Systems and Soft Computing 7 (2025) 200408

Contents lists available at ScienceDirect

Systems and Soft Computing

journal homepage: www.journals.elsevier.com/soft-computing-letters

OWA operators and probabilities under hypersoft set environments

Muhammad Saqlain [a,b], Jose M. Merigó [b], Poom Kumam [a,*], Dragan Pamucar [c,*]

[a] Department of Mathematics, King Mongkut's University of Technology Thonburi, 126 Pracha Uthit Rd, Bang Mot, Thung Khru, Bangkok 10140, Thailand
[b] School of Computer Science, Faculty of Engineering and Information Technology, University of Technology Sydney, 81 Broadway, Ultimo 2007, NSW, Australia
[c] Széchenyi István University, Győ, Hungary

ARTICLE INFO

Keywords:
Aggregation operator
Entropy weights
Hypersoft set
OWA operator
Probability
Similarity weights

ABSTRACT

This study proposes novel extensions to overcome the limitations of classical aggregation methods, namely ordered weighted averaging (OWA) and probabilistic OWA (POWA) operators, in handling hierarchical or subdivided attributes under uncertainty within the hypersoft set (HS) framework, resulting in the hypersoft set-based OWA (HS-OWA) and hypersoft set-based POWA (HS-POWA) operators. These extensions (HS-OWA and HS-POWA operators) preserve sub-attribute information, enhance decision accuracy, and handle uncertainty, including fuzzy, intuitionistic, and neutrosophic data. We formalize the mathematical definitions and theoretical properties of HS-OWA and HS-POWA, demonstrating their practical applicability through a case study of sustainable wastewater treatment method selection. Additionally, we generalize the proposed operators under various fuzzy extensions, including intuitionistic fuzzy sets (IFS), pythagorean fuzzy sets (PFS), q-Rung orthopair fuzzy sets (q-ROFS), and neutrosophic sets (NS), allowing flexible modeling of uncertainty, hesitation, and conflict in expert assessments. The results from our study validate the superiority of the proposed framework in aggregating distributed evaluations while preserving semantic depth and interoperability. The proposed operators are effective in complex multi-criteria and group decision-making problems, such as sustainable technology assessment and policy-making, and provide a robust framework for future research in dynamic and large-scale MCDM applications.

1. Introduction

Aggregation operators play a crucial role in decision-making as they provide a systematic way to combine multiple criteria, preferences, or pieces of information into a single representative value. In complex decision problems where data may be uncertain, imprecise, or derived from diverse sources, aggregation operators help synthesize this information to support rational and consistent choices. By capturing the decision maker's priorities, risk attitudes, and preferences, they enable more accurate evaluation of alternatives and improve the quality of final decisions. Their flexibility and adaptability make them fundamental tools in fields such as multi-criteria decision-making (MCDM), artificial intelligence, and data analysis. Despite many advances, classical OWA [1] and POWA operators [2] are not capable of handling the membership values of the subdivided attributive values. For example, in cases where decision elements have multiple criteria that are further subdivided, such as sustainable technology evaluation, policy-making, or large-scale infrastructure assessment, where criteria like "cost" or "efficiency" might be divided into sub-attributes like "initial cost," "maintenance cost," "low efficiency" or "high efficiency" thus simple OWA or POWA operators may lead to information loss. To resolve this, the hypersoft set theory provides a framework where multi-attribute evaluations are preserved through refined mappings (considering the sub-attribute values of the alternatives).

However, the absence of a formal aggregation mechanism within the hypersoft context limits its applicability in group decision-making and uncertainty analysis. This motivates the development of the hypersoft set-based ordered weighted averaging (HS-OWA) and probabilistic ordered weighted averaging (HS-POWA) operators.

The proposed HS-OWA and HS-POWA operators address this gap by:

- Integrating the structural advantages of hypersoft sets with the flexible weighting schemes of OWA and POWA operators.
- Allowing for context-aware aggregation by incorporating sub-attribute level information.
- Capturing both the attitudinal and probabilistic perspectives of decision-makers in uncertain environments.

* Corresponding authors.
E-mail addresses: muhammad.saql@kmutt.ac.th, Muhammad.Saqlain@uts.edu.au (M. Saqlain), Jose.Merigo@uts.edu.au (J.M. Merigó), poom.kum@kmutt.ac.th (P. Kumam), pamucar.dragan@sze.hu (D. Pamucar).

https://doi.org/10.1016/j.sasc.2025.200408
Received 5 August 2025; Received in revised form 6 October 2025; Accepted 22 October 2025
Available online 30 October 2025

- Extending classical aggregation frameworks to accommodate fuzzy, intuitionistic, and neutrosophic information under hypersoft mappings.

By utilizing hypersoft mappings, the HS-OWA operator preserves the semantic richness of multi-attribute evaluations, while the HS-POWA operator provides a probabilistic enhancement that supports risk-based decision-making. Together, these operators establish a robust decision-making framework suitable for a wide range of real-world multi-criteria decision-making (MCDM) and multi-criteria group decision-making (MCGDM) problems involving ambiguity, contradiction, and complex attribute hierarchies.

The layout of the paper is structured as follows. In section 2, we present the necessary definitions that are helpful to understand the rest of the paper, i.e, hypersoft set, OWA operators, and probabilistic OWA. Section 3 formally defines the HS-OWA and HS-POWA operators with a solved example. The theoretical properties, like boundedness, monotonicity, and commutativity with theorems, propositions, and remarks, are presented in section 4. Section 5 presents the entropy-weighted and similarity weighted variants of HS-OWA and HS-POWA. In section 6, we generalized the proposed operators under various fuzzy extensions, and practical implications are also suggested in various real-life problems. Section 7 consists of the proposed decision-making methodology with an illustrative example, and a result discussion subsection with comparison to the classical operators. Finally, section 8 concludes the present study with future research directions.

2. Literature review

Decision-making is a fundamental process in various fields, ranging from engineering and economics to healthcare and environmental management, where selecting the most suitable alternative among several options is often complex [3–7]. Hypersoft set [8], and aggregation operators like the ordered weighted averaging (OWA) Yager [1], and probabilistic ordered weighted averaging (POWA) operators [2] play a crucial role in handling such intricacies. These operators provide flexible frameworks for combining information by incorporating the relative importance and attitudinal characteristics of decision-makers.

Aggregation operators improve decision-making by incorporating contextual relevance through membership degrees, addressing uncertainty [9–11]. The studies explore the foundations and applications, demonstrating effectiveness through case studies and fuzzy frameworks [10,12].

The OWA operator emphasizes the ordered positions of inputs based on predefined weights, allowing for modeling optimism, pessimism, or neutrality in decision strategies. The POWA operator further enhances this by introducing probabilistic information, offering a more nuanced and realistic representation of uncertainty and variability in decision environments. Together, these operators enrich the decision-making process by enabling a balanced, adaptable, and insightful analysis of complex problems.

The ordered weighted averaging (OWA) operator, introduced by Yager [1], has become a fundamental tool in decision-making (DM) processes by incorporating the attitudinal character of decision-makers. Its ability to interpolate between optimistic and pessimistic strategies makes it highly flexible. Over the years, this operator has undergone significant evolution, resulting in a variety of extensions such as weighted OWA [13], heavy OWA [14], norms [15], and many other theoretical and practical developments [16,17]. The concept of immediate probabilities [18], consideration of decision attitude in probabilistic decision making [19], and generalization of probabilistic OWA [2] demonstrate the developments that enabled broader application of OWA in uncertain and multi-criteria decision-making (MCDM) environments.

Several studies have contributed to the development and application of ordered weighted averaging (OWA) operators in decision-making and aggregation methods. For instance, Merigó and Yager [20] studied the use of the moving average, and Anido [21] examined the performance evaluation procedure based on fuzzy mathematics and OWA operators. Stupňanová and Jin [22] and Mesiar et al. [23] introduced the BiOWA operator, a variant that incorporates bipolar preferences in decision-making.

Torres-Martínez et al. [24] explored the use of OWA operators with the boxplot method in time series analysis, demonstrating the versatility of OWA in handling temporal data. The concept of functional weights in OWA operators was further explored by [25], while [26] integrated covariances with OWA operators and Bonferroni means to enhance aggregation in fuzzy systems. Additionally, Wang and Chin [27] applied OWA operator weights in cross-efficiency aggregation, extending their use in multi-criteria decision-making. Furthermore, Dai and Zheng [28] investigated the orders of OWA operators, contributing to a deeper understanding of their properties in aggregation. Other notable works include [29] development of the SDOWA operator for decision-making and Jin et al. [30,31], who explored bipolar preferences in OWA weight vectors for aggregation in uncertain environments. These advancements highlight the evolving role of OWA operators in handling complex decision-making problems across various domains.

Verma and Mittal [32] discussed multiple attribute group decision-making under POWA operators. [33] proposed a modified soft-likelihood function based on the POWA operator, concept of the scatters of probability distributions, and OWA weights [34], focusing on its applications in decision-making, and aggregation of pragmatic operators [35] for probabilistic linguistic multi-criteria group decision-making problems. Chen et al. [36] explored an efficiency evaluation method for coordinated areas based on the HOWA operator, providing insights into its use in system performance analysis. Hussain et al. [37] integrated AHP with IOWA and POWA frameworks for ideal location selection, showcasing the application of these operators in real-world decision-making. The consideration of probabilistic information along with OWA operators [38] in group decision-making emphasizes their role in enhancing decision support systems. [39] proposed an intelligent quality-based fusion method for image defogging using the interval-valued intuitionistic fuzzy POWA operator, demonstrating its effectiveness in enhancing image processing. Espinoza-Audelo et al. [40] introduced the Bonferroni probabilistic ordered weighted averaging (BPOWA) operator and explored its application in decision-making problems, offering a robust framework for probabilistic aggregation. Zhu et al. [41] developed an improved model for fusing multi-source remote sensing image information using the IVIF-POWA operator, contributing to the integration of diverse data types for more accurate analysis.

The proposed HS-OWA and HS-POWA operators address a timely need in decision-making where traditional aggregation methods (OWA, POWA) fail to handle subdivided attribute values. By preserving sub-attribute information and incorporating both attitudinal and probabilistic perspectives, this framework enables more accurate and robust multi-criteria evaluations under uncertainty. It is expected to benefit the scientific community by providing a flexible, theoretically sound, and practical tool for complex decision problems in fields such as AI, sustainability, and infrastructure planning.

