Control of Non-Linear Vibrations using Three-to-One Internal Resonances

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Abstract
A weakly nonlinear vibration absorber is used to suppress the primary resonance vibrations of a single-degree-of-freedom weakly nonlinear oscillator subjected to periodic excitation. The linearized natural frequency (low frequency mode) of the nonlinear absorber is tuned to be approximately one-third of the linearized natural frequency (high frequency mode) of the primary nonlinear oscillator. The cubically nonlinear coupling of stiffness establishes the terms that develop three-to-one internal resonances. The low frequency mode required for the absorber can be achieved by a light-weight mass nonlinear attachment with small values of linear and nonlinear stiffness of coupling. The method of multiple scales is used to obtain the averaged equations that determine the amplitudes and phases of the first-order approximate solutions. Numerical results are given to show the effectiveness of the nonlinear absorber for suppressing nonlinear vibrations of the primary nonlinear oscillator under primary resonance conditions.

Key words: Nonlinear vibration absorber, single degree-of-freedom nonlinear oscillator, three-to-one internal resonances, primary resonance response, passive vibration control, vibration absorber.

1. Introduction

Linear and nonlinear dynamic vibration absorbers have been employed to suppress the vibrations of the nonlinear oscillators subjected to parametric or external excitations [1-17]. Conceptually a dynamic vibration absorber [18, 19] consists of a mass that is attached to the primary oscillator by a linear damper and a spring of linear or linear-plus-nonlinear characteristics. The addition of an absorber to a single-degree-of-freedom weekly nonlinear oscillator results in a two-degree-of-freedom weekly nonlinear system. The linearized natural frequency of the absorber can be tuned to be under non-internal resonances or internal resonances with the linearized natural frequency of the primary oscillator. Both non-internal resonances and internal resonances have been implemented to suppress the nonlinear vibrations of nonlinear oscillators.

Implementation of internal resonance control technique requires creating nonlinear coupling between the primary oscillator and nonlinear absorber and tuning the two linearized natural frequencies to be commensurable. Depending on the order of the coupling nonlinearity in the resultant system that is formed by the primary oscillator and absorber, the commensurable relationships of the linearized natural frequencies can cause the corresponding modes to be strongly coupled once an internal resonance exists. For example, if the coupling of the resultant system is of quadratic nonlinearities, then a one-to-two or two-to-one internal resonances exist in seeking the first-order approximate solutions. For a resultant system with cubic nonlinearities of coupling, one-to-one, one-to-three, or three-to-one internal resonances can exist. When an internal resonance exists in a nonlinear system, vibrational energy imparted to one of the modes will be continuously exchanged between two modes.

For a resultant system with cubic nonlinearities of coupling, there are three corresponding design options for the utilisation of internal resonances to suppress the vibrations of the primary nonlinear oscillator, namely $\omega_p : \omega_a = 1:1$, $1:3$, or $3:1$. Here ω_p and ω_a denote the linearized natural frequencies of the primary nonlinear oscillator and absorber respectively. One-to-one and one-to-three internal resonance techniques have been proposed to suppress the resonant vibrations of the nonlinear oscillators. In the present paper, three-to-one internal resonances will be tuned to suppress the primary resonance response of the nonlinear oscillator. Specifically, the linearized natural frequency of the absorber will be approximately one-third of the linearized natural frequency of the primary nonlinear oscillator, thereby requiring a weakly linear stiffness of coupling. The values of the absorber spring stiffness are significantly lower than those of the forced nonlinear oscillator. The linearized natural frequency and the forcing frequency interval for primary resonances of the primary nonlinear oscillator change only slightly after the nonlinear absorber is attached to. In this sense, the coupling stiffness of nonlinear vibration absorber can be considered as a small perturbation to those of the primary nonlinear oscillator. It can thus be easily implemented in practical applications.

The present paper is organised into four sections. Section 2 describes the mathematical modelling of a primary nonlinear oscillator attached by a nonlinear vibration absorber and perturbation analysis. Illustrative examples are presented in Section 3 and conclusion is given in Section 4.

