There is another approach to the question of finding analogous transformation formulae of the type explored in this thesis. This method uses the techniques of spherical kernels and spherical zonal functions to achieve similar results. We shall require the following definition for a spherical zonal decomposition:

Definition 7.3.15. A spherical zonal kernel separates the variables of a spherical function such that:

$$f(r)f(s) = \int k(r,s;z)f(z)dz \tag{7.93}$$

where $f(.)$ is a spherical function, with support on the domain of z, the kernel being given by $k(r,s;z)$. We refer to this as a spherical kernel formula.

We then have the following lemma, which follows as a consequence.

Lemma 7.3.16. *Every spherical kernel formula defines a product formula.*

Proof. A product formula is by definition some function given by the product $f(r)f(s)$, this is given by the left-hand side of 7.3.15, the result follows. □

Remark 140. The contrapositive argument unfortunately fails. For example, one can see that the existence of a product formula is clearly insufficient to establish the existence of a spherical product formula. These only exist in special circumstances, and as we shall show by construction of a contradictory example, there exist many different representations of kernels and integral transforms that satisfy the product law, without conforming to the spherical zonal law.

We shall now quote a result from [112], which gives a convolution formula associated to the index Whittaker transform. As is well known, there is a connection between convolution formulae and products. They obtained the following:

Lemma 7.3.17. *The product formula for the index Whittaker functions $W_{\alpha,\nu}(.)$, as in Sousa and Yakubovich [112] is given by the spherical kernel formula:*

$$W_{\alpha,\nu}(x)W_{\alpha,\nu}(y) = \int_0^\infty W_{\alpha,\nu}(\xi)k_\alpha(x,y;\xi)\frac{d\xi}{\xi^2} \tag{7.94}$$

where the kernel is defined through the parabolic cylinder functions $D_{2\alpha}(.)$:

$$k_\alpha(x,y;\xi) = 2^{-(1+\alpha)}\pi^{-1/2}(xy\xi)^{1/2}\exp\left(\frac{1}{2}(x+y+\xi) - \frac{\eta^2}{4}\right)D_{2\alpha}(\eta) \tag{7.95}$$

the argument of the kernel is given by:

$$\eta = \frac{xy+y\xi+x\xi}{\sqrt{2xy\xi}} \tag{7.96}$$

This gives a product formula for the modified Bessel functions, which can be readily evaluated.

Corollary 7.3.18. *The product formula for the modified Bessel functions $K_{i\nu}(.)$ can be derived using a special case of the product formula for the Whittaker functions, as in 7.3.17. It reads as:*

$$K_{ip}\left(\frac{x}{2}\right)K_{ip}\left(\frac{y}{2}\right)=\frac{1}{2}\int_0^\infty K_{ip}\left(\frac{\xi}{2}\right)\exp\left(-\frac{(x^2\xi^2+y^2\xi^2+x^2y^2)}{4xy\xi}\right)\frac{d\xi}{\xi} \tag{7.97}$$

Proof. This follows from the special case using the substitutions $\alpha=0,\nu=ip$. Writing the integral:

$$W_{0,ip}(x)W_{0,ip}(y)=\int_0^\infty W_{0,ip}(\xi)K_0(x,y;\xi)\frac{d\xi}{\xi^2} \tag{7.98}$$

As shown in 5.3.11 the Whittaker function and the Macdonald function are related by the conversion formulae:

$$W_{0,i\nu}(2y)=\sqrt{\frac{2y}{\pi}}K_{i\nu}(y) \tag{7.99}$$

Substituting into the expression, and using the parabolic cylinder identity $D_0(u)=\exp(-u^2/4)$ we obtain the result.

$$K_0(x,y;\xi)=\frac{\sqrt{xy\xi}}{2\sqrt{\pi}}\exp\left(-\frac{(x^2\xi^2+y^2\xi^2+x^2y^2)}{4xy\xi}\right) \tag{7.100}$$

hence

$$K_{ip}\left(\frac{x}{2}\right)K_{ip}\left(\frac{y}{2}\right)=\frac{1}{2}\int_0^\infty K_{ip}\left(\frac{\xi}{2}\right)\exp\left(-\frac{(x^2\xi^2+y^2\xi^2+x^2y^2)}{4xy\xi}\right)\frac{d\xi}{\xi} \tag{7.101}$$

as required. □

Another form of this type of product identity may be derived. Using a formula from [134], we have the following complimentary product formula representation:

Theorem 7.3.19. *The product formula for the modified Bessel functions can be written in the form:*

$$K_{ip}\left(\frac{x}{2}\right)K_{ip}\left(\frac{y}{2}\right)=2\cosh(\pi p)\int_0^\infty \exp\left(-\frac{(x+y)}{2}\cosh v\right)K_{2ip}\left(\sqrt{xy}\sinh v\right)dv \tag{7.102}$$

Proof. See Buchholz, [134], eq. 6.4α, pp.85 for the proof of the integral formula. □

From application of this formula, the following representation of the Yakubovich kernel, i.e. the solution to the heat-type PDE $u_t=x^2u_{xx}+xu_x-x^2u$ may be developed.

Corollary 7.3.20. *The heat kernel solution of modified Bessel type (Yakubovich kernel) has formula:*

$$K(x,y;t)=\frac{4}{\pi^2}\int_0^\infty dv.\exp\left(-(x+y)\cosh v\right)F(x,y,v;t) \tag{7.103}$$

and satisfies $u_t=x^2u_{xx}+xu_x-x^2u$.

Proof. This follows from the formula for the product from Buchholz [134]. We have the special case of the Whittaker function given by the modified Bessel functions:

$$K_{\mu/2}\left(\frac{x}{2}\right)K_{\mu/2}\left(\frac{y}{2}\right)=2\cos\left(\frac{\pi\mu}{2}\right)\int_0^\infty \exp\left(-\frac{(x+y)}{2}\cosh v\right)K_\mu\left(\sqrt{xy}\sinh v\right)dv \tag{7.104}$$

Taking $\mu = 2ip$ we may write:

$$K_{ip}\left(\frac{x}{2}\right)K_{ip}\left(\frac{y}{2}\right) = 2\cos\left(\pi ip\right)\int_0^\infty \exp\left(-\frac{(x+y)}{2}\cosh v\right)K_{2ip}\left(\sqrt{xy}\sinh v\right)dv \quad (7.105)$$

Using the complex translation of trigonometric functions:

$$\cos\left(\pi ip\right) = \cosh(\pi p) \quad (7.106)$$

we obtain:

$$K_{ip}\left(\frac{x}{2}\right)K_{ip}\left(\frac{y}{2}\right) = 2\cosh(\pi p)\int_0^\infty \exp\left(-\frac{(x+y)}{2}\cosh v\right)K_{2ip}\left(\sqrt{xy}\sinh v\right)dv \quad (7.107)$$

The Yakubovich kernel is given by the spectral expansion into the modified Bessel functions as demonstrated throughout this thesis 5.6.2:

$$K(x,y;t) = \frac{2}{\pi^2}\int_0^\infty e^{-p^2t}K_{ip}(x)K_{ip}(y)p\sinh(\pi p)dp \quad (7.108)$$

Using the product formula, we have immediately:

$$K_{ip}\left(x\right)K_{ip}\left(y\right) = 2\cosh(\pi p)\int_0^\infty \exp\left(-(x+y)\cosh v\right)K_{2ip}\left(2\sqrt{xy}\sinh v\right)dv \quad (7.109)$$

and therefore the kernel may be written:

$$K(x,y;t) = \frac{2}{\pi^2}\int_0^\infty e^{-p^2t}2\cosh(\pi p)p\sinh(\pi p)dp\int_0^\infty \exp\left(-(x+y)\cosh v\right)K_{2ip}\left(2\sqrt{xy}\sinh v\right)dv \quad (7.110)$$

$$= \frac{4}{\pi^2}\int_0^\infty dv.\exp\left(-(x+y)\cosh v\right)\int_0^\infty pe^{-p^2t}\cosh(\pi p)\sinh(\pi p)K_{2ip}\left(2\sqrt{xy}\sinh v\right)dp \quad (7.111)$$

$$= \frac{4}{\pi^2}\int_0^\infty dv.\exp\left(-(x+y)\cosh v\right)F(x,y,v;t) \quad (7.112)$$

as required, where the function from the corollary may be specified as the Kontorovich-Lebedev transform:

$$F(x,y,v;t) = \int_0^\infty pe^{-p^2t}\cosh(\pi p)\sinh(\pi p)K_{2ip}\left(2\sqrt{xy}\sinh v\right)dp \quad (7.113)$$

$$= \frac{1}{2}\int_0^\infty pe^{-p^2t}\sinh(2\pi p)K_{2ip}\left(2\sqrt{xy}\sinh v\right)dp \quad (7.114)$$

This establishes the corollary. □

Remark 141. Note that the argument of the function is the distance metric in the space. This is a natural consequence of the existence of positive definite functions via the group homomorphism, as given by Mercer's theorem [67], see also this work 3.3.11. Analysis of the different formulae we have uncovered for the kernel is likely to have some far-reaching consequences for the representation theory of these types of spherical functions. Indeed, the inner integral defines a kernel:

$$\int_0^\infty dp.e^{-p^2t}p\sinh(2\pi p)K_{2ip}(d(x,y)) \quad (7.115)$$

which is equivalent up to time scaling with the Yor integral calculated earlier 5.7.14.

7.3.6 Orthogonal Polynomials related to the Weierstrass Transform

The following brief calculation outlines how one may construct a set of polynomials related to the Weierstrass transformation, via the Gaussian kernel. These polynomials appear naturally as a consequence of the extension of this kernel to operator arguments. We shall begin with the basic observation familiar to analysis of Fourier integrals:

Lemma 7.3.21. *The Fourier transform of a Gaussian is also a Gaussian, i.e. it is an invariant distribution:*

$$e^{-ax^2/2} = \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{+\infty} e^{ixy} e^{-y^2/2a} dy \tag{7.116}$$

We assume in the following arguments that this equation is true for real a and complex x. We derive the fundamental solution via the kernel transform in the sequence of corollaries below.