3. Preliminaries

3.1. Hypersoft set

Definition 3.1 (*Hypersoft Set*)**.** *Consider a collection of attribute sets $\mathbb{L}_1^a, \mathbb{L}_2^b, \dots, \mathbb{L}_n^z$, then the Cartesian product $(\mathbb{L}_1^a \times \mathbb{L}_2^b \times \cdots \times \mathbb{L}_n^z)$ along with a mapping $\mathbb{F}$ is called a hypersoft set [8] over a universal set $\mathcal{U}$, and is formally defined as:*

$$\mathbb{F} : (\mathbb{L}_1^a \times \mathbb{L}_2^b \times \cdots \times \mathbb{L}_n^z) \to \mathbb{P}(\mathcal{U}) \tag{1}$$

Here, $\mathbb{P}(\mathcal{U})$ denotes the power set of $\mathcal{U}$. Each sub-attribute from the domain represents a combination of attribute values, and $\mathbb{F}$ maps such a sub-attribute to a subset of $\mathcal{U}$, reflecting the elements satisfying that combination.

Definition 3.2. *The ordered weighted averaging (OWA) operator was introduced by Yager [1], and is a powerful aggregation tool for decision-making under uncertainty. It is defined as:*

$$OWA(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{2}$$

Where:

- b_j *is the* j*th largest alternative after reordering in descending order.*
- $W = (w_1, w_2, \dots, w_n)$ *is a weighting vector satisfying:*

$$\sum_{j=1}^{n} w_j = 1, \quad 0 \le w_j \le 1. \tag{3}$$

Definition 3.3. *The probabilistic OWA (POWA) operator was defined by Merigó [2]. A classical probabilistic aggregation (PA) operator is a function that weights the aggregation process based on probability distributions, mathematically defined as:*

$$PA(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n} p_i a_i, \tag{4}$$

where a_i *represents the decision alternatives, and* $P = (p_1, p_2, \dots, p_n)$ *is the probability distribution satisfying:*

$$\sum_{i=1}^{n} p_i = 1, \quad 0 \le p_i \le 1. \tag{5}$$

The PA operator is widely used in decision-making under risk, where the alternatives are weighted based on their likelihood of occurrence.

4. OWA and POWA operators under hypersoft set environment

This section presents the formal definition of HS-OWA, HS-POWA, and HS-hybrid aggregation operators along with solved examples.

4.1. Definitions

Definition 4.1 (*Hypersoft Set OWA (HS-OWA) Operator*). *Let* $\mathcal{U}$ *be a universe of alternatives, and let* $(\mathbb{F}, \mathbb{L}_1^a \times \mathbb{L}_2^b \times \cdots \times \mathbb{L}_n^z)$ *be a hypersoft set over* $\mathcal{U}$*, where each* $\mathbb{L}_j^*$ *represents the sub-attributes for criterion* C_j*. Let* $x_{ij} \in [0,1]$ *be the score of alternative* A_i *under criterion* C_j*, and let* $\mu_{ij} \in [0,1]$ *be the associated membership degree derived from the hypersoft context.*

Define the adjusted values as:

$\gamma_{ij} = x_{ij} \cdot \mu_{ij}$

Reorder these adjusted values for each A_i in descending order:

$\gamma_{i(k)} \ge \gamma_{i(k+1)} \ge \cdots \ge \gamma_{i(n)} \quad \text{for } k = 1, 2, \dots, n$

Then, the hypersoft set OWA aggregation of alternative A_i is given by:

$$HS\text{-}OWA(A_i) = \sum_{k=1}^{n} w_k \cdot \gamma_{i(k)} \tag{6}$$

where:

- $w_k \in [0,1]$ is the OWA weight assigned to the kth largest adjusted value,
- $\sum_{k=1}^{n} w_k = 1$,
- $\gamma_{i(k)}$ denotes the kth largest adjusted score for alternative A_i.

This operator extends the classical OWA by incorporating the contextual relevance of each criterion via hypersoft membership degrees, enabling a more informative aggregation.

Definition 4.2 (*Hypersoft Set POWA (HS-POWA) Operator*). *Let* $\mathcal{U}$ *be a universe of alternatives, and let* $(\mathbb{F}, \mathbb{L}_1^a \times \mathbb{L}_2^b \times \cdots \times \mathbb{L}_n^z)$ *be a hypersoft set over* $\mathcal{U}$*, where each* $\mathbb{L}_j^*$ *represents the sub-attributes for criterion* C_j*. For each alternative* A_i*, define:*

- $x_{ij} \in [0,1]$ as the evaluation score under criterion C_j,
- $\mu_{ij} \in [0,1]$ as the corresponding membership degree from the hypersoft set context,
- $\gamma_{ij} = x_{ij} \cdot \mu_{ij}$ as the adjusted score incorporating contextual relevance.

Let $W = (w_1, w_2, \dots, w_n)$ be an OWA weight vector such that $\sum_{k=1}^{n} w_k = 1$ and $w_k \in [0,1]$, and let $P = (p_1, p_2, \dots, p_n)$ be a probability vector such that $\sum_{j=1}^{n} p_j = 1$ and $p_j \in [0,1]$.

1. Compute the adjusted scores: $\gamma_{ij} = x_{ij} \cdot \mu_{ij}$,
2. Sort the γ_{ij} values for each alternative A_i in descending order:

   $\gamma_{i(k)} \ge \gamma_{i(k+1)} \ge \cdots \ge \gamma_{i(n)}$

3. Reorder the original probability vector $P = (p_1, p_2, \dots, p_n)$ according to the positions of the sorted γ_{ij} values, resulting in a reordered vector $P^{(i)} = (p_{i(1)}, p_{i(2)}, \dots, p_{i(n)})$.

Then, the hypersoft set POWA aggregation for A_i is defined as:

$$HS\text{-}POWA(A_i) = \sum_{k=1}^{n} p_{i(k)} \cdot w_k \cdot \gamma_{i(k)} = \sum_{k=1}^{n} p_{i(k)} \cdot w_k \cdot (x_{i(k)} \cdot \mu_{i(k)}) \tag{7}$$

where:

- $p_{i(k)}$ is the reordered probability associated with the kth largest adjusted value,
- w_k is the OWA weight for the kth ordered position,
- $\gamma_{i(k)}$ is the kth largest adjusted score for alternative A_i.

This operator generalizes the hypersoft set OWA and the classical probabilistic OWA (POWA) by incorporating contextual relevance via membership degrees and positional uncertainty via probabilistic importance.

Definition 4.3 (*Hybrid Hypersoft Aggregation Operator (HS-Hybrid)*). *Let* $\mathcal{U}$ *be a universe of alternatives, and let* $(\mathbb{F}, \mathbb{L}_1^a \times \mathbb{L}_2^b \times \cdots \times \mathbb{L}_n^z)$ *be a hypersoft det over* $\mathcal{U}$*, where each* $\mathbb{L}_j^*$ *represents a sub-attribute under criterion* C_j*. For each alternative* A_i*, define:*

- $x_{ij} \in [0,1]$*: the score under criterion* C_j*,*
- $\mu_{ij} \in [0,1]$*: the corresponding hypersoft membership degree,*
- $\gamma_{ij} = x_{ij} \cdot \mu_{ij}$*: the adjusted score incorporating contextual relevance,*
- $\gamma_{i(k)}$*: the* k*th largest value in the descending ordered sequence* $\{\gamma_{ij}\}_{j=1}^{n}$*.*

Let:

- $W = (w_1, w_2, \dots, w_n)$*: an OWA weight vector such that* $w_k \in [0,1]$ *and* $\sum_{k=1}^{n} w_k = 1$*,*
- $P = (p_1, p_2, \dots, p_n)$*: a probabilistic weight vector such that* $p_k \in [0,1]$ *and* $\sum_{k=1}^{n} p_k = 1$*,*
- $\beta \in [0,1]$*: a convex combination parameter controlling the trade-off between probabilistic importance and ordered preference.*

Then, the hybrid hypersoft aggregation score of alternative A_i *is defined as:*

$$HS\text{-}Hybrid(A_i) = \sum_{k=1}^{n} h_k \cdot \gamma_{i(k)} \quad \text{where} \quad h_k = \beta p_k + (1-\beta) w_k \tag{8}$$

Properties:

- $h_k \in [0,1]$*, and* $\sum_{k=1}^{n} h_k = \beta \sum p_k + (1-\beta) \sum w_k = \beta + (1-\beta) = 1$*,*
- *When* $\beta = 0$*, the operator reduces to standard HS-OWA,*
- *When* $\beta = 1$*, the operator becomes HS-POWA,*
- *For* $0 < \beta < 1$*, it flexibly combines probabilistic and ordered decision-making.*

4.2. Solved examples

Now we demonstrate the applicability of the formal definitions of HS-OWA, HS-POWA, and HS-hybrid operators in the context of a hypersoft set. For this, consider the following case study:

Let the universe of alternatives be $\mathcal{U} = \{A_1, A_2, A_3\}$ representing economic policies. Consider a group of decision-makers evaluating these economic policies: A_1 (Policy A), A_2 (Policy B), and A_3 (Policy C), based on three criteria:

- C_1: Economic growth
- C_2: Employment rate
- C_3: Public support

$$\mathbb{F} : (\mathbb{L}_1^2 \times \mathbb{L}_2^2 \times \mathbb{L}_3^2) \rightarrow \mathcal{P}(\mathcal{U})$$

where $\mathbb{F}(A_i)$ returns the degree of suitability of alternative A_i under each sub-attribute related to the alternative.

- $t_1 = (\text{Low}, \text{High})$ for Economic Growth
- $t_2 = (\text{Medium}, \text{High})$ for Employment Rate
- $t_3 = (\text{Low}, \text{High})$ for Public Support

The hypersoft set $(\mathbb{F}^{(k)}, \mathbb{L}_1^2 \times \mathbb{L}_2^2 \times \mathbb{L}_3^2)$ maps to a fuzzy set over sub-attribute (High, Medium, Low) with associated membership degrees in [0,1] for each alternative A_i. Then the score matrix $X = [x_{ij}]$ and membership matrix $\mu = [\mu_{ij}]$ are given as:

$$X = \begin{bmatrix} 0.8 & 0.7 & 0.6 \\ 0.6 & 0.9 & 0.5 \\ 0.7 & 0.6 & 0.9 \end{bmatrix}, \quad \mu = \begin{bmatrix} 0.9 & 0.8 & 0.7 \\ 0.7 & 0.9 & 0.6 \\ 0.8 & 0.6 & 0.9 \end{bmatrix}$$

Here:

- Rows represent alternatives A_1, A_2, A_3,
- Columns represent criteria C_1, C_2, C_3,
- x_{ij} is the evaluation score of alternative A_i under criterion C_j,
- μ_{ij} is the membership degree of criterion C_j for alternative A_i in the hypersoft context.

