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Non-Commutative Kirillov Method and Coadjoint Orbits

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Certificate of Original Authorship

I, Tianpeng Gou, declare that this thesis, is submitted in fulfilment of the requirements for the award of Doctor of Philosophy, in the Faculty of Science at the University of Technology Sydney.

This thesis is wholly my own work unless otherwise referenced or acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

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Abstract

Élie Cartan and Hermann Weyl's theorem of highest weight states that every irreducible highest weight representation $d\pi$ of a compact real form $\mathfrak{u}$ of a complex semisimple Lie algebra $\mathfrak{g}$, where $\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u}$, integrates to an irreducible highest weight representation π of a compact, simply connected, semisimple Lie group U whose Lie algebra is $\mathfrak{u}$. However, there is no explicit computation method for this correspondence.

Building on the work of Raed Raffoul (2006) [44], we combine the Kirillov orbit method, the sum of adjoint orbits, the convexity theorem for moment maps, Nelson's formula for Weyl calculus, and the transversality condition and induced differential operators, to develop a *non-commutative Kirillov method*. This framework allows us to explicitly compute the exponential of $d\pi$, which corresponds to the lifted representation $\pi \circ \exp$ on $\mathfrak{u}$. It also enables the lifting of invariant vector fields on U, arising from root vectors in $\mathfrak{g}$, to differential operators induced by the adjoint action of U on $\mathfrak{u}$, with respect to the matrix coefficients of $\pi \circ \exp$. Moreover, we show that the Euclidean Fourier transform of $\pi \circ \exp$ on $\mathfrak{u}$ consists of polynomials of transversal differential operators acting on a U-invariant measure supported on the image of the moment map of π, in the dual Lie algebra $\mathfrak{u}^*$. We also extend this method to compute spherical functions on symmetric spaces of the compact type.

“The question for the ultimate foundations and the ultimate meaning of mathematics remains open; we do not know in which direction it will find its final solution nor even whether a final objective answer can be expected at all. ‘Mathematising’ may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalisation.”

Hermann Weyl

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Historical Overview

This thesis has integrated four subjects inspired by the representation theory of semisimple Lie groups, to solve a fundamental problem in the representation theory: Complete the theorem of highest weight by explicitly calculating the integration of an irreducible highest weight representation of the compact real form of a complex semisimple Lie algebra, corresponding to a compact simply connected semisimple Lie group.

In the following paragraphs, we briefly summarise the historical developments and key aspects of these subjects that are related to this thesis.

Representation Theory of Semisimple Lie Groups:

In representation theory, the **theorem of highest weight** classifies the irreducible representations of a complex semisimple Lie algebra $\mathfrak{g}$. It states that every irreducible representation of $\mathfrak{g}$ can be determined by an integral weight in the dual space $\mathfrak{g}^*$. This theorem was proved by Élie Cartan in 1913 [36]. Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$, such that $\mathfrak{g} = \mathfrak{u} + i\mathfrak{u}$. Suppose the corresponding compact semisimple Lie group U is simply connected, we can integrate the highest weight representation $d\pi$ of $\mathfrak{u}$ to obtain the representation π of the group U. We can also take the average over the group U to show that there is an inner product on the representation space V such that V is unitarily invariant under the actions of π of U. We may use the unitarity of π to see that V decomposes as a direct sum of irreducible representations. This is a consequence of **Peter-Weyl theorem** for general compact connected groups, which was proposed by Herman Weyl and his student Fritz Peter in 1927 [42].

The **Borel-Weil-Bott theorem** is a modern interpretation of the theorem of the highest weight. It was proved by Armand Borel and André Weil in the early 1950s [34], and was later extended to higher cohomology groups by Raoul Bott. It constructs an irreducible representation space as global sections of a line bundle on the flag manifold of a complex semisimple Lie group G. The theorem of the highest weight results as a consequence.

The **Kirillov orbit method** is a geometric method that associates each irreducible unitary representation of a Lie group G to an integral coadjoint orbit of G. It realises the representation space as the irreducible holomorphic sections of a line bundle over an integral coadjoint orbit. It was introduced by Alexandre Kirillov in the early 1960s [32]. It generalises the Borel-Weil-Bott theorem for complex semisimple Lie groups. This method has been shown to be able to describe the representations for nilpotent Lie groups, semisimple Lie groups, and semi-direct product of a Lie group with a vector group. Kirillov also derived an analytic version of the character formula of irreducible unitary representations, which is called the **Kirillov character formula**. This formula can be interpreted as the Fourier transform of a distribution supported on an integral coadjoint of G.

The **Cartan-Helgason theorem** (rigorously proved by Sigurdur Helgason in 1970 [27]) for the symmetric space of compact type of the pair (U, K), has inherited and expanded the theorem of the highest weight and the Peter-Weyl theorem, to the space $L^2(U/K)$. The zonal and non-zonal spherical functions can be calculated by the K-invariant vector v_K, which is a projection of the highest weight vector v_λ of a spherical representation π_λ of U of a highest weight λ.

Weyl Functional Calculus:

The **Weyl functional calculus** is a quantisation procedure proposed by Hermann Weyl [53], to associate a quantum observable to a classical observable, and the underlying symmetry group is the Heisenberg group. The Weyl functional calculus can be formulated in an abstract setting. Suppose that $\mathcal{A} = (A_1, ..., A_d)$ is a d-tuple of operators acting in a Banach space V. For each $\xi \in \mathbb{R}^d$, and for some $C > 0$ and $s \geqslant 0$, if the bound $||e^{i\xi\cdot\mathcal{A}}|| \leqslant C(1 + |\xi|)^s$ holds for every $\xi \in \mathbb{R}^d$, then the operator-valued integral

$$f(\mathcal{A}) = \int_{\mathbb{R}^d} \widehat{f}(\xi)\, e^{i\xi\cdot\mathcal{A}}\, d\xi,$$

is convergent, bounded and well defined for every $f \in \mathcal{S}(\mathbb{R}^d)$ (Schwartz spaces). The operators $A_1, ..., A_d$ do not necessarily commute.

By the Paley–Wiener-Schwartz theorem, the Weyl functional calculus given by $\mathcal{W}_{\mathcal{A}}(f) = f(\mathcal{A})$ is an operator-valued distribution with compact support if and only if there exists numbers $C, s, r \geqslant 0$ such that $||e^{i\xi\cdot\mathcal{A}}|| \leqslant C(1+|\xi|)^s e^{r|Im(\xi)|}$ for all $\xi \in \mathbb{C}^d$. Let $\mathcal{A}$ be a d-tuple of bounded self-adjoint operators, Michael Taylor [50] showed that such distributions exist for $C = 1, s = 0$ and $r = \sum_{j=1}^{d} ||A_j||^2$.

In this setting, the Weyl functional calculus has been developed by Robert Anderson [5], Edward Nelson [40] and Ernest Albrecht [3]. Nelson has introduced the concept of 'operant' and derived an explicit formula of $\mathcal{W}_{\mathcal{A}}$ for the finite-dimensional self-adjoint operator $\mathcal{A}$, with operator norm $||e^{i\xi\cdot\mathcal{A}}|| = 1$. $\mathcal{W}_{\mathcal{A}}$ is an operator of differential operators acting on the unitarily invariant measure of the joint numerical range of $\mathcal{A}$. Brian Jefferies [31] gave an alternative proof for this formula of $\mathcal{W}_{\mathcal{A}}$. Raed Raffoul [44] applied the Weyl functional calculus and formula for $\mathcal{W}_{\mathcal{A}}$ to study the unitary representations of compact connected Lie groups.

Convexity Theorem of the Image of Moment Map:

Let G be a connected Lie group, $\mathfrak{g}^*$ the dual of its Lie algebra $\mathfrak{g}$. A symplectic manifold (M, ω) with the G action preserving ω, equipped with a moment map $\Psi : M \to \mathfrak{g}^*$, is called a **$G$-Hamiltonian manifold** [7]. The fundamental problem of studying Hamiltonian manifolds is to compute the image of the moment map. Of particular interest is the determination of the convexity of the image of the moment map.

A coadjoint orbit $\mathcal{O}$ is naturally a G-Hamiltonian manifold. Let $\mathfrak{g}$ be the Lie algebra of a compact connected semisimple Lie group G, $\mathfrak{t}^*$ be the dual of a Cartan subalgebra $\mathfrak{t}$, and let $p : \mathfrak{g} \to \mathfrak{t}$ be the orthogonal projection map. Let $\mathcal{O}_\lambda$ be a coadjoint orbit passing $\lambda \in \mathfrak{t}^*$. Bertram Kostant [38] showed that the orthogonal projection $p(\mathcal{O}_\lambda)$ is equal to the the convex hull of $W \cdot \lambda$, where W is the Weyl group of $\mathfrak{t}$. The *convexity theorem* of a compact G-Hamiltonian manifold was proposed by Victor Guillemin and Shlomo Sternberg [25], and Frances Kirwan [35]. Let $\mathfrak{t}^*_+$ be the chosen fundamental dual Weyl chamber in $\mathfrak{t}^*$, the convexity theorem says the image $\Psi(M) \cap \mathfrak{t}^*_+$ is a convex polytope. Anthony Dooley and Norman Wildberger [18] showed that the sumset of a pair of coadjoint orbits $(\mathcal{O}_\lambda + \mathcal{O}_\eta) \cap \mathfrak{t}^*_+$ is also a convex polytope. Norman Wildberger [55] also showed that the image of the moment map of a finite-dimensional unitary representation π_λ of G of a highest weight λ, is contained in the convex hull of the integral coadjoint orbit $\mathcal{O}_\lambda$. This image is convex if and only if λ is root distinct.

Invariant Differential Operators:

The **Duflo isomorphism** [20], denoted by Φ, is an isomorphism between the centre of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the finite-dimensional Lie algebra $\mathfrak{g}$, which can be identified as the algebra of bi-invariant vector fields on its Lie group G, and the invariant elements of the symmetric algebra $\mathcal{S}(\mathfrak{g})$ of $\mathfrak{g}$, which can be identified as the G-invariant differential operators on $\mathfrak{g}$ with constant coefficients. Let μ, ν be the fundamental solutions to the G-invariant differential operators $\partial p, \partial q$ on $\mathfrak{g}$, supported at the origin of $\mathfrak{g}$. Then $\Phi(\mu), \Phi(\nu)$ are the fundamental solutions to the bi-invariant vector fields $\Phi(\partial p), \Phi(\partial q)$ on G.

Anthony Dooley and Norman Wildberger [19] showed that for a compact connect Lie group G, Φ is a homomorphism (also known as the 'wrapping map') between the Banach algebra of central distributions on G and the Banach algebra of G-invariant distributions on $\mathfrak{g}$ with compact support. The multiplication of these Banach algebras are the convolution operations on G and $\mathfrak{g}$, respectively. They also showed that the Kirillov character formula for compact connected semisimple Lie groups is a consequence of this homomorphism.

François Rouvière [45] developed a 'twisted' convolution for Riemannian symmetric spaces by introducing the e-function. It is related to the sectional curvature of the symmetric spaces. Anthony Dooley [16] derived an explicit formula for the e-function for compact symmetric spaces.

Sigurdur Helgason [27] introduced the concepts of transversality condition of a submanifold K of a G-vector space V and G-induced differential operator on V. That is, every differential operator on V can be decomposed as a polynomial of the transversal part and the G-induced part. When V is a semisimple Lie algebra $\mathfrak{g}$, K is the submanifold of the regular elements in the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. By implementing the transversality condition into the G-invariant differential operators on $\mathfrak{g}$ with constant coefficients, he derived a continuous version of the Kirillov character formula.

Chapter 1

Introduction

1.1 Background

The Kirillov orbit method [34], introduced by Russian mathematician A.A. Kirillov in the early 1960s, is a method to associate an integral coadjoint orbit $\mathcal{O}$ in the dual Lie algebra $\mathfrak{g}^*$ of a connected Lie group G, to an irreducible unitary representation π of G. He observed that each integral coadjoint orbit is naturally a symplectic manifold (which carries a symplectic 2-form) with G acting as a group of symplectomorphisms of $\mathcal{O}$. He also conjectured that the representation space of π can be realised as the irreducible polarised G-invariant holomorphic sections of the line bundle E on $\mathcal{O}$, denoted by $\Gamma(E)$. Let $\Gamma^2(E)$ denote the space $\Gamma(E)$ with a suitable L^2 condition implemented. Then the complex lines in $\Gamma^2(E)$ form a projective space of $\Gamma^2(E)$, denoted by $P\Gamma^2(E)$, which also carries a symplectic structure, and the corresponding 2-form is the differential of the moment map $P\Gamma^2(E) \to \mathfrak{g}^*$. Kirillov also proved an analytic version of the character formula for π, which says the product of an analytic square root of the Jacobian of the exponential map, the j-function, and the lift by the exponential map of a character formula χ of an integral weight is equal to the inverse Fourier transform of the Liouville measure μ over an integral coadjoint orbit $\mathcal{O}$ [33]. That is,

$$j(X)\chi_\lambda(\exp X) = \int_{\mathcal{O}_{\lambda+\delta}} e^{i\beta(X)}\, d\mu_{\lambda+\delta}(\beta), \quad X \in \mathfrak{g}. \tag{1.1.1}$$

When G is compact connected and semisimple, the Kirillov character formula is exact, and the integral weight of the character is a highest dominant weight λ and the integral coadjoint orbit is $\mathcal{O}_{\lambda+\delta}$, where δ is the half sum of all positive roots of the root system of the complexification $\mathfrak{g}^{\mathbb{C}}$ [33]. Also, by the Borel-Weil theorem, every $\Gamma^2(E)$ is the representation space of the finite-dimensional irreducible unitary representation π_λ. Let I_λ denote the image of the moment map of π_λ, which is also called the moment set of π_λ. Then I_λ is contained in the convex hull of $\mathcal{O}_\lambda$. A dominant weight λ is called *root distinct* if the set of the pairwise differences between λ and an element of the orbit of the Weyl group of λ does not contain any roots of $\mathfrak{g}$. Hence, a moment set I_λ is convex and is exactly the convex hull of $\mathcal{O}_\lambda$ if and only if λ is root distinct [55].

By following the Kirillov orbit method for compact connected semisimple Lie groups, a concrete construction and calculation for the irreducible unitary representations π of the simplest non-commutative compact Lie group $SU(2)$, can be formulated by combining the methods in [15], [29], [21]. $SU(2)$ can be parametrised by

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Each π is characterised by a positive integer n, and each matrix coefficient of π_n is a homogeneous polynomial of complex variables $(\alpha, \beta) \in \mathbb{C}^2$ for $|\alpha|^2 + |\beta|^2 = 1$. The Kirillov character formula $j \cdot \chi_n$ is equal to the inverse Fourier transform of the Liouville measure ν_{n+1} supported on the 2-sphere in the Lie algebra $\mathfrak{su}(2) \cong \mathbb{R}^3$ with radius $n+1$.

What if one replaces the character in $j \cdot \chi_n$ by an arbitrary matrix coefficient $\pi^n_{i,j} \circ \exp$ of $SU(2)$? After a close examination of the calculation of the Kirillov character formula $j \cdot \chi_n \circ \exp$ on $\mathfrak{su}(2)$, Franco Cazzaniga [9] found a closed form for the exponential of the standard form of $\mathfrak{su}(2)$, which also gives explicit calculations for $\alpha \circ \exp$ and $\beta \circ \exp$. He further noticed that the Fourier transform of $\alpha \circ \exp$ and $\beta \circ \exp$ are given by

$$(\alpha \circ \exp)^\wedge = \left(-r\frac{\partial}{\partial r} - 2 + \frac{\partial}{\partial \xi_3} \right) \nu_1, \quad (\beta \circ \exp)^\wedge = -i \left(\frac{\partial}{\partial \xi_1} + i \frac{\partial}{\partial \xi_2} \right) \nu_1, \tag{1.1.2}$$

where $\{\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial \xi_3}\}$ are differential operators in the dual Lie algebra $\mathfrak{su}(2) \cong \mathbb{R}^3$ and the non-constant-coefficient radial differential operator $r\frac{\partial}{\partial r} = \xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2} + \xi_3 \frac{\partial}{\partial \xi_3}$, and ν_1 is the surface measure of the unit 2-sphere in $\mathbb{R}^3$. The inverse

Fourier transform of ν_1 is given by

$$\check{\nu}_1(x) = \frac{\sin\sqrt{x_1^2+x_2^2+x_3^2}}{\sqrt{x_1^2+x_2^2+x_3^2}}, \quad x \in \mathbb{R}^3, \tag{1.1.3}$$

which is rotationally invariant. Because every matrix coefficient of the irreducible unitary representation π_n is a homogeneous polynomial of degree n of the products of α and β, and the Fourier transform of j, denoted by $\widehat{j}$, is the Liouville measure $\nu_\delta = \nu_1$. Therefore, $\left(j \cdot \pi_{i,j}^{(n)} \circ \exp\right)^{\wedge} = D_{i,j}^n (\nu_1)^{*n+1}$, where $(\nu_1)^{*n+1}$ is the $(n+1)$-fold convolution of ν_1, and each $D_{i,j}^n$ is a polynomial of differential operators in $\{r\frac{\partial}{\partial r}, \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial \xi_3}\}$. It is well known that the support of the k-fold convolution of the surface measure of the unit 2-sphere is the ball with radius k, and Cazzaniga also proved that the singular support of $(\nu_1)^{*n+1}$ is exactly the spheres with integer radius $\{n+1, n+1-2, n+1-4, ...\}^+$, and interestingly, this set coincides with all positive integer weights of π_{n+1}.

This novel finding for $SU(2)$ has led to the possibility of describing and calculating matrix coefficients of general compact connected semisimple Lie groups as distributions in $\mathfrak{g}^*$.

Raed Raffoul [44], [17], and Anthony Dooley [17] extended Cazzaniga's results on $SU(2)$ to general compact connected semisimple Lie groups, through an exposition of Weyl calculus [5] and the concept of operants introduced by Edward Nelson [40]. Let $\mathcal{L}(\mathcal{H})$ be a finite-dimensional Banach algebra of bounded operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{A} = (A_1, ..., A_d)$ be a d-tuple of the Hermitian operator in $\mathcal{L}(\mathcal{H})$, and let $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$. Then the Weyl calculus of $\xi \cdot \mathcal{A}$, denoted by $\mathcal{W}_{\xi\cdot\mathcal{A}}$, is a $\mathcal{L}(\mathcal{H})$-valued distribution on a Schwartz space, which is also the Fourier transform of $e^{i\xi\cdot\mathcal{A}}$. Nelson derived an explicit formula for $\mathcal{W}_{\xi\cdot\mathcal{A}}$, which is an operator of polynomials of constant-coefficient differential operators in $\{\frac{\partial}{\partial \xi_1}, ..., \frac{\partial}{\partial \xi_d}\}$ and a non-constant-coefficient differential operator $\xi\frac{\partial}{\partial \xi} = \xi_1\frac{\partial}{\partial \xi_1} + \cdots + \xi_d\frac{\partial}{\partial \xi_d}$ (which generalises the radial differential operator $r\frac{\partial}{\partial r}$ in Cazzaniga's results), acting on an unitarily invariant measure μ supported on the joint numerical range of $\xi \cdot \mathcal{A}$. This result can naturally be applied to the study of unitary representations, because when G is compact, simply connected and semisimple, the exponential of a skew-Hermitian irreducible representation $d\pi_\lambda$ of the Lie algebra $\mathfrak{g}$ of the highest weight λ is exactly the unitary representation π_λ of G. Let ν_δ be the Liouville measure of the coadjoint orbit of δ element, and $\mathcal{W}_{d\pi_\lambda}$ be the Weyl calculus of $d\pi_\lambda$, Raffoul proved that the convex hull of the support of $\nu_\delta * \mathcal{W}_{d\pi_\lambda}$ is equal to the convex hull of the coadjoint orbit

$\mathcal{O}_{\lambda+\delta}$, and the singular support of $\nu_\delta * \mathcal{W}_{d\pi_\lambda}$ is equal to $\bigcup_{w\in W}\mathcal{O}_{\lambda+w\,\delta}$, which is the union of the coadjoint orbits of λ shifted by the Weyl orbit of δ.

1.2 Research Questions and Challenges

We aim to extend the work of Cazzaniga, Raffoul, and Dooley to symmetric spaces of the compact type. For example, their approaches should enable the derivation of distributions associated with the spherical functions of the symmetric space $SU(2)/SO(2)$, which can be derived from the matrix coefficients of the irreducible unitary representation π_n of $SU(2)$ with respect to an even positive integer n, by the Cartan-Helgason theorem. However, during literature review and explicit calculations for $SU(2)/SO(2)$, several issues were encountered that prevent this extension. In the following, we summarise five major **Issues**:

1. In Cazzaniga's calculations, the Fourier transform of $\alpha \circ \exp$ contains a term $r\frac{\partial}{\partial r}\nu_1$. This means that the Fourier transform of $(\alpha \circ \exp)^n$ in the irreducible unitary representation $\pi_n \circ \exp$ contains a n-fold convolution $\left(r\frac{\partial}{\partial r}\nu_1\right)^{*n}$. Note that the non-constant-coefficient differential operator $r\frac{\partial}{\partial r}$ does not commute with the convolution operation, for example, $r\frac{\partial}{\partial r}\nu_1 * r\frac{\partial}{\partial r}\nu_1 \neq r\frac{\partial}{\partial r}r\frac{\partial}{\partial r}(\nu_1 * \nu_1)$. Hence, these are not the same distributions. However, Cazzaniga claimed that each matrix coefficient of $SU(2)$ is explicitly associated with a n-fold convolution $(\nu_1)^{*n}$. His result is convincing because he showed that the singular support of $(\nu_1)^{*n}$ exactly matches the weights of π_n.

2. By combining the orthonormal basis of the Hilbert space $\mathcal{H}_n$ of π_n derived in [21] and the method in [30], we can explicitly calculate the skew-Hermitian irreducible representations of $\mathfrak{su}(2)$. We put these representations into Nelson's explicit formula for the Weyl calculus $\mathcal{W}_{d\pi_n}$ and use $(\nu_1)^{*n}$ as the associated measure. However, the solutions become inconsistent when $n > 1$. For instance, the inverse Fourier transform of the Weyl calculus $\mathcal{W}_{d\pi_n}$ is not equal to $\pi_n \circ \exp$ in Cazzaniga's result.

3. Both Cazzaniga and Raffoul did not provide a general formula for the G-invariant measures associated with the irreducible unitary representations. However, Cazzaniga indicated that these measures are a n-fold convolution of a single coadjoint orbit for $SU(2)$. Does this also apply

to the general compact connected semisimple Lie groups? Also, Raffoul showed these measures are associated with the moment sets of irreducible unitary representations. Is there a general formula for these measures on the moment sets of irreducible unitary representations? The answers to these questions were not clear.

4. There is a lack of description of the relationships among the matrix coefficients of an irreducible unitary representation π. Let G be a compact simply connected semisimple Lie group, $\mathfrak{g}$ be the Lie algebra of $\mathfrak{g}$, π_λ be an irreducible unitary representation of G of highest weight λ, and $d\pi_\lambda$ be the infinitesimal version of π_λ. Since G is simply connected, $e^{d\pi_\lambda} = \pi_\lambda \circ \exp$, and they share the same representation space V and weight space classification. Also, if we fix a basis for V, then the matrix coefficients of each row or each column of π_λ form a basis of a copy of V [21]. By the theorem of highest weight, we can also choose a highest weight vector in a row or column of π_λ, and successively apply the invariant vector fields on G induced by the negative root vectors (lowering operators) in $\mathfrak{g}^{\mathbb{C}}$ to generate the entire row and column. Now, we have Nelson's formula for $\pi \circ \exp$. So, can we lift these vector fields that act on the matrix coefficient $\pi^\lambda_{i,j}$ on G to some differential operators that act on $\pi^\lambda_{i,j} \circ \exp$ on $\mathfrak{g}$?

5. Finally, the construction of irreducible highest weight representations $d\pi_\lambda$ of the compact real form of a complex semisimple Lie algebra can easily follow from the highest weight method [30], but the $d\pi_\lambda$ are not in general skew-Hermitian. This means that we may not apply Nelson's formula and explicitly work out the corresponding unitary representations $e^{d\pi_\lambda}$ of general compact simply connected semisimple Lie groups. This is another difficulty.

In this thesis, we resolve these issues.

1.3 Results

We present our new findings and main results. We let G be a compact simply connected semisimple Lie group, $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$, $\mathfrak{t}, \mathfrak{t}^*$ be a (dual) Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{g}^*$, respectively, and W be the (dual) Weyl group of $\mathfrak{t}$, $\mathfrak{t}^*$. In addition, we let $\mathfrak{t}^*_+$ be the fundamental dual Weyl chamber of $\mathfrak{t}^*$, $\mathfrak{t}^*_0$ be the interior of $\mathfrak{t}^*_+$, Φ be the set of roots o f $\mathfrak{g}$, Φ^+ be the subset of positive roots of Φ, Λ be the set of weights of $\mathfrak{t}^*$, Λ^+ be the set of dominant weights of Λ in $\mathfrak{t}^*_+$, and δ be the half-sum of all positive roots in Φ^+. In addition, we let $p : \mathfrak{g}^* \to \mathfrak{t}^*$ be the projection map with respect to the Killing form of $\mathfrak{g}^*$.

Based on the work of A. Dooley and N. Wildberger [18], which extensively studied the convolution structures of coadjoint orbits, we present a new formula for the convolution structure of coadjoint orbits of $SU(2)$ and also determine the singular support of these convolutions.

Result 1 (Proposition 4.4.1) *Let $G = SU(2)$, $\mathfrak{g} \cong \mathbb{R}^3$ be the Lie algebra of G, and $\mathfrak{t} \cong \mathbb{R}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $n \in \mathbb{Z}^+$ be an integer dominant weight of $SU(2)$, ν_1 be the Liouville measure of the coadjoint orbit $\mathcal{O}_1$ in the dual Lie algebra $\mathfrak{g}^* \cong \mathbb{R}^3$ with $\nu_1(\mathcal{O}_1) = 1$ (for $n = 1$). Denote the n-fold convolution of ν_1 by $(\nu_1)^{*(n)}$. Then $(\nu_1)^{*(n)}$ can be expressed as*

$$(\nu_1)^{*(n)} = \int_0^\infty \sum_{j=0}^{1} (-1)^j e_{(-1)^j} * \left((\nu_1^p)^{(*(n-1))} \right)(\lambda'') \cdot \nu_{\lambda''}\, d\lambda'', \tag{1.3.1}$$

where e_1 is the unit point mass function at 1, ν_1^p is the projection of ν_1 with respect to p, and

$$(\nu_1^p)^{*(n-1)} = \int_0^\infty \sum_{j=0}^{1} (-1)^j e_{(-1)^j} * \left((\nu_1^p)^{*(n-2)} \right)(\lambda'') \cdot \nu^p_{\lambda''}\, d\lambda'', \tag{1.3.2}$$

where

$$\nu^p_{\lambda''} = \sum_{j=0}^{1} (-1)^j e_{(-1)^j \lambda''} * F, \tag{1.3.3}$$

and F is the arc-length measure on $(0, \infty) \subset \mathbb{R}$. The support and singular support of $(\nu_1)^{*(n)}$ are:

$$\mathit{supp}\left((\nu_1)^{*(n)} \right) = \mathit{conv}\left(\mathcal{O}_n \right), \qquad \mathit{singsupp}\left((\nu_1)^{*(n)} \right) = \bigcup_{j=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{O}_{n-2j}, \tag{1.3.4}$$

where $\lfloor \cdot \rfloor$ is the floor function. So, the singular support of $(\nu_1)^{*(n)}$ is the union of coadjoint orbits of integers which coincides with the weights of the irreducible unitary representation π_n of $SU(2)$.

This singular support is identical to Cazzaniga's results for $SU(2)$. The formula (1.3.1) also generalises Cazzaniga's formula for the n-fold convolution of the surface measure of the unit 2-sphere in $\mathbb{R}^3$ and can be extended to arbitrary integer dominant weights of $SU(2)$.

For the defining representation π of $SU(n)$, the moment set I of π is exactly a single integral coadjoint orbit. For example, the moment set I_1 of π_1 of $SU(2)$ is exactly the unit sphere in $\mathbb{R}^3$. However, this analogue is not true for an arbitrary dominant weight of a compact connected semisimple Lie group. We wish to examine whether the measure $(\nu_1)^{*n}$ can describe the moment set I_n of π_n of $SU(2)$. So, we propose that there exists a G-invariant probability measure ν_{I_λ}, which is the pushforward of the unitarily probability measure of the unit sphere of the Hilbert space $\mathcal{H}_\lambda$, supported on the moment set $I_\lambda \subset \mathfrak{g}^*$ of the irreducible unitary representation π_λ of a highest weight $\lambda \in \Lambda^+$ of a compact simply connected semisimple Lie group G. In addition, there also exists a normalisation formula for the $\mathfrak{g}$-Fourier transform of ν_{I_λ}.

Result 2.1 (Proposition 5.2.2) *Let G be a compact simply connected semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$, T be a maximal torus of G, $\mathfrak{t}$ be the Lie algebra of T (Cartan subalgebra), and $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. Let $\lambda \in \Lambda^+ \subset \mathfrak{t}^*$ be a dominant highest weight. Let π_λ be a d-dimensional irreducible unitary representation of G of λ acting on a Hilbert space $\mathcal{H}_\lambda$, $d\pi_\lambda$ be its infinitesimal representation. Let $\Pi(\lambda) = \{\lambda_1, ..., \lambda_n\}$ be the set of weights of π_λ such that the multiplicities of the weights satisfy $\sum_{i=1}^{n} multi(\lambda_i) = d$. Let $p : \mathfrak{g}^* \to \mathfrak{t}^*$ be the orthogonal projection map with respect to the Killing form on $\mathfrak{g}^*$. Let $P\mathcal{H}_\lambda$ be the projective space of $\mathcal{H}_\lambda$ which can be identified by the unit sphere $\Omega_d = \{u \in \mathcal{H} : |u| = 1\}$, where $u = \sum_{j=1}^{d} u_j$ such that $\sum_{j=1}^{d} |u_j|^2 = 1$, $\{u_j\}_{j=1}^{d}$ is an orthonormal basis of $\mathcal{H}_\lambda$, and each u_j is also the weight vector of a weight $\lambda_j \in \Pi(\lambda)$. We define the moment set of π_λ as $I_\lambda = \{\langle d\pi_\lambda(H)u, u\rangle : u \in \Omega_d\}$ and a continuous, adjoint invariant function $\tilde{\Psi}_d$ on $\mathfrak{t}$ by*

$$\tilde{\Psi}_d(H) = \int_{\Omega_d} e^{\langle d\pi_\lambda(H)u,u\rangle}\, d\nu(u) = \int_{\Omega_d} e^{\sum_{j=1}^{d} i|u_j|^2\lambda_j(H)}\, d\nu(u), \qquad \forall H \in \mathfrak{t}. \quad (1.3.5)$$

where ν is the probability unitarily invariant measure on Ω_d. Suppose we let ν_{I_λ} be the probability G-invariant measure on the moment set I_λ in $\mathfrak{g}^*$ (pushforward of ν). Then

$$\tilde{\Psi}_d(H) = \int_{I_\lambda} e^{i\beta(H)}\, d\nu_{I_\lambda}(\beta) = \int_{conv(\Pi(\lambda))} e^{i\beta(H)}\, d\nu^p_{I_\lambda}(\beta), \qquad \forall H \in \mathfrak{t}, \qquad (1.3.6)$$

where $\nu^p_{I_\lambda}$ is the projection of ν_{I_λ} with respect to p.

So, the projection measure $\nu^p_{I_\lambda}$ is supported on the convex hull of the set of weights $\Pi(\lambda)$, and the $\mathfrak{g}$-Fourier transform of ν_{I_λ} restricted to $\mathfrak{t}$, denoted by $(\nu_{I_\lambda})^{\vee\mathfrak{g}}|_{\mathfrak{t}}$, is equal to the the $\mathfrak{t}$-Fourier transform of $\nu^p_{I_\lambda}$, denoted by $(\nu^p_{I_\lambda})^{\vee\mathfrak{t}}$. Therefore, if we can find the explicit formula for $(\nu^p_{I_\lambda})^{\vee\mathfrak{t}}$, then we can compare it with the convolution measure $((\nu^p_1)^{*n})^{\vee\mathfrak{t}}$ of $SU(2)$. Inspired by the work of Raffoul [44] and Nelson [40], we have found an explicit formula for $(\nu^p_{I_\lambda})^{\vee\mathfrak{t}}$.

Result 2.2 (Proposition 5.2.3) *Let G be a compact simply connected semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$, T be a maximal torus of G, $\mathfrak{t}$ be the Lie algebra (Cartan subalgebra) of T and $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. Let $\lambda \in \Lambda^+ \subset \mathfrak{t}^*$ be a dominant highest weight. Let π_λ be a d-dimensional irreducible unitary representation of G of highest weight λ acting on a Hilbert space $\mathcal{H}_\lambda$, $d\pi_\lambda$ be its infinitesimal representation. Let $\Pi(\lambda) = \{\lambda_1, \ldots, \lambda_n\}$ be the set of weights of π_λ such that the multiplicities of the weights satisfy $\sum_{i=1}^n multi(\lambda_i) = d$. Let $P\mathcal{H}_\lambda$ be the projective space of $\mathcal{H}_\lambda$ which can be identified with the unit sphere $\Omega_d = \{u \in \mathcal{H} : |u| = 1\}$, where $u = \sum_{j=1}^d u_j$ such that $\sum_{j=1}^d |u_j|^2 = 1$, $\{u_j\}_{j=1}^d$ is an orthonormal basis of $\mathcal{H}_\lambda$, and each u_j is also the weight vector of the weight $\lambda_j \in \Pi(\lambda)$. Suppose the multiplicity of each weight $\lambda_j \in \Pi(\lambda)$ is exactly 1, and all vectors in $\{u_j\}_{j=1}^d$ have* ***distinct weights*** *(in this case $d = n$). Then, the $\mathfrak{t}$-Fourier transform of the projection measure $\nu^p_{I_\lambda}$ is given by*

$$\left(\nu^p_{I_\lambda}\right)^{\vee\mathfrak{t}}(H) = \tilde{\Psi}_n(H) = (-1)^{n-1}(n-1)! \sum_{k=1}^n \frac{e^{i\lambda_k(H)}}{\prod_{\substack{j\neq k \\ 1\leqslant j\leqslant n}} i(\lambda_j - \lambda_k)(H)}, \qquad (1.3.7)$$

for $H \in \mathfrak{t}$.

The normalisation formula (1.3.7) is for the case of π_λ having distinct weights and $multi(\lambda_i) = 1$ for each $\lambda_i \in \Pi(\lambda)$. Suppose there exists a $\lambda_j \in \Pi(\lambda_j)$ such that $multi(\lambda_j) > 1$, then the right-hand side of (1.3.7) is undefined. However, in (1.3.5), the integral formula on the right-hand side is still continuous and integrable. By the formula for the projection measure ν^p_λ for a coadjoint orbit $\mathcal{O}_\lambda$ in [18], we have realised that each $1/i(\lambda_j - \lambda_k)(H)$ in (1.3.7) is a principal value distribution in $\mathfrak{t}$ and its Fourier transform is an arc-length measure $F_{\lambda_j-\lambda_k}$ along the ray of a linear combination of root vectors in Φ. Hence, when the multiplicity of a weight is greater than 1, there exists an arc-length measure $\lim_{\lambda''\to\lambda'} F_{\lambda''-\lambda'} = e_0$, which is the Dirac delta function in $\mathfrak{t}^*$. This means the principal value distribution $\lim_{\lambda''\to\lambda'} 1/i(\lambda'' - \lambda')(H) = 1$. Based on this observation, we present an explicit formula for $\nu^p_{I_\lambda}$.

Result 2.3 (Proposition 5.2.5) Let G be a compact simply connected semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$, T be a maximal torus of G, $\mathfrak{t}$ be the Lie algebra (Cartan subalgebra) of T, $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$, and W be the Weyl group of $\mathfrak{t}, \mathfrak{t}^*$. Let $\lambda \in \Lambda^+ \subset \mathfrak{t}^*$ be a dominant highest weight. Let π_λ be a finite-dimensional irreducible unitary representation of G of highest weight λ acting on a Hilbert space $\mathcal{H}_\lambda$. Let $\Pi(\lambda)$ be the set of weights of the irreducible unitary representation π_λ, d_λ be the dimension of π_λ. Let $m_\lambda(\mu)$ be the multiplicity of the weight $\mu \in \Pi(\lambda)$. Let $D = \Pi(\lambda) \cap \Lambda^+$ be the subset of dominant weights in $\Pi(\lambda)$ with cardinality $|D| = q$. Let I_λ be the moment set of π_λ. Then, the projection of the G-invariant probability measure of the moment set I_λ on $\mathfrak{t}^*$, denoted by $\nu^p_{I_\lambda}$, is given by

$$\nu^p_{I_\lambda} = (-1)^{d_\lambda - 1}(d_\lambda - 1)! \sum_{k=1}^{q} \sum_{w \in W/W_{\lambda_k}} m_\lambda(\lambda_k) e_{w\lambda_k} * \prod_{\substack{j=1 \\ j \neq k}}^{q} \prod_{w' \in W/W_{\lambda_j}} * \left(F_{w'\lambda_j - w\lambda_k}\right)^{*\, m(\lambda_j)}, \tag{1.3.8}$$

where each dominant weight λ_k, λ_j lie in D, and e_{λ_k} is the unit point mass at λ_k, F_{λ_k} is the arc-length measure along the ray of λ_k in $\mathfrak{t}^*$, and $w'\lambda_j - w\lambda_k = \sum_{i=1}^{l} t_i \alpha_i$, for $\alpha_i \in \Delta$, $t_i \in \mathbb{Z}$, and W_{λ_k} is the subgroup of W that stabilises λ_k.

Therefore, by the normalisation formula for $\left(\nu^p_{I_\lambda}\right)^{\vee \mathfrak{t}}$ in (1.3.7), we can compare it with $((\nu^p_1)^{*n})^{\vee \mathfrak{t}}$ of $SU(2)$. We have proved that $\left(\nu^p_{I_n}\right)^{\vee \mathfrak{t}}$ indeed is equal to $((\nu^p_1)^{*n})^{\vee \mathfrak{t}}$. Thus $(\nu_{I_n})^{\vee \mathfrak{g}}|_{\mathfrak{t}} = ((\nu_1)^{*n})^{\vee \mathfrak{g}}|_{\mathfrak{t}}$. Since they are both G-invariant, we have $\nu_{I_n} = (\nu_1)^{*n}$.

Result 2.4 (Proposition 5.2.4) *Let $G = SU(2)$, T be a maximal torus of G, $\mathfrak{g}, \mathfrak{t}$ be their Lie algebras, respectively, and $\mathfrak{g}^*, \mathfrak{t}^*$ be the duals of $\mathfrak{g}, \mathfrak{t}$. Let $n \in \Lambda^+ \cong \mathbb{Z}^+$ be an integer dominant weight, π_n be a $(n+1)$-dimensional irreducible unitary representation with respect to n, and I_n be the moment set of π_n. Also, let $\mathcal{O}_1$ be the coadjoint orbit of the integer 1, and $\Pi(n)$ be the set of the weights of π_λ. Denote $(\nu_1)^{*n}$ the n-fold convolution of the (unique) G-invariant Liouville measure of $\mathcal{O}_1$, and ν_{I_n} the G-invariant probability measure of I_n. Then,*

$$\nu_{I_n} = (\nu_1)^{*n}. \tag{1.3.9}$$

Let $n \geqslant 1$, $\Pi(n) = \{n, n-2, ..., -(n-2), -n\}$, *we have*

$$(\nu_{I_n})^{\vee\mathfrak{g}}|_{\mathfrak{t}}(H) = (-1)^n\, n! \sum_{k\in\Pi(n)} \frac{e^{ikH}}{\prod_{\substack{j\in\Pi(n)\\ j\neq k}} i(j-k)H} = \left(\frac{\sin(H)}{H}\right)^n = ((\nu_1)^{*n})^{\vee\mathfrak{g}}|_{\mathfrak{t}}(H) \tag{1.3.10}$$

for $H \in \mathfrak{t} \cong \mathbb{R}$.

This is an important observation of this project. So far, this correspondence in describing a moment set by convolutions of coadjoint orbits has only been found for $SU(2)$. We have resolved **Issue** 3. Additionally, based on the techniques developed in [18], we have also proved the density function for the intersection between the moment set and the fundamental Weyl chamber $I_\lambda \cap \mathfrak{t}^+$.

Let $\lambda \in \Lambda^+ \subset \mathfrak{t}^*_+$ be a highest weight, I_λ be the moment set of λ. Then the G-invariant measure of the moment set I_λ, denoted by ν_{I_λ}, is given by

$$\nu_{I_\lambda} = \int_{\mathfrak{t}^*_+} \varphi(\lambda, \lambda'')\nu_{\lambda''}\, d\lambda'', \tag{1.3.11}$$

where $\nu_{\lambda''}$ is the unique probability G-invariant measure of the coadjoint orbit $\mathcal{O}_{\lambda''}$, and $\varphi(\lambda, \lambda'')$ is a density function supported on $I_\lambda \cap \mathfrak{t}^*_+$, which relies on the choice of λ. Then

$$\nu^p_{I_\lambda} = \int_{\mathfrak{t}^*_+} \varphi(\lambda, \lambda'')\nu^p_{\lambda''}\, d\lambda''. \tag{1.3.12}$$

Result 2.5 (Proposition 5.2.7) *Let* $\mathfrak{t}*_0$ *be the interior of the fundamental dual Weyl chamber* $\mathfrak{t}^*_+$. *Suppose the intersection* $I_\lambda \cap \mathfrak{t}^*_0$ *is non-empty. Then the density function*

$$\varphi(\lambda, \eta) = C_\eta^{-1}\left(\prod_{\alpha\in\Phi^+} \partial_\alpha\right)\nu^p_{I_\lambda}(\eta), \qquad \eta \in \mathfrak{t}^*_0, \tag{1.3.13}$$

where ∂_α *is the directional derivative of the root vector* α, *and*

$$C_\eta = \prod_{\alpha\in\Phi^+}(\delta, \alpha)/\prod_{\alpha\in\Phi^+}(\eta, \alpha). \tag{1.3.14}$$

When λ lies on a wall of $\mathfrak{t}^+$, the intersection $I_\lambda \cap \mathfrak{t}^*_+$ could be fully contained in the closure of a minimal wall $\mathfrak{t}^*_\Xi$ of $\mathfrak{t}^*_+$, and we have a new formula for the density function $\varphi(\lambda, \cdot)$.

Result 2.6 (Proposition 5.2.8) *Suppose* $I_\lambda \cap \mathfrak{t}^*_+$ *is fully contained in the closure of a minimal wall* $\mathfrak{t}^*_\Xi$ *of* $\mathfrak{t}^*_+$*, and let* $\mathfrak{t}^*_{\Xi_0}$ *denote the interior of* $\mathfrak{t}^*_\Xi$*. Then the density function*

$$\varphi(\lambda,\eta) = C_\eta^{-1}\left(\prod_{\alpha\in\Phi^+\setminus\Phi^+_{\Xi_0}} \partial_\alpha\right)\nu^p_{I_\lambda}(\eta), \qquad \eta\in\mathfrak{t}^*_{\Xi_0}, \tag{1.3.15}$$

where $\Phi^+_{\Xi_0} = \{\alpha\in\Phi^+ \,:\, (\lambda,\alpha)=0, \forall\lambda\in\mathfrak{t}^*_{\Xi_0}\}$*, and*

$$C_\eta = \frac{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}{\prod_{\alpha\in\Phi^+\setminus\Phi^+_{\Xi_0}}(\eta,\alpha)\cdot\prod_{\alpha\in\Phi^+_{\Xi_0}}(\delta_\eta,\alpha)}, \tag{1.3.16}$$

where $\delta_\eta = \frac{1}{2}\sum_{\alpha\in\Phi^+_{\Xi_0}}\alpha$*.*

We have calculated examples of $SU(3)$ and $Spin(5)$ and used figures to illustrate these density functions. Now, we are certain that $(\nu_1)^{*n}$ is the measure supported on the moment set I_n in Nelson's formula for the Weyl calculus $\mathcal{W}_{d\pi_n}$. So, we have also conducted a thorough review of Nelson's original proof. It turns out that there is missing a recursive function ψ of the non-constant-coefficient differential operator $x\frac{\partial}{\partial x}$. The original formula worked for π_1 because ψ does not affect the outcome of a 2-dimensional representation. Here, we present this updated version of Nelson's formula for irreducible unitary representations of compact simply connected semisimple Lie groups. Since Cazzaniga's work [9] was original inspired by the Kirillov character formula, we call it *Kirillov-type non-commutative formula.*

Result 3.1 (Proposition 6.3.3) *Let* G *be a compact simply connected semisimple Lie group,* T *be a maximal torus of* G*, and* $\mathfrak{g},\mathfrak{t}$ *be the Lie algebras of* G, T*, respectively. Let* $\mathfrak{t}^*$ *be the dual of* $\mathfrak{t}$*, and* $\mathfrak{t}^*_+$ *be a chosen positive Weyl chamber of* $\mathfrak{t}^*$*. Let* $\lambda\in\Lambda^+\subset\mathfrak{t}^*_+$ *be a dominant highest weight,* π_λ *be the irreducible unitary representation of the highest weight* λ *acting on a Hilbert space* $\mathcal{H}$*,* n *be the dimension of* π_λ*,* $d\pi_\lambda$ *be the infinitesimal version of* π_λ*,* I_λ *be the moment set of* π_λ*, and* ν_{I_λ} *be the probability* G*-invariant measure on* I_λ*. Let* $\{X_1,\ldots,X_d\}$ *be a basis of* $\mathfrak{g}$*, and denote* $x\cdot X = \sum_{k=1}^d x_j X_j$*, for* $x\in\mathbb{R}^d$*. Then, the closed form of the exponential mapping* $d\pi_\lambda\mapsto e^{d\pi_\lambda}$ *is given by*

$$\begin{aligned} e^{d\pi_\lambda(x\cdot X)} = \frac{1}{(n-1)!}\sum_{k=0}^{n-1}\sum_{j=0}^{n-k-1}\sum_{m=0}^{j}(-1)^{n-k-j-1}\frac{\phi_{n-k-j-1}\left(x\cdot d\pi_\lambda(X)\right)}{(n-j+m-1)!} \\ \cdot\left(x\cdot d\pi_\lambda(X)\right)^k\,\psi(m-1)\,\check{\nu}_{I_\lambda}(x\cdot X), \end{aligned} \tag{1.3.17}$$

where $\phi_j(A)$ *is the sum of the principal minors of* A *of degree* j*,*

$$\check{\nu}_{I_\lambda}(x \cdot X) = \int_{\mathfrak{g}^*} e^{i\beta(x\cdot X)}\, d\nu_{I_\lambda}(\beta), \tag{1.3.18}$$

and

$$\psi(p) = \sum_{q=0}^{p} (-1)^{p-q} \frac{p!}{q!} \left(x \cdot \frac{\partial}{\partial x} \right) \cdot \psi(q-1), \tag{1.3.19}$$

with $\psi(-1) = 1$*.*

So, we have resolved **Issue** 2. We have also performed detailed calculations for $e^{d\pi_2}$ of $SU(2)$, and the defining representation of $SU(3)$ and $Spin(5)$ to illustrate this formula. Furthermore, we have examined the differential operator $x\frac{\partial}{\partial x}$ and established that it is a G-invariant differential operator which preserves the G-invariance of an adjoint invariant function.