Corollary 7.3.22. *The operator form of 7.3.21 is for* $\hat{H} = \hat{H}\left(x, \frac{\partial}{\partial x}\right)$

$$e^{\hat{H}^2} = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{+\infty} e^{y\hat{H}} e^{-y^2/4} dy \tag{7.117}$$

Proof. Assume that the expression 7.3.21 is true for operator a. If we take the simplest type of operator, we might have $\hat{x}f(x) = xf(x)$. In the formula given by 7.3.21, if we perform the transformation $x = ix$, and take $a = 2$, we obtain:

$$e^{x^2} = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{+\infty} e^{xy} e^{-y^2/4} dy \tag{7.118}$$

Formally, we may write in the operator $\hat{x}$ the operator equation equivalent. We are free to write symbolically for $\hat{H} = \hat{H}\left(x, \frac{\partial}{\partial x}\right)$

$$e^{\hat{H}^2} = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{+\infty} e^{y\hat{H}} e^{-y^2/4} dy \tag{7.119}$$

by the method of the solution operator. □

Remark 142. Note that we have a certain amount of freedom in choice of our operator, for example we may write alternatively:

$$e^{\hat{H}^2} = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{+\infty} e^{y\hat{H}} e^{-y^2/4} dy \tag{7.120}$$

which also solves the integral formula. The implication here is that the operator satisfies a relationship such that $\hat{H} = \sqrt{\hat{H}^2}$, which is only true for one-dimensional. For example, we can see that $\partial_x^2 = \partial_{xx} = (\partial_x)^2$ satisfies this.

We may now state a useful result that links the kernel method of solution for the heat equation, the integral transform method, and the solution via operator calculus as in the preceding corollary.

Lemma 7.3.23. *The solution to the heat equation satisfies the integral equation:*

$$e^{\partial_x^2} f(z) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{+\infty} f(z+y) e^{-y^2/4} dy \tag{7.121}$$

This is the Weierstrass transform of a shifted function.

Proof. The solution to the heat kernel 3.3.14 satisfies the operator equation:

$$e^{t\nabla^2} f(z) = u(x,t) = \int K(x,z) f(z) dz \tag{7.122}$$

Using this formula, with the operator equation for the Gaussian kernel 7.3.21, we take $\hat{H} = \dfrac{\partial}{\partial x}$, hence obtaining:

$$e^{\nabla^2} f(z) = e^{\partial_x^2} f(z) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{+\infty} e^{y\partial_x} f(z) e^{-y^2/4} dy \tag{7.123}$$

$$= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{+\infty} f(z+y) e^{-y^2/4} dy \tag{7.124}$$

as required, where the last line follows from the shift theorem of the Fourier transform. Note that the implied integral transformation is given by the Gaussian convolution, or Weierstrass transform. □

We shall now show how this type of analysis of the Weierstrass transform leads to a certain finite series transform given by the Hermite polynomials.

Lemma 7.3.24. *The Hermite and inverse Hermite transforms are given by the formulae:*

$$F(z) = \sum_n \frac{a_n}{\sqrt{\pi} 2^{n+1} n!} H_n(y/2) \tag{7.125}$$

and

$$a_n = \int_{-\infty}^{+\infty} F(y) H_n(y/2) e^{-y^2/4} dy \tag{7.126}$$

where $H_n(x)$ is the Hermite polynomial.

Proof. We take our basic decomposition over the Hermite polynomials as a linear combination, viz.:

$$F(y) = \sum_n c_n H_n(y/2) \tag{7.127}$$

Evaluating the orthogonality relations for the Hermite polynomials, we have:

$$\int_{-\infty}^{+\infty} H_m(x) H_n(x) e^{-x^2} dx = \sqrt{\pi} 2^n n! \delta_{mn} \tag{7.128}$$

Renormalising, we may write the orthogonality as:

$$\int_{-\infty}^{+\infty} H_m(x) H_n(x) e^{-x^2} dx = \frac{1}{2} \int_{-\infty}^{+\infty} H_m(y/2) H_n(y/2) e^{-y^2/4} dy \tag{7.129}$$

or

$$\int_{-\infty}^{+\infty} H_m(y/2) H_n(y/2) e^{-y^2/4} dy = \sqrt{\pi} 2^{n+1} n! \delta_{mn} \tag{7.130}$$

and we may easily deconstruct the function via:

$$F(z) = \sum_n \frac{a_n}{\sqrt{\pi} 2^{n+1} n!} H_n(y/2) \tag{7.131}$$

Using the orthonormality relation, we easily establish that the forward transform is defined through the sequence:

$$a_n = \int_{-\infty}^{+\infty} F(y) H_n(y/2) e^{-y^2/4} dy \tag{7.132}$$

as required. □

We shall now show how the Hermite transform and the related polynomial sequences can be developed. We require the following definition for the Hermite polynomials:

Definition 7.3.25. The generating function for the Hermite polynomials may be written as the operator equation:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{x^2}) \tag{7.133}$$

and further, the polynomials of scaled argument satisfy:

$$H_n(x/2) = e^{-\partial_x^2}(x^n) \tag{7.134}$$

Remark 143. We note that this essentially amounts to construction of a homogenous polynomial solution to the heat equation, which imposes a particular form of kernel via the solution operator $\exp(-\nabla^2)$.

We now prove a series of small lemmas that demonstrate the scaling properties of the Hermite polynomials and the relationship to the fundamental solution as given by the Gaussian kernel.

Lemma 7.3.26. *For the heat equation:*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{7.135}$$

there exist two solutions, given by the Gaussian kernel and the heat polynomials. These satisfy the following identities:

$$u(x,t) = \frac{1}{\sqrt{t}} u\left(\frac{x}{\sqrt{t}}, 1\right) \tag{7.136}$$

and

$$H_n(x,t) = t^{n/2} H_n(x/\sqrt{t}, 1) \tag{7.137}$$

These solutions are homogenous in the space and time variables.

Proof. The Gaussian heat kernel may be written:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right) \tag{7.138}$$

so, indeed we have:

$$u(x,1) = \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{x^2}{4}\right) \tag{7.139}$$

$$u\left(\frac{x}{\sqrt{t}}, 1\right) = \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{x^2}{4t}\right) = \sqrt{t} u(x,t) \tag{7.140}$$

$$u(x,t) = \frac{1}{\sqrt{t}} u\left(\frac{x}{\sqrt{t}}, 1\right) \tag{7.141}$$

which establishes the first part of the lemma. For the second claim, we write the heat polynomials in the form:

$$H_n(x,t) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n!}{j!(n-2j)!} x^{n-2j} t^j \tag{7.142}$$

Taking out a common factor from the polynomial expansion, we find:

$$H_n(x,t) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n!}{j!(n-2j)!} x^{n-2j} t^j = x^n \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n!}{j!(n-2j)!} x^{-2j} t^j \tag{7.143}$$

$$= x^n \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n!}{j!(n-2j)!} \left(\frac{x^2}{t}\right)^{-j} \tag{7.144}$$

and hence we find the special case:

$$H_n(x,1) = x^n \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n!}{j!(n-2j)!} x^{-2j} \tag{7.145}$$

Substituting back into the original identity for the heat polynomials, we find:

$$H_n(x/\sqrt{t},1) = t^{-n/2} x^n \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n!}{j!(n-2j)!} x^{-2j} t^j = t^{-n/2} H_n(x,t) \tag{7.146}$$

$$H_n(x,t) = t^{n/2} H_n(x/\sqrt{t},1) \tag{7.147}$$

as required. These are homogenous solutions of the heat equation. □

Remark 144. Note that there is an additional symmetry defined through

$$He_n(z) = 2^{-n/2} H_n\left(\frac{z}{\sqrt{2}}\right) = H_n(z,-1/2) \tag{7.148}$$

which gives the necessary connection between the physical, probabilistic and generalised Hermite polynomials. A complex rotation is therefore sufficient to recover the symmetric polynomials, the formula defined through:

$$H_n\left(\frac{x}{\sqrt{t}},1\right) = (-i)^n H_n\left(\frac{ix}{2\sqrt{t}}\right) \tag{7.149}$$

which also are solutions of the heat kernel equations in a similar way as the caloric polynomials. These two solutions span the solution space for the heat equation in the same way as the Hermite polynomials are composed of the functions of odd and even parity.

Many different formulae may be developed using the Hermite transform in this fashion. An apparent extension to use Gauss-Hermite quadrature to approximate the kernels would appear to be of interest. Another avenue of open questions regards the conversion of these formulae over into the continuous spectrum. In this case, the expected eigenfunctions are of parabolic cylinder form, which are defined through an analogous Rodrigues formula with non-integer coefficients. The Hermite polynomials have a well-characterised relationship with the fields of stochastic processes on the one hand, quantum physics on the other. One final point is to note that many of these properties may be deduced from the scaling property of Brownian motion, viz.

$$W_t = cB_{t/c^2} \tag{7.150}$$

We have shown using some simple methods how one can go about constructing the orthogonal polynomial sequences associated with the Weierstrass transform. This is of course the Gauss-Hermite transform pair familiar from classical mathematics. Interestingly, a similar approach may be defined in general, for any positive weighting function with the correct properties will define a series of orthogonal polynomials via the Sturm-Liouville theory. This is one way in which to associate the sets of polynomials with equivalence classes of systems of continuous spectra. As an example of this type of structure, it is possible to show that the Bessel polynomials are associated with the Whittaker function, the Laguerre polynomials with the Macdonald function, and the Legendre polynomials with the Mehler-Fock functions. The case involving the Hermite polynomials is then likely associated with its continuous extension, being the parabolic cylinder function of Whittaker.

7.4 Weierstrass Transform of the Wave Kernel

We shall briefly summarise some results that are discussed in Bragg and Dettman [98,99] that are of relevance to our discoveries regarding the wave kernel. The authors in these papers uncovered a number of useful relationships that allow the EPD equation in its simplest form to be transformed into the wave equation. They also utilised a transformation, known as the Weierstrass transform, to convert the heat equation into the wave equation and vice versa using the inverse of this transform. These results mean that many of the formulae we have developed in understanding EPD equations and wave kernels can be transferred over into heat kernels with the correct technique. The Gaussian or Weierstrass kernel is related to the transformation between the different kernels given by the heat, wave and EPD systems of differential operators. We review some relevant results used in the sequel to show that the Gaussian-sinh kernel, the Poisson kernel and the two dimensional Gaussian-sinh kernel may all be recovered using eigenfunction decompositions. Further analysis of the residue calculus given by the Weierstrass transform of the wave equation can be used to derive the Gaussian-sinh kernel. We shall prove the following series of lemmas and the consequent theorem. The basic forms of functional calculus we shall employ in the remainder of this chapter rely on the observation that any operator commutes with a function of the operator, if that function is expressible by a Taylor series expansion.

