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Multi-Criteria Decision-Making Method Based on Weighted and Geometric Aggregate Operators of Linguistic Fuzzy-Valued Hypersoft Set with Application

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Abstract

This study aims to fill the research gap and linguistic uncertainty associated with crypto mining. Recent environmental concerns such as increased carbon emissions have been raised due to cryptocurrency mining, and demand for energy production increases as cryptocurrency mining increases. The mathematical model can be used to evaluate and mitigate the environmental concerns associated with cryptocurrency mining activities. Thus, this paper presents the Linguistic Fuzzy-valued Hypersoft Set (LFHSs), its aggregate operators, and the unique mathematical technique for Multi-Criteria Decision-Making (MCDM) method based on proposed aggregation to evaluate the uncertainty and vagueness associated with crypto mining and its impacts on the environment. This analyzes the comprehensive understanding of environmental problems such as increased carbon emissions, higher demand for electricity associated with crypto mining, and the inherent uncertainty associated with data processing in the Decision-Making (DM) process using LFHSs. Linguistic terms are used to collect and express complicated, ambiguous, and amorphous information. The modeling and construction of the problem involve the identification of energy production as alternatives, and relevant environmental factors are in the form of further bifurcation as attributes. Subsequently, the fuzzy values are assigned to each linguistic attribute by considering individual impact. This allows for a more accurate representation of the many factors of the environmental effect of crypto mining. The proposed MCDM method, based on weighted and geometric aggregate operators, solves the case study. The results evaluated are then compared with the existing studies, and it shows that the study has the potential to assist researchers, industry stakeholders, and policymakers in creating strategies that will effectively address the environmental problems raised by crypto mining while minimizing energy use and promoting sustainable practices in changing digital environment.

Keywords: Renewable energy, CO_2 emission, Hypersoft set theory, Bitcoin mining, Linguistic quantifiers, Aggregate operators, Decision-making, Machine learning.

1 | Introduction

Bitcoin (BTC) mining creates significant environmental and energy challenges. It consumes an enormous

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amount of electricity often as much as an entire small country because it relies on complex mathematical calculations to verify transactions and secure the network. This energy-intensive process relies exclusively on fossil fuels like coal and natural gas, which greatly increases carbon emissions and exacerbates climate change. The demand for energy puts pressure on the local energy production units, and some of the customers may have to face power outages. The miners normally move to places where electricity is cheap and environmental policies are not strictly observed, this causes environmental concerns like, water pollution, air pollution, and habitat destruction. Thus, there is an urgent need to study the energy and environmental concerns related to BTC mining to lessen its negative effects on our planet and to find out the eco-friendly ways for energy production and site selections. The process of BTC mining involves the solution of complex puzzles with the help of many connected computers known as hash rate to verify the transaction on the blockchain network. So, in reward, the miner will be given some percentage of the transaction in exchange. Some nations have drawn a lot of interest from BTC miners hoping to take advantage of its advantageous conditions due to their abundance of clean energy resources and progressive regulations. However, problem have been raised regarding the environmental effects and energy usage of BTC mining facilities. The countries which are at the forefront of environment friendly energy practices, working to lower carbon emissions and increase its reliance on renewable energy sources. The country, which has a long history of using hydroelectric power, has taken use of its plentiful water resources to produce a sizable amount of its electricity [1]. According to a life-cycle analysis, depending on the type (solar, wind, hydro, geothermal, tidal, wave, biomass), renewable energy produces between 11 and 740 gCO_2 for every kWh produced. Switzerland is committed to eliminating nuclear power, promoting energy efficiency, and reducing CO_2 emissions across all sectors [2]. Mining process of BTC is associated with energy and blockchain networks. The amount of energy the network uses will depend on the hash rate of the entire BTC network. A blockchain network's hash rate rises as more computers join it and participate in processing hashes (guesses) on the network. Since there is a lower chance of an attack, a Proof of Work (PoW) blockchain network with a high hash rate is safer and more robust. Energy use will decrease with a lower network hash rate. The network will need more energy to mine each new block when the hash rate is higher. A BTC (coin) is produced using 2.7 quadrillion computed hashes. The production of one BTC can consume 663.68kWh of energy and it produces 370.17 $kgCO_2$ [3], [4]. Through the potential for financial gain, job development, infrastructure investment, and technical advancement, BTC mining may improve the economy. BTCs may be earned as rewards, and miners can also invest in cutting-edge gear and data centers, stabilize the energy markets, improve technology, and promote financial inclusion [5].

I. Determine which renewable energy source is feasible to use in BTC mining operations.

II. Which renewable energy source should be used if it complies with financial, environmental, and energy policies?

By effectively utilizing renewable energy sources for mining, countries may further set its status as a pioneer in environment friendly cryptocurrency activities by attracting more environmentally aware investors and businesses to the country's crypto ecosystem. In essence, the Swiss example of combining BTC mining with renewable energy provides a convincing example of how to reduce the negative environmental effects of cryptocurrency operations while promoting innovation in the blockchain industry. This move toward sustainability helps the country's standing as an eco-friendly center for blockchain technology.

Zadeh's influential 1974 papers [6] introduced the concept of a linguistic variable, and membership function. The linguistic quantifiers are then used in practical scenarios. Researchers explored relevant areas to apply these concepts to real-life problems. Nowadays many electronic gadgets are using this concept especially in MCDM to facilitate Decision-Making (DM) to evaluate the optimal choice that align with target goals [7]. To address DM and linguistic preferences [8] focused on linguistic DM techniques by introducing various approaches and methods to understand DM algorithms. The semantic model of computing with flexible linguistic expressions for DM, was proposed by [10], [11].

Issues such as doubt, uncertainty, vagueness, ambiguity, and indeterminacy were difficult to resolve. Researchers proposed several set theories to transform linguistic concepts into mathematical ones. The theories include the fuzzy set theory by Zadeh [12] to address uncertainty in information and control systems, ambiguity was addressed by Attanasov [13], who introduced the concept of intuitionistic set, where each alternative has a membership degree and a non-membership degree, with their sum not exceeding 1. To address indeterminacy [14–16] introduced the concept of indeterminacy value and known as neutrosophic set [17], [18]. The *Table 1*, shows the development made in linguistic set structure and its extensions with aggregation and MCDM application.

Table 1. The literature review of some fuzzy sets and their extensions.

Researcher	Set Structure	Aggregations	MCDM Technique/Application
Joyce [19]	Fuzzy set & linguistics	√	√
Sharma et al. [20]	Hesitant fuzzy linguistic term set	√	√
Garg and Kumar [21]	Linguistic intuitionistic fuzzy set	√	√
Li et al. [24]	Linguistic neutrosophic set	√	√

DM problems mean to deal with attributes and alternatives. The mathematical notation given by [25], the paper introduces basic ideas of soft set theory which forms a foundation for studying soft sets. Soft set theory and its applications in different areas including DM are then discussed; later Maji [26] applies soft sets to a real DM problem. The study illustrated the logical value of using soft sets as an effective tool for handling uncertainty in practical decision situations, leading other researchers to propose new operations or extensions: such as fuzzy soft set [27], intuitionistic soft set [28], neutrosophic soft set [29] and other hybrid structures [30] that can further enhance capabilities in both DM and data analysis. An innovative paper by [31] proposed a comprehensive framework for group DM that incorporates uncertainty. It introduces uncertain linguistic fuzzy soft sets and their role in group DM. Another study [32] puts forth the fuzzy-valued linguistic soft set theory for multi-attribute DM offering a new perspective on how to address such uncertainties and enhance decision support. In a different approach, the authors in [33] introduce a belief-based probabilistic linguistic term sets method for multi-attribute decision making. Similarly, Liu et al. [34] suggests a methodology for selection of medical waste treatment stations using linguistic q-rung orthopair fuzzy numbers. The introduction of Hypersoft Set (HSS) theory was proposed to address this issue HSS [35] is the generalization of soft set theory and, it consider further divided attributes or attributes bifurcation. The development of HSS theory has demonstrated to be useful in handling both MCDM and Multi-Attributive Decision-Making (MADM) problems because this new model can adapt itself according to Decision Maker's (DM) requirements. The HSS structure has seen various extensions such as fuzzy HSS [36–38] an intuitionistic HSS [39], and a neutrosophic HSS [40]. These contributions cover a wide range of research on linguistic variables, fuzzy sets, soft sets, and HSSs. They explore various methodologies for tackling DM challenges under uncertainty and for analyzing complex data. To make decisions, involving further subdivided attributes, one must rely on a combination of quantitative and qualitative aspects. However, current approaches to linguistic knowledge management cannot adequately handle that, since there is no standard methodology for allocation of numerical values to language.

By modeling elements like energy effectiveness, environmental impact, and the integration of renewable energy sources as linguistic variables, terms, rules, and fuzzy sets with membership degrees and human language as qualitative descriptions, fuzzy HSS theory can be used to address the energy and environmental difficulties associated with BTC mining. This method enables a detailed evaluation of mining operations considering data uncertainty and imprecision.

1.1 | The Research Question

Which renewable energy source is the most viable for BTC mining in terms of environmental, social, and economic concerns? Why the concept of linguistic fuzzy HSSs is utilized?

The present paper present the proposed Linguistic Fuzzy-valued Hypersoft Set (LFHSs) concept to address the issues. Also, this paper fills the gap by providing the aggregation and MCDM methods for LFHSs to solve the decision problems. Specifically; introducing fuzzy set theory in the assessment process involving linguistic ambiguity permits for a better understanding of the interrelations between crypto mining, energy consumption, and environmental impacts. The below subsections present the research gap, contribution, significance, and validity of this study.

1.2 | Research Gap

Despite the significant developments observed in the existing DM models, they still seem to feature the challenging integration of linguistic knowledge since language is generally the most abstract and context-dependent. This is mainly due to the general tendency of these models to correlate linguistic and mathematical factors randomly; as a result, the acquired mathematical evaluations are inaccurate as they do not characterize linguistic nuances. Thus, the existing approaches are not liable and cannot be comprehensively applied to all possible conditions. Developing a reliable and accurate methodologies system that will effectively convert linguistic expressions to numerical values is necessary. The research does not reveal any acceptable tool capable of solving this issue; the current literature does not provide viable and valid proven approaches to the most applied and studied areas like natural language processing, sentiment analysis, and artificial intelligence. This gap proves the need to generalize and develop DM tools in language-dependent uncertainty problems.

1.3 | Contribution and Significance

The development of the proposed paper significantly contributes to the scope of DM literature since it overcomes the identified difficulties in processing linguistic knowledge. The uniqueness of the LFHS model lies in its innovative approach to handling the abstraction and context dependency of linguistic expressions. The LFHSs define the aggregate operators and operational laws that allow generalizing linguistic quantifiers and converting them into numerical values; hence, the novel model enhances the accuracy and reliability of DM. The practical application of the aggregate operators and operational laws resolves the common issues with the comprehension and use of linguistic knowledge perceivable by applying DM processes based on natural language processing, sentiment analysis, and the implementation of artificial intelligence, among others.

