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Benchmark-neutral pricing

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The paper proposes benchmark-neutral pricing and hedging for long-term contingent claims. It employs the growth optimal portfolio of the stocks as numéraire and the new benchmark-neutral pricing measure for pricing. For the assumed 'natural' dynamics of a well-diversified stock portfolio, which are those of the continuous limit of a branching process of diversified wealth in some activity time, this pricing measure turns out to be an equivalent probability measure. This is not the case for the putative risk-neutral pricing measure. Benchmark-neutral pricing identifies the minimal possible prices of contingent claims. Risk-neutral prices of long-term contracts can be significantly more expensive than necessary. The extremely accurate hedge of a long-term zero-coupon bond illustrates the proposed pricing and hedging method.

Keywords: Long-term pricing; Benchmark approach; Change of numéraire; Activity time; Squared Bessel process; Hedging

JEL Classifications: G10, G11
Mathematics Subject Classifications: 62P05, 60G35, 62P20

1. Introduction

Risk-neutral pricing employs the savings account or cash account as numéraire and represents the preferred pricing method of the classical finance theory; see, e.g. Merton (1992), Cochrane (2001), and Jarrow (2022). The current paper assumes the 'natural' dynamics of a well-diversified stock portfolio as those of the continuous time limit of a branching process, see Feller (1971), of diversified wealth in some flexible activity time. The seminal *No Free Lunch with Vanishing Risk* (NFLVR) no-arbitrage condition of Delbaen and Schachermayer (1998) was already shown in Loewenstein and Willard (2000) and Platen (2001) to fail for some perfectly acceptable financial market models. In Platen and Fergusson (2025), the NFLVR condition has been falsified for real markets. It fails also for the model employed in this paper, which does not have an equivalent risk-neutral probability measure.

The *growth optimal portfolio* (GOP) of stocks, as described, e.g. in Kelly (1956) and Merton (1992), is approximated by well-diversified stock portfolios; see Platen and Rendek (2020). For the benchmark approach, see Platen (2006) and Platen and Heath (2006), the failure of the NFLVR condition does not represent a problem. The benchmark approach only requires the existence of the GOP and not the existence of an equivalent risk-neutral probability measure. This matters because when the putative risk-neutral measure is not an equivalent probability measure, it is shown in Platen (2002) and Platen (2006) that risk-neutral pricing leads to higher prices than necessary.

For a given market model, the existence of the GOP is an extremely weak and easily verifiable *no-arbitrage condition* because Karatzas and Kardaras (2007) and Karatzas and Kardaras (2021) have shown that the existence of the GOP is equivalent to their *No Unbounded Profit with Bounded Risk* (NUPBR) condition. This no-arbitrage condition is weaker than the NFLVR condition. The assumed model for a well-diversified stock portfolio follows a squared Bessel process of dimension four in some flexible activity time and satisfies the NUPBR condition. For this model, the hedge of a long-term zero-coupon bond is shown to be impressively accurate, which indicates in addition to other empirical properties that the model is realistic and the NFLVR condition seems to fail in reality.

The benchmark approach provides the pricing concept of real-world pricing, where the GOP of the entire market is taken as the numéraire and the real-world probability measure acts as the pricing measure; see Platen and Heath (2006). Real-world pricing avoids the additional assumptions that a change of the GOP of the entire market as numéraire to another numéraire would require.

For a typical market that consists of stocks and the savings account, the GOP of the entire market is a highly leveraged

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portfolio that goes short in the savings account and long in a portfolio of stocks; see, e.g. Theorem 7.1 in Filipović and Platen (2009). This GOP does not represent a suitable numéraire or even a desirable investment portfolio; see, e.g. Samuelson (1979). To generate it physically one must go short in the savings account. Since only discrete-time dynamic asset allocation is feasible, one could obtain a negative portfolio as a proxy of the GOP of the entire market, which would fail as a numéraire for pricing and hedging.

As a feasible alternative to real-world pricing, the current paper proposes employing as a numéraire the *stock GOP*, which is the GOP of the investment universe formed only by the stocks (without the savings account). As shown by Theorem A.3 in Platen and Rendek (2020), the stock GOP can be approximated by a well-diversified, guaranteed strictly positive portfolio of stocks. For instance, the MSCI-Total Return Stock Index (MSCI) of the developed markets could serve as a reasonable proxy of the stock GOP. However, better proxies are available, as shown in Platen and Rendek (2020).

The proposed new pricing method is called *benchmark-neutral* (BN) *pricing*. It has wider applicability than risk-neutral pricing and can be conveniently implemented, as will be demonstrated in the current paper. It provides the minimal possible prices of nonnegative contingent claims when the ratio of the stock GOP over the GOP of the entire market is a martingale. This ratio represents the Radon–Nikodym derivative of the respective *BN pricing measure*. It is a nonnegative local martingale and, therefore, a supermartingale. This means, its current value is greater than or equal to its expected future values given the current information. Under the assumed 'natural' dynamics of the stock GOP, the above-mentioned ratio turns out to be a martingale, and the BN prices of contingent claims coincide with the respective minimal possible prices obtained via the real-world pricing formula in Proposition 3.3 of Du and Platen (2016).

Motivated by the structure of the stochastic differential equation (SDE) of the stock GOP of a continuous market, the *minimal market model* (MMM) was proposed in Platen (2001) as a potential model for the stock GOP. When approximating the activity time as a linear function of the calendar time, the MMM coincides with the stock GOP model assumed in the current paper. As the paper will demonstrate, the MMM evolving in some flexible *activity time* captures impressively well the 'natural' evolution of well-diversified stock indexes. It models parsimoniously the volatility of the normalized stock GOP as a scalar diffusion process that is evolving in its flexible activity time. The latter captures the impact of the trading activity on the stock GOP dynamics and can be observed. It turns out that the activity time does not need to be modeled in detail and only its average value must be predicted for the pricing and hedging of a long-term zero-coupon bond.

It seems to be extremely difficult to model accurately the volatility of a stock index, as pointed out by the 'leverage effect puzzle' of Ait-Sahalia *et al.* (2013). The assumed model separates the modeling of the leverage effect from that of the trading activity and points in a direction where this puzzle could be resolved. The current paper applies the MMM in some flexible activity time for the stock GOP and illustrates the fitting of its few parameters, as well as, the BN pricing and hedging of a long-term zero-coupon bond that pays one unit of the savings account at maturity.

In previous works, long-term zero-coupon bonds and other derivatives have been priced and hedged using the MMM by assuming that it evolves in some activity time that is a linear function of the calendar time and the GOP of the entire market equals the stock GOP; see, e.g. Platen and Heath (2006), Fergusson and Platen (2023), and Barone-Adesi *et al.* (2024). The novelty of the current paper is that the GOP of the entire market is different from the stock GOP. Furthermore, the stock GOP dynamics are far more realistically modeled by assuming and observing a flexible activity time. The activity time can be observed but only its average value at maturity needs to be predicted for the pricing and hedging of a long-term zero-coupon bond. This is important in practice because long-term zero coupon bonds serve as the building blocks of long-term life insurance contracts and pension payment streams.

The paper is organized as follows: Section 2 introduces the market setting and real-world pricing under the benchmark approach. The new concept of benchmark-neutral pricing is presented in section 3. The model of the stock GOP is introduced in section 4. Section 5 illustrates the BN pricing and hedging of a long-term zero-coupon bond.

2. Market setting

2.1. *Primary security accounts*

The modeling is performed on a filtered probability space $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, P)$, satisfying the usual conditions; see, e.g. Karatzas and Shreve (1991) and Karatzas and Shreve (1998). The filtration $\underline{\mathcal{F}} = (\mathcal{F}_t)_{t\in[t_0,\infty)}$ describes the evolution of market information over time. For an investment universe consisting of stocks, we assume that a *growth optimal portfolio* (GOP) S_t^* exists, which is the strictly positive stock portfolio with maximum growth rate; see Kelly (1956) and Merton (1992). This portfolio we call the *stock GOP*. In the given market we model, for simplicity, one adapted, nonnegative, *risky primary security account* as Itô diffusion, denoted by S_t^1, where all dividends or other payments are reinvested. If not otherwise mentioned, the securities are assumed to be denominated in units of the savings account $S_t^0 = 1$. To focus in the current paper on the properties of the proposed benchmark-neutral pricing method, we set, for simplicity, the risky primary security account S_t^1 equal to the stock GOP S_t^*. We emphasize that the savings account $S_t^0 = 1$ is not included in the investment universe that has the stock GOP S_t^* as its GOP.