Lemma 7.4.1. *Let $\hat{A}$ be an operator, and $f(.)$ be some function with a Taylor series expansion that acts in the space of the operator. Then $\hat{A}$ and $f(\hat{A})$ commute, i.e. $[\hat{A}, f(\hat{A})] = 0$.*

Proof. We have the assumed Taylor series for the functional of the operator:

$$f(\hat{A}) = \sum_{j=0}^{\infty} c_j \hat{A}^j \tag{7.151}$$

Applying $\hat{A}$ to both sides, we find that $\hat{A}f(\hat{A}) = f(\hat{A})\hat{A}$, and the identity results. □

The following corollary for the wave kernel is a result of this lemma, we have:

Corollary 7.4.2. *Two examples of 7.4.1 are:*

$$\left[e^{t\nabla^2}, \nabla^2\right] = 0 \tag{7.152}$$

and

$$\left[\cos\left(it\partial_x\right), \partial_x^2\right] = 0 \tag{7.153}$$

The expressions inside the commutator bracket appearing as functions have significance, in that the first is the heat kernel, and the second is the wave kernel.

Corollary 7.4.3. *Assume we have a PDE with time-space separability:*

$$\hat{H}(x^i, \partial_x^i) f(x,t) = \hat{T}(t^i, \partial_t^i) f(x,t) \tag{7.154}$$

For the heat equation, the PDE satisfies:

$$\frac{\partial u}{\partial t} = \nabla^2 u \tag{7.155}$$

and hence is separable. The wave equation is likewise separable. PDEs with time-space separability have kernel solutions that are given by functionals of the operators, which can be evaluated due to the commutation relations.

Remark 145. In practice, this means that the kernels solve the PDEs in the Frechet sense for the time derivative component, and the commutation relations enable the kernel to pass through the Laplacian operator or otherwise, which allows the argument to be made that the operator may be treated as an effective constant in the sense of Heaviside [100]. One is required to generalise the argument of a function to an argument containing an operator, which is possibly continuous and infinite dimensional. The complexities of dealing with the delicate mathematics of this are far beyond the subject matter of this thesis and we direct the reader to Bragg and Dettman for further work on the topic [166], [167].

Lemma 7.4.4. *Assume the wave equation defined through:*

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} \tag{7.156}$$

Then there exists a kernel solution to this equation:

$$w(x,t) = \int K_w(x,y;t) w(y,0) dy \tag{7.157}$$

This solution may be written in terms of of the solution operator:

$$K_w(x,y;t) = \cos(it\partial_x) \tag{7.158}$$

Remark 146. Note that in making this proof work, we are required to extend the notion of a function of a variable, to the function of an operator argument, in this case given by ∂_x.

Proof. This is elementary. If we assume the functional form as given by the solution operator, the space derivatives commute as per 7.4.1:

$$\frac{\partial^2}{\partial x^2} \cos(it\partial_x) = \cos(it\partial_x) \frac{\partial^2}{\partial x^2} \tag{7.159}$$

and further:

$$\frac{\partial^2}{\partial t^2} \cos(it\partial_x) = \partial_x^2 \cos(it\partial_x) \tag{7.160}$$

□

The equivalent result for the heat kernel is derived in an identical way. We have:

Lemma 7.4.5. *For the heat equation:*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 w}{\partial x^2} \tag{7.161}$$

the solution operator may be written:

$$K_h(x,y;t) = \exp\left(t\nabla^2\right) \tag{7.162}$$

where $K_h(x,y;t)$ is the kernel solution to the heat equation.

Proof. Direct substitution, with the proviso that the operator ∇ can be treated as an effective constant when considering the Frechet derivative with respect to time. We have:

$$\frac{\partial^2}{\partial x^2} e^{t\partial_x^2} = e^{t\partial_x^2} \frac{\partial^2}{\partial x^2} \tag{7.163}$$

and

$$\frac{\partial^2}{\partial t^2} e^{t\partial_x^2} = \partial_x^2 e^{t\partial_x^2} \tag{7.164}$$

which solves the equation. □

Remark 147. Note that in both of these situations, we are considering boundary conditions on the kernel of the form $K(x;0) = 1$. Although these methods may seem somewhat doubtful, they give useful information about the structure of the kernel and have a long and illustrious history dating back to the original enquiries of Heaviside, who was the first to make such calculations when using the Laplace transform.

The following theorem expresses the connection between the wave and heat kernels directly through the use of operator calculus methods.

Theorem 7.4.6. *The connection between the heat and wave kernel solutions may be written as the Weierstrass transform:*

$$K_h(x,y;t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-s^2/4t} K_w(x,y;s)ds \tag{7.165}$$

where $K_h(.)$ is the heat kernel, likewise $K_w(..)$ the wave equivalent.

Proof. Using the Fourier transform of a Gaussian, and the symmetry between the cosine and exponential transform for even functions, we may write:

$$e^{-tb^2} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-s^2/4t} e^{isb} ds = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-s^2/4t} \cos(bs)ds \tag{7.166}$$

$$e^{t\nabla^2} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-s^2/4t} \cos(is\nabla)ds \tag{7.167}$$

Inserting the expressions from 7.4.5, 7.4.4 we obtain the result. □

We shall now outline for a simple application of this technique to demonstrate its utility.

Corollary 7.4.7. *Assume the heat equation:*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{7.168}$$

with boundary condition $u(0,t) = 0$. The solution to this problem, given by the heat kernel, may be found via eigenfunction decomposition:

$$K_h(x,y;t) = \int_0^{\infty} e^{-k^2 t} \sin(kx)\sin(ky)dk \tag{7.169}$$

Then the wave kernel associated with the Laplacian operator is given by:

$$K(x,y;t) = \int_0^{\infty} e^{-kt} \sin(kx)\sin(ky)dk \tag{7.170}$$

These kernels satisfy the Weierstrass transform identity:

$$K_h(x,y;t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \exp\left(-\frac{s^2}{4t}\right) K_w(x,y;s)ds \tag{7.171}$$

i.e. 7.4.6.

Proof. Assume we have the basic heat kernel, described through the integral in the corollary:

$$K_h(x,y;t) = \int_0^{\infty} e^{-k^2 t} \sin(kx)\sin(ky)dk \tag{7.172}$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{t}} \exp\left(-\frac{(x^2+y^2)}{4t}\right) \sinh\left(\frac{xy}{2t}\right) \tag{7.173}$$

and following this we obtain:

$$K_h(x,x;t) = K_h(x;t) = \frac{1}{2}\sqrt{\frac{\pi}{t}} \exp\left(-\frac{x^2}{2t}\right) \sinh\left(\frac{x^2}{2t}\right) \tag{7.174}$$

See e.g. GR eq. 4.133.1-2 [122]. A priori, we have found a valid solution to the heat equation given appropriate boundary conditions as per our assumptions, i.e. $u(0,t) = 0$. In terms of the wave equation, we have:

$$\frac{\partial^2 K_w}{\partial t^2} = \nabla^2 K_w \tag{7.175}$$

If we calculate a particular solution to the wave equation of the type specified in the corollary, we can use the Laplace transform defined through:

$$K(x,y;t) = \int_0^\infty e^{-kt} \sin(kx)\sin(ky)dk \tag{7.176}$$

This is permissible, as the eigenfunction decomposition of the heat equation has eigenfunction k^2, but this is first order in time, whereas for the wave equation, being second order in time, we must have eigenvalue $-k$, and using $\sin(-kx)\sin(ky) = \sin(kx)\sin(ky)$ we obtain the eigenfunction decomposition. Then the solution defined through $K_w = K(x,y;it)$, i.e. a complex rotation in time solves the wave equation, as is easily identified from the specified form. Evaluating the Laplace transform, we have [122] eq. 3.895.16:

$$K_w(x,y;t) = K(x,y;it) = -\frac{2ixyt}{(-t^2+(x+y)^2)(-t^2+(x-y)^2)} \tag{7.177}$$

which solves the wave equation by inspection. To apply the Weierstrass transformation between the kernels, we are required to determine that:

$$K_h(x,y;t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \exp\left(-\frac{s^2}{4t}\right) K_w(x,y;s)ds \tag{7.178}$$

To solve this, we use analytic continuation to the complex plane. The singularities of the function defined through

$$\frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{s^2}{4t}\right) K_w(x,y;s) \tag{7.179}$$

are located at the points $s = \pm(x \pm y)$. We use the residue theorem in the following form:

$$\oint_\Gamma f(s)ds = 2\pi i \sum \text{Residues} \tag{7.180}$$

where the curve encloses all four singularities, being below the x-axis in the complex plane, proceeding in a straight line past $\pm(x+y)$ and joined by a semicircular loop. In this case, we obtain:

$$K_h(x,y;t) = \frac{1}{\sqrt{4\pi t}} 2\pi i \left(\frac{2i}{4} \exp\left(-\frac{(x+y)^2}{4t}\right) - \frac{2i}{4} \exp\left(-\frac{(x-y)^2}{4t}\right)\right) \tag{7.181}$$

$$= -\sqrt{\frac{\pi}{t}}\frac{1}{2}\left(\exp\left(-\frac{(x+y)^2}{4t}\right) - \exp\left(-\frac{(x-y)^2}{4t}\right)\right) \tag{7.182}$$

$$= -\sqrt{\frac{\pi}{t}}\frac{1}{2} \exp\left(-\frac{x^2+y^2}{4t}\right)\left[\exp\left(-\frac{xy}{2t}\right) - \exp\left(\frac{xy}{2t}\right)\right] \tag{7.183}$$

$$= \sqrt{\frac{\pi}{t}} \exp\left(-\frac{x^2+y^2}{4t}\right) \sinh\left(\frac{xy}{2t}\right) \tag{7.184}$$

which is the solution to the heat equation as required.

□

Remark 148. If we had solved the wave equation through the method of eigenfunctions and separable solutions, it is not too hard to see that one can arrive at a similar result without a great degree of effort. So this is the most direct way in which the wave kernel can be used to derive solutions for PDEs in the same way that the heat kernel is used in diffusion problems, and indeed, given a solution in one domain, one may convert into a solution in the other domain by using the Weierstrass transformation or inverse as required. One final note is to observe that there exist multiple kernel solutions to e.g. the wave equation. These solutions are not unique, but specified through the boundary conditions.