1.4 | Scientific Validity

This paper is scientifically rigorous as it comprehensively addresses complex language problems in DM by introducing the new model of LFHSs. The definition of basic operations and operational laws ensures the integrity and the possibility of replication of the framework as examples and properties support the model, hence proving a robust and more reliable way of transforming linguistic quantifiers into numbers, improving the accuracy and reliability of DM. This method can be extensively applied across multiple fields and disciplines, especially in areas such as engineering and research scholarly studies, to enhance performance and bolster operational efficiencies of complex systems for better functioning, future technologies, and automation. Since fuzzy sets allow to cater to uncertainty regarding assumptions made about the five renewable energy options, as well as concerning the parameters of the study, such as carbon footprints, BTC hash rate, economic gains/losses, legislative uncertainty, and expert knowledge is implemented in modeling through a fuzzy inference system and a decision support system by appropriately weighing and aggregating the fuzzy inputs, the decision about the optimal renewable energy source is made. This entire approach is applied to provide insights into policymaking concerned with ensuring sustainable and responsible energy consumption for cryptocurrency-based activities. The graphical abstract of the paper is presented in *Fig. 1*.

Fig. 1. Graphical layout of the proposed paper.

1.5 | Organization of This Study

Section 2 presents preliminary definitions. Section 3 involves basic definitions, notions, properties, aggregate operators' operational laws, and examples of LFHSs. Section 4 covers the MCDM algorithm and numerical illustration. Section 5 includes discussions on results, comparison, summary, managerial implications, ethical consideration, and policy recommendations are presented. Section 6 concludes the proposed study with future directions.

2 | Preliminaries

This subsection presents the key definitions to understand the rest of the paper.

2.1 | Linguistic Quantifiers

The linguistic quantifiers were introduced by Zadeh [6-8], also known as absolute quantifiers. Let $\mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}$ where $t = 2n + 1 :\ n \geq 1$ and $n \in \mathbb{R}^+$, be a finite, strictly increasing set.

2.2 | Hypersoft Set

Let, $a^1, a^2, a^3, \dots, a^t$ for $t \geq 1$ be t distinct parameters whose corresponding parametric values are, respectively, the sets $\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3, \dots, \mathcal{L}^t$ with $\mathcal{L}^i \cap \mathcal{L}^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, t\}$ [35].

Then the pair $(\mathcal{F}, \mathbb{L})$ where $\mathbb{L} = \{\mathcal{L}^1 \times \mathcal{L}^2 \times \mathcal{L}^3 \times \dots \times \mathcal{L}^t$: t is finite and natural no.$\}$ is known as HSS over $\mho$ with mapping $\mathcal{F} :\ \mathbb{L} = \mathcal{L}^1 \times \mathcal{L}^2 \times \mathcal{L}^3 \times \dots \times \mathcal{L}^t \longrightarrow \mathrm{P}(\mho)$.

2.3 | Linguistic Hypersoft Set

Let, $\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t$ for $t \geq 1$ be t distinct parameters whose corresponding parametric values are, respectively, the sets $\Upsilon^1, \Upsilon^2, \Upsilon^3, \dots, \Upsilon^t$ with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, t\}$ [41].

Then the pair (Γ, Λ) where $\Lambda = \{\Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t$: t is finite and real valued$\}$ is known as HSS over Ω with mapping.

$$\Gamma : \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t \longrightarrow \mathrm{P}(\Omega).$$

Then, the linguistic HSS will be

$\Gamma(\{M(\Omega)(i)\}) : M \subseteq \Lambda \ \& \ i \in K = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}$ where $t = 2n + 1 : n \geq 1, n \in \mathbb{R}^+\}$.

3 | Linguistic Fuzzy Hypersoft Set

3.1 | Basic Definition and Operations

This subsection presents the definition of LFHSs, set structural properties, operational laws, and aggregate operators.

Definition 1 (Linguistic Fuzzy Hypersoft Set (LFHSs)). Let, $\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t$ for $t \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the sets $\Upsilon^1, \Upsilon^2, \Upsilon^3, \dots, \Upsilon^t$ with $\Upsilon^m \cap \Upsilon^n = \emptyset$, for $m \neq n$, and m, n ∈ {1, 2, …, t}. Then the pair (Γ, Λ) where $\Lambda = \{\Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t\}$ where t is finite and real valued} is known as HSS over Ω with mapping.

$$\Gamma : \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t \longrightarrow P(\Omega).$$

Then, the linguistic fuzzy HSS will be

$$\Gamma(\alpha(k)) = \{M(\alpha(k)) \mid k \in [0,1]\}.$$

Table 2 shows the linguistic quantifiers; if the linguistic term is between any two terms, then it can further be divided into decimals. The Python code to generate these membership values based on the decision-makers' choice (by selecting the values of n, where n is the number of linguistic quantifiers he made) is presented in Appendix A.

Table 2. Linguistic to fuzzy quantifiers.

Linguistic to Fuzzy Quantifiers										
Low					**Medium**	**High**				
None	V.V.V Low	V. V-Low	Very Low	Low	Medium	High	Very High	Very Very-High	V.V.V High	Perfect
0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1

Example 1. A Tech company, XYZ, is about to start a major project and must assemble a team of its best-reputed employees. To achieve this, its HR department uses a structured approach to measuring socio-cognitive factors, such as trends in diversity and experiences. Description attributes, Nationality, Gender, and Color are used because they are examples of the key socio-cognitive variables. For these variables, they will use the Language Fuzzy Hypersoft Set model to convert verbal ones to numerical values, allowing for exact evaluation and, eventually, selections. They would use this model because it seems that this fiercely competitive project will need both an experienced and a diverse team, and LFHSs are well-suited for DM on issues with complex, multifaceted, and context-dependent information.

Let $\Omega = \{\sigma^1, \sigma^2, \sigma^3, \sigma^4\}$ and set $M((\alpha(k)) = \{\sigma^2, \sigma^3\} \subset \Omega$.

Consider the parameters be: $\alpha^1 =$ nationality, $\alpha^2 =$gender, $\alpha^3 =$color, and their respective parametric values are:

Nationality = $\Upsilon^1 = \{\text{Pakistani}, \text{Chinese}, \text{American}\}$.

Gender = $\Upsilon^2 = \{\text{Male}, \text{Female}\}$.

Color = $\Upsilon^3 = \{\text{Pink}, \text{Black}, \text{Orange}\}$.

Then the function $\Gamma : \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \longrightarrow P(\Omega)$ and assume the HSS,

$\Gamma(\{\text{Pakistani}, \text{Male}, \text{Orange}\}) = \{\sigma^2, \sigma^3\} = M(\alpha(k))$.

The LFHSs, $\Gamma(\sigma^K) = \{M(\alpha(k)) \mid k \in [0,1]\}$.

$\Gamma(\{\text{Pakistani}, \text{Male}, \text{Orange}\}) = \{\sigma^2, \sigma^3\} = \{\sigma^2(0.8), \sigma^3(0.1)\} =$ LSimilarly,

$\Gamma_1(\{\text{Pakistani, Male, Pink}\}) = \{\sigma^2(0.7), \sigma^3(0.4)\} = L_1$.

$\Gamma_2(\{\text{Chinese, Female, Pink}\}) = \{\sigma^1(0.5), \sigma^4(0.3)\} = L_2$.

$\Gamma_3(\{\text{American, male, black}\}) = \{\sigma^1(0.8), \sigma^3(0.1)\} = L_3$.

Definition 2 (Subset of LFHSs). Let $(\Gamma_1, \Lambda_1) = L_1$ be an LFHS, then the subset L_s can be defined as. $\Gamma\big((\alpha(k))\big) = \{M(\alpha(k)) \mid k \in [0,1]\}$.

I. $L_s \subseteq L_1$.

II. For all $\sigma \in L_s, \Gamma_2(\sigma) \subseteq \Gamma_1(\sigma)$.

Example 2. Recall *Example 1*. The function Γ_2 :Λ_s=Υ^1×Υ^2 ⟶P(Ω) and assume the HSS, Γ_2 ({Pakistani,Male})={o^2 (0.5)}=L_s. Where Λ_s⊆Λ and L_s⊆L_1.

Definition 3 (Empty Linguistic Hypersoft Set (Empty LFHSs)). Consider a mapping,

$$\Gamma_1 : \Lambda_E = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^n \longrightarrow P(\Omega).$$

such that each Υ^i $(i \leq n)$ is empty. $\Gamma_1(\{L_E(\Omega)\})$

I. $(\Gamma_1, \Lambda_E)^{\phi} = L_E$ if For all $\Gamma_1(\sigma^k) = \phi$: For all $\sigma^k \in \Lambda_E$. Then, it is said to be an Empty LFHS.

Example 3. Recall *Example 1*. The function $\Gamma_1 : \Lambda_E = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \longrightarrow P(\Omega)$ and assume LFHSs, $\Gamma_1(\emptyset) = \emptyset = L_E$. Where $\Lambda_E \subseteq \Lambda$.

Definition 4. Consider two $(\Gamma_1, \Lambda_1) = L_1$ and $(\Gamma_2, \Lambda_2) = L_2$ LFHSs, then the AND operation can be defined as:

I. $L_1 \wedge L_2 = (\Gamma_3, \Lambda_3) = L_3$; max of (σ^k).

II. $(\sigma_i, \sigma_j) = \sigma_k = L_3$ where $\sigma_i \in \sigma_1$ and $\sigma_j \in L_2$ with $i \neq j$.

III. $\Gamma_3(\sigma_i, \sigma_j) = \Gamma_1(\sigma_i) \cup \Gamma_2(\sigma_j)$.

Definition 5. Consider two $(\Gamma_1, \Lambda_1) = L_1$ and $(\Gamma_2, \Lambda_2) = L_2$ LFHSs, then the OR operation can be defined as:

I. $L_1 \vee L_2 = (\Gamma_3, \Lambda_3) = L_3$.

II. $(\sigma_i, \sigma_j) = \sigma_k = L_3$ where $\sigma_i \in L_1$ and $\sigma_j \in L_2$ with $i \neq j$.

III. $\Gamma_3(\sigma_i, \sigma_j) = \Gamma_1(\sigma_i) \cap \Gamma_2(\sigma_j)$.

Definition 6. Consider (Γ, Λ) LFHSs, then the NOT operation can be defined as:

I. $\sim L = \sim(\Gamma, \Lambda) = \sim \Upsilon^1 \times \sim \Upsilon^2 \times \sim \Upsilon^3 \times \ldots \times \sim \Upsilon^n$.

II. $\sim L = \sim \prod \sigma_i : i = 1,2,3, \ldots, n$.

III. $|\sim L| = n - \text{Tuple}$.

Definition 7. Consider (Γ, Λ) LFHSs, then the complement can be defined as:

I. $(\Gamma, \Lambda)^{\sim} = (\Gamma^{\sim}, \sim L)$; $\Gamma^{\sim} : \sim L \longrightarrow P(\Omega)$.

II. $\Gamma^{\sim}(\sim \sigma) = \Omega \setminus \Gamma(\sigma)$; For all $\sigma \in L$.

Proposition 1. Let $(\Gamma, \Lambda) = L$, $(\Gamma_1, \Lambda_1) = L_1$, $(\Gamma_2, \Lambda_2) = L_2$ and $(\Gamma_3, \Lambda_3) = L_3$ be LFHSs, then following holds:

I. $(\Gamma_1, \Lambda_1) \subseteq (\Gamma_1, \Lambda_1)$.

II. $(\Gamma_1, \Lambda_E)^{\phi} \subseteq (\Gamma_1, \Lambda_1)$.

III. $\sim(\sim L) = L$.

IV. $\sim(\Gamma_1, \Lambda_E)^{\phi} = \Omega$.

V. If $(\Gamma_1, \Lambda_1) \subseteq (\Gamma_2, \Lambda_2)$ and $(\Gamma_2, \Lambda_2) \subseteq (\Gamma_2, \Lambda_2)$

Then $(\Gamma_1, \Lambda_1) = (\Gamma_2, \Lambda_2)$.