The stock GOP S_t^* in savings account denomination is assumed to be continuous and to satisfy according to Theorem 3.1 in Filipović and Platen (2009) the stochastic differential equation (SDE)

$$\frac{\mathrm{d}S_t^*}{S_t^*} = \lambda_t^* \mathrm{d}t + \theta_t(\theta_t \mathrm{d}t + \mathrm{d}W_t) \tag{1}$$

for $t \in [t_0, \infty)$ with $S_{t_0}^* > 0$. Here, λ_t^* denotes the *net risk-adjusted return* of S_t^*, θ_t its *volatility*, and $W = \{W_t, t \in [t_0, \infty)\}$ a $(P, \underline{\mathcal{F}})$-Brownian motion in calendar time.

When we add the savings account $S_t^0 = 1$ as an additional primary security account to the market and assume that the GOP S_t^{**} of the extended market exists, it follows directly by Theorem 7.1 in Filipović and Platen (2009) that the GOP S_t^{**} of the extended market satisfies the SDE

$$\frac{dS_t^{**}}{S_t^{**}} = \sigma_t^{**}(\sigma_t^{**}dt + dW_t) \tag{2}$$

with initial value $S_{t_0}^{**} = 1$ and *market price of risk*

$$\sigma_t^{**} = \frac{\lambda_t^*}{\theta_t} + \theta_t \tag{3}$$

for $t \in [t_0, \infty)$.

2.2. *Real-world pricing*

We denote by $\mathbf{E}^P(.|\mathcal{F}_t)$ the conditional expectation under the real-world probability measure P, conditional on the information $\mathcal{F}_t$ available at time t. Consider a bounded stopping time $T > t_0$, and let $\mathcal{L}^1(\mathcal{F}_T)$ denote the set of integrable, $\mathcal{F}_T$-measurable random variables in the given filtered probability space.

DEFINITION 2.1 *For a bounded stopping time $T \in (t_0, \infty)$, a nonnegative payoff H_T, denominated in units of the savings account is called a contingent claim if $\frac{H_T}{S_T^{**}} \in \mathcal{L}^1(\mathcal{F}_T)$.*

As shown in Section 10.4 in Platen and Heath (2006) and in Corollary 6.2 in Du and Platen (2016), for a contingent claim H_T with maturity at a bounded stopping time T the *real-world pricing formula*

$$H_t = S_t^{**}\mathbf{E}^P\left(\frac{H_T}{S_T^{**}}|\mathcal{F}_t\right) \tag{4}$$

determines its, so called, *fair* price H_t for all $t \in [t_0, T]$. The ratio $\frac{H_t}{S_t^{**}}$ forms a $(P, \underline{\mathcal{F}})$-martingale, which is P-almost surely unique for the given random payoff $\frac{H_T}{S_T^{**}}$ at maturity T. The real-world pricing formula uses the GOP S_t^{**} of the extended market as numéraire and the real-world probability measure P as pricing measure. It has been shown by Corollary 6.2 in Du and Platen (2016) that, when a contingent claim is replicable, its fair price process coincides with the value process of the minimal possible self-financing hedge portfolio that replicates its payoff.

The Law of One Price of the classical finance theory does no longer hold because there exist other pricing rules that can be applied to pricing and hedging, including the popular risk-neutral pricing rule. However, these pricing rules never provide lower prices for nonnegative replicable contingent claims than the real-world pricing formula because all self-financing portfolios that hedge a given contingent claim form $(P, \underline{\mathcal{F}})$-supermartingales when denominated in S_t^{**}. The $(P, \underline{\mathcal{F}})$-martingale among these $(P, \underline{\mathcal{F}})$-supermartingales coincides with the fair hedge portfolio value process in the denomination of the GOP S_t^{**} of the extended market. The fair hedge portfolio is the least expensive hedge portfolio that replicates the contingent claim; see Lemma A.1 in Du and Platen (2016). More expensive self-financing hedge portfolios can exist that replicate the contingent claim and one may not even realize that these are more expensive than necessary.

3. Benchmark-neutral pricing

3.1. *Change of numéraire*

The numéraire for real-world pricing is the GOP S_t^{**} of the extended market, which is, in reality, a highly leveraged portfolio that goes long in the stock GOP S_t^* and short in the savings account $S_t^0 = 1$. When hedging contingent claims, one needs to be able to trade the numéraire that one is using for pricing and hedging. For instance, when hedging a zero-coupon bond that pays one unit of the savings account at maturity, the hedging requires the trading of the numéraire and the savings account, as will be shown later.

A tradeable proxy of the highly leveraged GOP S_t^{**} of the extended market cannot be easily constructed as a guaranteed strictly positive, self-financing portfolio because such a highly leveraged portfolio can only be traded at discrete times and, therefore, faces the possibility of becoming negative. To avoid the above-mentioned difficulties, the paper suggests employing the stock GOP S_t^* as a numéraire.

As shown by Theorem A.3 in Platen and Rendek (2020), a well-diversified total return stock index is a reasonable proxy for the stock GOP and can be made, by construction, strictly positive. Total return stock indexes have been used traditionally as benchmarks in portfolio management. The current paper suggests employing such a benchmark as a numéraire for pricing and hedging. It calls the new pricing method *benchmark-neutral pricing* (BN pricing) and the proxy for the stock GOP the *benchmark*. Intuitively, in the denomination of the benchmark and under the respective BN pricing measure the expected returns of portfolios are zero and, in this sense, 'neutral' to the randomness that drives the stock market.

The stock GOP has, in the long run, a trajectory that is almost surely pathwise outperforming any other strictly positive stock portfolio; see Theorem 10.5.1 in Platen and Heath (2006). Intuitively, BN pricing centers the risk management around the long-run best-performing strictly positive stock portfolio, whereas risk-neutral pricing centers it around the rather poorly performing savings account.

By application of the Itô formula it follows that the stock GOP S_t^*, when denominated in units of the GOP S_t^{**}, satisfies a driftless SDE and is, therefore, a $(P, \underline{\mathcal{F}})$-local martingale. We make throughout the paper the following assumption, which we verify later for the realistic stock GOP model that we will employ:

ASSUMPTION 3.1 *The stock GOP S_t^*, when denominated in units of the GOP S_t^{**}, forms the $(P, \underline{\mathcal{F}})$-martingale $\frac{S^*}{S^{**}} = \left\{\frac{S_t^*}{S_t^{**}}, t \in [t_0, \infty)\right\}$, where $\frac{S_t^*}{S_t^{**}}$ is the unique strong solution of the SDE*

$$\frac{d\left(\frac{S_t^*}{S_t^{**}}\right)}{\frac{S_t^*}{S_t^{**}}} = -\sigma^{S^*}(t)dW_t \tag{5}$$

with finite integrated squared volatility

$$\int_{t_0}^{t} \sigma^{S^*}(s)^2 \mathrm{d}s < \infty$$

for $t \in [t_0, \infty)$.

Under Assumption 3.1, the proposed change of numéraire permits the application of the change of numéraire technique of Geman *et al.* (1995). This leads us to consider the Radon–Nikodym derivative

$$\Lambda_{S^*}(t) = \frac{\mathrm{d}Q_{S^*}}{\mathrm{d}P}\bigg|_{\mathcal{F}_t} = \frac{\frac{S_t^*}{S_t^{**}}}{\frac{S_{t_0}^*}{S_{t_0}^{**}}} = \exp\left\{-\int_{t_0}^{t} \sigma^{S^*}(s)\mathrm{d}W_s - \frac{1}{2}\int_{t_0}^{t} \sigma^{S^*}(s)^2 \mathrm{d}s\right\}, \tag{6}$$

which characterizes for the numéraire S_t^* the respective *BN pricing measure* Q_{S^*}. By application of the Itô formula we obtain from (1) and (2) the following result:

Corollary 3.2 *Under Assumption 3.1,* Λ_{S^*} *satisfies the SDE*

$$\frac{\mathrm{d}\Lambda_{S^*}(t)}{\Lambda_{S^*}(t)} = -\sigma^{S^*}(t)\mathrm{d}W_t \tag{7}$$

with

$$\sigma^{S^*}(t) = \frac{\lambda_t^*}{\theta_t} \tag{8}$$

for $t \in [t_0, \infty)$.

Under Assumption 3.1, for $T \in [t_0, \infty)$ we have for an event $A \in \mathcal{F}_T$ its BN probability

$$Q_{S^*}(A) = \mathbf{E}^P(\Lambda_{S^*}(t)\mathbf{1}_A) = \mathbf{E}^{Q_{S^*}}(\mathbf{1}_A). \tag{9}$$

Here $\mathbf{1}_A$ denotes the indicator function of the event A and $\mathbf{E}^{Q_{S^*}}(.)$ the expectation under Q_{S^*}. Two measures are *equivalent* if they have the same sets of events of measure zero. One says that a measure is an equivalent probabilty measure if it is equivalent to the real-world probability measure P.