2. Mathematical Modelling and Perturbation Analysis

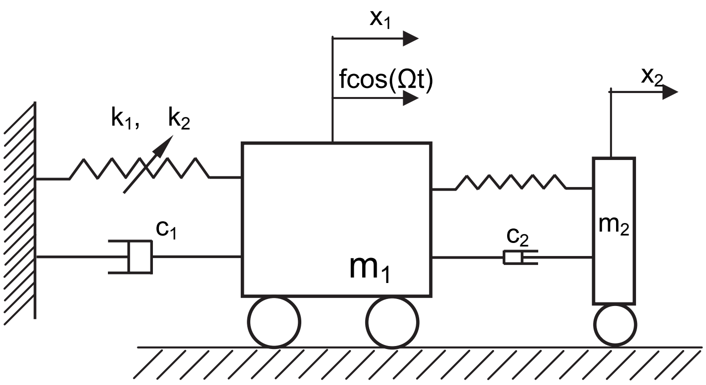


Fig. 1 A primary nonlinear oscillator attached by a nonlinear vibration absorber, where m_1 and m_2 represent the masses of the primary oscillator and absorber, respectively.

For simplicity, consider an externally excited nonlinear oscillator with cubic nonlinearity that is attached by a weakly nonlinear vibration absorber as shown in Figure 1. The equations of motion for the resultant system can be written as:

$$m_1\ddot{x}_1 = -c_1\dot{x}_1 - k_1x_1 - k_2x_1^3 + c_2(\dot{x}_2 - \dot{x}_1) + k_3(x_2 - x_1) + k_4(x_2 - x_1)^3 + f_0\cos(\Omega t) ,$$

$$m_2\ddot{x}_2 = -c_2(\dot{x}_2 - \dot{x}_1) - k_3(x_2 - x_1) - k_4(x_2 - x_1)^3 , \qquad (1)$$

where x_1 and x_2 are the displacements of the primary nonlinear oscillator and the absorber. m_1, k_1, k_2 and c_1 represent the mass, linear stiffness, nonlinear stiffness and damping coefficient of the primary nonlinear oscillator, respectively. Similarly, m_2, k_3, k_4 and c_2 denote the mass, linear stiffness, nonlinear stiffness and damping coefficient of the nonlinear absorber. The parameters f_0 and Ω are the amplitude and frequency of the external excitation. An overdot indicates the differentiation with respect to time t.

Equation (1) can be re-organized as:

$$\ddot{x}_1 + \mu_1\dot{x}_1 + \omega_1^2 x_1 + \alpha_1 x_1^3 - m\mu_2\dot{x}_2 - m\omega_2^2 x_2 + m\alpha_2(x_1 - x_2)^3 = f\cos(\Omega t)\,,$$

$$\ddot{x}_2 + \mu_2\dot{x}_2 + \omega_2^2 x_2 - \alpha_2(x_1 - x_2)^3 = \mu_2\dot{x}_1 + \omega_2^2 x_1\,, \tag{2}$$

where $\mu_1 = \dfrac{c_1 + c_2}{m_1} = \mu_{10} + m\mu_2$, $\omega_1^2 = \dfrac{k_1 + k_3}{m_1} = \omega_{10}^2 + m\omega_2^2$, $\alpha_1 = \dfrac{k_2}{m_1}$, $m = \dfrac{m_2}{m_1}$, $\mu_2 = \dfrac{c_2}{m_2}$, $\omega_2^2 = \dfrac{k_3}{m_2}$, $\alpha_2 = \dfrac{k_4}{m_2}$, $f = \dfrac{f_0}{m_1}$. Here μ_{10} and ω_{10} denote the damping and linearized natural frequency of the primary nonlinear oscillator without absorber.

Equation (2) can be interpreted in the context of nonlinear oscillations as a two-degree-of-freedom weakly nonlinear system subjected to periodic excitation. The closed form of the solutions to equation (2) cannot be found analytically thus an approximate solution will be sought using a perturbation method.