Result 3.2 (Proposition 6.2.5) Let G be a compact connected semisimple Lie group, T be a maximal torus of G. Let $\mathfrak{g}$, $\mathfrak{t}$ be the Lie algebras of G, T, respectively. Let $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. Suppose $dim(\mathfrak{g}) = d$, and we let $x \in \mathbb{R}^d$, and $\xi \in (\mathbb{R}^d)^*$. If $f \in \mathcal{S}(\mathfrak{g})$ is an adjoint invariant function, then

$$\widehat{\left(x\frac{\partial}{\partial x} f \right)} = -\left(\xi\frac{\partial}{\partial \xi} + dI \right) \widehat{f}, \tag{1.3.20}$$

and $x\frac{\partial}{\partial x}$ and $\xi\frac{\partial}{\partial \xi}$ are G-invariant differential operators.

When we restrict Nelson's formula to representations, we have found that the existence of G-invariant differential operator $x\frac{\partial}{\partial x}$ is closely related to the tensor products of representations. If we fix a basis for $\mathcal{H}_{\pi_\lambda}$, and let $\pi^\lambda_{1,1}$ be the highest weight matrix coefficient of π_λ, and let $\pi_\lambda, \pi_{\lambda'}$ for $\lambda, \lambda' \in \Lambda^+$, then the highest weight coefficient of $\pi^{\lambda+\lambda'}_{1,1}$ is $\pi^\lambda_{1,1}\pi^{\lambda'}_{1,1}$, by the Clebsch-Gordan theorem.

However, if we lift $\pi^{\lambda+\lambda'}_{1,1}$ to $\mathfrak{g}$ by the exponential mapping and take the Fourier transform, then it contain a term $\xi\frac{\partial}{\partial\xi}\nu_{I_\lambda} * \xi\frac{\partial}{\partial\xi}\nu_{I_{\lambda'}}$. Since $\xi\frac{\partial}{\partial\xi}$ does not commute with convolution, this means that this convolution is not equal to $\left(\xi\frac{\partial}{\partial\xi}\right)^2 \nu_{I_\lambda} * \nu_{I_{\lambda'}}$. Also, because in general the moment set $I_{\lambda+\lambda'}$ is not equal to $I_\lambda + I_{\lambda'}$, so $\xi\frac{\partial}{\partial\xi}$ ensures that the support of the Fourier transform of $\pi^{\lambda+\lambda'}_{1,1}$ is not the same as the support of $\nu_{I_\lambda} * \nu_{I_{\lambda'}}$, which is the sumset $I_\lambda + I_{\lambda'}$. Clearly, $SU(2)$ is a special case and this finding provides a reasonable clarification for **Issue 1**.

If we take the $\mathfrak{g}$-Fourier transform of $e^{d\pi_\lambda}$, with respect to the Killing form of $\mathfrak{g}$, then we have the following.

Result 3.3 (Proposition 6.3.5) *Let* G *be a compact simply connected semisimple Lie group,* T *be a maximal torus of* G*, and* $\mathfrak{g}, \mathfrak{t}$ *be the Lie algebras of* G, T*, respectively. Let* $\mathfrak{t}^*$ *be the dual of* $\mathfrak{t}$*, and* $\mathfrak{t}^*_+$ *a chosen fundamental Weyl chamber of* $\mathfrak{t}^*$*. Let* $\lambda \in \Lambda^+ \subset \mathfrak{t}^*_+$ *be a dominant highest weight,* π_λ *the irreducible unitary representation of the highest weight of* λ *acting on a Hilbert space* $\mathcal{H}$*,* n *be the dimension of* π_λ*,* $d\pi_\lambda$ *be the infinitesimal version of* π_λ*,* I_λ *be the moment set of* π_λ*, and* ν_{I_λ} *be the probability* G*-invariant measure on* I_λ*. Let* $\{X_1, ..., X_d\}$ *be a basis of* $\mathfrak{g}$*, and denote* $x \cdot X = \sum_{k=1}^{d} x_j X_j$*, for* $x \in \mathbb{R}^d$*. Let* C *be the Cartan matrix of* $\mathfrak{g}^{\mathbb{C}}$*, and* $\zeta = (\beta, \xi, \eta) \in \left(\mathbb{R}^d\right)^*$*. Then the* $\mathfrak{g}$*-Fourier transform of* $e^{d\pi_\lambda(x\cdot X)}$*, denoted by* $\left(e^{d\pi_\lambda(x\cdot X)}\right)^{\wedge_{\mathfrak{g}}}$*, can be written as*

$$\begin{aligned}\left(e^{d\pi_\lambda(x\cdot X)}\right)^{\wedge_{\mathfrak{g}}} = \frac{1}{(n-1)!} \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \sum_{m=0}^{j} (-1)^{n-k-j-1} & \frac{\phi_{n-k-j-1}\left(d\pi_\lambda(X) \cdot -i\frac{\partial}{\partial\zeta}\right)}{(n-j+m-1)!} \\ & \cdot \left(d\pi_\lambda(X) \cdot -i\frac{\partial}{\partial\zeta}\right)^k \psi(m-1) \cdot \nu_{I_\lambda} \\ & = D^\lambda \cdot \nu_{I_\lambda},\end{aligned} \tag{1.3.21}$$

where ϕ_k *is the sum of principal minors of order* k*, and* ψ *is a recursive function of the theta differential operator* $\zeta\frac{\partial}{\partial\zeta}$*, given by*

$$\psi(p) = \sum_{q=0}^{p} (-1)^{p-q} \frac{p!}{q!} \cdot \left(-\zeta\frac{\partial}{\partial\zeta} - dI\right) \cdot \psi(q-1)$$

where $\psi(-1) = 1$*, and*

$$-i\frac{\partial}{\partial\zeta} = -i\left(C^{-1} \cdot \frac{\partial}{\partial\beta}, \frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}\right).$$

Also, the support of $\left(e^{d\pi_\lambda(x\cdot X)}\right)^{\wedge_{\mathfrak{g}}}$ *satisfies*

$$\operatorname{supp}\left(D^\lambda \cdot \nu_{I_\lambda}\right) \subseteq \operatorname{conv}\left(\mathcal{O}_\lambda\right) \quad \text{and} \quad \operatorname{supp}\left(D^\lambda \cdot \nu_{I_\lambda}\right) \cap \mathfrak{t}^* \subseteq \operatorname{conv}\left(W \cdot \lambda\right).$$

The matrix of differential operators D^λ *consists of polynomial of Euclidean differential operator in some fundamental weight directions* $\frac{\partial}{\partial\lambda_i}$*, root directions*

$\frac{\partial}{\partial \alpha_j^+}$, $\frac{\partial}{\partial \alpha_k^-}$, *and differential operator* $\zeta \frac{\partial}{\partial \zeta}$. *We have*

$$\frac{\partial}{\partial \lambda_i} = C_i^{-1} \cdot \beta,$$

where C_i^{-1} *is the* i*-th row of the inverse of the Cartan matrix* C, $\beta = (\beta_1, ..., \beta_l)$ *is the tuple of simple roots* $\Delta \subset \Phi$. *Also, we have*

$$\frac{\partial}{\partial \alpha_j^+} = \frac{\partial}{\partial \xi_j} - i\frac{\partial}{\partial \eta_j}, \quad \text{and} \quad \frac{\partial}{\partial \alpha_k^-} = \frac{\partial}{\partial \xi_k} + i\frac{\partial}{\partial \eta_k},$$

where α_j^+ *is a positive root in* Φ, *and* α_k^- *is a negative root in* Φ.

Since the matrix coefficients of π_λ are the fundamental building blocks of the functions in $C(G)$ and $L^p(G)$ and now we have a good understanding of what each irreducible unitary representation $\pi_\lambda \circ \exp$ looks like, we can examine the difference between the convolutions of the pair of arbitrary matrix coefficients $\pi_{i,j}^\lambda * \pi_{k,l}^\lambda$ in G and $\pi_{i,j}^\lambda * \pi_{k,l}^\lambda \circ \exp$ in $\mathfrak{g}$. However, if we compare this to the criteria of *wrapping map* [19], then the lift of the matrix coefficients on $\mathfrak{g}$ are not compactly supported but are supported everywhere on $\mathfrak{g}$, and not G-invariant. However, the matrix coefficients on G are compactly supported but not central on G. Hence, the wrapping map does not apply directly to matrix coefficients.

But, we have observed that the convolution of any pair of matrix coefficients in the same representation (the same equivalent class) π_λ on G is either a matrix coefficient in the same π_λ or zero. Also, since G is compact, simply connected and semisimple, each row and column of π_λ form a copy of the representation space of π_λ, which can be generated by applying the vector fields induced by the root vectors of $\mathfrak{g}^{\mathbb{C}}$ to the highest weight matrix coefficient $\pi_{1,1}^\lambda$.

Sigurdur Helgason [27] showed that the differential operators in $\mathfrak{g}$ consist of transversal differential operators and differential operators induced by the action of G. In fact, the differential operators we found in Result 3.3 including $\zeta \frac{\partial}{\partial \zeta}$ are transversal differential operators. We have made connections between the matrix coefficients and the differential operators induced by the action of G, which allows us to lift the differentiation by vector fields on $\pi_{i,j}^\lambda$ to differentiation by induced differential operators on $\pi_{i,j}^\lambda \circ \exp$.

Result 4.1 (Proposition 3.2.5) *Let* G *be a compact simply connected semisimple Lie group,* $\mathfrak{g}$ *be its Lie algebra. Let* π_λ *be a finite-dimensional irreducible unitary* *representation of G with the highest dominant weight λ on the Hilbert space $\mathcal{H}_{\pi_\lambda}$. Let $d_{\pi_\lambda} = \dim(\pi_\lambda)$, and fix an orthonormal basis $\{\xi_1, ...\xi_{d_{\pi_\lambda}}\}$ of $\mathcal{H}_{\pi_\lambda}$ (where ξ_1 is the chosen highest weight vector). Define the matrix coefficient of π_λ as $T_{i,j}(\cdot) = \langle \pi_\lambda(\cdot)\xi_j, \xi_i\rangle$. Then the convolution of any pair of matrix coefficients is given by*

$$T_{i,j} * T_{k,l} = \frac{1}{d_\pi} D^{i,l}_{k,j} T_{1,1}, \quad 1 \leqslant i, j, k, l \leqslant d_{\pi_\lambda}, \tag{1.3.22}$$

where $D^{i,l}_{k,j}$ is a polynomial of left and right invariant vector fields on G induced by the root vectors in $\mathfrak{g}^{\mathbb{C}}$. Since G is simply connected, we can lift π_λ to its Lie algebra $\pi_\lambda \circ \exp$ by the exponential mapping, that is, $\pi_\lambda(\exp X) = \exp(d\pi_\lambda(X))$, for $X \in \mathfrak{g}$, where $d\pi_\lambda$ is the skew-Hermitian infinitesimal representation of $\mathfrak{g}$. Then, there exists a polynomial of vector fields induced by G on $\mathfrak{g}$, denoted by $(D^{i,l}_{k,j})^G$, such that

$$(T_{i,j} * T_{k,l})(\exp X) = \frac{1}{d_\pi} \left(D^{i,l}_{k,j}\right)^G T_{1,1}(\exp X), \quad 1 \leqslant i, j, k, l \leqslant d_{\pi_\lambda}. \tag{1.3.23}$$

This observation has resolved **Issue** 4. We have also constructed examples of $SU(2)$ and $SU(3)$ to illustrate the actions of these induced differential operators on the lift of matrix coefficients.

The representation theory of semisimple Lie algebra shows us that we can fix a highest weight $\lambda \in \Lambda^+$, and generate all the weight spaces of the representation $d\pi_\lambda(\mathfrak{g}^{\mathbb{C}})$ of a complex semisimple semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$, by applying the lowering operators induced by the negative roots in Φ to a chosen 1-dimensional highest weight vector v_λ. If we take a compact real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$, we can obtain $d\pi_\lambda(\mathfrak{g})$, which is equal to the infinitesimal version of the representation π_λ of a compact simply connected semisimple Lie group G. However, in general $d\pi_\lambda(\mathfrak{g})$ is not skew-Hermitian, so we may not apply the Kirillov-type non-commutative formula, which is only for skew-Hermitian matrices. However, we have observed that if we apply the *Weyl's unitarian trick*, we may extend the non-commutative formula to non-skew-Hermitian highest weight representations.

Result 4.2 (Proposition 6.4.1) *Let G be a compact simply connected semisimple Lie group, T be a maximal torus of G, and $\mathfrak{g}, \mathfrak{t}$ be their Lie algebras, respectively. Let $\tilde{\pi}_\lambda$ be an irreducible finite-dimensional highest weight representation (not necessarily unitary) of G of $\lambda \in \Lambda^+$, and $d\tilde{\pi}_\lambda$ be the infinitesimal version of $\tilde{\pi}_\lambda$ in $\mathfrak{g}$ (not necessarily skew-Hermitian). Then, for every $X \in \mathfrak{g}$, there exists*

$H \in \mathfrak{t}$ and $g \in G$ such that the skew-Hermitian representation $d\tilde{\pi}_\lambda(H)$ satisfies $d\tilde{\pi}_\lambda(H) = \tilde{\pi}(g)d\tilde{\pi}_\lambda(X)\tilde{\pi}(g^{-1})$, and the Kirillov-type non-commutative formula in Result 3.1 and 3.2 can be extended to the highest weight representation $d\tilde{\pi}_\lambda$.

Therefore, we have resolved **Issue 5**. We have also used the classical example of the representation of $\mathfrak{sl}_2(\mathbb{C})$ (with compact real form $\mathfrak{su}(2)$) [30], to illustrate this extension.

The detailed structure of each chapter is as follows.

In Chapter 2, we review the definition of a G-Hamiltonian symplectic manifold, and its two canonical examples: coadjoint orbits and projective spaces of unitary representations of Lie groups. We also study the orbit method on a compact connected semisimple Lie group G, which is equivalent to the Borel-Weil theorem. We give a comprehensive treatment of $SU(2)$. Firstly, we demonstrate the orbit method on $SU(2)$ [15], which realises the representation space as homogeneous polynomials P_n spanned by $\{z^n, z^{n-1}, z^1, 1\}$ for $z \in \mathbb{C}$. Then, we study the explicit construction for the irreducible unitary representation π_n of $SU(2)$ of highest integer weight n acting on P_n [29], [21]. Finally, we study Cazzaniga's result [9]. He showed that each matrix coefficient of the Euclidean Fourier transform of $\pi_n \circ \exp$ is a polynomial of differential operators acting on the rotationally invariant convolution measure $(\nu_1)^{*n}$, where ν_1 is the surface measure of the unit sphere in $\mathbb{R}^3$.

In Chapter 3, we study the properties of the matrix coefficients of unitary representations of compact connected semisimple Lie groups [21], [30], and we show that the convolution of any pair of matrix coefficients of an irreducible highest weight unitary representation π of a compact simply connected semisimple Lie group G is equivalent to applying a polynomial of invariant vector fields on G to the highest weight matrix coefficient $\pi_{1,1}$ of π. Then, we study the transversality condition [27] of a semisimple Lie algebra $\mathfrak{g}$, and we also study the general form of a differential operator in $\mathfrak{g}$, which contains transversal differential operators and vector fields induced by the action of G on $\mathfrak{g}$. Subsequently, we show that the convolution of any pair of the lift of matrix coefficients on $\mathfrak{g}$ by exponential mapping of an irreducible highest weight unitary representation π of a compact simply connected semisimple Lie group G is equivalent to applying a polynomial of vector fields induced by the action of G on $\mathfrak{g}$, to the lift of the highest weight matrix coefficient $\pi_{1,1} \circ \exp$ of $\pi \circ \exp$ on $\mathfrak{g}$. Furthermore, we study the general form of the radial part of the Laplacian differential operator and constant-coefficient G-invariant differential operators in a semisimple Lie algebra $\mathfrak{g}$, and one of its applications which it leads to a derivation of a continuous version of the Kirillov character formula for compact connected semisimple Lie groups.

In Chapter 4, we study the convolution structures of coadjoint orbits of compact connected semisimple Lie groups [18]. Firstly, we study the explicit formula for the projection of a G-invariant measure of a regular coadjoint orbit $\mathcal{O}_\lambda$ in $\mathfrak{t}^*$, denoted by μ_λ^p. The 'basic tent' is the projection of the coadjoint orbit $\mathcal{O}_\delta$, and it acts as a foundation for the derivation of μ_λ^p. We also study an explicit formula for the convolution of any pair of regular coadjoint orbits. Next, we study how to approximate the G-invariant Liouville measure of a singular coadjoint orbit from a regular one. Finally, we use the convolution formula developed to calculate the singular supports of the n-fold convolution of coadjoint orbits (unit 2-spheres) of $SU(2)$, and compare it with Cazzaniga's result. The techniques used in developing these formulas are important as we will use them to develop a G-invariant measure of moment sets of unitary representations.

In Chapter 5, we review the convexity theorem of a G-Hamiltonian manifold M, and study the properties of moment sets of unitary representations [55]. Then, we show the n-fold sumset of the moment set I_1 of the irreducible unitary representation π_1 of $SU(2)$ of the highest integer weight 1 is equal to the moment set I_n of π_n of $SU(2)$. In addition, we determine a subset of the intersection $I_\lambda \cap \mathfrak{t}^*$ for a non-root distinct dominant weight $\lambda \in \Lambda^+$. Next, we derive an explicit formula for the projection measure of $\nu_{I_\lambda}^p$ of the moment set I_λ, and we use this formula to show that the G-invariant measure ν_{I_n} of the moment set I_n of $SU(2)$ is equal to the n-fold convolution (ν_1^{*n}) of coadjoint orbit $\mathcal{O}_1$ of $SU(2)$. Finally, we use the techniques studied in Chapter 4 to derive the density function of $I_\lambda \cap \mathfrak{t}_+^*$ for two different scenarios: 1. $I_\lambda \cap \mathfrak{t}_0^*$ is non-empty; 2. $I_\lambda \cap \mathfrak{t}_+^*$ is fully contained in a minimal wall of $\mathfrak{t}_+^*$.

In Chapter 6, we review the definitions of Weyl calculus, Nelson's formula for Weyl calculus in a finite setting [40], and Raffoul's results of support and singular support of Weyl calculus for general compact Lie groups [44], [17]. Then, we discuss some interesting properties of the non-constant-coefficient differential operator $\zeta\frac{\partial}{\partial\zeta}$ (which generalises $r\frac{\partial}{\partial r}$ for $SU(2)$) in $\mathfrak{g}$. In addition, we derive generalised Bessel functions for compact connected semisimple Lie groups, and use these Bessel functions to show that $\zeta\frac{\partial}{\partial\zeta}$ is a G-invariant differential operator in

$\mathfrak{g}$. Lastly, we derive a Kirillov type non-commutative formula for an irreducible unitary highest weight representation π_λ, which is an updated version of Nelson's formula with a novel recursive function ψ of the differential operator $\zeta\frac{\partial}{\partial\zeta}$. Also, we show that the $\mathfrak{g}$-Fourier transform of $\pi_\lambda \circ \exp$ is an operator of polynomials of differential operators in fundamental weight directions, root directions, and $\zeta\frac{\partial}{\partial\zeta}$, acting on the G-invariant measure ν_{I_λ} of the moment set I_λ of π_λ. Additionally, we show that the Kirillov-type non-commutative formula can be extended to non-unitary highest weight representations.

In Chapter 7, we extend everything developed in previous chapters to the symmetric space of compact type (G, K). We quote the Cartan-Helgason theorem as the starting point of this extension. We show how to calculate the K-invariant vector v_K from the Kirillov-type non-commutative formula, which leads to the calculations of zonal and non-zonal spherical functions, as they correspond to K-invariant and non-K-invariant measures with compact support in the tangent space $\mathfrak{p}$ at eK of the symmetric space G/K.

In the course of this project, we have developed Wolfram Mathematica programs to visualise the projection measures of coadjoint orbits onto $\mathfrak{t}^*$, convolution of coadjoint orbits intersecting the $\mathfrak{t}^*$, and the moment sets intersecting $\mathfrak{t}^*$, for rank 2 simple Lie algebras. We have also calculated the matrix of polynomials in Kirillov-type non-commutative formula and tested their correctness with respect to the non-constant-coefficient differential operators induced by the actions of G on $\mathfrak{g}$. We have also discussed the complexity of calculating this formula, and written an efficient algorithm for calculations. These are detailed in Appendix *A*.2.

Chapter 2

Hamiltonian Manifolds and the Kirillov Orbit Method

A G-Hamiltonian manifold is a symplectic manifold M with a Lie group G action which supports a moment map and preserve the symplectic form on M. Two canonical examples of G-Hamiltonian manifolds are: a coadjoint orbit $\mathcal{O}$ in the dual Lie algebra $\mathfrak{g}^*$ by the adjoint action of G, and the projective space PV of the representation space V of a finite-dimensional irreducible unitary representation of G. The orbit method states that the holomorphic sections of a line bundle on an integral coadjoint orbit form a representation space of G. In this chapter, we examine the relationship between an integral coadjoint orbit $\mathcal{O}$ of a compact connected semisimple Lie group G and the image of the moment map associated with the finite-dimensional irreducible unitary representation of G induced from $\mathcal{O}$, utilizing the orbit method.

In Section 1.1, we give the definition of a G-Hamiltonian manifold, and study the examples of $\mathcal{O}$ and PV by defining the symplectic 2 forms.

In Section 1.2, we study the generalised flag varieties of a complex semisimple Lie group $G^{\mathbb{C}}$, given by the quotient of $G^{\mathbb{C}}/B$, where B is the Borel subgroup, and the corresponding full flag manifolds and degenerate manifolds of the compact real form G.

In Section 1.3, we study the orbit method, which contains procedures including: pre-quantisation, polarisation, quantisation and L^2 conditions. We focus on elaborating this method for compact connected semisimple Lie groups.

In Section 1.4, we propose a modern approach to the irreducible unitary representations of the simplest non-commutative compact Lie group $G = SU(2)$.

We perform quantisation on the integral coadjoint orbits of $SU(2)$. Then, we present the calculations of the irreducible unitary representation π_n, where each π_n is identified by a positive integer n. We also show that the Euclidean Fourier transform of $\pi_n \circ \exp$ in the Lie algebra $\mathfrak{su}(2)$ (which can be realised as $\mathbb{R}^3$), consists of a polynomial of differential operators acting on a unique measure supported in the convex hull of the coadjoint orbit $\mathcal{O}_n$.

All content in this chapter is gathered from various sources. Additionally, a few propositions are drawn from implicitly mentioned external resources and are included to enhance coherence and logical consistency.

2.1 Hamiltonian Manifolds and Coadjoint Orbits

The contents of this section are extracted from [7], [51] and [55].

We introduce the concept of Hamiltonian manifolds and give two key examples. Our main objective in studying Hamiltonian manifolds is to study the image of the moment map.

Definition 2.1.1. Let M be a differential manifold of even dimension. Then M carries a *symplectic structure* if there is a closed and non-degenerate differential 2-form ω on M that satisfies

$$d\omega = 0 \quad \text{and} \quad \omega_p(X, Y) = 0, \quad \forall Y \in T_pM \text{ implies } X = 0.$$

where T_pM is the tangent space at $p \in M$. The pair (M, ω) is called a **symplectic manifold**.

Definition 2.1.2. Let T^*M be the cotangent bundle of M. The symplectic form ω induces an isomorphism $TM \to T^*M$, between vector fields and also a **differential 1-form** in M, that is,

$$\omega_X(Y) = \omega(X, Y), \tag{2.1.1}$$

where X, Y are vector fields of M.

Definition 2.1.3. Let G be a Lie group, e be its identity element, and $\mathfrak{g}$ be the Lie algebra of G, which is also the tangent space T_eG. Let $t \in \mathbb{R}^+, X \in \mathfrak{g}$, $t \mapsto \exp(tX)$ be the one-parameter subgroup of G that has derivative equal to X at $t = 0$. The action of G on M is defined by $G \times M \to M$. Every $X \in \mathfrak{g}$ gives rise to a **vector field** X^M in M, which is defined by

$$(X^M.f)(m) = \frac{d}{dt}\bigg|_{t=0} f(\exp(-tX) \cdot m), \quad f \in C^\infty(M). \tag{2.1.2}$$

The map $X \mapsto X^M$ is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of the vector fields on M.

Definition 2.1.4. A vector field X on M is called a **Hamiltonian vector field** if it is induced by a function $H \in C^\infty(M)$, and X corresponds to a differential 1-form dH by the identification $\omega : TM \to T^*M$. For each vector field Y, we have

$$dH(Y) = \omega(X, Y). \tag{2.1.3}$$

Let X, Y be Hamiltonian vector fields with respect to functions $H_X, H_Y \in C^\infty(M)$, then $\{H_X, H_Y\} = \omega(X, Y) = H_{[X,Y]}$, where $\{\cdot, \cdot\}$ is the Poisson bracket on $C^\infty(M)$, and $X \to H_X$ is a Lie homomorphism from the Lie algebra of vector fields to the Lie algebra of Hamiltonian vector fields.

Theorem 2.1.5 ([51], Theorem 38.A)**.** *Let a Lie group G act on a symplectic manifold (M, ω), and let $H : M \to \mathbb{R}$ be a smooth function on M. Let $X \in \mathfrak{g}$ and X_H be the Hamiltonian vector field corresponding to the smooth function H. Suppose X_H is given by*

$$X_H(m) = \frac{d}{dt}\bigg|_{t=0} \exp(tX) \cdot m, \quad m \in M, \tag{2.1.4}$$

*where $t \mapsto \exp(tX)$ is the one-parameter subgroup of G. Then, $\exp(tX)$ preserves the symplectic structure ω, i.e., $\exp(tX) \cdot \omega = \omega$, and the action of G is called the **Hamiltonian action** and acts on M as a **group of symplectomorphisms**.*

Definition 2.1.6. A **moment map** Ψ for the symplectomorphism of the G action on M is an equivariant map

$$\Psi : M \to \mathfrak{g}^*, \tag{2.1.5}$$

such that for every $X \in \mathfrak{g}$, there is a Hamiltonian vector field X_H on M induced by a smooth function H_X that gives $\Psi(m)(X) = H_X(m)$ and $d\Psi(X) = dH_X = \omega_{X_H}$. The equivariance condition is given by $\Psi(g \cdot m)(X) = \Psi(m)(Ad(g^{-1}) \cdot X)$ where Ad is the adjoint representation of G on $\mathfrak{g}$.

Definition 2.1.7. A symplectic manifold (M, ω) with the action of G preserving ω, also equipped with a moment map Ψ, is called a ***G*-Hamiltonian manifold**.

Here we look at three examples of G-Hamiltonian manifold.

Example 2.1.7.1. Let $M = \mathbb{R}^2$, $G = SO(2)$. The natural symplectic form is $\omega = dx \wedge dy$. The Lie algebra $\mathfrak{so}(2)$ and $SO(2)$ themselves are given by

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad e^{\theta X} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in \mathbb{R}. \tag{2.1.6}$$

The vector field on $\mathbb{R}^2$ induced by $SO(2)$ can be derived as follows. The associated one-parameter subgroup with respect to X is given by the matrix exponential:

$$e^{tX} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad t \in \mathbb{R} \tag{2.1.7}$$

To find the vector field, we compute the derivative of the rotated coordinates with respect to t, evaluated at $t = 0$. That is,

$$\left.\frac{d}{dt}\right|_{t=0} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \left.\begin{pmatrix} -\sin(t)x + \cos(t)y \\ -\cos(t)x - \sin(t)y \end{pmatrix}\right|_{t=0} = \begin{pmatrix} y \\ -x \end{pmatrix}. \tag{2.1.8}$$

With respect to the partial derivatives $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, the vector field $X^{\mathbb{R}^2}$ can be written as:

$$X^{\mathbb{R}^2} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{2.1.9}$$

The symplectic 2-form ω acts on the vector field $X^{\mathbb{R}^2}$. To find the 1-form $\omega_{X^{\mathbb{R}^2}}$, we use the definition of the contraction of 2-form with a vector field:

$$\omega_{X^{\mathbb{R}^2}} = i_{X^{\mathbb{R}^2}}(\omega) = i_{y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}}(dx \wedge dy) = ydy - x(-dx) = xdx + ydy. \tag{2.1.10}$$

We can also define a moment map Ψ for $\mathbb{R}^2$, that is

$$\Psi(x, y)\left(X^{\mathbb{R}^2}\right) = \frac{1}{2}(x^2 + y^2), \tag{2.1.11}$$

which reflects the angular momentum in $\mathbb{R}^2$ for $SO(2)$ rotations that is proportional to the distance to the origin. And, the exterior derivative $d\Psi$ is

$$d\Psi(x,y)\left(X^{\mathbb{R}^2}\right) = \frac{\partial\Psi}{\partial x}dx + \frac{\partial\Psi}{\partial y}dy = xdx + ydy. \tag{2.1.12}$$

So, the differential of the moment map $d\Psi$ is equal to the one-form $\omega_{X^{\mathbb{R}^2}}$.

Example 2.1.7.2. Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$ with respect to the Killing form $\langle\cdot,\cdot\rangle$. Define the **coadjoint representation** of G on $\mathfrak{g}^*$ by $\langle Ad(g)^*\beta, X\rangle = \langle\beta, Ad(g^{-1})X\rangle$, for $X \in \mathfrak{g}$, $\beta \in \mathfrak{g}^*$. For $\beta \in \mathfrak{g}^*$, define the coadjoint orbit $\mathcal{O}_\beta = \{Ad(g)\beta : g \in G\}$. Let G_β be the subgroup of G that stabilises β and $\mathfrak{g}_\beta = \{X \in \mathfrak{g} : \langle\beta, [X,Y]\rangle = 0, \forall Y \in \mathfrak{g}\}$. Then $\mathcal{O}_\beta \cong G/G_\beta$, and $T\mathcal{O}_\beta$ is the tangent space of $\mathcal{O}_\beta$ at β, which is given by $T\mathcal{O}_\beta = \mathfrak{g}/\mathfrak{g}_\beta$.

Notice that $\mathfrak{g}_\beta$ is the kernel of the map $X \mapsto \langle\beta,[X,Y]\rangle$, for $X \in \mathfrak{g}_\beta$ and for all $Y \in \mathfrak{g}$. Hence, it induces a real non-degenerate 2-form $(X,Y) \mapsto \omega_\beta(X,Y) = \langle\beta,[X,Y]\rangle$ on $\mathcal{O}_\beta$. The 2-form ω_β is also closed, that is, $d\omega = 0$. To see this, notice that for every 2-form ω,

$$\begin{aligned} d\omega(X,Y,Z) &= X.\omega(Y,Z) - Y.\omega(X,Z) + Z.\omega(X,Y) \\ &\quad - \omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X), \end{aligned} \tag{2.1.13}$$

by ([39], Prop 14.32). Now $X.\omega_\beta(Y,Z)$ can be written as $Y.X.\langle\beta,Z\rangle$. Hence, by the Jacobi identity, $d\omega = 0$. Therefore, a coadjoint orbit $\mathcal{O}$ is a symplectic manifold, which implies it is even dimensional. The moment map is given by the identity map $\Psi(\beta) = \beta$, and $\Psi(\beta)(X) = \langle\beta, X\rangle$. Let $Y^{\mathcal{O}}$ be a Hamiltonian vector field on $\mathcal{O}$ induced by the action of G. Then at $\beta \in \mathcal{O}$,

$$\begin{aligned} d\Psi(\beta)(X)(Y^{\mathcal{O}}) &= Y^{\mathcal{O}}.\langle\beta, X\rangle \\ &= \frac{d}{dt}\bigg|_{t=0} \langle\beta, Ad(\exp^{-tY})X\rangle \\ &= \langle\beta,[X,Y]\rangle, \end{aligned} \tag{2.1.14}$$

which is equal to $\omega_{X^{\mathcal{O}}}$. The 2-form ω is also called *Kirillov-Kostant-Souriau* symplectic form. Coadjoint orbits play a key role in the subject of geometric quantisation, which will be explained in detail in Section 2.3.

Example 2.1.7.3. ([55], Example.1.1) Let π be a finite-dimensional unitary representation of a Lie group G acting on a complex Hilbert space V, where

V is isomorphic to $\mathbb{C}^n$, for certain integers n. Let $p : V\backslash\{0\} \to PV$ be the projection from V to the complex projective space PV, which takes a non-zero v to the complex line $[v]$ passing through v. Then PV has a 2-form defined in the following way: Let η_1, η_2 be tangent vectors of PV at $[v]$. Then there exists v_1, v_2 in the tangent space of V at v, $T_vV \cong V$, such that $dp(v_i) = \eta_i$, for $i = 1, 2$. Thus the 2-form ω is given by

$$\omega_{[v]}(\eta_1, \eta_2) = \text{Imaginary part of } 2\frac{\langle v_1, v_2\rangle\langle v, v\rangle - \langle v_1, v\rangle\langle v, v_2\rangle}{\langle v, v\rangle^2}, \tag{2.1.15}$$

which is real, anti-symmetric and bilinear. Note the action of G on V induces an action of G on PV, which preserves ω. Thus G acts as a group of symplectomorphisms. The moment map on PV is defined by

$$\Psi([v])(X) = \frac{1}{i}\frac{\langle d\pi(X)v, v\rangle}{\langle v, v\rangle}, \tag{2.1.16}$$

for $X \in \mathfrak{g}, [v] \in PV$, where $d\pi$ is the skew-Hermitian infinitesimal version of π. By the definition of moment map in Definition 2.1.6, for each $X \in \mathfrak{g}$, the smooth function H_X on PV is defined by

$$H_X([v]) = \frac{1}{i}\frac{\langle d\pi(X)v, v\rangle}{\langle v, v\rangle}. \tag{2.1.17}$$

For $\eta \in T_{[v]}PV$, there is a $\xi \in T_vV$ such that $dp(\xi) = \eta$. Then, $(dH_X)_{[v]}(\eta) = (\eta.H_X)([v])$, which is equal to

$$\begin{aligned}
&= \frac{d}{dt}\bigg|_{t=0} \frac{1}{i}\frac{\langle d\pi(X)(v + t\xi), (v + t\xi)\rangle}{\langle (v + t\xi), (v + t\xi)\rangle} \\
&= \frac{1}{i}\frac{(\langle d\pi(X)v, \xi\rangle + \langle d\pi(X)\xi, v\rangle)\langle v, v\rangle - \langle d\pi(X)v, v\rangle(\langle v, \xi\rangle + \langle \xi, v\rangle)}{\langle v, v\rangle^2} \\
&= \frac{2\text{Im}\langle d\pi(X)v, \xi\rangle\langle v, v\rangle + 2i\langle d\pi(X)v, v\rangle\text{Re}\langle v, \xi\rangle}{\langle v, v\rangle^2} \\
&= \omega_{[v]}(X^{PV}([v]), \eta),
\end{aligned} \tag{2.1.18}$$

where $X \mapsto X^{PV}$ is a Hamiltonian vector field on PV. Note that $\langle v, v\rangle$ is real, $\langle d\pi(X)v, v\rangle$ is imaginary. Let X^{PV}, Y^{PV} be Hamiltonian vector fields with respect to smooth functions H_X, H_Y. Then by Definition 2.1.4, the Poisson bracket operation gives

$$\begin{aligned}\{H_X, H_Y\} &= 2\mathrm{Im}\frac{\langle d\pi(X)v, d\pi(Y)v\rangle\langle v, v\rangle - \langle d\pi(X)v, v\rangle\langle v, d\pi(Y)v\rangle}{\langle v, v\rangle^2}\\ &= 2\mathrm{Im}\frac{\langle d\pi(X)v, d\pi(Y)v\rangle}{\langle v, v\rangle}\\ &= \frac{1}{i}\frac{\langle d\pi(X)v, d\pi(Y)v\rangle - \langle d\pi(Y)v, d\pi(X)v\rangle}{\langle v, v\rangle}\\ &= \frac{1}{i}\frac{\langle d\pi([X,Y])v, v\rangle}{\langle v, v\rangle} = H_{[X,Y]}([v]),\end{aligned} \tag{2.1.19}$$

so that $X \to H_X$ is a Lie homomorphism. Notice that the image of the moment map, $PV \to \mathfrak{g}^*$ is adjoint invariant, so the image of the moment map of a unitary representation is a union of coadjoint orbits in $\mathfrak{g}^*$.

2.2 Generalised Flag Varieties and Coadjoint Orbits

In this section, we study the complete and degenerate flag manifolds of compact connected semisimple Lie groups and their relationships with coadjoint orbits.

Definition 2.2.1. Let G be an algebraic group. A subgroup B is called a **Borel subgroup** if it is maximal among all Zariski closed connected solvable subgroups. All Borel subgroups are mutually conjugate and the intersection of any two contains a maximal torus in G.

Definition 2.2.2. Let $\mathbb{T}$ be the 1-dimensional circle group. A **torus** T of a compact Lie group G is a compact connected abelian subgroup of G, which is isomorphic to $\mathbb{T}^n \cong \mathbb{R}^n/\mathbb{Z}^n$. A torus T is maximal in G if for any torus T' containing T, we have $T' = T$.

Theorem 2.2.3 ([2] Theorem 4.21, [8], Theorem IV.1.6)**.** *Let T be a fixed maximal torus in a compact Lie group G, then every element of G is conjugate to an element of T.*

Definition 2.2.4. Let G be a semisimple Lie group, and let $P \subset G$ be a **parabolic subgroup**, which is a closed subgroup of G that contains a Borel subgroup B. A **generalised flag variety** is a homogeneous space of the form G/P ($B \subseteq P$). The choice of P determines the structure of the flag variety. For $P = B$, G/P is the **complete flag variety**, parametrising the most detailed breakdown of subspaces. For larger P, G/P is a **coarser flag variety**, parametrising partial decomposition.

Example 2.2.4.1. Let $G = SL_3(\mathbb{C})$ be the group of 3×3 complex matrices with determinant 1. A flag in $\mathbb{C}^3$ is a nested sequence of subspaces:

$$V_1 \subset V_2 \subset \mathbb{C}^3, \quad \dim(V_1) = 1 \text{ and } \dim(V_2) = 2. \tag{2.2.1}$$

The Borel subgroup $B \subset G$ is the group of upper triangular matrices in $SL_3(\mathbb{C})$. The G/B parametrises the complete flags in $\mathbb{C}^3$. That is:

$$\{0\} \subset V_1 \subset V_2 \subset \mathbb{C}^3. \tag{2.2.2}$$

Let P be the parabolic subgroup that stabilises a one-dimensional subspace in $\mathbb{C}^3$. G/P is known as the Grassmannian $Gr(1,3)$, which parametrises all lines (one-dimensional subspaces) in $\mathbb{C}^3$, corresponding to the partial flag

$$\{0\} \subset V_1 \subset \mathbb{C}^3. \tag{2.2.3}$$

The explicit matrix form of P, the stabiliser of the one-dimensional subspace $V_1 \subset \mathbb{C}^3$, is as follows:

$$P = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \in SL_3(\mathbb{C}) \,\middle|\, a_{11}(a_{22}a_{33} - a_{23}a_{32}) = 1 \right\}. \tag{2.2.4}$$

Definition 2.2.5. Let G be a Lie group, and let $\mathfrak{g}$ be its Lie algebra. Let $(\cdot,\cdot)$ be the Killing form of $\mathfrak{g}$. The coadjoint action Ad^* of G in the dual Lie algebra $\mathfrak{g}^*$ is defined by $(Ad^*(g)\beta, X) = (\beta, Ad(g^{-1})X)$, for $\beta \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. Define the **coadjoint orbit** passing through β by $\mathcal{O}_\beta = \{Ad^*(g)\beta : \forall g \in G\}$. In addition, define the stabiliser subgroup of β as $G_\beta = \{g \in G : Ad^*(g)\beta = \beta\}$. Then, $\mathcal{O}_\beta \cong G/G_\beta$. In addition, $\mathcal{O}_\beta$ is a symplectic (flag) manifold with the 2-form given by $\omega_\beta(X,Y) = (\beta, [X,Y])$, for $X, Y \in \mathfrak{g}$.

Example 2.2.5.1. Let $G = SU(2)$, the Lie algebra $\mathfrak{su}(2) \cong \mathbb{R}^3$. The Lie algebra of a maximal torus $\mathfrak{t} \cong \mathbb{R}$. Let $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$, and for every nonzero $\beta \in \mathfrak{t}^*$, the coadjoint orbit $\mathcal{O}_\beta$ is isomorphic to a 2-sphere in $\mathbb{R}^3$. This can also be realised by the construction of *Hopf fibration.*

Theorem 2.2.6 ([34], Theorem.3.1)**.** *Let G be a compact connected semisimple Lie group, then there are finitely many types of coadjoint orbits as a homogeneous manifold under the action of G. That is, if T is a maximal torus of G, then there exists a subgroup G_i, for $1 \leqslant i \leqslant k$ for some finite number k, such that $T \subset G_i \subset G$ and every subgroup between T and G is conjugate to some G_i, and every coadjoint orbit is isomorphic to a homogeneous manifold $X_i = G/G_i$.*

Definition 2.2.7. Let G be a compact connected semisimple Lie group, the homogeneous manifold $X = G/T$ is called the **complete flag manifold** of G, and the others $X_i = G/G_i$ are called the **degenerate flag manifolds**, which can be obtained from the X by a projection whose fibres are isomorphic to lower-dimensional flag manifolds G_i/T.

Example 2.2.7.1. Let $G = SU(3)$. A maximal torus T of $SU(3)$ can be realised as

$$T = \left\{ \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{-i(\theta_1+\theta_2)} \end{pmatrix} \in SU(3) \,\middle|\, \theta_1 \in \mathbb{R}, \theta_2 \in \mathbb{R} \right\}. \tag{2.2.5}$$

The homogeneous manifold $SU(3)/T$ describes the complete unitary flags in $\mathbb{C}^3$. $SU(3)/T$ is also a real form of the complex homogeneous manifold $SL_3(\mathbb{C})/B$ in Example 2.2.4.1. Another real form of $SL_3(\mathbb{C})/B$ is the real homogeneous manifold $SL_3(\mathbb{R})/T_\mathbb{R}$ (which describes the complete orthogonal flags in $\mathbb{R}^3$), where $T_\mathbb{R}$ is a maximal torus of $SL_3(\mathbb{R})$ that has explicit matrix form as follows:

$$T_\mathbb{R} = \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \in SL_3(\mathbb{R}) \,\middle|\, t_1 t_2 t_3 = 1 \right\}. \tag{2.2.6}$$

Let $S(U(2) \times \mathbb{T})$ be a subgroup of $SU(3)$ ('S' is the special condition for the determinant being 1). Then, $SU(3)/S(U(2) \times \mathbb{T})$ is a degenerate flag manifold that parametrises the partial flag

$$\{0\} \subset V_2 \subset \mathbb{C}^3, \tag{2.2.7}$$

where $\dim(V_2) = 2$. Here, $U(2)$ preserves V_2, and $\mathbb{T}$ rotates the orthogonal direction $V_2^{\perp}$, and it ensures all possible ways that a 2-dimensional subspace can sit inside $\mathbb{C}^3$. $SU(3)/S(U(2) \times \mathbb{T})$ is also known as $Gr(2,3)$, which parametrises all planes in $\mathbb{C}^3$.

Definition 2.2.8. Let G be a compact connected semisimple Lie group, T be a maximal torus. Define the normaliser subgroup of G in T by $N(T) = \{g \in G : gTg^{-1} \subset T\}$, and the centraliser $C(T) = \{g \in G : gtg^{-1} = t, \forall t \in T\}$. Then the **Weyl group** of G is given by the quotient group $W = N(T)/C(T)$. Since T is a maximal abelian subgroup of G, it follows that $C(T) = T$. Hence, $W = N(T)/T$.

Example 2.2.8.1. Let $G = SU(2)$. A maximal torus T is defined by

$$T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \middle| \ \theta \in \mathbb{R} \right\}. \tag{2.2.8}$$

The Weyl group of $SU(2)$ acts on T by conjugation and permutes the elements on the diagonal. It follows that the Weyl group has representatives

$$W = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}. \tag{2.2.9}$$

Definition 2.2.9. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{t}$ be a **Cartan subalgebra** of $\mathfrak{g}$. Let Φ be the chosen root system of $\mathfrak{g}$. The **root space decomposition** of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha, \tag{2.2.10}$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X, \forall H \in \mathfrak{t}\}$, and $\alpha(H) = (H_\alpha, H)$ is a linear functional with respect to the Killing form $(\cdot,\cdot)$. In addition, every $\mathfrak{g}_\alpha$ is one-dimensional, and every root space is in pairs, that is, for each $\alpha \in \Phi$, $-\alpha$ is also in Φ.

Example 2.2.9.1. Let the complex semisimple Lie algebra $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. The root system Φ of $\mathfrak{sl}_3(\mathbb{C})$ contains 6 roots:

$$\Phi = \{\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \alpha_1 + \alpha_2, -(\alpha_1 + \alpha_2)\}. \tag{2.2.11}$$

Let $(\cdot,\cdot)$ be the Killing form and $\mathfrak{t}^*$ be the dual Cartan subalgebra. Define

$$\langle \xi, \gamma \rangle = \frac{2(\xi,\gamma)}{(\gamma,\gamma)}, \quad \xi, \gamma \in \mathfrak{t}^*. \tag{2.2.12}$$

For the roots α_1, α_2 above, they satisfy $\langle \alpha_1, \alpha_2 \rangle = \langle \alpha_2, \alpha_1 \rangle = -1$.

Definition 2.2.10. Let $\mathfrak{g}^{\mathbb{C}}$ be a complex semisimple Lie algebra, $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, Φ be the chosen root system of $\mathfrak{g}^{\mathbb{C}}$, Φ^+ be the set of positive roots in Φ, and $\Delta \subset \Phi$ be the subset of **simple roots** (which spans Φ). Let X_α, Y_α be the one-dimensional positive and negative root vectors with respect to a root $\alpha \in \Phi^+$. Also, let H_α be the dual of a root $\alpha \in \Phi$ in $\mathfrak{t}$. The unique (up to isomorphism) **compact real form** $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$ is given by

$$\mathfrak{g} = \sum_{\alpha\in\Delta} \mathbb{R}(iH_\alpha) + \sum_{\alpha\in\Phi^+} \mathbb{R}(X_\alpha - Y_\alpha) + \sum_{\alpha\in\Phi^+} \mathbb{R}i(X_\alpha + Y_\alpha). \tag{2.2.13}$$

Example 2.2.10.1. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_3(\mathbb{C})$. The compact real form of $\mathfrak{sl}_3(\mathbb{C})$ is $\mathfrak{su}(3)$. The root system Φ of $\mathfrak{sl}_3(\mathbb{C})$ is

$$\Phi = \{\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \alpha_1 + \alpha_2, -(\alpha_1 + \alpha_2)\}. \tag{2.2.14}$$

With respect to the root system Φ, the standard basis of $\mathfrak{sl}_3(\mathbb{C})$ ([26], Section 6.5) is given by

$$\begin{gathered} H_{\alpha_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ X_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ Y_{\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y_{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{gathered} \tag{2.2.15}$$

Now, by the compact real form (2.2.13), the standard basis of $\mathfrak{su}(3)$ is given by

$$\begin{gathered}
H_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \\
X_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad (2.2.16) \\
Y_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}.
\end{gathered}$$

Definition 2.2.11. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$, and Φ be the chosen root system of $\mathfrak{g}$. For any $\alpha \in \Phi$, define $k_\alpha = \{H \in \mathfrak{t} : \alpha(H) = 0\}$, a subspace of $\mathfrak{t}$. It is clear that $k_\alpha = k_{-\alpha}$. Let Φ^+ be the subset of all positive roots, and let $\Phi_0^+ \subseteq \Phi^+$, and define the subspace

$$k_{\Phi_0^+} = \{H \in \mathfrak{t} : \alpha(H) = 0, \forall \alpha \in \Phi_0^+\} = \bigcap_{\alpha \in \Phi_0^+} k_\alpha. \qquad (2.2.17)$$

A **wall** of $\mathfrak{t}$ is any subspace of the form $k_{\Phi_0^+}$ for some $\Phi_0^+ \subseteq \Phi^+$.