Iff each σ^k of $(\Gamma_1, \Lambda_1) = \sigma^k$ of (Γ_2, Λ_2).

Subject to the conditions for each σ^k of $(\Gamma_1, \Lambda_1) = \sigma^k$ of (Γ_2, Λ_2).

VI. If $(\Gamma_1, \Lambda_1) \subseteq (\Gamma_2, \Lambda_2)$ and $(\Gamma_2, \Lambda_2) \subseteq (\Gamma_3, \Lambda_3)$ then $(\Gamma_1, \Lambda_1) \subseteq (\Gamma_3, \Lambda_3)$.

Subject to the conditions, i.e., for each σ^k of $(\Gamma_1, \Lambda_1) = \sigma^k$ of $(\Gamma_2, \Lambda_2) = \sigma^k$ of (Γ_3, Λ_3).

Proof: recall L, L_1, L_2 and L_3 from *Example 1*.

I. $\Gamma_1(\{\text{Pakistani, Male, Pink}\}) = \{\sigma^2, \sigma^3\} = \{\sigma^2(1), \sigma^3(0.4)\} = L_1 \quad \because \sigma^2(1) \in L_1$ also $\sigma^3(0.4) \in L_1 \Rightarrow \sigma^2, \sigma^3 \in L_1$.

Thus $(\Gamma_1, \Lambda_1) \subseteq L_1 = (\Gamma_1, \Lambda_1)$.

II. Consider $L_1 = (\Gamma_1, \Lambda_1)$.

$\because \phi \in L_1 \Rightarrow (\Gamma_1, \Lambda_E)^\phi \in L_1$.

Thus $(\Gamma_1, \Lambda_E)^\phi \subseteq L_1 = (\Gamma_1, \Lambda_1)\ (\Gamma_1, \Lambda_E)^\phi \subseteq (\Gamma_1, \Lambda_1)$.

III. Consider $L = \{\sigma^2(1), \sigma^3(0)\}$, apply *Definition* 6, we get, $(\sim L) = \{\sigma^1(0), \sigma^4(1)\}$ again, apply *Definition* 6, we get; $\sim(\sim L) = \{\sigma^2(1), \sigma^3(0)\} = L$.

IV. Consider $(\Gamma_1, \Lambda_E)^\phi = \phi \Rightarrow \phi \in L_E$ taking complement, $\sim(L_E) = \Omega \setminus \Gamma_1(\sigma^k) = \phi; \Rightarrow \sim(L_E) = \Omega$,

hence $\sim(\Gamma_1, \Lambda_E)^\phi = \Omega$.

V. Consider, $(\Gamma_1, \Lambda_1) = \{\sigma^1(0.6), \sigma^3(0.4)\}$, $(\Gamma_2, \Lambda_2) = \{\sigma^1(0.6), \sigma^3(0.4)\}$,

$(\Gamma_1, \Lambda_1) \subseteq (\Gamma_2, \Lambda_2)$ *also* $(\Gamma_2, \Lambda_2) \subseteq (\Gamma_1, \Lambda_1)$,

Thus $(\Gamma_2, \Lambda_2) = (\Gamma_1, \Lambda_1)$.

Counter Example 4. Consider, $(\Gamma_1, \Lambda_1) = \{\sigma^2(0.6), \sigma^3(0.3)\}$ and $(\Gamma_2, \Lambda_2) = \{\sigma^2(1), \sigma^3(0.4)\}$.

$(\Gamma_1, \Lambda_1) \subseteq (\Gamma_2, \Lambda_2)$ But $(\Gamma_2, \Lambda_2) \nsubseteq (\Gamma_1, \Lambda_1)$ since the linguistic variable of $(\Gamma_2, \Lambda_2) >$ linguistic variable of (Γ_1, Λ_1).

$(\Gamma_2, \Lambda_2) \neq (\Gamma_1, \Lambda_1)$.

VI. Same as 5.

3.2 | Operational Laws on LFHSs

In this subsection, we propose some operational laws and theorems on LFHSs.

Let $(\Gamma_1, \Lambda_1) = L_1$ and $(\Gamma_2, \Lambda_2) = L_2$ be two LFHSs and $\mu \geq 0$, where $\Lambda_1 = \{\Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^n$: n is finite and real valued$\}$ over Ω with mapping $\Gamma : \Lambda_1 = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^n \longrightarrow P(\Omega)$ and $\Lambda_2 = \{\Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^m$: m is finite and real valued$\}$ over Ω with mapping

$$\Gamma_2 : \Lambda_2 = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^m \longrightarrow P(\Omega). \tag{1}$$

such that

$$\Gamma(\alpha(k)) = \{M(\alpha(k)) \mid k \in [0,1]\}. \tag{1}$$

Note: in coming definitions, the operational laws defined are subject to some conditions.

Definition 8. Unin of LFHSs.

Case 1: $L_1 \cup L_2 = \{\prod \alpha^i(K^i) \times \prod \alpha^j(K^j) \in \prod_{i=1}^{n} \Upsilon^i \times \prod_{j=1}^{n} \Upsilon^j\}$, where, $\alpha^i(k^i) \in \prod_{i=1}^{n} \Upsilon^i$, and $\alpha^j(k^j) \in \prod_{j=1}^{n} \Upsilon^j$ should be distinct with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \ldots, t\}$ and $k \in [0,1]$.

Case 2: $L_1 \cup L_2 = \{\alpha^i(k^i) \in \prod_{i=1}^{n} \Upsilon^i \times \prod_{j=1}^{n} \Upsilon^j\}$,

with $i = j$, and fuzzy variable k^i of σ^i should be the same.

Example 5. Consider,

Case 1: $\Gamma_1(\{\text{Pakistani, male, black}\}) = \{\sigma^2(1), \sigma^3(0.4)\} = L_1$.

$\Gamma_2(\{\text{American, Female, Pink}\}) = \{\sigma^1(0.6), \sigma^4(0.4)\} = L_2$.

$\because\ \Upsilon^i \cap \Upsilon^j = \emptyset$.

$L_1 \cup L_2 = \{\sigma^2(1), \sigma^3(0.4), \sigma^1(0.6), \sigma^4(0.4)\}$.

Case 2: $\Gamma_1(\{\text{Pakistani, male, black}\}) = \{\sigma^2(1), \sigma^3(0.4)\} = L_1$.

$\Gamma_2(\{\text{Pakistani, female, pink}\}) = \{\sigma^2(1), \sigma^3(0.4)\} = L_2$.

$\because\ \Upsilon^i \cap \Upsilon^j \neq \emptyset$ with $i = j$.

$L_1 \cup L_2 = \{\sigma^2(1), \sigma^3(0.4)\}$.

Case 3: (Counter example)\Restriction: $\Gamma_1(\{\text{Pakistani, male, black}\}) = \{\sigma^2(0.6), \sigma^3(0.3)\} = L_1$.

$\Gamma_2(\{\text{Pakistani, female, pink}\}) = \{\sigma^2(1), \sigma^3(0.4)\} = L_2 \because\ \Upsilon^i \cap \Upsilon^j \neq \emptyset$ with $i = j$

Example 6.

$L_1 = \{\sigma^2(0.6), \sigma^3(0.3)\}$, $L_2 = \{\sigma^2(1), \sigma^3(0.4)\}$.

As, $\sigma^2(0.6) < \sigma^2(1)$, and $\sigma^3(0.3) < \sigma^3(0.4)$.

Then $L_1 \cup L_2 = \{\sigma^2(1), \sigma^3(0.4)\}$.

3.2.4. | Intersection of LFHSs

Let $(\Gamma_1, \Lambda_1) = L_1$ and $(\Gamma_2, \Lambda_2) = L_2$ be two LFHSs and $\mu \geq 0$; then the intersection can be defined as;

$$L_1 \cap L_2 = \left\{ \prod \alpha^i(k^i) \times \prod \alpha^j(k^j) \in \prod_{i=1}^{n} \Upsilon^i \times \prod_{j=1}^{n} \Upsilon^j \right\} = \emptyset, \tag{1}$$

where, $\alpha^i(k^i) \in \prod_{i=1}^{n} \Upsilon^i$, and $\alpha^j(k^j) \in \prod_{j=1}^{n} \Upsilon^j$ should be distinct with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i = j$, and $i, j \in \{1, 2, \ldots, t\}$ and $k \in [0,1]$.

Case 2: $L_1 \cap L_2 = \{\alpha^i(k^i) \in \prod_{i=1}^{n} \Upsilon^i \times \prod_{j=1}^{n} \Upsilon^j\}$,

with $i = j$, and fuzzy value k^i of σ^i Then $L_1 \cap L_2 = L_1$ or L_2.

Example 7. Consider,

Case 1: $\Gamma_1(\{\text{Pakistani, male, black}\}) = \{\sigma^2(1), \sigma^3(0.4)\} = L_1$,

$\Gamma_2(\{\text{American, Female, Pink}\}) = \{\sigma^1(0.6), \sigma^4(0.4)\} = L_2$, $\because\ \Upsilon^i \cap \Upsilon^j = \emptyset$,

$L_1 \cap L_2 = \{\emptyset\}$.

Case 2: $\Gamma_1(\{\text{Pakistani, male, black}\}) = \{\sigma^2(1), \sigma^3(0.4)\} = L_1$,

$\Gamma_2(\{\text{Pakistani, female, pink}\}) = \{\sigma^2(1), \sigma^3(0.4)\} = L_2$, $\because\ \Upsilon^i \cap \Upsilon^j \neq \emptyset$ with $i = j$,

$L_1 \cap L_2 = \{\sigma^2(1), \sigma^3(0.4)\}$.

Case 3: (Counter example) \ Restriction $\Gamma_1(\{\text{Pakistani, male, black}\}) = \{\sigma^2(0.6), \sigma^3(0.3)\} = L_1$,

$\Gamma_2(\{\text{Pakistani, female, pink}\}) = \{\sigma^2(1), \sigma^3(0.4)\} = L_2, \because \Upsilon^i \cap \Upsilon^j \neq \emptyset$ with $i = j$.

Example 8. Consider two LFHSs,

$L_1 = \{\sigma^2(0.6), \sigma^3(0.3)\}$ and $L_2 = \{\sigma^2(1), \sigma^3(0.4)\}$.

Such that,

$$\sigma^2(0.6) < \sigma^2(1), \text{and } \sigma^3(0.3) < \sigma^3(0.4). \quad (1)$$

Then $L_1 \cap L_2 = \emptyset$.