As a self-financing portfolio that is denominated in units of the GOP S_t^{**} of the extended market, the Radon–Nikodym derivative $\Lambda_{S^*}(t)$ forms a $(P, \underline{\mathcal{F}})$-local martingale. Under Assumption 3.1 it is assumed to be a true martingale. Let $\mathbf{E}^{Q_{S^*}}(.|\mathcal{F}_t)$ denote the conditional expectation with respect to the BN pricing measure Q_{S^*} under the information available at time $t \in [t_0, \infty)$. By Theorem 9.5.1 in Platen and Heath (2006) one obtains directly the following Bayes Theorem:

Corollary 3.3 *Under Assumption 3.1, for some bounded stopping time* $T \in [t_0, \infty)$ *and any* $\mathcal{F}_T$*-measurable random variable* $Y = \frac{S_{t_0}^* H_T}{S_T^*}$*, satisfying the integrability condition* $\mathbf{E}^{Q_{S^*}}(|Y|) < \infty$*, one has the Bayes rule*

$$\mathbf{E}^{Q_{S^*}}(Y|\mathcal{F}_s) = \frac{\mathbf{E}^P(\Lambda_{S^*}(T)Y|\mathcal{F}_s)}{\mathbf{E}^P(\Lambda_{S^*}(T)|\mathcal{F}_s)} \tag{10}$$

for $s \in [t_0, T]$ *and, therefore,*

$$\mathbf{E}^{Q_{S^*}}\left(\frac{S_{t_0}^* H_T}{S_T^*}|\mathcal{F}_{t_0}\right) = \mathbf{E}^P\left(\frac{S_{t_0}^{**} H_T}{S_T^{**}}|\mathcal{F}_{t_0}\right). \tag{11}$$

We obtain directly from Corollary 3.3 and Theorem 9.5.2 in Platen and Heath (2006) the following result that includes a BN version of Girsanov's Theorem:

Theorem 3.4 (BN Pricing Formula) *Under the Assumption 3.1, for some bounded stopping time* $T \in [t_0, \infty)$ *and an* $\mathcal{F}_T$*-measurable contingent claim* H_T*, satisfying the integrability condition* $\mathbf{E}^{Q_{S^*}}(\frac{H_T}{S_T^*}) < \infty$*, the BN pricing measure* Q_{S^*} *is an equivalent probability measure, and the fair price* H_t *at time* $t \in [t_0, T]$*, which the real-world pricing formula identifies for* H_T*, is obtained via the BN pricing formula*

$$H_t = S_t^* \mathbf{E}^{Q_{S^*}}\left(\frac{H_T}{S_T^*}|\mathcal{F}_t\right) \tag{12}$$

for $t \in [t_0, T]$*. The process* $\bar{W} = \{\bar{W}_t, t \in [t_0, \infty)\}$*, satisfying the SDE*

$$\mathrm{d}\bar{W}_t = \sigma^{S^*}(t)\mathrm{d}t + \mathrm{d}W_t \tag{13}$$

for $t \in [t_0, \infty)$ *with* $\bar{W}_{t_0} = 0$*, is under* Q_{S^*} *a Brownian motion with respect to calendar time.*

This result is of practical importance because it allows one to use the stock GOP S_t^* as a numéraire for pricing and hedging under the BN pricing measure Q_{S^*}. We will see later for the pricing of a zero-coupon bond that BN pricing is capturing the dynamics under Q_{S^*} as if under P the net risk-adjusted return λ_t^* were zero.

3.2. Portfolios

The market participants can combine primary security accounts to form portfolios. Denote by $\delta = \{\delta_t = (\delta_t^0, \delta_t^1)^\top, t \in [t_0, \infty)\}$ the strategy, where $\delta_t^j, j \in \{0, 1\}$, represents the number of units of the jth primary security account that are held at time $t \in [t_0, \infty)$ in a corresponding portfolio S_t^δ. When denominated in units of the stock GOP S_t^*, this portfolio is captured by the process $\tilde{S}^\delta = \{\tilde{S}_t^\delta = \frac{S_t^\delta}{S_t^*}, t \in [t_0, \infty)\}$, where

$$\tilde{S}_t^\delta = (\delta_t)^\top \tilde{\mathbf{S}}_t \tag{14}$$

for $t \in [t_0, \infty)$ with $\tilde{\mathbf{S}}_t = (\tilde{S}_t^0, \tilde{S}_t^1)^\top$. If changes in the value of a portfolio are only due to changes in the values of the primary security accounts, then no extra funds flow in or out of the portfolio, and the corresponding portfolio and strategy are called *self-financing*. The self-financing property of a portfolio is expressed by the equation

$$\tilde{S}_t^\delta = \tilde{S}_{t_0}^\delta + \int_{t_0}^{t} (\delta_s)^\top \mathrm{d}\tilde{\mathbf{S}}_s \tag{15}$$

for $t \in [t_0, \infty)$ with $\tilde{S}_{t_0}^\delta = (\delta_{t_0})^\top \tilde{\mathbf{S}}_{t_0}$, where the stochastic integral in (15) is assumed to be a vector-Itô integral; see Shiryaev and Cherny (2002).

To introduce a class of admissible strategies for forming portfolios, denote by $[\tilde{\mathbf{S}}_.]_t = ([\tilde{S}_.^i, \tilde{S}_.^j]_t)_{i,j=0}^1$ the matrix-valued optional covariation of the vector of stock GOP-denominated primary security accounts $\tilde{\mathbf{S}}_t$ for $t \in [t_0, \infty)$.

Definition 3.5 *An admissible self-financing strategy* $\delta = \{\delta_t = (\delta_t^0, \delta_t^1)^\top, t \in [t_0, \infty)\}$*, initiated at the time* t_0*, is an*

$\mathbf{R^2}$*-valued, predictable stochastic process, satisfying the condition*

$$\int_{t_0}^{t} \delta_u^\top [\tilde{\mathbf{S}}_\cdot]_u \delta_u \mathrm{d}u < \infty \tag{16}$$

for $t \in [t_0, \infty)$.

An admissible self-financing strategy generates the stock GOP-denominated gains from trade

$$\int_{t_0}^{t} \delta_s^\top \mathrm{d}\tilde{\mathbf{S}}_\mathbf{s} = \int_{t_0}^{t} \mathrm{d}\tilde{S}_s^\delta = \tilde{S}_t^\delta - \tilde{S}_{t_0}^\delta \tag{17}$$

for $t \in [t_0, \infty)$. It does this without requiring outside funds or generating extra funds. The predictability of the integrand in the above stock GOP-denominated gains from trade expresses the real informational constraint that the allocation of units of primary security accounts in the admissible self-financing strategy δ is not allowed to anticipate the movements of the stock GOP-denominated primary security account vector $\tilde{\mathbf{S}}_\mathbf{t}$.

3.3. Contingent claims

In the following, we consider contingent claims that can be replicated by using self-financing portfolios under BN pricing. Let for a bounded stopping time T the set $\mathcal{L}^1_{Q_{S^*}}(\mathcal{F}_T)$ denote the set of $\mathcal{F}_T$-measurable and Q_{S^*}-integrable random variables in the filtered probability space $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, Q_{S^*})$.

Definition 3.6 *We call for a bounded stopping time* $T \in [t_0, \infty)$ *a stock GOP-denominated contingent claim* $\tilde{H}_T = \frac{H_T}{S_T^*} \in \mathcal{L}^1_{Q_{S^*}}(\mathcal{F}_T)$ *BN-replicable if it has for all* $t \in [t_0, T]$ *a representation of the form*

$$\tilde{H}_T = \mathbf{E}^{Q_{S^*}}(\tilde{H}_T|\mathcal{F}_t) + \int_t^T \delta_{\tilde{H}_T}(s)^\top \mathrm{d}\tilde{\mathbf{S}}_\mathbf{s} \tag{18}$$

Q_{S^*}*-almost surely, involving some predictable vector process* $\delta_{\tilde{H}_T} = \{\delta_{\tilde{H}_T}(t) = (\delta^0_{\tilde{H}_T}(t), \delta^1_{\tilde{H}_T}(t))^\top, t \in [t_0, T]\}$ *satisfying the condition* (16).