The OWA weights are given by $w = [0.5, 0.3, 0.2]$, and the probabilistic weight vector is $p = [0.4, 0.4, 0.2]$. We now compute results for the hypersoft set OWA (HS-OWA), and the hypersoft set POWA (HS-POWA).

Example 4.1 (*Solved Example using (HS-OWA)*)**.** *For this, assume the above case study. Now we calculate the adjusted scores, then, after sorting in descending order we compute the final HS-OWA values.*

Step 1: Calculate adjusted score $\gamma_{i(j)}$

- A_1: $\gamma_{11} = 0.8 \cdot 0.9 = 0.72, \quad \gamma_{12} = 0.7 \cdot 0.8 = 0.56, \quad \gamma_{13} = 0.6 \cdot 0.7 = 0.42$
- A_2: $\gamma_{21} = 0.6 \cdot 0.7 = 0.42, \quad \gamma_{22} = 0.9 \cdot 0.9 = 0.81, \quad \gamma_{23} = 0.5 \cdot 0.6 = 0.30$
- A_3: $\gamma_{31} = 0.7 \cdot 0.8 = 0.56, \quad \gamma_{32} = 0.6 \cdot 0.6 = 0.36, \quad \gamma_{33} = 0.9 \cdot 0.9 = 0.81$

Step 2: Sort the adjusted scores in descending order $\gamma_{i(k)}$

- A_1: $\gamma_{1(1)} = 0.72, \gamma_{1(2)} = 0.56, \gamma_{1(3)} = 0.42$
- A_2: $\gamma_{2(1)} = 0.81, \gamma_{2(2)} = 0.42, \gamma_{2(3)} = 0.30$
- A_3: $\gamma_{3(1)} = 0.81, \gamma_{3(2)} = 0.56, \gamma_{3(3)} = 0.36$

Step 3: Compute HS-OWA Using $w = [0.5, 0.3, 0.2]$:

$$HS\text{-}OWA(A_1) = 0.5 \cdot 0.72 + 0.3 \cdot 0.56 + 0.2 \cdot 0.42 = 0.612$$
$$HS\text{-}OWA(A_2) = 0.5 \cdot 0.81 + 0.3 \cdot 0.42 + 0.2 \cdot 0.30 = 0.591$$
$$HS\text{-}OWA(A_3) = 0.5 \cdot 0.81 + 0.3 \cdot 0.56 + 0.2 \cdot 0.36 = 0.645$$

Step 4: Ranking by HS-OWA: $A_3 \succ A_1 \succ A_2$

Example 4.2 (*Solved Example using (HS-POWA)*)**.** *For this, assume the above case study. Now we reorder the probabilistic weights and compute the final HS-POWA values.*

Step 1: Reorder the Probabilistic Weights
Reorder $p = [0.4, 0.4, 0.2]$ based on the ranking of γ_{ij} for each A_i:

- A_1: Order is $[0.72, 0.56, 0.42] \Rightarrow p^{(1)} = [0.4, 0.4, 0.2]$
- A_2: Order is $[0.81, 0.42, 0.30] \Rightarrow p^{(2)} = [0.4, 0.2, 0.4]$
- A_3: Order is $[0.81, 0.56, 0.36] \Rightarrow p^{(3)} = [0.4, 0.4, 0.2]$

Step 2: Compute HS-POWA

$$\begin{aligned} HS\text{-}POWA(A_1) &= 0.4 \cdot 0.5 \cdot 0.72 + 0.4 \cdot 0.3 \cdot 0.56 + 0.2 \cdot 0.2 \cdot 0.42 \\ &= 0.144 + 0.0672 + 0.0168 = 0.228 \\ HS\text{-}POWA(A_2) &= 0.4 \cdot 0.5 \cdot 0.81 + 0.2 \cdot 0.3 \cdot 0.42 + 0.4 \cdot 0.2 \cdot 0.30 \\ &= 0.162 + 0.0252 + 0.024 = 0.2112 \\ HS\text{-}POWA(A_3) &= 0.4 \cdot 0.5 \cdot 0.81 + 0.4 \cdot 0.3 \cdot 0.56 + 0.2 \cdot 0.2 \cdot 0.36 \\ &= 0.162 + 0.0672 + 0.0144 = 0.2436 \end{aligned}$$

Step 3: Ranking by HS-POWA: $A_3 \succ A_1 \succ A_2$

Example 4.3 (*Solved Example using HS-Hybrid Operator*)**.**
Let the convex combination parameter be $\beta = 0.6$*, which controls the trade-off between probabilistic and ordered importance.*

Step 1: Compute HS-Hybrid Aggregation Scores The HS-Hybrid aggregation for an alternative A_i is computed as:

$$HS\text{-}Hybrid(A_i) = \sum_{k=1}^{n} h_k \cdot \gamma_{i(k)} \quad \text{where} \quad h_k = \beta p_k + (1-\beta) w_k$$

Alternatively, if the HS-OWA and HS-POWA scores have already been computed individually, the hybrid score can also be represented as a convex combination $\beta = 0.6$:

$$HS\text{-}Hybrid(A_i) = \beta \cdot HS\text{-}POWA(A_i) + (1-\beta) \cdot HS\text{-}OWA(A_i)$$

- A_1: $HS\text{-}Hybrid(A_1) = 0.6 \cdot 0.228 + 0.4 \cdot 0.612 = 0.1368 + 0.2448 = 0.3816$
- A_1: $HS\text{-}Hybrid(A_2) = 0.6 \cdot 0.2112 + 0.4 \cdot 0.591 = 0.1267 + 0.2364 = 0.3631$
- A_1: $HS\text{-}Hybrid(A_3) = 0.6 \cdot 0.2436 + 0.4 \cdot 0.645 = 0.14616 + 0.258 = 0.4041$

Step 2: Final Ranking:

$A_3 \succ A_1 \succ A_2$

Comparison with Individual Methods:

- **HS-OWA Ranking:** $A_3 \succ A_1 \succ A_2$
- **HS-POWA Ranking:** $A_3 \succ A_1 \succ A_2$
- **HS-Hybrid Ranking:** $A_3 \succ A_1 \succ A_2$

This example demonstrates how the hybrid hypersoft aggregation method combines the benefits of both OWA and POWA, reflecting the probabilistic and ordered importance of the alternatives for their adjusted scores.

5. Theoretical properties of HS-owa, HS-powa, and HS-hybrid operators

This section presents the basic properties of HS-OWA, HS-POWA, and HS-hybrid operators, like monotonicity, commutativity, boundedness, and additional property of special cases, along with their proofs.

5.1. Monotonicity

The HS-OWA, HS-POWA, and HS-hybrid aggregation operators are monotonic with respect to the input scores. That is, if the score x_{ij} of any alternative A_i increases for any criterion C_j, while the membership degrees and weights remain fixed, then the corresponding aggregated value will not decrease.

Proof. Let $x_{ij} \leq x'_{ij}$ for all j, and $\mu_{ij} \in [0,1]$. Then:

$\gamma_{ij} = x_{ij} \cdot \mu_{ij} \leq x'_{ij} \cdot \mu_{ij} = \gamma'_{ij}$

After ordering γ_{ij} in descending order to obtain $\gamma_{i(k)}$, and applying non-negative weights w_k, p_k, and convex combination parameter $\beta \in [0,1]$, we have:

$$HS\text{-}OWA(A_i) = \sum_{k=1}^{n} w_k \cdot \gamma_{i(k)} \leq \sum_{k=1}^{n} w_k \cdot \gamma'_{i(k)} = HS\text{-}OWA(A'_i)$$

$$HS\text{-}POWA(A_i) = \sum_{k=1}^{n} p_k \cdot \gamma_{i(k)} \leq \sum_{k=1}^{n} p_k \cdot \gamma'_{i(k)} = HS\text{-}POWA(A'_i)$$

$$HS\text{-}Hybrid(A_i) = \sum_{k=1}^{n} h_k \cdot \gamma_{i(k)} \leq \sum_{k=1}^{n} h_k \cdot \gamma'_{i(k)} = HS\text{-}Hybrid(A'_i)$$

Hence, monotonicity is proved.

5.2. Commutativity

The HS-OWA, HS-POWA, and HS-hybrid aggregation operators are commutative for alternatives having the same input vector (scores and membership degrees) in any order.

Proof. Let alternatives A_i and A_j have the same set of score-membership products γ_{ij}. Since the aggregation depends only on the sorted values $\gamma_{i(k)}$, not their original positions: sorting $\{\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{in}\}$ yields the same result for both alternatives. Therefore:

$$HS\text{-}OWA(A_i) = HS\text{-}OWA(A_j),$$

$$HS\text{-}POWA(A_i) = HS\text{-}POWA(A_j),$$

$$HS\text{-}Hybrid(A_i) = HS\text{-}Hybrid(A_j)$$

Thus, commutativity is preserved.

5.3. Boundedness

The HS-OWA, HS-POWA, and HS-hybrid operators are bounded within the minimum and maximum of the adjusted scores $\gamma_{ij} = x_{ij} \cdot \mu_{ij}$, provided that all values lie within the interval $[0,1]$.

Proof. Since $x_{ij}, \mu_{ij} \in [0,1]$, the product $\gamma_{ij} \in [0,1]$. Let $\gamma_i^{\min} = \min_j \gamma_{ij}$, $\gamma_i^{\max} = \max_j \gamma_{ij}$, and $h_k \in [0,1]$, $\sum h_k = 1$. Then:

$$HS\text{-}OWA(A_i) = \sum_{k=1}^{n} w_k \gamma_{i(k)} \in [\gamma_i^{\min}, \gamma_i^{\max}]$$

$$HS\text{-}POWA(A_i) = \sum_{k=1}^{n} p_k \gamma_{i(k)} \in [\gamma_i^{\min}, \gamma_i^{\max}]$$

$$HS\text{-}Hybrid(A_i) = \sum_{k=1}^{n} h_k \gamma_{i(k)} \in [\gamma_i^{\min}, \gamma_i^{\max}]$$

Hence, boundedness is confirmed.

Theorem 5.1. *When all hypersoft membership degrees $\mu_{ij} = 1$, the HS-OWA and HS-POWA operators reduce to the classical OWA and POWA aggregations, respectively.*

Proof. If $\mu_{ij} = 1$, then $\gamma_{ij} = x_{ij}$. So:

$$HS\text{-}OWA(A_i) = \sum_{k=1}^{n} w_k \cdot x_{i(k)} = \text{Classical OWA}$$

$$HS\text{-}POWA(A_i) = \sum_{k=1}^{n} p_k \cdot x_{i(k)} = \text{Classical POWA}$$

$$HS\text{-}Hybrid(A_i) = \sum_{k=1}^{n} (\beta p_k + (1-\beta) w_k) \cdot x_{i(k)} = \text{Hybrid of Classical OWA and POWA}$$

Hence, the operators reduce to classical forms under full membership.