Theorem 1. If L_1, L_2 and L_3 be three LFHSs, then the following holds:

I. $L_1 \cup L_1 = L_1$.

II. $L_1 \cup \emptyset = L_1$.

III. $L_1 \cap L_1 = L_1$.

IV. $L_1 \cap \emptyset = \emptyset$.

V. $L_1 \cup L_2 = L_2 \cup L_1$.

VI. $L_1 \cap L_2 = L_2 \cap L_1$.

VII. $L_1 \cup (L_2 \cup L_3) = (L_1 \cup L_2) \cup L_3)$.

VIII. If $L_1 \subset L_2$ and $L_2 \subset L_1$ the $L_1 = L_2$.

IX. $\mu(L_1) = \mu L_1$; $\mu \geq 0$.

X. $\mu(L_1 \cup L_2) = \mu(L_2 \cup L_1)$.

The proofs are straightforward.

Theorem 2. If L_1, L_2 be two LFHSs, then the operations are given as follows:

I. $\mu \times L_1 = L_{\mu\times 1}$; μ (Fuzzy variable).

II. $L_1 \oplus L_2 = L_{1\oplus 2}$.

III. $L_1 \otimes L_2 = L_{1\otimes 2}$.

IV. $(L_1)^\mu = L_{1^\mu}$.

Proof:

I. Consider, $\Gamma_1(\{\text{Pakistani, male, black}\}) = \{\sigma^2(1), \sigma^3(0.4)\} = L_1$ and $\mu = 0.4$.

L.H.S

$$\mu \times L_1 = \{ 0.4 \times \sigma^2(1), 0.4 \times \sigma^3(0.4)\} = \{\sigma^2(0.4), \sigma^3(0.16)\}.$$

R.H.S

$$L_{0.4\times 1} = L_{0.4} = \{\sigma^2(0.4), \sigma^3(0.16)\}.$$

Thus, we can conclude that

$$\mu \times L_1 = L_{\mu\times 1}.$$

II. Assume $L_1 = \{\sigma^2(1), \sigma^3(0.4)\}$ and $L_2 = \{\sigma^2(0.5), \sigma^3(0.6)\}$.

L.H.S

$$L_1 \oplus L_2 = \left\{\max\left(\sigma^i(k) \text{ of } L_1, \left(\sigma^i(k) \text{ of } L_2\right)\right)\right\}$$

$$= \{\max(\sigma^2\ (1, 0.5)\ , \sigma^3\ (0.4, 0.6))\}$$

$= \{(\sigma^2\ (1)\ , \sigma^3\ (\ 0.6))\}.$

R.H.S

$L_{1\oplus 2}\ =\ \{(\sigma^2\ (1)\ , \sigma^3\ (\ 0.6))\}.$

Thus, we can conclude that

$L_1 \oplus\ L_2 = L_{1\oplus 2.}$

III. Assume $L_1 = \{\sigma^2(1), \sigma^3(0.4)\}$ and $L_2 = \{\sigma^2(0.5), \sigma^3(0.6)\}$.

L.H.S

$L_1 \otimes\ L_2 =\ \left\{\min\left(\sigma^{i}(k) \text{ of } L_1\ , \left(\sigma^{i}(k) \text{ of } L_2\right)\right)\right\}$

$= \{\min(\sigma^2\ (1, 0.5)\ , \sigma^3\ (0.4, 0.6))\}$

$= \{(\sigma^2\ (0.5)\ , \sigma^3\ (\ 0.4))\}.$

R.H.S

$L_{1\otimes 2}\ =\ \{(\sigma^2\ (0.5)\ , \sigma^3\ (\ 0.4))\}.$

Thus, we can conclude that

$L_1 \otimes\ L_2 = L_{1\otimes 2.}$

IV. Consider $L_1 = \{\sigma^2(1), \sigma^3(0.4)\}$ and $\mu = 0.4$.

L.H.S

$(L_1)^{\mu} =\ \{\sigma^2(1), \sigma^3(0.4)\}^{0.4}$

$= \{\sigma^2(1^{0.4}), \sigma^3(0.4^{0.4})\}$

$= \{\sigma^2(1), \sigma^3(0.6)\}.$

R.H.S

$L_{1^{\mu}} =\ L_{1^{0.4}} = L_1 =\ \{\sigma^2(1), \sigma^3(0.6)\}.$

Thus, we can conclude that

$(L_1)^{\mu} =\ L_{1^{\mu}}.$

3.3 | Some Aggregation Operators

In this subsection, aggregate operators are presented. Aggregate operators are critical in the DM process and should not be underestimated under any conditions. They are responsible for combining and aggregating individual linguistic quantifiers or numeric values that result in a general assessment of all the factors and features presented. Thus, the application of aggregate operators enables decision-makers to analyze the volume of information received efficiently, providing grounds for DM. Aggregate operator has the advantage of being able to manage various criteria simultaneously. Several factors come into play in making decisions, such as cost, quality, reliability, and customer satisfaction. These aggregate operators allow decision-makers to combine these factors in a proportionality manner with respect to their respective significance. This way, it becomes possible to evaluate different choices or alternatives comprehensively.

Definition 9 (Linguistic Fuzzy Hypersoft Weighted Geometric Averaging Operator (LFHSWGAO)). Consider, $\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t$ for $t\ \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the sets $\Upsilon^1, \Upsilon^2, \Upsilon^3, \dots, \Upsilon^t$ with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, t\}$.

Let $\mathfrak{A}$: $\Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t\ \longrightarrow P(\Omega) = \Gamma\big(\alpha(k)\big) = \big\{M\big(\alpha(k)\big)\ \big|\ k \in [0,1]\big\}$.(1)

If $\mathfrak{A}^{\omega}(\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(k))^{(\omega^t)}$.

Such that

$$\mathfrak{A}^{\omega}(\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) =$$

$$\alpha^1{}_i^{\omega^1} \otimes \alpha^2{}_i^{\omega^2} \otimes \alpha^3{}_i^{\omega^3} \otimes \dots \otimes \alpha^t{}_i^{\omega^t} = \sigma_i(k),$$

where $\omega = (\omega^1, \omega^2, \omega^3, \dots, \omega^t)^T$ is the exponential weighting vector of the $\alpha^t(k) \in \{M(\alpha(k))\}$ and $\omega^t \in [0, 1]$ with $\sum_{t=1}^{n} \omega^t = 1$, and $k \in [0,1]$. Then, $\mathfrak{A}$ is called LFHSWGAO.

Example 9. Assume $\omega = (0.4, 0.3, 0.3)^T$ then LFHSWGAO $\{\sigma^2(\text{Pakistani, Male, Orange}),$

$\sigma^3(\text{Pakistani, Male, Orange})\}$

$$= \sigma^2 \begin{pmatrix} \text{Pakistani(low), Male(medium),} \\ \text{Orange(high)} \end{pmatrix}$$

$$\because \mathfrak{A}^{\omega}(\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(k))^{(\omega^t)}$$

$$= \alpha^1{}_i^{\omega^1} \otimes \alpha^2{}_i^{\omega^2} \otimes \alpha^3{}_i^{\omega^3} \otimes \dots \otimes \alpha^t{}_i^{\omega^t} = \sigma_i$$

$$= \{\text{Pakistani}(0.4)^{0.4}, \text{Male}(0.5)^{0.3}, \text{Orange}(0.6)^{0.3}\}$$

$$= \sigma^2\{(0.4)^{0.4} + (0.5)^{0.3} + (0.6)^{0.3}\}$$

$$= \sigma^2(0.256 + 0.125 + 0.216)$$

$$= \sigma^2(0.597) = \sigma^2(\text{High}).$$

Now,

$$= \sigma^3 \begin{pmatrix} \text{Pakistani(none), Male(none),} \\ \text{Orange(none)} \end{pmatrix}$$

$$\because \mathfrak{A}^{\omega}(\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(k))^{(\omega^t)}$$

$$= \alpha^1{}_i^{\omega^1} \otimes \alpha^2{}_i^{\omega^2} \otimes \alpha^3{}_i^{\omega^3} \otimes \dots \otimes \alpha^t{}_i^{\omega^t} = \sigma_i$$

$$= \{\text{Pakistani}(0)^{0.4}, \text{Male}(0)^{0.3}, \text{Orange}(0)^{0.3}\}$$

$$= \sigma^2\{(0)^{0.4} + (0)^{0.3} + (0)^{0.3}\}$$

$$= \sigma^2(0) = \sigma^2(\text{none}).$$

Definition 10 (Linguistic Fuzzy Hypersoft Ordered Weighted Geometric Averaging Operator (LFHSOWGAO)). Consider, $\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t$ for $t \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the sets $\Upsilon^1, \Upsilon^2, \Upsilon^3, \dots, \Upsilon^t$ with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, t\}$.

Let $\mathfrak{O}: \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t \longrightarrow P(\Omega) = \Gamma(\alpha(k)) = \{M(\alpha(k)) \mid k \in [0,1]\}$.(2)

If $\mathfrak{O}^{\omega}(\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(k))^{(\omega^t)}$.

Such that $\mathfrak{O}^{\omega}(\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \alpha^1{}_i^{\omega^1} \otimes \alpha^2{}_i^{\omega^2} \otimes \alpha^3{}_i^{\omega^3} \otimes \dots \otimes \alpha^t{}_i^{\omega^t} = \sigma_i(k)$.

Subject to the condition, the linguistic values of α_i should be in ascending order. Where $\omega = (\omega^1, \omega^2, \omega^3, \dots, \omega^t)^T$ is the exponential weighting vector of the $\alpha^t(k) \in \{\{M(\alpha(k)) \mid k \in [0,1]\}\}$ and $\omega^t \in [0, 1]$ with $\sum_{t=1}^{n} \omega^t = 1$, and $k \in [0,1]$, then $\mathfrak{O}$ is called LFHSOWGAO.

Example 10. Assume $\omega = (0.4, 0.3, 0.3)^T$ then LHSOWGAO $\{\sigma^2(\text{Pakistani, Male, Orange}),$

$\sigma^3(\text{Pakistani, Male, Orange})\}$

$= \sigma^2 \begin{pmatrix} \text{Pakistani(low), Male(medium),} \\ \text{Orange(high)} \end{pmatrix}$

$$\because \mathfrak{D}^\omega\,(\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(k))^{(\omega^t)}$$

$= {\alpha^1}_i^{\omega^1} \otimes {\alpha^2}_i^{\omega^2} \otimes {\alpha^3}_i^{\omega^3} \otimes \dots \otimes {\alpha^t}_i^{\omega^t} = \sigma_i$

$= \{\text{Pakistani}(0.6)^{0.4}, \text{Male}(0.5)^{0.3}, \text{Orange}(0.4)^{0.3}\}$

$= \sigma^2\{(0.6)^{0.4} + (0.5)^{0.3} + (0.4)^{0.3}\}$

$= \sigma^2(0.129 + 0.125 + 0.064)$

$= \sigma^2(0.318) = \sigma^2(\text{v.low}).$

Now,

$= \sigma^3 \begin{pmatrix} \text{Pakistani(none), Male(none),} \\ \text{Orange(none)} \end{pmatrix}$

$$\because \mathfrak{D}^\omega\,(\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(k))^{(\omega^t)}$$

$= {\alpha^1}_i^{\omega^1} \otimes {\alpha^2}_i^{\omega^2} \otimes {\alpha^3}_i^{\omega^3} \otimes \dots \otimes {\alpha^t}_i^{\omega^t} = \sigma_i(k)$

$= \{\text{Pakistani}(0)^{0.4}, \text{Male}(0)^{0.3}, \text{Orange}(0)^{0.3}\}$

$= \sigma^2\{(0)^{0.4} + (0)^{0.3} + (0)^{0.3}\}$

$= \sigma^2(0)$

$= \sigma^2(\text{none}).$

Theorem 3.