To capture the replication of a targeted contingent claim we introduce the following notion:

Definition 3.7 *We say, an admissible self-financing strategy* $\delta = \{\delta_t = (\delta_t^0, \delta_t^1)^\top, t \in [t_0, T]\}$ *delivers the stock GOP-denominated BN-replicable contingent claim* $\tilde{H}_T$ *at a bounded stopping time* T *if the equality*

$$\tilde{S}_T^\delta = \tilde{H}_T \tag{19}$$

holds Q_{S^*}*-almost surely.*

Combining Definitions 3.5, 3.6, 3.7, and the SDE (15), leads directly to the following statement:

Corollary 3.8 *For a BN-replicable stock GOP-denominated contingent claim* $\tilde{H}_T$ *with representation* (18), *there exists an admissible self-financing strategy* $\delta_{\tilde{H}_T} = \{\delta_{\tilde{H}_T}(t) = (\delta^0_{\tilde{H}_T}(t), \delta^1_{\tilde{H}_T}(t))^\top, t \in [t_0, T]\}$ *with corresponding stock GOP-denominated price process* $\tilde{S}_t^{\delta_{\tilde{H}_T}} = \tilde{H}_t$ *given by the BN pricing formula*

$$\tilde{H}_t = \mathbf{E}^{Q_{S^*}}(\tilde{H}_T|\mathcal{F}_t), \tag{20}$$

which delivers the stock GOP-denominated contingent claim

$$\tilde{H}_T = \tilde{S}_T^\delta \tag{21}$$

Q_{S^*}*-almost surely.*

The stock GOP-denominated price $\tilde{H}_t$ at time $t \in [t_0, T]$ represents, within the set of admissible self-financing strategies, the minimal possible self-financing portfolio process that delivers the stock GOP-denominated contingent claim $\tilde{H}_T$.

3.4. Hedging replicable claims

Recall that the jth stock GOP-denominated primary security account process $\tilde{S}_t^j$, $j \in \{0, 1\}$, is a $(Q_{S^*}, \underline{\mathcal{F}})$-local martingale under the BN pricing measure Q_{S^*}. Consequently, a stock GOP-denominated self-financing portfolio $\tilde{S}_t^\delta$ is a $(Q_{S^*}, \underline{\mathcal{F}})$-local martingale.

Consider a stock GOP-denominated BN-replicable contingent claim $\tilde{H}_T$ with a bounded stopping time T as maturity, where under Q_{S^*} its entire randomness is driven by the $(Q_{S^*}, \underline{\mathcal{F}})$-Brownian motion $\bar{W}$. Each stock GOP-denominated primary security account $\tilde{S}_t^j$, $j \in \{0, 1\}$, is assumed to satisfy an SDE of the form

$$\frac{\mathrm{d}\tilde{S}_t^j}{\tilde{S}_t^j} = -\Phi_t^{j,1}\mathrm{d}\bar{W}_t \tag{22}$$

for $t \in [t_0, \infty)$ with $\tilde{S}_{t_0}^j > 0$. We assume that $\Phi_t^{j,1} = \{\Phi_t^{j,1}, t \in [t_0, \infty)\}$ forms for each $j \in \{0, 1\}$ a predictable process such that the above stochastic differentials are well defined; see Karatzas and Shreve (1998). For $t \in [t_0, \infty)$ we denote by $\Phi_t = [\Phi_t^{j,k}]_{j,k=0}^{1,1}$ the matrix with elements $\Phi_t^{j,1}$ for $j \in \{0, 1\}$, and

$$\Phi_t^{j,0} = 1 \tag{23}$$

for $j \in \{0, 1\}$. Let us impose the following assumption:

Assumption 3.9 *We assume that a BN-replicable stock GOP-denominated contingent claim* $\tilde{H}_T$ *has for its fair stock GOP-denominated price at time* $t \in [t_0, T]$ *under* Q_{S^*} *a unique martingale representation of the form*

$$\tilde{H}_t = \tilde{H}_{t_0} + \int_{t_0}^{t} x_s \mathrm{d}W_s, \tag{24}$$

where x_t *is predictable and the integral*

$$\int_{t_0}^{t} x_s^2 \mathrm{d}s < \infty \tag{25}$$

is Q_{S^*}*-almost surely finite for* $t \in [t_0, \infty)$.

As in the proof of Proposition 7.1 in Du and Platen (2016), we obtain the following result:

THEOREM 3.10 *If the matrix* Φ_t *is Lebesgue-almost everywhere invertible, then the strategy* $\delta_{\tilde{H}_T}$ *is given by the vector*

$$\delta_{\tilde{H}_T}(t) = diag(\tilde{\mathbf{S}}_t)^{-1}(\Phi_t^\top)^{-1}\xi_t \tag{26}$$

with

$$\xi_t = (-x_t, \tilde{H}_t)^\top \tag{27}$$

for all $t \in [t_0, T)$.

Here diag($\mathbf{S}$) denotes the diagonal matrix with the elements of a vector $\mathbf{S}$ as its diagonal. In the case when the dynamics of the extended market are modeled using state variables that satisfy the SDEs of a Markovian system of diffusions, one can systematically identify for a given stock GOP-denominated BN-replicable contingent claim $\tilde{H}_T$ the respective representation (18). The price $\tilde{H}_t$ at the time t can be obtained, e.g. by explicit calculation of the conditional expectation, by application of the Feynman–Kac formula, or via some numerical method. The price results as a function of the state variables that satisfies a respective partial differential equation (PDE).

The integrands in the representation (24) and the predictable vector ξ_t emerge when applying the Itô formula to the price function and matching the respective terms in the martingale part of the resulting SDE. The PDE operator follows by setting the drift part in the resulting SDE for the price function to zero. Consequently, the price function satisfies a Kolmogorov-backward PDE.

The boundary conditions of the PDE need to be specified such that the solution of the PDE, as a function of the evolving state variables, becomes a $(Q_{S^*}, \underline{\mathcal{F}})$-martingale. When only fixing the PDE operator and the boundary conditions that are determined by the payoff structure of the contingent claim, there may exist several price functions that solve the PDE. All these solutions yield nonnegative $(Q_{S^*}, \underline{\mathcal{F}})$-local martingales. Since these processes form $(Q_{S^*}, \underline{\mathcal{F}})$-supermartingales that deliver the targeted contingent claim, they yield price processes that are larger than or equal to the one that forms the *fair* price process, which is the $(Q_{S^*}, \underline{\mathcal{F}})$-martingale. We emphasize, it is the fair price process that delivers at time T the stock GOP-denominated contingent claim $\tilde{H}_T$ by starting from the most economical minimal possible stock GOP-denominated initial price $\tilde{H}_{t_0}$.

4. Stock GOP dynamics

4.1. Equivalent BN pricing measure

The question arises whether Assumption 3.1, which provides the martingale property of the Radon–Nikodym derivative of the BN pricing measure, is realistic. This means, whether it is realistic for existing stock markets to model the Radon–Nikodym derivative $\Lambda_{S^*}(t)$ of the putative BN pricing measure Q_{S^*} as a true $(P, \underline{\mathcal{F}})$-martingale. For the realistic dynamics of the stock GOP that we will assume, this answer will be positive. It is closely related to the boundary behavior of the volatility $\sigma^{S^*}(t) = \frac{\lambda_t^*}{\theta_t}$, see (8), of the Radon–Nikodym derivative $\Lambda_{S^*}(t)$; see, e.g. Andersen and Piterbarg (2007), Mijatovic and Urusov (2012), Hulley and Platen (2012), and Hulley and Ruf (2012). Intuitively, according to the mentioned references, the numéraire S_t^*, when denominated in the GOP S_t^{**} of the extended market, is a martingale when its volatility $-\sigma^{S^*}(t)$ remains finite for finite values of S_t^*, including its asymptotic value at the boundary where it approaches zero.

To illustrate BN pricing, the current paper assumes a model where the stock GOP evolves in some *activity time* $\tau = \{\tau_t, t \in [t_0, \infty)\}$ with *activity* $a_t = \frac{d\tau_t}{dt} \in (0, \infty)$ for $t \in [t_0, \infty)$ starting with the *initial activity time* τ_{t_0}, where

$$\tau_t = \tau_{t_0} + \int_{t_0}^t a_s ds. \tag{28}$$

The activity is the derivative of the time in which the market dynamics evolve. The net risk-adjusted return λ_t^* is a Lagrange multiplier, see Theorem 3.1 in Filipović and Platen (2009), and does not need to be modeled under BN pricing because it is not relevant under the proposed change of measure. For simplicity, the current paper makes the simplifying assumption that the net risk-adjusted return is proportional to the activity, which means

$$\lambda_t^* = \bar{\lambda} a_t \tag{29}$$

with $\bar{\lambda} > 0$ for $t \in [t_0, \infty)$. Since the net risk-adjusted return is notoriously difficult to estimate, the fact that its value does not matter when applying BN pricing is of practical importance and simplifies the implementation. Another simplification for the implementation of BN pricing and hedging for long-term contingent claims, like long-term zero-coupon bonds, arises from the fact that one does not have to model the random evolution of the activity time. Only its average value at the maturity date has to be predicted.

4.2. Minimal market model in activity time

A branching process is a continuous time model for the size of a population where individuals independently give birth to new individuals or die; see Feller (1971). The current paper assumes that the 'natural' continuous evolution of diversified wealth in some activity time is that of the continuous limit of a branching process because wealth units independently generate new ones or vanish. This assumption, when empirically confirmed, provides a basis for the understanding of the 'natural' dynamics (in some activity time) of well-diversified stock portfolios, including the stock GOP.