7.4.1 Solution Operator for the Wave and Heat Equations

The kernel for the heat and wave equations can be understood from a solution operator perspective, in the spirit of Heaviside. This allows definition of an operator calculus via integral identities. In the following segment, we shall show using the standard trigonometric eigenfunctions, we recover a Poisson kernel solution which satisfies the wave equation. Dawson's integral may also be recovered from the operator calculus using an integral identity and a Laplace transform. Bragg and Dettman [98,99] analysed the link between wave and heat kernels using a different type of fundamental solution in the following way.

Example 7.4.8. Assume we have a differential equation that can be described as:

$$\frac{\partial^2 f}{\partial t^2} = \hat{J} f \tag{7.185}$$

Then a solution operator is easily written down, we have:

$$K_w(\hat{H}, t) = \frac{\sinh\left(t\sqrt{\hat{J}}\right)}{\sqrt{\hat{J}}} \tag{7.186}$$

which obviously solves the wave equation by inspection, with a different boundary condition to the one we considered before.

The following representation of the Green's function in terms of the solution operator is readily accessible:

Lemma 7.4.9. *The Green's function for the wave kernel 7.4.4 satisfies:*

$$\frac{1}{s^2 - \hat{J}} = \int_0^\infty e^{-st} K_w(\hat{J}, t) ds \tag{7.187}$$

Proof. This follows from the definition of the Green's function and the kernel. See e.g. 6.8.1 for application to the EPD system. □

We also have the following operator extension of the standard Gaussian integrals, which may be found in Prudnikov, Vol. III, 2.3.16.7 [148]:

Lemma 7.4.10. *There exists an operator equivalent for the integral:*

$$2\sqrt{\frac{\pi}{A}} e^{-t\sqrt{A}} = \int_0^\infty e^{-st^2} s^{-3/2} e^{-A/4s} ds \tag{7.188}$$

In particular, $\sqrt{\hat{A}}$ *is well defined, i.e. is positive definite.*

This gives us the kernel equivalence for the wave and heat systems in a new way. We have:

Theorem 7.4.11. *The kernel solutions for the wave and heat equations can be written in terms of one another via:*

$$K_w(\hat{A}, t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-su} s^{-3/2} K_h\left(\hat{A}, s\right) ds \tag{7.189}$$

where $K_w(\hat{A}, t)$ is the wave kernel, $K_h(\hat{A}, s)$ the heat kernel.

Proof. This is a slight modification of the notation originally appearing in Bragg and Dettman [98,99], as it is not necessary to indicate the inverse and forward variables in this fashion within this work. The solution operator for the wave equation defined through:

$$2\sqrt{\frac{\pi}{A}} e^{-t\sqrt{A}} \tag{7.190}$$

solves the wave equation in an identical way as before, we therefore can write:

$$K_w(\hat{A}, t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-su} s^{-3/2} e^{-\hat{A}/4s} ds \tag{7.191}$$

Noting that the expression $e^{-\hat{A}/4s} = K_h\left(\hat{A}, s\right)$, we obtain the result in the theorem. □

Remark 149. This is using the forward Laplace transform. If we use the inverse transform, we find the complementary set of identities:

Theorem 7.4.12.

$$4i\sqrt{\pi} \frac{\sin(t\sqrt{A})}{\sqrt{A}} = \int_{-i\infty}^{+i\infty} e^{st^2} s^{-3/2} \exp\left(-\frac{A}{4s}\right) ds \tag{7.192}$$

This is the Bragg-Dettman form [98] of the solution operator for the wave equation.

We take the following as an integral form of the solution operator.

Definition 7.4.13. A kernel solution to the equivalent heat equation is defined through

$$K_h(\hat{A}; t) = \int_0^\infty e^{-k^2 t} \cos(k\sqrt{\hat{A}}) dk = \frac{1}{2}\sqrt{\frac{\pi}{t}} \exp\left(-\frac{\hat{A}}{4t}\right) \tag{7.193}$$

In an analogous fashion to the previous transformation between the wave and heat kernels, we obtain:

Corollary 7.4.14. *The solution operator for the wave equation in terms of the heat equation is defined by the inverse Fourier transform:*

$$\frac{i\sin(t\sqrt{A})}{\sqrt{A}} = \frac{\sinh(it\sqrt{A})}{\sqrt{A}} = \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} s^{-1} e^{st^2} K_h(\hat{A}; s) ds \tag{7.194}$$

where K_h is the heat kernel.

Proof. We have the solution operator for the heat equation:

$$K_h(\hat{A}; t) = \int_0^\infty e^{-k^2 t} \cos(k\sqrt{\hat{A}}) dk = \frac{1}{2}\sqrt{\frac{\pi}{t}} \exp\left(-\frac{\hat{A}}{4t}\right) \tag{7.195}$$

Evaluating the identity, we find:

$$\exp\left(-\frac{\hat{A}}{4s}\right) = 2\sqrt{\frac{s}{\pi}}K_h(\hat{A};s) \tag{7.196}$$

$$4i\sqrt{\pi}\frac{\sin(t\sqrt{A})}{\sqrt{A}} = \int_{-i\infty}^{+i\infty} 2\sqrt{\frac{s}{\pi}}e^{st^2}s^{-3/2}K_h(\hat{A};s)ds \tag{7.197}$$

and hence:

$$\frac{i\sin(t\sqrt{A})}{\sqrt{A}} = \frac{\sinh(it\sqrt{A})}{\sqrt{A}} = \frac{1}{2\pi}\int_{-i\infty}^{+i\infty} s^{-1}e^{st^2}K_h(\hat{A};s)ds \tag{7.198}$$

as required. □

We shall now show how one may alter the boundary conditions on the solution and gain a type of modified error function known as Dawson's integral. If we change the heat kernel such that on the boundary at y the eigenfunction is equal to 1, we then derive immediately the following extension of the heat kernel:

Lemma 7.4.15. *For the heat kernel solution, there exists eigenfunction decomposition:*

$$K_h(x,y;t) = \int_0^\infty e^{-k^2t}\sin(kx)\cos(ky)dk = k(x-y,t) - k(x+y,t) \tag{7.199}$$

$$= -\frac{i\sqrt{\pi}}{4\sqrt{t}}\exp\left(-\frac{(x+y)^2}{4t}\right)\left(e^{xy/t}\Phi\left(\frac{i(x-y)}{2\sqrt{t}}\right) + \Phi\left(\frac{i(x+y)}{2\sqrt{t}}\right)\right) \tag{7.200}$$

The function $\Phi\left(\frac{i(x+y)}{2\sqrt{t}}\right)$ *is Dawson's integral, or the complex error function.*

Remark 150. Although this appears as if it is complex, it is actually real valued.

Proof. This integral may be found in tables, see GR eq. 3.891.1-2 [122], also 3.323.1-2 for a similar result. It may also be evaluated using computer algebra packages such as Maple. □

Corollary 7.4.16. *For the special case of a particle with initial position at the origin, we have a heat kernel of the form:*

$$K_h(x,0;t) = -\frac{i\sqrt{\pi}}{2\sqrt{t}}\Phi\left(\frac{ix}{2\sqrt{t}}\right)e^{-x^2/(4t)} \tag{7.201}$$

Proof. This evidently solves the heat equation by inspection. Note that the function, which appears complex, is actually Dawson's integral and is a real-valued, well characterised special function. □

Remark 151. This is also an operator equation which we may use to express the kernel more generally via:

$$K_h(\hat{A};t) = -\frac{i\sqrt{\pi}}{2\sqrt{t}}\Phi\left(\frac{i\sqrt{\hat{A}}}{2\sqrt{t}}\right)e^{-\hat{A}/(4t)} \tag{7.202}$$

To find the equivalent wave equation solution, one direct way in which to proceed is to use the Laplace transform. We shall show the following representation of Dawson's integral:

Lemma 7.4.17. *Dawson's integral can be given as a Laplace transform:*

$$\mathcal{L}\left[\Phi\left(\frac{ix}{2\sqrt{t}}\right)\right] = \frac{1}{u}\int_0^\infty e^{-us-x^2/(4s)}s^{-3/2}ds \tag{7.203}$$

This takes the same form of the equivalence between wave and heat equations, as in 7.4.6.

Proof. Writing the Laplace transform directly, and applying 7.4.6, we have:

$$\mathcal{L}\left[\Phi\left(\frac{\hat{A}}{2\sqrt{t}}\right)\right] = \int_0^\infty e^{-ut}\Phi\left(\frac{\hat{A}}{2\sqrt{t}}\right)dt \tag{7.204}$$

$$= \frac{1}{u}\int_0^\infty e^{-ut+a^2/(4t)}t^{-3/2}dt \tag{7.205}$$

where we used integration by parts and the fact that the function vanishes on the boundary. Inserting the factor $\hat{A} = i\hat{x} = ix$, we obtain:

$$\mathcal{L}\left[\Phi\left(\frac{ix}{2\sqrt{t}}\right)\right] = \frac{1}{u}\int_0^\infty e^{-us-x^2/(4s)}s^{-3/2}ds \tag{7.206}$$

i.e. a form of Dawson's function, the complex error function. □

Remark 152. If, instead of applying the Laplace transform directly, we chose the eigenfunction route, it is not too hard to see that the wave kernel is given by:

$$K_w(\hat{A}, t) = \int_0^\infty e^{-kt}\sin(kx)\cos(ky)dk = \frac{x(t^2+x^2-y^2)}{(t^2+(x+y)^2)(t^2+(x-y)^2)} \tag{7.207}$$

Proceeding via the residue calculus, we are faced with the integral:

$$\int_{-\infty}^{+\infty} \frac{x(s^2+x^2-y^2)}{(s^2+(x+y)^2)(s^2+(x-y)^2)}\exp\left(-\frac{s^2}{4t}\right)ds \tag{7.208}$$

Many of these different types of infinite integrals may be generated from this formulation. Although we have only briefly skimmed it in this work, it is a valuable technique that could be used to derive many interesting and useful integral formulae. The residue calculus will naturally lead to consideration of the Green's function and kernel singularities.

We can see how these various different representations come about, by replacing scalars by self-adjoint Hermitian operators, we are able to generalise these types of integral relations to operators and matrices. This is the key precept of the solution operator method, as we have shown, it is a useful method for developing kernel representations and linking dynamical systems.