The main purpose of the present paper is to utilise a three-to-one internal resonance (i.e., $\omega_p : \omega_a = 3:1$) to suppress the primary nonlinear oscillator. The values of the absorber spring stiffness are significantly lower than the stiffness of the primary oscillator as the low frequency of the absorber requires a low value of linear stiffness. As a result, for the primary nonlinear oscillator, the coupled damping and spring stiffness between the primary nonlinear oscillator and absorber can be considered as a small perturbation to the corresponding parameters of the primary nonlinear oscillator. On the other hand, for the absorber, the linear part of its stiffness is comparable with its mass, though both are smaller than their counterpart of the primary nonlinear oscillator. As such, equation (2) can be re-scaled as

$$\ddot{x}_1 + \varepsilon\mu_1\dot{x}_1 + \omega_1^2 x_1 + \varepsilon\alpha_1 x_1^3 - \varepsilon m\mu_2\dot{x}_2 - \varepsilon m\omega_2^2 x_2 + \varepsilon m\alpha_2(x_1 - x_2)^3 = f\cos(\Omega t)\,,$$

$$\ddot{x}_2 + \varepsilon\mu_2\dot{x}_2 + \omega_2^2 x_2 - \varepsilon\alpha_2(x_1 - x_2)^3 = \varepsilon\mu_2\dot{x}_1 + \omega_2^2 x_1\,, \tag{3}$$

where ε is a non-dimensional small parameter, the coefficients of the damping terms and nonlinear terms, μ_i and α_i ($i = 1,2$) in equation (2) have been rescaled in terms of $\mu_i = \varepsilon\,\overline{\mu}_i$ and $\alpha_i = \varepsilon\,\overline{\alpha}_i$, the overbars in $\overline{\mu}_i$ and $\overline{\alpha}_i$ have been removed for brevity. In particular, all damping terms and nonlinear terms are assumed to be small and in the order of O(ε). The amplitude of the excitation has been re-scaled in terms of $f = \varepsilon\bar{f}$ to account for the primary resonances and the overbar in $\bar{f}$ has been removed for the sake of brevity. Equation (3) is a system of two weakly nonlinear oscillators with coupling terms in the context of nonlinear oscillations. Though the resonant response of certain systems of coupled weakly nonlinear oscillators has been studied for the systems under one-to-one, three-to-one, or one-to-three international resonances [i.e., 20], equation (3) is different from those equations as the coupled linear terms act as excitations in the second equation. The coupled linear term $\omega_2^2 x_1$ cannot be considered as a small perturbation term and has the same order as the linear term of the second equation, while the existing studies usually considered the coupling terms as small perturbations.

It is assumed that the two-DOF nonlinear system given by equation (3) is simultaneously under both three-to-one internal resonance and external primary resonance of the high vibration mode (the primary nonlinear oscillator). This means that the linearized natural frequencies satisfy the relation

$$\omega_1 = 3\omega_2 + \varepsilon\sigma_2\,,$$

and the forcing frequency is such that

$$\Omega = \omega_1 + \varepsilon\sigma \ .$$

where σ is an external detuning parameter to express the nearness of Ω to ω_1 and σ_2 is an internal detuning parameter to express the nearness of ω_1 to $3\omega_2$.

According to the method of multiple scales [20], the approximate solutions of the equations are sought in the form:

$$x_1(t;\varepsilon) = x_{10}(T_0,T_1) + \varepsilon x_{11}(T_0,T_1) + O(\varepsilon^2)\ ,$$

$$x_2(t;\varepsilon) = x_{20}(T_0,T_1) + \varepsilon x_{21}(T_0,T_1) + O(\varepsilon^2)\ , \tag{4}$$

where $T_0 = t$ is the so-called fast time scale, and $T_1 = \varepsilon t$ is a slow time scale related to modulations in the amplitude and phase caused by the non-linearity, damping and resonances.