The continuous limit of a branching process is known to be that of a squared Bessel process or squared radial Ornstein–Uhlenbeck process; see Feller (1971) and Göing-Jaeschke and Yor (2003). For our modeling the characteristic property of a squared Bessel process is important, which is that its squared diffusion coefficient is proportional to its value. To understand heuristically the reason for this property, one notices that if the constituents of a diversified portfolio evolve independently and follow similar dynamics, then the variance of the increment of the portfolio value equals the sum of the variances of the increments of the constituents. Consequently, in a respective diffusion limit, the square of the diffusion coefficient of such a branching process turns out to be proportional to the sum of these variances and, therefore, proportional to the portfolio value.

Only the diffusion coefficient of the SDE (1) for the stock GOP S_t^* needs to be modeled for BN pricing, which we assume to be of the form

$$\theta_t S_t^* = \sqrt{4e^{\tau_t} a_t}\sqrt{S_t^*} \tag{30}$$

for $t \in [t_0, \infty)$.

Due to the structure of the SDE for the stock GOP, the *minimal market model* (MMM) has been suggested by Platen (2001) as a realistic model for the long-term stock GOP dynamics. The MMM models the stock GOP as a time transformed squared Bessel process of dimension four, see (8.7.1) in Platen and Heath (2006). This is consistent with the above assumed 'natural' dynamics of the stock GOP. For the MMM, described, e.g. in Section 13.2 in Platen and Heath (2006), when evolving in activity time, the volatility of the stock GOP dynamics in calendar time can be directly obtained from (30) in the form

$$\theta_t = \sqrt{\frac{4e^{\tau_t} a_t}{S_t^*}} \tag{31}$$

for $t \in [t_0, \infty)$. Note that it is not the volatility that is here the fundamental quantity that determines the structure of the diffusion coefficient of the MMM dynamics. It is the diffusion coefficient (30) itself that is proportional to the square root of the stock GOP as a consequence of the continuous limit of a branching process. By applying this model, the stock GOP is according to (1) and (13) assumed to satisfy under the BN pricing measure the SDE

$$dS_t^* = 4e^{\tau_t} d\tau_t + \sqrt{S_t^* 4e^{\tau_t}}\, d\bar{W}(\tau_t) \tag{32}$$

with $S_{t_0}^* > 0$ for $t \in [t_0, \infty)$. The process $\bar{W}(\tau_t)$ represents a Brownian motion under Q_{S^*} in activity time with stochastic differential

$$d\bar{W}(\tau_t) = \sqrt{a_t}\, d\bar{W}_t$$

for $t \in [t_0, \infty)$. The SDE (32) shows that S_t^* represents under Q_{S^*} a time-transformed squared Bessel process of dimension four; see Definition 1.1 of Chapter XI in Revuz and Yor (1999), or equation (8.7.1) in Platen and Heath (2006). One notes that its volatility in activity time exhibits a leverage effect and is, as a 3/2 volatility model, a constant elasticity of variance model, see, e.g. Cox (1975), Platen (1997), Heston (1997), and Lewis (2000). It is worth noting that the stock GOP S_t^* is in activity time under the real-world probability measure a branching process with imigration rate $\bar{\lambda}$; see Feller (1971). This means that BN pricing removes the immigration of wealth in the dynamics of S_t^*.

In Platen and Rendek (2008) it was shown with high significance that the log-returns of a well-diversified stock portfolio are Student-t distributed with four degrees of freedom, which coincides with what one would estimate when the underlying stock GOP dynamics were those of the MMM. This empirical fact supports the choice of the MMM in activity time as the model for the stock GOP. The current paper will add another supporting empirical fact by showing that under the MMM with flexible activity time, a long-term zero-coupon bond can be hedged with an impressively small hedge error.

4.3. Observed activity time

To observe the activity time, we consider the square root of the stock GOP $\sqrt{S_t^*}$ and obtain by a straightforward application of the Itô formula, (32), (29), and (13) the SDE

$$d\sqrt{S_t^*} = \frac{3e^{\tau_t}}{2\sqrt{S_t^*}} d\tau_t + \sqrt{e^{\tau_t}}\, d\bar{W}(\tau_t) \tag{33}$$

for $t \in [t_0, \infty)$. Since the measure change does not affect the diffusion coefficient in (33), the quadratic variation of $\sqrt{S_t^*}$ becomes

$$[\sqrt{S_\cdot^*}]_{t_0,t} = \int_{\tau_{t_0}}^{\tau_t} e^s ds = e^{\tau_t} - e^{\tau_{t_0}} \tag{34}$$

and the activity time results in the form

$$\tau_t = \ln\left([\sqrt{S_\cdot^*}]_{t_0,t} + e^{\tau_{t_0}}\right) \tag{35}$$

for $t \in [t_0, \infty)$.

For illustration, the current paper demonstrates the pricing and hedging of a zero-coupon bond by using the market capitalization-weighted total return stock index (MCI) that is displayed in figure 1. The MCI was generated in Platen and Rendek (2020) from stock data to match the daily observed US Dollar savings account-denominated MSCI-Total Return Stock Index for the developed markets. The US Dollar savings account was approximated by a roll-over account of 3-months US T-Bills. As shown in Theorem A.3 of Platen and Rendek (2020), one can interpret the MCI as a reasonable proxy for the stock GOP.

For some tentative initial activity time τ_{t_0}, one can observe the trajectory of the resulting respective activity time, which turns out to evolve approximately linearly. Only at the beginning of this trajectory, one typically notices some deviation from its approximate linearity, which changes with the choice of the tentative value of the initial activity time. We exploit the observed linearity and assume that the average of the activity time remains always a linear function of calendar time. Therefore, one can estimate the initial activity time by standard linear regression, which yields the activity time that is closest to a straight line. The resulting estimated *trendline*

$$\bar{\tau}_t = \bar{\tau}_{t_0} + \bar{a}(t - t_0) \tag{36}$$

is exhibited in figure 2 together with the MCI for the period from t_0 =2 January 1984 until T =1 November 2014. We observe for the trendline its *slope* $\bar{a} \approx 0.053$ and *initial value* $\bar{\tau}_{t_0} \approx 2.15$. The R^2-value of 0.98 confirms that the observed activity time evolves approximately linearly. Because of this finding, the current paper employs first the trendline $\bar{\tau}_t$ as a substitute for the activity time in BN pricing and hedging, and later the observed activity time as it evolves in an enhanced method of pricing and hedging.

Of course, when pricing contingent claims, one cannot look into the future. However, as the paper assumes, one can approximate the activity time of future maturity dates by using the trendline of the activity time when assuming that the trendline remains similar also in the future.

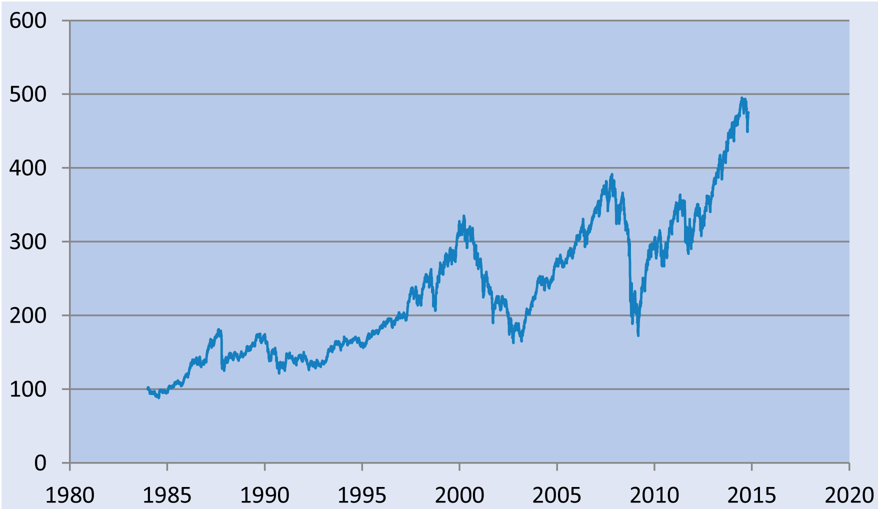

Figure 1. US Dollar savings account-denominated MCI.