Definition 2.2.12. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{t}$ be a **Cartan subalgebra** of $\mathfrak{g}$, and Φ be the chosen root system of $\mathfrak{g}$. The **fundamental Weyl chamber** of $\mathfrak{t}$ can be defined by $\mathfrak{t}_+ = \{H \in \mathfrak{t} : \alpha(H) \geqslant 0, \forall \alpha \in \Delta\}$, and the **interior** of $\mathfrak{t}_+$, denoted by $\mathfrak{t}_0$, can be defined by $\mathfrak{t}_0 = \{H \in \mathfrak{t} : \alpha(H) > 0, \forall \alpha \in \Delta\}$. The intersection of a wall with $\mathfrak{t}_+$ is called a **face** of $\mathfrak{t}_+$. Let l be the rank of $\mathfrak{t}$, and $\{H_i\}_{i=1}^l$ be the set of all one-dimensional faces of $\mathfrak{t}_+$, the fundamental Weyl chamber

$$\mathfrak{t}_+ = \left\{ \sum_{i=1}^{l} c_i H_i : \forall c_i \geqslant 0 \right\}. \qquad (2.2.18)$$

In fact, each H_i is called a **fundamental weight vector** of $\mathfrak{t}$.

Example 2.2.12.1. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_3(\mathbb{C})$. The rank of Cartan subalgebra $\mathfrak{t}$ is equal to 2. The **Cartan matrix** C ([30] Sec.13) of $\mathfrak{sl}_3(\mathbb{C})$ is given by

$$C = \begin{pmatrix} \langle \alpha_1, \alpha_1 \rangle & \langle \alpha_1, \alpha_2 \rangle \\ \langle \alpha_2, \alpha_1 \rangle & \langle \alpha_2, \alpha_2 \rangle \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{2.2.19}$$

The inverse of C is

$$C^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}. \tag{2.2.20}$$

Let $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$, and the two fundamental weight vectors of $\mathfrak{t}^*$, λ_1 and λ_2 can be written as

$$\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \quad \lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2. \tag{2.2.21}$$

Note that $\{\lambda_1, \lambda_2\}$ are also the dual basis of $\{\alpha_2, \alpha_1\}$ that spans $\mathfrak{t}^*$. So, the fundamental Weyl chamber of $\mathfrak{t}^*$ is given by

$$\mathfrak{t}^*_+ = \left\{ \sum_{i=1}^{2} c_i \lambda_i \;:\; \forall c_i \geqslant 0 \right\}. \tag{2.2.22}$$

In addition, $\{c_1\lambda_1 : \forall c_1 \geqslant 0\}$ and $\{c_2\lambda_2 : \forall c_2 \geqslant 0\}$ are the minimal walls of $\mathfrak{t}^*_+$.

Lemma 2.2.13 ([18], Lemma 1.2)**.** *Suppose $k \subset \mathfrak{t}$ is a wall, let $\Phi^+_k = \{\alpha \in \Phi^+ : k \subseteq k_\alpha\}$, and $\mathfrak{g}_k = \sum_{\alpha \in \Phi^+_k} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$. If we denote the **centraliser** of k in $\mathfrak{g}$ by $C_{\mathfrak{g}}(k)$, then it is given by*

$$C_{\mathfrak{g}}(k) = \{X \in \mathfrak{g} \;:\; [X, Y] = 0, \forall Y \in k\}. \tag{2.2.23}$$

Then $C_{\mathfrak{g}}(k) = \mathfrak{t} \oplus \mathfrak{g}_k$.

Lemma 2.2.14 ([18], Proposition 4.2)**.** *Fix a wall k in $\mathfrak{t}$, let $k^\perp \subseteq \mathfrak{t}$ be its orthogonal complement in $\mathfrak{t}$, and let $\mathfrak{h}_k = k^\perp \oplus \mathfrak{g}_k$. Then $\mathfrak{h}_k$ is a subalgebra of $C_{\mathfrak{g}}(k)$, is semisimple, commutes with k, and $k^\perp$ is a maximal abelian subalgebra. Define $W_k = \{w \in W \;:\; wH = H, \forall H \in k\}$. Then W_k is the Weyl group of $\mathfrak{h}_k$.*

Definition 2.2.15. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$, and W be the Weyl group of $\mathfrak{t}$. Let $\mathfrak{t}_+$ be the fundamental Weyl chamber of $\mathfrak{t}$, and let $\mathfrak{t}_0$ be the interior of $\mathfrak{t}_+$. Suppose an element $\gamma \in W \cdot \mathfrak{t}_0$, then it is called a **regular element** of $\mathfrak{t}$, otherwise it is called a **singular element** of $\mathfrak{t}$.

Proposition 2.2.16. *Let G be a compact connected semisimple Lie group, T be a maximal torus of G, $\mathfrak{g}$ and $\mathfrak{t}$ be their Lie algebras and $\mathfrak{g}^*, \mathfrak{t}^*$ be their respective duals. The complexification of $\mathfrak{t}$ is a Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$. Let $\beta \in \mathfrak{t}^*$, and G_β be the stabiliser subgroup of β. If β is regular, then $G_\beta = T$. Otherwise, β is singular and lies on a wall of dual Weyl chambers, and $G_\beta \supset T$. Let $\mathfrak{g}_\beta$ be the Lie algebra of G_β. Then, the complexification of $\mathfrak{g}_\beta$, denoted by $\mathfrak{g}_\beta^{\mathbb{C}}$, is given by*

$$\mathfrak{g}_\beta^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\alpha \in \Phi_\beta^+} \left(\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}\right), \tag{2.2.24}$$

where $\Phi_\beta^+ = \{\alpha \in \Phi^+ : \alpha(H_\beta) = 0\}$, and H_β is the dual of β in $\mathfrak{t}$.

Proof. Because $\mathfrak{t}^* \cong \mathfrak{t}$, we can identify $\beta \in \mathfrak{t}^*$ with its dual $H_\beta \in \mathfrak{t}$. The Lie algebra of the stabiliser G_β is given by $\mathfrak{g}_\beta = \{X \in \mathfrak{g} : [X, H_\beta] = 0\}$. If H_β lies off the wall of $\mathfrak{t}$ (in an interior of a Weyl chamber of $\mathfrak{t}$) and because $\mathfrak{t}$ is a maximal abelian subalgebra, it follows that $\mathfrak{g}_\beta = \mathfrak{t}$. Suppose now that H_β lies on a minimal wall k of (a Weyl chamber of) $\mathfrak{t}$, then by Definition 2.2.12, k is spanned by some fundamental weight vectors. Since k is a minimal wall that contains H_β (so they have the same dimension), hence $\Phi_k^+ = \{\alpha \in \Phi^+ : \alpha(H_\beta) = 0\} = \Phi_\beta^+$, the proposition then follows by Lemma 2.2.13. □

Example 2.2.16.1. Let $\mathfrak{g} = \mathfrak{su}(3)$, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_3(\mathbb{C})$ be its complexification, and $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{su}(3)$. Let λ_1 be the first fundamental weight vector in a dual Cartan subalgebra $\mathfrak{t}^*$, and suppose that β lies on the wall $\{c_1\lambda_1 : \forall c_1 \geqslant 0\}$ of $\mathfrak{t}_+^*$ (Example 2.2.12.1). So, $\beta = c\lambda_1$ for some $c \in \mathbb{R}^+$. Let G_β be the stabiliser of β. Since λ_1 is dual to α_2, i.e., $(\lambda_1, \alpha_2) = 0$. So, the complexification of the Lie algebra of the stabiliser $\mathfrak{g}_\beta^{\mathbb{C}}$, is given by

$$\mathfrak{g}_\beta^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \left(\mathfrak{g}_{\alpha_2} + \mathfrak{g}_{-\alpha_2}\right), \tag{2.2.25}$$

where $\mathfrak{t}^{\mathbb{C}}$ contains two copies of the complexification of the Lie algebra of the 1-torus $\mathbb{T}$. Thus, $\mathfrak{g}_\beta^{\mathbb{C}}$ is also the complexification of the Lie algebra of $U(2) \times \mathbb{T}$.

Definition 2.2.17. Let V be a finite-dimensional vector space over $\mathbb{R}$ with a positive definite symmetric bilinear form (α, β), for $\alpha, \beta \in V$. Any nonzero vector α determines a **reflection** σ_α, with respect to a hyperplane $P_\alpha = \{\beta \in V : (\beta, \alpha) = 0\}$. The formula for σ_α is given by

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha, \quad \text{for } \alpha, \beta \in V. \tag{2.2.26}$$

Proposition 2.2.18. *Let β lie on a wall of $\mathfrak{t}^*$. The Weyl orbit of β is isomorphic to the quotient group W/W_β, where $W_\beta = \{w \in W : w\beta = \beta\}$.*

Proof. This follows from the properties of the Weyl group ([30], Theorem 10.3), and an element lying on a wall of $\mathfrak{t}^*$ is orthogonal to a subset of simple roots (with respect to the dual Killing form) corresponding to these simple reflections and the combination of these simple reflections that constitute a Weyl group element. □

Example 2.2.18.1. Let $\Delta = \{\alpha_1, \alpha_2\} \subset \Phi$ be the set of simple roots of $\mathfrak{sl}_3(\mathbb{C})$. Let σ_{α_1} and σ_{α_2} be the simple reflections with respect to α_1 and α_2. The Weyl group W of $\mathfrak{sl}_3(\mathbb{C})$ contains 6 elements and is given by

$$W = \{e,\, \sigma_{\alpha_1},\, \sigma_{\alpha_2},\, \sigma_{\alpha_2}\sigma_{\alpha_1},\, \sigma_{\alpha_1}\sigma_{\alpha_2},\, \sigma_{\alpha_2}\sigma_{\alpha_1}\sigma_{\alpha_2}\}. \tag{2.2.27}$$

Let $\beta = \lambda_1$, which lies on the wall spanned by λ_1 (a fundamental weight vector). So, the Weyl group W_β stabilising β, is given by $W_\beta = \{e, \sigma_{\alpha_2}\}$. Hence, the quotient group W/W_β is given by

$$W/W_\beta = \{e, \sigma_{\alpha_1}, \sigma_{\alpha_2}\sigma_{\alpha_1}\}, \tag{2.2.28}$$

which is isomorphic to the the Weyl orbit of λ_1. That is,

$$W \cdot \beta = \{\lambda_1,\, \lambda_1 - \alpha_1,\, \lambda_1 - \alpha_1 - \alpha_2\}. \tag{2.2.29}$$

Proposition 2.2.19. *The coadjoint orbits of a compact connected semisimple Lie group G are even-dimensional.*

Proof. The dimension of $\mathcal{O}_\beta$ is equal to $\dim(\mathfrak{g}/\mathfrak{g}_\beta)$, and since $\dim(\mathfrak{g}/\mathfrak{g}_\beta) = \dim(\mathfrak{g}^\mathbb{C}/\mathfrak{g}_\beta^\mathbb{C})$, and whether β lies on a wall or not, the complexification of the tangent space $T\mathcal{O}_\beta^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}/\mathfrak{g}_\beta^{\mathbb{C}}$ is always isomorphic to the sum of the pairs of root spaces (positive and negative), by Proposition 2.2.16 and Definition 2.2.9. Hence, $T\mathcal{O}_\beta^{\mathbb{C}}$ is always even-dimensional. □

Remark. *In fact, every coadjoint orbit is even-dimensional, since every $\mathcal{O}$ is a symplectic manifold with a non-degenerate* 2*-form ω. Let P be the maximal isotropic subspace of $T\mathcal{O}^{\mathbb{C}}$, i.e., $\omega(P,P) = 0$, then $\dim P = \frac{1}{2}\dim T\mathcal{O}^{\mathbb{C}}$ by ([1], Proposition 5.3.2).*

Definition 2.2.20. Let G be a compact connected semisimple Lie group, T be a maximal torus of G. Let $\mathfrak{g}, \mathfrak{t}$ be the Lie algebras of G and T, respectively. Let $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. Let $\beta \in \mathfrak{t}^*$, and β is called **integral** if $i\beta(\cdot)$ is the differential of a character χ of T, that is, $d\chi : \mathfrak{t} \to \mathbb{C}$. To be precise, β lies in the integral lattice L in $\mathfrak{t}^*$, for which every $\beta \in L$ satisfies

$$2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z},\ \forall\alpha \in \Phi. \tag{2.2.30}$$

Every character χ of T can be uniquely extended to a holomorphic (non-unitary) character of the Borel subgroup B of $G^{\mathbb{C}}$ that contains T, given by the same formula on $T^{\mathbb{C}}$ and is trivial on the commutator subgroup $N = [B,B]$.

Remark. *A coadjoint orbit containing an element that is in the integral lattice of $\mathfrak{t}^*$ is called an **integral coadjoint orbit**.*

Proposition 2.2.21. *Let G be a compact connected semisimple Lie group, and T be a maximal torus of G. Let $\mathfrak{g}, \mathfrak{t}$ be the Lie algebras of G and T, respectively. Let $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. If $\beta \in \mathfrak{t}^*$ is integral, then the characters of G_β coincide with the characters of T.*

Proof. When β lies on a wall of a Weyl chamber in $\mathfrak{t}$, the stabiliser of β, $G_\beta \supset T$, and since $i\beta$ is trivial on $\sum_{\alpha\in\Phi_\beta^+}(\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$ with respect to the Killing form, so $i\beta$ is well defined in $\mathfrak{g}_\beta$ by Proposition 2.2.16. Otherwise, $G_\beta = T$, so the proposition follows. □

2.3 Holomorphic Sections of Line Bundles

Let $\mathfrak{g}^{\mathbb{C}}$ be a complex semisimple Lie algebra, $\mathfrak{g}$ be its compact real form (unique up to isomorphism), which corresponds to a compact simply connected Lie group

G. The theorem of highest weight classifies the irreducible representations of $\mathfrak{g}^{\mathbb{C}}$ and G. It says that there is a bijection between the set of dominant integral weights and the set of equivalence classes of irreducible representations of $\mathfrak{g}^{\mathbb{C}}$ or G. This theorem was initially developed by E. Cartan, and the version for compact Lie group is due to Hermann Weyl.

The Borel-Weil theory is a modern interpretation of the theorem of the highest weight. It says that an integral weight λ determines a $G^{\mathbb{C}}$-equivariant holomorphic line bundle L_λ on the flag manifold $X = G^{\mathbb{C}}/B$, and $G^{\mathbb{C}}$ acts on its space of global sections $\Gamma(G^{\mathbb{C}}/B, L_\lambda)$. This is an irreducible holomorphic representation if λ is dominant weight.

The orbit method shows how to associate an integral coadjoint orbit of G with an irreducible unitary representation of G. It was introduced by A.A Kirillov [32] to describe the irreducible representations of nilpotent groups, and it generalises the Borel-Weil theorem for compact connected semisimple Lie groups. It can also be applied to classify representations of a semi-direct product of a Lie group with a vector group.

In this section, we look at how to perform quantisation to obtain the representations in holomorphic sections of line bundles of compact connected semisimple Lie groups. However, some definitions are not necessarily limited to compact cases if otherwise specified. The main content of this section is adopted from [13], [15].

Definition 2.3.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{t}$ be a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$, $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$, and Φ be the chosen root system of $\mathfrak{g}$. Let Λ denote the set of **integral element** of $\mathfrak{t}^*$. An integral element $\lambda \in \Lambda$ is said to be **dominant** if $\langle \lambda, \alpha \rangle \geqslant 0, \forall \alpha \in \Phi^+$. Each integral element $\lambda \in \Lambda$ is also called a **weight** if it is associated with an eigenspace of a representation of $\mathfrak{g}$.

Theorem 2.3.2 ([26], Theorem 9.5). ***(E. Cartan's Theorem of Highest Weight)*** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$, and $\mathfrak{t}^*_+$ be the fundamental Weyl chamber of $\mathfrak{t}^*$. Let $\Lambda^+ \subset \mathfrak{t}^*_+$ be the set of all dominant integral weights in $\Lambda \subset \mathfrak{t}^*$. Then for each element $\lambda \in \Lambda^+$, there exists an irreducible finite-dimensional representation of $\mathfrak{g}$ with the highest weight λ.*

Example 2.3.2.1 ([30], Lemma 7.2)**.** Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$. It has a standard basis which is given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.3.1}$$

They satisfy Lie bracket operations $[H, X] = 2X, [H, Y] = -2Y, [X, Y] = H$. Assume V is an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module. H acts diagonally on V and decomposes V into a direct sum of the eigenspaces $V_\mu = \{v \in V \mid H.v = \mu \cdot v\}$, for $\mu \in \Lambda$. Let $\lambda \in \Lambda^+$ be a dominant highest weight. Choose v_λ to be a maximal vector in V_λ, and let $v_{-1} = 0, v_i = (1/i!)Y^i.v_\lambda$ $(i \geqslant 0)$. Then,

1. $H.v_i = (\lambda - 2i)v_i$,
2. $Y.v_i = (i+1)v_{i+1}$,
3. $X.v_i = (\lambda - i + 1)v_{i-1}$ $(i \geqslant 0)$,

where X and Y are called **raising** and **lowering** operators. In this case, the action of Y can be applied successively to v_λ to generate the entire eigenspaces of H. These eigenspaces are also called the **weight spaces** of H, and these weights are from the set $\{\lambda, \lambda - 2, \lambda - 4, ..., -\lambda + 2, -\lambda\} \subset \Lambda$.

Theorem 2.3.3 ([34], Theorem 3.1)**.** *(Borel-Weil) The space $\Gamma(G^{\mathbb{C}}/B, L_\lambda)$ is nonzero exactly when the integral element $\lambda \in \Lambda^+$ is a dominant weight, and in this case $\Gamma(G^{\mathbb{C}}/B, L_\lambda)$ corresponds to an irreducible representation σ_λ of $G^{\mathbb{C}}$ with highest weight λ. Its restriction to the compact real form G is an irreducible unitary representation π_λ with highest weight λ.*

Remark. *The Borel-Weil-Bott theorem [22] is built on the Borel-Weil theorem, and is an extension of the space of sections (the zero cohomology group, denoted by $H^0(G^{\mathbb{C}}/B, L_\lambda)$), to higher cohomology groups $H^k(G^{\mathbb{C}}/B, L_\lambda)$, due to R. Bott.*

Next, we introduce the general procedures of the orbit method.

Definition 2.3.4. Let G be a semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. Let $\mathcal{O}_\beta$ be an integral coadjoint orbit passing through β, and $\chi_\beta : G_\beta \to \mathbb{C}$ the character of the stabiliser subgroup G_β. Define a G-module induced from χ_β as $V_\beta = \{f \in C^\infty(G) : f(g^{-1}x) = \chi_\beta(g)f(x), \forall g \in G_\beta, \forall x \in G\}$, with the action of G_β on the product group $G \times \mathbb{C}$ by $g.(x, \lambda) = (g^{-1}x, \chi_\beta(g)\lambda)$ for

$g \in G_\beta, x \in G, \lambda \in \mathbb{C}$. The orbit of G_β in G is isomorphic to the coset space G/G_β which may be identified by $\mathcal{O}_\beta$ through $gG_\beta \mapsto g \cdot \beta$. If E is the G_β orbit of $G \times \mathbb{C}$, then a smooth function $f \in V_\beta$ induces a section (graph morphism) $\tilde{f} : \mathcal{O}_\beta \to E$ of the line bundle E over $\mathcal{O}_\beta$. We denote the space of sections by $\Gamma(E)$. This procedure is called **pre-quantisation**.

Definition 2.3.5. Let G be a semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$, and $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. For $\beta \in \mathfrak{g}^*$, let $\mathfrak{g}_\beta$ be the Lie algebra of the stabiliser subgroup G_β of β. Let $f \in V_\beta$, as defined above. We define the **β-derivative** in a complex direction $X = X_1 + iX_2 \in \mathfrak{g}^{\mathbb{C}}$ by

$$\tilde{\nabla}_X f(g) = \frac{d}{dt} f(\exp(-tX_1)g)\bigg|_{t=0} + i\,\frac{d}{dt} f(\exp(-tX_2)g)\bigg|_{t=0} - i\beta(X)f(g), \quad (2.3.2)$$

where $\tilde{\nabla}_X f(g)$ again belongs to V_β, since the G-action $g \cdot \tilde{\nabla}_X(f) = \chi(g)\tilde{\nabla}_{g \cdot X}(f) \in V_\beta$, $\forall g \in G_\beta$. Note that if $X \in \mathfrak{g}_\beta^{\mathbb{C}}$, then $\tilde{\nabla}_X f(g) = 0$.

Definition 2.3.6. Let G be a semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$, and $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. For $\beta \in \mathfrak{g}^*$, let $\mathfrak{g}_\beta$ be the Lie algebra of the stabiliser subgroup G_β of β. The β-derivative $\tilde{\nabla}$ induces a linear map $\nabla_X : \Gamma(E) \to \Gamma(E)$. Let $\mathfrak{g}_\beta^{\mathbb{C}}$ be the complexification of the Lie algebra $\mathfrak{g}_\beta$ of G_β. For each $X \in \mathfrak{g}_\beta^{\mathbb{C}}$, $s \in \Gamma(E)$, $\nabla_X s = 0$, and $\nabla_X s$ depends on the coset $X + \mathfrak{g}_\beta^{\mathbb{C}}$. Hence, ∇_X is a well-defined linear map for each $X \in \mathfrak{g}^{\mathbb{C}}/\mathfrak{g}_\beta^{\mathbb{C}} \cong T\mathcal{O}_\beta{}^{\mathbb{C}}$, which is the complexification of the tangent space at β. Since X induces a vector field on $\mathcal{O}$, ∇_X is a **connection**. It is G-invariant because $\nabla_X(g \cdot s) = g \cdot \nabla_{g^{-1} \cdot X}(s)$, $\forall g \in G$ and $\forall s \in \Gamma(E)$.

Definition 2.3.7. Let G be a semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, and $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. Let $\beta \in \mathfrak{g}^*$, and a **polarisation** for the coadjoint orbit $\mathcal{O}_\beta$ equipped with a symplectic form $(\mathcal{O}, \omega)$ is an assignment of a subspace P_β of the complexification of the tangent space at β, denoted by $T\mathcal{O}_\beta^{\mathbb{C}}$, satisfying the following conditions:

1. P_β is a **maximal isotropic subspace** of $T\mathcal{O}_\beta^{\mathbb{C}}$, meaning that $\omega(P_\beta, P_\beta) = 0$ and P_β has maximal dimension among such spaces.
2. The assignment $\beta \to P_\beta$ is **integrable**. If X, Y are complex vector fields over $\mathcal{O}$ taking value in P, then their Lie bracket $[X, Y]$ also takes value in P. That is, for all $\beta \in \mathcal{O}$, if $X(\beta), Y(\beta) \in P_\beta$, then $[X(\beta), Y(\beta)] \in P_\beta$.

3. The assignment $\beta \to P_\beta$ is **smooth**, meaning that if $X \in P_\beta$, then there exists a smooth complex vector field ξ on $\mathcal{O}$ such that $\xi(\beta) = X$.

4. The sum of subspaces $\beta \to P_\beta + \overline{P}_\beta$ is also integrable over $\mathcal{O}$.

Furthermore, a polarisation P is called **G-invariant** if for every $g \in G$, $\beta \in \mathcal{O}$, the polarisation satisfies the equivariance condition:

$$P_{g\cdot\beta} = A_g(P_\beta) \tag{2.3.3}$$

where $A_g : T_\beta\mathcal{O}^{\mathbb{C}} \to T_{g\cdot\beta}\mathcal{O}^{\mathbb{C}}$ is the differential of the G-action of conjugation.

Remark. *The space of sections* $\Gamma(E)$ *is not necessarily irreducible. So, the purpose of polarisation is to help identify an irreducible subspace* $\Gamma(E)' \subseteq \Gamma(E)$ *such that* $\Gamma(E)' = \{s \in \Gamma(E) : \tilde{\nabla}_X s = 0, \forall X \in P\}$*. This procedure of finding* $\Gamma(E)'$ *is called* ***quantisation****.*

Example 2.3.7.1. Let G be a semisimple Lie group, $\mathfrak{g}$ be its Lie algebra. Let $\beta \in \mathfrak{g}$, and any polarisation P_β satisfying

1. $P_\beta \cap \overline{P}_\beta = 0$,

2. $P_\beta + \overline{P}_\beta = T\mathcal{O}^{\mathbb{C}}_\beta$,

is called a **Kähler polarisation**. (There are also non-Kähler Polarisation, e.g., nilpotent Lie groups.)

In fact, assigning a polarisation to a coadjoint orbit is equivalent to finding a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}}$, which is also a maximal isotropic subspace of $\mathfrak{g}^{\mathbb{C}}$ that contains P_β.

Definition 2.3.8. Let G be a semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, and $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. A **complex Lie subalgebra** $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$ corresponds to a polarisation P satisfies the following conditions:

1. $\mathfrak{h} \supseteq \mathfrak{g}^{\mathbb{C}}_\beta$ and $Ad(G_\beta)\mathfrak{h} \subseteq \mathfrak{h}$.

2. $(\beta, [\mathfrak{h}, \mathfrak{h}]) = 0$ and $\mathfrak{h}$ is a maximal isotropic subspace of $\mathfrak{g}^{\mathbb{C}}$.

3. $\tilde{\nabla}_X f = 0$ for $X \in \mathfrak{h}$, $f \in \Gamma(E)'$.

4. $\mathfrak{h} \cap \overline{\mathfrak{h}} = \mathfrak{g}^{\mathbb{C}}_\beta$ and $\mathfrak{h} + \overline{\mathfrak{h}} = \mathfrak{g}^{\mathbb{C}}$.

Example 2.3.8.1. The polarisations for compact connected semisimple Lie groups are Kähler polarisations. Let G be a compact connected semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. Let β be a regular element in the Lie subalgebra of a maximal torus $\mathfrak{t}^* \subset \mathfrak{g}^*$. The Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}}$ corresponding to P_β can be written as

$$\mathfrak{h} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \tag{2.3.4}$$

where each $\mathfrak{g}_\alpha$ is a one-dimensional eigenspace of the adjoint representation of $\mathfrak{t}^{\mathbb{C}}$. Since $[\mathfrak{h}, \mathfrak{h}] \subset \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$, $\mathfrak{h}$ is a maximal isotropic subspace of $\mathfrak{g}^{\mathbb{C}}$. Similarly, if β is not regular, the Lie subalgebra $\mathfrak{h}' \subset \mathfrak{g}^{\mathbb{C}}$ corresponding to the polarisation P_β can be written as

$$\mathfrak{h}' = \mathfrak{g}_\beta^{\mathbb{C}} \oplus \sum_{\alpha \in (\Phi^+)'} \mathfrak{g}_\alpha, \tag{2.3.5}$$

where $(\Phi^+)'$ contains all the positive roots in Φ^+ so that for each $\alpha \in (\Phi^+)'$, $\alpha(H_\beta) \neq 0$, for which H_β is the dual element of β in $\mathfrak{t}^*$. This is because

$$[\mathfrak{h}', \mathfrak{h}'] \subset \sum_{\alpha \in \Phi_\beta^+} \mathbb{C} H_\alpha \oplus \sum_{\alpha \in (\Phi^+)'} \mathfrak{g}_\alpha. \tag{2.3.6}$$

In addition, note that $\mathfrak{g}^{\mathbb{C}}/\mathfrak{h} = \sum_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}$ and $\mathfrak{g}^{\mathbb{C}}/\mathfrak{h}' = \sum_{\alpha \in (\Phi^+)'} \mathfrak{g}_{-\alpha}$. Since each root in Φ appears in a pair, we have $\dim(\mathfrak{g}^{\mathbb{C}}/\mathfrak{h}) = \frac{1}{2} \dim\left(\mathfrak{g}^{\mathbb{C}}/\mathfrak{g}_\beta^{\mathbb{C}}\right) = \frac{1}{2}\dim\left(\mathfrak{g}/\mathfrak{g}_\beta\right)$, and $\dim(\mathfrak{g}^{\mathbb{C}}/\mathfrak{h}') = \frac{1}{2} \dim\left(\mathfrak{g}^{\mathbb{C}}/\mathfrak{g}_\beta^{\mathbb{C}}\right) = \frac{1}{2}\dim\left(\mathfrak{g}/\mathfrak{g}_\beta\right)$.

Proposition 2.3.9. *A real even-dimensional symplectic manifold $(\mathcal{O}, \omega)$ with Kähler polarisation admits a complex structure.*

Proof. Since each coadjoint orbit $\mathcal{O}$ is even-dimensional, so it admits an almost complex structure J, where the linear map $J : T\mathcal{O}^{\mathbb{C}} \to T\mathcal{O}^{\mathbb{C}}$, satisfies $J^2 = -1$. Explicitly, we let $J(X) = iX_1 - iX_2$ for $X_1 \in P$ and $X_2 \in \overline{P}$, where $X = X_1 + X_2 \in T\mathcal{O}^{\mathbb{C}}$. The Newlander–Nirenberg theorem [41] says that an almost complex structure J on a differential manifold M is integrable and induced from a holomorphic structure on M if and only if the Nijenhuis tensor $N_J = 0$. The Nijenhuis tensor is defined by

$$N_J(X, Y) = [X, Y] + J([JX, Y] + [X, JY]) - [JX, JY]. \tag{2.3.7}$$

Since P is integrable, we can show that $N_J(X, Y) = 0$ for $X, Y \in P + \overline{P} = T\mathcal{O}^{\mathbb{C}}$. Hence, the space of sections $\Gamma(E, P)$ is holomorphic. $\square$

2.4 L^2 Condition and Image of Moment Map

Definition 2.4.1. Let G be a compact connected semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, and $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. Let $\mathcal{O}$ be a coadjoint orbit in $\mathfrak{g}^*$ with even dimension n, and ω be the symplectic form on $\mathcal{O}$. The **Liouville measure** $\mu_{\mathcal{O}}$ on a coadjoint orbit $\mathcal{O}$ is defined by $\mu_{\mathcal{O}} = \omega \wedge ... \wedge \omega$, $\frac{1}{2}n$-times. Let $\Gamma^2(E, P)$ denote the finite-dimensional square-integrable irreducible holomorphic sections of a line bundle E with polarisation P (a G-invariant Hilbert space), then the inner product on $\Gamma^2(E, P)$ is defined by

$$\int_{\mathcal{O}} |s(\beta)|^2 \, d\mu_{\mathcal{O}}(\beta) < \infty, \quad \text{for all } s \in \Gamma(E, P). \tag{2.4.1}$$

$\mathcal{O}$ is always compact as it is a continuous image of a compact subset of G.

Remark. *By the Borel-Weil theorem, the representation of a compact Lie group G on $\Gamma^2(E, P)$ is irreducible and unitary.*

Kirillov [33] also proved an analytic version of the character formula, which demonstrates strong connections between irreducible unitary representations and integral coadjoint orbits.

Theorem 2.4.2 ([33], Theorem 2)**.** *Let G be a compact semisimple Lie group, $\mathfrak{g}$ be its Lie algebra. Let λ be a dominant weight in Λ^+, δ be the half-sum of positive roots in Φ^+, and $\mu_{\lambda+\delta}$ be the Liouville measure of the coadjoint orbit passing through $\lambda + \delta$. Let j be the square root of the Jacobian of the exponential map, $\chi_\lambda \circ exp$ be the lift of the character of the irreducible unitary representation with respect to a highest weight λ. Then the* ***Kirillov character formula*** *is given by*

$$j(X)\chi_\lambda(\exp X) = \int_{\mathcal{O}_{\lambda+\delta}} e^{i\beta(X)} \, d\mu_{\lambda+\delta}(\beta), \quad X \in \mathfrak{g}. \tag{2.4.2}$$

Remark. *The Kirillov character formula is exact when G is compact. However, when G is not compact, this formula needs more interpretation, since the coadjoint orbit is no longer compact and $\mu_{\mathcal{O}_{\lambda+\delta}}$ is not compactly supported. In Chapter 3, we give an alternative interpretation of this formula, which is a special case of the result of an G-invariant differential operator with constant coefficient acting on a G-invariant function in $\mathfrak{g}$.*

Example 2.4.2.1. Let $G = SU(3)$, its Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$, $\mathfrak{t}$ be the subalgebra of a maximal torus of $\mathfrak{su}(3)$, and $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{su}(3)^{\mathbb{C}}$. Let $\lambda = \lambda_1$ be a dominant weight in Λ^+, equal to the first fundamental weight of $\mathfrak{h}$. Also, let $\{\alpha_1, \alpha_2\}$ be the set of simple roots of $\mathfrak{h}$, and the weights of the irreducible highest weight representation $d\pi_{\lambda_1}$ of λ_1 are in the set $\{\lambda_1, \lambda_1 - \alpha_1, \lambda_1 - \alpha_1 - \alpha_2\}$. The character formula for λ_1 is given by

$$\chi_{\lambda_1}(\exp H) = \frac{\sum_{w \in W} sgn(w) e^{iw(\lambda_1+\delta)(H)}}{\sum_{w \in W} sgn(w) e^{iw(\delta)(H)}}, \quad H \in \mathfrak{t}, \tag{2.4.3}$$

where W is the Weyl group of $\mathfrak{h}$, and $\delta = \alpha_1 + \alpha_2$. The j-function is given by

$$j(H) = \frac{\sum_{w \in W} sgn(w) e^{iw(\delta)(H)}}{\prod_{\alpha \in \Phi^+} i\alpha(H)}, \quad H \in \mathfrak{t}, \tag{2.4.4}$$

where $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. Hence, the Kirillov character formula for λ_1 of $SU(3)$ is given by

$$\int_{\mathcal{O}_{\lambda_1+\delta}} e^{i\beta(H)}\, d\mu_{\lambda_1+\delta}(\beta) = \frac{\sum_{w \in W} sgn(w) e^{iw(\lambda_1+\delta)(H)}}{\prod_{\alpha \in \Phi^+} i\alpha(H)}, \quad H \in \mathfrak{t}, \tag{2.4.5}$$

where $\mu_{\lambda_1+\delta}$ is the Liouville measure of the coadjoint orbit $\mathcal{O}_{\lambda_1+\delta}$.

Proposition 2.4.3. *Let G be a compact connected semisimple Lie group. Let $\Gamma^2(E,P)$ be a finite-dimensional square-integrable irreducible holomorphic sections of line bundle E with polarisation P such that the representation π of G on $\Gamma^2(E,P)$ is irreducible and unitary. Also, let $P\Gamma^2(E,P)$ denote the projective space of all complex lines in $\Gamma^2(E,P)$. Then $P\Gamma^2(E,P)$ is a G-Hamiltonian manifold, and the moment map $\Psi : P\Gamma^2(E,P) \to \mathfrak{g}^*$ is given by*

$$\Psi([s])(X) = \frac{1}{i}\frac{\langle d\pi(X)s, s\rangle}{\langle s, s\rangle}, \tag{2.4.6}$$

for $X \in \mathfrak{g}$, $[s] \in P\Gamma^2(E,P)$, where $d\pi$ is the infinitesimal version of the representation π of G.

Proof. See Example 2.1.7.3. □

Definition 2.4.4. Let G be a compact connected semisimple Lie group, $\mathfrak{g}$ be its Lie algebra. The image of the moment map Ψ of a representation π of G in the dual Lie algebra $\mathfrak{g}^*$ is defined by

$$I_\pi = \left\{ \Psi([s]) \;:\; [s] \in P\Gamma^2(E, P) \right\}, \tag{2.4.7}$$

and I_π is called the **moment set** of the representation π.

Theorem 2.4.5 ([55], Theorem 3.4)**.** *Let G be a compact semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, and $\mathfrak{h}^*$ be the dual of $\mathfrak{h}$. Let $(\mathcal{H}, \pi_\lambda)$ be an irreducible unitary representation of G with respect to a highest weight $\lambda \in \Lambda^+ \subset \mathfrak{h}^*$. Then the extreme set of the convex hull of the moment set I_λ is a single coadjoint orbit $\mathcal{O}_\lambda$, and $I_\pi \subseteq conv(\mathcal{O}_\lambda)$.*

Remark. *The core idea here is to relate irreducible holomorphic sections of the line bundles of an integral coadjoint orbit to the moment set of an irreducible unitary representation.*

2.5 An Exposition of $SU(2)$

We demonstrate the orbit method by first performing geometric quantisation on $SU(2)$. Then, we show how to explicitly calculate all irreducible unitary representations of $SU(2)$ acting on homogeneous polynomials. Lastly, we associate each irreducible unitary representation of $SU(2)$ to a measure with the support contained in the convex hull of a coadjoint orbit of $SU(2)$, in the dual of its Lie algebra $\mathfrak{su}(2)^*$.

The matrix form of the 2 × 2 special unitary group $SU(2)$ is defined by

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \;:\; |\alpha|^2 + |\beta|^2 = 1,\ \alpha, \beta \in \mathbb{C} \right\}, \tag{2.5.1}$$

and the Lie algebra of $SU(2)$, denoted by $\mathfrak{su}(2)$, is defined by the skew-Hermitian matrices with zero trace,

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ix_3 & x_1 + ix_2 \\ -x_1 + ix_2 & -ix_3 \end{pmatrix} \;:\; (x_1, x_2, x_3) \in \mathbb{R}^3 \right\}, \tag{2.5.2}$$

and the standard basis is given by

$$X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{2.5.3}$$

Now $SU(2)$ acts on $\mathfrak{su}(2)$ by the adjoint action, which can be realised as the group of rotations in $\mathbb{R}^3$ (each rotation has determinant 1). The orbit of this rotation always intersects an element x_3X_3 for $x_3 > 0$. The coadjoint action can also be realised on the dual $(\mathbb{R}^3)^* \cong \mathbb{R}^3$, and we denote the dual basis by $\{X_i^*\}$ for $i = 1, 2, 3$. Hence, the coadjoint orbit contains a unique point of the form $\xi_{\mathbb{R}} = \xi X_3^*$, $\xi > 0$, on the real line $\mathbb{R}$.

Pre-quantisation:

The stabiliser of every $\xi_{\mathbb{R}}$ is the 1-torus of $SU(2)$ defined by

$$\mathbb{T} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix} \ : \ |\alpha| = 1 \right\}, \tag{2.5.4}$$

and its Lie algebra is $\mathfrak{t} = \{rX_3 \ : \ r \in \mathbb{R}\}$. The characters of $\mathbb{T}$ are exactly $\chi_n(\alpha) = \alpha^n$, for integers n, such that they have derivatives ξ_n. Therefore, every integral coadjoint orbit of $SU(2)$ is exactly a 2-sphere $\mathcal{O}_{\xi_n}$ in $\mathbb{R}^3$ with an integer radius of n.

Polarisation (Kähler):

The complexification $\mathfrak{su}(2)^{\mathbb{C}}$ has the basis

$$H = \frac{1}{i}X_3, \ \ X = \frac{1}{2}(X_1 - iX_2), \ \ Y = -\frac{1}{2}(X_1 + iX_2), \tag{2.5.5}$$

and H diagonalises $\mathfrak{su}(2)^{\mathbb{C}}$ as $[H, H] = 0$, $[H, X] = 2X$ and $[H, Y] = -2Y$. Hence, we choose the Lie subalgebra

$$\mathfrak{h} = \{\lambda_1 H + \lambda_2 X \ : \ \lambda_1, \lambda_2 \in \mathbb{C}\}, \tag{2.5.6}$$

to be the polarisation for $\mathfrak{su}(2)$. Note that $\dim P_\xi = \dim\{\lambda X\} = \frac{1}{2}\dim T\mathcal{O}_\xi^{\mathbb{C}}$, and $\dim \mathfrak{su}(2)^{\mathbb{C}}/\mathfrak{h} = \frac{1}{2}\dim \mathfrak{su}(2)/\mathfrak{su}(2)_\xi$. Also, $\xi([\mathfrak{h}, \mathfrak{h}]) = \xi(\{\lambda X\}) = \{0\}$ for $\lambda \in \mathbb{C}$, so that $\mathfrak{h}$ is the maximal isotropic subspace of $\mathfrak{su}(2)^{\mathbb{C}}$, and $\mathfrak{h} + \overline{\mathfrak{h}} = \mathfrak{su}(2)^{\mathbb{C}}$.

Induced representations and holomorphic sections:

Let $f(\alpha, \beta)$ be a function on $SU(2)$. Given a character χ_n, we define a $SU(2)$-module that satisfies

$$\mathcal{H}_n = \{f(\alpha, \beta) \in C^\infty(G) \ : \ f(\alpha e^{it}, \beta e^{-it}) = e^{int} f(\alpha, \beta)\}. \tag{2.5.7}$$

This definition can also be extended from $|\alpha|^2 + |\beta|^2 = 1$ to $\mathbb{C}^2$. Let $(z, w) \in \mathbb{C}^2$,

$e^{int}f(\alpha,\beta) \to u^n f(z,w)$. This is equivalent to $f(z,w^{-1}) = w^n f(\frac{z}{w},1)$. Let $f(z,1) = \tilde{f}(z)$, and $\tilde{f}(z)$ be a function such that $\lim_{z\to\infty} z^{-n}\tilde{f}(z)$ must exist.

Holomorphic condition:

By the polarisation, we can further cut down $\mathcal{H}_n$, by letting f satisfy

$$X.f(\alpha,\beta) = (X_2 - iX_3).f(\alpha,\beta) = 0, \tag{2.5.8}$$

and this translates into $(X_2 - iX_3).\tilde{f}(z) = (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})\tilde{f}(z) = 0$, or $\frac{\partial}{\partial x}\tilde{f}(z) = i\frac{\partial}{\partial y}\tilde{f}(z)$. Hence, $\tilde{f}(z)$ satisfies the Cauchy-Riemann equations, so it is a holomorphic function on $\mathbb{C}$, i.e., an entire function. It is also meromorphic since $\lim_{z\to\infty} z^{-n}\tilde{f}(z)$ must exist.

By Liouville's theorem, if f is meromorphic at infinity, in this case the point at infinity is a pole of order n if $n > 0$. If an entire function has a pole of order n at infinity, that is, it grows in magnitude comparably to z^n in some neighborhood of ∞, then f is a polynomial of degree n. Therefore, let π be a right regular representation, we have

$$\begin{aligned} \pi\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}\tilde{f}(z) &= f(\alpha z - \overline{\beta}, \beta z + \overline{\alpha}) \\ &= (\beta z + \overline{\alpha})^n \tilde{f}\left(\frac{\alpha z - \overline{\beta}}{\beta z + \overline{\alpha}}\right), \end{aligned} \tag{2.5.9}$$

and with respect to z^k for $0 \leqslant k \leqslant n$,

$$\pi\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} z^k = (\alpha z - \overline{\beta})^k(\beta z + \overline{\alpha})^{n-k}. \tag{2.5.10}$$

L^2 conditions:

We can also implement an L^2 condition on the space of complex homogeneous polynomials of degree n, so we can retrieve the well-known matrices of the irreducible unitary representations of $SU(2)$ which were explicitly calculated in [29].

Theorem 2.5.1 ([29], Theorem 29.18, [21], Lemma 5.33)**.** *Let l be a nonnegative half integer in the set $\{0, \frac{1}{2}, 1, \frac{3}{2}, ...\}$. Let P_l be the space of homogeneous polynomials of a single indeterminate z spanned by monomials $\{z^{2l}, z^{2l-1}, ..., 1\}$. Let π_l be an irreducible representation of $SU(2)$ with respect to l, acting on P_l.*

There exists an inner product $\langle , \rangle$ *on* P_l *making* P_l *into a Hilbert space, so that the irreducible representation* π_l *of* $SU(2)$ *is unitary on* P_l*. Let* $\{\xi_j\}$ *be a basis of* P_l*, where each* $\xi_j = z^{l-j}$*, for* $j = -l, -l+1, ..., l$*. For* $j, k \in \{-l, -l+1, ..., l\}$*, then*

$$\langle \xi_j, \xi_k \rangle = \begin{cases} (l-j)!(l+j)!, & \text{if } j = k. \\ 0, & \text{otherwise.} \end{cases} \tag{2.5.11}$$

Irreducibility and explicit calculations:

Definition 2.5.2. ([29], Definition 29.14) Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra. Let π be a representation of G, the **infinitesimal representation** $d\pi$ is defined by

$$d\pi(X) = \lim_{t \to 0} \frac{\pi(\exp tX) - I}{t}, \quad X \in \mathfrak{g}. \tag{2.5.12}$$

Lemma 2.5.3 ([29], Lemma 29.16)**.** *Let* $f(z) \in P_l$*, and let* H, X, Y *be in* (2.5.5)*. Then the infinitesimal representations* $d\pi(X), d\pi(Y), d\pi(H)$ *can be represented by the following complex differential operators:*

$$\begin{aligned} d\pi(X)f(z) &= \left(-2lz + z^2 \frac{\partial}{\partial z}\right) f(z), \\ d\pi(Y)f(z) &= -\frac{\partial}{\partial dz} f(z), \\ d\pi(H)f(z) &= \left(l - z\frac{\partial}{\partial z}\right) f(z). \end{aligned} \tag{2.5.13}$$

Proof. $d\pi$ is a linear map (and also a Lie homomorphism), so it suffices to work out $d\pi(X_i)$ for $i = 1, 2, 3$ in (2.5.3), acting on polynomials z^k, that is,

$$\begin{aligned} d\pi(X_i)z^k &= \lim_{t \to 0} \frac{\pi(\exp tX)z^k - z^k}{t} \\ &= \frac{d}{dt}\bigg|_{t=0} \left(X_i^{11}(t)z + X_i^{21}(t)\right)^k \left(X_i^{12}(t)z + X_i^{22}(t)\right)^{2l-k}, \end{aligned} \tag{2.5.14}$$

where $X_i^{j,k}$ is a matrix entry of $\exp(tX_i)$. Hence, the result follows. ☐

Theorem 2.5.4 ([21], Theorem 5.37, [26], Proposition 4.11)**.** π_l *is irreducible for all* $l \in \{0, \frac{1}{2}, 1, \frac{3}{2}, ...\}$.

Proof. Suppose that M is a π-invariant subspace of P_l. Let $f \in M$, then $t^{-1}\pi(\exp(tX_i))f - f$ also belongs to M. As $t \to 0$, this polynomial approaches

$d\pi(X_i)f$, which is still in M because P_l is finite dimensional, so M is a closed subspace in P_l. Let $f \neq 0 \in M$, and write $f(z) = \sum_{j=0}^{2l} c_j z^j$. Let J be the largest of j such that $c_j \neq 0$, then $d(Y)f(z) = c_J J!$ for $d(Y)$ in (2.5.13). Therefore, $1 \in M$, and if we successively apply $d\pi(X)$ to it, we obtain polynomials of $z, z^2, ..., z^{2l}$, which all belong to M. Therefore, $M = P_l$. ☐

Theorem 2.5.5 ([29], Theorem 29.18). *Let $(\alpha, \beta) \neq 0$, then for any nonnegative half integer $l \in \{0, \frac{1}{2}, 1, \frac{3}{2}, ...\}$, the (j,k)-th matrix coefficient of the irreducible unitary representation π_n is given by*

$$\pi_{j,k}^{(l)}(\alpha,\beta) = (-1)^{j-k} A_{j,k}^{(l)} \sum_{s=max\{0,k-l\}}^{min\{l+k,l-j\}} (-1)^s \binom{l+k}{s} \binom{l-k}{l-j-s} \cdot \alpha^{l-j-s} \overline{\alpha}^{l+k-s} \beta^s \overline{\beta}^{j-k+s}, \tag{2.5.15}$$

where $j,k \in \{-l, -l+1, ..., l\}$ and

$$A_{j,k}^{(l)} = \left(\frac{(l-j)!(l+j)!}{(l-k)!(l+k)!} \right)^{\frac{1}{2}}. \tag{2.5.16}$$

For $\beta = 0$, $\pi_{j,k}^{(l)}(\alpha, 0) = \alpha^{-2k}\delta_{j,k}$, and for $\alpha = 0$, $\pi_{j,k}^{(l)}(0,\beta) = \beta^{2k}(-1)^{l-k}\delta_{j,-k}$.