Substituting the approximate solutions (4) into (3) and then balancing the like powers of ε results in the following ordered perturbation equations:

$$\varepsilon^0 \quad D_0^2 x_{10} + \omega_1^2 x_{10} = 0\ , \quad D_0^2 x_{20} + \omega_2^2 x_{20} = \omega_2^2 x_{10}\ , \tag{5}$$

$$\varepsilon \qquad D_0^2 x_{11} + \omega_1^2 x_{11} = -2D_0D_1x_{10} - \mu_1 D_0 x_{10} + m\mu_2 D_0 x_{20} + m\omega_2^2 x_{20} - \alpha_1 x_{10}^3$$

$$- m\alpha_2(x_{10} - x_{20})^3 + f\cos(\Omega T_0)\ ,$$

$$D_0^2 x_{21} + \omega_2^2 x_{21} = -2D_0D_1x_{20} - \mu_2 D_0 x_{20} + \mu_2 D_0 x_{10} + \omega_2^2 x_{11} + \alpha_2(x_{10} - x_{20})^3\ . \tag{6}$$

where $D_0 = \partial / \partial T_0$, $D_0^2 = \partial^2 / \partial T_0^2$, $D_0 D_1 = \partial^2 / \partial T_0 \partial T_1$.

The general solutions of equation (5) can be expressed in complex form as:

$$x_{10} = A\exp(i\omega_1 T_0) + cc\ ,$$

$$x_{20} = FA\exp(i\omega_1 T_0) + B\exp(i\omega_2 T_0) + cc\ , \tag{7}$$

where the amplitudes A and B are unknown functions of the time scale T_1 which will be determined by imposing the solvability conditions. $F = 1/(1-\omega_1^2/\omega_2^2)$, and $'cc'$ stands for the complex conjugates of the preceding terms.

Substituting equation (7) into equation (6) and eliminating the secular terms from the resultant equations for the second-order approximate solutions yields:

$$-2i\omega_1 A' + (-i\mu_1\omega_1 + i\mu_2 m\omega_1 F + mF\omega_2^2)A - 3[\alpha_1 - m\alpha_2(F-1)^3]A^2\overline{A} +$$

$$+ 6m\alpha_2(F-1)A\overline{B}B + m\alpha_2 B^3 e^{-i\sigma_2 T_1} + \frac{1}{2} f e^{i\sigma T_1} = 0\ ,$$

$$-2i\omega_2 B' + -i\mu_2\omega_2 B - 3\alpha_2 B^2\overline{B} - 6\alpha_2(F-1)^2 A\overline{A}B + 3(m-1)(F-1)\alpha_2 A\overline{B}^2 e^{i\sigma_2 T_1} = 0\ , \tag{8}$$

where $\overline{A}$ and $\overline{B}$ are the complex conjugates of A and B, and primes stand for differentiation with respect to the slow time scale T_1. The amplitude functions A and B can be expressed in the polar form as

$$A = \tfrac{1}{2} a\exp(i\beta)\ , \qquad B = \tfrac{1}{2} b\exp(i\theta)\ , \tag{9}$$

where a, b, β and θ are real functions of the time scale T_1.

Substituting equation (9) into equation (8) and then separating real and imaginary parts of the resulting equations lead to the following equations:

$$a' = -c_1 a + s_1 b^3 \sin(\gamma - 3\phi) - s_2 \sin\gamma ,$$

$$a\gamma' = -g_1 a + s_3 a^3 + s_4 a b^2 + s_1 b^3 \cos(\gamma - 3\phi) - s_2 \cos\gamma ,$$

$$b' = -c_2 b + h_1 a b^2 \sin(\gamma - 3\phi) ,$$

$$b\phi' = -g_2 b + h_2 a^2 b + h_3 b^3 - h_1 a b^2 \cos(\gamma - 3\phi) , \tag{10}$$

where $c_1 = \frac{1}{2}(\mu_1 - m\mu_2 F)$, $s_1 = -\frac{m\alpha_2}{8\omega_1}$, $s_2 = \frac{f}{2\omega_1}$, $g_1 = \sigma + \frac{mF\omega_2^2}{2\omega_1}$, $F = -1/8$,