7.4.2 Solution Operator for EPD Equation

The solution operator method for deriving the kernel remains valid for the EPD equation. This is a natural consequence of the Weierstrass transform theory between the heat, wave and EPD equations. As the following calculation shows, the EPD equations are solved through use of modified Bessel solution operator kernels, which may be found through a formula which contains Mellin, Gaussian and Laplace transforms. This method can be generalised to the EPD equation in the following way.

Theorem 7.4.18. *For the time-Bessel EPD equation written as:*

$$\frac{\partial^2 f}{\partial t^2} + \frac{(1+2\eta)}{t}\frac{\partial f}{\partial t} = \hat{A}f \tag{7.209}$$

where η is some constant, $\hat{A}$ is an operator in x, the solution operator form of the kernel gives two representations:

$$K_{e,1}(\hat{A}, t) = Ct^{-\eta}K_{\eta}(t\sqrt{\hat{A}}) \tag{7.210}$$

$$K_{e,2}(\hat{A}, t) = C(\sqrt{\hat{A}}t)^{-\eta}I_{\eta}(t\sqrt{\hat{A}}) \tag{7.211}$$

where $I_{\eta}(.), K_{\eta}(.)$ are the modified Bessel functions.

Proof. This follows directly as a consequence from treating the operator $\hat{A}$ as a constant and solving the differential equation, as per the solution operator method. See [98,99] for analysis of this type. □

We now prove two simple lemmas that follow from the integral representation of the modified Bessel function:

Lemma 7.4.19. *The integral representation of the modified Bessel function is given through the Gaussian integral:*

$$K_{\nu}(z) = \frac{1}{2}\int_0^{\infty} x^{-\nu-1}\exp\left(-x - \frac{z^2}{4x}\right)dx \tag{7.212}$$

Proof. See e.g. GR 8.432.6 [122]. □

Applying this, we have the following solution operator representations for the Bessel function:

Lemma 7.4.20. *The solution operator form of the kernel to the time-Bessel EPD equations can be written as the integral transform:*

$$K_{e,1}(\hat{A}, t) = \frac{Ct^{-\eta}}{2}\int_0^{\infty} x^{-\eta-1}\exp\left(-x - \frac{t^2\hat{A}}{4x}\right)dx \tag{7.213}$$

where $\hat{A}$ is the space part of the system operator.

Proof. Substituting in the values, we find:

$$K_{e,1}(\hat{A}, t) = Ct^{-\eta}K_{\eta}(t\sqrt{\hat{A}}) \tag{7.214}$$

$$= \frac{Ct^{-\eta}}{2}\int_0^{\infty} x^{-\eta-1}\exp\left(-x - \frac{(t\sqrt{\hat{A}})^2}{4x}\right)dx \tag{7.215}$$

$$= \frac{Ct^{-\eta}}{2}\int_0^{\infty} x^{-\eta-1}\exp\left(-x - \frac{t^2\hat{A}}{4x}\right)dx \tag{7.216}$$

thereby establishing the lemma. □

We have the following simple application, which gives an eigenfunction related to the kernel expansion of the time-EPD diffusion equation.

Lemma 7.4.21. *The eigenfunction solution for the EPD-diffusion equation can be written as the Weierstrass-Gauss transform:*

$$t^{-\eta}K_{\eta}(xt) = \left(\frac{\sqrt{x}}{t}\right)^{2\eta}\int_0^{\infty} e^{-st^2}s^{-\eta-1}e^{-x^2/(4s)}ds \tag{7.217}$$

where $K_{\eta}(.)$ is the modified Bessel function.

Proof. The proof follows as a simple application of the preceding lemma 7.4.20, with $\hat{A} = x^2$. We have already discussed this type of function in 6.8.2. □

We now shall state the equivalent type of solution operator that applies for the second type of modified Bessel function. In the same way that the solution operator for the wave equation has parts which depend on the sine or the cosine, in this context we have the Basset and Macdonald functions.

Lemma 7.4.22. *A second solution operator for the time-Bessel EPD equation is given by the operator integral:*

$$\pi I_\eta(t\sqrt{\hat{A}}) = \int_0^\infty e^{t\sqrt{\hat{A}}\cos s}\cos(\eta s)ds - \sin(\eta\pi)\int_0^\infty e^{-t\sqrt{\hat{A}}\cosh s - \eta s}ds \tag{7.218}$$

where $I_\eta(.)$ is the modified Bessel function.

Proof. This merely uses the solution operator approach, we have access to the following integral representation for the modified Bessel function (see GR 8.431.5 [122]):

$$\pi I_\nu(z) = \int_0^\infty e^{z\cos t}\cos(\nu t)dt - \sin(\nu\pi)\int_0^\infty e^{-z\cosh t - \nu t}dt \tag{7.219}$$

Substituting in the argument of the function, and assuming that the operator is an effective constant, we obtain the result. □

Remark 153. One can see from this perspective how the various different kernels are related. We can transform the Weierstrass kernel, which is the fundamental solution to the heat equation, into either the EPD or the wave equation by using the Gaussian or generalised Gauss-Mellin transform. There are relationships in the complex plane that are associated with these types of operator equations which we discuss in the conclusions of this chapter.

7.4.3 Symmetries of the Bessel-EPD Equation

There is a direct relationship between the Bessel EPD equation and the wave equation with an inverse square potential. The Bose invariant theory of the Bessel EPD equation may be used to show this directly. We shall prove the following theorem:

Theorem 7.4.23. *The EPD equation, given by the differential system:*

$$\frac{\partial^2 f}{\partial t^2} + \frac{(1+2\eta)}{t}\frac{\partial f}{\partial t} = \nabla^2 f \tag{7.220}$$

is equivalent to the system given by the wave equation with potential:

$$\nabla^2 g(x,t) = \frac{\partial^2 g}{\partial t^2} - \frac{(\eta^2 - 1/4)}{t^2}g(x,t) \tag{7.221}$$

Proof. The equivalence is through the Bose invariant potential. We have the formula for the Bose invariant EPD equation (see e.g. 6.7.21), given through the Sturm-Liouville transformation:

$$\frac{\partial^2 f}{\partial t^2} + \frac{(2\eta+1)}{t}\frac{\partial f}{\partial t} = t^{-(\eta+1/2)}\frac{\partial^2}{\partial t^2}\left(t^{\eta+1/2}f\right) - \frac{(\eta^2 - 1/4)}{t^2}f \tag{7.222}$$

Using $g(x,t) = t^{\eta+1/2}f$, we readily obtain the following:

$$\frac{\partial^2 f}{\partial t^2} + \frac{(2\eta+1)}{t}\frac{\partial f}{\partial t} = t^{-(\eta+1/2)}\frac{\partial^2 g}{\partial t^2} - \frac{(\eta^2 - 1/4)}{t^2}t^{-(\eta+1/2)}g(x,t) \tag{7.223}$$

$$= t^{-(\eta+1/2)}\left(\frac{\partial^2 g}{\partial t^2} - \frac{(\eta^2-1/4)}{t^2}g(x,t)\right) \tag{7.224}$$

On the other side of the EPD equation, we have:

$$\nabla^2 f = t^{-(\eta+1/2)}\nabla^2 g(x,t) = t^{-(\eta+1/2)}\left(\frac{\partial^2 g}{\partial t^2} - \frac{(\eta^2-1/4)}{t^2}g(x,t)\right) \tag{7.225}$$

hence we arrive at the Bose invariant form of the EPD equations:

$$\nabla^2 g(x,t) = \frac{\partial^2 g}{\partial t^2} - \frac{(\eta^2-1/4)}{t^2}g(x,t) \tag{7.226}$$

□

We can state a more general form of this result:

Lemma 7.4.24. *Under any continuous function $\phi(x)$, the Laplacian transforms as:*

$$[\phi(x)]^{-1}\nabla^2(\phi(x)f(x)) = \nabla^2 f + \left(\frac{2\nabla\phi}{\phi}\right)\cdot\nabla f + (\nabla^2\phi)f \tag{7.227}$$

This is the product rule for vector calculus. In this context, we may apply it to any Laplacian operator in this thesis.

Proof. Using elementary differential calculus, we have:

$$[\phi(x)]^{-1}\nabla^2(\phi(x)f(x)) = [\phi(x)]^{-1}\nabla\cdot(\nabla(\phi(x)f(x))) \tag{7.228}$$

$$= [\phi(x)]^{-1}\left[\nabla\cdot(\phi\nabla f + f\nabla\phi)\right] \tag{7.229}$$

$$= [\phi(x)]^{-1}\left[\phi\nabla^2 f + f\nabla^2\phi + 2\nabla f\cdot\nabla\phi\right] \tag{7.230}$$

$$= \nabla^2 f + \left(\frac{2\nabla\phi}{\phi}\right)\cdot\nabla f + (\nabla^2\phi)f \tag{7.231}$$

as required □

Remark 154. It is straightforward to see how the inherent dispersion in the co-ordinate transformation adds as a potential term $\nabla^2\phi$, and a linear drift is contributed of magnitude $\frac{2\nabla\phi}{\phi}$.

We note the following basic corollary of this calculation, which we summarise in the result below.

Corollary 7.4.25. *There exists a transformation between the EPD equation and the wave equation with potential. This transformation takes the form:*

$$x^\alpha\frac{\partial^2}{\partial x^2}\left(x^{-\alpha}f(x,t)\right) = \frac{\partial^2 f}{\partial x^2} - \frac{2\alpha}{x}\frac{\partial f}{\partial x} + \frac{\alpha(\alpha+1)}{x^2}f \tag{7.232}$$

or equivalently:

$$x^{-(\eta+1/2)}\frac{\partial^2}{\partial x^2}\left(x^{\eta+1/2}f(x,t)\right) = \frac{\partial^2 f}{\partial x^2} + \frac{(2\eta+1)}{x}\frac{\partial f}{\partial x} + \frac{(\eta^2-1/4)}{x^2}f \tag{7.233}$$

Proof. The first relationship follows as a result from basic calculus. The equivalence follows upon setting $-2\alpha = 1+2\eta$ i.e. $\alpha = -(1/2+\eta)$. □

Remark 155. These calculations raise the question of symmetry transformations more directly. For insight into how these types of transformations may be derived from the kernel perspective, note that the form of the kernel is given by the solution operator:

$$x^{-\eta}K_{e,1}(\hat{x},t) = Ct^{-2\eta}\int_0^\infty e^{-st^2}s^{-\eta-1}e^{-x^2/(4s)}ds \tag{7.234}$$

from which we can recover the scalar function. Generally, we can see that this is essentially the scale factor of the system as given in 6.7.21.