Theorem 5.2. *The HS-OWA, HS-POWA, and HS-hybrid operators provide flexibility to model uncertainty, since the hypersoft membership degrees μ_{ij}, probabilistic weights p_j, and OWA weights w_j can be varied based on context.*

Proof.

- The membership degree $\mu_{ij} \in [0,1]$ allows partial relevance of criterion C_j for alternative A_i.
- Probabilistic weights p_j reflect the stochastic nature or importance of each criterion.
- OWA weights w_j capture attitudinal preferences (e.g., optimism, pessimism).
- Convex combination parameter $\beta \in [0,1]$ enables blending of two perspectives.

By modifying μ_{ij}, γ_{ij}, $(p_j$, w_j, and β, decision-makers can tailor the aggregation process to suit uncertain or vague decision environments.

Theorem 5.3. *Given a finite set of alternatives $A = \{A_1, A_2, \ldots, A_m\}$ evaluated across a finite set of criteria $C = \{C_1, C_2, \ldots, C_n\}$, where each alternative A_i is associated with a set of hypersoft elements $\gamma_{ij} = x_{ij} \cdot \mu_{ij}$, and the weights $w_j, p_j \in [0,1]$ satisfy $\sum_j w_j = 1$, $\sum_j p_j = 1$, the HS-OWA, HS-POWA, and HS-hybrid aggregation functions will always produce a unique aggregated value for each alternative.*

Proof. For each alternative A_i, we compute the vector of adjusted evaluations:

$$\gamma_i = \{\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{in}\}, \quad \text{where } \gamma_{ij} = x_{ij} \cdot \mu_{ij}, \quad x_{ij}, \mu_{ij} \in [0,1]$$

These values are sorted in non-increasing order to obtain $\gamma_{i(k)}$, the kth largest adjusted value.

Now, consider the aggregation formulas:

$$HS\text{-}OWA(A_i) = \sum_{k=1}^{n} w_k \cdot \gamma_{i(k)}, \quad HS\text{-}POWA(A_i) = \sum_{k=1}^{n} p_k \cdot \gamma_{i(k)},$$

$$HS\text{-}Hybrid(A_i) = \sum_{k=1}^{n} h_k \cdot \gamma_{i(k)}$$

where $h_k = \beta p_k + (1-\beta) w_k$ and $\beta \in [0,1]$. Each component in the summation involves a product of two real numbers in the interval $[0,1]$, and all weights sum to 1. The sorting process is deterministic and well-defined, and the weighted sum of a finite number of real numbers yields a unique real number.

Thus, for each alternative A_i, a unique aggregated score is produced under HS-OWA, HS-POWA, and HS-hybrid.

Theorem 5.4. *The HS-OWA, HS-POWA, and HS-hybrid aggregation functions are symmetric for permutations of criteria when the distributions of membership degrees μ_{ij} and corresponding weights w_j, p_j are symmetric. That is, if the multiset of μ_{ij} and the weight vector are invariant under permutations, then permuting the order of criteria will not affect the aggregated value.*

Proof. Let us define the set of adjusted evaluations for an alternative A_i as:

$$\Gamma_i = \{\gamma_{i1}, \gamma_{i2}, \dots, \gamma_{in}\}, \quad \text{where } \gamma_{ij} = x_{ij} \cdot \mu_{ij}$$

For each alternative, the aggregated score is computed using a sorted version of Γ_i, denoted $\gamma_{i(k)}$, where the values are arranged in descending order:

$$\gamma_{i(1)} \geq \gamma_{i(2)} \geq \cdots \geq \gamma_{i(n)}$$

Now, consider a permutation π of the indices $\{1, 2, \dots, n\}$, resulting in a reordered version of the input:

$$\Gamma_i^{\pi} = \{\gamma_{i\pi(1)}, \gamma_{i\pi(2)}, \dots, \gamma_{i\pi(n)}\}$$

Since the aggregation functions use only the sorted values $\gamma_{i(k)}$, and not their original positions, we have:

$$\text{Sort}(\Gamma_i) = \text{Sort}(\Gamma_i^{\pi})$$

Next, consider the symmetric nature of the weights. A weight vector $W = \{w_1, w_2, \dots, w_n\}$ is symmetric if for every permutation π, the permuted vector $W^{\pi} = \{w_{\pi(1)}, \dots, w_{\pi(n)}\}$ satisfies:

$$\text{Sort}(W) = \text{Sort}(W^{\pi})$$

This implies that the aggregation operation uses weights in sorted order, aligned with sorted $\gamma_{i(k)}$, and the alignment is invariant to the permutation of inputs.

Therefore, for HS-OWA:

$$\begin{aligned} HS\text{-}OWA(A_i) &= \sum_{k=1}^{n} w_k \cdot \gamma_{i(k)} \\ &= \sum_{k=1}^{n} w_{\pi(k)} \cdot \gamma_{i(k)} \quad (\text{since Sort}(W) = \text{Sort}(W^{\pi})) \end{aligned}$$

Similarly, for HS-POWA:

$$HS\text{-}POWA(A_i) = \sum_{k=1}^{n} p_k \cdot \gamma_{i(k)} = \sum_{k=1}^{n} p_{\pi(k)} \cdot \gamma_{i(k)} \quad (\text{as Sort}(P) = \text{Sort}(P^{\pi}))$$

And for HS-Hybrid:

$$HS\text{-}Hybrid(A_i) = \sum_{k=1}^{n} h_k \cdot \gamma_{i(k)}, \quad h_k = \beta p_k + (1-\beta) w_k$$

The vector $H = \{h_1, h_2, \dots, h_n\}$ inherits symmetry from P and W. Hence:

$$\text{Sort}(H) = \text{Sort}(H^{\pi})$$

which ensures:

$$HS\text{-}Hybrid(A_i) = \sum_{k=1}^{n} h_k \cdot \gamma_{i(k)} = \sum_{k=1}^{n} h_{\pi(k)} \cdot \gamma_{i(k)}$$

Therefore, under symmetric distributions of weights and membership degrees (i.e., equal multisets), the aggregation results remain invariant under permutations of the criteria:

$$HS\text{-}OWA(A_i) = HS\text{-}OWA(A_i^{\pi}),$$

$$HS\text{-}POWA(A_i) = HS\text{-}POWA(A_i^{\pi}),$$

$$HS\text{-}Hybrid(A_i) = HS\text{-}Hybrid(A_i^{\pi})$$

Theorem 5.5. *Let $A_i = \{a_{i1}, a_{i2}, \dots, a_{in}\}$ be the set of decision elements for alternative A_i, where each $a_{ij} = x_{ij} \cdot \mu_{ij}$ and $x_{ij} = r$, $\mu_{ij} = \mu$ are constant for all $j \in \{1, 2, \dots, n\}$. Then, the HS-POWA aggregation of A_i satisfies:*

$$HS\text{-}POWA(A_i) = r \cdot \mu$$

Proof. Given $x_{ij} = r$ and $\mu_{ij} = \mu$, we have:

$$\gamma_{ij} = x_{ij} \cdot \mu_{ij} = r \cdot \mu \quad \forall j$$

Therefore, the adjusted input vector is:

$$\Gamma_i = \{\gamma_{i1}, \gamma_{i2}, \dots, \gamma_{in}\} = \{r\mu, r\mu, \dots, r\mu\}$$

Since all entries are identical, the ordered vector $\gamma_{i(k)}$ remains:

$$\gamma_{i(1)} = \gamma_{i(2)} = \cdots = \gamma_{i(n)} = r\mu$$

Now, applying the HS-POWA operator:

$$HS\text{-}POWA(A_i) = \sum_{k=1}^{n} p_k \cdot \gamma_{i(k)} = \sum_{k=1}^{n} p_k \cdot (r\mu)$$

Factoring out the constant value:

$$HS\text{-}POWA(A_i) = r\mu \cdot \sum_{k=1}^{n} p_k$$

Since $\sum_{k=1}^{n} p_k = 1$, we conclude:

$$HS\text{-}POWA(A_i) = r\mu$$

Corollary 5.5.1. *If all decision values $a_i = r$, then:*

$$HS\text{-}OWA(a) = HS\text{-}POWA(a) = r$$

Proof. In this case, the reordered vector $\gamma = \{r, r, \dots, r\}$. Thus:

$$HS\text{-}OWA = \sum_{i=1}^{n} w_i \cdot r = r \cdot \sum w_i = r, \quad HS\text{-}POWA = \sum_{i=1}^{n} p_i \cdot r = r \cdot \sum p_i = r$$

Corollary 5.5.2. *If any weight $w_i = 0$ or $p_i = 0$, the corresponding element γ_i in the ordered vector does not affect the final HS-OWA or HS-POWA result.*

Proof. By the definition of the aggregators, elements with zero weight contribute nothing:

$$w_i \cdot \gamma_i = 0, \quad p_i \cdot \gamma_i = 0$$

Hence, they can be excluded without affecting the final value.

Theorem 5.6. *Let $A = \{(a_1, t_1), (a_2, t_2), \dots, (a_n, t_n)\}$ be the set of decision values with associated sub-attributes. Then, the result of HS-OWA or HS-POWA aggregation is invariant under any permutation of A.*

Proof. Both HS-OWA and HS-POWA operate on the sorted vector $\gamma = \{\gamma_1, \dots, \gamma_n\}$, which is obtained by sorting the original decision values a_i, regardless of their original position. Since any permutation of the input does not change the sorted values, and the aggregation depends only on γ, the result remains unchanged:

$$HS\text{-}OWA(A) = \sum_{i=1}^{n} w_i \cdot \gamma_i, \quad HS\text{-}POWA(A) = \sum_{i=1}^{n} p_i \cdot \gamma_i$$

Hence, both operators are permutation invariant.

Theorem 5.7. *Let $a_i \in [L, U]$ for all i, then the aggregated value under HS-OWA and HS-POWA lies within the same interval:*

$$L \leq HS\text{-}OWA(A),\ HS\text{-}POWA(A) \leq U$$

Proof. Since $\gamma_i \in [L, U]$ and the weights w_i and p_i are non-negative and sum to 1, the aggregation is a convex combination of the elements in the interval $[L, U]$. Therefore:

$$\sum_{i=1}^{n} w_i \cdot \gamma_i \in [L, U], \quad \sum_{i=1}^{n} p_i \cdot \gamma_i \in [L, U]$$

Thus, the aggregated result of HS-OWA or HS-POWA must also lie within $[L, U]$.