$s_3 = \frac{3[\alpha_1 + m\alpha_2(1-F)^3]}{8\omega_1}$, $s_4 = \frac{3m\alpha_2(1-F)}{4\omega_1}$, $c_2 = \frac{1}{2}\mu_2$, $h_1 = \frac{3(1-m)(1-F)\alpha_2}{8\omega_2}$,

$g_2 = \frac{1}{3}(\sigma + \sigma_2)$, $h_2 = \frac{3\alpha_2(1-F)^2}{4\omega_2}$, $h_3 = \frac{3\alpha_2}{8\omega_2}$, $\gamma = \beta - \sigma T_1$, $\phi = \theta - \frac{1}{3}(\sigma + \sigma_2)T_1$.

The steady-state response of the nonlinear system given by equation (1) corresponds to the constant solutions of a set of four non-linear algebraic equations that can be obtained by letting $a' = b' = 0$ and $\gamma' = \phi' = 0$ in equation (10).

Elimination of the trigonometric terms in the resulting equations leads to the following two nonlinear algebraic equations that determine the amplitudes *a* and *b* of the steady-state response:

$$s_2^2 - c_1^2 a^2 + \frac{2c_1 c_2 s_1 b^2}{h_1} - s_1^2 b^6 - \frac{2s_1(g_2 - h_2 a^2 - h_3 b^2)(g_1 - s_3 a^2 - s_4 b^2)b^2}{h_1} -$$

$$(g_1 - s_3 a^2 - s_4 b^2)^2 a^2 = 0 ;$$

$$h_1^2 a^2 b^4 - c_2^2 b^2 - (g_2 - h_2 a^2 - h_3 b^2)^2 b^2 = 0 , \tag{11}$$

Equation (11) admits two types of solutions that *b* can be zero while *a* is nonzero or both *a* and *b* are nonzero, namely, $a \neq 0$, $b = 0$, and $a \neq 0$, $b \neq 0$, which will be referred to here as uncoupled mode and coupled mode of the amplitudes. For the uncoupled mode amplitudes, equation (11) reduces to

$$s_2^2 - c_1^2 a^2 - (g_1 - s_3 a^2)^2 a^2 = 0 , \tag{12}$$

which is similar to the amplitude equation of frequency-response curve of an uncoupled Duffing's equation under primary resonances[20]. It is worth noting that the coefficients have been changed due to the addition of the absorber. This indicates that the addition of absorber can modify the dynamic behaviour of the primary nonlinear oscillator even in the case of uncoupled mode vibrations.

The first-order approximate solutions for the steady-state response of the primary nonlinear oscillator and absorber, by combining equations (4), (5) and (7), can be written as:

$$x_1 = a\cos(\Omega t + \gamma) + O(\varepsilon) ,$$

$$x_2 = -\tfrac{1}{8} a\cos(\Omega t + \gamma) + b\cos[\tfrac{1}{3}\Omega t + \phi] + O(\varepsilon) , \tag{13}$$

where the amplitudes *a* and *b*, phases γ and ϕ are given by equation (8).

For uncoupled mode vibrations with *b* being zero, the approximate solutions are given by:

$$x_1 = a\cos(\Omega t + \gamma) + O(\varepsilon) ,$$

$$x_2 = -\tfrac{1}{8} a \cos(\Omega t + \gamma) + O(\varepsilon) \ . \tag{14}$$

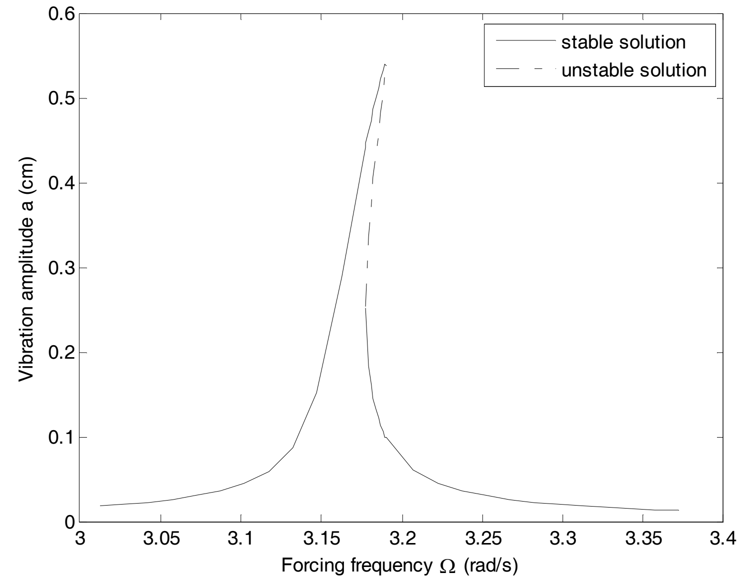
It is easy to note that the steady-state response of the absorber consists of one frequency (forcing frequency) for uncoupled mode vibrations and two frequencies (forcing frequency and one-third of the forcing frequency) for coupled mode vibrations.

The stability of the steady-state solutions can be examined by computing the eigenavlues of the coefficient matrix of the characteristic equation, which can be derived from equation (10) in terms of small disturbances to the steady-state solutions. If the real parts of all the eigenvalues are negative, the steady-state solution is stable. If at least one eignevalue has positive real part, the solution is unstable. A bifurcation occurs when parameters leading to eigenvalues with zero real part.

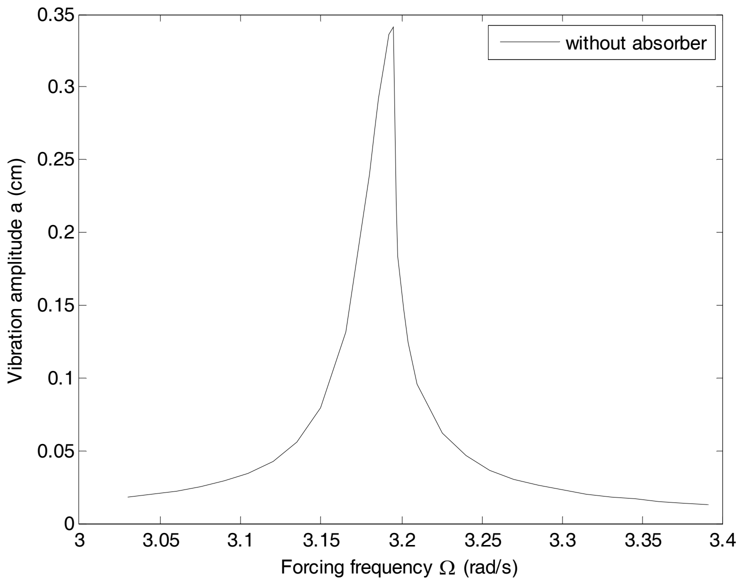
3. Numerical Simulations

Numerical simulations have been performed under the following values of the system parameters; $m_1 = 10.0$ kg , $m_2 = 1.0$ kg , $c_1 = 0.1$ Ns/m , $c_2 = 0.05$ Ns/m , $k_1 = 100.0$ N/m , $k_2 = 8.0$ N/m^3 , $k_3 = 1.1236$ N/m , $k_4 = 0.4$N/m^3 , $f_0 = 0.171$N , unless otherwise specified. The linear stiffness of the absorber is selected based on the three-to-one internal resonance conditions. The linearized natural frequencies of the primary nonlinear oscillator before and after the nonlinear absorber is attached to are $\omega_{10} = 3.16228$ rad/s , $\omega_1 = 3.17999$ rad/s , and the linearized natural frequency of the nonlinear absorber is $\omega_2 = 1.06$ rad/s , respectively. The change in linearized natural frequencies of the primary nonlinear oscillator after and before the nonlinear absorber is attached is by approximately 0.56%. This set of system parameters confirms that the nonlinear absorber can be regarded as a small perturbation to the primary nonlinear oscillator with the mass ratio of m_2 / m_1 being 0.1, the quotient of linear stiffness k_3 / k_1 being 0.011236, and the ratio of nonlinear stiffness k_4 / k_2 being 0.05.