There are other extensions that may be made to this basic expression, many of the forms of generalised EPD equations explored in this chapter can be seen in this context. In a sense the radial term we have extracted in order to find these simplified equations represents a symmetry of the kernel. We shall discuss in following sections some ways in which these types of spherical factors may be understood with respect to the heat, wave and EPD equations.

7.4.4 Kernel Solutions to the EPD Equation

The kernel solution to the EPD equation may be recovered through use of an eigenfunction decomposition. In particular, we recover a kernel given by a Mehler-Fock function of the hyperbolic distance. In the following, we take our fundamental differential equation equal to the EPD equation (EPD PDE):

Definition 7.4.26.

$$\frac{\partial^2 g}{\partial t^2} + \frac{(1+2\eta)}{t}\frac{\partial g}{\partial t} = \hat{A}g \tag{7.235}$$

with the operator $\hat{A} = \nabla^2$ given by the Laplacian.

Applying the theory of the solution operator, we have:

Lemma 7.4.27. *The solution operator for the differential system 7.4.26 is given by:*

$$g(x,t) = \cosh\left(x\sqrt{\hat{A}}\right) \tag{7.236}$$

Proof. Using the solution function in the form where the Laplacian is the action operator, we write:

$$\nabla^2 g(x,t) = \hat{A}g(x,t) \tag{7.237}$$

$$g(x,t) = \cosh\left(x\sqrt{\hat{A}}\right) \tag{7.238}$$

□

We can write down the form of the eigenfunction:

Lemma 7.4.28. *There exists an eigenfunction solution to the EPD equation 7.4.26. In terms of the standing wave methodology, this can be written:*

$$g(x,t) = C\sin(kx)t^{-\eta}J_\eta(kt) \tag{7.239}$$

Proof. By inspection, this solves the equation as written. □

We then can formulate the kernel using the eigenfunction expansion. We have:

Theorem 7.4.29. *Assume the EPD equation:*

$$\frac{\partial^2 g}{\partial t^2} + \frac{(1+2\eta)}{t}\frac{\partial g}{\partial t} = \hat{A}g \tag{7.240}$$

with the operator $\hat{A} = \nabla^2$. *Then there exists a kernel solution of the form:*

$$K(t,\tau;x) = \frac{C}{2(t\tau)^{\eta+1/2}}\mathcal{P}_{\eta-1/2}\left(\frac{\tau^2+t^2-x^2}{2t\tau}\right) \tag{7.241}$$

where $\mathcal{P}_{\eta-1/2}()$ *is the Mehler-Fock function, a special case of the Legendre function.*

Proof. We have the kernel representation via the eigenfunction decomposition:

$$K(t,\tau;x) = (t\tau)^{-\eta}\int_0^\infty \sin(kx)J_\eta(kt)J_\eta(k\tau)dk \tag{7.242}$$

Inserting the expression for the integral formula eq. 6.672.1 from [122], we obtain the result. □

Remark 156. It is extremely interesting that integral formulae of the type:

$$e^{-tb^2} = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty} e^{-s^2/4t}e^{isb}ds = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty} e^{-s^2/4t}\cos(bs)ds \tag{7.243}$$

may be manipulated into a condition whereby the argument of the function is a self-adjoint operator. In this case, there is a consistent meaning between the right and left-hand sides of this equation, and we may apply it in order to find various solution operators. Other such formulae may obviously be found for the modified Bessel functions, as used in the previous section, we had the expression:

$$\int_0^\infty e^{-st^2}s^{-\eta-1}e^{-x^2/(4s)}ds = 2^{1+\nu}\left(\frac{x}{t}\right)^{-\eta}K_\eta(xt) \tag{7.244}$$

which generalised to give a solution operator we used in the EPD equation. It is interesting to think of higher order special functions where the argument is a matrix. The theory of such functions has a known relationship with certain eigenvalue problems in random matrices. This method of solution operators is more powerful than one would expect at face value. Such techniques have great potential to open up the study of kernels via direct results that may be readily derived using this method. Dettman, in [166,167] gives the following transmutation operator between the solution to the EPD equation and the heat equation:

$$\frac{\partial^2 u}{\partial t^2} + \frac{(1+2\eta)}{t}\frac{\partial u}{\partial t} = \nabla_N^2 u \tag{7.245}$$

$$\frac{\partial v}{\partial t} = \nabla_N^2 v \tag{7.246}$$

$$v(\mathbf{x},t) = \frac{2}{(4t)^{m+1}\Gamma(m+1)}\int_0^\infty e^{-\xi^2/4t}\xi^{2m+1}u(\mathbf{x},\xi)d\xi \tag{7.247}$$

It is these types of relationships we shall now calculate. However, there are several key differences. Firstly, the EPD equations we shall be dealing with modify this approach by allowing other operators instead of this division of time and space. In particular, we may modify the time and space variables. However, the underlying result remains the same as we shall demonstrate. In general, for equations of this class we are able to find integral transforms that relate the solutions to wave, EPD and heat kernels in a very similar way to this important result.

7.4.5 Kernel Solution to Bessel Heat Equation

For completeness, we shall now outline how to derive similar kernels using other results that are available in integral tables. Specifically, we shall show that the heat kernel equivalent to the EPD equation can be found using the eigenfunction decomposition. In this case, we shall take the EPD component in space, and the derivative in time, which represents the operator $\hat{A}$ in the previous calculation. All the work we have done previously on equivalences for EPD equations and so on is amenable to permutation of the time variable with that of space. If we take the basic heat equation equivalent to the EPD equation as written, we will have the expression:

Definition 7.4.30. The space-EPD heat system is given by the partial differential equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{(1+2\eta)}{x}\frac{\partial f}{\partial x} = \frac{\partial f}{\partial t} \tag{7.248}$$

We then have the following simple application of the eigenfunction form of the kernel.

Lemma 7.4.31. *The kernel of the space-EPD heat system is given by the eigenfunction decomposition:*

$$K(x,y;t) = (xy)^{-\eta} \int_0^\infty e^{-k^2 t} J_\eta(kx) J_\eta(ky) d\mathfrak{m}(k) \tag{7.249}$$

This has compact form:

$$K(x,y;t) = \frac{1}{2t(xy)^\eta} \exp\left(-\frac{x^2+y^2}{4t}\right) I_\eta\left(\frac{xy}{2t}\right) \tag{7.250}$$

where $J_\eta(.), I_\eta(.)$ are the standard Bessel function of the first kind and modified Bessel function of the second kind (Basset function) respectively, see 3.4.2 in this work.

Proof. We take the space-EPD heat equation from the definition:

$$\frac{\partial^2 f}{\partial x^2} + \frac{(1+2\eta)}{x}\frac{\partial f}{\partial x} = \frac{\partial f}{\partial t} \tag{7.251}$$

This obviously implies the eigenfunction problem which solves the system:

$$\frac{\partial f}{\partial t} = -\lambda f = \frac{\partial^2 f}{\partial x^2} + \frac{(1+2\eta)}{x}\frac{\partial f}{\partial x} \tag{7.252}$$

If we examine the eigenfunction structure of this expression, we can satisfy this through:

$$\phi_{\eta,k}(x) = x^{-\eta} J_\eta(kx) \tag{7.253}$$

To calculate the kernel, we require the measure of the space. The kernel may be written using the eigenfunction decomposition:

$$K(x,y;t) = (xy)^{-\eta} \int_0^\infty e^{-k^2 t} J_\eta(kx) J_\eta(ky) d\mathfrak{m}(k) \tag{7.254}$$

Consulting integral tables, we have formula [122] GR 6.633.2:

$$\int_0^\infty e^{-k^2 t} J_\nu(kx) J_\nu(ky) k dk = \frac{1}{2t} \exp\left(-\frac{x^2+y^2}{4t}\right) I_\nu\left(\frac{xy}{2t}\right) \tag{7.255}$$

Making the necessary identifications, we have the heat kernel for the Bessel equation defined through:

$$K(x,y;t) = \frac{1}{2t(xy)^\eta} \exp\left(-\frac{x^2+y^2}{4t}\right) I_\eta\left(\frac{xy}{2t}\right) \tag{7.256}$$

which solves the PDE system by construction. □

7.4.6 Kernel Solution to Bessel Wave Equation

The Bessel wave equation equivalent to the EPD system has a solution given by an eigenfunction decomposition. This solution is a Mehler-Fock function in the hyperbolic distance. In this subsection, we shall assume the following PDE system:

Definition 7.4.32. The space-EPD wave equation is given by the PDE system:

$$\frac{\partial^2 f}{\partial x^2} + \frac{(1+2\eta)}{x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial t^2} \tag{7.257}$$

This is simply addressed, as we may permute the time and space co-ordinates from the problem previously solved to write down the solution straight away. However, as we shall show, similar results using eigenfunction decompositions solve this problem.

Lemma 7.4.33. *The solution to the PDE system given by 7.4.32 is:*

$$K(x,y;t) = \frac{C}{2(xy)^{\eta+1/2}}\mathcal{P}_{\eta-1/2}\left(\frac{x^2+y^2-t^2}{2xy}\right) \tag{7.258}$$

where $\mathcal{P}_{\eta-1/2}(.)$ is the Mehler-Fock function.

Proof. It is also straightforward to see that any solution of this equation must also be given by a kernel of the type:

$$K(x,y;t) = (xy)^{-\eta}\int_0^\infty \sin(kt)J_\eta(kx)J_\eta(ky)dk \tag{7.259}$$

$$= \frac{1}{2(xy)^{\eta+1/2}}\mathcal{P}_{\eta-1/2}\left(\frac{x^2+y^2-t^2}{2xy}\right) \tag{7.260}$$

which solves the singular Bessel wave equation by inspection. Note the integral formulae GR 6.672.1 from [122]

$$\int_0^\infty \sin(kt)J_\nu(kx)J_\nu(ky)dk = \frac{1}{2\sqrt{xy}}\mathcal{P}_{\nu-1/2}\left(\frac{x^2+y^2-t^2}{2xy}\right) \tag{7.261}$$

and also the Laplace transform 6.612.3 [122]:

$$\int_0^\infty e^{-kt}J_\nu(kx)J_\nu(ky)dk = \frac{1}{\pi\sqrt{xy}}Q_{\nu-1/2}\left(\frac{t^2+x^2+y^2}{2xy}\right) \tag{7.262}$$

which relate the hyperbolic distance, given by the argument of the function on the right, with the kernel decomposition formula on the left. One can see how the modification of the exponent from real exponential to purely complex exponential changes the solution space, so these expressions can be understood in the same way. □

7.4.7 Fourier-Laplace Transforms of the EPD Equation

We shall now discuss some applications of Fourier and Laplace transforms to EPD type equations. This is a useful way in which to view these expressions, as the transformed equations are often quite simple to understand. The generalised EPD equation may be solved using successive transformations of Fourier and Laplace type. As the following computation shall show, the EPD equation is associated with a first order inhomogenous hyperbolic PDE in the Green's function.