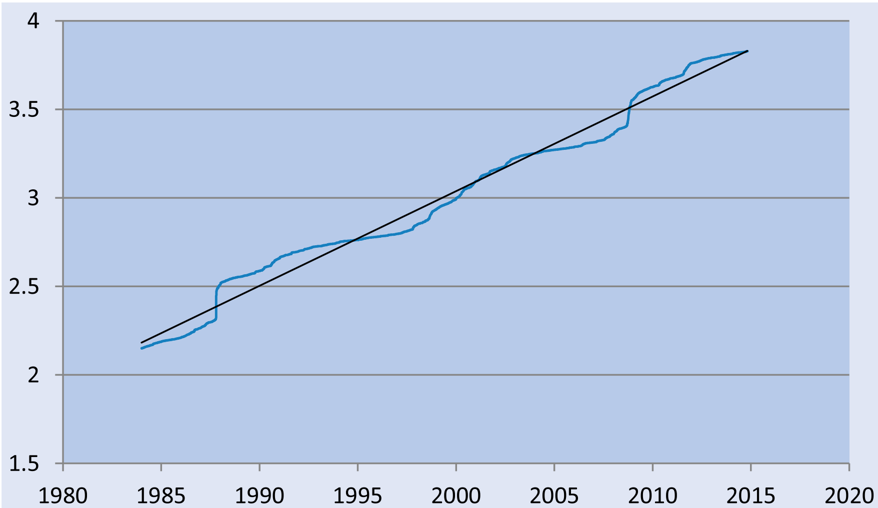

Figure 2. Activity time τ_t and trendline $\bar{\tau}_t$.

4.4. BN pricing measure

When using the trendline $\bar{\tau}_t$ with constant slope $\bar{a} \in (0, \infty)$ as a model for the activity time, it follows by (8) and (31) that the negative volatility $\sigma^{S^*}(t)$ of the Radon–Nikodym derivative of the BN pricing measure equals

$$\sigma^{S^*}(t) = \bar{\lambda}\sqrt{\frac{\bar{a}S_t^*}{4e^{\bar{\tau}_t}}} \tag{37}$$

for $t \in [t_0, \infty)$. The following statement is derived below:

THEOREM 4.1 *Under the MMM in activity time with the stock GOP satisfying the SDE* (32)*, where the activity time τ_t is assumed to equal its trendline $\bar{\tau}_t$, the Radon–Nikodym derivative of the BN pricing measure Q_{S^*} is a $(P, \underline{\mathcal{F}})$-martingale, the measure Q_{S^*} is an equivalent probability measure, and Assumption 3.1 is satisfied.*

Proof To prove the above statement we employ Theorem 2.1 in Mijatovic and Urusov (2012). It follows from (1) and (36) by application of the Itô formula that

$$Y_t = \frac{S_t^*}{e^{\bar{\tau}_t}} \tag{38}$$

satisfies the SDE

$$dY_t = (4 - (1 - \bar{\lambda})Y_t)\bar{a}dt + \sqrt{Y_t 4\bar{a}}dW_t, \tag{39}$$

which yields in the notation of Theorem 2.1 in Mijatovic and Urusov (2012) the drift coefficient function $\mu(x) = (4 - (1 - \bar{\lambda})x)\bar{a}$ and the diffusion coefficent function $\sigma(x) = \sqrt{x4\bar{a}}$ for $x \in J = (0, \infty)$. Both coefficients satisfy the Engelbert–Schmidt conditions. The SDE (38) characterizes a stationary radial Ornstein–Uhlenbeck process of dimension four, as studied in Göing-Jaeschke and Yor (2003), where it is shown that it has a unique strong solution and not only a unique weak solution, as requested. Furthermore, it is shown in the mentioned reference that this process does not exit P-almost surely the open interval $J = (0, \infty)$.

By (7), (37), and (38) the diffusion coefficient of the SDE for the Radon–Nikodym derivative emerges as $b(x) = -\frac{\bar{\lambda}}{2}\sqrt{\bar{a}x}$. Since the ratio

$$\frac{(b(x))^2}{(\sigma(x))^2} = \frac{\bar{\lambda}^2}{16} \tag{40}$$

is constant, condition (8) of Theorem 2.1 in Mijatovic and Urusov (2012) is satisfied.

As requested by this theorem, one must consider the J-valued process $\tilde{Y}$ that is characterized by the SDE

$$d\tilde{Y}_t = (4 - \tilde{Y}_t)\bar{a}dt + \sqrt{\tilde{Y}_t 4\bar{a}}dW_t, \tag{41}$$

which is again a stationary radial Ornstein–Uhlenbeck process of dimension four. As shown in Göing-Jaeschke and Yor (2003), this process does not exit the open interval $J = (0, \infty)$ P-almost surely. Consequently, the conditions (a) and (c) of Theorem 2.1 in Mijatovic and Urusov (2012) are satisfied, which proves that the Radon–Nikodym derivative process Λ_{S^*} of the BN pricing measure Q_{S^*} is a true martingale.

It follows by Geman *et al.* (1995) that Q_{S^*} is an equivalent probability measure. Indeed, the stock GOP dynamics before and after the measure change are given by squared radial Ornstein–Uhlenbeck processes of dimension four. Therefore, the sets of events with measure zero, in particular, the event of hitting zero, are the same. This completes the proof of Theorem 4.1. ■

4.5. *Putative risk-neutral pricing measure*

Risk-neutral pricing employs the savings account $S_t^0 = 1$ as numéraire and the classical finance theory postulates that the putative risk-neutral pricing measure Q_{S^0} is an equivalent probability measure, see, e.g. Cochrane (2001) and Delbaen and Schachermayer (1998). The following result shows that Q_{S^0} is not an equivalent probability measure under the above-described model:

THEOREM 4.2 *Under the MMM in activity time with the stock GOP satisfying the SDE* (32), *where the activity time τ_t equals its trendline $\bar{\tau}_t$, the Radon–Nikodym derivative Λ_{S^0} of the putative risk-neutral pricing measure Q_{S^0} is a strict $(P, \underline{\mathcal{F}})$-supermartingale and Q_{S^0} is not an equivalent probability measure.*

Proof The Radon–Nikodym derivative of the putative risk-neutral pricing measure Q_{S^0} equals

$$\Lambda_{S^0}(t) = \left.\frac{dQ_{S^0}}{dP}\right|_{\mathcal{F}_t} = \frac{S_t^0}{S_t^{**}} = \frac{1}{S_t^{**}} \tag{42}$$

for $t \in [t_0, \infty)$. It follows by (4) and Theorem 4.1 that

$$\mathbf{E}^P(\Lambda_{S^0}(s)|\mathcal{F}_t) = \mathbf{E}^P\left(\frac{1}{S_s^{**}}|\mathcal{F}_t\right) = \frac{H_t}{S_t^{**}} = \frac{S_t^*}{S_t^{**}}\mathbf{E}^{Q_{S^*}}\left(\frac{1}{S_s^*}|\mathcal{F}_t\right) \tag{43}$$

for $t_0 \le t < s < \infty$. Under Q_{S^*} the savings account-denominated stock GOP S_t^* is a time-transformed squared Bessel process of dimension four satisfying the SDE (32). Its inverse $\frac{1}{S_t^*}$ is a strict $(Q_{S^*}, \underline{\mathcal{F}})$-supermartingale; see (8.7.21) in Platen and Heath (2006). Therefore, we have

$$\mathbf{E}^P(\Lambda_{S^0}(s)|\mathcal{F}_t) < \frac{S_t^*}{S_t^{**}S_t^*} = \frac{1}{S_t^{**}} = \Lambda_{S^0}(t) \tag{44}$$

for $t_0 \le t < s < \infty$, which shows that Λ_{S^0} is a strict $(P, \underline{\mathcal{F}})$-supermartingale. In this case, the SDE of the savings account-discounted stock GOP under the putative risk-neutral measure turns by (32) out to be of the form

$$dS_t^* = \sqrt{S_t^* 4e^{\tau_t}}d\bar{W}^{RN}(\tau_t), \tag{45}$$

where $\bar{W}_{\tau_t}^{RN}$ would denote a Brownian motion in activity time under the putative risk-neutral measure. Under this measure, S_t^* would be a squared Bessel process of dimension zero, which has a strictly positive probability to reach zero; see (8.7.8) in Platen and Heath (2006). However, under the real-world probability measure P the squared Bessel process S_t^* is of dimension four and does never reach the value zero; see (8.7.7) in Platen and Heath (2006). Therefore, Q_{S^0} is not an equivalent probability measure, which proves Theorem 4.2. ■

5. Pricing and hedging of a zero-coupon bond

5.1. *BN pricing of a zero-coupon bond*

Under the above model, which employs the trendline of the activity time as model for the activity time, the transition probability density for the stock GOP S^* under Q_{S^*} has, according to Corollary 1.4 in Chapter XI of Revuz and Yor (1999), or Equation (8.7.9) in Platen and Heath (2006), the form

$$p(\bar{\tau}_t, S_t^*; \bar{\tau}_s, S_s^*) = \frac{1}{2(e^{\bar{\tau}_s} - e^{\bar{\tau}_t})}\left(\frac{S_s^*}{S_t^*}\right)^{\frac{1}{2}} \exp\left\{-\frac{S_t^* + S_s^*}{2(s^{\bar{\tau}_s} - e^{\bar{\tau}_t})}\right\} I_1\left(\frac{\sqrt{S_t^* S_s^*}}{e^{\bar{\tau}_s} - e^{\bar{\tau}_t}}\right) \tag{46}$$

for $t_0 \le t \le s < \infty$, where $I_1(.)$ denotes the modified Bessel function of the first kind with index 1; see, e.g. Abramowitz and Stegun (1972). Consequently, we know the transition probability density of the stock GOP under the BN pricing measure, where its key characteristic is the trendline of the activity time.