I. $\min_i(\alpha^t(k)) \leq \mathfrak{A}^\omega\,(\alpha^1, \alpha^2, \dots, \alpha^t) \leq \max_i(\alpha^t(k)).$

II. $\min_i(\alpha^t(k)) \leq \mathfrak{D}^\omega\,(\alpha^1, \alpha^2, \dots, \alpha^t) \leq \max_i(\alpha^t(k)).$

Proof:

I. $\min_i(\alpha^t(k)) \leq \mathfrak{A}^\omega\,(\alpha^1, \alpha^2, \dots, \alpha^t) \leq \max_i(\alpha^t(k)).$

Proof for lower bounds.

Since $0 < \omega^t \leq 1$ with $\sum_{t=1}^{n} \omega^t = 1$ and each $(\alpha^t(k))^{(\omega^t)}$ is bounded by $(\alpha^t(k))$, thus the product $\prod_{t=1}^{n}(\alpha^t(k))^{(\omega^t)}$ will be bounded below, given as.

$\min_i(\alpha^t(k)) \leq \mathfrak{A}^\omega\,(\alpha^1, \alpha^2, \dots, \alpha^t)$ (a)

Similarly, for upper bounds.

$(\alpha^t(k))^{(\omega^t)}$ cannot exceed the max of $(\alpha^t(k))$, thus

$\mathfrak{A}^\omega\,(\alpha^1, \alpha^2, \dots, \alpha^t) \leq \max_i(\alpha^t(k))$ (b)

From (a) and (b)

$$\min_i(\alpha^t(k)) \leq \mathfrak{A}^\omega(\alpha^1, \alpha^2, .., \alpha^t) \leq \max_i(\alpha^t(k)).$$

$$\min_i(\alpha^t(k)) \leq \mathfrak{D}^\omega(\alpha^1, \alpha^2, .., \alpha^t) \leq \max_i(\alpha^t(k)).$$

The proof is similar to (i), only the operator $\mathfrak{D}^\omega$ is changing rest all the conditions are same, results in (ii) holds.

Theorem 4.

I. $\mathfrak{D}^\omega(\alpha^t(k)) = \mathfrak{D}^\omega(\alpha^t(k'))$ where $(\alpha^t(k'))$ is any permutation of $(\alpha^t(k))$.

II. If For all $(\alpha^t(k)) = (\alpha(k))$ for all t, then $\mathfrak{D}^\omega(\alpha^t(k)) = \sigma_i(k)$.

III. If $(\alpha^t(k)) \leq (\widehat{\alpha}^t(k))$ for all t, then $\mathfrak{D}^\omega(\alpha^t(k)) \leq \mathfrak{D}^\omega(\widehat{\alpha}^t(k))$.

Proof:

I. $\mathfrak{D}^\omega(\alpha^t(k)) = \mathfrak{D}^\omega(\alpha^t(k'))$ where $(\alpha^t(k'))$ is any permutation of $(\alpha^t(k))$.

Since $(\alpha^t(k'))$ is any permutation of $(\alpha^t(k))$, we have

$$\mathfrak{D}^\omega(\alpha^t(k)) = \prod_{t=1}^{n}(\alpha^t(k))^{(\omega^t)}.$$

Permuting the values of $(\alpha^t(k))$ does not affect the product results, as multiplication is commutative in meaning. Hence

$$\mathfrak{D}^\omega(\alpha^t(k)) = \mathfrak{D}^\omega(\alpha^t(k')).$$

2 | If For all $(\alpha^t(k)) = (\alpha(k))$ for all t, then $\mathfrak{D}^\omega(\alpha^t(k)) = \sigma_i(k)$

If $(\alpha^t(k)) = (\alpha(k))$ for all t, then each term in $\mathfrak{D}^\omega(\alpha^t(k))$ becomes the same, specifically.

$$\mathfrak{D}^\omega(\alpha^t(k)) = \prod_{t=1}^{n}(\alpha(k))^{(\omega^t)}.$$

Since $0 < \omega^t \leq 1$ with $\sum_{t=1}^{n}\omega^t = 1$, thus it becomes

$$\mathfrak{D}^\omega(\alpha^t(k)) = \prod_{t=1}^{n}(\alpha(k))^1 = \prod_{t=1}^{n}\alpha(k),$$

$$\mathfrak{D}^\omega(\alpha^t(k)) = \sigma_i(k).$$

3 | If $(\alpha^t(k)) \leq (\widehat{\alpha}^t(k))$ for all t, then $\mathfrak{D}^\omega(\alpha^t(k)) \leq \mathfrak{D}^\omega(\widehat{\alpha}^t(k))$

Assuming $(\alpha^t(k)) \leq (\widehat{\alpha}^t(k))$ for each t, each term $(\alpha^t(k))^{(\omega^t)}$ is bounded by $(\widehat{\alpha}^t(k))^{(\omega^t)}$when raised to the non-negative weight ω^t, giving:

$$(\alpha^t(k))^{(\omega^t)} \leq (\widehat{\alpha}^t(k))^{(\omega^t)}.$$

Taking the product, we obtain:

$$\mathfrak{D}^\omega(\alpha^t(k)) = \prod_{t=1}^{n}(\alpha(k))^{(\omega^t)} \leq \prod_{t=1}^{n}(\widehat{\alpha}^t(k))^{(\omega^t)} = \mathfrak{D}^\omega(\widehat{\alpha}^t(k)).$$

It implies that

$\mathfrak{D}^{\omega}(\alpha^{t}(k)) \leq \mathfrak{D}^{\omega}(\widehat{\alpha}^{t}(k))$.

4 | Proposed LFHSs Algorithm

Based on the Linguistic Fuzzy Hypersoft Weighted Geometric Averaging Operator (LFHSWGAO), a method for LFHSs-based Multi-Criteria Decision-Making (MCDM) (LFHSs algorithm to solve MCDM problem) was designed using a DM technique. The following *Fig. 2* represents the graphic illustration of the LFHSs algorithm proposed.

4.1 | Proposed Algorithm

Step 1. Consider, $\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t$ for $t \geq 1$ be t distinct parameters, whose corresponding parametric values are, respectively, the sets $\Upsilon^1, \Upsilon^2, \Upsilon^3, \ldots, \Upsilon^t$ with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \ldots, t\}$. Let $\omega = (\omega^1, \omega^2, \omega^3, \ldots, \omega^t)^T$ be the exponential weighting vector. Where $\omega^t \geq 0$, and $\sum_{t=1}^{n} \omega^t = 1$.

Let $\mathfrak{A}: \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^t \longrightarrow P(\Omega) = \{M(\alpha(k)) \mid k \in [0,1]\}$. The decision-makers $\mathcal{D}^m$ compare the values with the linguistic quantifiers and assign linguistic variables to each alternative as $H_i = \{(\alpha^t(k) \mid i = 1, 2, \ldots, t \text{ and } k \in [0,1]\}$, and construct a linguistic fuzzy preference table for $(\alpha^t(k))^{(\omega^t)}$.

Step 2. Construct a matrix $[\alpha_{ij}]_{i \times j}$ for each $\mathcal{D}^m$ using LFHSWGAO,

$$\sigma_i^t = {\alpha^1}_i^{\omega^1} \otimes {\alpha^2}_i^{\omega^2} \otimes {\alpha^3}_i^{\omega^3} \otimes \ldots \otimes {\alpha^t}_i^{\omega^t}.$$

Step 3. List max value among all the decision-makers.

$$\max\left[\mathcal{D}^1(\alpha_{ij}), \mathcal{D}^2(\alpha_{ij}), \ldots, \mathcal{D}^m(\alpha_{ij})\right]_{i \times j}.$$

Step 4. Finally, list the alternatives with total scores H_i and rank the highest value.

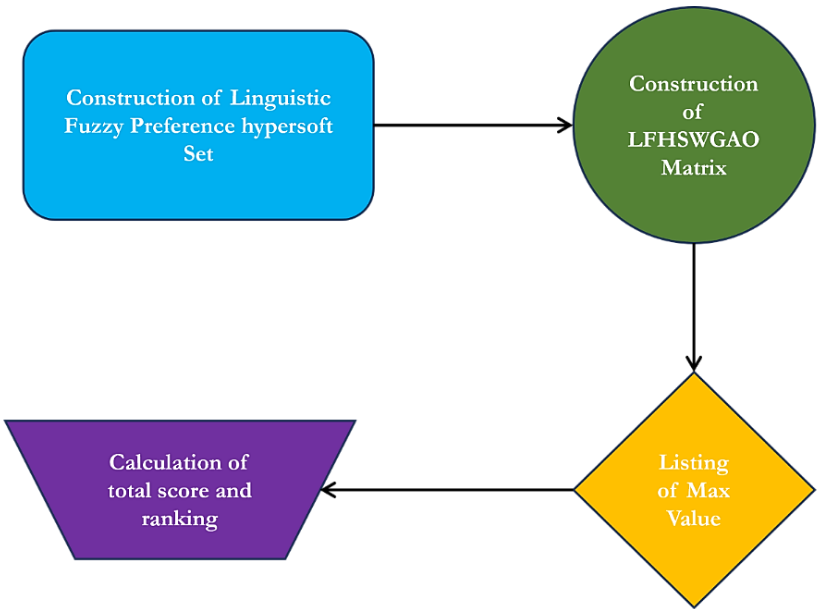

Fig. 2. Flowchart of proposed LFHSs algorithm based on aggregate operators.

4.2 | Case Study

The country has been at the forefront of environmentally friendly energy practices, working to lower carbon emissions and increase its reliance on renewable energy sources. The country, which has a long history of using hydroelectric power, has used its plentiful water resources to produce a sizable amount of electricity [1].

According to a life-cycle analysis, depending on the type (solar, wind, hydro, geothermal, tidal, wave, biomass), renewable energy produces between 11 and 740 gCO_2 for every kWh produced. Switzerland is committed to eliminating nuclear power, promoting energy efficiency, and reducing CO_2 emissions across all sectors [2]. The mining process of BTC is associated with energy and blockchain networks. The amount of energy the network uses will depend on the hash rate of the entire BTC network. A blockchain network's hash rate rises as more computers join it and participate in processing hashes (guesses) on the network. Since there is a lower chance of an attack, a PoW blockchain network with a high hash rate is safer and more robust. Energy use will decrease with a lower network hash rate.

When the hash rate is higher, the network will need more energy to mine each new block. A BTC (a cryptocurrency) is produced using 2.7 quadrillion computed hashes. The production of one BTC can consume 663.68kWh of energy, and it produces 370.17 $kgCO_2$. Through the potential for financial gain, job development, infrastructure investment, and technical advancement, BTC mining may improve the economy. BTCs may be earned as rewards, and miners can also invest in cutting-edge gear and data centers, stabilize the energy markets, improve technology, and promote financial inclusion.

4.3 | Data Collection and Preprocessing

This study's data collection involved gathering comprehensive datasets related to renewable energy sources, BTC mining operations, and environmental impact metrics. Key features such as the cost of energy production, carbon footprints, economic benefits, and regulatory measures were extracted from various reliable sources, including the World Bank, government reports, relevant source data websites, and academic research. Key features such as the cost of energy production, carbon footprints, economic benefits, and regulatory measures were extracted from various reliable sources, including government reports, industry publications, and expert opinions from decision-makers.