Definition 7.4.34. The general form of a separable EPD equation can be written as:

$$\hat{A}h = \mathscr{H}(t)h + \hat{B}h \tag{7.263}$$

where the EPD operator is defined through the singular Bessel derivative:

$$\frac{\partial^2 f}{\partial t^2} + \frac{\alpha}{t}\frac{\partial f}{\partial t} = \hat{\mathscr{H}}(t)f \tag{7.264}$$

We now take the following lemma:

Lemma 7.4.35. *Assume the separable EPD system given by the expression:*

$$\frac{\partial^2 f}{\partial t^2} + \frac{\alpha}{t}\frac{\partial f}{\partial t} = \hat{\mathscr{H}}(t)f = \left(\hat{A} - \hat{B}\right)f = \hat{\mathscr{A}}f \tag{7.265}$$

In the situation where the operators are separable, we have

$$\hat{\mathscr{H}}(t)f = \hat{\mathscr{A}}(x)f \tag{7.266}$$

i.e. one operator is solely a function of time, the other solely a function of space, then the application of Fourier and Laplace transforms results in the intertwining relation:

$$\mathcal{L}\left[\hat{\mathscr{H}}(t)\mathcal{F}\left[f(x,t)\right]\right] = \mathcal{F}\left[\hat{\mathscr{A}}(x)\mathcal{L}\left[f(x,t)\right]\right] \tag{7.267}$$

Proof. From the definition of the lemma, we have the operator:

$$\frac{\partial^2 f}{\partial t^2} + \frac{\alpha}{t}\frac{\partial f}{\partial t} = \hat{\mathscr{H}}(t)f = \hat{\mathscr{A}}f \tag{7.268}$$

The operators are separable, hence:

$$\hat{\mathscr{H}}(t)f = \hat{\mathscr{A}}(x)f \tag{7.269}$$

Taking a Laplace transform in time, and a Fourier transform in space, we arrive at:

$$\mathcal{F}\left[\mathcal{L}\left[\hat{\mathscr{H}}(t)f(x,t)\right]\right] = \mathcal{F}\left[\mathcal{L}\left[\hat{\mathscr{A}}(x)f(x,t)\right]\right] \tag{7.270}$$

Assuming that we can rearrange the integrals under the conditions of Fubini-Tonelli, we may write:

$$\mathcal{L}\left[\hat{\mathscr{H}}(t)\mathcal{F}\left[f(x,t)\right]\right] = \mathcal{F}\left[\hat{\mathscr{A}}(x)\mathcal{L}\left[f(x,t)\right]\right] \tag{7.271}$$

as required. □

We now show the result of the application of this lemma to a certain type of EPD equation.

Theorem 7.4.36. *Assume the EPD system defined by:*

$$\frac{\partial^2 f}{\partial t^2} + \frac{\alpha}{t}\frac{\partial f}{\partial t} = \nabla^2 f \tag{7.272}$$

where the solution is $f = f(\boldsymbol{x}, t)$. *Green's function for this problem satisfies the hyperbolic PDE:*

$$-(E^2 + |\boldsymbol{k}|^2)\frac{\partial G}{\partial E} + f(\boldsymbol{x}, 0)\left[1 - \alpha\right] + (\alpha E + 2)G(\boldsymbol{k}, E) = 0 \tag{7.273}$$

Proof. We have the Green's function as the application of Fourier and Laplace transforms of the solution:

$$G = G(\boldsymbol{k}, E) = \mathcal{F}_{\boldsymbol{x}}\left[\mathcal{L}_t\left[f(\boldsymbol{x}, t)\right]\right] = \mathcal{L}_t\left[\mathcal{F}_{\boldsymbol{x}}\left[f(\boldsymbol{x}, t)\right]\right] \tag{7.274}$$

i.e. the solution in energy-momentum space. We know that the Laplace operator transforms according to

$$\mathcal{F}_{\boldsymbol{x}}\left[\nabla^2 f\right] = -|\boldsymbol{k}|^2 \mathcal{F}_{\boldsymbol{x}}\left[f\right] \tag{7.275}$$

$$\mathcal{F}_{\boldsymbol{x}}\left[f\right] = \int e^{i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}, t) d^3x \tag{7.276}$$

and the Green's function can be written as the integral over space and time:

$$G(\boldsymbol{k}, E) = \int_0^\infty \int_{-\infty}^{+\infty} e^{-Et + i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}, t) d^3x.dt \tag{7.277}$$

We have the transformation laws

$$\mathcal{F}_{\boldsymbol{x}}\left[\mathcal{L}_t\left[t\frac{\partial^2 f}{\partial t^2}\right]\right] = \mathcal{L}_t\left[\mathcal{F}_{\boldsymbol{x}}\left[t\frac{\partial^2 f}{\partial t^2}\right]\right] = -2G(\boldsymbol{k}, E) - E^2\frac{\partial G}{\partial E} + f(\boldsymbol{x}, 0) \tag{7.278}$$

$$\mathcal{F}_{\boldsymbol{x}}\left[\mathcal{L}_t\left[\alpha\frac{\partial f}{\partial t}\right]\right] = \mathcal{L}_t\left[\mathcal{F}_{\boldsymbol{x}}\left[\alpha\frac{\partial f}{\partial t}\right]\right] = \alpha\left[EG(\boldsymbol{k}, E) - f(\boldsymbol{x}, 0)\right] \tag{7.279}$$

We therefore obtain the hyperbolic differential equation:

$$-2G(\boldsymbol{k}, E) - E^2\frac{\partial G}{\partial E} + f(\boldsymbol{x}, 0) + \alpha\left[EG(\boldsymbol{k}, E) - f(\boldsymbol{x}, 0)\right] = \mathcal{L}_t\left[\mathcal{F}_{\boldsymbol{x}}\left[t\nabla^2 f\right]\right] \tag{7.280}$$

$$= \mathcal{L}_t\left[t\mathcal{F}_{\boldsymbol{x}}\left[\nabla^2 f\right]\right] = -|\boldsymbol{k}|^2\mathcal{L}_t\left[tU(\boldsymbol{k}, t)\right] \tag{7.281}$$

$$= |\boldsymbol{k}|^2\frac{\partial G}{\partial E} \tag{7.282}$$

Consolidating the coefficients, we find the result:

$$-(E^2 + |\boldsymbol{k}|^2)\frac{\partial G}{\partial E} + f(\boldsymbol{x}, 0)\left[1 - \alpha\right] + (\alpha E + 2)G(\boldsymbol{k}, E) = 0 \tag{7.283}$$

□

This ends our calculations regarding EPD systems and their connection to hyperbolic PDEs. We shall now recap the results and go over the major findings of this chapter.

7.5 Comments

We have shown in this work that many of the relationships known within various fields of integral calculus, special functions and so on can be understood as different aspects of the same phenomena. The Euler-Poisson-Darboux equations give one way in which we can understand many of these connections. Our major results are to generalise the class of kernels, eigenfunctions and associated PDEs from the EPD equation, and reformulate it in a way in which the fundamental solutions and kernels may be written down without a great deal of effort beyond consulting integral tables. This method is powerful and allows us to sidestep many of the difficulties involved in evaluating group representations of special functions.

The kernel functionals we have derived have an obvious relationship with relativity principles. However, the novelty here lies in the fact that a number of the systems we have encountered are of the type SO(1,2), i.e. a single space dimension, with two times. The results are complementary to other calculations related to diffusion on a hyperbolic plane; extensions to path integration are straightforward via the standard expression giving the connection between the Green's function and the kernel/transition probability density. The symmetries between known results for space diffusive systems and those where time is given by a stochastic variable are hopefully put on a stronger footing through these simple forms of analysis.

In terms of the classification of singular hyperbolic equations, we have opened a new chapter in the study of these objects through the use of simple considerations using spherical functions, group theory and transformations of PDEs. This work is definitely related to the ground-breaking work carried out in [98,99,166,167], where the transmutation operators between the wave kernels, heat kernel and EPD equations were calculated. The approach we have taken is more geometric in nature, and many of the operators familiar from this earlier analysis arise as a consequence of the application of this updated method. Possible new avenues of investigation include the extension of the Radon-type transform to the new hyperbolic realms defined through the various families of PDEs. These might have application in tomography; as other authors have calculated various types of scattering and imaging transforms, this seems to be an interesting and scientifically useful place to examine.

As a touchstone in the universe of special functions, this simple example of Euler, Poisson and Darboux is surprisingly descriptive of a large number of different phenomena. From simple precepts, we have been able to describe a large number of significant and important special functions that are familiar from other contexts in the analysis of special functions, kernels and integral transformations. Of course, the underlying reason for this is that this system is equivalent to the heat equation on a group, which in this case is given through the spherical surface. The transmutation equations give this integral transform, which is equivalent to the Weierstrass or generalised Weierstrass transform of the solution operator. The altered perspective in this case means that in the new co-ordinates the heat equation appears as the wave equation or the EPD equation.

Part II

Concluding Remarks

We shall now comment on the results contained in this thesis, and analyse the various complementary methodologies that have been utilised within this work.

We have shown that one can derive many of the known groups of special functions from the axiom of the quantum brachistochrone, when coupled with the representation theory of the Fubini-Study metric via the Laplacian. The Fubini-Study metric appears as a result of the projective geometry we associate to any diffusion problem; with the assistance of this insight we are able to write down the induced Laplacian on the group.

In doing so, we have shown how the representation theory of the quantum and hyperbolic brachistochrone can be constructed through use of the Laplacian operator on a curved space. The connection is, of course, the metric, which we derived in this thesis through use of the unitary operators that flow from the hyperbolic and/or quantum brachistochrone equation. This allows the construction of the hyperbolic heat kernel in a direct, intuitive fashion.