To illustrate BN pricing and hedging, we consider a zero-coupon bond with fixed maturity $T \in (t_0, \infty)$ and contingent claim $H_T = 1$. The respective fair zero-coupon bond

$$P(t,T) = \tilde{P}(t,T)S_t^* \tag{47}$$

pays at maturity one unit of the savings account $H_T = S_T^0 = 1$. Its value, obtained via the BN pricing formula (12), is given in the denomination of the savings account by the explicit formula

$$P(t,T) = S_t^* \mathbf{E}^{Q_{S^*}}\left(\frac{1}{S_T^*}|\mathcal{F}_t\right) = 1 - \exp\left\{-\frac{S_t^*}{2(e^{\bar{\tau}_T} - e^{\bar{\tau}_t})}\right\} \tag{48}$$

for $t \in [t_0, T)$; see Platen (2002), or Equation (13.3.5) in Platen and Heath (2006).

We display in figure 3 the savings account-discounted zero-coupon bond price with maturity at the end of our observation period, which emerges when using the MCI as a proxy for the stock GOP. One notes that there exist at least two self-financing portfolios that hedge the payment of one unit of the savings account at the maturity date T. The classical hedge portfolio would simply purchase one unit of the savings account at the initial time and hold it until maturity. The one that the proposed BN pricing suggests (with the MCI as proxy for the stock GOP) is less expensive; see figure 3. It requests only about three-quarters of the risk-neutral price. For longer time periods and better proxies of the stock GOP, this price can become significantly smaller, as forthcoming work will document.

5.2. *BN hedging of a zero-coupon bond*

The payoff of the zero-coupon bond can be replicated through hedging: We have for the zero-coupon bond $\tilde{P}(t,T)$, when denominated in the stock GOP, the SDE

$$\mathrm{d}\tilde{P}(t,T) = \delta_t^0 \mathrm{d}\tilde{S}_t^0 \tag{49}$$

with the hedge ratio

$$\begin{aligned} \delta_t^0 &= \frac{\partial \tilde{P}(t,T)}{\partial \tilde{S}_t^0} \\ &= 1 - \exp\left\{-\frac{S_t^*}{2(e^{\bar{\tau}_T} - e^{\bar{\tau}_t})}\right\}\left(1 + \frac{S_t^*}{2(e^{\bar{\tau}_T} - e^{\bar{\tau}_t})}\right) \end{aligned} \tag{50}$$

for the investment in the savings account $\tilde{S}_t^0 = \frac{1}{S_t^*}$, when denominated in the stock GOP, at time $t \in [t_0, T)$. Figure 4 shows the weight

$$\begin{aligned} \pi_t^{S^*} &= 1 - \pi_t^{S^0} = 1 - \frac{\delta_t^0 \tilde{S}_t^0}{\tilde{P}(t,T)} \\ &= \left(1 - \frac{1}{P(t,T)}\right)\ln(1 - P(t,T)) \end{aligned} \tag{51}$$

of the value of the hedge portfolio invested in the stock GOP. One notes that when the time to maturity is long, the fraction invested in the stock GOP is rather high and can get close to one. This fraction declines with decreased time to maturity according to the prescribed strategy and becomes finally zero at maturity. One could interpret its trajectory as a rigorous description for the glide path of a common financial planning strategy when it targets at maturity one unit of the savings account. This strategy invests when one is young mostly in stocks and closer to retirement more and more in the savings account.

Note in equation (50) that the strategy sells units of the savings account and buys units of the stock GOP when the stock GOP value declines and vice versa. Since the normalized stock index, which is the ratio of the benchmark over the exponential of the activity time, is mean-reverting under the MMM, this strategy is rational. However, the reaction of many investors, who invest for retirement, appears to be different in situations when the stock market crashes. The above rational strategy, when widely implemented, e.g. by pension funds and life insurance companies, has the potential to help stabilizing the stock market in times of major market drawdowns or extremely high stock prices.

One can only trade at discrete times, which creates hedge errors. According to the above-described hedging strategy, the hedge portfolio process $V = \{V_t, t \in [t_0, T]\}$ reallocates, say daily, the holdings in the savings account and the stock GOP in a self-financing manner. The difference between the BN price and the hedge portfolio value is shown in figure 5, which we call the *profit and loss*

$$C_t = V_t - P(t,T), \tag{52}$$

of the hedge portfolio V_t formed when replicating the zero-coupon bond, which was initiated at the initial time $t_0 = 1$ January 1984, reallocated daily in a self-financing manner, and matured at the maturity date $T = 1$ November 2014.

The absolute value of the profit and loss turns out to be rather small and remains in figure 5 always below 1.5% of one unit of the savings account value.

5.3. *Enhanced pricing and hedging of zero-coupon bond*

The current paper proposes a new method that allows one to reduce significantly the hedge error when pricing and hedging a zero-coupon bond. As can be seen in figure 2, the observed activity time τ_t is different from its trendline $\bar{\tau}_t$. Let us price a zero-coupon bond at the initial time t_0, by assuming that the trendline of the activity time when estimated in the future is similar to the one initially estimated. In practice, we do not know the random activity time at the maturity date T and have only a prediction $\bar{\tau}_T$ for this activity time. However, we can observe the activity until the current time. This allows us to introduce for the case $\tau_{t_0} < \bar{\tau}_T$ the stopping time

$$\rho = \sup\{t \in (t_0, T] : \tau_t < \bar{\tau}_T\} \tag{53}$$

as the supremum of all times t where the activity time τ_t is still smaller than the value $\bar{\tau}_T$ of the trendline at maturity.

During hedging, one can exploit the information that becomes available through the evolving observed activity time τ_t and one knows when one has reached the stopping time ρ. For $t \in [t_0, \rho)$, the current paper proposes the following formula for the *enhanced zero-coupon bond price* $\bar{P}(t,T)$ in the form

$$\bar{P}(t,T) = 1 - \exp\left\{-\frac{S_t^*}{2(e^{\bar{\tau}_T} - e^{\tau_t})}\right\}, \tag{54}$$

which yields the *enhanced fraction*

$$\bar{\pi}_t^{S^*} = \left(1 - \frac{1}{\bar{P}(t,T)}\right)\ln(1 - \bar{P}(t,T)) \tag{55}$$

to be invested in the stock GOP. One stops the hedge at the time ρ, where one exchanges all the wealth in the hedge portfolio into units of the savings account. The latter value can be expected to be close to 1.0 when the stopping time ρ occurs before the maturity date and the time step size of the hedge is sufficiently small.

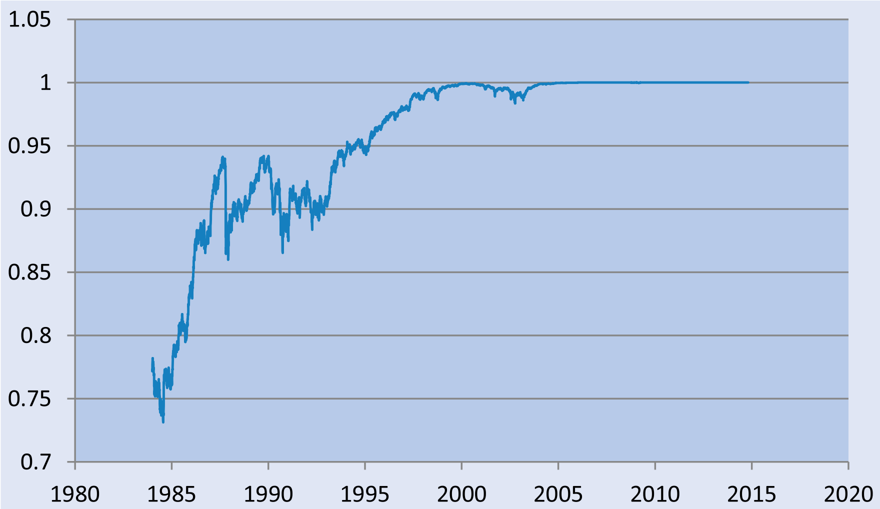

Figure 3. Discounted zero-coupon bond $P(t,T)$.