Remark 5.1. *In the hypersoft environment, even if two decision vectors a and a' yield the same values after aggregation, the underlying semantic meaning may differ due to sub-attribute associations t_i. This reinforces the idea that explainability must be sub-attribute-aware.*

Remark 5.2. *HS-POWA introduces probabilistic weights based on induced orderings, making the aggregation result sensitive to small perturbations in input values.*

Remark 5.3. *The rank of a value a_i in the sorted vector γ_{ij} can be influenced not just by its numerical value but by the semantic or contextual interpretation of its sub-attribute t_i. This highlights the non-triviality of ranking in hypersoft settings.*

6. Entropy weighted and similarity weighted variants of HS-OWA and HS-POWA

Definition 6.1. *Let $a = \{a_1, a_2, \dots, a_n\}$ be decision scores associated with sub-attributes $t_1, t_2, \dots, t_n$. Let $H(t_i)$ denote the entropy of the sub-attribute t_i. Define the entropy-based weights as:*

$$w_i = \frac{H(t_i)}{\sum_{j=1}^{n} H(t_j)} \tag{9}$$

Let γ_{ij} be the reordered values of a_i in descending order. Then the Entropy-Weighted HS-OWA operator is defined as:

$$HS\text{-}OWA_H(a) = \sum_{i=1}^{n} w_i \cdot \gamma_{ij} \tag{10}$$

Example 6.1. *Consider decision scores $a = \{0.6, 0.8, 0.5\}$ with corresponding sub-attributes t_1, t_2, t_3. Let their entropies be:*

$$H(t_1) = 0.3, \quad H(t_2) = 0.5, \quad H(t_3) = 0.2$$

Total entropy:

$$\sum H(t_i) = 0.3 + 0.5 + 0.2 = 1.0$$

Weights:

$$w = \left(\frac{0.3}{1}, \frac{0.5}{1}, \frac{0.2}{1}\right) = \{0.3, 0.5, 0.2\}$$

Reordered decision scores γ_{ij} in descending order:

$$\gamma = \{0.8, 0.6, 0.5\}$$

Apply Entropy-weighted HS-OWA:

$$HS\text{-}OWA_H(a) = 0.3 \cdot 0.8 + 0.5 \cdot 0.6 + 0.2 \cdot 0.5 = 0.24 + 0.30 + 0.10 = 0.64$$

Definition 6.2 (*Entropy-Weighted HS-POWA*)**.** *Let p_i be the probabilistic weights induced based on entropy ranks, defined as:*

$$p_i = \frac{H(t_{\pi(i)})}{\sum_{j=1}^{n} H(t_{\pi(j)})} \tag{11}$$

where $\pi(i)$ denotes the index after ordering a in descending form. Then the Entropy-weighted HS-POWA operator is given by:

$$HS\text{-}POWA_H(a) = \sum_{i=1}^{n} p_i \cdot \gamma_{ij} \tag{12}$$

Example 6.2. *Using the same scores $a = \{0.6, 0.8, 0.5\}$ and entropies $H(t_1) = 0.3,\ H(t_2) = 0.5,\ H(t_3) = 0.2$, we first reorder the scores:*

$$\gamma = \{0.8, 0.6, 0.5\}, \quad \text{corresponding to } t_2, t_1, t_3$$

Now compute entropy-based probabilistic weights:

$$\sum H(t_{\pi(i)}) = H(t_2) + H(t_1) + H(t_3) = 0.5 + 0.3 + 0.2 = 1.0$$

$$p = \left(\frac{0.5}{1}, \frac{0.3}{1}, \frac{0.2}{1}\right) = \{0.5, 0.3, 0.2\}$$

Apply entropy-weighted HS-POWA:

$$HS\text{-}POWA_H(a) = 0.5 \cdot 0.8 + 0.3 \cdot 0.6 + 0.2 \cdot 0.5 = 0.40 + 0.18 + 0.10 = 0.68$$

Definition 6.3 (*Similarity-Weighted HS-OWA*)**.** *Let $sim(t_i, t^*)$ be a similarity measure between sub-attribute t_i and a reference or ideal sub-attribute t^*. Define the similarity-based weights as:*

$$w_i = \frac{sim(t_i, t^*)}{\sum_{j=1}^{n} sim(t_j, t^*)} \tag{13}$$

Then, the similarity-weighted HS-OWA operator is defined as:

$$HS\text{-}OWA_{sim}(a) = \sum_{i=1}^{n} w_i \cdot \gamma_{ij} \tag{14}$$

Example 6.3. *Let decision scores be $a = \{0.7, 0.5, 0.9\}$, and assume the similarity of each sub-attribute with the ideal sub-attribute t^* is:*

$$sim(t_1, t^*) = 0.4, \quad sim(t_2, t^*) = 0.3, \quad sim(t_3, t^*) = 0.8$$

Total similarity:

$$\sum sim(t_i, t^*) = 0.4 + 0.3 + 0.8 = 1.5$$

Weights:

$$w = \left(\frac{0.4}{1.5}, \frac{0.3}{1.5}, \frac{0.8}{1.5}\right) = \{0.267, 0.2, 0.533\}$$

Reordered scores $\gamma = \{0.9, 0.7, 0.5\}$ Apply similarity-weighted HS-OWA:

$$HS\text{-}OWA_{sim}(a) = 0.267 \cdot 0.9 + 0.2 \cdot 0.7 + 0.533 \cdot 0.5$$

$$= 0.2403 + 0.14 + 0.2665 = 0.6468$$

Definition 6.4 (*Similarity-Weighted HS-POWA*)**.** *Let p_i be the similarity-induced probabilistic weights, assigned according to reordered sub-attribute similarities $sim(t_{\pi(i)}, t^*)$, where $\pi(i)$ corresponds to the permutation based on sorting a in descending order. Then:*

$$p_i = \frac{sim(t_{\pi(i)}, t^*)}{\sum_{j=1}^{n} sim(t_{\pi(j)}, t^*)} \tag{15}$$

Hence, the similarity-weighted HS-POWA operator becomes:

$$HS\text{-}POWA_{sim}(a) = \sum_{i=1}^{n} p_i \cdot \gamma_{ij} \tag{16}$$

Example 6.4. *Using the same input $a = \{0.7, 0.5, 0.9\}$ and similarities:*

$$sim(t_1, t^*) = 0.4, \quad sim(t_2, t^*) = 0.3, \quad sim(t_3, t^*) = 0.8$$

Reordered scores $\gamma = \{0.9, 0.7, 0.5\}$ correspond to t_3, t_1, t_2. Now reorder the similarities accordingly:

$$sim_{\pi(i)} = \{0.8, 0.4, 0.3\}, \quad \sum = 1.5$$

$$p = \left(\frac{0.8}{1.5}, \frac{0.4}{1.5}, \frac{0.3}{1.5}\right) = \{0.533, 0.267, 0.2\}$$

Apply similarity-weighted HS-POWA:

$$HS\text{-}POWA_{sim}(a) = 0.533 \cdot 0.9 + 0.267 \cdot 0.7 + 0.2 \cdot 0.5$$

$$= 0.4797 + 0.1869 + 0.1 = 0.7666$$

7. Extension of HS-OWA and HS-POWA

7.1. Extended sets with definitions

The hypersoft set OWA (HS-OWA) and hypersoft set POWA (HS-POWA) operators can be extended to handle more expressive uncertainty models such as Fuzzy [42], Intuitionistic [43], Neutrosophic [44], Pythagorean [45], and q-Rung fuzzy sets [46], offering enhanced flexibility in uncertain environments.

Definition 7.1 (*HS-OWA and HS-POWA under Intuitionistic Sets*). *An intuitionistic fuzzy set A over a universe X is defined as:*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X,\ \mu_A(x), \nu_A(x) \in [0,1],\ \mu_A(x) + \nu_A(x) \le 1\}$$

where $\mu_A(x)$ and $\nu_A(x)$ denote the membership and non-membership degrees of x, respectively.

Now, consider each score x_{ij} and membership μ_{ij} be intuitionistic fuzzy numbers, represented as:

$$x_{ij} = \langle \mu_{x_{ij}}, \nu_{x_{ij}}\rangle, \quad \mu_{ij} = \langle \mu_{\mu_{ij}}, \nu_{\mu_{ij}}\rangle,$$

where μ is the degree of membership and ν is the degree of non-membership, satisfying $0 \le \mu + \nu \le 1$. Define the adjusted value as:

$$\gamma_{ij} = \langle \mu_{x_{ij}} \cdot \mu_{\mu_{ij}}, \nu_{x_{ij}} + \nu_{\mu_{ij}} - \nu_{x_{ij}} \cdot \nu_{\mu_{ij}}\rangle$$

Then, reorder the values γ_{ij} based on the score function $S(\gamma_{ij}) = \mu_{\gamma_{ij}} - \nu_{\gamma_{ij}}$, and compute the HS-OWA aggregation as:

$$HS\text{-}OWA(A_i) = \sum_{k=1}^{n} w_k \cdot \gamma_{i(k)}$$

where $\gamma_{i(k)}$ are sorted adjusted intuitionistic fuzzy values. Then HS-POWA aggregation of alternative A_i is computed by using:

$$HS\text{-}POWA(A_i) = \sum_{k=1}^{n} p_k \cdot \gamma_{i(k)}$$

where:

- $w_k \in [0,1]$ are OWA weights such that $\sum w_k = 1$,
- $p_k \in [0,1]$ are probabilistic weights such that $\sum p_k = 1$,
- $\gamma_{i(k)}$ are reordered adjusted intuitionistic fuzzy values based on the score function.