Proof. See [29], Section 29. The proof is also reworked in the author's honours thesis [24]. ☐

Remark. *The half-integer parametrisation of representations is inspired by the half integer spin in physics. There is also an analytic version of this formula in* ([21], Section 5.4) *with respect to the non-negative whole integers. Let n be a nonnegative integer, the (j,k)-th matrix coefficient of the irreducible unitary representation $\pi^{(n)}$ of $SU(2)$ is given by*

$$\pi_{j,k}^{(n)}(\alpha,\beta) = \sqrt{\frac{j!(n-j)!}{k!(n-k)!}} \int_0^1 (\overline{\alpha}e^{i2\pi t} + \beta)^k (-\overline{\beta}e^{i2\pi t} + \alpha)^{n-k} e^{-i2\pi jt}\, dt, \tag{2.5.17}$$

where $j,k \in \{0,1,...,n\}$ and $\pi^{(n)}$ acts on a Hilbert space $\mathcal{H}_n$ with an orthonormal basis given by

$$\left\{ \sqrt{\frac{(n+1)!}{j!(n-j)!}} z^{n-j} w^j \ : \ 0 \leqslant j \leqslant n \right\}, \tag{2.5.18}$$

for $(z, w) \in \mathbb{C}^2$*, and this set of basis is derived with respect to the inner product*

$$\langle z^p w^r, z^q w^s \rangle = \int_{\mathbb{S}^3} z^p \overline{z}^q w^r \overline{w}^s \, d\sigma(z, w) = \begin{cases} p!r!/(p+r+1)! & \text{if } p = q \text{ and } s = r, \\ 0 & \text{if } p \neq q \text{ or } w \neq s, \end{cases} \tag{2.5.19}$$

where $d\sigma$ *is the Euclidean surface measure of the* 3*-sphere in* $\mathbb{C}^2$*. In addition, the derivation of this analytic formula is obtained by finding the Fourier coefficient of a finite Fourier series.*

Example 2.5.5.1. We can write down $\pi^{(n)}$ for $n = 1, 2, 3$:

$$\begin{aligned} \pi^{(1)} &= \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \\ \pi^{(2)} &= \begin{pmatrix} \alpha^2 & \sqrt{2}\alpha\beta & \beta^2 \\ -\sqrt{2}\alpha\bar{\beta} & \alpha\bar{\alpha} - \beta\bar{\beta} & \sqrt{2}\bar{\alpha}\beta \\ \bar{\beta}^2 & -\sqrt{2}\bar{\alpha}\bar{\beta} & \bar{\alpha}^2 \end{pmatrix} \\ \pi^{(3)} &= \begin{pmatrix} \alpha^3 & \sqrt{3}\alpha^2\beta & \sqrt{3}\alpha\beta^2 & \beta^3 \\ -\sqrt{3}\alpha^2\bar{\beta} & \alpha^2\bar{\alpha} - 2\alpha\beta\bar{\beta} & 2\alpha\bar{\alpha}\beta - \beta^2\bar{\beta} & \sqrt{3}\bar{\alpha}\beta^2 \\ \sqrt{3}\alpha\bar{\beta}^2 & -2\alpha\bar{\alpha}\bar{\beta}^2 + \beta\bar{\beta}^2 & \alpha\bar{\alpha}^2 - 2\bar{\alpha}\beta\bar{\beta} & \sqrt{3}\bar{\alpha}^2\beta \\ -\bar{\beta}^3 & \sqrt{3}\bar{\alpha}\bar{\beta}^2 & -\sqrt{3}\bar{\alpha}^2\bar{\beta} & \bar{\alpha}^3 \end{pmatrix} \end{aligned} \tag{2.5.20}$$

Measures in $\mathfrak{su}(2)^*$:

F. Cazzaniga [9] derived a formula which calculates the irreducible unitary representations of $SU(2)$ from its Lie algebra $\mathfrak{su}(2)$, and made a connection between the irreducible unitary representations in Theorem 2.5.4 and non-G-invariant measures supported in the convex hull of integral coadjoint orbits of $SU(2)$. These coadjoint orbits can be realised as 2-spheres in the dual space $\mathfrak{su}(2)^* \cong (\mathbb{R}^3)^*$ with integer radius. We briefly discuss Cazzaniga's calculations in the following part.

Theorem 2.5.6 ([9], Proposition 1.2)**.** *Let* X_1, X_2, X_3 *be the* 2×2 *matrices defined in* (2.5.3)*. Let* $X = x_1X_1 + x_2X_2 + x_3X_3$*, for* $(x_1, x_2, x_3) \in \mathbb{R}^3$*, and* *define* $|X| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, *and note that* $X^2 = -|X|^2 I$. *Then*

$$\begin{aligned} \exp X &= \cos|X|\, I + \frac{\sin|X|}{|X|} X \\ &= \begin{pmatrix} \cos|X| + ix_3 \frac{\sin|X|}{|X|} & (x_1 + ix_2)\frac{\sin|X|}{|X|} \\ (-x_1 + ix_2)\frac{\sin|X|}{|X|} & \cos|X| - ix_3\frac{\sin|X|}{|X|} \end{pmatrix}. \end{aligned} \tag{2.5.21}$$

Proof. The Taylor expansion of the matrix exponential of X is

$$\begin{aligned} \exp X &= \sum_{n=0}^{\infty} \frac{X^n}{n!} \\ &= \sum_{n=0}^{\infty}\left((-1)^n \frac{|X|^{2n}}{(2n)!}\right) I + \sum_{n=0}^{\infty}\left((-1)^{n+1}\frac{|X|^{2n+1}}{(2n+1)!}\right)\frac{X}{|X|} \\ &= \cos|X|\, I + \frac{\sin|X|}{|X|} X. \end{aligned} \tag{2.5.22}$$

$\square$

Remark. *Every* 2×2 *matrix* X *with zero trace satisfies* $X^2 = -|X|^2 I$ ([26], Exercise 2.6.6).

Theorem 2.5.7 ([9], Proposition 1.4). *Let* μ_1 *be the Liouville measure of the coadjoint orbit* $\mathcal{O}_1$ *of* $SU(2)$, *which is the unit sphere in* $\mathbb{R}^3$. *Then, by the Kirillov character formula, the* j*-function is given by*

$$j(X) = \int_{\mathcal{O}_1} e^{i\beta(X)}\, d\mu_1(\beta) = \frac{\sin|X|}{|X|}, \quad \forall X \in \mathfrak{g}. \tag{2.5.23}$$

Proof. By the Kirillov character formula (2.4.2), if we let $\lambda = 0, \delta = 1$, then we can obtain this formula. Also, since every coadjoint orbit is a 2-sphere passing through a point $r > 0 \in \mathbb{R}$, hence,

$$j(r) = \int_0^{\infty} \delta_1(\eta)\frac{\sin r\eta}{r\eta}\eta^2\, d\eta, \tag{2.5.24}$$

where the right-hand side is the radial Fourier transform in $\mathbb{R}^3$ ([49], Sec.7.7). Hence, the result follows. $\square$

Proposition 2.5.8. *Let* α, β *be the monomials that span the matrix coefficients in Theorem* 2.5.5. *Let* $r\frac{\partial}{\partial r}$ *be a radial differential operator in* $\mathbb{R}^3$ *defined by*

$r\frac{\partial}{\partial r} = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}$, for $(x_1, x_2, x_3) \in \mathbb{R}^3$. Then,

$$\alpha\left(\exp(X)\right) = \left(r\frac{\partial}{\partial r} + 1 + ix_3\right)\frac{\sin(|X|)}{|X|}, \quad \beta\left(\exp(X)\right) = (x_1 + ix_2)\frac{\sin(|X|)}{|X|}, \tag{2.5.25}$$

and the Fourier transform of $r\frac{\partial}{\partial r}$ is $-r^*\frac{\partial}{\partial r*}-3$, where $\frac{\partial}{\partial r*}$ is the radial differential operator in the dual space $(\mathbb{R}^3)^*$.

Proof. These formulas are straightforward from direct computations. □

Proposition 2.5.9. *Let $\pi^{(n)}$ be an irreducible unitary representation of $SU(2)$ with respect to an integer n. Then, the Fourier transform of $j(X)\pi^{(n)}_{i,j}(\exp X)$ of $SU(2)$ in the Lie algebra $\mathfrak{su}(2)$, is a non-G-invariant measure $D^{(n)}_{i,j}(\mu_1)^{*n+1}$, where $D^{(n)}_{i,j}$ is a polynomial of differential operators consisting of $r^*\frac{\partial}{\partial r^*}, \frac{\partial}{\partial x_1^*}, \frac{\partial}{\partial x_2^*}, \frac{\partial}{\partial x_3^*}$ in $(\mathbb{R}^3)^*$, and $(\mu_1)^{*k}$ is the k-fold convolution of the surface measure μ_1 of 2-sphere in $\mathbb{R}^3$.*

Proof. From Theorem 2.5.5 and Proposition 2.5.8, we know that every matrix coefficient of an irreducible unitary representation of $SU(2)$ is a polynomial of the form $c\,\alpha^p\beta^q$, for $c \in \mathbb{C}, p, q \in \mathbb{Z}^+$. The Fourier transforms of $\alpha\left(\exp(X)\right)$ and $\beta\left(\exp(X)\right)$ in $\mathfrak{su}(2)$ are

$$\left(-r\frac{\partial}{\partial r^*} - 2 + \frac{\partial}{\partial x_3^*}\right)\mu_1 \quad \text{and} \quad \left(-i\frac{\partial}{\partial x_1^*} + \frac{\partial}{\partial x_2^*}\right)\mu_1, \tag{2.5.26}$$

respectively. Also, because μ_1 is a radial distribution supported on the 2-sphere in $(\mathbb{R}^3)^*$ with radius 1, so $r^*\frac{\partial}{\partial r^*}\mu_1 = \frac{\partial}{\partial r^*}\mu_1$ as a distribution. Therefore, the Fourier transform of each j-multiplied matrix coefficient $j(X)\pi^{(n)}_{i,j}(\exp X)$ is a non-G-invariant measure that has the form

$$c \cdot \left(-r^*\frac{\partial}{\partial r^*} - 2 + \frac{\partial}{\partial x_3^*}\right)^p \left(-i\frac{\partial}{\partial x_1^*} + \frac{\partial}{\partial x_2^*}\right)^q (\mu_1)^{*(n+1)}. \tag{2.5.27}$$

where $p + q = n$. □

Remark. *This proposition interprets Cazzaniga's perspective of these non-G-invariant measures $D^{(n)}_{i,j}(\mu_1)^{*n+1}$. However, the author has a different point of view. Notice that the radial differential operator $r^*\frac{\partial}{\partial r^*}$ does not commute with the convolution operator, i.e., $r^*\frac{\partial}{\partial r^*}\mu_1 * r^*\frac{\partial}{\partial r^*}\mu_1 \neq r^*\frac{\partial}{\partial r^*}r^*\frac{\partial}{\partial r^*}(\mu_1 * \mu_1)$. Hence, these are not the same measures and the differential operators $D^{(n)}_{i,j}$ here are not* *accurately represented. It should keep its original form, which is*

$$c \cdot \mu_1 * \left(\left(-r^* \frac{\partial}{\partial r^*} - 2 + \frac{\partial}{\partial x_3^*} \right) \mu_1 \right)^{*p} * \left(\left(-i \frac{\partial}{\partial x_1^*} + \frac{\partial}{\partial x_2^*} \right) \mu_1 \right)^{*q}. \tag{2.5.28}$$

Cazzaniga also worked out the support and singular support of $D_{i,j}^{(n)}(\mu_1)^{*n+1}$.

Theorem 2.5.10 ([9], Lemma 5)**.** *The support and singular support of the measure $D_{i,j}^{(n)}(\mu_1)^{*n+1}$ are*

$$supp \left(D_{i,j}^{(n)}(\mu_1)^{*n+1} \right) \subseteq B_{n+1}, \quad singsupp \left(D_{i,j}^{(n)}(\mu_1)^{*n+1} \right) = \bigcup_{j=0}^{\lfloor \frac{n+1}{2} \rfloor} \partial B_{n+1-2j}, \tag{2.5.29}$$

where B_k is a ball in $(\mathbb{R}^3)^*$ with radius k, and ∂B_k is its boundary, the 2-sphere with radius k.

Remark. *In fact,*

- *The k-fold convolution of unit spheres produces a measure that is supported in the ball with radius k. To determine its singular support, F. Cazzaniga used a recursive method to show where it is not differentiable. For more detail, see [9].*
- *Let I_{π_n} denote the moment set of the irreducible unitary representation π_n of $SU(2)$ with respect to an integer n. By Theorem 2.4.5, $I_{\pi_n} \subseteq conv(\mathcal{O}_n)$. The support of $(\mu_1)^{*n}$ is exactly the $conv(\mathcal{O}_n)$.*

However, R. Raffoul [17] has a different result for the singular support of these measures in Proposition 2.5.9.

Theorem 2.5.11 ([17], Theorem 3)**.** *Let π_n be an irreducible unitary representation of $SU(2)$ with respect to a positive integer n. Let T_n denote the Fourier transform of $j(X)\pi_n(\exp X), X \in \mathfrak{su}(2)$. Then*

$$supp\,(T_n) \subseteq B_{n+1}, \quad singsupp\,(T_n) = \partial B_{n-1} \cup \partial B_{n+1}. \tag{2.5.30}$$

Intuitively, Cazzaniga's result of singular support is more ideal. Because he proved that the singular support of $(\mu_1)^{*(n+1)}$ is exactly the union of coadjoint orbits where each of them intercepts with an integer weight of π_n which belongs to the weight lattice of the highest weight $n+1$. However, Raffoul used the explicit formula for the Weyl calculus for the finite-dimensional skew-Hermitian operator $d\pi$ [40] supported on the moment set of an irreducible unitary representation π of a compact Lie group [55] to study the singular support for these measures. However, we have a different observation and a different point of view from Raffoul's result of singular support.

Proposition 2.5.12. *Let π_n be an irreducible unitary representation of $SU(2)$ with respect to a positive integer n. Let T_n denote the Fourier transform of $j(X)\pi_n(\exp X)$. Then,*

$$\mathit{singsupp}\,(T_n) \subset \bigcup_{j=0}^{\lfloor \frac{n+1}{2} \rfloor} \partial B_{n+1-2j}. \tag{2.5.31}$$

Proof. Recall the general form of the non-G-invariant measures in $\mathfrak{su}(2)^*$ in (2.5.28):

$$c \cdot \mu_1 * \left(\left(-r^* \frac{\partial}{\partial r^*} - 2 + \frac{\partial}{\partial x_3^*} \right) \mu_1 \right)^{*p} * \left(\left(-i \frac{\partial}{\partial x_1^*} + \frac{\partial}{\partial x_2^*} \right) \mu_1 \right)^{*q}. \tag{2.5.32}$$

where c is a constant term. Clearly, this measure contains a term $c \cdot (\mu_1)^{n+1}$, or a term

$$\left(\frac{\partial}{\partial x_1^*} \right)^i \left(\frac{\partial}{\partial x_2^*} \right)^j \left(\frac{\partial}{\partial x_1^*} \right)^k \cdot (\mu_1)^{n+1}. \tag{2.5.33}$$

Because a constant-coefficient differential operator generally decreases the support of the measure, and differentiation does not introduce new singularities, so the result follows. □

In addition, our interest is to calculate the measures corresponding to the irreducible unitary representations of any other compact connected semisimple Lie groups. For example, even those as simple as the defining representation of $SU(n)$.

So, what is the explicit formula for the measure supported in a single coadjoint orbit of $SU(n)$ or a general compact Lie group? What is the explicit formula for convolutions of measures of coadjoint orbits of general compact connected semisimple Lie groups? Does the support of the moment set of π_n of $SU(2)$ coincide with the n-fold convolution of single coadjoint orbits for $SU(2)$? What is the explicit formula for this measure of the moment set for general compact connected semisimple Lie groups? These questions were not answered by Cazzaniga and Raffoul, and we aim to make them clearer and try to answer them in the following chapters.

Chapter 3

Matrix Coefficients and Differential Operators

Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra. Let μ be an adjoint invariant distribution on $\mathfrak{g}$ with compact support, and define a central distribution $\Phi(\mu)$ on G to be

$$\langle \Phi(\mu), f \rangle_G = \langle \mu, j\tilde{f} \rangle_{\mathfrak{g}} \tag{3.0.1}$$

where f is a test function on G, j is the square root of the Jacobian, and $\tilde{f}(X) = f(\exp X)$. It is a fundamental question in analysis on Lie groups ([45], Ch.I) to show

$$\Phi(\mu *_{\mathfrak{g}} \nu) = \Phi(\mu) *_G \Phi(\nu) \tag{3.0.2}$$

such that Φ is a homomorphism from the algebra of G-invariant distributions on $\mathfrak{g}$ to the algebra of central distributions on G.

- M. Duflo [20] proved (3.0.2) for μ, ν supported at the origin of $\mathfrak{g}$, which is called the **Duflo isomorphism**, between the centre of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the finite-dimensional Lie algebra $\mathfrak{g}$, identified as the algebra of bi-invariant differential operators on G, and the invariant elements of its symmetric algebra $\mathfrak{g}$, identified as the G-invariant differential operators with constant coefficients on $\mathfrak{g}$. So μ, ν are fundamental solutions of G-invariant differential operators $p(\partial)$ and $q(\partial)$ in $\mathfrak{g}$, and $\Phi(p(\partial)), \Phi(q(\partial))$ are elements in the centre of $\mathcal{U}(\mathfrak{g})$.

- A. Dooley and N. Wildberger [19] proved (3.0.2) for compact connected semisimple Lie groups, for G-invariant distributions μ, ν with compact supports. In this case, Φ is also called the **wrapping map**. The proof only requires fundamental results of central harmonic analysis and the

Kirillov character formula. In fact, it is a homomorphism from the Banach algebra of the G-invariant distributions on $\mathfrak{g}$ to the Banach algebra of central distributions on G. Additionally, the Kirillov character formula is a consequence of this homomorphism [14].

- F. Rouvière [45] developed a 'twisted convolution' for Riemannian symmetric spaces by introducing the e-function, which is related to the sectional curvature. A. Dooley found an explicit formula for the e-function for compact symmetric spaces [16].

Based on the wrapping map of G-invariant distributions, does there exist a version of the wrapping map between non-G-invariant distributions on $\mathfrak{g}$ and non-central distributions on G? The difficulty is that convolution on G is generally not commutative, but convolution is commutative on $\mathfrak{g}$. However, for a compact simply connected semisimple Lie group G, we can study the relationships between convolutions of the matrix coefficients of unitary representations and the vector fields (differential operators) induced by the action of G on $\mathfrak{g}$.

In Section 3.1, we study the properties of the matrix coefficients of unitary representations of compact connected semisimple Lie groups. In Proposition 3.1.7 and Proposition 3.1.11, we show that the convolution of any pair of matrix coefficients of an irreducible highest weight unitary representation π of a compact simply connected semisimple Lie group G is equivalent to applying a polynomial of invariant vector fields on G to the highest weight matrix coefficient $\pi_{1,1}$ of π. If we fix a basis of the Hilbert space $\mathcal{H}_\pi$, then the highest weight matrix coefficient $\pi_{1,1}$ is the $(1,1)$-th element of the matrix of π.

In Section 3.2, we study the *transversality condition* of a semisimple Lie group where the group acts by the adjoint action on its Lie algebra $\mathfrak{g}$, and we also study the general form of a differential operator in $\mathfrak{g}$, which contains transversal differential operators and vector fields induced by the action of G on $\mathfrak{g}$. Subsequently, in Proposition 3.2.5, we show that the convolution of any pair of the lift of matrix coefficients on $\mathfrak{g}$ (by the exponential mapping) of an irreducible highest weight unitary representation π of a compact simply connected semisimple Lie group G can be calculated by applying a polynomial of vector fields induced by the action of G on $\mathfrak{g}$, to the lift of the highest weight matrix coefficient $\pi_{1,1} \circ \exp$ of $\pi \circ \exp$ on $\mathfrak{g}$.

In Section 3.3, we study the general form of the *radial part* of the Laplacian differential operator and constant coefficients G-invariant differential operator in a semisimple Lie algebra $\mathfrak{g}$. In addition, we study an application of this general form in Section 3.4 in which it leads to a derivation of a continuous version of the Kirillov character formula for compact connected semisimple Lie groups.

The majority of the content in this chapter is adapted from [21] and [28].

3.1 Orthogonal 'Rotations' of Matrix Coefficients

Let $G^{\mathbb{C}}$ be a complex semisimple Lie group and let $\Gamma(G^{\mathbb{C}}/B, L_\lambda)$ denote the irreducible holomorphic sections of the flag manifold $G^{\mathbb{C}}/B$ with respect to a highest weight λ (Theorem 2.3.3). When we restrict it to a compact real form G, the representation π_λ of G on $\Gamma(G^{\mathbb{C}}/B, L_\lambda)$ is irreducible and unitary. In addition, if we fix a basis for the Hilbert space $\mathcal{H}_{\pi_\lambda}$, then each row and column of π_λ is naturally a copy of the representation space of π_λ. To see this, we note the following theorem.

Definition 3.1.1. Let G be a compact connected non-abelian semisimple Lie group. If π is any unitary representation of G in a Hilbert space $\mathcal{H}_\pi$, then the smooth functions

$$T_{u,v}(x) = \langle \pi(x)u, v\rangle, \qquad \text{for } u, v \in \mathcal{H}_\pi, x \in G, \tag{3.1.1}$$

are called **matrix coefficients** of π. If $\mathcal{H}_\pi$ is finite dimensional and has dimension d_π, and u and v are members of an orthonormal basis $\{e_j\}$ in $\mathcal{H}_\pi$, then $T_{u,v}(x)$ is one of the entries of the matrix for $\pi(x)$ with respect to that basis, that is

$$\pi_{i,j}(x) = T_{e_i,e_j}(x) = \langle \pi(x)e_j, e_i\rangle, \quad x \in G. \tag{3.1.2}$$

for $i, j \in \{1, ..., d_\pi\}$.

Theorem 3.1.2 ([21], Theorem 5.9)**.** *Suppose π is an irreducible unitary representation of a compact connected non-abelian semisimple Lie group G acting on a finite-dimensional Hilbert space $\mathcal{H}_\pi$. Let $d_\pi = dim(\pi)$. Fix an orthonormal basis $\{e_1, ..., e_{d_\pi}\}$ in $\mathcal{H}_\pi$, and let $\pi_{i,j}$ be the (i, j)-th entry of π. For $i = 1, ..., d_\pi$, let $\mathcal{R}_i$ be the linear span of $\pi_{i,1}, ..., \pi_{i,d_\pi}$ (the i-th row of the matrix $(\pi_{i,j})$) and let $\mathcal{L}_i$ be the linear span of $\pi_{1,i}, ..., \pi_{d_\pi,i}$ (the i-th column of the matrix $(\pi_{i,j})$). Then*

$\mathcal{R}_i$ (resp. $\mathcal{L}_i$) is invariant under the right (resp. left) regular representation, and the right regular representation $R^{\mathcal{R}_i}$ (resp. the left regular representation $L^{\mathcal{L}_i}$) is equivalent to π (resp. $\overline{\pi}$, which is the contragradient representation). That is, each $\mathcal{R}_i$ (resp. $\mathcal{L}_i$) is a copy of the Hilbert space $\mathcal{H}_\pi$, and the linear span of $e_1, ...e_{d_\pi}$, i.e., $\sum_{j=1}^{d_\pi} c_j e_j$, which can also be written as

$$\sum_{j=1}^{d_\pi} c_j \pi_{i,j}, \qquad \left(resp. \ \sum_{j=1}^{d_\pi} c_j \pi_{j,i} \right). \tag{3.1.3}$$

Example 3.1.2.1. Let π_3 be the irreducible unitary representation of $SU(2)$ with respect to the highest integer weight 3. The matrix form of π_3 (in Example 2.5.5.1) is given by

$$\pi_3 = \begin{pmatrix} \alpha^3 & \sqrt{3}\alpha^2\beta & \sqrt{3}\alpha\beta^2 & \beta^3 \\ -\sqrt{3}\alpha^2\bar{\beta} & \alpha^2\bar{\alpha} - 2\alpha\beta\bar{\beta} & 2\alpha\bar{\alpha}\beta - \beta^2\bar{\beta} & \sqrt{3}\bar{\alpha}\beta^2 \\ \sqrt{3}\alpha\bar{\beta}^2 & -2\alpha\bar{\alpha}\bar{\beta}^2 + \beta\bar{\beta}^2 & \alpha\bar{\alpha}^2 - 2\bar{\alpha}\beta\bar{\beta} & \sqrt{3}\bar{\alpha}^2\beta \\ -\bar{\beta}^3 & \sqrt{3}\bar{\alpha}\bar{\beta}^2 & -\sqrt{3}\bar{\alpha}^2\bar{\beta} & \bar{\alpha}^3 \end{pmatrix}, \tag{3.1.4}$$

and each row and column of π_3 forms an orthonormal basis which constitutes a copy of the Hilbert space $\mathcal{H}_{\pi_3}$. The inner product of $\mathcal{H}_{\pi_3}$ is given in (2.5.19).

Definition 3.1.3. Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra (which is the tangent space T_eG at the identity e). For $g, h \in G$, define $L_g : G \to G$ to be the left translation by g^{-1}, as $L_g(h) = g^{-1}h$. Denote $\Gamma(TG)$ the space of smooth sections of tangle bundles of G. A **left invariant vector field** is a smooth vector field $X \in \Gamma(TG)$ that is preserved under the left translations. That is

$$(L_g)_* X(h) = X(g^{-1}h), \tag{3.1.5}$$

for all $g, h \in G$, where $(L_g)_* : T_hG \to T_{g^{-1}h}G$ is the differential of the left translation L_g. Since $X(e) \in T_eG \cong \mathfrak{g}$, we can extend it to all of G by $X(g^{-1}) = (L_g)_* X(e)$ for all $g \in G$. The space of the left-invariant vector field forms a Lie algebra under the Lie brackets of the vector field, which is also naturally isomorphic to $\mathfrak{g}$.

Definition 3.1.4. Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra. The **left regular infinitesimal representation** of $\mathfrak{g}$ in the space of smooth functions $C^\infty(G)$ is given by the map

$$d\sigma : \mathfrak{g} \to \operatorname{Diff}(C^\infty(G)). \tag{3.1.6}$$

The action of $d\sigma$ on $f \in C^\infty(G)$ is defined by

$$d\sigma(X)f(g) = \frac{d}{dt}\bigg|_{t=0} f\left(\exp(-tX)g\right), \quad X \in \mathfrak{g}, \tag{3.1.7}$$

where $\exp(-tX)$ is the one-parameter subgroup of G generated by $X \in \mathfrak{g}$. Hence, $d\sigma(X)$ acts as a first-order differentiable operator on functions, and is given by the left invariant vector field

$$\left(d\sigma(X)\right) f(g) = (Xf)(g), \quad \text{for } X \in \mathfrak{g},\ g \in G. \tag{3.1.8}$$

The differential operators also satisfy commutation relation

$$[d\sigma(X), d\sigma(Y)] = d\sigma([X, Y]), \quad \text{for } X, Y \in \mathfrak{g}. \tag{3.1.9}$$

The left regular infinitesimal representation realises $\mathfrak{g}$ as differential operators on G, with these operators being precisely the left invariant vector fields.

Theorem 3.1.5 ([30], Theorem 20.2)**.** *Let $\mathfrak{g}^{\mathbb{C}}$ be a complex semisimple Lie algebra, and let $\lambda \in \Lambda^+$ be a dominant weight. Let V be a standard cyclic $\mathfrak{g}^{\mathbb{C}}$-module, with the highest weight vector $v_\lambda \in V$. Let $\Phi^+ = \{\alpha_1, ..., \alpha_m\}$ be the set of all positive roots. Then V is spanned by the vectors of the form*

$$Y_{\alpha_1}^{i_1} ... Y_{\alpha_m}^{i_m} \cdot v_\lambda, \quad i_j \in \mathbb{Z}^+, \tag{3.1.10}$$

where each Y_{α_i} spans a negative root space in $\mathfrak{g}^{\mathbb{C}}$. In particular, V is the direct sum of its weight spaces, where the dimension of the weight space of λ has $\dim V_\lambda = 1$, and the weights of V are of the form $\mu = \lambda - \sum_i k_1 \alpha_i$, for $k_i \in \mathbb{Z}^+$ and $\alpha_i \in \Phi^+$.

Theorem 3.1.6 ([37], Proposition 7.1, 7.2. [26], Theorem 5.6)**.** *Let $\mathfrak{g}^{\mathbb{C}}$ be a complex semisimple Lie algebra. Let $\mathfrak{g}$ be a compact real form of $\mathfrak{g}^{\mathbb{C}}$ ($\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$) such that the corresponding Lie group G of $\mathfrak{g}$ is simply connected. Let $\lambda \in \Lambda^+$ be a dominant highest weight, and let $d\sigma_\lambda$, $d\pi_\lambda$ be the corresponding finite-dimensional irreducible highest weight representation of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{g}$, respectively.*

Then $d\sigma_\lambda$ and $d\pi_\lambda$ are in one-to-one correspondence, and they share the same weight spaces classification, thus the same representation space V. Also, since G is simply connected, the integration of $d\pi_\lambda$ is equal to the irreducible highest weight representation π_λ of G. If π_λ is not unitary, the method of taking the average over G shows that there is an inner product on V that is invariant under the action of G, and this action is unitary on V. This is also known as Weyl's unitarian trick.

Example 3.1.6.1. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$, and let $\mathfrak{su}(2)$ denote the compact real form of $\mathfrak{sl}_2(\mathbb{C})$. Choose a highest weight of $\mathfrak{sl}_2(\mathbb{C})$ to be $\lambda = 3$, and the $\mathfrak{sl}_2(\mathbb{C})$-module V can be realised as the span of $\{2z^3, 2\sqrt{3}z^2w, 2\sqrt{3}zw^2, 2w^3\}$ (which is the orthonormal basis in (2.5.18) with $\lambda = 3$), which corresponds to the set of weights $\{3, 1, -1, -3\}$. The set of positive roots is $\Phi^+ = \{2\}$. Let the highest weight vector be z^3, then V is spanned by the vectors

$$\left\{z^3,\ (X)\cdot z_3,\ (X)^2\cdot z_3,\ (X)^3\cdot z_3\right\}, \tag{3.1.11}$$

where the action of 'X' is the left regular infinitesimal representation $d\pi(X)^L$ acting on V (where X is in (2.3.1)), and it can be represented by the complex differential operator

$$d\pi(X)^L = -w\frac{\partial}{\partial z}. \tag{3.1.12}$$

To see this, let $f(z, w)$ be a differentiable function in $\mathbb{C}^2$, the action of the left regular infinitesimal representation $d\pi$ (left invariant vector field) on f is given by

$$d\pi(X)^L f(z, w) = \frac{d}{dt} f\left(\exp\left(-tX\right)\cdot(z, w)\right)\bigg|_{t=0}, \qquad X \in \mathfrak{su}(2). \tag{3.1.13}$$

Now, let $(z(t), w(t)) = \exp\left(-tX\right)\cdot(z, w)$ be a curve in $\mathbb{C}^2$, and by the chain rule,

$$d\pi(X)^L f = \frac{\partial f}{\partial z}\frac{dz}{dt}\bigg|_{t=0} + \frac{\partial f}{\partial w}\frac{dw}{dt}\bigg|_{t=0}. \tag{3.1.14}$$

Also, since $\frac{d(z,w)}{dt}\Big|_{t=0} = -X\cdot(z, w)$, we have

$$d\pi(X)^L f = -\frac{\partial f}{\partial z}(X_{11}z + X_{12}w) - \frac{\partial f}{\partial w}(X_{21}z + X_{22}w). \tag{3.1.15}$$

We may take the complex linear extension of $d\pi$ to $\mathfrak{sl}_2(\mathbb{C})$. Hence, for the basis elements H, X, Y of $\mathfrak{sl}_2(\mathbb{C})$ in (2.3.1), we can write down the left regular infinitesimal representations $d\pi(H)^L, d\pi(X)^L, d\pi(Y)^L$ as complex differential operators:

$$\begin{aligned} d\pi(H)^L &= -z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}, \\ d\pi(X)^L &= -w\frac{\partial}{\partial z}, \\ d\pi(Y)^L &= -z\frac{\partial}{\partial w}. \end{aligned} \tag{3.1.16}$$

Also, the right regular infinitesimal representations of $d\pi(H)^R, d\pi(X)^R, d\pi(Y)^R$ as complex differential operators are given by

$$\begin{aligned} d\pi(H)^R &= z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}, \\ d\pi(X)^R &= z\frac{\partial}{\partial w}, \\ d\pi(Y)^R &= w\frac{\partial}{\partial z}. \end{aligned} \tag{3.1.17}$$

This approach of finding these differential operators on $SU(2)$ is similar to Lemma 2.5.3, and this example also illustrates how to generate a $\mathfrak{sl}_2$-module (Example 2.3.2.1) at the group level by applying complex differentiable operators to polynomials. But this method is restricted to $SU(2)$ because the representation space is explicitly known.

With respect to the real basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ in (2.5.3), and the representation of H, X, Y above, and the relations:

$$X_1 = iH,\ X_2 = X - Y,\ X_3 = i(X - Y), \tag{3.1.18}$$

we can work out the the irreducible skew-Hermitian highest weight representation $d\pi_3(x_1X_1 + x_2X_2 + x_3X_3)$ as

$$d\pi_3 = \begin{pmatrix} i3x_3 & \sqrt{3}(x_1 + ix_2) & 0 & 0 \\ -\sqrt{3}(x_1 - ix_2) & ix_3 & 2(x_1 + ix_2) & 0 \\ 0 & -2(x_1 - ix_2) & -ix_3 & \sqrt{3}(x_1 + ix_2) \\ 0 & 0 & -\sqrt{3}(x_1 - ix_2) & -i3x_3 \end{pmatrix}, \tag{3.1.19}$$

for $(x_1, x_2, x_3) \in \mathbb{R}^3$. Since $SU(2)$ is simply connected, we can take the integration of $d\pi_3$ by the exponential mapping to obtain the irreducible unitary highest weight representation π_3 in (3.1.4).

Proposition 3.1.7. *Let G be a compact simply connected semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, and $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. Let π_λ be an finite-dimensional irreducible unitary representation of G acting on a Hilbert space $\mathcal{H}_{\pi_\lambda}$ with respect to a dominant highest weight $\lambda \in \Lambda^+$. Fix a basis of $\mathcal{H}_{\pi_\lambda}$ and choose a highest weight vector v_λ (which is unique up to a complex multiple), and define the highest weight matrix coefficient T_{v_λ,v_λ} of π_λ by $T_{v_\lambda,v_\lambda}(g) = \langle \pi_\lambda(g)v_\lambda, v_\lambda\rangle$, $g \in G$. Then, every other matrix coefficient of π_λ can be obtained by applying a polynomial of left and right invariant vector fields induced by the root vectors in $\mathfrak{g}^{\mathbb{C}}$, to the matrix coefficient T_{v_λ,v_λ}.*

Proof. For $X \in \mathfrak{g}$, the action of the left invariant vector field X^L on $T_{v_\lambda,v_\lambda}(g)$, for $g \in G$, can be written as

$$\begin{aligned} X^L \cdot T_{v_\lambda,v_\lambda}(g) &= \frac{d}{dt}\bigg|_{t=0} \langle \pi_\lambda(\exp(-tX)g)v_\lambda, v_\lambda\rangle \\ &= \frac{d}{dt}\bigg|_{t=0} \langle \pi_\lambda(\exp(-tX))\pi_\lambda(g)v_\lambda, v_\lambda\rangle \\ &= \frac{d}{dt}\bigg|_{t=0} \langle \pi_\lambda(g)v_\lambda, \pi_\lambda(\exp(-tX))^* v_\lambda\rangle \\ &= \langle -d\pi_\lambda(X)\pi_\lambda(g)v_\lambda, v_\lambda\rangle \\ &= \langle \pi_\lambda(g)v_\lambda, (-d\pi_\lambda(X))^* v_\lambda\rangle \\ &= \langle \pi_\lambda(g)v_\lambda, d\pi_\lambda(X)v_\lambda\rangle, \end{aligned} \tag{3.1.20}$$

where $d\pi_\lambda(X)$ is the skew-Hermitian infinitesimal version of π_λ. Similarly, the action of the right invariant field X on $T_{v_\lambda,v_\lambda}(g)$ can be written as

$$X^R \cdot T_{v_\lambda,v_\lambda}(g) = \langle \pi_\lambda(g) d\pi_\lambda(X)v_\lambda, v_\lambda\rangle. \tag{3.1.21}$$

Now, we take the complex extension of the invariant vector fields of $\mathfrak{g}$ to $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$. A vector in $X \in \mathfrak{g}^{\mathbb{C}}$ can be written as $X = X_1 + iX_2$, for $X_1, X_2 \in \mathfrak{g}$. This means that for each root vector X_α of $\mathfrak{g}^{\mathbb{C}}$, there exists $X_1, X_2 \in \mathfrak{g}$ such that $X_\alpha = X_1 + iX_2$. In addition, the root vectors in $\mathfrak{g}^{\mathbb{C}}$ come in pairs (positive and negative), and each one of the pairs is conjugate to the other. That is, for the pair (X_α, Y_α), $X_\alpha = X_1 - iX_2$ and $Y_\alpha = X_1 + iX_2$.

Hence, by Theorem 3.1.6, the irreducible highest weight representation $d\pi_\lambda$ of $\mathfrak{g}$ shares the same weight spaces classification as the irreducible highest weight representation of $\mathfrak{g}^{\mathbb{C}}$ with respect to the highest weight λ. So, we can take the complex linear extension of $d\pi$ to $\mathfrak{g}^{\mathbb{C}}$, and by Theorem 3.1.5, $\mathcal{H}_{\pi_\lambda}$ can be spanned by the vectors of the form

$$d\pi_\lambda(Y_{\alpha_1}^{i_1})...d\pi_\lambda(Y_{\alpha_m}^{i_m}) \cdot v_\lambda, \quad i_j \in \mathbb{Z}^+, \tag{3.1.22}$$

where $\{\alpha_1, ..., \alpha_m\}$ is the set of positive roots of $\mathfrak{g}^{\mathbb{C}}$. This leads to the actions of the right and left invariant fields induced by the root vectors of $\mathfrak{g}^{\mathbb{C}}$ on T_{v_λ,v_λ}, are given by

$$(Y_{\alpha_1}^{i_1})^R...(Y_{\alpha_m}^{i_m})^R \cdot T_{v_\lambda,v_\lambda}(g) = \langle \pi_\lambda(g) d\pi_\lambda(Y_{\alpha_1}^{i_1})...d\pi_\lambda(Y_{\alpha_m}^{i_m})v_\lambda, v_\lambda\rangle, \tag{3.1.23}$$

and

$$(Y_{\alpha_1}^{i_1})^L...(Y_{\alpha_m}^{i_m})^L \cdot T_{v_\lambda,v_\lambda}(g) = \langle \pi_\lambda(g)v_\lambda, d\pi_\lambda(X_{\alpha_1}^{i_1})...d\pi_\lambda(X_{\alpha_m}^{i_m})v_\lambda\rangle, \tag{3.1.24}$$

for each pair of root vectors $(X_{\alpha_j}, Y_{\alpha_j})$. The left invariant field of a negative root vector $(Y_{\alpha_j})^L$ translates to the $d\pi_\lambda(X_{\alpha_j})$ in the second argument of the inner product is due to the complex conjugation. Hence, if v_λ is chosen, then every other matrix coefficient of π_λ can be obtained by applying a polynomial of left and right invariant vector fields induced by root vectors of $\mathfrak{g}^{\mathbb{C}}$. ☐

Remark. *Suppose π_λ is not unitary on $\mathcal{H}_\lambda$ (equipped with an inner product $(\cdot,\cdot)$), then the first row of π_λ can still be obtained by applying polynomials of right invariant vector field Y, that is,*

$$(Y_{\alpha_1}^{i_1})^R...(Y_{\alpha_m}^{i_m})^R \cdot T_{v_\lambda,v_\lambda}(g) = (\pi_\lambda(g) d\pi_\lambda(Y_{\alpha_1}^{i_1})...d\pi_\lambda(Y_{\alpha_m}^{i_m})v_\lambda, v_\lambda). \tag{3.1.25}$$

Example 3.1.7.1. Let π_3 in (3.1.4) be the irreducible unitary representation of $SU(2)$ with the highest integer weight 3. Let $\{2z^3, 2\sqrt{3}z^2w, 2\sqrt{3}zw^2, 2w^3\}$ span the Hilbert space $\mathcal{H}_{\pi_3}$, and let H, X, Y in (2.5.5) be the basis of $\mathfrak{su}(2)^{\mathbb{C}}$. If we choose the highest weight vector to be $2z^3$, then the highest weight matrix coefficient $T_{v_\lambda,v_\lambda}(g)$ for $g \in SU(2)$ (which is given in (2.5.1)), is given by

$$T_{v_\lambda,v_\lambda}(g) = \langle \pi_3(g)\, 2z^3,\, 2z^3\rangle = \alpha^3. \tag{3.1.26}$$

Since $\mathcal{H}_{\pi_3}$ is also a $\mathfrak{su}(2)^{\mathbb{C}}$-module, and by Example 2.3.2.1 and the complex differential operators in (3.1.17) , we have

$$d\pi_3(Y) \cdot z^3 = 3z^2w,\ d\pi_3(Y) \cdot z^2w = 2zw^2,\ d\pi_3(Y) \cdot zw^2 = w^3. \tag{3.1.27}$$

The action of the right invariant vector field Y on $T_{v_\lambda,v_\lambda}(g)$ are given by

$$\begin{aligned} Y^R \cdot T_{v_\lambda,v_\lambda}(g) &= \langle \pi_3(g)\, d\pi_3(Y) 2z^3,\, 2z^3 \rangle \\ &= \langle \pi_3(g)\, 3 \cdot 2\sqrt{3} z^2 w,\, 2z^3 \rangle \\ &= 3\sqrt{3}\alpha^2\beta, \end{aligned} \tag{3.1.28}$$

which is equal to the $(1,2)$-th matrix coefficient $3\,T_{1,2}$ in π_3. Similarly, $(Y^R)^2 \cdot T_{v_\lambda,v_\lambda}(g) = 6\sqrt{3}\alpha\beta^2$, and $(Y^R)^3 \cdot T_{v_\lambda,v_\lambda}(g) = 6\beta^3$. These are $6\,T_{1,3}$ and $6\,T_{1,4}$. Also, by (2.5.5), X is the complex conjugate of $-Y$, and the action of the left invariant vector field X on $T_{v_\lambda,v_\lambda}(g)$ is given by

$$\begin{aligned} X^L \cdot T_{v_\lambda,v_\lambda}(g) &= \langle -d\pi_3(X)\pi_3(g)\, 2z^3,\, 2z^3 \rangle \\ &= \langle \pi_3(g)\, 2z^3,\, -d\pi_3(Y) 2z^3 \rangle \\ &= \langle \pi_3(g)\, 2z^3,\, -3 \cdot 2\sqrt{3} z^2 w \rangle \\ &= -3\sqrt{3}\alpha^2\overline{\beta}, \end{aligned} \tag{3.1.29}$$

which is equal to the $(2,1)$-th matrix coefficient $T_{2,1}$ in π_3. In similar manners, we can apply polynomials of invariant vector fields X^L, Y^R to $T_{v_\lambda,v_\lambda}(g)$ to obtain every other matrix coefficient (with a real multiple) of π_3.

Definition 3.1.8. Let G be a compact connected group. If π_1 and π_2 are unitary representations of G on Hilbert spaces $\mathcal{H}_{\pi_1}$ and $\mathcal{H}_{\pi_2}$, an **intertwining operator** for π_1 and π_2 is a bounder linear map $T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ such that $T\pi_1(g) = \pi_2(g)T$ for all $g \in G$. The set of all such operators is denoted by $\vartheta(\pi_1, \pi_2)$:

$$\vartheta(\pi_1, \pi_2) = \{T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2} \,:\, T\pi_1(g) = \pi_2(g)T \text{ for all } g \in G\}. \tag{3.1.30}$$

The π_1 and π_2 are **unitarily equivalent** if $\vartheta(\pi_1, \pi_2)$ contains a unitary operator U, so that $\pi_2(g) = U\pi_1(g)U^{-1}$.

Theorem 3.1.9 ([21], Theorem 5.8). ***(Schur's Orthogonality Relations)*** *Let G be a compact connected group. Let π and $\tilde{\pi}$ be irreducible unitary representations of G on Hilbert spaces $\mathcal{H}_\pi$ and $\mathcal{H}_{\tilde{\pi}}$, respectively. Fix an basis for both $\mathcal{H}_\pi$ and $\mathcal{H}_{\tilde{\pi}}$ so that each matrix coefficient of π and $\tilde{\pi}$ can be worked out explicitly. Let M_π and $M_{\tilde{\pi}}$ be the subspaces of $L^2(G)$, spanned by the matrix coefficients of π and $\tilde{\pi}$, respectively. Denote the equivalence classes of π and*

$\tilde{\pi}$ by $[\pi]$ and $[\tilde{\pi}]$. If $[\pi]$ and $[\tilde{\pi}]$ are distinct, then M_π is orthogonal to $M_{\tilde{\pi}}$. Furthermore, let $d_\pi = \dim(\pi)$, $\{v_1, ..., v_{d_\pi}\}$ be the chosen basis of $\mathcal{H}_\pi$ (and let $d_{\tilde{\pi}} = \dim(\tilde{\pi})$, $\{\tilde{v}_1, ..., \tilde{v}_{d_{\tilde{\pi}}}\}$ be the chosen basis of $\mathcal{H}_{\tilde{\pi}}$), and define the matrix coefficient $T^{\pi}_{v_i,v_j}(g) = \langle \pi(g)v_j, v_i\rangle$ for $g \in G$. Then, the set

$$\{\sqrt{d_\pi} T^{\pi}_{v_i,v_j} \,:\, 1 \leqslant i, j \leqslant d_\pi\}, \tag{3.1.31}$$

forms an orthonormal basis for M_π. In addition, let dg be a left invariant Haar measure on G with $\int_G dg = 1$. Let $v_i, v_j \in \mathcal{H}_\pi$ and $\tilde{v}_i, \tilde{v}_j \in \mathcal{H}_{\tilde{\pi}}$, then the matrix coefficients $T^{\pi}_{v_i,v_j}$ and $T^{\tilde{\pi}}_{\tilde{v}_i,\tilde{v}_j}$ satisfy the relation

$$\langle T^{\pi}_{v_i,v_j}, T^{\tilde{\pi}}_{\tilde{v}_i,\tilde{v}_j}\rangle = \int_G T^{\pi}_{v_i,v_j}(g)\overline{T^{\tilde{\pi}}_{\tilde{v}_i,\tilde{v}_j}}(g)\, dg = \begin{cases} \frac{1}{d\pi}\langle v_i, \tilde{v}_i\rangle\langle v_j, \tilde{v}_j\rangle, & \text{if } [\pi] = [\tilde{\pi}], \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.32}$$

Remark. *Matrix coefficients are the fundamental building blocks of the functions in $C(G)$ and $L^p(G)$ ([21], Ch.5). To examine the convolution structures in these spaces, we can start by looking at the convolutions of matrix coefficients, which can be seen as a form of 'orthogonal rotations'.*

Definition 3.1.10. Let G be a locally compact group with a fixed left invariant Haar measure dx. If $f, g \in L^1(G)$, then the convolution of f and g is the function defined by

$$f * g(x) = \int_G f(y)g(xy^{-1})\, dx. \tag{3.1.33}$$

Proposition 3.1.11. *Let G be a compact simply connected semisimple Lie group, $\mathfrak{g}$ be its Lie algebra. Let π_λ be a finite-dimensional irreducible unitary representation of G with highest dominant weight λ on the Hilbert space $\mathcal{H}_{\pi_\lambda}$. Let $d_{\pi_\lambda} = \dim(\pi_\lambda)$, and fix an orthonormal basis $\{\xi_1, ...\xi_{d_{\pi_\lambda}}\}$ of $\mathcal{H}_{\pi_\lambda}$ (where ξ_1 is the chosen highest weight vector). Define the matrix coefficient of π_λ as $T_{i,j}(\cdot) = \langle \pi_\lambda(\cdot)\xi_j, \xi_i\rangle$. Then, the convolution of any pair of arbitrary matrix coefficients is given by*

$$T_{i,j} * T_{k,l} = \frac{1}{d_\pi} D^{i,l}_{k,j} T_{1,1}, \quad 1 \leqslant i, j, k, l \leqslant d_{\pi_\lambda}, \tag{3.1.34}$$

where $D^{i,l}_{k,j}$ is a polynomial of left and right invariant vector fields on G induced by the root vectors in $\mathfrak{g}^{\mathbb{C}}$.