Complementary to this analysis, we have shown how the addition laws of special relativity may be seen as a consequence of the theorems of hyperbolic geometry, in particular the addition formula of hyperbolic triangles, deriving this in full from the hyperbolic brachistochrone equation and associated Hamiltonian and time evolution operators. Extending this, we have also demonstrated how one may construct a singular projective state which possesses a metric that is analogous to the Friedmann-Robertson-Walker metric, familiar in general relativity. This is a novel result, which shows the utility of this geometric perspective.

Underlying many of these results is the powerful method of the Bose invariant. This technique allows us to transform any second order PDE into a diagonal form where analysis is most easily achieved. We have applied this in order to achieve a close analysis of the speed, scale and invariant measures related to the systems in this thesis. In doing so, we have sidestepped many of the difficulties associated with transformation groups and symmetries of differential equations, directly deriving the eigenfunctions and using that to build up the kernels via decomposition formulae in a bottom-up analytical approach.

We have seen how the projection-slice theorem may be generalised to hyperbolic systems. In the same way that the Abel, Fourier and Hankel transforms are connected through this theorem, we have shown how the Mehler-Fock (associated Legendre/toroidal harmonics), Whittaker and modified Bessel functions are related in a similar fashion. Direct application of this method allows solution of a number of different index integrals via the Fubini-Tonelli theorem. In particular, we were able to find several mixed index integrals, composition formulae for the Kontorovich-Lebedev functions, the cosine transform of Whittaker functions, product lemmas, and a number of generalised Yor integrals that reduce to known functions for special cases. These newly accessible formulations give clarity to the general theory of index integration, and point to the way forward for the development of innovative analysis in this field. The process centres on the application of transformation formulae between the different hyperbolic eigenfunctions, as given by the modified Bessel, Mehler-Fock (associated Legendre) and Whittaker functions. As we have demonstrated in detail, this can be understood either through transformation of the integrals which define the kernel, or the PDE systems which describe its dynamics. The connection is, of course, the projection-slice theorem in the hyperbolic plane.

The Radon transformation presents as a useful extension of the method of the Fubini-Study metric and hyperbolic brachistochrone to the complex projective sphere. We have shown how this is related to the Poincare sphere, and analysed the connection between this set of projective metrics and the Euler-Poisson-Darboux equation. Radon transforms are an important application of analysis, tomography having many uses within non-destructive testing, medicine and other areas. We have used this entry point to derive a number of time dispersive kernels related to the Euler-Poisson-Darboux equation, generalised appropriately to hyperbolic systems. In doing so, we have found a large class of singular hyperbolic equations and essentially solvable operators related to special functions and index transforms.

Finally, we have shown how the methods developed from analysis of the heat kernel are extended to the wave and Euler-Poisson-Darboux kernels through use of the Weierstrass transform. Using the technique of solution operators, we were able to derive a number of solutions for the Euler-Poisson-Darboux formula in time, and found the transformation laws between these systems in the hyperbolic plane. The Fourier-Laplace transform shows that the Euler-Poisson-Darboux problem can be related to a certain hyperbolic PDE for the Green's function.

On a deeper level, we have shown in this thesis how one may approach the question of kernel solutions to diffusion problems through use of eigenfunction decompositions, and a clever application of ideas from differential geometry. This is classically expressed through the representation theory of positive definite functions on groups, and the spherical function theory which gives the product law on the group. In this thesis, we have taken the perspective that these ideas necessarily flow from the solution to the differential geometry, which stems from the solutions for the unitary or pseudounitary operators defined by the quantum or hyperbolic brachistochrone equation. This sidesteps the difficulty of identification of group actions, and allows us direct access to the eigenfunction representations via the Laplacian in a curved space. From knowledge of the group invariants, such as the Laplacian, the metric, and the distance function, we can build up decompositions of the kernel via the eigenfunctions, and develop our model of the system in a novel and intuitive way. This gives us insight into many complicated results that are previously only known to be accessible using path integrals and stochastic processes.

We have shown how the various different forms of special functions arise in this representation of the hyperbolic plane. We can see how only simple modifications are required in order to cover large classes of functions, all of which share similar relationships to the same underlying problems. This theory of construction for the kernel will work in any situation where the spectrum is imiscid as in 3.1.17, containing a continuous and a discrete part with a single crossover point. It is interesting to think of extensions to this simple question of eigenfunction decomposition, of bounded states and the nature of eigenstate continuity.

In this thesis, we have seen how the question of continuous states reduces to the problem of a single continuous eigenvalue, which in quantum mechanics can be seen as the wavefunction for a scattered particle. In finance, these solutions are most often seen due to index integration, which appear in problems related to Asian option pricing. We have shown how these questions can be all directly related to the analysis of diffusion on the hyperbolic plane. One would hope that future endeavours would focus on a program of reductive analysis to uncover further connections between the worlds of projective geometry, group homomorphisms and representation theory. It is hoped that this work offers some concrete proof as to the utility of examining these links between the branches of differential analysis.

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7.6 Summary of Important Formulae

Time Optimal Quantum Control

Quantum Brachistochrone Equation

$$i\frac{d}{dt}(\tilde{H}(t)+\tilde{F}(t))=[\tilde{H}(t),\tilde{F}(t)] \tag{7.284}$$

Quantum Fermat Principle

$$\delta t=\int 1dt+\text{constraints} \tag{7.285}$$

$$\int\frac{\sqrt{\left\langle d\Psi\left|\left(1-\hat{P}\right)\right|d\Psi\right\rangle}}{\Delta E}+\int\lambda_1\mathrm{Tr}[\tilde{H}\tilde{F}]dt+\int\lambda_2\left(\mathrm{Tr}[\tilde{H}^2/2]-k\right)dt=\min \tag{7.286}$$

Fubini-Study Metric

$$F_{\alpha\beta}=\left\langle\bar{\Psi}_\alpha\right|\left.\Psi_\beta\right\rangle-\left\langle\bar{\Psi}_\alpha\right|\left.\Psi\right\rangle\left\langle\Psi\right|\left.\Psi_\beta\right\rangle=\left\langle\bar{\Psi}_\alpha\right|\left(\mathbf{1}-\hat{P}(\tau,\varphi,\psi)\right)\left|\Psi_\beta\right\rangle \tag{7.287}$$

$$\left|\Psi_\beta\right\rangle=\frac{\partial}{\partial x_\beta}\left|\Psi\right\rangle \tag{7.288}$$

$$g_{\alpha\beta}=\mathfrak{Re}\left[F_{\alpha\beta}\right] \tag{7.289}$$

Differential Geometry

Differential Geometry of the Hyperboloid

$$\frac{4dz.dz^*}{(1-|z|^2)^2}=\sinh^2\tau d\phi^2+d\tau^2 \tag{7.290}$$

$$ds^2=\frac{4(dX^2+dY^2)}{(1-(X^2+Y^2))^2} \tag{7.291}$$

Curvilinear Laplace Operator

$$\nabla^2 f=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\alpha}\left(\sqrt{g}g^{\alpha\beta}\frac{\partial f}{\partial x_\beta}\right) \tag{7.292}$$

$$g=\det g_{\alpha\beta} \tag{7.293}$$

Other Forms of Hyperbolic Metrics

$$g_{\alpha\beta}=\begin{bmatrix} y^{-2} & 0 \\ 0 & y^{-2}\end{bmatrix} \tag{7.294}$$

$$\nabla^2 f=y^2\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right) \tag{7.295}$$

Theory of Kernels

Positive Definite Kernels

Mercer's Theorem

$$\sum_{u,v\in G} \left\langle F(u^{-1}v)\psi(v), \psi(u)\right\rangle \geq 0 \tag{7.296}$$

with $K(u,v) = F(u^{-1}v)$ as kernel.

Unitary Homomorphism

$$\Phi(u^{-1}) = \Phi^*(u) \tag{7.297}$$

Projective Representation

$$F(u^{-1}v) = \hat{P}\Phi(u^{-1}v) \tag{7.298}$$

Principles of Invariants

Kernel as Invariant Function

$$K(x,y) = K(y,x) = K(d(x,y)) \tag{7.299}$$

Kernel as Group Invariant of Distance

$$K(\hat{\Omega}\hat{g}, \hat{\Omega}\hat{g}_l) = K(\hat{g}, \hat{g}_l) \tag{7.300}$$

$$K(\hat{g}, \hat{g}_l) = K(d(\hat{g}^{-1}\hat{g}_l, \mathbf{1})) \tag{7.301}$$

Spectral Theorems and Partition Functions

Spectral Decomposition of Kernel

$$K(\mathbf{x},\mathbf{y};t) = \int_{\Omega_c} e^{-E_p t}\phi_p^*(\mathbf{x})\phi_p(\mathbf{y})d\mathfrak{m}(p) + \sum_{\Omega_d} |c_n|^2 e^{-E_n t}\psi_n^*(\mathbf{x})\psi_n(\mathbf{y}) \tag{7.302}$$

Quantum Zeta Function

$$Z(s) = \sum_n \frac{1}{E_n^s} = \frac{i^s}{\Gamma(s)}\int_0^\infty t^{s-1}\int K(x,t|x,0)dxdt \tag{7.303}$$

Trace Kernel/Partition Function

$$K(x,t|x,0) = \sum_n e^{-E_n t}|\psi_n(x)|^2 = \sum_n e^{-E_n t} \tag{7.304}$$

Spherical Functions

Spherical Function on a Group

$$\int_K f(gkh)dk = f(g)f(h) \tag{7.305}$$

Spherical Decomposition of Distance Function

$$\int_K f(d_K(x,y))dk = f(x)f(y) \tag{7.306}$$

Transforms of PDEs

Bose Invariant

$$\mathcal{L}f = a(y)\frac{\partial^2 f}{\partial y^2} + b(y)\frac{\partial f}{\partial y} + c(y)f(y) \tag{7.307}$$

$$\mathcal{I}(y) = \frac{2(a'b - b'a) + 4ac - b^2}{4a^2} \tag{7.308}$$

Speed and Scale Distributions

$$\mathfrak{m}(x) = \frac{2}{\sigma^2(x)} \exp\left(\int^x \frac{2b(s)}{\sigma^2(s)} ds\right) = \frac{1}{a} \exp\left(\int^x \frac{b(s)}{a(s)} ds\right) \tag{7.309}$$

$$\mathfrak{s}'(x) = \exp\left(-\int^x \frac{b(s)}{a(s)} ds\right) \tag{7.310}$$