Definition 7.2 (*HS-OWA and POWA under Neutrosophic Sets*). *A neutrosophic set A over a universe X is defined as:*

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X,\ T, I, F \in [0,1]\}$$

where T, I, and F represent degrees of truth, indeterminacy, and falsity of x, respectively with $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Now, consider each score and membership be neutrosophic numbers:

$$x_{ij} = \langle T_{x_{ij}}, I_{x_{ij}}, F_{x_{ij}}\rangle, \quad \mu_{ij} = \langle T_{\mu_{ij}}, I_{\mu_{ij}}, F_{\mu_{ij}}\rangle$$

Then the adjusted value can be defined as:

$$\gamma_{ij} = \langle T_{x_{ij}} \cdot T_{\mu_{ij}},\ I_{x_{ij}} + I_{\mu_{ij}} - I_{x_{ij}} \cdot I_{\mu_{ij}},\ F_{x_{ij}} + F_{\mu_{ij}} - F_{x_{ij}} \cdot F_{\mu_{ij}}\rangle$$

The aggregation is carried out by sorting the neutrosophic adjusted scores using a ranking function such as:

$$S(\gamma_{ij}) = T_{\gamma_{ij}} - F_{\gamma_{ij}} - \alpha \cdot I_{\gamma_{ij}}, \quad \alpha \in [0,1]$$

and calculated as:

$$HS\text{-}OWA(A_i) = \sum_{k=1}^{n} w_k \cdot \gamma_{i(k)}$$

And

HS-POWA operator can be calculated as:

$$HS\text{-}POWA(A_i) = \sum_{k=1}^{n} p_k \cdot \gamma_{i(k)}$$

where:

- $w_k \in [0,1]$ are OWA weights such that $\sum w_k = 1$,
- $p_k \in [0,1]$ are probabilistic weights such that $\sum p_k = 1$,
- $\gamma_{i(k)}$ are reordered adjusted neutrosophic values based on the score function.

Definition 7.3 (*HS-OWA and HS-POWA under Pythagorean Fuzzy Sets*). *A Pythagorean fuzzy set A over a universe X is defined as:*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X,\ \mu_A(x), \nu_A(x) \in [0,1],\ \mu_A(x)^2 + \nu_A(x)^2 \le 1\}$$

where $\mu_A(x)$ and $\nu_A(x)$ represent the membership and non-membership degrees of element x, with a looser constraint than in intuitionistic fuzzy sets.

Consider each score x_{ij} and membership μ_{ij} be Pythagorean fuzzy numbers:

$$x_{ij} = \langle \mu_{x_{ij}}, \nu_{x_{ij}}\rangle, \quad \mu_{ij} = \langle \mu_{\mu_{ij}}, \nu_{\mu_{ij}}\rangle,$$

where $\mu^2 + \nu^2 \le 1$ for each pair. Then the adjusted value is calculated using:

$$\gamma_{ij} = \left\langle \mu_{x_{ij}} \cdot \mu_{\mu_{ij}},\ \sqrt{1 - (1 - \nu_{x_{ij}}^2)(1 - \nu_{\mu_{ij}}^2)}\right\rangle$$

The adjusted score function is defined as:

$$S(\gamma_{ij}) = \mu_{\gamma_{ij}}^2 - \nu_{\gamma_{ij}}^2$$

Sort the adjusted values γ_{ij} based on $S(\gamma_{ij})$ in descending order. Then compute:

$$HS\text{-}OWA(A_i) = \sum_{k=1}^{n} w_k \cdot \gamma_{i(k)}, \quad HS\text{-}POWA(A_i) = \sum_{k=1}^{n} p_k \cdot \gamma_{i(k)}$$

where:

- $w_k \in [0,1]$ are OWA weights such that $\sum w_k = 1$,
- $p_k \in [0,1]$ are probabilistic weights such that $\sum p_k = 1$,
- $\gamma_{i(k)}$ are reordered adjusted Pythagorean fuzzy values based on the score function.

Definition 7.4 (*HS-OWA and HS-POWA under q-Rung Orthopair Fuzzy Sets*). *A q-Rung orthopair fuzzy set (q-ROFS) "A" over a universe X is defined as:*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X,\ \mu_A(x), \nu_A(x) \in [0,1],\ \mu_A(x)^q + \nu_A(x)^q \le 1\}$$

where $\mu_A(x)$ and $\nu_A(x)$ are the membership and non-membership degrees of x, and $q \ge 1$ controls the flexibility of the fuzzy representation—larger q allows greater uncertainty representation than Pythagorean or Intuitionistic sets.

Consider each score and membership be q-Rung orthopair fuzzy numbers:

$$x_{ij} = \langle \mu_{x_{ij}}, \nu_{x_{ij}}\rangle, \quad \mu_{ij} = \langle \mu_{\mu_{ij}}, \nu_{\mu_{ij}}\rangle,$$

where $\mu^q + \nu^q \le 1$ for all elements.Then the adjusted value is computed as:

$$\gamma_{ij} = \left\langle \mu_{x_{ij}} \cdot \mu_{\mu_{ij}},\ \left(1 - (1 - \nu_{x_{ij}}^q)(1 - \nu_{\mu_{ij}}^q)\right)^{1/q}\right\rangle$$

The score function is defined as:

$$S(\gamma_{ij}) = \mu_{\gamma_{ij}}^q - \nu_{\gamma_{ij}}^q$$

Sort the adjusted values γ_{ij} based on $S(\gamma_{ij})$ in descending order. Then compute the HS-OWA aggregation as:

$$HS\text{-}OWA(A_i) = \sum_{k=1}^{n} w_k \cdot \gamma_{i(k)}$$

and the HS-POWA aggregation as:

$$HS\text{-}POWA(A_i) = \sum_{k=1}^{n} p_k \cdot \gamma_{i(k)}$$

where:

- $w_k \in [0,1]$ are OWA weights such that $\sum w_k = 1$,
- $p_k \in [0,1]$ are probabilistic weights such that $\sum p_k = 1$,
- $\gamma_{i(k)}$ denotes the kth largest adjusted q-Rung fuzzy value sorted by score.

7.2. Practical implications of HS-OWA and HS-POWA operators

The hypersoft set-based OWA (HS-OWA) and POWA (HS-POWA) operators provide flexible and context-aware aggregation frameworks, making them highly applicable in various practical domains. Their ability to incorporate both semantic relevance (via hypersoft membership degrees) and positional uncertainty (via probabilistic weighting) enhances decision-making processes in complex environments. The following are some practical implications across different fields:

- **Medical Diagnosis:** In healthcare, patient symptoms and clinical indicators often vary in importance and relevance depending on subcategories (e.g., age group, severity level). HS-OWA can integrate diagnostic weights while accounting for symptom relevance, and HS-POWA can further model uncertainty in symptom reporting or probabilistic outcomes of test results.
- **Environmental Management:** In sustainable decision-making (e.g., selecting wastewater treatment technologies), HS-OWA allows aggregation of multiple criteria (efficiency, cost, impact) while incorporating the contextual importance of environmental factors. HS-POWA enhances this by introducing probabilistic importance to account for variability in environmental data or stakeholder preferences.
- **Supply Chain and Logistics:** In vendor selection or route optimization, HS-OWA can be used to combine scores related to cost, time, and reliability with context-aware membership degrees representing qualitative criteria. HS-POWA is useful where probabilistic models are needed to capture uncertainties like demand fluctuation or delivery risk.
- **Social Sciences and Policy Making:** Complex decisions involving qualitative judgments, such as public policy evaluations or educational reform prioritization, can benefit from HS-OWA by accounting for nuanced attribute relevance (e.g., regional impact, demographic specificity). HS-POWA is ideal for integrating survey-based probabilistic judgments or opinion distributions.
- **Engineering Design and Evaluation:** In product development or multi-criteria engineering assessments, HS-OWA allows for an interpretable aggregation of design criteria with varying contextual relevance. HS-POWA helps when design choices involve probabilistic risks or user-preference uncertainties.
- **Information Retrieval and Recommendation Systems:** HS-OWA can be applied in ranking documents or recommending items by incorporating user profile relevance. HS-POWA can further refine the model when user preferences are uncertain or based on historical probabilistic behavior.

These applications highlight how HS-OWA and HS-POWA are robust decision tools that extend classical aggregation schemes into real-world, uncertain, and semantically rich decision environments.

8. Proposed group decision-making methodology

This section proposes a group decision-making methodology, an illustrative case study, result discussions, and a comparison section.

8.1. Group decision-making methodology

The step-by-step methodology for group decision-making is illustrated below:

Step 1: Define the Decision Problem

Let $A = \{A_1, A_2, \dots, A_m\}$ be the set of m alternatives, and $P = \{p_1, p_2, \dots, p_n\}$ be the set of parameters (criteria). Each parameter may have associated sub-attributes forming sub-attribute-based evaluations. A group of k experts provides their evaluations in the form of hypersoft sets, which include:

- Evaluation scores $a_{ij}^{(k)}$ of alternative A_i under parameter P_j by expert k
- Corresponding attribute sub-attributes $t_{ij}^{(k)}$
- Expert-defined OWA weights $w_{ij}^{(k)}$ or probabilities $p_{ij}^{(k)}$

Step 2: Construct the Hypersoft Decision Matrix

For each expert k, construct a hypersoft decision matrix $D^{(k)}$:

$$D^{(k)} = \left[a_{ij}^{(k)}, t_{ij}^{(k)}\right]_{m\times n}$$

Aggregate the evaluations per alternative using HS-OWA or HS-POWA.

Step 3: Apply Aggregation Operators For each expert:

- Sort each row based on the scores $a_{ij}^{(k)}$ in descending order to obtain reordered scores $\gamma_{ij}^{(k)}$
- Apply HS-OWA:

$$HS\text{-}OWA_i^{(k)} = \sum_{j=1}^{n} w_j^{(k)} \cdot \gamma_{ij}^{(k)}$$

- Or apply HS-POWA:

$$HS\text{-}POWA_i^{(k)} = \sum_{j=1}^{n} p_j^{(k)} \cdot \gamma_{ij}^{(k)}$$

Step 4: Aggregation of Expert Opinions

After computing individual scores for each alternative using HS-OWA and HS-POWA operators for all experts, the group decision is derived by aggregating the evaluations across the K experts. This is achieved by averaging the scores per alternative under each operator.

For a given alternative A_i, the final group scores using HS-OWA and HS-POWA are calculated as:

$$\bar{a}_i^{OWA} = \frac{1}{K}\sum_{k=1}^{K} HS\text{-}OWA_i^{(k)}, \quad \bar{a}_i^{POWA} = \frac{1}{K}\sum_{k=1}^{K} HS\text{-}POWA_i^{(k)} \tag{17}$$

Step 5: Rank Alternatives Rank alternatives based on their aggregated scores.

8.2. Illustrative example

A municipal body seeks to select the most sustainable wastewater treatment technology from five alternatives: A_1 (Activated Sludge), A_2 (Membrane Bioreactor), A_3 (Constructed Wetlands), A_4 (SBR), and A_5 (UASB). Evaluation is based on four criteria: cost, efficiency, maintenance, and environmental impact. Three experts provide hypersoft evaluations using sub-attribute-based attributes.

Step 1: Alternatives and Criteria

For this decision-makers (Expert) has considered five alternatives A_1 to A_5 and four criteria i.e. C_1 = Cost, C_2 = Efficiency, C_3 = Maintenance, C_4 = Environmental Impact.