The performance of the nonlinear vibration absorber on vibration suppression of nonlinear oscillator can be shown in the frequency-response curves in the neighbourhood of primary resonances of the primary nonlinear oscillator. The minimum forcing amplitude (i.e. the critical forcing amplitude) that would lead to jumps in the frequency-response curve of the primary nonlinear oscillator without absorber was found to be $f_{0critical} = 0.09008$ N . The frequency-response curve exhibit saddle-node bifurcations, coexistence of three solutions and jump phenomena if the amplitude of excitation is larger than the critical forcing amplitude. Figure 2a shows the frequency-response curve of the primary nonlinear oscillator without absorber. The horizontal axis represents an interval of external detuning $\sigma_{10} \in [-0.15, 0.21]$ rad/s , which corresponds to the interval of forcing frequency $\Omega_{10} \in [3.01228, 3.37228]$ rad/s . Saddle-node bifurcations occur at $\Omega_{10} = 3.1778$ rad/s and 3.19025 rad/s , respectively, where result in jump-up phenomenon from the low-amplitude branch to the high-amplitude branch and jump-down from the high-amplitude branch to the low-amplitude branch. The maximum amplitude of primary resonance vibrations is 0.54038 cm, occurs at $\Omega_{10} = 3.1898$ rad/s .



(a)



(b)

Fig. 2. Frequency-response curves of the primary nonlinear oscillator with and without nonlinear vibration absorber for the amplitude of excitation $f_0 = 0.171$ N.

After the nonlinear absorber is attached to the primary nonlinear oscillator, the resonant vibrations of the primary nonlinear oscillator have been greatly suppressed. Figure 2b shows the frequency-response curve in the region of the frequency of excitation $\Omega_{10} \in [3.02999, 3.38999]$ rad/s. For this combination of system parameters, the amplitude of the absorber *b* is zero and only the amplitude *a* exists. Saddle-node bifurcations, jump and hysteresis phenomena that have appeared in the frequency-response curve of the primary nonlinear oscillator before the nonlinear vibration absorber is attached have been eliminated. The maximum amplitude of vibrations has been reduced to 0.341226 cm. The forcing frequency at which the amplitudes of the primary resonance vibrations reach their maximums has shifted from $\Omega = 3.1899$ rad/s for the primary nonlinear oscillator alone to

$\Omega = 3.19499\,\text{rad/s}$ for the primary nonlinear oscillator attached by nonlinear absorber. This suggests that the nonlinear absorber makes a small change of the forcing frequency interval for the primary resonances of the primary nonlinear oscillator.

4. Conclusion

A weakly nonlinear absorber was found to be effective in suppressing the primary resonance vibrations and eliminating saddle-node bifurcations of the nonlinear oscillator, as the frequency-response curves can be modified by the nonlinear absorber attached. The nonlinear absorber referred in the present paper consists of a relatively light-mass attached to a vibrating nonlinear oscillator by a linear damper and a spring of linear-plus-nonlinear characteristic, from which the light-mass can absorb vibrational energy without significantly modifying the primary nonlinear system and adversely affecting its performance. The linearized natural frequencies of the primary nonlinear oscillator and the absorber are tuned to be under three-to-one internal resonances, as such the linear stiffness of the absorber is much lower than the stiffness of the primary system itself. The vibrations of primary nonlinear system act as an external excitations to excite the vibrations of the absorber oscillator formed by the light mass. Vibrational energy input to the primary system is then transferred to the nonlinear absorber. The absorber can not only effectively suppress the amplitude of oscillations of the primary oscillator, but also eliminate saddle-node bifurcations and jump phenomena.

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