Proof. Let dh be a left invariant Haar measure on G. The convolution of $T_{i,j}$ and $T_{k,l}$ is given by

$$T_{i,j} * T_{k,l}(g) = \int_G T_{i,j}(h) T_{k,l}(gh^{-1})\, dh, \quad g \in G. \tag{3.1.35}$$

Since π_λ is a representation, $\pi_\lambda(gh^{-1}) = \pi_\lambda(g)\pi_\lambda(h^{-1}) = \pi_\lambda(g)\pi_\lambda(h)^*$ (complex conjugate). This leads to

$$T(gh^{-1})_{k,l} = \sum_{m=1}^{d_{\pi_\lambda}} T_{k,m}(g) T_{m,l}(h)^*, \tag{3.1.36}$$

where $T_{m,l}(h)^* = \langle \pi_\lambda(h)^* e_l, e_m \rangle = \overline{T_{l,m}}(h)$. Substitute and rearrange, we get

$$T_{i,j} * T_{k,l}(g) = \sum_{m=1}^{d_{\pi_\lambda}} T_{k,m}(g) \int_G \overline{T_{l,m}}(h) T_{i,j}(h)\, dh. \tag{3.1.37}$$

By Schur's orthogonality relations, the integral above is nonzero only when $l = i$ and $m = j$. In that case,

$$\int_G \overline{T_{l,m}}(h) T_{i,j}(h)\, dh = \frac{1}{d_{\pi_\lambda}} \delta_{l,i} \delta_{m,j} \tag{3.1.38}$$

where $\delta_{p,q}$ is the Kronecker delta function. If we further simplify this convolution, we get

$$\begin{aligned} T_{i,j} * T_{k,l}(g) &= \frac{1}{d_{\pi_\lambda}} \sum_{m=1}^{d_{\pi_\lambda}} T_{k,m}(g) \delta_{m,j}\, \delta_{l,i} \\ &= \frac{1}{d_{\pi_\lambda}} T_{k,j}(g)\, \delta_{l,i}. \end{aligned} \tag{3.1.39}$$

Therefore, if $i \neq j$, then the convolution is zero. If $i = j$, the result simplifies to

$$T_{i,j} * T_{k,l}(g) = \frac{1}{d_{\pi_\lambda}} T_{k,j}(g). \tag{3.1.40}$$

Now, by Proposition 3.1.7, any matrix coefficient of π_λ can be obtained by applying a polynomial of left and right invariant vector fields induced by root vectors in $\mathfrak{g}^{\mathbb{C}}$ to $T_{1,1}$. Hence the result follows. □

Example 3.1.11.1. Let π_3 be the irreducible unitary representation of $SU(2)$

with highest integer weight 3 given in (3.1.4),

$$\pi_3(g) = \begin{pmatrix} \alpha^3 & \sqrt{3}\alpha^2\beta & \sqrt{3}\alpha\beta^2 & \beta^3 \\ -\sqrt{3}\alpha^2\bar{\beta} & \alpha^2\bar{\alpha} - 2\alpha\beta\bar{\beta} & 2\alpha\bar{\alpha}\beta - \beta^2\bar{\beta} & \sqrt{3}\bar{\alpha}\beta^2 \\ \sqrt{3}\alpha\bar{\beta}^2 & -2\alpha\bar{\alpha}\bar{\beta}^2 + \beta\bar{\beta}^2 & \alpha\bar{\alpha}^2 - 2\bar{\alpha}\beta\bar{\beta} & \sqrt{3}\bar{\alpha}^2\beta \\ -\bar{\beta}^3 & \sqrt{3}\bar{\alpha}\bar{\beta}^2 & -\sqrt{3}\bar{\alpha}^2\bar{\beta} & \bar{\alpha}^3 \end{pmatrix}, \tag{3.1.41}$$

where $g \in SU(2)$ (which is given in (2.5.1)), $(\alpha, \beta) \in \mathbb{C}^2$ and satisfies $|\alpha|^2 + |\beta|^2 = 1$. Recall that the inner product in polynomials of (α, β) is given by

$$\langle \alpha^p\beta^r, \alpha^q\beta^s \rangle = \int_{\mathbb{S}^3} \alpha^p\overline{\alpha}^q\beta^r\overline{\beta}^s \, d\sigma(\alpha,\beta) = \begin{cases} p!r!/(p+r+1)! & \text{if } p = q \text{ and } r = s, \\ 0 & \text{if } p \neq q \text{ or } r \neq s, \end{cases} \tag{3.1.42}$$

where $d\sigma$ is the Euclidean surface measure of the 3-sphere $\mathbb{S}^3$ in $\mathbb{C}^2$. If the matrix coefficient is in the form of $\alpha^i\overline{\alpha}^j\beta^k\overline{\beta}^l$, then

$$\begin{aligned} \left\langle \alpha^p\overline{\alpha}^q\beta^r\overline{\beta}^s, \alpha^i\overline{\alpha}^j\beta^k\overline{\beta}^l \right\rangle &= \int_{\mathbb{S}^3} \alpha^{p+j}\overline{\alpha}^{q+i}\beta^{r+l}\overline{\beta}^{s+k} \, d\sigma(\alpha,\beta) \\ &= \begin{cases} \frac{(p+j)!(r+l)!}{(p+j+r+l+1)!} & \text{if } p+j = q+i \text{ and } r+l = s+k, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.1.43}$$

Since $\dim(\pi_3) = 4$, and let M_3 be the subspaces of $L^2(SU(2))$ spanned by the matrix coefficients of π_3. Denote by $T_{i,j}$ the (i,j)-element of π_3. A brief calculation shows that the set

$$\{\sqrt{4}T_{i,j} \,:\, 1 \leqslant i, j \leqslant 4\}, \tag{3.1.44}$$

forms an orthonormal basis for M_3 with respect to the inner product defined above.

Now, let $\{2z^3, 2\sqrt{3}z^2w, 2\sqrt{3}zw^2, 2w^3\}$ be an orthonormal basis of the Hilbert space $\mathcal{H}_{\pi_3}$. By Proposition 3.1.11, we can write the convolution of a pair of matrix coefficients as differentiation, for example,

$$T_{1,4} * T_{4,1} = \frac{1}{4}T_{4,4} = \beta^3 * \left(-\bar{\beta}^3\right) = \frac{1}{4}\bar{\alpha}^3. \tag{3.1.45}$$

This is equivalent to defining a polynomial of invariant vector fields induced by

X, Y given in (2.5.5), that is,

$$D^{1,1}_{4,4} = \frac{1}{4} \cdot \frac{1}{6^2} \left(-X^L\right)^3 \cdot \left(Y^R\right)^3, \tag{3.1.46}$$

acting on the matrix coefficient $T_{1,1} = \alpha^3$.

Next, we aim to lift this convolution of (or in this case the differentiation of) matrix coefficients from G to $\mathfrak{g}$. That is, there should exist a differentiable operator $\left(D^{i,l}_{j,k}\right)_{\mathfrak{g}}$ in $\mathfrak{g}$ such that $D^{i,l}_{j,k} T_{1,1}(g), g \in G$ can be written as

$$\left(D^{i,l}_{j,k}\right)_{\mathfrak{g}} (X) \cdot T_{1,1}(\exp X), \quad X \in \mathfrak{g}. \tag{3.1.47}$$

We show the existence of $\left(D^{i,l}_{j,k}\right)_{\mathfrak{g}} (X)$ in the next section.

3.2 Transversality Condition and Differential Operators

In this section, we study the transversal condition and differential operators induced by the action of G , and show how to combine these to define the general form of differential operators of a semisimple Lie algebra $\mathfrak{g}$ acted upon by its corresponding Lie group G.

Definition 3.2.1 ([27], Sec II.3.2)**.** Let V be a manifold satisfying the second axiom of countability and suppose that G is a Lie transformation group of V. If $v \in V$, let G_v denote the subgroup of G that fixes v. Let $\mathfrak{g}$ denote the Lie algebra of G, and let X^G denote the vector field on V that is **induced by the action of G on X**, that is,

$$(X^G \cdot f)(v) = \left.\frac{d}{dt}\right|_{t=0} f(\exp tX \cdot v) \tag{3.2.1}$$

for $v \in V$ and $f \in C^\infty(V)$.

A C^∞-function f on an open subset of V is said to be **locally invariant** if $X^G \cdot f = 0$ for each $X \in \mathfrak{g}$. A submanifold $B \subset G$ is called a **local cross section** over an open set $U \subset G/G_v$ if the natural map $p : G \to G/G_v$ gives a diffeomorphism of B onto U.

Example 3.2.1.1. Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra. Let $X \in \mathfrak{g}$, $f \in C^\infty(\mathfrak{g})$. The action of X^G on f is given by

$$(X^G \cdot f)(Y) = \frac{d}{dt}\bigg|_{t=0} f(Ad(\exp tX) \cdot Y), \quad Y \in \mathfrak{g}, \tag{3.2.2}$$

where $Ad(\cdot)$ is the adjoint representation of G in its Lie algebra $\mathfrak{g}$. In addition, any adjoint G-invariant function $f \in C^\infty(\mathfrak{g})$ satisfies $X^G \cdot f = 0$. Fix a $X \in \mathfrak{g}$, the homogeneous manifold G/G_X is isomorphic to an adjoint orbit $\mathcal{O}_X$ in $\mathfrak{g}$.

S. Helgason [28] introduced the concept of *transversality condition* to help define the general form of differential operators of a Riemannian manifold V, acted on by a Lie transformation group G.

Lemma 3.2.2 ([28], Lemma II.3.3)**.** *(S.Helgason) Suppose that K is a submanifold of V such that for every $k \in K$, the tangent space at k satisfies the following* ***transversality condition****:*

$$V_k = (G \cdot k)_k + K_k \quad \textit{(direct sum)}, \tag{3.2.3}$$

where $K_k \cong K$, and $G \cdot k$ is the G-orbit of k. Fix a $k_0 \in K$. Then there exists an open, relatively compact neighborhood K_0 of k_0 in K, and a relatively compact submanifold B of G forming a local cross section over a neighborhood U_0 of eG_{k_0} in the homogeneous manifold G/G_{k_0} such that the map $\gamma : (b, k) \mapsto b \cdot k$ is a diffeomorphism of $B \times K_0$ onto an open neighborhood of k_0 in V.

Example 3.2.2.1 ([28], Ch.II Ex.4.vii)**.** Let G be a complex semisimple Lie group, $\mathfrak{g}$ be its Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{h}' \subset \mathfrak{h}$ the subset of regular elements (which lies off the walls of the Weyl chambers of $\mathfrak{h}$). Let Φ be the set of roots of $\mathfrak{g}$, recall that the root space decomposition of $\mathfrak{g}$ (Definition 2.2.9) is given by $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$. If $H \in \mathfrak{g}$, the adjoint orbit $G \cdot H$ has tangent space $[\mathfrak{g}, H]$ at H. This means that if $H \in \mathfrak{h}' \subset \mathfrak{g}$, the tangent space of $G \cdot H$ at H is equal to $\sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$. This also shows that the tangent space V_H at H satisfies the transversality condition and is given by

$$V_H = (G \cdot H)_H + \mathfrak{h}'_H \quad \text{(direct sum)}. \tag{3.2.4}$$

where $\mathfrak{h}'_H \cong \mathfrak{h}'$, and $G \cdot H$ is the G-orbit of H. Note that in this case the singular elements of $\mathfrak{h}$ are excluded from the transversality condition. This is because if

$H' \in \mathfrak{h}$ lies on a wall of a Weyl chamber of $\mathfrak{h}$, the homogeneous manifold $G/G_{H'}$ is no longer the same as (and does not have the same dimension as) G/G_H of any regular element $H \in \mathfrak{h}$. This is due to Proposition 2.2.16, where the Lie algebra $\mathfrak{g}_{H'}$ of the stabiliser $G_{H'}$ of H' is given by

$$\mathfrak{g}_{H'} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi^+_{H'}} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}), \tag{3.2.5}$$

where $\Phi^+_{H'} = \{\alpha \in \Phi^+ : \alpha(H') = 0\}$.

Definition 3.2.3. Let G be a Lie transformation group of isometries of a Riemannian manifold V. Suppose that K is a submanifold of V satisfying the transversality condition. Let $s_0 \in V$ and let $G \cdot s_0$ be the G-orbit of s_0. Put $k_0 = s_0$, and choose B and K_0 as in Lemma 3.2.2. This means that the disjoint union $\bigcup_{k \in K_0} B \cdot k$ is a neighborhood V_0 of s_0 in V. If $f \in C^\infty(V)$, we can restrict it to K_0, and then extend it to a function f_{s_0} on V_0 by

$$f_{s_0}(b \cdot k) = f(k), \quad b \in B,\, k \in K_0, \tag{3.2.6}$$

so that f_{s_0} is G-invariant and depends on K_0. Let D be a differential operator on V, and define a new differential operator D_T on V whose action is 'transversal to the orbits' as

$$(D_T f)(s_0) = (D f_{s_0})(s_0). \tag{3.2.7}$$

Since f_{s_0} is smooth near s_0, the right-hand side makes sense; also, since $B \cdot w$ is a neighborhood of w in the orbit $G \cdot w$. Thus, $D_T f$ is a well-defined C^∞-function on V. If f vanishes on an open subset U of V, then so does $D_T f$. Hence, the mapping $f \mapsto D_T f$ decreases supports, so D_T is a differential operator. The operator D_T is called the **transversal part** of D. If $D = D_T$, then D is said to be **transversal**.

Theorem 3.2.4 ([27], Theorem II.3.4)**.** *Let G be a Lie transformation group of isometries of a Riemannian manifold V. Suppose all orbits of G on V have same dimension. Let $X_1, \ldots, X_l$ be a basis of the Lie algebra $\mathfrak{g}$ of G and let $Y_i = X_i^G, (1 \leqslant i \leqslant l)$ be the vector fields induced by G on V. Then each differential operator D on V can be written as a locally finite sum*

$$D = D_T + \sum_{(i)} D_{(i)} Y_{i_1} \cdots Y_{i_l}, \tag{3.2.8}$$

where D_T is the transversal part of D and each $D_{(i)}$ ($(i) = (i_1, ..., i_r)$) is transversal.

Example 3.2.4.1 ([27], Example II.3.4.1)**.** Let V be the plane $\mathbb{R}^2$ and G be the group of translations $T_t : (x, y) \mapsto (x, y + t), t \in \mathbb{R}$. Fix a y_0 on the y-axis, and the G-orbit of y_0 is the entire y-axis. So $\frac{\partial}{\partial x}$ is transversal to $G \cdot y_0$. In this case, the operators of the form

$$\sum_i b_i(x, y) \left(\frac{\partial}{\partial x}\right)^i, \tag{3.2.9}$$

are the transversal operators, $\left(\frac{\partial}{\partial y}\right)$ is the vector field induced by G, and (3.2.8) reduces to the representation

$$\sum_j \left(\sum_i a_{i,j}(x, y) \left(\frac{\partial}{\partial x}\right)^i\right) \left(\frac{\partial}{\partial y}\right)^j \tag{3.2.10}$$

of an arbitrary differential operator on $\mathbb{R}^2$.

Example 3.2.4.2. Let the Lie group $G = SU(2)$, and $\mathfrak{g} = \mathfrak{su}(2)$ be its Lie algebra. Let V be $\mathfrak{su}(2) \cong \mathbb{R}^3$, G acts by the adjoint action on $\mathfrak{g}$. The G-orbits in $\mathfrak{g}$ are 2-spheres in $\mathbb{R}^3$. Recall that the standard basis of $\mathfrak{su}(2)$ is

$$X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{3.2.11}$$

Let $(x_1, x_2, x_3) \in \mathbb{R}^3$, and each element Y in $\mathfrak{su}(2)$ can be written as $Y = x_1X_1 + x_2X_2 + x_3X_3$. Let $r\frac{\partial}{\partial r} = x_1\frac{\partial}{\partial X_1} + x_2\frac{\partial}{\partial X_2} + x_3\frac{\partial}{\partial X_3}$ be the radial differential operator defined in Proposition 2.5.8. The subspace spanned by $\{X_3\}$ satisfies the transversality condition (since every element in it is regular). Hence, the differential operators of the form

$$\sum_k \left(\sum_j \left(\sum_i c_{i,j,k}(x_1, x_2, x_3) \left(\frac{\partial}{\partial x_1}\right)^i\right) \left(\frac{\partial}{\partial x_2}\right)^j\right) \left(\frac{\partial}{\partial x_3}\right)^k, \tag{3.2.12}$$

are transversal differential operators. We can also calculate the vector fields X_1, X_2, X_3 induced by the action of G in $\mathfrak{g}$.

For example, let

$$Z = \begin{pmatrix} ix_3 & x_1 + ix_2 \\ -x_1 + ix_2 & -ix_3 \end{pmatrix} \in \mathfrak{su}(2). \tag{3.2.13}$$

Let $f \in C^\infty(\mathfrak{g})$, and define

$$(X_1^G \cdot f)(Z) = \frac{d}{dt}\bigg|_{t=0} f(Ad(\exp tX_1) \cdot Z), \quad Z \in \mathfrak{su}(2), \tag{3.2.14}$$

where

$$\begin{aligned} Ad(\exp tX_1) \cdot Z &= \exp(tX_1) Z \exp(-tX_1) \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ix_3 & x_1 + ix_2 \\ -x_1 + ix_2 & -ix_3 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \end{aligned} \tag{3.2.15}$$

and by the chain rule,

$$\frac{d}{dt}\bigg|_{t=0} Ad(\exp tX_1) \cdot Y = \begin{pmatrix} i2x_2 & -i2x_3 \\ -i2x_3 & -i2x_2 \end{pmatrix}. \tag{3.2.16}$$

Comparing this with Z, we conclude that

$$X_1^G = 2\left(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}\right). \tag{3.2.17}$$

Similarly, we can work out X_2^G, X_3^G:

$$X_2^G = 2\left(x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}\right), \quad X_3^G = \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right). \tag{3.2.18}$$

We may also extend X_1^G, X_2^G, X_3^G to the basis of their complexification (as in (2.5.5)):

$$\begin{aligned} H_\alpha^G &= \frac{1}{i} X_3^G : & & \frac{1}{i}\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right), \\ X_\alpha^G &= \frac{1}{2}(X_1^G - iX_2^G) : & & (ix_1 + x_2)\frac{\partial}{\partial x_3} - x_3(i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}), \\ Y_\alpha^G &= -\frac{1}{2}(X_1^G + iX_2^G) : & & (ix_1 - x_2)\frac{\partial}{\partial x_3} - x_3(i\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}). \end{aligned} \tag{3.2.19}$$

With respect to the Lie bracket operation, a brief calculation shows that

$$[H_\alpha^G, X_\alpha^G] = 2X_\alpha^G, \quad [H_\alpha^G, Y_\alpha^G] = -2Y_\alpha^G, \quad [X_\alpha^G, Y_\alpha^G] = H_\alpha^G. \tag{3.2.20}$$

Hence, $H_\alpha^G, X_\alpha^G, Y_\alpha^G$ form a basis of the Lie algebra $\mathfrak{su}_2^{\mathbb{C}}$.

Example 3.2.4.3. Let $G = SU(3)$, and its Lie algebra $\mathfrak{g} = \mathfrak{su}(3) \cong \mathbb{R}^8$. The standard basis of $\mathfrak{su}(3)$ contains eight elements, these are

$$\{H_1, H_2, X_1, X_2, X_3, Y_1, Y_2, Y_3\}, \tag{3.2.21}$$

and their matrix forms can be found in (2.2.16).

If we let $(h_1, h_2, x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^8$, then we can follow the same method as in Example 3.2.4.2 and work out the vector fields induced by the action of G: X_1^G, X_2^G, Y_1^G, Y_2^G as

$$\begin{aligned}
X_1^G &= 2y_1\frac{\partial}{\partial h_1} - (2h_1 - h_2)\frac{\partial}{\partial y_1} + x_2\frac{\partial}{\partial x_3} + y_2\frac{\partial}{\partial y_3} - x_3\frac{\partial}{\partial x_2} - y_3\frac{\partial}{\partial y_2} \\
Y_1^G &= -2x_1\frac{\partial}{\partial h_1} + (2h_1 - h_2)\frac{\partial}{\partial x_1} - y_2\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial y_2} \\
X_2^G &= 2y_2\frac{\partial}{\partial h_2} + (h_1 - 2h_2)\frac{\partial}{\partial y_2} + x_3\frac{\partial}{\partial x_1} + y_3\frac{\partial}{\partial y_1} - x_1\frac{\partial}{\partial x_3} - y_1\frac{\partial}{\partial y_3} \\
Y_2^G &= -2x_2\frac{\partial}{\partial h_2} - (h_1 - 2h_2)\frac{\partial}{\partial x_2} + y_3\frac{\partial}{\partial x_1} - x_3\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial y_3}
\end{aligned} \tag{3.2.22}$$

Let $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2\}$ be the set of positive roots of $\mathfrak{su}(3)^{\mathbb{C}}$, and the standard basis of $\mathfrak{su}(3)^{\mathbb{C}}$ is given by

$$\begin{aligned}
&H_{\alpha_1} = -iH_1, \quad H_{\alpha_2} = -iH_2, \\
&X_{\alpha_1} = \frac{1}{2}(X_1 - iY_1), \quad X_{\alpha_2} = \frac{1}{2}(X_2 - iY_2), \quad X_{\alpha_3} = \frac{1}{2}(X_3 - iY_3), \\
&Y_{\alpha_1} = -\frac{1}{2}(X_1 + iY_1), \quad Y_{\alpha_2} = -\frac{1}{2}(X_2 + iY_2), \quad Y_{\alpha_3} = -\frac{1}{2}(X_3 + iY_3).
\end{aligned} \tag{3.2.23}$$

If we know $X_{\alpha_1}^G, Y_{\alpha_1}^G, X_{\alpha_2}^G, Y_{\alpha_2}^G$, then we can work out $X_{\alpha_3}^G = \left[X_{\alpha_1}^G, X_{\alpha_2}^G\right]$, $Y_{\alpha_3}^G = \left[Y_{\alpha_1}^G, Y_{\alpha_2}^G\right]$, $H_{\alpha_1}^G = \left[X_{\alpha_1}^G, Y_{\alpha_1}^G\right]$, and $H_{\alpha_2}^G = \left[X_{\alpha_2}^G, Y_{\alpha_2}^G\right]$.

Proposition 3.2.5. *Let G be a compact simply connected semisimple Lie group, $\mathfrak{g}$ be its Lie algebra. Let π_λ be a finite-dimensional irreducible unitary representation of G with highest dominant weight λ on the Hilbert space $\mathcal{H}_{\pi_\lambda}$.*

Let $d_{\pi_\lambda} = \dim(\pi_\lambda)$*, and fix an orthonormal basis* $\{\xi_1, ...\xi_{d_{\pi_\lambda}}\}$ *of* $\mathcal{H}_{\pi_\lambda}$ *(where* ξ_1 *is the chosen highest weight vector). Define the matrix coefficient of* π_λ *as* $T_{i,j}(\cdot) = \langle \pi_\lambda(\cdot)\xi_j, \xi_i\rangle$*. By Proposition 3.1.11, the convolution of any pair of arbitrary matrix coefficients is given by*

$$T_{i,j} * T_{k,l} = \frac{1}{d_\pi} D^{i,l}_{k,j} T_{1,1}, \quad 1 \leqslant i, j, k, l \leqslant d_{\pi_\lambda}, \tag{3.2.24}$$

where $D^{i,l}_{k,j}$ *is a polynomial of left and right invariant vector fields on* G *induced by the root vectors in* $\mathfrak{g}^{\mathbb{C}}$*. Since* G *is simply connected, we can lift* π_λ *to its Lie algebra* $\pi_\lambda \circ \exp$ *by exponential mapping, that is,* $\pi_\lambda(\exp X) = \exp(d\pi_\lambda(X))$*, for* $X \in \mathfrak{g}$*, where* $d\pi_\lambda$ *is the skew-Hermitian infinitesimal representation of* $\mathfrak{g}$*. Then, there exists a polynomial of vector fields induced by* G *on* $\mathfrak{g}$*, denoted by* $(D^{i,l}_{k,j})^G$*, such that*

$$(T_{i,j} * T_{k,l})(\exp X) = \frac{1}{d_\pi} \left(D^{i,l}_{k,j}\right)^G T_{1,1}(\exp X), \quad 1 \leqslant i, j, k, l \leqslant d_{\pi_\lambda}. \tag{3.2.25}$$

Proof. Let $X \in \mathfrak{g}$, and let X^L, X^R denote the left and right invariant vector field on G by X, respectively, and let X^G denote the vector field induced by the G action of X on $\mathfrak{g}$. Then,

$$\left(X^L + X^R\right) \cdot T_{i,j}\left(\exp Y\right) = -X^G \cdot T_{i,j}\left(\exp Y\right), \quad Y \in \mathfrak{g}. \tag{3.2.26}$$

To see this, we can write X^L acting on $T_{i,j}$ as

$$\begin{aligned}
X^L \cdot T_{i,j}\left(\exp Y\right) &= \left.\frac{d}{dt}\right|_{t=0} T_{i,j}\left(\exp -tX \exp Y \exp tX \exp -tX\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} T_{i,j}\left(\exp Ad(\exp -tX)Y \exp -tX\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \sum_{m=1}^{d\pi_\lambda} T_{i,m}\left(\exp Ad(\exp -tX)Y\right) T_{m,j}\left(\exp -tX\right) \\
&= \sum_{m=1}^{d\pi_\lambda} -X^G \cdot T_{i,m}\left(\exp Y\right) T_{m,j}\left(\exp 0 \cdot X\right) \\
&+ \left.\frac{d}{dt}\right|_{t=0} \sum_{m=1}^{d\pi_\lambda} T_{i,m}\left(\exp Ad(\exp 0 \cdot X)Y\right) T_{m,j}\left(\exp -tX\right),
\end{aligned} \tag{3.2.27}$$

by the product rule and the definition of a representation. Note that

$$T_{m,j}\left(\exp 0 \cdot X\right) = \begin{cases} 1, & m = j, \\ 0, & \text{otherwise,} \end{cases} \qquad T_{i,m}\left(\exp 0 \cdot X\right) = \begin{cases} 1, & m = i, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2.28}$$

It follows that

$$X^L \cdot T_{i,j}\left(\exp Y\right) = -X^G \cdot T_{i,j}\left(\exp Y\right) - X^R \cdot T_{i,j}\left(\exp Y\right). \tag{3.2.29}$$

Now, let Φ^+ be the set of positive roots of $\mathfrak{g}^{\mathbb{C}}$. For each $\alpha \in \Phi^+$, let X_α (Y_α) denote the positive (negative) root vector in $\mathfrak{g}^{\mathbb{C}}$. By Proposition 3.1.7, the first row of the matrix coefficients of π_λ can be obtained as the functions of the form

$$(Y_{\alpha_1}^{i_1})^R ... (Y_{\alpha_m}^{i_m})^R \cdot T_{1,1}(g) = \langle \pi_\lambda(g) d\pi_\lambda(Y_{\alpha_1}^{i_1}) ... d\pi_\lambda(Y_{\alpha_m}^{i_m}) v_\lambda, v_\lambda \rangle, \; g \in G. \tag{3.2.30}$$

However, if we replace each right invariant vector field $(Y_{\alpha_j})^R$ by a left one, $(Y_{\alpha_j})^L$, then

$$(Y_{\alpha_1}^{i_1})^L ... (Y_{\alpha_m}^{i_m})^L \cdot T_{1,1}(g) = \langle \pi_\lambda(g) v_\lambda, d\pi_\lambda(X_{\alpha_1}^{i_1}) ... d\pi_\lambda(X_{\alpha_m}^{i_m}) v_\lambda \rangle. \tag{3.2.31}$$

Notice that since v_λ is the highest weight vector in $\mathcal{H}_\lambda$, so each raising operator $d\pi_\lambda(X_{\alpha_j})$ annihilates v_λ, that is, $d\pi_\lambda(X_{\alpha_j})v_\lambda = 0$. Therefore, by (3.2.26), we conclude that

$$(Y_{\alpha_1}^{i_1})^R ... (Y_{\alpha_m}^{i_m})^R \cdot T_{1,1}(\exp Z) = (-1)^{i_1+\cdots+i_m} (Y_{\alpha_1}^{i_1})^G ... (Y_{\alpha_m}^{i_m})^G \cdot T_{1,1}(\exp Z), \tag{3.2.32}$$

for $Z \in \mathfrak{g}$. Similarly, the first column of the matrix coefficients of π_λ can be obtained as the functions of the form

$$(X_{\alpha_1}^{i_1})^L ... (X_{\alpha_m}^{i_m})^L \cdot T_{1,1}(g) = \langle \pi_\lambda(g) v_\lambda, d\pi_\lambda(Y_{\alpha_1}^{i_1}) ... d\pi_\lambda(Y_{\alpha_m}^{i_m}) v_\lambda \rangle, \tag{3.2.33}$$

and we also have

$$(X_{\alpha_1}^{i_1})^L ... (X_{\alpha_m}^{i_m})^L \cdot T_{1,1}(\exp Z) = (-1)^{i_1+\cdots+i_m} (X_{\alpha_1}^{i_1})^G ... (X_{\alpha_m}^{i_m})^G \cdot T_{1,1}(\exp Z). \tag{3.2.34}$$

To generate the rest of the matrix coefficients of π_λ, observe that for each row

i, with $i \geqslant 2$, we have

$$Y^R_{\alpha_j} T_{i,j}(\exp Z) = -Y^L_{\alpha_j} T_{i,j}(\exp Z) - Y^G_{\alpha_j} T_{i,j}(\exp Z), \tag{3.2.35}$$

and $Y^L_{\alpha_k}$ either 'raises' $T_{i,j}$ to a row l, for $l < i$, which is previously generated, or annihilates $T_{i,j}$. Hence, the proposition follows. ☐

Example 3.2.5.1. Let π_1 be the irreducible unitary representation of $SU(2)$ with integer highest weight 1 given in Example (2.5.5.1),

$$\pi_1 = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \tag{3.2.36}$$

where $(\alpha, \beta) \in \mathbb{C}^2$ and it satisfies $|\alpha|^2 + |\beta|^2 = 1$. Let $\mathfrak{su}(2)$ be the Lie algebra of $SU(2)$. Recall that by Proposition 2.5.8, the lift of α and β by the exponential mapping of $\mathfrak{su}(2)$ are given by

$$\alpha\left(\exp(X)\right) = \left(r\frac{\partial}{\partial r} + 1 + ix_3\right)\frac{\sin(|X|)}{|X|}, \; \beta\left(\exp(X)\right) = (x_1 + ix_2)\frac{\sin(|X|)}{|X|}, \tag{3.2.37}$$

for $X = x_1X_1 + x_2X_2 + x_3X_3 \in \mathfrak{su}(2)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$, $r\frac{\partial}{\partial r} = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}$, and $|X| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Notice that $\frac{\sin|X|}{|X|}$ is both differentiable and G-invariant, and $r\frac{\partial}{\partial r}$ preserves the G-invariance since

$$r\frac{\partial}{\partial r}\frac{\sin(|X|)}{|X|} = \cos(|X|) - \frac{\sin(|X|)}{|X|}. \tag{3.2.38}$$

Let $H^G_\alpha, X^G_\alpha, Y^G_\alpha$ in (3.2.19) be the complexification of the vector fields induced by the $SU(2)$ actions in $\mathfrak{su}(2)$. So, we have

$$\begin{aligned} Y^G_\alpha \cdot \alpha(\exp X) &= -\frac{1}{2}\left(X^G_1 + iX^G_2\right) \cdot \alpha(\exp X) \\ &= \left((ix_1 - x_2)\frac{\partial}{\partial x_3} - x_3(i\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2})\right) \cdot \left(\left(r\frac{\partial}{\partial r} + 1 + ix_3\right)\frac{\sin(|X|)}{|X|}\right) \\ &= -(x_1 + ix_2)\frac{\sin(|X|)}{|X|} = -\beta(\exp X), \end{aligned} \tag{3.2.39}$$

and

$$\begin{aligned}X_\alpha^G \cdot \alpha(\exp X) &= \frac{1}{2}\left(X_1^G - iX_2^G\right)\cdot \alpha(\exp X)\\ &= \left((ix_1+x_2)\frac{\partial}{\partial x_3} - x_3(i\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2})\right)\cdot\left(\left(r\frac{\partial}{\partial r}+1+ix_3\right)\frac{\sin(|X|)}{|X|}\right)\\ &= -(x_1-ix_2)\frac{\sin(|X|)}{|X|} = -\overline{\beta}(\exp X).\end{aligned} \tag{3.2.40}$$

Let $\{\xi_1,\xi_2\}$ be the orthonormal basis of the Hilbert space $\mathcal{H}_{\pi_1}$, choose ξ_1 to be the highest weight vector. Also, let $T_{1,1}(\exp X) = \langle \pi_1(\exp X)\xi_1,\xi_1\rangle = \alpha(\exp X)$. Then by the identity in (3.2.26), we have

$$\begin{aligned}&-\frac{1}{2}\left(X_1^R + iX_2^R\right)\cdot\langle\pi_1(\exp X)\xi_1,\xi_1\rangle\\ &\quad= \frac{1}{2}\left(X_1^L + iX_2^L\right)\cdot\langle\pi_1(\exp X)\xi_1,\xi_1\rangle + \frac{1}{2}\left(X_1^G+iX_2^G\right)\langle\pi_1(\exp X)\xi_1,\xi_1\rangle.\end{aligned} \tag{3.2.41}$$

This translates to

$$\begin{aligned}&\langle\pi_1(\exp X)d\pi_1(Y_\alpha)\xi_1,\xi_1\rangle\\ &\quad= \langle\pi_1(\exp X)\xi_1, d\pi_1(X_\alpha)\xi_1\rangle - Y_\alpha^G\cdot\langle\pi_1(\exp X)\xi_1,\xi_1\rangle.\end{aligned} \tag{3.2.42}$$

Since $d\pi_1(Y_\alpha)\xi_1 = \xi_2$ and $d\pi_1(X_\alpha)\xi_1 = 0$, we have $T_{1,2}(\exp X) = -Y_\alpha^G \cdot T_{1,1}(\exp X) = \beta(\exp X)$. Also, we have

$$\begin{aligned}&\frac{1}{2}\left(X_1^R - iX_2^R\right)\cdot\langle\pi_1(\exp X)\xi_1,\xi_1\rangle\\ &\quad= -\frac{1}{2}\left(X_1^L - iX_2^L\right)\cdot\langle\pi_1(\exp X)\xi_1,\xi_1\rangle - \frac{1}{2}\left(X_1^G - iX_2^G\right)\cdot\langle\pi_1(\exp X)\xi_1,\xi_1\rangle,\end{aligned} \tag{3.2.43}$$

and this translates to

$$\begin{aligned}&\langle\pi_1(\exp X)d\pi_1(X_\alpha)\xi_1,\xi_1\rangle\\ &\quad= \langle\pi_1(\exp X)\xi_1, d\pi_1(X_\alpha)\xi_1\rangle - X_\alpha^G\cdot\langle\pi_1(\exp X)\xi_1,\xi_1\rangle,\end{aligned} \tag{3.2.44}$$

and this means $T_{2,1}(\exp X) = X_\alpha^G \cdot T_{1,1}(\exp X) = -\overline{\beta}(\exp X)$. To generate the last element $\overline{\alpha}$, first we have

$$\begin{aligned} Y^G_\alpha \cdot -\overline{\beta}(\exp X) &= -\frac{1}{2}\left(X_1^G + iX_2^G\right) \cdot -\beta(\exp X) \\ &= \left((ix_1 - x_2)\frac{\partial}{\partial x_3} - x_3(i\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2})\right) \cdot \left(-\left(x_1 - ix_2\right)\frac{\sin(|X|)}{|X|}\right) \\ &= 2ix_3\frac{\sin(|X|)}{|X|}. \end{aligned} \tag{3.2.45}$$

Since $T_{2,1}(\exp X)$ has been generated, we have

$$\begin{aligned} &-\frac{1}{2}\left(X_1^R + iX_2^R\right) \cdot \langle \pi_1(\exp X)\xi_1, \xi_2\rangle \\ &\quad = \frac{1}{2}\left(X_1^L + iX_2^L\right) \cdot \langle \pi_1(\exp X)\xi_1, \xi_2\rangle + \frac{1}{2}\left(X_1^G + iX_2^G\right)\langle \pi_1(\exp X)\xi_1, \xi_2\rangle, \end{aligned} \tag{3.2.46}$$

and this translates to

$$\begin{aligned} &\langle \pi_1(\exp X)d\pi_1(Y_\alpha)\xi_1, \xi_2\rangle \\ &\quad = \langle \pi_1(\exp X)\xi_1, d\pi_1(X_\alpha)\xi_2\rangle - X^G_\alpha \cdot \langle \pi_1(\exp X)\xi_1, \xi_2\rangle. \end{aligned} \tag{3.2.47}$$

Thus, we have $T_{2,2}(\exp X) = T_{1,1}(\exp X) - Y^G_\alpha \cdot T_{2,1}(\exp X)$. This is exactly

$$\alpha(\exp X) - \overline{\alpha}(\exp X) = 2ix_3\frac{\sin(|X|)}{|X|}. \tag{3.2.48}$$

In this way, every lift of the matrix coefficient of π_1 can be generated by applying a polynomial of induced vector fields by G action on $T_{1,1}(\exp X)$. We can also explicitly write down these operators for convolutions of matrix coefficients of π_1, for example,

$$\begin{aligned} T_{1,2} * T_{1,1}(\exp X) &= \frac{1}{2}T_{1,2}(\exp X) \\ &= -\frac{1}{2}Y^G_\alpha \cdot T_{1,1}(\exp X) = \beta(\exp X), \\ T_{1,2} * T_{2,1}(\exp X) &= \frac{1}{2}T_{2,2}(\exp X) \\ &= \frac{1}{2}(1 - Y^G_\alpha X^G_\alpha) \cdot T_{1,1}(\exp X) = \overline{\alpha}(\exp X) \\ T_{1,2} * T_{1,2}(\exp X) &= 0 \cdot T_{1,1}(\exp X) = 0, \end{aligned} \tag{3.2.49}$$

etc. Similarly, we can work out these differential operators for irreducible unitary representations π_n of $SU(2)$, for $n > 1$.

3.3 Radial Part of G-Invariant Differential Operators

In this section, we study the radial part of a G-invariant differential operator, with respect to the transversality condition.

Theorem 3.3.1 ([27], Theorem II.3.6)**.** *Let V be a manifold satisfying the second axiom of countability and suppose that G is a Lie transformation group of V. Let K be a submanifold of V that satisfies the transversality condition. Let D be a differential operator on V. Then there exists a unique differential operator $\Theta(D)$ on K such that*

$$(D \cdot f)^{-} = \Theta(D) \cdot \overline{f}, \tag{3.3.1}$$

*for each locally invariant function f on an open subset of V. The 'overline denotes the restriction to K, and $\Theta(D)$ is called the **'radial part'** of D.*

Theorem 3.3.2 ([27], Theorem II.3.7)**.** *Let V be a Riemannian manifold, G a unimodular isometry group of V. Assume that a submanifold $K \subset V$ satisfies the transversality condition. Then the radial part of the Laplacian operator L_V is given by*

$$\Theta(L_V) = \delta^{-\frac{1}{2}} L_K \circ \delta^{\frac{1}{2}} - \delta^{-\frac{1}{2}} L_K(\delta^{\frac{1}{2}}). \tag{3.3.2}$$

where $\circ$ is the composition of differential operators and δ is the density function given by proportionality of the measures: $d\nu_v = \delta(v) d\mu_v, v \in V$, between the orbit $G \cdot v$ and the homogeneous manifold G/G_v.

Example 3.3.2.1 ([27], Sec.III.3.4.(i))**.** *'Radial part' of the Laplacian in $\mathbb{R}^n$:*

Let $V = \mathbb{R}^n$, and $G = O(n)$ be the orthogonal group acting on $\mathbb{R}^n$. Let $L_{\mathbb{R}^n}$ be the Laplacian operator in $\mathbb{R}^n$. Then the submanifold $W = \mathbb{R}^+ \backslash \{0\} \subset \mathbb{R}^n$ satisfies the transversality condition. Hence,

$$\Theta(L_{\mathbb{R}^n}) = r^{-\frac{1}{2}(n-1)} \frac{d^2}{dr^2} \circ r^{\frac{1}{2}(n-1)} - r^{-\frac{1}{2}(n-1)} \frac{d^2}{dr^2} \left(r^{\frac{1}{2}(n-1)} \right). \tag{3.3.3}$$

where the density function $\delta(r) = r^{(n-1)}$, for $r \in W$. This can also be written as the familiar expression,

$$\Theta(L_{\mathbb{R}^n}) = \frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr}. \tag{3.3.4}$$

Example 3.3.2.2 ([27], Example.II.3.4.(vii))**.** *'Radial part' of the complex Laplacian of semisimple Lie algebras.*

Let G be a complex semi-simple Lie group. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{h}' \subset \mathfrak{h}$ the subset of regular elements. The Killing form $\kappa(\cdot,\cdot)$ on $\mathfrak{g}$ is non-degenerate on both $\mathfrak{g}$ and $\mathfrak{h}$, so we can use it to identify their duals $\mathfrak{g}^*$ and $\mathfrak{h}^*$.

If we treat $\mathfrak{g}$ as a vector space, we can define polynomial functions on it in the usual sense of polynomial functions on a vector space. That is, if $\mathfrak{g}$ is a n-dimensional Lie algebra with a basis $\{X_1, X_2, ..., X_n\}$, then a polynomial function $p : \mathfrak{g} \to \mathbb{C}$ is a function in the coordinate function $x_1, x_2, ..., x_n$:

$$p(x_1, x_2, ..., x_n) = \sum a_{i_1 i_2 ... i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, \tag{3.3.5}$$

Let p be a polynomial function on $\mathfrak{g}$, $p(\partial)$ be the constant coefficient differential operator on $\mathfrak{g}$, which can be written as

$$p(\partial)(x_1, x_2, ..., x_n) = \sum a_{i_1 i_2 ... i_k} \frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_k}}. \tag{3.3.6}$$

Let $\omega(X) = \kappa(X,X) = trace(ad_X, ad_X)$, for $X \in \mathfrak{g}$ (also known as complex Laplacian). Suppose we let $X = H + \sum_{\alpha\in\Phi^+} X_\alpha \in \mathfrak{h}' \oplus \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$, and let $\overline{\omega}(H) = \kappa(H,H), H \in \mathfrak{h}'$. Then, given that $\mathfrak{h}'$ satisfies the transversality condition (Example 3.2.2.1), the radial part of the operator $\omega(\partial)$ is given by

$$\Theta(\omega(\partial))(H) = \Omega^{-1}(H)\overline{\omega}(\partial)(H) \circ \Omega(H), \quad H \in \mathfrak{h}', \tag{3.3.7}$$

where $\Omega(H) = \prod_{\alpha\in\Phi^+} \alpha(H)$, is the product of all positive roots of $\mathfrak{g}$, as linear functionals on $\mathfrak{h}$, and $\overline{\omega}$ is the restriction of ω to $\mathfrak{h}'$. In fact, this proposition can be extended to all G-invariant polynomial functions on $\mathfrak{g}$.

Theorem 3.3.3 ([27], Theorem II.5.33)**.** *Let G be a complex semisimple Lie group and $\mathfrak{g}$ be its Lie algebra. Let p be a G-invariant polynomial function and $p(\partial)$ be the corresponding constant coefficient G-invariant differential operator on $\mathfrak{g}$. Let $\mathfrak{h}' \subset \mathfrak{h}$ be the subset of regular elements in the Cartan subalgebra $\mathfrak{h}$. Given that $\mathfrak{h}'$ satisfies the transversality condition, then the radial part of the differential operator $p(\partial)$ is given by*

$$\Theta(p(\partial)) = \Omega^{-1}\overline{p}(\partial) \circ \Omega. \tag{3.3.8}$$

Example 3.3.3.1. Let $G^{\mathbb{C}} = SL_2(\mathbb{C})$, and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$. Let $G = SU(2)$ be the compact real form (and also a subgroup) of $SL_2(\mathbb{C})$, $\mathfrak{g} = \mathfrak{su}(2) \cong \mathbb{R}^3$ be the Lie algebra of $SU(2)$. Let $\{X_1, X_2, X_3\}$ be the standard basis of $\mathfrak{su}(2)$ in (3.2.11). Suppose $X = x_1X_1 + x_2X_2 + x_3X_3 \in \mathfrak{su}(2)$, for $(x_1, x_2, x_3) \in \mathbb{R}^3$, we have the Killing form $\omega(X) = \kappa(X, X) = x_1^2 + x_2^2 + x_3^2$. This means that if we restrict to $\mathfrak{su}(2)$, then the Laplacian of $\mathfrak{su}(2)$ is given by

$$L(\mathfrak{su}(2)) = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}. \tag{3.3.9}$$

Since the only positive root $\{\alpha\} \in \Phi^+$ has value $\alpha = 1$, the radial part of $L(\mathfrak{su}(2))$ is given by

$$\Theta(L(\mathfrak{su}(2))) = r^{-1}\frac{\partial^2}{\partial r^2} \circ r = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}, \quad r \in \mathbb{R}^+\backslash\{0\}, \tag{3.3.10}$$

which is the exactly the same as (3.3.4) when $n = 3$.

This interesting correspondence in (3.3.8) has led to the derivation of a continuous version of the Kirillov character formula for compact Lie groups. We will explain it in detail in the next section.

3.4 Liouville Measures on Coadjoint Orbits

Definition 3.4.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$. Also, let Φ be the root system of $\mathfrak{g}$. Let $\alpha \in \Phi$ be a linear functional (with respect to the Killing form) on $\mathfrak{h}$, denoted by $\alpha(H)$, for $H \in \mathfrak{h}$, and let $\alpha(\partial)$ be the **directional derivative of** α. If $\alpha_i, \alpha_j \in \Phi$, then for $H \in \mathfrak{h}$,

$$\alpha_i(\partial) \cdot \alpha_j(H) = \alpha_j(H_{\alpha_i}). \tag{3.4.1}$$

where H_{α_i} and α_i form a dual pair.