Step 2: Construct a hypersoft decision matrix

Let the universe of alternatives be $\mathcal{U} = \{A_1, A_2, A_3, A_4, A_5\}$, and let the attribute space be as: Thus, the hypersoft set mapping for each expert becomes:

$$\mathbb{F}^{(k)} : (\mathbb{L}_1^2 \times \mathbb{L}_2^2 \times \mathbb{L}_3^2 \times \mathbb{L}_4^2) \to \mathcal{P}(\mathcal{U})$$

where $\mathbb{F}^{(k)}(A_i)$ returns the degrees of membership of alternative A_i under the sub-attribute t_1, t_2, t_3, t_4.

- $t_1 = (\text{Low cost}, \text{High cost})$
- $t_2 = (\text{Low efficiency}, \text{High efficiency})$
- $t_3 = (\text{High maintenance}, \text{Low maintenance})$
- $t_4 = (\text{High environmental Impact}, \text{Low environmental Impact})$

The hypersoft set $(\mathbb{F}^{(k)}, \mathbb{L}_1^2 \times \mathbb{L}_2^2 \times \mathbb{L}_3^2)$ maps each expert E_k to a fuzzy set over sub-attribute (Low cost, High efficiency, Low Maintenance) with associated membership degrees in [0,1] for each alternative A_i. This results in:

$$D^{(k)} = \left[\mu_{ij}^{(k)}\right]_{5\times 3}, \quad \mu_{ij}^{(k)} \in [0,1]$$

Each row corresponds to an alternative A_i, and each column to a sub-attribute t_j. For example:

$$\mathbb{F}^{(1)}(A_1) = \{(t_1, 0.8), (t_2, 0.5), (t_3, 0.2)\} \quad \mathbb{F}^{(2)}(A_1) = \{(t_1, 0.9), (t_2, 0.4), (t_3, 0.3)\}$$

$$D^{(1)} = \begin{bmatrix} 0.8 & 0.5 & 0.2 \\ 0.7 & 0.6 & 0.3 \\ 0.6 & 0.7 & 0.4 \\ 0.5 & 0.6 & 0.5 \\ 0.6 & 0.5 & 0.3 \end{bmatrix}$$

Expert 2 (E_2)

$$D^{(2)} = \begin{bmatrix} 0.9 & 0.4 & 0.3 \\ 0.6 & 0.7 & 0.2 \\ 0.7 & 0.6 & 0.4 \\ 0.6 & 0.5 & 0.5 \\ 0.5 & 0.4 & 0.2 \end{bmatrix}$$

Expert 3 (E_3)

$$D^{(3)} = \begin{bmatrix} 0.85 & 0.45 & 0.25 \\ 0.65 & 0.6 & 0.35 \\ 0.55 & 0.7 & 0.45 \\ 0.45 & 0.55 & 0.5 \\ 0.55 & 0.45 & 0.3 \end{bmatrix}$$

Each row represents an alternative A_i, and each column represents a sub-attribute t_j. These values reflect the degree of suitability of each alternative under the respective sub-attribute from each expert's perspective.

Step 3: Apply Aggregation Operators

Each expert provided a 3-column decision matrix based on sub-attributes. We now sort each row in descending order to get reordered scores $\gamma_{ij}^{(k)}$, the weight vector for HS-OWA: $w = \{0.4, 0.3, 0.3\}$, the probability vector for HS-POWA: $p = \{0.35, 0.35, 0.3\}$, and finally compute both HS-OWA and HS-POWA for each alternative and each expert.

Reordered Scores $\gamma_{ij}^{(1)}$ and aggregated values for Expert 1:

$$A_1 : \{0.8, 0.5, 0.2\} \quad \Rightarrow HS\text{-}OWA = 0.56, \quad HS\text{-}POWA = 0.545$$
$$A_2 : \{0.7, 0.6, 0.3\} \quad \Rightarrow HS\text{-}OWA = 0.61, \quad HS\text{-}POWA = 0.595$$
$$A_3 : \{0.7, 0.6, 0.4\} \quad \Rightarrow HS\text{-}OWA = 0.64, \quad HS\text{-}POWA = 0.625$$
$$A_4 : \{0.6, 0.5, 0.5\} \quad \Rightarrow HS\text{-}OWA = 0.56, \quad HS\text{-}POWA = 0.545$$
$$A_5 : \{0.6, 0.5, 0.3\} \quad \Rightarrow HS\text{-}OWA = 0.53, \quad HS\text{-}POWA = 0.515$$

Reordered Scores $\gamma_{ij}^{(2)}$ and aggregated values for Expert 2:

$$A_1 : \{0.9, 0.4, 0.3\} \quad \Rightarrow HS\text{-}OWA = 0.63, \quad HS\text{-}POWA = 0.615$$
$$A_2 : \{0.7, 0.6, 0.2\} \quad \Rightarrow HS\text{-}OWA = 0.55, \quad HS\text{-}POWA = 0.535$$
$$A_3 : \{0.7, 0.6, 0.4\} \quad \Rightarrow HS\text{-}OWA = 0.64, \quad HS\text{-}POWA = 0.625$$
$$A_4 : \{0.6, 0.5, 0.5\} \quad \Rightarrow HS\text{-}OWA = 0.56, \quad HS\text{-}POWA = 0.545$$
$$A_5 : \{0.5, 0.4, 0.2\} \quad \Rightarrow HS\text{-}OWA = 0.41, \quad HS\text{-}POWA = 0.395$$

Reordered Scores $\gamma_{ij}^{(3)}$ and aggregated values for Expert 3:

$$A_1 : \{0.85, 0.45, 0.25\} \quad \Rightarrow HS\text{-}OWA = 0.575, \quad HS\text{-}POWA = 0.56$$
$$A_2 : \{0.65, 0.6, 0.35\} \quad \Rightarrow HS\text{-}OWA = 0.57, \quad HS\text{-}POWA = 0.555$$
$$A_3 : \{0.7, 0.55, 0.45\} \quad \Rightarrow HS\text{-}OWA = 0.61, \quad HS\text{-}POWA = 0.595$$
$$A_4 : \{0.55, 0.5, 0.45\} \quad \Rightarrow HS\text{-}OWA = 0.515, \quad HS\text{-}POWA = 0.5025$$
$$A_5 : \{0.55, 0.45, 0.3\} \quad \Rightarrow HS\text{-}OWA = 0.475, \quad HS\text{-}POWA = 0.46$$

Step 4: Aggregation of Expert Opinions

We now calculate each alternative's final aggregated HS-OWA and HS-POWA scores by averaging the individual expert scores.

$$\bar{a}_1^{OWA} = \frac{1}{3}(0.56 + 0.63 + 0.575) = 0.588$$
$$\bar{a}_2^{OWA} = \frac{1}{3}(0.61 + 0.55 + 0.57) = 0.577$$
$$\bar{a}_3^{OWA} = \frac{1}{3}(0.64 + 0.64 + 0.61) = 0.6267$$
$$\bar{a}_4^{OWA} = \frac{1}{3}(0.56 + 0.56 + 0.515) = 0.545$$
$$\bar{a}_5^{OWA} = \frac{1}{3}(0.53 + 0.41 + 0.475) = 0.4713$$

$$\bar{a}_1^{POWA} = \frac{1}{3}(0.545 + 0.615 + 0.56) = 0.5733$$
$$\bar{a}_2^{POWA} = \frac{1}{3}(0.595 + 0.535 + 0.555) = 0.5617$$
$$\bar{a}_3^{POWA} = \frac{1}{3}(0.625 + 0.625 + 0.595) = 0.615$$
$$\bar{a}_4^{POWA} = \frac{1}{3}(0.545 + 0.545 + 0.5025) = 0.5308$$
$$\bar{a}_5^{POWA} = \frac{1}{3}(0.515 + 0.395 + 0.46) = 0.4567$$

Step 5: Final Ranking of Alternatives

The final ranking of alternatives based on the aggregated HS-OWA and HS-POWA scores is presented in the following table:

Alternative	HS-OWA Score	HS-POWA Score
A_3 (Constructed Wetlands)	0.6267	0.615
A_1 (Activated Sludge)	0.588	.5733
A_2 (Membrane Bioreactor)	0.577	0.5617
A_4 (SBR)	0.545	0.5308
A_5 (UASB)	0.4713	0.4567

Based on the group decision-making process using HS-OWA and HS-POWA in a hypersoft setting, A_3 (Constructed Wetlands) emerges as the most sustainable option.

8.3. Comparison and result discussion

The hypersoft set-based OWA (HS-OWA) and POWA (HS-POWA) operators offer substantial theoretical enhancements over their classical counterparts by incorporating contextual relevance through membership degrees. In the classical OWA operator, aggregation is purely based on positional weights assigned to ordered evaluation scores, without considering the semantic or contextual strength of the criteria involved. Similarly, the classical POWA adds an element of probabilistic weighting but still operates on raw scores. In contrast, the HS-OWA operator introduces contextual evaluation by multiplying each score x_{ij} with a corresponding hypersoft membership degree μ_{ij}, producing adjusted values $\gamma_{ij} = x_{ij} \cdot \mu_{ij}$. These adjusted values are then ordered, and the aggregation is carried out using standard OWA weights. This contextualized adjustment ensures that the evaluations' magnitude and relevance influence the outcome (see Table 1).

The HS-POWA operator further generalizes this approach by integrating both OWA weights and a reordered probabilistic vector, making it sensitive to the uncertainty or variability in the importance of the positions. This dual weighting mechanism, which contextualizes adjustment through membership degrees and probabilistic reinterpretation of importance, results in a more nuanced and informed aggregation

Table 1
Classical vs hypersoft set-based OWA/POWA Operators.

Operator	Score used	Mathematical formula
Classical OWA	x_{ij}	$\sum_{k=1}^{n} w_k \cdot x_{i(k)}$
Classical POWA	x_{ij}	$\sum_{k=1}^{n} p_{i(k)} \cdot w_k \cdot x_{i(k)}$
HS-OWA	$\gamma_{ij} = x_{ij} \cdot \mu_{ij}$	$\sum_{k=1}^{n} w_k \cdot \gamma_{i(k)}$
HS-POWA	$\gamma_{ij} = x_{ij} \cdot \mu_{ij}$	$\sum_{k=1}^{n} p_{i(k)} \cdot w_k \cdot \gamma_{i(k)}$

strategy. Thus, while classical operators focus solely on numeric ranking or probabilistic bias, the hypersoft counterparts align aggregation with the underlying semantics of the decision context, making them particularly suitable for complex, attribute-rich decision-making environments such as medical diagnosis, environmental sustainability, and social sciences. The main limitation of HS-OWA and HS-POWA operators is the subjectivity in selecting membership degrees, which can affect the robustness of results. Additionally, these operators may face computational challenges as the number of criteria increases, limiting scalability in large datasets.