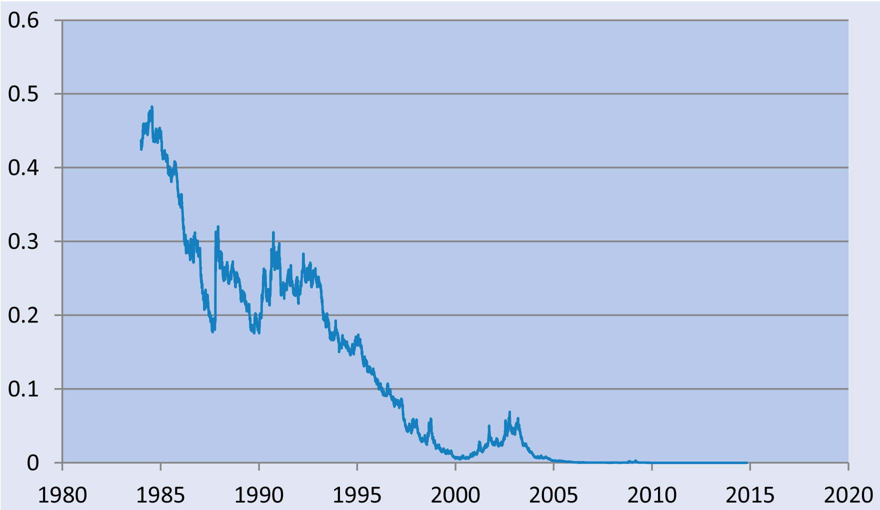

Figure 4. Fraction $\pi_t^{S^*}$ invested in the stock GOP.

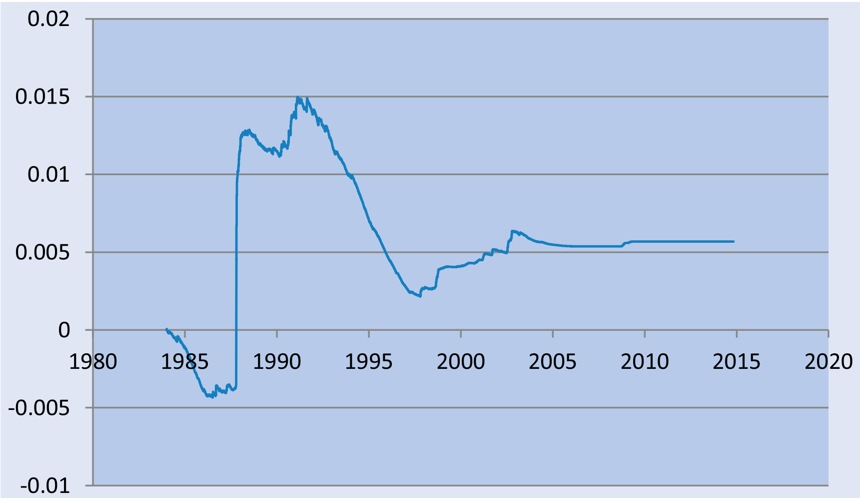

Figure 5. Profit and loss C_t for the hedge of a zero-coupon bond.

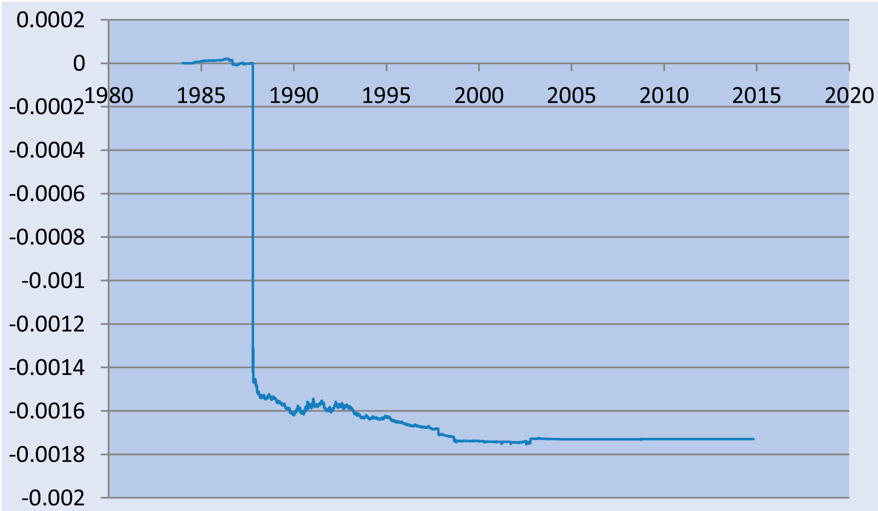

Figure 6. Profit and loss $\bar{C}_t$ for the enhanced hedge.

In the other case, the value of the hedge portfolio at the maturity date depends on the distance of the predicted activity time $\bar{\tau}_T$ at maturity and the activity time τ_T at maturity. Fortunately, the zero-coupon bond price $\bar{P}(t, T)$ is not very sensitive to the activity time at maturity and the value of the hedge portfolio turns out to be close to one if the predicted activity time and the activity time at maturity are not too far from each other.

We denote the resulting hedge portfolio process by $\bar{V} = \{\bar{V}_t, t \in [t_0, T]\}$. In comparison to the previous formulas (48) and (50), in the above two formulas (54) and (55), we substituted the trendline of the activity time $\bar{\tau}_t$ by the observed current activity time τ_t as long as we have $t < \rho$. The resulting profit and loss

$$\bar{C}_t = \bar{V}_t - \bar{P}(t, T) \tag{56}$$

of the enhanced hedge portfolio $\bar{V}_t$ is shown in figure 6.

The maximum of its absolute value remains smaller than 0.0018 of one unit of the savings account for the 30-year daily hedge. The enhanced hedge provides an impressively accurate replication of the payout of the zero-coupon bond. Only during the 1987 stock market crash one notices a sudden increase in the absolute value of the hedge error. Such a sudden increase would be typical when the stock index values were simulated under the MMM and the time step size in activity time would be much larger than usual. Indeed, the activity time step size was significantly larger than usual during the 1987 stock market crash because several days of data were missing and the market went extremely fast during that period. The otherwise almost perfect enhanced hedge supports the assumption of the paper that the MMM in activity time models well the 'natural' dynamics of a well-diversified stock portfolio. Forthcoming work will show for other well-diversified stock portfolios that the hedge errors for the enhanced hedge of long-term zero-coupon bonds are similarly small and the initial fair prices of the zero-coupon bonds can be made significantly smaller through the choice of a faster growing benchmark. It remains to remark that the small hedge error in the above illustration is also a result of the fact that the predicted activity time $\bar{\tau}_T$ at maturity and the observed activity time τ_T at maturity are reasonably close to each other. Forthcoming work will analyze the hedge error when the activity time at maturity remains significantly smaller than the predicted activity time at maturity is fixed, which makes it a random time in calendar time. In this case the hedge error can be expected to be extremely small for all scenarios of the activity time under the assumed 'natural' dynamics of the benchmark.

The risk-neutral price before maturity for the above zero-coupon bond, which pays at maturity one unit of the savings account, equals always one unit of the savings account. Formula (48) shows that when there is some strictly positive time to maturity, its BN price is lower than the risk-neutral price. The respective risk-neutral hedging portfolio is self-financing and delivers the targeted payoff of one unit of the savings account at maturity. However, due to the strict supermartingale property of the Radon–Nikodym derivative process Λ_{S^0} of the putative risk-neutral measure Q_{S^0}, the respective risk-neutral price is higher than the BN price.

There seems to exist no economic reason for producing the zero-coupon bond payoff more expensively than necessary by following the popular risk-neutral pricing rule. The BN price and hedge offer a more economical way of replicating the targeted long-term payoff, as illustrated in figure 3.

6. Conclusion

The paper proposes the new method of benchmark-neutral pricing and hedging, which employs as numéraire the growth optimal portfolio of stocks. For a long-term zero-coupon bond, it is demonstrated that the proposed benchmark-neutral price is lower than the risk-neutral price and the payoff can be impressively accurately hedged over several decades. The paper assumes the 'natural' dynamics of well-diversified stock portfolios as those of squared Bessel processes in respective activity times. By applying and extending the proposed benchmark-neutral pricing methodology, it should be possible to develop accurate quantitative methods for a wide range of long-term contracts. Such new quantitative methods should allow reducing the costs of pension and insurance contracts.

7. Open Scholarship

This article has earned the Center for Open Science badges for Open Data and Open Materials through Open Practices Disclosure. The data and materials are openly accessible at https://doi.org/10.7910/DVN/R9I0AB and https://doi.org/10.7910/DVN/R9I0AB.

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Disclosure statement

No potential conflict of interest was reported by the author(s).

Data availability statement

The EXCEL file that generates the figures of the paper is available at Harvard Dataverse https://doi.org/10.7910/DVN/R9I0AB.

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