Theorem 3.4.2 ([27], Theorem II.5.35). *Let G be a compact and semisimple Lie group with a maximal torus T, $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{t}^{\mathbb{C}}$ (a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$) be the complexification of their Lie algebras $\mathfrak{g}$ and $\mathfrak{t}$. Let $\Omega = \prod_{\alpha\in\Phi^+}\alpha$, the product of all positive roots. Let dg be a normalised Haar measure on G such that*

$\int_G dg = 1$. *Then, for* $H, H' \in \mathfrak{t}^{\mathbb{C}}$,

$$\Omega(H)\Omega(H')\int_G e^{(Ad(g)H,H')}\,dg = \frac{1}{|W|}\Omega(\partial)(\Omega)\sum_{w\in W} sgn(w)e^{(wH,H')}, \quad X \in \mathfrak{g} \tag{3.4.2}$$

where $\Omega(\partial)(\Omega)$ *is the product of the directional derivatives on the function* Ω *in the directions of all positive roots,* W *is the Weyl group, and* $|W|$ *is the order of* W*.*

Proof. Let $f \in C^\infty(\mathfrak{g})$. Define a G-invariant function F on $\mathfrak{g}$ as follows:

$$F(X) = \int_G f(Ad(g)X)dg\,, \quad X \in \mathfrak{g}. \tag{3.4.3}$$

If p is a G-invariant polynomial function, then we have

$$(p(\partial)F)(H) = (\Omega^{-1}\overline{p}(\partial))(\Omega\overline{F}))(H), \quad H \in \mathfrak{t}^{\mathbb{C}}. \tag{3.4.4}$$

This means, the function $\phi_f(H) = \Omega(H)F(H)$ satisfies

$$(\overline{p}(\partial)\phi_f)(H) = \Omega(H)(p(\partial)F)(H), \tag{3.4.5}$$

by Theorem 3.3.3, which is equal to $\phi_{p(\partial)f}(H)$. Suppose we let $H' \in \mathfrak{t}^{\mathbb{C}}$ such that $\Omega(H') \neq 0$, and let $f(X) = e^{(X,H')}$, then $p(\partial)f = p(H')f$ and $\overline{p}(\partial)\phi_f = p(H')\phi_f$. So, by ([27], Theorem III.3.13), ϕ_f has the form

$$\phi_f(H) = \sum_{w\in W} c_w e^{(w\cdot H,H')} \tag{3.4.6}$$

where $c_w \in \mathbb{C}$. Also, since $\Omega(w\cdot H) = \mathrm{sgn}(w)\Omega(H)$, which leads to $\phi_f(w\cdot H) = \mathrm{sgn}(w)\phi_f(H)$, hence, each $c_w = c\cdot\mathrm{sgn}(w)$, and

$$\phi_f(H) = c\sum_{w\in W} sgn(w)e^{(w\cdot H,H')}. \tag{3.4.7}$$

To determine the constant c, apply $\Omega(\partial)$ to $\phi_f(H)$, and evaluate it at $H = 0$. By the definition of ϕ_f, we have

$$(\Omega(\partial)\phi_f)(0) = \Omega(\partial)(\Omega) = c|W|\Omega(H'). \tag{3.4.8}$$

Hence, the integral formula (3.4.2) is proved, for $H \in \mathfrak{t}$, $\Omega(H') \neq 0$, and hence for all $H, H' \in \mathfrak{t}^{\mathbb{C}}$ by holomorphic continuation.

Let δ denote the half-sum of positive roots. Notice that if $H' = H_\delta$, then

$$\sum_{w\in W} sgn(w)e^{(wH,H_\delta)} = \prod_{\alpha\in\Phi^+}(e^{i\frac{1}{2}\alpha(H)} - e^{-i\frac{1}{2}\alpha(H)}). \tag{3.4.9}$$

Suppose now we restrict the equation above to $\mathfrak{t}$, we have

$$\frac{\sum_{w\in W} sgn(w)e^{(wH,H_\delta)}}{\Omega(H)} = \frac{\prod_{\alpha\in\Phi^+}\sin\frac{1}{2}\alpha(H)}{\prod_{\alpha\in\Phi^+}\frac{1}{2}\alpha(H)}. \tag{3.4.10}$$

So, we have a product of *sinc* functions, and it converges to 1 as $H \to 0$. Hence we conclude that

$$|W| = \frac{\Omega(\partial)(\Omega)}{\Omega(H_\delta)}, \tag{3.4.11}$$

if we let $H \to 0$ in both sides of (3.4.2). ☐

Corollary 3.4.2.1. *Let H' be a regular element in $\mathfrak{t}^{\mathbb{C}}$, then*

$$\frac{\Omega(H')}{\Omega(H_\delta)}\int_G e^{(Ad(g)H,H')}\,dg = \frac{\sum_{w\in W} sgn(w)e^{(wH,H')}}{\Omega(H)}, \quad \forall H\in\mathfrak{t}^{\mathbb{C}} \tag{3.4.12}$$

Hence, if we let $H' = H_{\lambda+\delta}$ for a dominant weight λ, we recover the Kirillov character formula in (2.4.2).

Example 3.4.2.1. Let the compact connected semisimple Lie group G be $SU(2)$, $\mathbb{T}$ be a maximal torus of $SU(2)$. Also, let $\mathfrak{su}(2)$ and $\mathfrak{t}$ be their Lie algebras. Choose an integral element $\lambda \in \mathfrak{t}^{\mathbb{C}}$ with $\lambda = 1$, and the only positive root $\alpha = 2$, also the δ element is equal to $\delta = \frac{1}{2}\alpha = 1$. Therefore, we have

$$\frac{\Omega(H_1)}{\Omega(H_1)}\int_G e^{i(Ad(g)H,H_1)}\,dg = \frac{\sum_{w\in W} sgn(w)e^{i(wH,H_1)}}{\Omega(H)}, \quad \forall H\in\mathfrak{t}. \tag{3.4.13}$$

We can map the Haar measure dg (which is unique up to a scalar multiple) on G to a G-invariant probability measure in the dual Lie algebra $\mathfrak{su}(2)^*$ that is supported on the coadjoint orbit passing through $\lambda = 1$ (by $p : G \to \mathcal{O}_1$), denoted by ν_1. Since each coadjoint orbit carries a symplectic structure, this measure ν_1 is also the *Liouville measure* on $\mathcal{O}_1$, which is also unique up to a scalar multiple. Therefore, we have

$$\int_{\mathcal{O}_1} e^{i\beta(H)}\,d\nu_1(\beta) = \frac{\sin H}{H}, \quad H\in\mathfrak{t}\cong\mathbb{R}. \tag{3.4.14}$$

So, this formula generalises Cazzaniga's calculations for $SU(2)$ in Proposition 2.5.8.

Example 3.4.2.2. Let G be $SU(3)$, T be a maximal torus of $SU(3)$. Also, let $\mathfrak{su}(3)$ and $\mathfrak{t}$ be their Lie algebras. Let $\mathfrak{t}^{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{su}(3)^{\mathbb{C}}$. Let $\{\alpha_1, \alpha_2\}$ be the set of simple roots of $\mathfrak{su}(3)^{\mathbb{C}}$. By (2.2.21), the 'first' fundamental weight vector $\lambda_1 = \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$. Denote $\mathcal{O}_{\lambda_1}$ the coadjoint orbit passing through λ_1, and also denote ν_{λ_1} the Liouville measure on $\mathcal{O}_{\lambda_1}$. In this case, we have

$$\frac{\Omega(H_{\lambda_1})}{\Omega(H_\delta)} \int_{\mathcal{O}_{\lambda_1}} e^{i\beta(H)}, d\nu_{\lambda_1}(\beta) = \frac{\sum_{w \in W} sgn(w) e^{i(wH, H_{\lambda_1})}}{\Omega(H)}, \quad \forall H \in \mathfrak{t}. \tag{3.4.15}$$

However, both sides above are equal to zero. To see this, on the left hand side,

$$\Omega(H_{\lambda_1}) = \prod_{\alpha \in \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}} \alpha(H_{\lambda_1}), \tag{3.4.16}$$

and because λ_1 is perpendicular to α_2 with respect to the Killing form, it follows that $\Omega(H_{\lambda_1}) = 0$. Also, from Example 2.2.18.1, we can see that

$$\sum_{w \in W} sgn(w) e^{i(wH, H_{\lambda_1})} = 0. \tag{3.4.17}$$

Hence, in general, this theorem is not valid for any non regular elements in $\mathfrak{t}$ which also lie on the walls of the Weyl chambers of $\mathfrak{t}$.

Therefore, in order to extend Cazzaniga's method beyond $SU(2)$, for example, as simple as the defining representation of $SU(3)$ (the irreducible representation π_{λ_1} with highest weight λ_1), we also need to have a normalisation formula for the *Liouville measures* for coadjoint orbits of general non-regular elements in $\mathfrak{t}$. This is a subject which will be studied in Chapter 4.

Chapter 4

Convolution Structures of Coadjoint Orbits

We have described the convolutions of coadjoint orbits of $SU(2)$ in Chapter 2, and we wish to generalise this to convolution structures for an arbitrary pair of coadjoint orbits of a compact connected semisimple Lie group G. Let T be a maximal torus of G, $\mathfrak{t}$ be the Lie algebra of T, and $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. In this chapter, we study the convolution structure of the sumset of a pair of coadjoint orbits $\mathcal{O}_\lambda + \mathcal{O}_\gamma$, for arbitrary $\lambda, \gamma \in \mathfrak{t}^*$. The original proof was given by A. Dooley and N. Wildberger in [18]. In fact, the set of adjoint invariant probability measures on $\mathfrak{g}$ is a commutative *hypergroup* [54]. This also leads to finding the explicit structural convolution equation for the hypergroup of coadjoint orbits of G.

In Section 4.1, we study how to use Theorem 3.4.2 and the *Kostant convexity theorem* [38] to find a formula for the projection of a G-invariant measure of a regular coadjoint orbit $\mathcal{O}_\lambda$ in $\mathfrak{t}^*$, denoted by μ_λ^p. The 'basic tent' is the projection of the coadjoint orbit $\mathcal{O}_\delta$, and it acts as a foundation for the derivation of μ_λ^p.

In Section 4.2, we study an explicit formula for the convolution of any pair of regular coadjoint orbits.

In Section 4.3, we study how to approximate the G-invariant Liouville measure of a singular coadjoint orbit from a regular one.

In Section 4.4, in Proposition 4.4.1, we calculate the singular supports of the n-fold convolution of coadjoint orbits (unit 2-spheres) of $SU(2)$ by using the formulas developed in previous sections.

In addition, all **lemmas** and **theorems** included in this chapter are interpreted and reworked from the original content of the article 'Sum of Adjoint Orbits' by A. Dooley and N. Wildberger [18], for the purpose of coherence and also the completeness of the development of the content in the next chapter.

In this chapter, we let G be a compact semisimple Lie group, T be a maximal torus of G, and $\mathfrak{g}$, $\mathfrak{t}$ be their Lie algebras. Also, let $\mathfrak{g}^*, \mathfrak{t}^*$ be the duals of $\mathfrak{g}$, $\mathfrak{t}$, and let $\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}, \mathfrak{t}$, respectively. Additionally, we let Φ be the chosen root system of $\mathfrak{g}^{\mathbb{C}}$, and let Φ^+ be the subset of all positive roots of Φ, and let Δ be the set of simple roots. Let W denote the Weyl group of $\mathfrak{t}^*$, and let $\mathfrak{t}^*_+$ denote a chosen fundamental dual Weyl chamber with respect to W, $\mathfrak{t}^*_0$ be the interior of $\mathfrak{t}^*_+$, Λ be the set of weights of $\mathfrak{t}^*$, and Λ^+ be the set of dominant weights of Λ in $\mathfrak{t}^*_+$.

4.1 Basic Tent and Projection of Coadjoint Orbit

The Kostant convexity theorem [38] says that the orthogonal projection of the coadjoint orbit $\mathcal{O}_\lambda$ of $\lambda \in \mathfrak{t}^*$ onto $\mathfrak{t}^*$ is the convex hull of $W \cdot \lambda$, the Weyl orbit of λ. Let $p : \mathfrak{g}^* \to \mathfrak{t}^*$ be the projection map. We can show that the $\mathfrak{g}$-Fourier transform of μ_λ restricted to $\mathfrak{t}$ is equal to the $\mathfrak{t}$-Fourier transform of μ^p_λ, that is, $\mu_\lambda^{\vee_{\mathfrak{g}}}\big|_{\mathfrak{t}} = (\mu^p_\lambda)^{\vee_{\mathfrak{t}}}$. To see this, we first state the **disintegration theorem**.

Theorem 4.1.1 ([12], Theorem III.70)**.** *(Disintegration Theorem) Let X, Y be Radon spaces, and $\mathcal{P}(X)$, $\mathcal{P}(Y)$ be the collections of Radon probability measures on X and Y. Let $\pi : X \to Y$ be a Borel measurable function, such that $\{\pi^{-1}(y) : y \in Y\}$ form a partition of X. Let $\mu \in \mathcal{P}(X)$, and $\nu \in \mathcal{P}(Y)$ be the measure $\nu = \mu \circ \pi^{-1}$. Then, there exists a ν-almost everywhere uniquely determined family of probability measures $\{\mu_y\}_{y \in Y} \subseteq \mathcal{P}(X)$, which provides a disintegration of μ into $\{\mu_y\}_{y \in Y}$, such that*

- *the function $y \mapsto \mu_y(B)$ is a Borel-measurable function for each Borel-measurable set $B \subseteq X$.*
- *μ_y is assigned to the fibre $\pi^{-1}(y)$, that is, for ν-almost all $y \in Y$, we have $\mu_y(X \backslash \pi^{-1}(y)) = 0$ and $\mu_y(B) = \mu_y(B \cap \pi^{-1}(y))$.*
- *for every Borel-measurable function $f : X \to [0, \infty]$,*

$$\int_X f(x)d\mu(x) = \int_Y \int_{\pi^{-1}(y)} f(x)d\mu_y(x)d\nu(y). \tag{4.1.1}$$

Theorem 4.1.2 ([38], Theorem 4.1)**.** *(Kostant convexity theorem) Let $\lambda \in \mathfrak{t}^*$, $\mathcal{O}_\lambda$ be the coadjoint orbit passing through λ. Let $p : \mathfrak{g}^* \to \mathfrak{t}^*$ be the orthogonal projection map relative to the Killing form. Then the image of the projection $p : \mathcal{O}_\lambda \to \mathfrak{t}^*$ is the convex hull of the Weyl orbit of λ in $\mathfrak{t}^*$, denoted by conv$(W \cdot \lambda)$ (a convex polytope with vertices $W \cdot \lambda$).*

Remark. *This is a special case of a more general result for symmetric spaces. An elementary new proof of this theorem for compact semisimple Lie groups based on the Jacobi eigenvalue algorithm was given by N. Wildberger [56].*

Proposition 4.1.3. *If $\beta \in \mathfrak{g}^*$, then there exists a unique (up to a scalar multiple) G -invariant measure μ_β on the coadjoint orbit $\mathcal{O}_\beta$.*

Proof. G acts transitively on $\mathcal{O}_\beta$. For each $\beta \in \mathfrak{g}^*$, the coadjoint orbit $\mathcal{O}_\beta$ is a homogeneous space of the form G/G_β, where G_β is the stabiliser subgroup of β. A Haar measure on G can be used to define an invariant measure on G/G_β.

Since G is compact, it is unimodular ([21], Corollary 2.28) and carries an invariant measure μ_G, which is both left and right invariant. In addition, μ_G is unique up to a scalar multiple ([21], Theorem 2.20).

Let $\sigma : G \to G/G_\beta$ be the projection such that $\sigma(g) = gG_\beta$, for $g \in G$. We can define a measure μ_{G/G_β} by

$$\mu_{G/G_\beta}(E) = \mu_G(\sigma^{-1}(E)), \tag{4.1.2}$$

for any measurable subset $E \subseteq G/G_\beta$. Hence, μ_{G/G_β} is also unique (up to a scalar multiple). Since $\mathcal{O}_\beta \cong G/G_\beta$, the result follows. In addition, G is compact, so μ_β is finite. Hence, we can normalise it so that $\mu_\beta(\mathcal{O}_\beta) = 1$. □

Definition 4.1.4. Let $\Phi^+ \subset \Phi$ be the set of all positive roots and let $\Delta \subset \Phi^+$ be the set of simple roots. Recall that from the definition of the decomposition of the **compact real form** $\mathfrak{g}$ of a complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ (Definition 2.2.10), we have

$$\mathfrak{g} = \underbrace{\sum_{\alpha\in\Delta} \mathbb{R}(iH_\alpha)}_{\mathfrak{t}} + \underbrace{\sum_{\alpha\in\Phi^+} \mathbb{R}(X_\alpha - Y_\alpha) + \sum_{\alpha\in\Phi^+} \mathbb{R}i(X_\alpha + Y_\alpha)}_{\mathfrak{t}^\perp}, \tag{4.1.3}$$

where $\mathfrak{t}^\perp$ is **orthogonal** to $\mathfrak{t}$ with respect to the *Killing form* of $\mathfrak{g}$.

If we apply the disintegration theorem and the Kostant convexity theorem to the coadjoint orbits of a compact connected semisimple Lie group, then we have the following result.

Lemma 4.1.5. *Let $p : \mathfrak{g}^* \to \mathfrak{t}^*$ be the orthogonal projection with respect to the Killing form, and let $p : \mathcal{O}_\lambda \mapsto conv(W \cdot \lambda)$, for $\lambda \in \mathfrak{t}^*$ (Theorem 4.1.2). Let $(\beta, \gamma) \in p^{-1}(\beta)$ for $\beta \in \mathfrak{t}^*$ and $\gamma \in (\mathfrak{t}^*)^\perp$. Let f be a Borel-measurable function in $\mathfrak{g}^*$, μ_λ be the G-invariant Liouville measure on $\mathcal{O}_\lambda$, μ_β be a probability measure on the fibre $p^{-1}(\beta)$ for $\beta \in conv(W \cdot \lambda)$, then by the disintegration theorem, we have*

$$\int_{\mathcal{O}_\lambda} f(\xi)\, d\mu_\lambda(\xi) = \int_{conv(W\cdot\lambda)} \int_{p^{-1}(\beta)} f(\beta, \gamma)\, d\mu_\beta(\gamma)\, d\mu_\lambda^p(\beta), \tag{4.1.4}$$

where μ_λ^p is the projection of μ_λ with respect to p.

Remark. *Because the dual $\mathfrak{t}^*$ is degenerate on $\mathfrak{t}^\perp$ with respect to the Killing form, for $(\beta, \gamma) \in p^{-1}(\beta)$, we have*

$$(\beta + \gamma)(H) = \beta(H) + \gamma(H) = \beta(H), \quad \textit{for } H \in \mathfrak{t}. \tag{4.1.5}$$

Therefore,

$$\int_{\mathcal{O}_\lambda} e^{i\beta(H)}\, d\mu_\lambda(\beta) = \int_{conv(W\cdot\lambda)} e^{i\beta(H)}\, d\mu_\lambda^p(\beta), \tag{4.1.6}$$

for $H \in \mathfrak{t}$. That is,

$$\mu_\lambda^{\vee_{\mathfrak{g}}}\big|_{\mathfrak{t}} = (\mu_\lambda^p)^{\vee_{\mathfrak{t}}}. \tag{4.1.7}$$

Theorem 4.1.6 ([30], Theorem 24.3)**.** *(Weyl dimension formula) Let $\lambda \in \Lambda^+$. Then the dimension of the irreducible highest weight representation π_λ of G is*

$$d_\lambda = \frac{\prod_{\alpha\in\Phi^+}(\lambda, \alpha)}{\prod_{\alpha\in\Phi^+}(\delta, \alpha)}, \tag{4.1.8}$$

δ is the half-sum of all positive roots.

We also have an explicit formula for $\mu_\lambda^{\vee_{\mathfrak{g}}}\big|_{\mathfrak{t}}$, and it is given in the following lemma.

Lemma 4.1.7. *Let $\lambda \in \mathfrak{t}_0^*$, then the volume of a coadjoint orbit $\mathcal{O}_\lambda$, and the Fourier transform of the G-invariant measure μ_λ restricted to $\mathfrak{t}$ are given by*

$$\mu_\lambda(\mathcal{O}_\lambda) = \prod_{\alpha\in\Phi^+} \frac{(\lambda,\alpha)}{(\delta,\alpha)} \quad \textit{and} \quad \mu_\lambda^{\vee\mathfrak{g}}|_{\mathfrak{t}}(H) = \frac{\sum_{w\in W} sgn(w) e^{iw(\lambda)(H)}}{\prod_{\alpha\in\Phi^+} i\alpha(H)}, \quad H \in \mathfrak{t}. \tag{4.1.9}$$

The measure μ_λ is called a ***Liouville measure*** *of $\mathcal{O}_\lambda$.*

Remark. *We can see the second formula from the continuous version of the Kirillov character formula in Corollary 3.4.2.1 at the end of previous chapter. Notice that,*

$$\mu_{\lambda+\delta}(\mathcal{O}_{\lambda+\delta}) = \prod_{\alpha\in\Phi^+} \frac{(\lambda+\delta,\alpha)}{(\delta,\alpha)} = d_\lambda, \tag{4.1.10}$$

is the Weyl dimension formula for integral weight λ, and the normalisation for $\mathcal{O}_\delta$ is always 1, that is, $\mu_\delta(\mathcal{O}_\delta) = \prod_{\alpha\in\Phi^+} \frac{(\delta,\alpha)}{(\delta,\alpha)} = 1$. In this case, the Weyl dimension formula has been extended to a continuous version, as a normalisation for the Liouville measure of $\mathcal{O}_\lambda$, for arbitrary $\lambda \in \mathfrak{t}_0^*$.

The formula for $\mu_\lambda^{\vee\mathfrak{g}}\big|_{\mathfrak{t}}$ is trivial when λ lies on a wall of $\mathfrak{t}_+$ (and this will be explained in Lemma 4.1.11). We will study a new normalisation formula for the singular elements in Section 4.3.

Also, if we take the $\mathfrak{t}$-Fourier transform of the right-hand side of (4.1.7), we have

$$\mu_\lambda^p = \left(\sum_{w\in W} \mathrm{sgn}(w) e^{iw\cdot\lambda}\right)^{\widehat{\mathfrak{t}}} * \left(\prod_{\alpha\in\Phi^+} \frac{1}{i\alpha}\right)^{\widehat{\mathfrak{t}}}. \tag{4.1.11}$$

Intuitively, the Fourier transform of $e^{i\beta(\cdot)}$ is the unit point mass e_β, and the Fourier transform of $\frac{1}{i\beta(\cdot)}$ is the arc-length measure F_β on the ray of the vector β. Therefore, we should express

$$\mu_\lambda^p \sim \sum_{w\in W} \mathrm{sgn}(w) e_{w\cdot\lambda} * \prod_{\alpha\in\Phi^+} {*F_\alpha}. \tag{4.1.12}$$

In the following part, we prove that the measure in (4.1.12) actually exists and is the formula for μ_λ^p.

We start the proof by introducing a few lemmas with regard to the properties of **spreads** (introduced by N.Wildberger [18]), and a construction of the 'basic tent' of $\mathfrak{t}^*$.

Lemma 4.1.8. *Define $S \subset \Phi$ to be a **spread** of the root system Φ, if for all $\alpha \in \Phi^+$, S contains exactly one of $\{\alpha, -\alpha\}$. Let $\mathcal{S}$ denote the set of spreads. If $S \in \mathcal{S}$ is the set of roots with respect to some Weyl chamber (i.e., If S contains Δ and all other positive roots, then S is a spread with respect to the positive Weyl chamber), then S is said to be **pure**. Let $\mathcal{S}_p \subset \mathcal{S}$ denote the subset of pure spreads. Then $|\mathcal{S}_p| = |W|$. In other words, the Weyl group W acts on $\mathcal{S}$ and acts transitively on $\mathcal{S}_p$. Also, define $sgn(S) = (-1)^r$ where $r = |\{\alpha \in \Phi^+ : -\alpha \in S\}|$, then for all $S \in \mathcal{S}, w \in W$,*

$$sgn(wS) = sgn(w)sgn(S). \tag{4.1.13}$$

Proof. Since each $w \in W$ preserves Φ, so they also preserve the hyperplanes that are perpendicular to the roots, hence the action of W on $\mathcal{S}_p$ is a single W-orbit.

By ([30], Exercise 10.6), $\mathrm{sgn}(w) = (-1)^{n(w)}$, for $w \in W$, where $n(w)$ is the number of positive roots $\alpha \in \Phi^+$ such that $w \cdot \alpha$ is a negative root.

Let $w \in W$, and k be the number of positive roots in S which turn negative in $w \cdot S$, and l be the number of negative roots in S which turn positive in $w \cdot S$, and because S contains exactly one of $\{\alpha, -\alpha\}$, then $n(w) = k + l$, and

$$\mathrm{sgn}(wS) = (-1)^{r+k-l} = (-1)^{r+n(w)-2l} = (-1)^{r+n(w)} = \mathrm{sgn}(w)\mathrm{sgn}(S). \tag{4.1.14}$$

□

Example 4.1.8.1. Let $\mathfrak{g} = \mathfrak{su}(3)$, $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{g}$. Let Φ be the chosen root system of $\mathfrak{t}$. The set of positive roots $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. The pure spread $\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ corresponds to the fundamental Weyl chamber of $\mathfrak{t}$. Recall the Weyl group W in Example 2.2.18.1, and the rest of the pure spreads are:

$$\begin{aligned} &\{-\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}, \quad \{-\alpha_1 - \alpha_2, \alpha_1, -\alpha_2\}, \quad \{-\alpha_2, -\alpha_1, -\alpha_1 - \alpha_2\}, \\ &\{\alpha_1 + \alpha_2, -\alpha_2, \alpha_1\}, \quad \{\alpha_2, -\alpha_1 - \alpha_2, -\alpha_1\}. \end{aligned} \tag{4.1.15}$$

Definition 4.1.9. Let $\mathfrak{t} \cong \mathbb{R}^d$ be a Cartan subalgebra , and $\mathfrak{t}^* \cong \mathbb{R}^d$ be the dual of $\mathfrak{t}$. Let $\alpha \in \Phi$, and define the map $\phi : \mathbb{R} \to \mathfrak{t}^* \cong \mathbb{R}^d, t \mapsto t\alpha \in \mathbb{R}^d$. Let $\mathcal{L}^d$ be the Lebesgue measure in $\mathbb{R}^d$, and we can define an **arc-length measure**

$$L(A) = \mathcal{L}^1(\phi^{-1}(A)), \tag{4.1.16}$$

for any Borel set $A \subset \mathfrak{t}^*$. Hence, L measures how much of the segment of α (in terms of t) lands in the set A, and assigns that as the measure of A. This measure ν is singular with respect to $\mathcal{L}^d$. We also define F_α to be the arc-length measure along the ray of a root vector $\alpha \in \Phi$ with a density function of 1. F_α is also a distribution that satisfies

$$F_\alpha(\varphi) = \int_0^\infty \varphi(t\alpha)\, dt, \tag{4.1.17}$$

where $\varphi \in \mathcal{S}(\mathfrak{t}^*)$ is a Schwartz function. Since F_α is not locally integrable, its Fourier transform is interpreted in the sense of distributions. The Fourier transform of F_α is given by

$$(F_\alpha)^{\wedge_{\mathfrak{t}^*}}(H) = \text{p.v.}\left(\frac{1}{i\alpha(H)}\right), \quad H \in \mathfrak{t}, \tag{4.1.18}$$

which is the principal value distribution of $1/i\alpha(H)$.

Lemma 4.1.10. *Let α be a root in Φ. Let $L_{\pm\frac{1}{2}\alpha}$ be the arc-length measure on the line segment $[-\frac{1}{2}, \frac{1}{2}] \cdot \alpha$. We define a measure T_δ by*

$$T_\delta = \prod_{\alpha \in \Phi^+} {*L_{\pm\frac{1}{2}\alpha}}. \tag{4.1.19}$$

We call the measure T_δ, the ***basic tent*** *of $\mathfrak{t}^*$. This measure is a piecewise polynomial that is absolute continuous with respect to the Lebesgue measure on $\mathfrak{t}^*$, supported in the convex hull of $W \cdot \delta$, δ is the half sum of all positive roots. Let L_α be the arc-length measure on the line segment $[0, 1] \cdot \alpha$ in $\mathfrak{t}^*$. For any spread $S = \{\alpha_1, ..., \alpha_l\}$, let $\delta(S) = \sum_{\alpha \in S} \frac{1}{2}\alpha$, then*

$$\prod_{\alpha \in S} {*L_\alpha} = e_{\delta(S)} * T_\delta, \tag{4.1.20}$$

which is the $\delta(S)$-shift of the basic tent.

Proof. By ([4], Lemma 5.3), the Minkowski sum of two compact convex sets is also a compact convex set, and by the Krein-Milman theorem, a compact and convex set is the closed convex hull of its extreme points. In particular, such a set has extreme points.

We claim $W \cdot \delta$ is the extremal set of the support D of T_δ. To see this, notice that D is a compact convex polytope, and the extreme points of D coincide with its vertices, denoted by K, and $K \subseteq \sum_{\alpha\in\Phi^+}\frac{1}{2}\{\alpha,-\alpha\}$, which is the set of $\delta(S)$, of all $S \in \mathcal{S}$. Also, let $N_S = \{\alpha \in \Phi^+ : -\alpha \in S\}$, and since every $\delta(S) = \delta - \sum_{N_S}\alpha$. Then every $\delta(S)$ is strictly smaller than δ, and because W is a group of isometries, hence $W \cdot \delta = \delta(\mathcal{S}_p)$ is the extremal set of D.

We then show $\prod_{\alpha\in S} *L_\alpha = e_{\delta(S)} * T_\delta$. Note that the support of $e_{\delta(S)} * T_\delta$ is $D + \delta(S)$, and

$$\begin{aligned} D + \delta(S) &= \sum_{\alpha\in\Phi^+}\left[-\frac{1}{2},\frac{1}{2}\right]\cdot\alpha + \sum_{\alpha'\in S}\frac{1}{2}\alpha' \\ &= \sum_{\alpha'\in S}[0,1]\cdot\alpha' \\ &= \operatorname{supp}\left(\prod_{\alpha'\in S} *L_\alpha\right), \end{aligned} \tag{4.1.21}$$

where $\sum$ denotes the Minkowski sum. $\square$

Let A_2 denote the root system of $\mathfrak{su}(3)$, and B_2 denote the root system of $\mathfrak{so}(5)$. Here we show the picture of the 'basic tent' T_δ of A_2 (linear sides) in Figure 4.1 and B_2 (quadratic sides) in Figure 4.2.

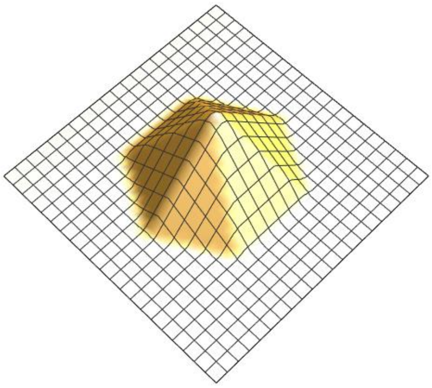

Figure 4.1: T_δ of A_2

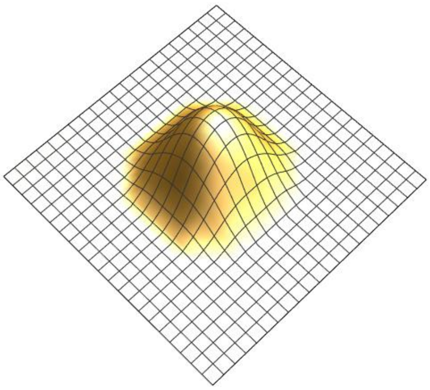

Figure 4.2: T_δ of B_2

The basic tent of A_2 in Figure 4.1 is supported in a hexagon centered at the origin in a real vector space of dimensions 2. The edges on top of the 'tent' coincide with the root vectors of A_2. This measure is a linear piecewise polynomial and its highest density is concentrated at the origin. Interestingly, this measure is also the image of the projection of a 3-dimensional cube (with one edge pointing downward) onto a 2-dimensional space.

The basic tent of B_2 in Figure 4.2 is supported in an octagon centered at the origin in a 2-dimensional real vector space. This measure is a piecewise quadratic polynomial, and its highest density is concentrated at the origin.

These piece-wise polynomials are quite complicated to write down here, but can be found in the programs referenced in Appendix *A*.

Lemma 4.1.11. *If $S \in \mathcal{S}$ is not a pure spread, then $\delta(S)$ lies on a wall of $\mathfrak{t}^*$, and*

$$\sum_{w \in W} sgn(w) e_{w\delta(S)} = 0. \tag{4.1.22}$$

Proof. Let $\{\lambda_j\}_{j=1}^{l}$ be the set of fundamental weights of $\mathfrak{g}$ in $\mathfrak{t}^*$, by ([30], Lemma 13.3A), we have:

$$\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{j=1}^{l} \lambda_j. \tag{4.1.23}$$

So, δ is the 'smallest' strongly dominant weight in Λ^+ (any dominant weights 'smaller' ($\prec$) than δ lie on the walls, a pair of weights $\lambda \prec \lambda'$ if and only if $\lambda' - \lambda$ is an integral sum of positive roots). Since each spread $\delta(S)$ lies on the Weyl-orbit of a spread in $\mathcal{S}$ that lies in Λ^+, and also every $\delta(S)$ can be written as $\delta(S) = \delta - \sum_{N_S} \alpha$, and suppose $\delta(S) \in \Lambda^+$, then $\delta(S)$ is not strongly dominant, so $\delta(S)$ lies on a wall of $\mathfrak{t}^*_+$. In addition, $W \cdot \delta(S)$ lies on some walls of $\mathfrak{t}^*$. To see this, we first show that if λ lies on a wall of $\mathfrak{t}^*$, then

$$\sum_{w \in W} \mathrm{sgn}(w) e_{w\lambda} = 0. \tag{4.1.24}$$

If λ lies on a wall of $\mathfrak{t}^*$, then $\lambda = \sum_{j=1}^{l} m_j \lambda_j$, with some elements of $\{m_{j_1}, ..., m_{j_k}\}$ being zero. Let $\{\alpha_{j_1}, ...\alpha_{j_k}\}$ be the corresponding simple roots, then the simple reflections $\{\sigma_{\alpha_{j_1}}, ..., \sigma_{\alpha_{j_k}}\}$ generate a subgroup of W that fixes λ, denoted by W_λ, (W_λ is the Weyl group of the semisimple Lie subalgebra $\mathfrak{h} = k^\perp \oplus \sum_{\alpha \in \Phi^+_\lambda} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$, where k is the minimal wall that contains λ, and $\Phi^+_\lambda = \{\alpha \in \Phi^+ : (\lambda, \alpha) = 0\}$, by Lemma 2.2.14), and as a Weyl group, W_λ is of even order and has alternating signs. Hence (4.1.24) follows. Also, the Weyl orbit of λ is isomorphic to the quotient W/W_λ. ☐

Lemma 4.1.12. *Let L_α be the arc-length measure defined above in Lemma 4.1.10, $\mathcal{S}$ be the set of spreads and $\mathcal{S}_p \subset \mathcal{S}$ the subset of pure spreads. Define a measure E_1 on $\mathfrak{t}^*$ as*

$$E_1 = \prod_{\alpha \in \Phi^+} *(L_\alpha - L_{-\alpha}) = \sum_{S \in \mathcal{S}} sgn(S) L_{\alpha_{j_1}} * ... * L_{\alpha_{j_k}}, \tag{4.1.25}$$

where $S = \{\alpha_{j_1}, ..., \alpha_{j_k}\}$*. Then,*

$$E_1 = \sum_{S \in \mathcal{S}_p} sgn(S) L_{\alpha_{j_1}} * ... * L_{\alpha_{j_k}} = \sum_{w \in W} sgn(wS_\delta) e_{w\delta} * T_\delta, \tag{4.1.26}$$

where S_δ *is the spread of all positive roots in* Φ^+*.*

Proof. Let $\mathcal{S}^+$ denote the subset of all spreads that for each $S \in \mathcal{S}^+$, $\delta(S)$ is contained in $\mathfrak{t}^*_+$. Let $W_{\delta(S)}$ be the Weyl subgroup of the Weyl orbit of $\delta(S)$, and by the $\delta(S)$-shift of the basic tent, E_1 can be written as

$$\begin{aligned} E_1 &= \sum_{S \in \mathcal{S}^+} \frac{|W_{\delta(S)}|}{|W|} \sum_{w \in W} \mathrm{sgn}(wS)\, e_{w\delta(S)} * T_\delta \\ &= \sum_{S \in \mathcal{S}^+} \frac{|W_{\delta(S)}|}{|W|} \mathrm{sgn}(S) \sum_{w \in W} \mathrm{sgn}(w)\, e_{w\delta(S)} * T_\delta \qquad \text{by (4.1.13)} \\ &= \sum_{w \in W} \mathrm{sgn}(wS_\delta) e_{w\delta} * T_\delta, \end{aligned} \tag{4.1.27}$$

by Lemma 4.1.11. □

Example 4.1.12.1. Let $\mathfrak{g} = \mathfrak{su}(3)$, and $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ be the set of positive roots $\mathfrak{g}$, and $\delta = \alpha_1 + \alpha_2$. Also, let S_δ be the spread with respect to δ, so that $sgn(S_\delta) = (-1)^0 = 1$. Let T_δ be the basic tent in Figure 4.1, then E_1 measure can be written as

$$E_1 = \sum_{w \in W} sgn(w) e_{w\delta} * T_\delta, \tag{4.1.28}$$

where $W = \{e,\, \sigma_{\alpha_1},\, \sigma_{\alpha_2},\, \sigma_{\alpha_2}\sigma_{\alpha_1},\, \sigma_{\alpha_1}\sigma_{\alpha_2},\, \sigma_{\alpha_2}\sigma_{\alpha_1}\sigma_{\alpha_2}\}$ (Example 2.2.18.1), and respectively,

$$sgn(W) = \{+1, -1, -1, +1, +1, -1\}, \tag{4.1.29}$$

and

$$W \cdot \delta = \{\delta, \alpha_2, \alpha_1, -\alpha_2, -\alpha_1, -\delta\}. \tag{4.1.30}$$

Therefore, E_1 is a measure of the summation of alternating sign of the shifts of the basic tent to the points of the Weyl orbit $W \cdot \delta$.

Next, we combine everything developed above to give a formula for μ^p_λ.

Theorem 4.1.13 ([18], Proposition 3.1). *Let $\lambda \in \mathfrak{t}_0^*$. For $n > 0$, define $P_n = \prod_{\alpha\in\Phi^+} *L_{n\alpha} = e_{n\delta} * T_{n\delta}$ and $E_n = \sum_{w\in W} sgn(w)\, wP_n$, where $wP_n = e_{wn\delta} * T_{n\delta}$. Let $P = \lim_{n\to\infty} P_n$ and $E = \lim_{n\to\infty} E_n$ be distributions on the Euclidean space $\mathfrak{t}^*$. Then*

$$\mu_\lambda^p = \frac{1}{|W|}\sum_{w\in W} sgn(w) e_{w\lambda} * E. \tag{4.1.31}$$

Proof. Let $n > 0$, and by (4.1.26),

$$E_n = \prod_{\alpha\in\Phi^+} *(L_{n\alpha} - L_{-n\alpha}) = \sum_{w\in W} \mathrm{sgn}(w) wP_n. \tag{4.1.32}$$

For all $H \in \mathfrak{t}$, the inverse $\mathfrak{t}$-Fourier transform of E_n is

$$\begin{aligned} E_n^{\vee_\mathfrak{t}}(H) &= \int_{\mathfrak{t}^*} \prod_{\alpha\in\Phi^+} *(L_{n\alpha} - L_{-n\alpha})(\beta) e^{i\beta(H)}\, d\beta \\ &= \prod_{\alpha\in\Phi^+} \int_0^1 (e^{in\alpha(sH)} - e^{-in\alpha(sH)})\, ds \\ &= \prod_{\alpha\in\Phi^+} \int_0^1 2\sin n\alpha(sH)\, ds \\ &= \prod_{\alpha\in\Phi^+} 2\,\frac{1-\cos n\alpha(H)}{in\alpha(H)} = \prod_{\alpha\in\Phi^+} 2^2\,\frac{\sin^2 n\alpha(H)/2}{in\alpha(H)}. \end{aligned} \tag{4.1.33}$$

Hence, $E_n^{\vee_\mathfrak{t}} = \prod_{\alpha\in\Phi^+} 2^2 \frac{\sin n\alpha/2}{in\alpha}$ is a distribution in $\mathfrak{t}$. Let $|\Phi| = m$. We claim $2^m \prod_{\alpha\in\Phi^+} \sin^2\frac{\alpha}{2}$ has a constant term equal to $|W|$. To see this, notice that by a variation of (4.1.26), we have

$$2^m \prod_{\alpha\in\Phi^+} \sin^2\frac{\alpha}{2} = \left|\sum_{w\in W} \mathrm{sgn}(w) e^{iw\delta}\right|^2 = \sum_{w\in W}\sum_{w'\in W} \mathrm{sgn}(ww') e^{iw\delta - iw'\delta}, \tag{4.1.34}$$

and by W-invariance, it only has a constant term $|W|$, and all other terms of $E_n^{\vee_\mathfrak{t}}$ can be written as

$$\cos n\alpha_1 \cdots \cos n\alpha_k = \sum \pm \cos n(\pm\alpha_1 \pm \ldots \pm \alpha_k), \tag{4.1.35}$$

by the cosine angle addition formula. Hence, let $f \in L^1(\mathfrak{t})$, and by the *Riemann-Lebesgue lemma*, we have

$$\lim_{n\to\infty} \frac{2^m}{|W|} \int_\mathfrak{t} \prod_{\alpha\in\Phi^+} \sin^2\frac{n\alpha(H)}{2} f(H)\, dH = \int_\mathfrak{t} f(H)\, dH, \tag{4.1.36}$$

and it follows that

$$\begin{aligned}\left\langle \frac{1}{|W|}E^{\vee_{\mathfrak{t}}}, f \right\rangle &= \lim_{n\to\infty} \frac{2^m}{|W|} \int_{\mathfrak{t}} \prod_{\alpha\in\Phi^+} \frac{\sin^2 \frac{n\alpha(H)}{2}}{i\alpha(H)} f(H)\, dH \\ &= \left\langle \prod_{\alpha\in\Phi^+} \frac{1}{i\alpha(H)}, f \right\rangle.\end{aligned} \tag{4.1.37}$$

Hence, $\frac{1}{|W|}E^{\vee_{\mathfrak{t}}}$ is a regularisation of $\prod_{\alpha\in\Phi^+}\frac{1}{i\alpha}$. Combining this observation with (4.1.11) and taking the inverse $\mathfrak{t}$-Fourier transform, we obtain (4.1.31). $\square$

In Figures 4.3 and 4.4, we illustrate the W-invariant measures μ_λ^p. Since E is the limit of E_n as $n \to \infty$, we choose a large n to approximate these projection measures μ_λ^p with respect to some dominant weights of A_2: $\lambda_1 + 3\lambda_2$ and $2\lambda_1 + 3\lambda_2$ (where λ_1, λ_2 are the fundamental weights of A_2).

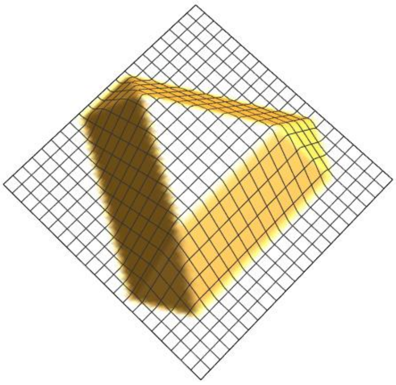

Figure 4.3: $\mu_\lambda^p\,(\lambda_1 + 3\lambda_2)$ of A_2

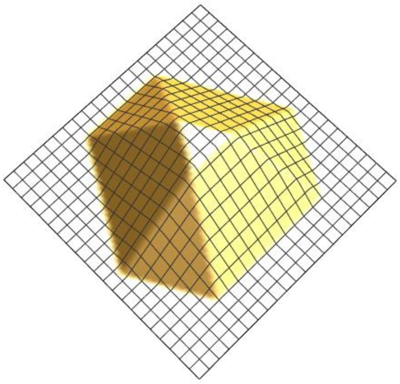

Figure 4.4: $\mu_\lambda^p\,(2\lambda_1 + 3\lambda_2)$ of A_2

As shown in Figures 4.3 and 4.4, these projection measures are also linear piecewise polynomials in the real vector space of dimension 2.

Corollary 4.1.13.1. *Let* $\lambda \in \mathfrak{t}_0^*$, *then*

$$\mu_\lambda^p = \sum_{w\in W} sgn(w) e_{w\lambda} * P, \tag{4.1.38}$$

where $P = \prod_{\alpha\in\Phi^+} F_\alpha$, *and* F_α *is the arc-length measure along the ray of the root vector* α *in* $\mathfrak{t}^*$.

Proof. By the projection measure (4.1.31),

$$\begin{aligned}
\mu_\lambda^p &= \frac{1}{|W|}\sum_{w\in W}\operatorname{sgn}(w)e_{w\lambda} * E \\
&= \frac{1}{|W|}\sum_{w\in W}\sum_{\overline{w}\in W}\operatorname{sgn}(w\overline{w})e_{w\lambda} * \overline{w}P \\
&= \frac{1}{|W|}\sum_{w\in W}\sum_{\overline{w}\in W}\operatorname{sgn}(w\overline{w})\overline{w}\left(e_{\overline{w}^{-1}w\lambda} * P\right) \\
&= \frac{1}{|W|}\sum_{\tilde{w}\in W}\sum_{\overline{w}\in W}\operatorname{sgn}(\tilde{w})\overline{w}\left(e_{\tilde{w}\lambda} * P\right) \\
&= \frac{1}{|W|}\sum_{\overline{w}\in W}\overline{w}\left(\sum_{\tilde{w}\in W}\operatorname{sgn}(\tilde{w})e_{\tilde{w}\lambda} * P\right).
\end{aligned} \tag{4.1.39}$$

We then claim that $T_\lambda = \sum_{w\in W}\operatorname{sgn}(w)e_{w\lambda} * P$ is in fact W-invariant. Let σ_α be the reflection generated by a simple root $\alpha \in \Delta \subset \Phi^+$. By ([30], Lemma 10.2.B), σ_α permutes the positive roots other than α, and $\sigma(\alpha) = -\alpha$, by the definition of reflection. Hence,

$$\begin{aligned}
\sigma_\alpha(T_\lambda) &= \sum_{w\in W}\operatorname{sgn}(w)e_{\sigma_\alpha w\lambda} * \sigma_\alpha P \\
&= \operatorname{sgn}(w)e_{\sigma_\alpha w\lambda} * \left(\prod_{\beta\in\Phi^+,\,\beta\neq\alpha} *F_\beta\right) * F_{-\alpha},
\end{aligned} \tag{4.1.40}$$

and because σ_α is a generator of W, so we have

$$T_\lambda - \sigma_\alpha(T_\lambda) = \left(\prod_{\beta\in\Phi^+,\,\beta\neq\alpha} *F_\beta\right) * \left(\sum_{w\in W}\operatorname{sgn}(w)e_{w\lambda} * F_\alpha + \sum_{w\in W}\operatorname{sgn}(w)e_{w\lambda} * F_{-\alpha}\right), \tag{4.1.41}$$

where $e_{w\lambda} * F_\alpha$ is a arc-length measure of the ray given by $w\lambda + t\alpha$, for $t \in (0, \infty)$. Since σ_α reflects an element in $W \cdot \lambda$ in the hyperplane k_α. So, $W \cdot \lambda$ are symmetrically arranged with respect to k_α, and each pair of symmetrical elements has different signs. Suppose that $w\lambda, w'\lambda$ form a symmetrical pair. Then the summand

$$\operatorname{sgn}(w\lambda)(e_{w\lambda} * F_\alpha + e_{w\lambda} * F_{-\alpha}) \qquad \text{and} \qquad \operatorname{sgn}(w'\lambda)(e_{w'\lambda} * F_\alpha + e_{w'\lambda} * F_{-\alpha}), \tag{4.1.42}$$

are both the arc-length measures of the line that is perpendicular to k_α and passing through $w\lambda, w'\lambda$, with opposite signs. Therefore, the sum in the second bracket of (4.1.41) is 0, and $T_\lambda = \sigma_\alpha(T_\lambda)$ for all $\alpha \in \Delta$. Because $w \in W$ is made of the products of simple reflections, it follows that T_λ is W-invariant. ☐

4.2 Sum of Coadjoint Orbits

We study an explicit formula for the convolution of coadjoint orbits.