The collected data underwent thorough preprocessing, which included normalization to ensure uniformity across different scales, handling missing values through imputation techniques, and filtering out outliers to enhance data quality. For instance, the cost of energy production was categorized into ranges (<\$20/MWh, <\$40/MWh, <\$80/MWh, <\$100/MWh), while carbon footprints were classified (10-200 g CO_2, 201-400 g CO_2, etc.). Additionally, economic benefits were measured in BTC (100BTC, 1000BTC, 10000BTC), and regulatory measures included aspects like consumer protection and financial stability. This meticulous approach ensured that the datasets were robust and suitable for subsequent analysis, allowing for accurate assessments of the proposed model's effectiveness in identifying the most suitable renewable energy source that meets Switzerland's environmental, social, and economic needs while optimizing the BTC mining process.

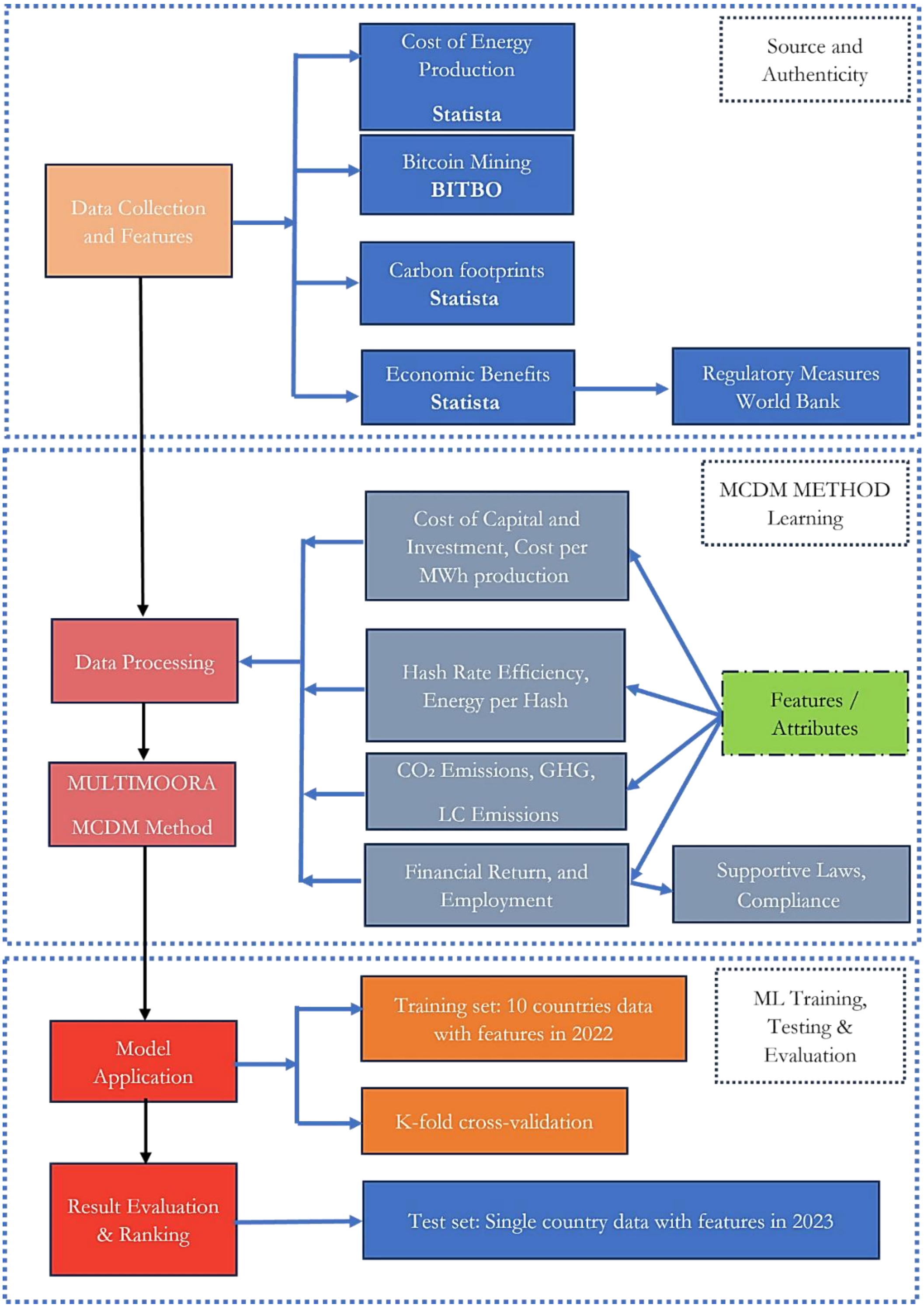

Fig. 3. Data collection and preprocessing.

4.4 | Mathematical Modelling

Five renewable energy options are assumed: hydrogen, wind, hydro, solar, and geothermal. The parameters used in the model include the following: carbon footprints, BTC hashing power, economic gains/losses, and legislative uncertainty. Each parameter is represented as a fuzzy set to incorporate uncertainty. Fuzzy sets are employed to address linguistic ambiguity, which is inherent when conducting an onsite or field study of the modeled object. The fuzzy inference system gives the process of demonstration and exploration. Consider five renewable energy resources RE^1 (Hydrogen), RE^2 (Wind and Hydro), RE^3 (Solar), and RE^4 (Geothermal), RE^5 (Nuclear) as alternatives $\mathrm{R} = \{\mathrm{RE}^1, \mathrm{RE}^2, \mathrm{RE}^3, \mathrm{RE}^4, \mathrm{RE}^5\}$. Switzerland has the potential to gain from the cryptocurrency business commercially, and with this study, we want to determine which renewable energy should be used to increase the BTC mining process to meet its economic targets. The DM $\mathcal{D} = \{\mathcal{D}^{\mathrm{m}}\ ; \mathrm{m} = 1,2,3\}$ will assign the fuzzy values based on their expertise. The goal should be to identify a renewable energy source that meets the country's environmental, social, and economic needs. Consider the parameters: $\Upsilon^1 = \text{Cost of energy production}$, $\Upsilon^2 = \text{Carbon footprints}$, $\Upsilon^3 =$ BTC mining, $\Upsilon^4 =$ economic benefits, $\Upsilon^5 = \text{regulatory measures}$, and their respective parametric values are:

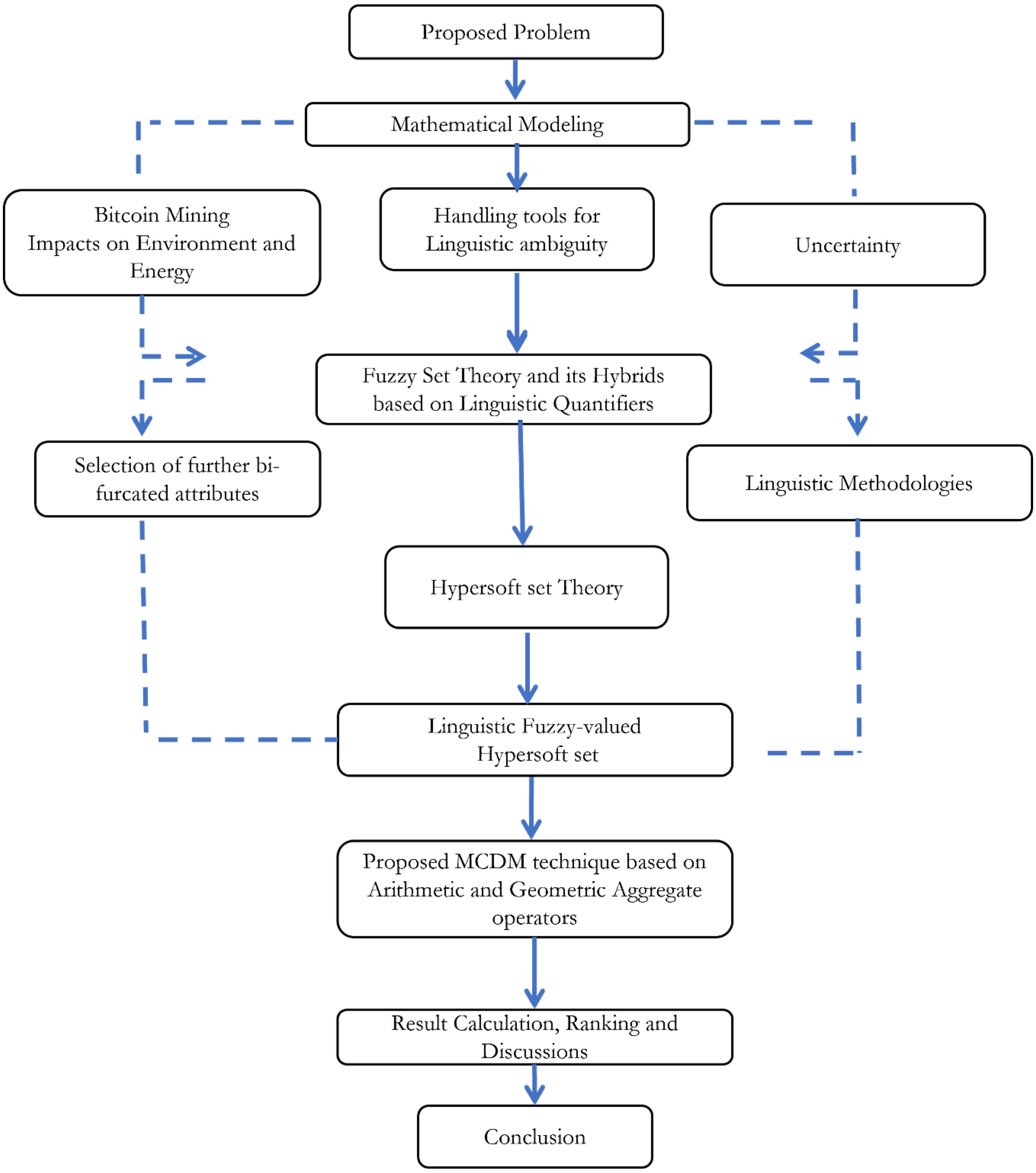

Fig. 4. Flow chart of problem formulation and MCDM technique selection.

Cost of energy production $= \Upsilon^1 = \{< \$20/\mathrm{MWh}, < \$40/\mathrm{MWh}, < \$80/\mathrm{MWh}, < \$100/\mathrm{MWh}\}$.

Carbon footprints $= \Upsilon^2 = \{10 - 200\ \mathrm{gCO_2},\ 201 - 400\ \mathrm{gCO_2},\ 401 - 600\ \mathrm{gCO_2},\ 601 - 800\ \mathrm{gCO_2}\ \}$.

BTC mining (hash rate per second) $= \Upsilon^3 = \{1\ \text{kilo}, 1\ \text{mega}, 1\ \text{gega}, 1\ \text{tera}, 1\ \text{peta}, 1\ \text{exa}\}$.

Economic benefits (per day) $= \Upsilon^4 = \{100\mathrm{BTC}, 1000\mathrm{BTC}, 10000\mathrm{BTC}\}$.