9. Conclusion

This study introduced the HS-OWA and HS-POWA operators to address the limitations of classical aggregation methods in handling hierarchical or subdivided attributes under uncertainty. The findings show that these operators preserve sub-attribute information, capture decision-makers' preferences and risk attitudes, and provide robust evaluations in complex multi-criteria problems such as sustainable technology assessment, policy-making, and large-scale infrastructure planning. Key advantages include enhanced accuracy, flexibility in uncertain environments, and applicability to fuzzy, intuitionistic, and neutrosophic information in both single- and group-decision-making scenarios. The study presents the formal definitions, examples, monotonicity, boundedness, relevant theorems, propositions, and remarks. The entropy and similarity-based weighting schemes are also proposed to further enhance the objectivity and adaptability of the aggregation process. To demonstrate the practical applicability of the definitions, a group decision-making methodology has been proposed and illustrated with a case study in both environmental and economic decision contexts. The practical implications of many real-life problems, as well as the extension of HS-OWA and HS-POWA to advanced fuzzy environments, such as Intuitionistic, Pythagorean, q-Rung, and Neutrosophic sets, highlight the model's versatility in representing imprecise and incomplete expert opinions.

Future directions include extending this framework to dynamic decision environments, where correlation coefficients, distance, and similarity measures can be defined to improve the decision processes. Researchers can focus on applying HS-OWA and HS-POWA to multi-criteria decision-making in fields like healthcare and environmental sustainability. Studies could also explore adaptive membership functions and optimize these operators for large-scale, real-time applications.

CRediT authorship contribution statement

Muhammad Saqlain: Writing – review & editing, Writing – original draft, Methodology, Formal analysis, Conceptualization. **Jose M. Merigó:** Writing – review & editing, Supervision, Methodology, Conceptualization. **Poom Kumam:** Writing – review & editing, Supervision, Resources, Conceptualization.

Consent for publication

This article does not contain any studies with human participants or animals performed by any authors. Each author contributed to and agreed to this publication.

Funding

The authors acknowledge the funding support provided by the King Mongkut's University of Technology Thonburi (KMUTT). This research was supported by the Petchra Pra Jom Klao Ph.D. research scholarship from King Mongkut's University of Technology Thonburi (KMUTT) under Grant No. (49/2565).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors would like to sincerely thank the Editor-in-Chief, the Associate Editor, and the reviewers for their valuable comments and constructive suggestions, which have significantly improved the quality of this paper.

Data availability

No data was used for the research described in the article.

References

[1] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. Syst. Man Cybern. 18 (1) (1988) 183–190.
[2] J.M. Merigó, Probabilities in the OWA operator, Expert Syst. Appl. 39 (13) (2012) 11456–11467.
[3] H.A. Simon, The New Science of Management Decision, Prentice-Hall, 1977.
[4] P.A. Frensch, J. Funke (Eds.), Complex Problem Solving: The European Perspective, Lawrence Erlbaum Associates, 1995.
[5] P. Liu, A weighted aggregation operators multi-attribute group decision-making method based on interval-valued trapezoidal fuzzy numbers, Expert Syst. Appl. 38 (2011) 1053–1060.
[6] J.M. Merigó, A.M. Gil-Lafuente, The induced generalized OWA operator, Inform. Sci. 179 (2009) 729–741.
[7] E.K. Zavadskas, Z. Turskis, Multiple criteria decision making (MCDM) methods in economics: An overview, Technol. Econ. Dev. Econ. 17 (2011) 397–427.
[8] F. Smarandache, Extension of soft set to hypersoft set, and then to plithogenic hypersoft set, Neutrosophic Sets Syst. 22 (2018) 168–170.
[9] M. Akram, A. Luqman, J.C.R. Alcantud, An integrated ELECTRE-I approach for risk evaluation with hesitant pythagorean fuzzy information, Expert Syst. Appl. 200 (2022) 116945.
[10] M. Akram, S. Zahid, Group decision-making method with pythagorean fuzzy rough number for the evaluation of best design concept, Granul. Comput. 8 (2023) 1121–1148.
[11] M. Saqlain, J.M. Merigó, P. Kumam, Neutrosophic probabilistic ordered weighted averaging (NPOWA) operator, Res. Math. 12 (1) (2025) 2535135.
[12] M. Akram, N. Ramzan, M. Deveci, Linguistic pythagorean fuzzy CRITIC-EDAS method for multiple-attribute group decision analysis, Eng. Appl. Artif. Intell. 119 (2023) 105777.
[13] V. Torra, The weighted OWA operator, Int. J. Intell. Syst. 12 (1997) 153–166.
[14] R.R. Yager, Heavy OWA operators, Fuzzy Optim. Decis. Mak. 1 (2002) 379–397.
[15] R.R. Yager, Norms induced from OWA operators, IEEE Trans. Fuzzy Syst. 18 (1) (2010) 57–66.
[16] R.R. Yager, J. Kacprzyk, The Ordered Weighted Averaging Operators: Theory and Applications, Kluwer Academic Publishers, Norwell, MA, 1997.
[17] R.R. Yager, J. Kacprzyk, G. Beliakov, Recent Developments on the Ordered Weighted Averaging Operators: Theory and Practice, Springer-Verlag, Berlin, Germany, 2011.
[18] R.R. Yager, K.J. Engemann, D.P. Filev, On the concept of immediate probabilities, Int. J. Intell. Syst. 10 (1995) 373–397.
[19] R.R. Yager, Including decision attitude in probabilistic decision making, Internat. J. Approx. Reason. 21 (1999) 1–21.

[20] J.M. Merigó, R.R. Yager, Aggregation operators with moving averages, Soft Comput. 23 (21) (2019) 10601–10615.
[21] L.S. Anido, Performance evaluation procedure based on fuzzy mathematics and OWA operators, Lect. Notes Networks Syst. 388 (2022) 100–120.
[22] A. Stupňanová, L.S. Jin, Biowa operators, Commun. Comput. Inf. Sci. 1238 (2020) 419–425.
[23] R. Mesiar, A. Stupňanová, L. Jin, Bipolar ordered weighted averages: BiOWA operators, Fuzzy Sets and Systems 433 (2022) 108–121.
[24] A. Torres-Martínez, A.M. Gil-Lafuente, J.M. Merigó, The OWA operator with boxplot method in time series, Adv. Intell. Syst. Comput. 894 (2020) 13–21.
[25] J. Medina, R.R. Yager, OWA operators with functional weights, Fuzzy Sets and Systems 414 (2021) 38–56.
[26] F. Blanco-Mesa, E. León-Castro, J.M. Merigó, Covariances with OWA operators and Bonferroni means, Soft Comput. 24 (19) (2020) 14999–15014.
[27] Y.-M. Wang, K.-S. Chin, The use of OWA operator weights for cross-efficiency aggregation, Omega 39 (5) (2011) 493–503.
[28] S. Dai, J. Zheng, On the orders of OWA operators, J. Intell. Fuzzy Syst. 44 (2) (2023) 1613–1623.
[29] M. Cardin, S. Giove, SDOWA: A new OWA operator for decision making, Smart Innov. Syst. Technol. 184 (2021) 305–315.
[30] L. Jin, R.R. Yager, J. Špirková, R. Mesiar, D. Paternain, H. Bustince, OWA aggregation with dual preferences for basic uncertain information, J. Intell. Fuzzy Syst. 40 (3) (2021) 4535–4544.
[31] L.S. Jin, R.R. Yager, Z.-S. Chen, J. Špirková, D. Paternain, R. Mesiar, H. Bustince, Some bipolar-preferences-involved aggregation methods for a sequence of OWA weight vectors, Soft Comput. 25 (2) (2021) 895–902.
[32] R. Verma, A. Mittal, Multiple attribute group decision-making based on Bonferroni mean under Fermatean fuzzy environment, Granul. Comput. 8 (1) (2023) 111–129.
[33] X. Mi, Y. Tian, B. Kang, A modified soft-likelihood function based on POWA operator, Int. J. Intell. Syst. 35 (5) (2020) 869–890.
[34] L. Jin, R. Mesiar, R. Yager, On scatters of probability distributions and OWA weights collections, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems 27 (5) (2019) 773–788.
[35] X. Feng, Q. Zhang, L. Jin, Aggregation of pragmatic operators to support probabilistic linguistic multi-criteria group decision-making problems, Soft Comput. 24 (10) (2020) 7735–7755.
[36] J. Chen, Y. Du, X.-T. Yang, Efficiency evaluation method for coordinated areas based on OWA operator, Xi Tong Gong Cheng Yu Dian Zi Ji Shu/Systems Eng. Electron. 35 (3) (2013) 527–531.
[37] W. Hussain, J.M. Merigó, H. Gao, A.M. Alkalbani, A. Mardani, Integrated AHP-IOWA, POWA framework for ideal location selection, IEEE Trans. Serv. Comput. 16 (1) (2023) 370–382.
[38] J.M. Merigó, Y. Xu, S. Zeng, Group decision making with distance measures and OWA operators, Knowl.-Based Syst. 40 (2013) 81–87.
[39] Y. Li, R. Zhu, X. Mi, B. Kang, An intelligent quality-based fusion method for image defogging using interval-valued intuitionistic fuzzy POWA operator, Eng. Appl. Artif. Intell. 109 (2022) 104618.
[40] L.F. Espinoza-Audelo, M. Olazabal-Lugo, F. Blanco-Mesa, J.M. Merigó, A.M. Gil-Lafuente, Bonferroni probabilistic ordered weighted averaging (BPOWA) operator and its application to decision-making problems, Mathematics 8 (8) (2020) 1350.
[41] R. Zhu, Y. Li, R. Cheng, B. Kang, An improved model in fusing multi-source information of remote sensing image based on IVIF-powa, Comput. Appl. Math. 41 (1) (2022) 16.
[42] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 353–388.
[43] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.
[44] F. Smarandache, Neutrosophic set, a generalization of the intuitionistic fuzzy sets, Int. J. Pure Appl. Math. 24 (2005) 287–297.
[45] R.R. Yager, Pythagorean membership grades in multi-criteria decision making, IEEE Trans. Fuzzy Syst. 22 (4) (2013) 958–965.
[46] R.R. Yager, Generalized orthopair fuzzy sets, IEEE Trans. Fuzzy Syst. 25 (5) (2017) 1222–1230.