Lemma 4.2.1. *Let* $R(\lambda)(H) = \sum_{w \in W} sgn(w) e^{iw\lambda(H)}$*, and* $\overline{R(\lambda)(H)}$ *be the complex conjugate of* $R(\lambda)(H)$*. Also, let* $f \in L^1(\mathfrak{t}^*_+)$*. Then, for* $\lambda \in \mathfrak{t}^*_0$*, the integral*

$$\int_{\mathfrak{t}^*_+} \int_{\mathfrak{t}} f(\lambda') R(\lambda')(H) \overline{R(\lambda)(H)} \, dH \, d\lambda' = |W| f(\lambda). \tag{4.2.1}$$

Proof. Let $f \in L^1(\mathfrak{t}^*_+)$. Then by the Fourier inversion theorem,

$$\begin{aligned} f(\lambda) &= \int_{\mathfrak{t}^*_+} \int_{\mathfrak{t}} f(\lambda') e^{i(\lambda - \lambda')(H)} \, dH \, d\lambda' \\ &= \int_{\mathfrak{t}^*_+} f(\lambda') e_\lambda(\lambda') \, d\lambda', \qquad \text{for } \lambda \in \mathfrak{t}^*_0, \end{aligned} \tag{4.2.2}$$

where e_λ is the unit point mass at λ. In (4.2.1), only the summand

$$\int_{\mathfrak{t}^*_+} \int_{\mathfrak{t}} f(\lambda') \sum_{\substack{w \in W \\ w' = w}} \operatorname{sgn}(ww') e^{i(w\lambda - w'\lambda')(H)} \, dH \, d\lambda', \tag{4.2.3}$$

is nonzero, and it is equal to

$$\int_{\mathfrak{t}^*_+} \int_{\mathfrak{t}} f(\lambda') \sum_{w \in W} e^{iw(\lambda - \lambda')(H)} \, dH \, d\lambda'. \tag{4.2.4}$$

By the W-invariance of $\mathfrak{t}$, the above is equal to

$$\int_{\mathfrak{t}^*_+} \int_{\mathfrak{t}} f(\lambda') \sum_{w \in W} e^{i(\lambda - \lambda')(H)} \, dH \, d\lambda' = |W| f(\lambda). \tag{4.2.5}$$

The other summands are zero because when $\lambda \in \mathfrak{t}^*_0$ lies off the wall, the elements $w\lambda$ and $w'\lambda'$ for $w \neq w'$ do not lie in the same Weyl chamber. So, we have the result. ☐

Lemma 4.2.2. *For* $H \in \mathfrak{t}$*, and let* $w \in W$*, then*

$$\prod_{\alpha \in \Phi^+} \frac{1}{w^{-1}\alpha(H)} = sgn(w) \prod_{\alpha \in \Phi^+} \frac{1}{\alpha(H)}, \tag{4.2.6}$$

and

$$R(w^{-1}\lambda)(H) = sgn(w)R(\lambda)(H). \tag{4.2.7}$$

Proof. By ([30], Theorem 10.3), let $n(w)$ be the number of positive roots such that $w\alpha$, for all $\alpha \in \Phi^+$ is a negative root, then $sgn(w) = (-1)^{n(w)}$, and because $sgn(w) = sgn(w^{-1})$, so the lemma follows. Hence, we say $R(\lambda)(H)$ and $\prod_{\alpha\in\Phi^+} \alpha(H)$ are *W-anti-invariant.* □

Lemma 4.2.3. *Let $\mu_\lambda, \mu_{\lambda'}$ be the G-invariant Liouville measures on the coadjoint orbits passing through $\lambda, \lambda' \in \mathfrak{t}_0^*$ respectively, p be the orthogonal projection defined in Lemma 4.1.5, then*

$$(\mu_\lambda * \mu_{\lambda'})^p = \mu_\lambda^p * \mu_{\lambda'}^p. \tag{4.2.8}$$

Remark. *This relation is true because μ_λ and $\mu_{\lambda'}$ are bounded measures, and the projection $p : \mathfrak{g}^* \to \mathfrak{t}^*$ is a linear map.*

Theorem 4.2.4 ([18], Theorem 2.5)**.** *Let $\lambda, \gamma \in \mathfrak{t}_+^*$. Then $(\mathcal{O}_\lambda + \mathcal{O}_\gamma)$ is G-invariant and is a union of coadjoint orbits. Furthermore, $(\mathcal{O}_\lambda + \mathcal{O}_\gamma) \cap \mathfrak{t}_+^*$ is a convex polytope.*

Remark. *This result was derived from the ideas of sympletic geometry and image of moment map.*

Theorem 4.2.5 ([18], Theorem 3.4)**.** *Let $\lambda, \lambda' \in \mathfrak{t}_0^*$. Then the convolution formula is given by*

$$\mu_\lambda * \mu_{\lambda'} = \int_{\mathfrak{t}_+^*} \phi(\lambda, \lambda', \lambda'')\, \mu_{\lambda''}\, d\lambda'', \tag{4.2.9}$$

where

$$\phi(\lambda, \lambda', \lambda'') = \sum_{w\in W} sgn(w) e_{w\lambda} * \mu_{\lambda'}^p(\lambda'') \qquad \textit{for all } \lambda'' \in \mathfrak{t}_+^*. \tag{4.2.10}$$

Proof. Let I be the support of $\mathcal{O}_\lambda + \mathcal{O}_{\lambda'}$ in $\mathfrak{g}^*$. By the disintegration theorem (Theorem 4.1.1),

$$\mu_\lambda * \mu_{\lambda'}(I) = \int_{\mathfrak{t}_+^*} \int_{\mathcal{O}_{\lambda''}} \phi(\lambda, \lambda', \lambda'') d\mu_{\lambda''}(\beta)\, d\lambda'', \tag{4.2.11}$$

where $\phi(\lambda, \lambda', \lambda'')$ is a density function of $I \cap \mathfrak{t}^*_+$, which depends on the choices of λ, λ'. By the projection of convolution in (4.2.8),

$$\mu^p_\lambda * \mu^p_{\lambda'} = \int_{\mathfrak{t}^*_+} \phi(\lambda, \lambda', \lambda'')\, \mu^p_{\lambda''}\, d\lambda'', \tag{4.2.12}$$

is a new projection measure on $\mathfrak{t}^*$. If we take the $\mathfrak{t}$-Fourier transform of both sides using the normalisation formula $(\mu^p_\lambda)^{\vee_\mathfrak{t}}$ in (4.1.11), we obtain a function on $\mathfrak{t}$,

$$\int_{\mathfrak{t}^*_+} \phi(\lambda, \lambda', \lambda'') R(\lambda'')(H)\, d\lambda'' = \prod_{\alpha \in \Phi^+} \frac{1}{\alpha(H)} R(\lambda)(H) R(\lambda')(H). \tag{4.2.13}$$

Letting $\zeta \in \mathfrak{t}^*_+$, we multiply both sides of the equation above by $\overline{R(\zeta)(H)}$ and integrate with respect to $\mathfrak{t}$. By Lemma 4.2.1, we obtain

$$\begin{aligned}
|W|\phi(\lambda, \lambda', \zeta) &= \int_\mathfrak{t} \prod_{\alpha \in \Phi^+} \frac{1}{\alpha(H)} R(\lambda)(H) R(\lambda')(H) \overline{R(\zeta)(H)}\, dH \\
&= \sum_{w \in W} \operatorname{sgn}(w) \int_\mathfrak{t} \prod_{\alpha \in \Phi^+} \frac{1}{\alpha(H)} R(\lambda)(H) R(\lambda')(H) e^{-iw\zeta(H)}\, dH \\
&= \sum_{w \in W} \operatorname{sgn}(w) \int_\mathfrak{t} \prod_{\alpha \in \Phi^+} \frac{1}{\alpha(wH)} R(\lambda)(wH) R(\lambda')(wH) e^{-i\zeta(H)}\, dH \\
&= \sum_{w \in W} \operatorname{sgn}(w) \int_\mathfrak{t} \prod_{\alpha \in \Phi^+} \frac{1}{w^{-1}\alpha(H)} R(w^{-1}\lambda)(H) R(w^{-1}\lambda')(H) e^{-i\zeta(H)}\, dH \\
&= |W| \int_\mathfrak{t} \prod_{\alpha \in \Phi^+} \frac{1}{\alpha(H)} R(\lambda)(H) R(\lambda')(H) e^{-i\zeta(H)}\, dH,
\end{aligned} \tag{4.2.14}$$

by Lemma 4.2.2. Therefore, if we solve the right-hand side of the last equality above, we obtain:

$$\begin{aligned}
\phi(\lambda, \lambda', \lambda'') &= \left(R(\lambda) \frac{R(\lambda')}{\prod_{\alpha \in \Phi^+} \alpha} \right)^{\wedge_\mathfrak{t}} (\lambda'') \\
&= \sum_{w \in W} \operatorname{sgn}(w) e_{w\lambda} * \mu^p_{\lambda'}(\lambda''),
\end{aligned} \tag{4.2.15}$$

by the projection measure in (4.1.38). The λ, λ' are interchangeable, and results are identical. □

Examples of convolutions of A_2, are given in Figures 4.5 and 4.6. Notice that

1. When $\lambda' \prec \lambda$, then the convolution formula is the sum of the shifts of the projection measure $\mu^p_{\lambda'}$ to the points of the Weyl orbit $W \cdot \lambda$.

2. Let $D_{\lambda'}$ be the support of $\mu^p_{\lambda'}$. When the size of $D_{\lambda'}$ is fairly small and λ is away from the origin, $w \cdot \lambda + D_{\lambda'}$ can be fully contained in a Weyl chamber (e.g., Figure.4.6), so its density function is

$$\phi(\lambda, \lambda', \lambda'') = e_\lambda * \mu^p_{\lambda'}(\lambda''), \qquad \lambda'' \in \mathfrak{t}^*_+. \tag{4.2.16}$$

Otherwise, $w \cdot \lambda + D_{\lambda'}$ can intersect the walls of the Weyl chambers and the support of $\phi(\lambda, \lambda', \lambda'')$ becomes complicated (e.g., Figure 4.5).

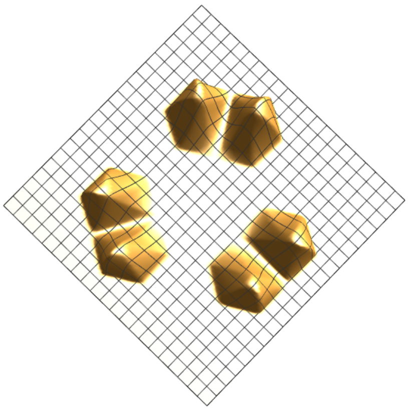

Figure 4.5: $\mu_\delta * \mu_\lambda|_{\mathfrak{t}^*} (\lambda_1 + 3\lambda_2)$ of A_2

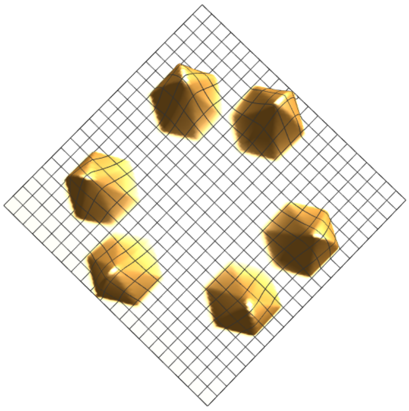

Figure 4.6: $\mu_\delta * \mu_\lambda|_{\mathfrak{t}^*} (2\lambda_1 + 3\lambda_2)$ of A_2

Corollary 4.2.5.1. *Let* $\lambda_1, ...\lambda_k \in \mathfrak{t}^*_0$. *Then the convolution formula for coadjoint orbits is given by*

$$\mu_{\lambda_1} * ... * \mu_{\lambda_k} = \int_{\mathfrak{t}^*_+} \sum_{w \in W} sgn(w) e_{w\lambda_1} * \mu^p_{\lambda_2} * ... * \mu^p_{\lambda_k}(\lambda'') \mu_{\lambda''} \, d\lambda'', \tag{4.2.17}$$

and the convolution formula for the projection measures is given by

$$\mu^p_{\lambda_1} * ... * \mu^p_{\lambda_k} = \int_{\mathfrak{t}^*_+} \sum_{w \in W} sgn(w) e_{w\lambda_1} * \mu^p_{\lambda_2} * ... * \mu^p_{\lambda_k}(\lambda'') \mu^p_{\lambda''} \, d\lambda''. \tag{4.2.18}$$

Proof. Let $\phi(\lambda_1, ..., \lambda_k, \lambda'')$ be the density function of support $I \cap \mathfrak{t}^*_+$, $k =$

$|\{\lambda_1, ..., \lambda_k\}|$, then

$$\begin{aligned}
&|W|\phi(\lambda_1, ..., \lambda_k, \zeta) \\
&\quad= \sum_{w\in W} \operatorname{sgn}(w) \int_{\mathfrak{t}} \left(\prod_{\alpha\in\Phi^+} \frac{1}{w^{-1}\alpha(H)} \right)^{k-1} \left(\prod_{j=1}^{k} R(w^{-1}\lambda_j)(H) \right) e^{-i\zeta(H)}\, dH \\
&\quad= \sum_{w\in W} \operatorname{sgn}(w)^{2k} \int_{\mathfrak{t}} \left(\prod_{\alpha\in\Phi^+} \frac{1}{\alpha(H)} \right)^{k-1} \left(\prod_{j=1}^{k} R(\lambda_j)(H) \right) e^{-i\zeta(H)}\, dH.
\end{aligned} \tag{4.2.19}$$

Hence, by Lemma 4.2.2, the result follows. □

4.3 Singular Coadjoint Orbits

When λ lies on a wall of one of the Weyl chambers in $\mathfrak{t}^*$, the normalisation in Lemma 4.1.7 does not extend to the singular coadjoint orbit $\mathcal{O}_\lambda$. We study how to obtain the Liouville measure of a singular coadjoint orbit by approximating it from a regular coadjoint orbit.

Lemma 4.3.1. *Let $f \in L^1(\mathfrak{t}^*)$, then*

$$\left[\left(\prod_{\alpha\in\Phi^+} \partial_\alpha \right) f \right]^{\wedge_{\mathfrak{t}^*}} (H) = \prod_{\alpha\in\Phi^+} i\alpha(H) f^{\wedge_{\mathfrak{t}^*}}(H), \qquad \text{for all } H \in \mathfrak{t}, \tag{4.3.1}$$

where ∂_α is the directional derivative of a positive root α.

Proof. This follows from the basic properties of the Fourier transform of linear functionals. □

Lemma 4.3.2. *Let $\lambda \in \mathfrak{t}^*$ and $f \in C^\infty(\mathfrak{t}^*)$, and define $g_\lambda(H) = \frac{R(\lambda)(H)}{\prod_{\alpha\in\Phi^+}(\lambda,\alpha)}$, for all $H \in \mathfrak{t}$. Then the following limit holds*

$$\lim_{\lambda\to 0} g_\lambda^{\vee_{\mathfrak{t}^*}} * f = \lim_{\lambda\to 0} \frac{\sum_{w\in W} sgn(w) e_{w\lambda}}{\prod_{\alpha\in\Phi^+}(\lambda,\alpha)} * f = \frac{\left(\prod_{\alpha\in\Phi^+} \partial_\alpha\right) f}{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}, \tag{4.3.2}$$

in the sense of distributions.

Proof. Let ν_λ be the probability measure of $\mathcal{O}_\lambda$ passing through $\lambda \in \mathfrak{t}^*$, which lies in the interior $\mathfrak{t}_0^*$. Then by the normalisation of μ_λ,

$$\nu_\lambda = \frac{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}{\prod_{\alpha\in\Phi^+}(\lambda,\alpha)}\mu_\lambda. \tag{4.3.3}$$

Hence, if we take the $\mathfrak{g}$-Fourier transform, we obtain

$$\nu_\lambda^{\vee_\mathfrak{g}}(H) = \frac{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}{\prod_{\alpha\in\Phi^+}(\lambda,\alpha)}\frac{R(\lambda)(H)}{\prod_{\alpha\in\Phi^+} i\alpha(H)}, \qquad \text{for all } H \in \mathfrak{t}, \tag{4.3.4}$$

which leads to

$$g_\lambda(H) = \frac{\prod_{\alpha\in\Phi^+} i\alpha(H)}{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}\nu_\lambda^{\vee_\mathfrak{g}}(H), \qquad \text{for all } H \in \mathfrak{t}, \tag{4.3.5}$$

and by Lemma 4.3.1, we have

$$\lim_{\lambda\to 0} g_\lambda^{\vee_{\mathfrak{t}^*}} = \lim_{\lambda\to 0}\frac{\left(\prod_{\alpha\in\Phi^+}\partial_\alpha\right)}{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}\,\nu_\lambda^p, \tag{4.3.6}$$

where ν_λ^p is the probability projection measure of $\mathcal{O}_\lambda$. Therefore,

$$\lim_{\lambda\to 0}\frac{\left(\prod_{\alpha\in\Phi^+}\partial_\alpha\right)}{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}\,\nu_\lambda^p * f = \frac{\left(\prod_{\alpha\in\Phi^+}\partial_\alpha\right) f}{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}, \tag{4.3.7}$$

given that $\lim_{\lambda\to 0}\nu_\lambda^p$ is the Dirac delta function at the origin. ☐

Lemma 4.3.3. *Suppose that $\lambda \in \mathfrak{t}_+^*\backslash\mathfrak{t}_0^*$ lies on a wall of the positive Weyl chamber, and let $W_\lambda = \{w \in W \,|\, w\cdot\lambda = \lambda\}$, $\Phi_\lambda^+ = \{\alpha \in \Phi^+ \,|\, (\lambda,\alpha) = 0\}$, and $\delta_\lambda = \frac{1}{2}\sum_{\alpha\in\Phi_\lambda^+}\alpha$. Then for any $\eta \in \mathfrak{t}_0^*$,*

$$\lim_{\eta\to 0}\frac{\sum_{w\in W_\lambda} sgn(w)e_{w\eta}}{\prod_{\alpha\in\Phi_\lambda^+}(\eta,\alpha)} * f = \frac{\left(\prod_{\alpha\in\Phi_\lambda^+}\partial_\alpha\right) f}{\prod_{\alpha\in\Phi_\lambda^+}(\delta_\lambda,\alpha)}. \tag{4.3.8}$$

Proof. Denote by k the minimal wall in $\mathfrak{t}^*$ that contains λ, and decompose $\eta = \eta_k + \eta_{k^\perp}$, where η_k is the part lying on k. Let $\mathfrak{h}_k$ be the subalgebra of $\mathfrak{g}$ given by

$$\mathfrak{h}_k = k^\perp \oplus \sum_{\alpha\in\Phi_\lambda^+}\mathfrak{g}_\alpha, \tag{4.3.9}$$

where $k^\perp$ is the orthogonal complement to k, and it is also the Cartan subalgebra of $\mathfrak{h}_k$, which commutes with k. Note that W_λ is the Weyl group of $k^\perp$, by Lemma

2.2.14. Thus, we have:

$$\begin{aligned}\frac{\sum_{w\in W_\lambda}\mathrm{sgn}(w)e^{iw\eta(H)}}{\prod_{\alpha\in\Phi_\lambda^+}(\eta,\alpha)} &= e^{i\eta_k(H)}\frac{\sum_{w\in W_\lambda}\mathrm{sgn}(w)e^{iw\eta_{k^\perp}(H)}}{\prod_{\alpha\in\Phi_\lambda^+}(\eta_{k^\perp},\alpha)}\\ &= e^{i\eta_k(H)}\frac{\prod_{\alpha\in\Phi_\lambda^+}i\alpha(H)}{\prod_{\alpha\in\Phi_\lambda^+}(\delta_\lambda,\alpha)}\,\frac{\prod_{\alpha\in\Phi_\lambda^+}(\delta_\lambda,\alpha)}{\prod_{\alpha\in\Phi_\lambda^+}(\eta_{k^\perp},\alpha)}\,\frac{\sum_{w\in W_\lambda}\mathrm{sgn}(w)e^{iw\eta_{k^\perp}(H)}}{\prod_{\alpha\in\Phi_\lambda^+}i\alpha(H)}\\ &= e^{i\eta_k(H)}\frac{\prod_{\alpha\in\Phi_\lambda^+}i\alpha(H)}{\prod_{\alpha\in\Phi_\lambda^+}(\delta_\lambda,\alpha)}\,\nu_{\eta_{k^\perp}}^{\vee\mathfrak{g}}(H),\end{aligned} \tag{4.3.10}$$

for all $H \in \mathfrak{t}$. Hence, $\nu_{\eta_{k^\perp}}$ is the probability measure of the coadjoint orbit of the subgroup H_k of G associated with $\mathfrak{h}_k$, passing through $\eta_{k^\perp} \in k^\perp$. Therefore,

$$\lim_{\eta\to 0} e_{\eta_k} * \frac{\left(\prod_{\alpha\in\Phi_\lambda^+}\partial_\alpha\right)}{\prod_{\alpha\in\Phi_\lambda^+}(\delta_\lambda,\alpha)}\,\nu_{\eta_{k^\perp}}^p * f = \frac{\left(\prod_{\alpha\in\Phi_\lambda^+}\partial_\alpha\right)f}{\prod_{\alpha\in\Phi_\lambda^+}(\delta_\lambda,\alpha)}, \tag{4.3.11}$$

by Lemma 4.3.2. ☐

Now we prove the formula for the projection measure of $\mathcal{O}_\lambda$, with λ lying on a wall of $\mathfrak{t}_+^*$.

Theorem 4.3.4 ([18], Proposition 4.3)**.** *Let $\lambda \in \mathfrak{t}_+^*\backslash\mathfrak{t}_0^*$ lies on a wall of the positive Weyl chamber, and let $W_\lambda = \{w \in W \mid w\cdot\lambda = \lambda\}$, $\Phi_\lambda^+ = \{\alpha\in\Phi^+ \mid (\lambda,\alpha) = 0\}$, and $\delta_\lambda = \frac{1}{2}\sum_{\alpha\in\Phi_\lambda^+}\alpha$. Define the quotient group $W/W_\lambda = \{w_j\}_{j=1}^m$ (where W/W_λ is in a natural bijection with the Weyl orbit of λ) so that each $w\in W$ can be written as $w = w_j u$ for a $w_j \in W/W_\lambda$ and a $u\in W_\lambda$. Then,*

$$\nu_\lambda^p = C_\lambda \sum_{j=1}^m sgn(w_j)e_{w_j\lambda} * \left(\prod_{\alpha\in\Phi_\lambda^+}\partial_{w_j\alpha}\right)P, \tag{4.3.12}$$

where

$$C_\lambda = \frac{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}{\prod_{\alpha\in\Phi^+\smallsetminus\Phi_\lambda^+}(\lambda,\alpha)\prod_{\alpha\in\Phi_\lambda^+}(\delta_\lambda,\alpha)},\quad \left(\prod_{\alpha\in\Phi_\lambda^+}\partial_{w_j\alpha}\right)P = sgn(\tau_{w_j})\prod_{\alpha\in\Phi^+\backslash\Phi_{w_j\lambda}^+} * F_\alpha, \tag{4.3.13}$$

where $sgn(\tau_{w_j}) = (-1)^t$, and t is the number $\left|\left\{\beta\in\Phi_\lambda^+ \mid w_j\beta\notin\Phi^+\right\}\right|$. Hence, the *normalisation* $\nu_\lambda^{\vee\mathfrak{g}}|_{\mathfrak{t}}$ *can be written as*

$$\nu_\lambda^{\vee\mathfrak{g}}|_{\mathfrak{t}}(H) = C_\lambda \sum_{j=1}^{m} \frac{sgn(w_j)e^{iw_j\lambda(H)}}{sgn(\tau_{w_j})\prod_{\alpha\in\Phi^+\backslash\Phi^+_{w_j\lambda}} i\alpha(H)}, \qquad \textit{for all } H \in \mathfrak{t}. \tag{4.3.14}$$

Proof. Let $w_1, ..., w_q \in W$ be the unique elements such that for every $w \in W$, $w = w_j u$, for some w_j and $u \in W_\lambda$. Suppose that λ lies on a wall. By continuity, we have

$$\begin{aligned}\lim_{\lambda'\to\lambda} \nu^p_{\lambda'} &= \lim_{\lambda'\to\lambda} \frac{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}{\prod_{\alpha\in\Phi^+}(\lambda',\alpha)} \sum_{w\in W} \mathrm{sgn}(w)e_{w\lambda'} * P \\ &= \frac{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}{\prod_{\alpha\in\Phi^+\backslash\Phi^+_\lambda}(\lambda,\alpha)} \sum_{j=1}^{m} \mathrm{sgn}(w_j) \lim_{\lambda'\to\lambda} \sum_{u\in W_\lambda} \mathrm{sgn}(u) \frac{\sum_{u\in W_\lambda}\mathrm{sgn}(u)e_{w_j u\lambda'} * P}{\prod_{\alpha\in\Phi^+_\lambda}(\lambda',\alpha)}.\end{aligned} \tag{4.3.15}$$

Now, we let $\lambda' = \lambda + \eta$, for $\eta \in \mathfrak{t}_0^*$. The above becomes

$$\frac{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}{\prod_{\alpha\in\Phi^+\backslash\Phi^+_\lambda}(\lambda,\alpha)} \sum_{j=1}^{m} \mathrm{sgn}(w_j)e_{w_j\lambda} * \lim_{\eta\to 0} \sum_{u\in W_\lambda} \mathrm{sgn}(u) \frac{\sum_{u\in W_\lambda}\mathrm{sgn}(u)e_{w_j u\eta} * P}{\prod_{\alpha\in\Phi^+_\lambda}(\eta,\alpha)}, \tag{4.3.16}$$

and by Lemma 4.3.3, and the W-invariance of ν^p_η, the formula for ν^p_λ follows. Also, in the sense of distributions, for $\alpha \in \Phi^+$

$$\partial_\alpha F_\alpha = e_0, \qquad \text{and} \qquad \partial_{-\alpha} F_\alpha = -e_0\,. \tag{4.3.17}$$

Define $\Phi^+_{w_j\lambda} = \{\alpha \in \Phi^+ \,|\, (w_j\lambda, \alpha) = 0\}$, these lead to the second identity in (4.3.13). □

Next, we derive the convolution formula for singular coadjoint orbits.

Theorem 4.3.5 ([18], Proposition 4.3)**.** *Let* $\eta \in \mathfrak{t}_0^*$ *and* λ *lie on a wall. Then*

$$\nu_\eta * \nu_\lambda = \frac{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}{\prod_{\alpha\in\Phi^+}(\eta,\alpha)} \int_{\mathfrak{t}^*_+} \sum_{w\in W} sgn(w)e_{w\eta} * \nu^p_\lambda(\lambda'')\mu_{\lambda''}\, d\lambda''\,. \tag{4.3.18}$$

Proof. Let $\lambda' \in \mathfrak{t}_0^*$,

$$\begin{aligned}\nu_\eta * \nu_\lambda &= \lim_{\lambda'\to\lambda} \frac{\prod_{\alpha\in\Phi^+}(\delta,\alpha)^2}{\prod_{\alpha\in\Phi^+}(\eta,\alpha)(\lambda',\alpha)}\mu_\eta * \mu_{\lambda'} \\ &= \lim_{\lambda'\to\lambda} \frac{\prod_{\alpha\in\Phi^+}(\delta,\alpha)^2}{\prod_{\alpha\in\Phi^+}(\eta,\alpha)(\lambda',\alpha)} \int_{\mathfrak{t}_+^*} \varphi(\eta,\lambda',\lambda'')\mu_{\lambda''}\,d\lambda'' \\ &= \frac{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}{\prod_{\alpha\in\Phi^+}(\eta,\alpha)} \lim_{\lambda'\to\lambda} \int_{\mathfrak{t}_+^*} \sum_{w\in W} \operatorname{sgn}(w) e_{w\eta} * \nu^p_{\lambda'}(\lambda'')\mu_{\lambda''}\,d\lambda'' \\ &= \frac{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}{\prod_{\alpha\in\Phi^+}(\eta,\alpha)} \int_{\mathfrak{t}_+^*} \sum_{w\in W} \operatorname{sgn}(w) e_{w\eta} * \nu^p_{\lambda}(\lambda'')\mu_{\lambda''}\,d\lambda''.\end{aligned} \tag{4.3.19}$$

□

By Theorem 4.3.5 and Lemma 4.3.3, we derive a formula for the convolution of singular coadjoint orbits.

Corollary 4.3.5.1. *Suppose that η, λ both lie on a wall of $\mathfrak{t}_+^*$. Then*

$$\begin{aligned}&\nu_\eta * \nu_\lambda \\ &= \frac{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}{\prod_{\alpha\in\Phi^+\smallsetminus\Phi_\eta^+}(\eta,\alpha)\prod_{\alpha\in\Phi_\eta^+}(\delta_\eta,\alpha)} \int_{\mathfrak{t}_+^*} \sum_{j=1}^{q} sgn(w_j) e_{w_j\eta} * \left(\prod_{\alpha\in\Phi_\eta^+} \partial_{w_j\alpha}\right) \nu^p_\lambda(\lambda'')\,\mu_{\lambda''}\,d\lambda''.\end{aligned} \tag{4.3.20}$$

4.4 Convolutions of Unit 2-Spheres

Now, we determine the singular supports for the convolutions of coadjoint orbits $\mathcal{O}_1$ of $SU(2)$.

Proposition 4.4.1. *Let $G = SU(2)$, n be an integer highest weight of $SU(2)$, ν_1 be the Liouville measure of the coadjoint orbit $\mathcal{O}_1$ (for $n = 1$). Denote the n-fold convolution of ν_1 by $(\nu_1)^{*n}$. Then, the support and singular support of $(\nu_1)^{*n}$ are:*

$$supp\left((\nu_1)^{*n}\right) = conv\left(\mathcal{O}_n\right), \qquad singsupp\left((\nu_1)^{*n}\right) = \bigcup_{j=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{O}_{n-2j}. \tag{4.4.1}$$

where $\lfloor\cdot\rfloor$ is the floor function. So, the singular support of $(\nu_1)^{*n}$ is the union of coadjoint orbits of integers. For instance, $singsupp\,(\nu_1) = \mathcal{O}_1$, also

$$singsupp\left((\nu_1)^{*2}\right) = \mathcal{O}_0 \cup \mathcal{O}_2, \quad singsupp\left((\nu_1)^{*3}\right) = \mathcal{O}_1 \cup \mathcal{O}_3, \tag{4.4.2}$$

etc.

Proof. The support of $((\nu_1)^{*n})$ is clear, which is the ball with radius n. For the singular support, by Corollary 4.2.5.1, we have

$$(\nu_1)^{*n} = \int_0^\infty \sum_{j=0}^{1} (-1)^j e_{(-1)^j} * \left((\nu_1^p)^{*(n-1)}\right)(\lambda'') \cdot \lambda'' \, \nu_{\lambda''} \, d\lambda'', \tag{4.4.3}$$

and the density function with respect to $\alpha \in \mathfrak{t}^*$ is

$$d\left((\nu_1)^{*n}\right)|_{\mathfrak{t}^*}(\alpha) = \left(\sum_{j=0}^{1} (-1)^j e_{(-1)^j} * \left((\nu_1^p)^{*(n-1)}\right)(\alpha) \cdot \alpha\right) d\alpha, \tag{4.4.4}$$

and

$$d\left((\nu_1^p)^{*(n-1)}\right)(\alpha) = \left(\int_0^\infty \sum_{j=0}^{1} (-1)^j e_{(-1)^j} * \left((\nu_1^p)^{*(n-2)}\right)(\lambda'') \cdot \nu_{\lambda''}^p(\alpha) \, d\lambda''\right) d\alpha. \tag{4.4.5}$$

We may simplify (4.4.4) to obtain

$$d\left((\nu_1)^{*n}\right)|_{\mathfrak{t}^*}(\alpha) = \left[\left((\nu_1^p)^{*(n-1)}\right)(\alpha - 1) - \left((\nu_1^p)^{*(n-1)}\right)(\alpha + 1)\right] \cdot \alpha \, d\alpha. \tag{4.4.6}$$

This means that $d\left((\nu_1)^{*n}\right)|_{\mathfrak{t}^*}$ is differentiable at α in $\mathfrak{t}^*$ if and only if $(\nu_1^p)^{*(n-1)}$ is differentiable at $\alpha - 1$ and $\alpha + 1$. Also, if we differentiate (4.4.5) with respect to α, since $\nu_{\lambda''}^p$ is the characteristic function on $[-\lambda'', \lambda''] \in \mathfrak{t}^*$, we have

$$\begin{aligned}
&\frac{d}{d\alpha} d\left((\nu_1^p)^{*(n-1)}\right)(\alpha) \\
&\quad = \left(\int_0^\infty \sum_{j=0}^{1} (-1)^j e_{(-1)^j} * \left((\nu_1^p)^{*(n-2)}\right)(\lambda'') \cdot \left(-e_{\lambda''}(\alpha) + e_{-\lambda''}(\alpha)\right) d\lambda''\right) d\alpha \\
&\quad = \pm \left[\left((\nu_1^p)^{*(n-2)}\right)(\alpha - 1) - \left((\nu_1^p)^{*(n-2)}\right)(\alpha + 1)\right] d\alpha.
\end{aligned} \tag{4.4.7}$$

Therefore, we can recursively apply differentiation to calculate the singular support of $(\nu_1)^{*n}$, which is exactly $\bigcup_{j=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{O}_{n-2j}$.

Remark. *This proof is inspired by Cazzaniga's proof in [9]. By finding the support and singular support of the n-fold convolution of ν_1, we are able to compare this result with the explicit formula for the G-invariant measure of moment sets of $SU(2)$ in Chapter 5.*

□

Chapter 5

Invariant Measures On Moment Sets

Let G be a Lie group, $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. Let M be a G-Hamiltonian manifold, Ψ be the moment map on M. The general image and convexity of a moment map $\Psi : M \to \mathfrak{g}^*$ has been greatly generalised by M. Atiyah [6], V. Guillemin and S. Sternberg [25], and F. Kirwan [35]. N. Wildberger [55] explicitly studied the convexity of the image of the moment map of an unitary representation of a compact connected semisimple Lie group. His theorem says that the moment set I_λ of an irreducible unitary representation of highest weight λ is convex if and only if λ is root distinct (that is, the set of the pairwise difference between λ and an element of the orbit of the Weyl group of λ does not contain any roots of $\mathfrak{g}$), and the intersection of the moment set with the dual Cartan subalgebra $\mathfrak{t}^*$ of $\mathfrak{g}^*$, $I_\lambda \cap \mathfrak{t}^*$, is the convex hull of the set of weights of π_λ.

The projection of the G-invariant (Liouville) measure for a coadjoint orbit μ_λ^p was derived in Chapter 4. We are also interested in working out a projection measure for a moment set I_λ of G, denoted by $\nu_{I_\lambda}^p$. Especially when λ is not root distinct, then I_λ is non-convex and we are also interested in determining the support of I_λ in $\mathfrak{t}^*$, that is, $I_\lambda \cap \mathfrak{t}^*$, for an arbitrary integral dominant weight $\lambda \in \Lambda^+$.

In this chapter, we aim to derive an explicit formulas for a G-invariant measure of a moment set I_λ, which is the pushforward of the unitarily invariant probability measure of the unit sphere of the Hilbert space $\mathcal{H}_\lambda$, denoted by ν_{I_λ}, for an arbitrary dominant weight $\lambda \in \Lambda^+$.

In this chapter, we let G be a compact connected semisimple Lie group, $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$, $\mathfrak{t}, \mathfrak{t}^*$ be the (dual) Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{g}^*$, respectively, and W be the Weyl group of $\mathfrak{t}$, $\mathfrak{t}^*$. In addition, we let $\mathfrak{t}^*_+$ be the fundamental dual Weyl chamber of $\mathfrak{t}^*$, $\mathfrak{t}^*_0$ be the interior of $\mathfrak{t}^*_+$, Φ be the set of roots of $\mathfrak{g}$, Φ^+ be the subset of positive roots of Φ, Λ be the set of weights of $\mathfrak{t}^*$, and Λ^+ be the set of dominant weights of Λ in $\mathfrak{t}^*_+$. Furthermore, we let $p : \mathfrak{g}^* \to \mathfrak{t}^*$ be the projection map with respect to the Killing form of $\mathfrak{g}^*$.

In Section 5.1, we review the convexity theorem for G-Hamiltonian manifolds, and study the properties of moment sets of unitary representations. In Proposition 5.1.8, we show the n-fold sumset of the moment set I_1 of the irreducible unitary representation π_1 of $SU(2)$ of the highest integer weight 1 is equal to the moment set I_n of π_n of $SU(2)$. Also, in Proposition 5.1.9, we determine a subset of the intersection $I_\lambda \cap \mathfrak{t}^*$ for a non-root distinct $\lambda \in \Lambda^+$.

In Section 5.2, we derive an explicit formula $\nu^p_{I_\lambda}$, in Proposition 5.2.2 and Proposition 5.2.5. Also, in Proposition 5.2.4, for $SU(2)$, we show $\nu_{I_n} = (\nu_1)^{*n}$. In addition, using the results of Chapter 4, we derive the density function of $I_\lambda \cap \mathfrak{t}^*_+$ for two different scenarios: 1. $I_\lambda \cap \mathfrak{t}^*_0$ is non-empty; (Proposition 5.2.7); 2. $I_\lambda \cap \mathfrak{t}^*_+$ is fully contained in a minimal wall of $\mathfrak{t}^*_+$ (Proposition 5.2.8).

5.1 Highest Weights and Convexity of Moment Sets

We have discussed properties and examples of a G-Hamiltonian manifold M and its moment map Ψ in Chapter 2. Now, we look at the convexity of the image of the moment sets of irreducible unitary representations of G.

Theorem 5.1.1 ([25], [35]). *(Convexity Theorem) Let G be a compact connect Lie group, T be its maximal torus, $\mathfrak{g}^*, \mathfrak{t}^*$ be the duals of their Lie algebras, respectively. Let M be a compact G-Hamiltonian manifold, and $\Psi : M \to \mathfrak{g}^*$ be the associated moment map. Then the intersection of the image of the moment map with the fundamental dual positive Weyl chamber, $\Psi(M) \cap \mathfrak{t}^*_+$, is a convex polytope.*

Example 5.1.1.1. Let $\lambda \in \mathfrak{t}^*_+$, $\mathcal{O}_\lambda$ be the coadjoint orbit passing through λ, then $\Psi(\mathcal{O}_\lambda) \cap \mathfrak{t}^*_+$ is exactly $\{\lambda\}$, which is a convex polytope of dimensions zero.

Definition 5.1.2. A dominant weight $\lambda \in \Lambda^+$ is called **root distinct** if the set $\{\lambda - w\lambda \,:\, w \in W\}$ does not contain any roots in Φ. It is equivalent to the condition that λ is root distinct if

$$\frac{(\lambda, \alpha)}{(\alpha, \alpha)} \neq 1, \tag{5.1.1}$$

for all $\alpha \in \Phi$.

Example 5.1.2.1. Let $G = SU(2)$, and its Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$, the Cartan subalgebra $\mathfrak{t} \cong \mathbb{R}$ (and its dual $\mathfrak{t}^* \cong \mathbb{R}$). The Weyl group $W \cong C_2 = \{e, \sigma\}$ (the cyclic group of order 2), so that $e \cdot \lambda = \lambda$ and $\sigma \cdot \lambda = -\lambda$ for all $\lambda \in \mathfrak{t}^*$. The only positive root of $SU(2)$ is $\alpha = 2$. The set $S_\lambda = \{\lambda - w\lambda \,:\, w \in \{e, \sigma\}\}$. Let $\lambda \in \Lambda^+$ be a dominant weight. If $\lambda = 1$, then $S_1 = \{0, 2\}$, which contains α. So $\lambda = 1$ is <u>not</u> root distinct. However, if $\lambda > 1$, every set S_λ does not contain any roots. So, λ is root distinct if the integer $\lambda > 1$.

Example 5.1.2.2. Let $G = SU(3)$, and its Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$. The Cartan subalgebra $\mathfrak{t} \cong \mathbb{R}^2$ (and its dual $\mathfrak{t}^* \cong \mathbb{R}^2$). Let $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ be the set of positive roots of $\mathfrak{g}$. Let the Weyl group

$$W = \{e,\, \sigma_{\alpha_1},\, \sigma_{\alpha_2},\, \sigma_{\alpha_2}\sigma_{\alpha_1},\, \sigma_{\alpha_1}\sigma_{\alpha_2},\, \sigma_{\alpha_2}\sigma_{\alpha_1}\sigma_{\alpha_2}\}, \tag{5.1.2}$$

(Example 2.2.18.1), where $\sigma_{\alpha_1}, \sigma_{\alpha_2}$ are simple reflections with respect to α_1, α_2. Let $\lambda \in \Lambda^+$ be a dominant weight. If $\lambda = \lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$, the first fundamental weight of $\mathfrak{t}^*$, and the Weyl orbit of λ_1 is the set $W \cdot \lambda_1 = \{\lambda_1, \lambda_1 - \alpha_1, \lambda_1 - \alpha_1 - \alpha_2\}$, then $S_{\lambda_1} = \{0, \alpha_1, \alpha_1 + \alpha_2\}$, which contains roots of $\mathfrak{g}$. Hence, λ_1 is not root distinct. Also, if $\lambda = \delta = \alpha_1 + \alpha_2$, the half-sum of all the positive roots, then δ is also not root distinct, since $W \cdot \delta$ contains all the roots of $\mathfrak{g}$, so S_δ does contains some roots of $\mathfrak{g}$, so δ is also not root distinct. For other dominant weights, $\lambda = 2\alpha_1 + 2\alpha_2$ is an example of the dominant weight being root distinct.

When M is the projective representation space of an irreducible unitary representation π_λ of a compact connected semisimple Lie group G (Example 2.1.7.3), the convexity of $\Psi(M)$ can be determined by the dominant highest weight λ.

Lemma 5.1.3 ([55], Proposition 3.2, [44], Lemma 6.1.4)**.** *Let $\lambda \in \Lambda^+$ be a dominant weight. Suppose that the set of weights of a d-dimensional irreducible* *highest weight unitary representation* π_λ *is* $\Pi(\lambda) = \{\lambda_1, ..., \lambda_n\}$ *(where each* λ_j *lies in* Λ*), so that multiplicities of the weights satisfy* $\sum_{i=1}^{n} multi(\lambda_i) = d$*. Let* I_λ *be the moment set of* π_λ*. If* $D = conv\{\lambda_1, ..., \lambda_n\}$*, then* $p(I_\lambda) = D$*, and* $I_\lambda \cap \mathfrak{t}^* \subseteq Ad(G) \cdot D$*.*

Theorem 5.1.4 ([55], Theorem 3.6)**.** *Let* π_λ *be a d-dimensional irreducible highest weight unitary representation of* λ *of a* G *acting on a Hilbert space* V*. Let* $\Pi(\lambda) = \{\lambda_1, ..., \lambda_n\}$ *be the set of weights of* π_λ *so that multiplicities of the weights satisfy* $\sum_{i=1}^{n} multi(\lambda_i) = d$*. Let* PV *be the projective space of* V *identified by the unit sphere* $\Omega_d = \{v \in V : ||v|| = 1\}$*, so that* PV *is a G-Hamiltonian manifold with a moment map* Ψ *(Example 2.1.7.3). Let* $I_\lambda = \Psi(\Omega_d)$ *be the moment set of* I_λ*. Then* $I_\lambda \cap \mathfrak{t}^*$ *is* ***convex*** *if and only if* λ *is* ***root distinct****, and in this case* $I_\lambda \cap \mathfrak{t}^* = conv(\{\lambda_1, ..., \lambda_n\})$*.*

Remark. *Unlike the general convexity theorem (Theorem 5.1.1), this convexity theorem by N. Wildberger for irreducible unitary representations of compact connected semisimple Lie group describes the convexity of the intersection of* I_λ *with the whole* $\mathfrak{t}^*$*, rather than only focusing on the intersection* I_λ *with only the fundamental dual Weyl chamber* $\mathfrak{t}^*_+$*.*

Example 5.1.4.1. Let $G = SU(2)$, with $W \cong C_2 = \{e, \sigma\}$, so that $e \cdot \lambda = \lambda$ and $\sigma \cdot \lambda = -\lambda$ for all $\lambda \in \mathfrak{t}^* \cong \mathbb{R}$. From Example 5.1.2.1, if $\lambda = 1$, then S_1 contains the only positive root, and λ is not root distinct, and the moment set I_1 is not convex. Since the unitary representation π_1 is the defining representation of $SU(2)$, π_{λ_1} acts transitively on the projective representation space PV_1, it follows that I_1 is exactly the coadjoint orbit $\mathcal{O}_1$. But when $\lambda > 1$, every set S_λ does not contain any root, so I_λ is the convex hull of $\{-\lambda, \lambda\}$. In terms of the Lie algebra $\mathfrak{su}(2)$, I_λ is the closure of the ball in $\mathbb{R}^3$ with radius λ.

Example 5.1.4.2. Let $G = SU(3)$. From Example 5.1.2.2, if $\lambda = \lambda_1$, the first fundamental weight of $\mathfrak{t}^*$, then λ_1 is not root distinct, and the moment set I_1 is not convex. Also, the unitary representation π_{λ_1} is the defining representation of $SU(3)$, so π_{λ_1} acts transitively on the projective representation space PV_{λ_1}, so it follows that I_1 is exactly the coadjoint orbit $\mathcal{O}_{\lambda_1}$. If $\lambda = \delta$, then δ is not a convex set as well (Later in the chapter, we plot the region of this non-convex set in Figure 5.3), but the image of the projection $p(I_\lambda)$ lies in the convex hull of the Weyl Orbit $W \cdot \delta$ (which coincides with the region in Figure 4.1).