Regulatory measures $= \Upsilon^5 = \{\ \text{consumer protection (cp)}, \text{financail stability (fs)}\}$.

Then the function $\Gamma : \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \Upsilon^5 \times \Upsilon^5 \longrightarrow P(\Omega)$ and assume the $M(\alpha(k)) \mid k \in [0,1]\}$.

$M = \{RE^2, RE^3, RE^4\} \subset \Omega$ where $\Omega = R = \{RE^1, RE^2, RE^3, RE^4, RE^5\}$ be the universal set. The sub-divided parametric function can be given below.

$\Gamma(\$80/\text{MWh}, 201 - 400\ \text{gCO}_2, 1\ \text{tera hash rate}, 10000\text{BTC}, \text{cp}) = \{RE^2, RE^3, RE^4\}$.

$\mathcal{D}^1$ will assign a linguistc values to the parametric choices.

$$= \left\{ \begin{array}{l} RE^2 < \dfrac{\$80 \text{ per MWh}}{(\text{v.v.high})}, \dfrac{201 - 400\ \text{gCO}_2}{(\text{medium})}, \dfrac{1 \text{ tera hash rate}}{(\text{v.v.v.high})}, \dfrac{10000\text{BTC}}{(\text{v.v.high})}, \dfrac{\text{cp}}{(\text{v.high})} >, \\ RE^3 < \dfrac{\$80 \text{ per MWh}}{(\text{high})}, \dfrac{201 - 400\ \text{gCO}_2}{(\text{high})}, \dfrac{1 \text{ tera hash rate}}{(\text{v.v.high})}, \dfrac{10000\text{BTC}}{(\text{high})}, \dfrac{\text{cp}}{(\text{v.v.high})} >, \\ RE^4 < \dfrac{\$80 \text{ per MWh}}{(\text{medium})}, \dfrac{201 - 400\ \text{gCO}_2}{(\text{v.high})}, \dfrac{1 \text{ tera hash rate}}{(\text{medium})}, \dfrac{10000\text{BTC}}{(\text{v.high})}, \dfrac{\text{cp}}{(\text{medium})} > \end{array} \right\}$$
$$= L_1.$$

$\mathcal{D}^2$ will assign a linguistc values to the parametric choices.

$$= \left\{ \begin{array}{l} RE^2 < \dfrac{\$80 \text{ per MWh}}{(\text{v.high})}, \dfrac{201 - 400\ \text{gCO}_2}{(\text{high})}, \dfrac{1 \text{ tera hash rate}}{(\text{medium})}, \dfrac{10000\text{BTC}}{(\text{high})}, \dfrac{\text{cp}}{(\text{v.high})} >, \\ RE^3 < \dfrac{\$80 \text{ per MWh}}{(\text{low})}, \dfrac{201 - 400\ \text{gCO}_2}{(\text{v.v.high})}, \dfrac{1 \text{ tera hash rate}}{(\text{v.low})}, \dfrac{10000\text{BTC}}{(\text{v.v.high})}, \dfrac{\text{cp}}{(\text{v.v.low})} >, \\ RE^4 < \dfrac{\$80 \text{ per MWh}}{(\text{low})}, \dfrac{201 - 400\ \text{gCO}_2}{(\text{low})}, \dfrac{1 \text{ tera hash rate}}{(\text{v.high})}, \dfrac{10000\text{BTC}}{(\text{v.v.high})}, \dfrac{\text{cp}}{(\text{v.v.high})} > \end{array} \right\}$$
$$= L_2.$$

$\mathcal{D}^3$ will assign a linguistc values to the parametric choices.

$$= \left\{ \begin{array}{l} RE^2 < \dfrac{\$80 \text{ per MWh}}{(\text{v.v..high})}, \dfrac{201 - 400\ \text{gCO}_2}{(\text{v.high})}, \dfrac{1 \text{ tera hash rate}}{(\text{v.high})}, \dfrac{10000\text{BTC}}{(\text{low})}, \dfrac{\text{cp}}{(\text{medium})} >, \\ RE^3 < \dfrac{\$80 \text{ per MWh}}{(\text{low})}, \dfrac{201 - 400\ \text{gCO}_2}{(\text{low})}, \dfrac{1 \text{ tera hash rate}}{(\text{medium})}, \dfrac{10000\text{BTC}}{(\text{v.v.high})}, \dfrac{\text{cp}}{(\text{v.high})} >, \\ RE^4 < \dfrac{\$80 \text{ per MWh}}{(\text{low})}, \dfrac{201 - 400\ \text{gCO}_2}{(\text{v.v.low})}, \dfrac{1 \text{ tera hash rate}}{(\text{perfect})}, \dfrac{10000\text{BTC}}{(\text{perfect})}, \dfrac{\text{cp}}{(\text{v.v.high})} > \end{array} \right\}$$
$$= L_3.$$

The DM $\mathcal{D} = \{\mathcal{D}^m ; m = 1,2,3\}$ will assign fuzzy values to $M(\alpha(k))$ variables from reference *Table 2*.

$\mathcal{D}^1$ will assign a linguistc values to the parametric choices.

$$= \left\{ \begin{array}{l} RE^2 < \dfrac{\$80 \text{ per MWh}}{(0.8)}, \dfrac{201 - 400\ \text{gCO}_2}{(0.5)}, \dfrac{1 \text{ tera hash rate}}{(0.9)}, \dfrac{10000\text{BTC}}{(0.8)}, \dfrac{\text{cp}}{(0.7)} >, \\ RE^3 < \dfrac{\$80 \text{ per MWh}}{(0.6)}, \dfrac{201 - 400\ \text{gCO}_2}{(0.6)}, \dfrac{1 \text{ tera hash rate}}{(0.8)}, \dfrac{10000\text{BTC}}{(0.6)}, \dfrac{\text{cp}}{(0.8)} >, \\ RE^4 < \dfrac{\$80 \text{ per MWh}}{(0.5)}, \dfrac{201 - 400\ \text{gCO}_2}{(0.7)}, \dfrac{1 \text{ tera hash rate}}{(0.5)}, \dfrac{10000\text{BTC}}{(0.7)}, \dfrac{\text{cp}}{(0.5)} > \end{array} \right\} = L_1.$$

$\mathcal{D}^2$ will assign a linguistc values to the parametric choices.

$$= \left\{ \begin{array}{l} RE^2 < \dfrac{\$80 \text{ per MWh}}{(0.7)}, \dfrac{201 - 400\ \text{gCO}_2}{(0.6)}, \dfrac{1 \text{ tera hash rate}}{(0.5)}, \dfrac{10000\text{BTC}}{(0.6)}, \dfrac{\text{cp}}{(0.7)} >, \\ RE^3 < \dfrac{\$80 \text{ per MWh}}{(0.4)}, \dfrac{201 - 400\ \text{gCO}_2}{(0.8)}, \dfrac{1 \text{ tera hash rate}}{(0.3)}, \dfrac{10000\text{BTC}}{(0.8)}, \dfrac{\text{cp}}{(0.2)} >, \\ RE^4 < \dfrac{\$80 \text{ per MWh}}{(0.4)}, \dfrac{201 - 400\ \text{gCO}_2}{(0.4)}, \dfrac{1 \text{ tera hash rate}}{(0.7)}, \dfrac{10000\text{BTC}}{(0.8)}, \dfrac{\text{cp}}{(0.8)} > \end{array} \right\} = L_2.$$

$\mathcal{D}^3$ will assign a linguistc values to the parametric choices.

$$= \begin{Bmatrix} RE^2 < \dfrac{\$80 \text{ per MWh}}{(0.8)}, \dfrac{201-400 \text{ gCO}_2}{(0.7)}, \dfrac{1 \text{ tera hash rate}}{(0.7)}, \dfrac{10000\text{BTC}}{(0.4)}, \dfrac{\text{cp}}{(0.5)} >, \\ RE^3 < \dfrac{\$80 \text{ per MWh}}{(0.4)}, \dfrac{201-400 \text{ gCO}_2}{(0.4)}, \dfrac{1 \text{ tera hash rate}}{(0.5)}, \dfrac{10000\text{BTC}}{(1)}, \dfrac{\text{cp}}{(0.7)} >, \\ RE^4 < \dfrac{\$80 \text{ per MWh}}{(0.4)}, \dfrac{201-400 \text{ gCO}_2}{(0.2)}, \dfrac{1 \text{ tera hash rate}}{(1)}, \dfrac{10000\text{BTC}}{(1)}, \dfrac{\text{cp}}{(0.8)} > \end{Bmatrix} = L_3.$$

Step 1. The fuzzy valued decision matrix is presented above.

Step 2. Construct a matrix using LFHSWGAO and LFHSOWGAO and shown in *Table 2*. The DM choice and expertise-based weights for each attribute are $\omega = (0.342, 0.276, 0.122, 0.223, 0.037)$.

Table 3. Decision-makers preference based LFHSOWGAO matrix.

Alternatives	$\mathcal{D}^1$	$\mathcal{D}^2$	$\mathcal{D}^3$
RE^2	4.677	4.4551	4.5800
RE^3	4.565	4.4282	4.4133
RE^4	4.512	4.4081	4.3640

Step 3. List max value among all the decision-makers, i.e., max $[\mathcal{D}^1(\alpha_{ij}), \mathcal{D}^2(\alpha_{ij}), \ldots, \mathcal{D}^m(\alpha_{ij})]_{i\times j}$ of *Step 2*.

$$\text{Score} = \{RE^2 < 4.677 >, RE^3 < 4.565 >, RE^4 < 4.512 >\}.$$

Step 4. Finally, list the alternatives with total scores $\mathcal{S}_i$ and rank highest value and shown in *Table 3*.

Table 4. Final score and ranking.

Alternative	Score Value	Rank
RE^2	4.677	1
RE^3	4.565	2
RE^4	4.512	3

According to ranking findings $RE^4 < RE^3 < RE^2$, using renewable energy sources to power BTC mining has the potential to have two positive effects: a significant decrease in carbon footprints and a boost to the economy. BTC mining companies may allay concerns regarding the energy-intensive nature of cryptocurrency mining by using sustainable energy sources like solar, wind, or hydroelectric power to lessen their environmental effect dramatically.

Fig. 5. Rankings of renewable energy methods.

In addition, using renewable energy may have economic benefits as it frequently costs less than conventional fossil fuels and may draw investors and consumers who care about the environment. By encouraging innovation and employment growth in the renewable energy industry and aligning with larger global initiatives to battle climate change, this trend towards sustainability will eventually lead to a more ecologically friendly world. The final ranking of renewable energy resources is presented in *Fig. 5*.

5 | Result Discussion and Comparison

In dedication to lowering carbon emissions and raising the proportion of renewable energy in its overall production, Switzerland stands out as a global leader. *Fig. 6* shows its standing globally (https://www.worldometers.info/co2-emissions/co2-emissions-by-country/ (Accessed on 11 September 2023). Switzerland understands the essential need to protect its natural beauty while addressing the urgent problem of climate change because of its stunning alpine scenery and clean ecology. The nation is now a role model for sustainable development thanks to several ambitious policies and programs it has implemented to accomplish these objectives. Switzerland's significant support for renewable energy sources like solar, wind, and hydroelectric power is a notable feature of the country's green agenda. The Swiss government has made significant investments in developing renewable energy infrastructure and research, encouraging organizations and people to embrace sustainable energy practices. *Table 5* presents the actual national standing of renewable energy production and the CO_2 emissions associated with it.

Along with lowering the country's carbon footprint, this also generates employment and encourages innovation in the field of green technology. (CO_2 emission) https://impactful.ninja/the-carbon-footprint-of-renewable-energy/ (accessed on 11 September 2023).

Table 5. Renewable energy production in percentage within Switzerland and Carbon emission.

Renewable Energy	Production Percentage	CO_2 Emission (g) per kWh	CO_2 Emission (kg) per BTC
RE^2 = Hydro & Wind	68%	24	15928.32
RE^3 = Solar	12%	48	31856.64
RE^4 = Geothermal	1%	38	25219.84

In our research on resource allocation and optimization in mining operations, we found that LFHSs effectively solve different attributes. Models using linguistic fuzzy consider things like environmental effects and energy efficiency to help in resource allocation. The compromises and trade-offs between various variables were clearly described using language concepts and variables, allowing for more intelligent resource allocation judgments. Recognizing this study's constraints, including data availability and modeling presumptions, is critical. LFHSs can be improved by combining real-time data streams, more complicated linguistic models, and solutions to energy and environmental issues in BTC mining. Interestingly, the existing approaches in *Table 6* use a completely different method in calculating the final rankings of alternatives as per our suggested strategy. *Table 6* presents optimal alternatives in existing approaches. The same information fusion techniques are used to generate the final ranking as in current approaches. Due to this, data is lost throughout the processing technique. On the other hand, LFHSs reduce data loss by using more accurate and objective data. Additionally, the LFHS technique is distinct in considering the trade-off between group utility maximization and individual regret reduction, increasing its accuracy and applicability in MCDM problems.

Table 6. Renewable energy alternatives ranking using existing approaches.

Method	Ranking of Alternatives	Optimal Alternative(S)
FHSs ([37])	$RE^2 \succ RE^3 \succ RE^4 \succ RE^1$	RE^2
FHSs WO ([38])	$RE^2 \succ RE^3 \succ RE^4 \succ RE^1$	RE^2
IFHSs ([39])	$RE^2 \succ RE^3 \succ RE^4 \succ RE^1$	RE^2
NHSS-TOPSIS ([40])	$RE^2 \succ RE^3 \succ RE^4 \succ RE^1$	RE^2
LHSs ([41])	$RE^2 \succ RE^3 \succ RE^4 \succ RE^1$	RE^2
LFHSs (Proposed)	$RE^2 \succ RE^3 \succ RE^4 \succ RE^1$	RE^2

The machine learning algorithm MULTIMOORA was employed to validate and strengthen the results, and the ranking of the alternatives was calculated. The correlation features are used in this study to enhance the model's accuracy, reliability, and interpretability and support rational DM and interpretation capabilities. It is used for predicting and DM as it might serve as meaningful predictors or inputs for statistical models, machine learning algorithms, or decision support systems.

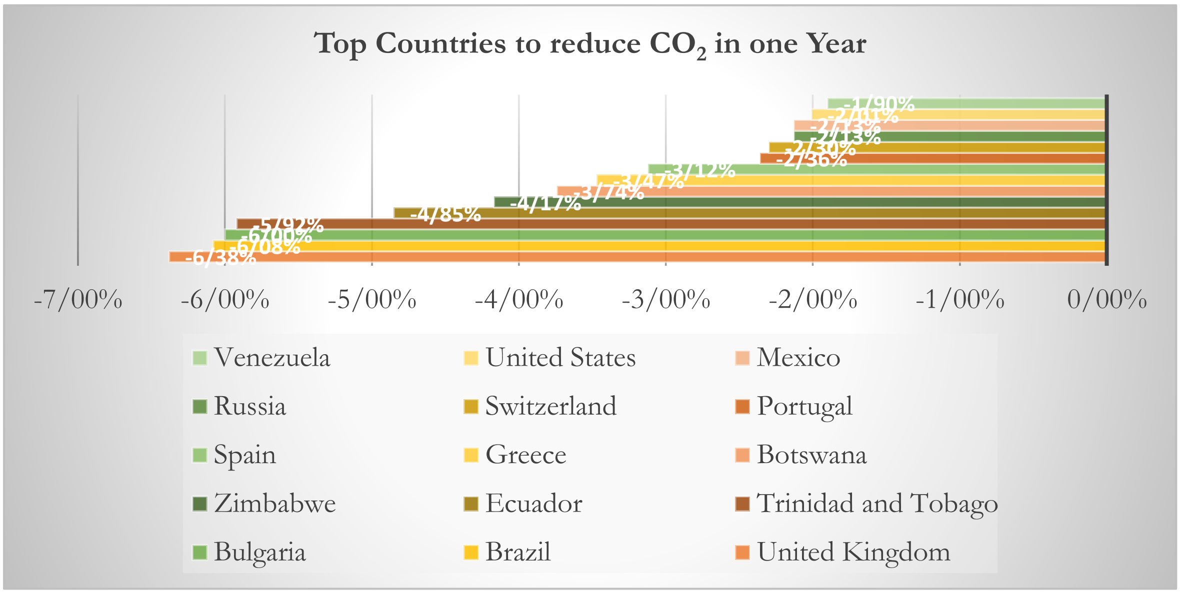

Fig. 6. Globally standings of the Top ranked countries contributing the lower carbon emission.

The concept of correlation analysis is used to understand how different attributes and alternatives correlate and identify the most significant factors that determine the phenomenon. Fuzzy logic can support DM under ambiguous circumstances [42], optimize resource allocation based on various criteria, and dynamically adjust mining practices in response to fluctuating renewable energy availability, ultimately assisting in making more sustainable and informed decisions in the cryptocurrency mining industry. The machine learning technique has been used to evaluate uncertainties [43]. The ranking of the alternatives calculated is presented in *Table 7*. The result shows that both the theoretical and ML are approximately the same.

Table 7. Comparison of rankings.

Method	Alternative Scores Ranking
LFHS MCDM algorithm (proposed)	$RE_2 > RE_3 > RE_4$
ML using (MULTIMOORA)	$RE_2 > RE_3 > RE_4$

The advancements in fuzzy set theory, DM models, and control systems across various sectors and applications[44–46]. Additionally, innovations in fuzzy control systems significantly enhance complex systems under uncertainty, emphasizing their practical impacts in machine learning [47]. The work presents various advanced mathematical models and DM frameworks across fields such as environmental sustainability [48], [49], renewable energy resources [50], pattern recognization [51], methods for fixed point study [52], medical diagnosis [53], [54], application of IoT & industry 4.0 in environmental sustainability [55]. Therefore, future research can be extended by choosing the best machine learning algorithms, building models by exploring the case studies of renewable energy resources [23], [24], [31] testing them in cryptocurrency industry [9], [22], and then integrating these models within DM system or monitoring systems.

5.1 | Managerial Insights

The linguistic parameter represents uncertainty in crypto-mining's environmental impact by introducing a layer of ambiguity into DM variables. Environmental factors can be uncertain in crypto-mining due to their dispersed and complex nature. Linguistic parameters are used to express qualitative data in the fuzzy form, which is common for environmental factors due to their assessment's diverse, uncertain, and imprecise character. As a result, certain DM variables should be characterized by the level of ambiguity that determines which optimal strategies or resource allocation will be selected to address environmental issues and promote environmental sustainability in crypto mining. Finally, renewable energy production for crypto mining is a parameter in the range of feasible factors. Such parameters include the availability of energy production capacity, which is necessary for crypto mining, as well as technological advancement and the location of this technology. Solar radiance, wind speed, and the potential of hydro production in each region are among the factors affecting the production feasibility of renewable energy for crypto mining.

Meanwhile, the feasibility of renewable energy depends on this factor's ability to produce significant amounts of energy without major environmental harm. Different energy storage possibilities and complex grid integration also expand the factors' feasible scope as an increasing number of these sources become viable compared to fossil fuel. As mentioned, the feasible range of renewable energy production supporting cryptocurrency mining encompasses variable parameters. Among the considered parameters are energy availability, prospects of technological development, and geographical location. Solar, wind, and hydroelectric power are the existing and sustainable alternatives to fossil fuels currently integrated into cryptocurrency mining. Such measures attainably define them as solar irradiance, wind and hydroelectric potentials, and velocity to identify the reasonable value within the feasible range. It is possible due to certain innovations in available and prospective energy storage and grid solutions that continue expanding the feasible range of renewable energy sources. In such a way, the considered sources have become acceptable support for cryptocurrency mining due to the reduced environmental impact. Other approaches to the complex issue of DM linked to cryptocurrency mining challenges include conventional fuzzy logic models or existing mathematical optimization methods, as well as probabilistic modeling. However, the suggested linguistic fuzzy valued HSS environment is advantageous by specific characteristics. It helps generate and accumulate information more directly by comprehensively representing linguistic ambiguity. Furthermore, it also systematizes the qualitative translation of linguistic knowledge into quantitative value due to a bifurcated function.

5.2 | Policy Recommendations

Overall, when it comes to formulating a government's policy recommendations, the state must develop a strong regulatory framework regarding cryptocurrency mining, energy consumption laws, waste management requirements, incentivizing renewable sources of energy, and carbon tax provisions to adopt environmental policies.

5.3 | Ethical Consideration

In the domain of industry ethics, cryptocurrency miners need to conduct their business sustainably, fairly, and ethically and adopt the latest technologies, assuming compliance with the latest regulations, etc.

6 | Conclusion

This study fills a research gap by providing a framework based on LFHSs. Its applicability in the MCDM scenario has been presented as a case study related to BTC mining, which has non-neglectable impacts on the environment and energy production. It defines LFHSs, describes their fundamental functions and aggregate operators, and proposes an MCDM method specifically designed for the LFHSs environment. By minimizing data loss by converting linguistic concepts into fuzzy representations, the article demonstrates the applicability of LFHSs in assessing the environmental impact of crypto mining. The ability of LFHSs to consider more precise attribute subdivisions and individual language expertise is particularly noteworthy since it improves accuracy in challenging MCDM tasks. This research, which helps to solve linguistic uncertainty, helps to provide accurate assessments of energy sources, environmental effects, economic benefits, and regulatory compliance, assisting in developing sustainable policies. Its authenticity is strengthened by validation and comparison with established approaches, and investigating machine learning integration. The importance of working with data scientists and subject experts is emphasized to handle the environmental and energy difficulties associated with BTC mining and arrive at sustainable conclusions. Regarding future directions and implications, research must be improved to refine the methodologies to assess environmental impacts in emerging technological areas such as cryptocurrency mining. Enhanced computational techniques, including deep learning algorithms, need to be employed to improve accuracy and efficiency further. Finally, the LFHSs model can be applied quite generally to various knowledge domains in addition to cryptocurrency mining; for example, industry, identifying specific applications, such as environmental monitoring, finance, healthcare, and energy management, could demonstrate the model's versatility and relevance in addressing complex, real-world issues.

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Author Contributaion

Muhammad Saqlain carried out the research conceptualization, data collection, methodology, data analysis, writing, and manuscript formatting. Wiyada Kumam has proofread it. Poom Kumam provided the overall supervision, resources, and funding.

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Data Availability

No data is associated with this submission.

Conflicts of Interest

The authors declare no conflict of interest.

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