Theorem 5.1.5 ([55], Lemma 3.1)**.** *Let G be a compact connected semisimple Lie group, T be a maximal torus of G, $\mathfrak{t}$ be the Lie algebra of T, and $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. Let $\lambda \in \Lambda^+ \subset \mathfrak{t}^*_+$, and π_λ be an irreducible unitary representation of highest weight λ. If we restrict π_λ to T, denoted by $\pi_\lambda|_T$. Then, the moment set $I_\lambda|_T$ is always convex and satisfies $I_\lambda|_T \cap \mathfrak{t}^* = conv(\{\lambda_1, ..., \lambda_n\})$.*

Example 5.1.5.1 ([44], Sec 6.1)**.** Let $\pi_\lambda|_T$ be a d-dimensional irreducible unitary representation of the highest weight λ restricted to T, acting on a Hilbert space V. Let $\Pi(\lambda) = \{\lambda_1, ..., \lambda_n\}$ be the set of weights of π_λ such that the multiplicities of the weights satisfy $\sum_{i=1}^n multi(\lambda_i) = d$. Let PV be the projective space of V identified by the unit sphere $\Omega_d = \{v \in V : |v| = 1\}$, so that PV is a G-Hamiltonian manifold with a moment map Ψ. Let $\{v_1, ..., v_d\}$ be an orthonormal basis of PV, and $d\pi_\lambda$ be the infinitesimal version of π_λ. By the Definition 2.4.4 and Proposition 2.4.3, we have

$$\Psi(\Omega_d) = \{\langle d\pi_\lambda(H)v, v\rangle : |v| = 1\}, \quad H \in \mathfrak{t}, \tag{5.1.3}$$

and for each $v \in PV$,

$$\begin{aligned} \langle d\pi_\lambda(H)v, v\rangle &= \langle d\pi_\lambda(H)(v_1 + \cdots + v_d), (v_1 + \cdots + v_d)\rangle && (5.1.4)\\ &= \langle(\lambda_1(H)v_1 + \cdots + \lambda_d(H)v_d), (v_1 + \cdots + v_d)\rangle && (5.1.5)\\ &= |v_1|\lambda_1(H) + \cdots + |v_d|\lambda_d(H), && (5.1.6) \end{aligned}$$

where each $\lambda_j \in \Pi(\lambda)$, and $|v_1| + \cdots + |v_d| = 1$. So, we have

$$I_\lambda \subseteq conv(\{\lambda_1, ..., \lambda_n\}). \tag{5.1.7}$$

On the other hand, let $a_1\lambda_1 + \cdots + a_d\lambda_d$ be a convex combination of weights. Let each v_j be the vector of the weight λ_j with $|v_j| = 1$. So, there exists a $v \in \Omega_d$ that can be written as $v = \sqrt{a_1}v_1 + \cdots + \sqrt{a_d}v_d$ and satisfies $|v| = 1$. This shows $conv(\{\lambda_1, ..., \lambda_n\}) \subseteq I_\lambda$. Hence, $I_\lambda = conv(\{\lambda_1, ..., \lambda_n\})$.

We can also build new representations from the tensor products of existing representations.

Theorem 5.1.6 ([30], Exercise 21.7)**.** *(Clebsch-Gordan Theory) Let $\lambda, \mu \in \Lambda^+$ be dominant weights, and $V(\lambda), V(\mu)$ be the irreducible standard cyclic modules* *of* λ, μ *with respect to the irreducible highest weight representations* $d\pi_\lambda, d\pi_\mu$*, respectively. Let* $\Pi(\lambda), \Pi(\mu)$ *be the set of weights of* $V(\lambda), V(\mu)$*, then*

$$\begin{aligned} \Pi\left(V(\lambda)\otimes V(\mu)\right) &= \{\nu+\nu' \mid \nu\in\Pi(\lambda), \nu'\in\Pi(\mu)\}, \\ \dim\left(V(\lambda)\otimes V(\mu)\right)_{\nu+\nu'} &= \sum_{\xi+\xi'=\nu+\nu'} \dim V_\xi \cdot \dim V_{\xi'}. \end{aligned} \tag{5.1.8}$$

In particular, the weight $\lambda+\mu$ *occurs once with multiplicity equal to 1, and the standard cyclic module* $V(\lambda+\mu)$ *occurs exactly* ***once*** *as a direct summand of* $V(\lambda)\otimes V(\mu)$*.*

Corollary 5.1.6.1 ([30], Exercise 21.8)**.** *Let* $(\lambda_1, ..., \lambda_l)$ *be the* l*-tuple of fundamental dominant weights with respect to a chosen root system* Φ *of* $\mathfrak{g}$*. There is a construction for an arbitrary standard cyclic module* $V(\lambda)$*,* $\lambda\in\Lambda^+$*, as a direct summand in a suitable tensor product of modules* $V(\lambda_1), ..., V(\lambda_l)$*. Choose an arbitrary* $\lambda\in\Lambda^+$*. If*

$$\lambda = \sum_{i=1}^{l} m_i\lambda_i, \qquad \textit{for } m_i\in\mathbb{Z}^+, \tag{5.1.9}$$

then $V(\lambda)$ *occurs exactly once as a summand in*

$$\underbrace{V(\lambda_1)\otimes...\otimes V(\lambda_1)}_{m_1 \textit{ number of times}} \otimes......\otimes \underbrace{V(\lambda_l)\otimes...\otimes V(\lambda_l)}_{m_l \textit{ number of times}} \tag{5.1.10}$$

where the weight vector space of the weight λ *has dimension 1, and* $V(\lambda = \sum_{i=1}^{l} m_i\lambda_i)$ *occurs exactly once in this tensor products.*

Example 5.1.6.1 ([30], Exercise 22.7)**.** Let $G = SU(2)$, $\mathfrak{g} = \mathfrak{su}(2)$, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_2\mathbb{C}$, and λ_1 be the only fundamental weight of $\mathfrak{sl}_2\mathbb{C}$. For $m\in\mathbb{Z}^+$, we identify a dominant weight $m\lambda_1$ by m, so the Clebsch-Gordan formula for both $\mathfrak{su}(2)$ and $\mathfrak{sl}_2(\mathbb{C})$ is given by: If $n \leqslant m$, then

$$V(m)\otimes V(n) \cong V(m+n)\oplus V(m+n-2)\cdots\oplus V(m-n), \tag{5.1.11}$$

$n+1$ summands in total. Subsequently, $V(n)$ occurs exactly once in

$$\underbrace{V(1)\otimes...\otimes V(1)}_{n \text{ number of times}}. \tag{5.1.12}$$

Lemma 5.1.7 ([55], Lemma 2.4)**.** *Let* $(V_1, \pi_1), (V_2, \pi_2)$ *be unitary representations of* G*, and* I_{π_1}, I_{π_2} *be their moment sets respectively. Let* $V = V_1 \otimes V_2$*, and let* π *be representation on* V *given by* $\pi(g)(v_1 \otimes v_2) = \pi_1(g)v_1 \otimes \pi_2(g)v_2$*. Then*

$$I_{\pi_1} + I_{\pi_2} \subseteq I_\pi \subseteq conv(I_{\pi_1}) + conv(I_{\pi_2}), \tag{5.1.13}$$

and when $I_{\pi_1} + I_{\pi_2} = conv(I_{\pi_1}) + conv(I_{\pi_2})$*, we have* $I_\pi = I_{\pi_1} + I_{\pi_2}$*.*

We can apply this lemma to study the moment sets of unitary representations of $SU(2)$, which has an interesting result.

Proposition 5.1.8. *Let* $G = SU(2)$*,* $\mathfrak{g} = \mathfrak{su}(2) \cong \mathbb{R}^3$*, the dual* $\mathfrak{g}^* = \mathfrak{su}(2)^* \cong \mathbb{R}^3$*. Let* $n \in \mathbb{Z}^+$ *be a dominant highest weight of* $SU(2)$*,* π_n *be an irreducible highest weight unitary representation of* $SU(2)$*, and let* I_n *be the moment set of* π_n*. Then the moment set* I_n *of* π_n *satisfies*

$$I_n = \sum_{j=1}^{n} I_1, \tag{5.1.14}$$

where the right-hand side is the sumset of n*-copies of the moment set* I_1*, where* I_1 *is the unit sphere in* $\mathbb{R}^3$ *and* I_n *is the ball in* I_n *with radius* n*.*

Proof. By the Clebsch–Gordan theorem (Theorem 5.1.6, Corollary 5.1.6.1), we can find exactly one copy of the irreducible unitary representation π_n inside the reducible unitary representation given by the tensor products of n-copies of π_1. As for the moment sets, by Lemma 5.1.7,

$$\sum_{j=1}^{n} I_1 \subseteq I_n \subseteq \sum_{j=1}^{n} conv(I_1). \tag{5.1.15}$$

Since the set I_1 is a unit sphere in $\mathbb{R}^3$ (Example 5.1.4.1), the convex set $conv(I_1)$ is the unit ball in $\mathbb{R}^3$, and the fact that sumset of n-fold unit spheres coincides with the sumset of n-fold unit balls, which is the ball in $\mathbb{R}^3$ with radius n, so the result follows. ☐

For general compact connected semisimple Lie groups, when λ is not root distinct, it can be challenging to determine the exact moment set I_λ. If we let $D_\lambda = conv\{\lambda_1, ..., \lambda_d\}$ be the convex hull of the set of weights of the highest weight λ, then the moment set I_λ satisfies I_λ is $I_\lambda \cap \mathfrak{t}^* \subseteq G \cdot D_\lambda$ (Lemma 5.1.3). In fact, we can determine a subset of $I_\lambda \cap \mathfrak{t}^*$ as follows.

Proposition 5.1.9. *Let $\lambda \in \Lambda^+$ be a dominant highest weight, $(\pi_\lambda, V(\lambda))$ be the irreducible highest weight unitary representation with respect to λ, $\Pi(\lambda)$ be the set of the weights of π_λ. Let $\{v_1, \dots, v_m\}$ be a subset of weight vectors of the representation space $V(\lambda)$ of the highest weight representation π_λ. If $S = \{\lambda_1 \dots, \lambda_m\}$ is the set of weights corresponding to $\{v_1, \dots, v_m\}$ such that S contains the largest possible number of weights with the property that the pairwise difference of any two weights in S is* ***not*** *a root, then $W \cdot conv(S) \subseteq I_\lambda \cap \mathfrak{t}^*$.*

Proof. Let $D' = conv\{\lambda_1 \dots, \lambda_m\}$, and suppose $\lambda' \in D'$. Then there exists $c_j \in \mathbb{C}$ for $j = 1, ..., m$ such that $\sum_{j=1}^m |c_j|^2 = 1$, and

$$\lambda' = \sum_{j=1}^{m} |c_j|^2 \lambda_j \, . \tag{5.1.16}$$

Let $v = \sum_{j=1}^m c_j v_j$ be in the unit sphere of the representation space $V(\lambda)$ with respect to a given norm $\langle \cdot, \cdot \rangle$. We claim that $\Psi(v) = \lambda'$. First, let $H \in \mathfrak{t}$, then

$$\Psi(v)(H) = \langle d\pi(H)v, v\rangle = \sum_{j=1}^{m} |c_j|^2 \lambda_j(H) = \lambda'(H), \tag{5.1.17}$$

Now, let $X_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in \Phi$ as in the root space decomposition defined in Definition 2.2.9, be a root space of $\mathfrak{g}$, then

$$\Psi(v_j)(X_\alpha) = \langle d\pi(X_\alpha)v_j, v_j\rangle, \tag{5.1.18}$$

where $d\pi(X_\alpha)$ acts as a raising operator and $d\pi(X_\alpha)v_j$ is a weight vector with weight $\lambda_j + \alpha$. By assumption, none of vectors of the weight $\lambda_j + \alpha$ is in $\{\lambda_1, \dots, \lambda_m\}$. Thus, $\Psi(v)(X_\alpha) = \lambda'(X_\alpha) = 0$. Also, $\lambda'(X_\alpha) = \langle H_{\lambda'}, X_\alpha\rangle$ is the Killing form, and $\lambda'(\cdot)$ is only non-degenerate on $\mathfrak{t}$. Hence, $\Psi(v) = \lambda'$.

It follows that, $D' \subseteq I_\lambda \cap \mathfrak{t}^*$, and because of W-invariance of $I_\lambda \cap \mathfrak{t}^*$, hence, $W \cdot D' \subseteq I_\lambda \cap \mathfrak{t}^*$. ☐

Corollary 5.1.9.1 ([47], Lemma 7.1)**.** *Let v be any unit vector in V, then $\Psi(v) \in I_\lambda \cap \mathfrak{t}^*$ if and only if $\langle d\pi(X_\alpha)v, v\rangle = 0$ for all $\alpha \in \Phi$.*

Example 5.1.9.1. Let $G = SU(3)$, and its Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$. The Cartan subalgebra $\mathfrak{t} \cong \mathbb{R}^2$ (and its dual $\mathfrak{t}^* \cong \mathbb{R}^2$). Let $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ be the set of positive roots of $\mathfrak{g}$. Let W be the Weyl group (5.1.2) of $\mathfrak{t}, \mathfrak{t}^*$. Let

$\delta = \alpha_1 + \alpha_2 \in \Lambda^+$ be the half-sum of positive roots. By Example 5.1.2.2, we know that δ is not root distinct, so the moment set I_δ is not a convex set. But we can determine a subset of $I_\delta \cap \mathfrak{t}^*$ as follows:

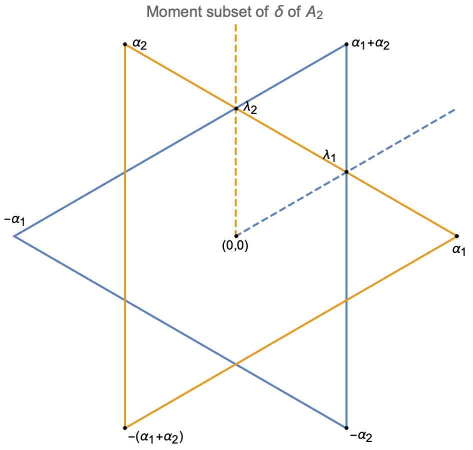

Figure 5.1: $I_\delta \cap \mathfrak{t}^*$ of A_2

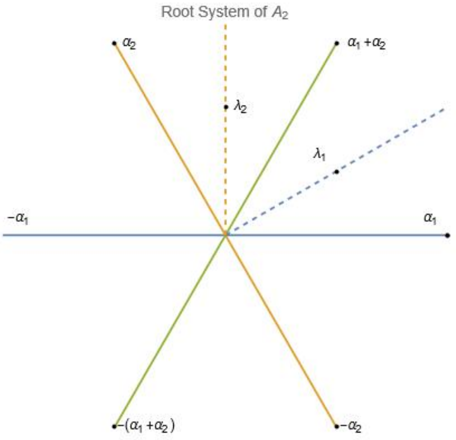

Figure 5.2: Root system of A_2

Let $\Pi(\delta)$ be the set of weights of the 8-dimensional irreducible unitary representation π_δ (which is also the adjoint representation of $SU(3)$), and

$$\Pi(\delta) = \{\delta, \alpha_2, \alpha_1, 0, -\alpha_1, -\alpha_2, -\delta\}, \tag{5.1.19}$$

where the multiplicity of the weight '0' is 2, while other ones have multiplicity 1. The largest possible subset S of the weights in $\Pi(\delta)$ with the property that pairwise difference of any two weights in S is not a root, which also contains δ, is the set $\{\delta, -\alpha_1, -\alpha_2\}$. If we let $D = conv\{\delta, -\alpha_1, -\alpha_2\}$, then

$$W \cdot D = conv\{\delta, -\alpha_1, -\alpha_2\} \cup conv\{-\delta, \alpha_1, \alpha_2\} \subseteq I_\delta \cap \mathfrak{t}^* \tag{5.1.20}$$

We can also see $W \cdot D$ in Figure 5.1, which is the area enclosed by the two equilateral triangles. (It is not a convex set. In fact, we will show $W \cdot D = I_\delta \cap \mathfrak{t}^*$ in the next section).

5.2 G-Invariant Measures on Moment Sets

In this section, we develop an explicit formula for the projection measure of the moment set I_λ. In addition, we use the techniques in Chapter 4 to develop an explicit formula for the G-invariant measure of I_λ.

Proposition 5.2.1. *Let G be a compact connected semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$, T be a maximal torus of G, $\mathfrak{t}$ be the Lie algebra (Cartan subalgebra) of T, and $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. Let $\lambda \in \Lambda^+ \subset \mathfrak{t}^*$ be a dominant highest weight. Let π_λ be a finite-dimensional irreducible unitary representation of G of λ. Let $I_\lambda \subset \mathfrak{g}^*$ be the moment set of π_λ. Then there exists a G-invariant measure on the moment set I_λ.*

Proof. G acts on I_λ, but the action of G in general is not transitive on I_λ, so we can treat I_λ as a disjoint union of coadjoint orbits of G. For each $\beta \in I_\lambda \subset \mathfrak{g}^*$, the coadjoint orbit $\mathcal{O}_\beta$ is a homogeneous space of the form G/G_β, where G_β is the stabiliser subgroup of β. By Proposition 4.1.3, there exists a unique (up to a scalar multiple) and finite G-invariant measure on $\mathcal{O}_\beta$. We denote this measure by ν_β and normalise it so that $\nu_\beta(\mathcal{O}_\beta) = 1$. Since each coadjoint orbit of G intersects with $\mathfrak{t}^* \subset \mathfrak{g}^*$, we denote the set of this intersection by $A = I_\lambda \cap \mathfrak{t}^*$. Hence, $I_\lambda = \bigcup_{\beta \in A} \mathcal{O}_\beta$, where A is an index set of coadjoint orbits. We can also assign a continuous measure ν_A on $A \subset \mathfrak{t}^*$. Finally, we can obtain a global measure on I_λ, denoted by ν_{I_λ}, by combining invariant measures on each coadjoint orbit $\mathcal{O}$, that is,

$$\nu_{I_\lambda}(E) = \int_A \nu_\beta(E \cap O_\beta)\, d\nu_A(\beta), \tag{5.2.1}$$

where $E \subseteq I_\lambda$ is a measurable subset. This is also an application of the *disintegration theorem* (Theorem 4.1.1). Since each measure ν_β of the coadjoint orbit $\mathcal{O}_\beta$ is unique (up to a scalar multiple), the total measure ν_{I_λ} of the union of coadjoint orbits I_λ is also unique once the normalisation is fixed. For example, if we let $\nu_{I_\lambda}(I_\lambda) = 1$, and each $\nu_\beta(\mathcal{O}_\beta) = 1$, then the weights of ν_A are fixed to satisfy this normalisation. □

Proposition 5.2.2. *Let G be a compact simply connected semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$, T be a maximal torus of G, $\mathfrak{t}$ be the Lie algebra (Cartan subalgebra) of T, and $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. Let $\lambda \in \Lambda^+ \subset \mathfrak{t}^*$ be a dominant highest weight. Let π_λ be a d-dimensional irreducible unitary representation of G of λ acting on a Hilbert space $\mathcal{H}_\lambda$, $d\pi_\lambda$ be its infinitesimal representation. Let $\Pi(\lambda) = \{\lambda_1, ..., \lambda_n\}$ be the set of weights of π_λ so that multiplicities of the weights satisfy $\sum_{i=1}^{n} multi(\lambda_i) = d$. Let $p : \mathfrak{g}^* \to \mathfrak{t}^*$ be the orthogonal projection map with respect to the Killing form on $\mathfrak{g}^*$. Let $P\mathcal{H}_\lambda$ be the projective space of $\mathcal{H}_\lambda$ which can be identified by the unit sphere*

$\Omega_d = \{u \in \mathcal{H} : |u| = 1\}$, where $u = \sum_{j=1}^{d} u_j$ such that $\sum_{j=1}^{d} |u_j|^2 = 1$, $\{u_j\}_{j=1}^{d}$ is an orthonormal basis of $\mathcal{H}_\lambda$, and each u_j is also the weight vector of the weight $\lambda_j \in \Pi(\lambda)$. We define the moment set of π_λ as $I_\lambda = \{\langle d\pi_\lambda(H)u, u\rangle : u \in \Omega_d\}$ and a continuous, adjoint invariant function $\tilde{\Psi}_d$ on $\mathfrak{t}$ by

$$\tilde{\Psi}_d(H) = \int_{\Omega_d} e^{\langle d\pi_\lambda(H)u,u\rangle}\, d\nu(u) = \int_{\Omega_d} e^{\sum_{j=1}^{d} i|u_j|^2\lambda_j(H)}\, d\nu(u), \qquad \forall H \in \mathfrak{t}. \tag{5.2.2}$$

where ν is the unitarily invariant probability measure supported on Ω_d. Suppose we let ν_{I_λ} be the probability G-invariant measure on the moment set I_λ of π_λ in $\mathfrak{g}^*$, which is the pushforward of ν. Then

$$\tilde{\Psi}_d(H) = \int_{I_\lambda} e^{i\xi(H)}\, d\nu_{I_\lambda}(\xi) = \int_{conv(\Pi(\lambda))} e^{i\beta(H)}\, d\nu^{p}_{I_\lambda}(\beta), \qquad \forall H \in \mathfrak{t}, \tag{5.2.3}$$

where $\nu^{p}_{I_\lambda}$ is the projection of ν_{I_λ} with respect to p.

Proof. The moment set I_λ is the union of coadjoint orbits of G, so I_λ is G-invariant. Let $H \in \mathfrak{t}$, for each $u \in \Omega_d$, there exists a $\beta \in \mathfrak{t}^*$ such that $\langle d\pi_\lambda(H)u, u\rangle = i\beta(H)$. Let $p : I_\lambda \mapsto conv(\Pi(\lambda))$ (Lemma 5.1.3). Let $(\beta, \gamma) \in \mathfrak{g}^*$ where $\beta \in \mathfrak{t}^*$ and $\gamma \in (\mathfrak{t}^*)^\perp$ as defined in Definition 4.1.4. Let ν_β be the probability measure on the fibre $p^{-1}(\beta)$, for $\beta \in conv(\Pi(\lambda))$. By the *disintegration theorem* (Theorem 4.1.1) and Lemma 4.1.5, we have

$$\begin{aligned}\int_{I_\lambda} e^{i\xi(H)}\, d\nu_{I_\lambda}(\xi) &= \int_{conv(\Pi(\lambda))}\int_{p^{-1}(\beta)} e^{i(\beta+\gamma)(H)}\, d\nu_\beta(\gamma)\, d\nu^{p}_{I_\lambda}(\beta) \\ &= \int_{conv(\Pi(\lambda))} e^{i\beta(H)}\, d\nu^{p}_{I_\lambda}(\beta).\end{aligned} \tag{5.2.4}$$

where $\nu^{p}_{I_\lambda}$ is the projection of ν_{I_λ} with respect to p. Hence, the proposition follows. □

Proposition 5.2.3. *Let G be a compact simply connected semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$, T be a maximal torus of G, $\mathfrak{t}$ be the Lie algebra (Cartan subalgebra) of T, and $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. Let $\lambda \in \Lambda^+ \subset \mathfrak{t}^*$ be a dominant highest weight. Let π_λ be a d-dimensional irreducible unitary representation of G of λ acting on a Hilbert space $\mathcal{H}_\lambda$, $d\pi_\lambda$ be its infinitesimal representation. Let $\Pi(\lambda) = \{\lambda_1, ..., \lambda_n\}$ be the set of weights of π_λ so that multiplicities of the weights satisfy $\sum_{i=1}^{n} multi(\lambda_i) = d$. Let $P\mathcal{H}_\lambda$ be the projective space of $\mathcal{H}_\lambda$ which can be identified by the unit sphere $\Omega_d = \{u \in \mathcal{H} : |u| = 1\}$,* *where* $u = \sum_{j=1}^{d} u_j$ *such that* $\sum_{j=1}^{d} |u_j|^2 = 1$, $\{u_j\}_{j=1}^{d}$ *is an orthonormal basis of* $\mathcal{H}_\lambda$, *and each* u_j *is also the weight vector of the weight* $\lambda_j \in \Pi(\lambda)$. *Suppose the multiplicity of each weight* $\lambda_j \in \Pi(\lambda)$ *is exactly* 1, *and all vectors in* $\{u_j\}_{j=1}^{d}$ *have* ***distinct weights*** *(in this case* $d = n$*). Then, the* $\mathfrak{t}$*-Fourier transform of the projection measure* $\nu^p_{I_\lambda}$ *is given by*

$$\left(\nu^p_{I_\lambda}\right)^{\vee_{\mathfrak{t}}}(H) = \tilde{\Psi}_n(H) = (-1)^{n-1}(n-1)! \sum_{k=1}^{n} \frac{e^{i\lambda_k(H)}}{\prod_{\substack{j \neq k \\ 1 \leqslant j \leqslant n}} i(\lambda_j - \lambda_k)(H)}, \tag{5.2.5}$$

for all $H \in \mathfrak{t}$.

Proof. Based on the calculations in ([40], Theorem 9) and ([44], Lemma 4.16), we let χ_{Ω_n} be the characteristic function on Ω_n, so we first evaluate the right-hand side of (5.2.2),

$$\int_{\mathbb{C}^n} \chi_{\Omega_n}(u)\, e^{\sum_{j=1}^{n} i|u_j|^2 \lambda_j(H)}\, du_1 \cdots du_n\,, \tag{5.2.6}$$

where each complex variable du_j can be re-written in polar form as $|u_j| d|u_j| d\theta_j$, so (5.2.6) is equal to

$$\begin{aligned} &\int_{\mathbb{R}^n_+} \int_{\mathbb{T}^n} \chi_{\Omega_n}(u)\, e^{\sum_{j=1}^{n} i|u_j|^2 \lambda_j(H)}\, |u_1| d|u_1| d\theta_1 \cdots |u_n| d|u_n| d\theta_n \\ &= \frac{(2\pi)^n}{2^n} \int_{\mathbb{R}^n_+} \chi_{\Omega_n}(u)\, e^{\sum_{j=1}^{n} i|u_j|^2 \lambda_j(H)}\, d|u_1|^2 \cdots d|u_n|^2\,. \end{aligned} \tag{5.2.7}$$

Now, we can consider $dw_j = \pi\, d|u_j|^2$ to be the change from the thickness of the annulus to the thickness of a wall. Also, by the definition of a simplex, (5.2.7) is equal to

$$\begin{aligned} &\frac{1}{\pi^n} \int_{\mathbb{R}^n_+} \chi_{\Delta^{n-1}}(w)\, \exp^{\sum_{j=1}^{n} \lambda_j w_j}\, dw_n, ..., dw_1 \\ &= e^{i\lambda_n(H)} \int_0^1 \int_0^{1-w_1} \cdots \int_0^{1-w_1-\cdots-w_{n-2}} e^{\sum_{j=1}^{n-1} i(\lambda_j - \lambda_n)(H) w_j}\, dw_{n-1}, ..., dw_1\,, \end{aligned} \tag{5.2.8}$$

where we have integrated with respect to w_n. Define the right hand side of (5.2.8) by $\Psi_n(\lambda_1, ..., \lambda_n)(H)$. We further integrate (5.2.8) with respect to w_{n-1}, we obtain the recursive relation

$$\Psi_n(\lambda_1, ..., \lambda_n) = \frac{\Psi_{n-1}(\lambda_1, ..., \lambda_{n-1}) - \Psi_{n-1}(\lambda_1, ..., \lambda_{n-2}, \lambda_n)}{\lambda_{n-1} - \lambda_n}\,. \tag{5.2.9}$$

By induction, we can show

$$\Psi_n(\lambda_1, ..., \lambda_n) = (-1)^{n-1} \sum_{k=1}^{n} \frac{e^{\lambda_k}}{\prod_{\substack{j \neq k \\ 1 \leqslant j \leqslant n}} (\lambda_j - \lambda_k)} \,. \tag{5.2.10}$$

We have $\Phi_1(\lambda_1) = e^{\lambda_1}$ and $\Phi_2(\lambda_1, \lambda_2) = \frac{e^{\lambda_1}}{\lambda_1 - \lambda_2} - \frac{e^{\lambda_2}}{\lambda_1 - \lambda_2}$. So it is true for $n = 1$ and 2. Now, we assume it is true for all $n = N$, then it is also true when $n = N + 1$, that is

$$\Psi_{N+1}(\lambda_1, ..., \lambda_{N+1}) = \frac{\Psi_N(\lambda_1, ..., \lambda_N) - \Psi_N(\lambda_1, ..., \lambda_{N-1}, \lambda_{N+1})}{\lambda_N - \lambda_{N+1}} \tag{5.2.11}$$

The right hand-side is equal to

$$\begin{aligned}
&\sum_{k=1}^{N-1} (-1)^{N-1} e^{\lambda_k} \prod_{\substack{j \leqslant N-1, N \\ j \neq k}} \frac{(\lambda_j - \lambda_k)^{-1}}{\lambda_N - \lambda_{N+1}} - \sum_{k=1}^{N-1} (-1)^{N-1} e^{\lambda_k} \prod_{\substack{j \leqslant N-1, N+1 \\ j \neq k}} \frac{(\lambda_j - \lambda_k)^{-1}}{\lambda_N - \lambda_{N+1}} \\
&+ (-1)^{N-1} e^{\lambda_N} \prod_{j \neq N} \frac{(\lambda_j - \lambda_N)^{-1}}{\lambda_N - \lambda_{N+1}} - (-1)^{N-1} e^{\lambda_{N+1}} \prod_{j \neq N+1} \frac{(\lambda_j - \lambda_{N+1})^{-1}}{\lambda_N - \lambda_{N+1}} \\
= &\sum_{k=1}^{N-1} (-1)^{N-1} e^{\lambda_k} \frac{[(\lambda_{N+1} - \lambda_k) - (\lambda_N - \lambda_k)]}{(\lambda_N - \lambda_{N+1}) \prod_{j \neq k} (\lambda_j - \lambda_k)} \\
&+ (-1)^{N} e^{\lambda_N} \prod_{j \neq N} \frac{(\lambda_j - \lambda_N)^{-1}}{\lambda_{N+1} - \lambda_N} + (-1)^{N} e^{\lambda_{N+1}} \prod_{j \neq n} \frac{(\lambda_j - \lambda_{N+1})^{-1}}{\lambda_N - \lambda_{N+1}} \\
&= \sum_{k=1}^{N+1} (-1)^{N} e^{\lambda_k} \prod_{j \neq k} (\lambda_j - \lambda_k)^{-1} \\
&= \Psi_{N+1}(\lambda_1, ..., \lambda_{N+1}) \,.
\end{aligned}$$

Also, by induction, we can prove that the simplex has volume

$$\nu(\Delta^{n-1}) = \int_0^1 \int_0^{1-w_1} \cdots \int_0^{1-w_1-\cdots-w_{n-2}} dw_{n-1}, ..., dw_1 = \frac{1}{(n-1)!} \tag{5.2.12}$$

Therefore, the function in (5.2.2) satisfies $\tilde{\Psi}_n = (n-1)!\, \Psi_n$, and the proposition follows. ☐

Remark. *Note that if we define*

$$\tilde{\Psi}_d(\lambda_1, ...\lambda_d)(\cdot) = \int_{\Omega_d} e^{\sum_{j=1}^{d} i |u_j|^2 \lambda_j(\cdot)} \, d\nu(u), \tag{5.2.13}$$

to be a function in $(\mathfrak{t}^*)^d$ *instead, then* $\tilde{\Psi}_d$ *is still continuous and well-defined if any multi*$(\lambda_j) > 1$*. Also, suppose* λ_m *lies in a neighborhood of an element* $\widehat{\lambda} \in \mathfrak{t}^*$*, then*

$$\lim_{\widehat{\lambda}_m \to \widehat{\lambda}} \tilde{\Psi}_d(..., \widehat{\lambda}_m, \widehat{\lambda}, ...)(\cdot) = \tilde{\Psi}_d(..., \widehat{\lambda}, \widehat{\lambda}, ...)(\cdot), \tag{5.2.14}$$

by the dominated convergence theorem.

Proposition 5.2.4. *Let* $G = SU(2)$*,* T *be a maximal torus of* G*,* $\mathfrak{g}, \mathfrak{t}$ *be their Lie algebras respectively, and* $\mathfrak{g}^*, \mathfrak{t}^*$ *be the duals of* $\mathfrak{g}, \mathfrak{t}$*. Let* $n \in \Lambda^+ \cong \mathbb{Z}^+$ *be an integer dominant weight,* π_n *be a* $(n+1)$*-dimensional irreducible unitary representation with respect to* n*, and* I_n *be the moment set of* π_n*. Also, let* $\mathcal{O}_1$ *be the coadjoint orbit of the integer* 1*, and* $\Pi(n)$ *be the set of the weights of* π_λ*. Denote* $(\nu_1)^{*n}$ *the* n*-fold convolution of the* G*-invariant Liouville measure of* $\mathcal{O}_1$*, and* ν_{I_n} *the* G*-invariant probability measure of* I_n*. Then,*

$$\nu_{I_n} = (\nu_1)^{*n}. \tag{5.2.15}$$

Proof. Since both ν_{I_n} and $(\nu_1)^{*n}$ and their Fourier transforms are G-invariant, hence, we can show $(\nu_{I_n})^{\vee\mathfrak{g}}|_{\mathfrak{t}} = ((\nu_1)^{*n})^{\vee\mathfrak{g}}|_{\mathfrak{t}}$. Also, because we have $(\nu_{I_n})^{\vee\mathfrak{g}}|_{\mathfrak{t}} = (\nu_{I_n}^p)^{\vee\mathfrak{t}}$ and $(\nu_1^{*n})^{\vee\mathfrak{g}}|_{\mathfrak{t}} = ((\nu_1^p)^{*n})^{\vee\mathfrak{t}}$, so we can show $(\nu_{I_n}^p)^{\vee\mathfrak{t}} = ((\nu_1^p)^{*n})^{\vee\mathfrak{t}}$.

Notice that by Theorem 2.5.7, the $\mathfrak{t}$-Fourier transform of $(\nu_1^p)^{*n}$ is given by

$$((\nu_1^p)^{*n})^{\vee\mathfrak{t}} = \left(\frac{\sin(H)}{H}\right)^n, \quad H \in \mathfrak{t} \cong \mathbb{R}. \tag{5.2.16}$$

Let $n \geqslant 1$, $\Pi(n) = \{n, n-2, ..., -(n-2), -n\}$, and each weight in $\Pi(n)$ is distinct. So, by Proposition 5.2.3, we have

$$(\nu_{I_n}^p)^{\vee\mathfrak{t}} = (-1)^n\, n! \sum_{k \in \Pi(n)} \frac{e^{ikH}}{\prod_{\substack{j \in \Pi(n) \\ j \neq k}} i(j-k)H}, \quad H \in \mathfrak{t} \cong \mathbb{R}. \tag{5.2.17}$$

So, we want to show

$$\left(\frac{\sin(H)}{H}\right)^n = (-1)^n\, n! \sum_{k \in \Pi(n)} \frac{e^{ikH}}{\prod_{\substack{j \in \Pi(n) \\ j \neq k}} i(j-k)H}, \quad H \in \mathfrak{t} \cong \mathbb{R}. \tag{5.2.18}$$

Firstly, we note that

$$\sin(H) = \frac{1}{2i}\left(e^{iH} - e^{-iH}\right), \quad H \in \mathfrak{t} \cong \mathbb{R}. \tag{5.2.19}$$

Next, by *binomial theorem*, we have

$$(\sin(H))^n = \sum_{r=0}^{n} \frac{1}{(2i)^n}\binom{n}{r}(-1)^r e^{i(n-2r)H}, \quad H \in \mathfrak{t} \cong \mathbb{R}. \tag{5.2.20}$$

If we let $\Pi(n) = \{n-2s : s \in \{0, 1, \cdots, n\}\}$, and let $k = n-2r$, then the denominator $\prod_{\substack{j\in\Pi(n)\\ j\neq k}}(j-k)$ becomes

$$\begin{aligned}\prod_{\substack{j\in\Pi(n)\\ j\neq k}}(j-k) &= \prod_{\substack{s\in\{0,1,\ldots,n\}\\ s\neq r}}(n-2s-(n-2r))\\ &= 2^n \prod_{\substack{s\in\{0,1,\ldots,n\}\\ s\neq r}}(r-s)\\ &= 2^n(-1)^{n-r}r!(n-r)!.\end{aligned} \tag{5.2.21}$$

Hence,

$$\begin{aligned}\frac{\sin^n(H)}{H^n} &= \sum_{r=0}^{n}\frac{1}{(2iH)^n}\binom{n}{r}(-1)^r e^{i(n-2r)H}\\ &= \sum_{r=0}^{n}\frac{1}{(2iH)^n}(-1)^n\frac{n!}{(-1)^{n-r}r!(n-r)!}e^{i(n-2r)H}\\ &= (-1)^n n!\sum_{r=0}^{n}\frac{1}{(iH)^n}\frac{e^{i(n-2r)H}}{2^n(-1)^{n-r}r!(n-r)!}\\ &= (-1)^n\, n! \sum_{k\in\Pi(n)}\frac{e^{ikH}}{\prod_{\substack{j\in\Pi(n)\\ j\neq k}} i(j-k)H},\end{aligned} \tag{5.2.22}$$

for $H \in \mathfrak{t} \cong \mathbb{R}$. Therefore, the proposition follows. □

Remark.

- *This is a key observation in this project. So far, this correspondence of describing the moment set of a highest weight unitary representation of a compact semisimple Lie group G, by the sumset of coadjoint orbits of G, is only known for $SU(2)$. It is unclear if this correspondence is true for other G.*
- *This correspondence is also related to the non-constant coefficient differential operator $r\frac{\partial}{\partial r}$, which will be discussed further in Chapter 6.*

Example 5.2.4.1. If $n = 1$, then $\Pi(1) = \{1, -1\}$, so 5.2.17 is equal to

$$\frac{e^{iH} - e^{-iH}}{i2H} = \frac{\sin(H)}{H}. \tag{5.2.23}$$

If $n = 2$, then $\Pi(2) = \{2, 0, -2\}$, so 5.2.17 is equal to

$$\frac{1}{2H^2} - \frac{e^{i2H} + e^{-i2H}}{4H^2} = \left(\frac{\sin(H)}{H}\right)^2. \tag{5.2.24}$$

If $n = 3$, then $\Pi(3) = \{3, 1, -1, -3\}$, so 5.2.17 is equal to

$$-\frac{i3(e^{iH} - e^{-iH})}{8H^3} + \frac{i(e^{i3H} - e^{-i3H})}{8H^3} = \left(\frac{\sin(H)}{H}\right)^3. \tag{5.2.25}$$

If $n = 4$, then $\Pi(3) = \{4, 2, 0, -2, -4\}$, so 5.2.17 is equal to

$$\frac{3}{8H^4} - \frac{e^{i2H} + e^{-i2H}}{4H^4} + \frac{e^{i4H} + e^{-i4H}}{16H^4} = \left(\frac{\sin(H)}{H}\right)^4. \tag{5.2.26}$$

If $n = 5$, then $\Pi(4) = \{5, 3, 1, -1, -3, -5\}$, so 5.2.17 is equal to

$$-\frac{i5(e^{iH} - e^{-iH})}{16H^5} + \frac{i5(e^{i3H} - e^{-i3H})}{32H^5} - \frac{i(e^{i5H} - e^{-i5H})}{32H^5} = \left(\frac{\sin(H)}{H}\right)^5, \tag{5.2.27}$$

etc.

Proposition 5.2.5. *Let G be a compact simply connected semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$, T be a maximal torus of G, $\mathfrak{t}$ be the Lie algebra (Cartan subalgebra) of T, $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$, and W be the Weyl group of $\mathfrak{t}, \mathfrak{t}^*$. Let $\lambda \in \Lambda^+ \subset \mathfrak{t}^*$ be a dominant highest weight. Let π_λ be a finite-dimensional irreducible unitary representation of G of highest weight λ acting on a Hilbert space $\mathcal{H}_\lambda$. Let $\Pi(\lambda)$ be the set of weights of the irreducible unitary representation π_λ, d_λ be the dimension of π_λ. Let $m_\lambda(\mu)$ be the multiplicity of the weight $\mu \in \Pi(\lambda)$. Let $D = \Pi(\lambda) \cap \Lambda^+$ be the subset of dominant weights in $\Pi(\lambda)$ with cardinality $|D| = q$. Let I_λ be the moment set of π_λ, and $p : \mathfrak{g}^* \to \mathfrak{t}^*$ be the orthogonal projection map. Then, the projection of the G-invariant probability measure of the moment set I_λ on $\mathfrak{t}^*$ (pushforward of the unitarily probability* *measure of the unit sphere of $\mathcal{H}_\lambda$), denoted by $\nu^p_{I_\lambda}$, is given by*

$$\nu^p_{I_\lambda} = (-1)^{d_\lambda - 1}(d_\lambda - 1)! \sum_{k=1}^{q} \sum_{w \in W/W_{\lambda_k}} m_\lambda(\lambda_k) e_{w\lambda_k} * \prod_{\substack{j=1 \\ j \neq k}}^{q} \prod_{w' \in W/W_{\lambda_j}} * \left(F_{w'\lambda_j - w\lambda_k}\right)^{*\, m(\lambda_j)} \tag{5.2.28}$$

where each dominant weight λ_k, λ_j lie in D, and e_{λ_k} is the unit point mass at λ_k, F_{λ_k} is the arc-length measure along the ray of λ_k in $\mathfrak{t}^*$, and $w'\lambda_j - w\lambda_k = \sum_{i=1}^{l} t_i \alpha_i$, for $\alpha_i \in \Delta$, $t_i \in \mathbb{Z}$, and W_{λ_k} is the subgroup of W that stabilises λ_k.

Proof. By Theorem 4.1.13, Theorem 4.3.4 and Definition 4.1.9, we see that for every root $\alpha \in \Phi$,

$$\int_{\mathfrak{t}} \frac{1}{i\alpha(H)}\, e^{-i\eta(H)}\, dH = F_\alpha \tag{5.2.29}$$

in the sense of distributions, where $1/i\alpha(H)$ is a principal value distribution and F_α is the arc-length measure (Definition 4.1.9) on the ray along the root vector α. We may extend this identity to an arbitrary linear combination of root vectors. Because every weight in $\Pi(\lambda)$ can be written as $\lambda - \sum_{j=1}^{l} k_j \alpha_j$, $\alpha_j \in \Delta$, $k_j \in \mathbb{Z}^+$ ([30], Theorem 20.2), the difference of any pair of weights is the integral linear combination of root vectors in Φ. Hence, if we let $\alpha'_j \in \Phi, k'_j \in \mathbb{Z}$, then we have

$$\int_{\mathfrak{t}} \frac{1}{i\sum_{j=1}^{l} k'_j \alpha'_j(H)}\, e^{-i\eta(H)}\, dH = F_{\sum_{j=1}^{l} k'_j \alpha'_j}. \tag{5.2.30}$$

Suppose π_λ contains repeated weights and that $\lambda'' \in \mathfrak{t}^*$ lies in neighborhood of a weight $\lambda' \in \Lambda$ such that $\lambda' - \lambda'' = \varepsilon \sum_{j=1}^{l} k_j \alpha_j$, $\alpha_j \in \Delta$, $\varepsilon > 0$, for arbitrary choice of integer k_j. Let $f \in \mathcal{S}(\mathfrak{t}^*)$ be a Schwartz function, then we have

$$\lim_{\varepsilon \to 0} \int_{\mathfrak{t}} \frac{f^{\vee\mathfrak{t}}(H)}{i\varepsilon \sum_{j=1}^{l} k_j \alpha_j(H)}\, e^{-i\eta(H)}\, dH = \lim_{\varepsilon \to 0} F_{\varepsilon \sum_{j=1}^{l} k_j \alpha_j} * f = e_0 * f. \tag{5.2.31}$$

in the sense of distributions, where e_0 is the unit point mass at the origin of $\mathfrak{t}^*$. This also explains the continuity of $\tilde{\Psi}_n$ in (5.2.13) with repeated weights (Proposition 5.2.2).

Let W be the Weyl group of $\mathfrak{t}^*$, we have $\Pi(\lambda) = W \cdot D$. Let $m_\lambda(\lambda_k)$ be the multiplicity of the dominant weight λ_k. The continuous function $\Psi_n(\lambda_1, \ldots \lambda_n)$ in (5.2.10) contains $m_\lambda(\lambda_k)$ copies of exponential function of $\lambda_k \in D$, and $m_\lambda(\lambda_k)$ copies of exponential function of $w \cdot \lambda_k$, for each $w \in W$. Because we have $m(\lambda_k) = m(w \cdot \lambda_k)$ by ([30], Theorem 21.2). Also, by the identity (5.2.31), the denominator of every term in $\Psi_n(\lambda_1, \ldots \lambda_n)$ is non-zero.

Hence, the proposition follows by considering the multiplicities, Weyl orbits and singularities of the dominant weights in D.

$\square$

Example 5.2.5.1. Let $G = SU(3)$, and $\mathfrak{g} = \mathfrak{su}(3)$ be its Lie algebra. The Cartan subalgebra $\mathfrak{t} \cong \mathbb{R}^2$ (and its dual $\mathfrak{t}^* \cong \mathbb{R}^2$). Let $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ be the set of positive roots of $\mathfrak{g}$. Let W be the Weyl group (5.1.2) of $\mathfrak{t}, \mathfrak{t}^*$. Also, let $W_{\lambda_1} = \{e, \sigma_{\alpha_2}\}$ be the stabiliser subgroup. Let $\lambda \in \Lambda^+$ be a dominant weight. If $\lambda = \lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$, the first fundamental weight of $\mathfrak{t}^*$ (in Figure 5.2), then the Weyl orbit of λ_1 is the set

$$W \cdot \lambda_1 = \{\lambda_1, \lambda_1 - \alpha_1, \lambda_1 - \alpha_1 - \alpha_2\} \cong W/W_{\lambda_1} = \{e, \sigma_{\alpha_1}, \sigma_2\sigma_1\}. \tag{5.2.32}$$

Let π_{λ_1} be the irreducible unitary representation of λ_1, $\Pi(\lambda_1)$ be the set of weights of π_{λ_1} which coincides with $W \cdot \lambda_1$, and I_{λ_1} be the moment set of π_{λ_1}. Hence, by (5.2.5), we have

$$\begin{aligned} \left(\nu^p_{I_{\lambda_1}}\right)^{\vee\mathfrak{t}}(H) &= (-1)^2(2)! \sum_{k=1}^{3} \frac{e^{i\lambda_k(H)}}{\prod_{\substack{j\neq k \\ 1\leqslant j\leqslant 3}} i(\lambda_j - \lambda_k)(H)} \\ &= 2\frac{e^{i\lambda_1(H)}}{i^2(-\alpha_1)(H)(-\alpha_1-\alpha_2)(H)} + 2\frac{e^{i(\lambda_1-\alpha_1)(H)}}{i^2(\alpha_1)(H)(-\alpha_2)(H)} \\ &+ 2\frac{e^{i(\lambda_1-\alpha_1-\alpha_2)(H)}}{i^2(\alpha_1+\alpha_2)(H)(\alpha_2)(H)}, \end{aligned} \tag{5.2.33}$$

for $H \in \mathfrak{t}$. Since π_{λ_1} is also the defining representation of $SU(3)$, so I_{λ_1} is exactly the single coadjoint orbit $\mathcal{O}_{\lambda_1}$. This means if we let $\nu^p_{\lambda_1}$ be the projection of the G-invariant probability measure of $\mathcal{O}_{\lambda_1}$, then $\left(\nu^p_{I_{\lambda_1}}\right)^{\vee\mathfrak{t}}(H) = \left(\nu^p_{\lambda_1}\right)^{\vee\mathfrak{t}}(H)$.

Recall that by Theorem 4.3.4, if λ lies on a wall of $\mathfrak{t}^*_+$, then

$$\nu_\lambda^{\vee\mathfrak{g}}|_{\mathfrak{t}}(H) = C_\lambda \sum_{j=1}^{m} \frac{\mathrm{sgn}(w_j)e^{iw_j\lambda(H)}}{sgn(\tau_{w_j})\prod_{\alpha\in\Phi^+\backslash\Phi^+_{w_j\lambda}} i\alpha(H)}, \qquad \text{for } H \in \mathfrak{t}. \tag{5.2.34}$$

where

$$C_\lambda = \frac{\prod_{\alpha\in\Phi^+}(\delta,\alpha)}{\prod_{\alpha\in\Phi^+\smallsetminus\Phi^+_\lambda}(\lambda,\alpha)\prod_{\alpha\in\Phi^+_\lambda}(\delta_\lambda,\alpha)}, \tag{5.2.35}$$

and $\mathrm{sgn}(\tau_{w_j}) = (-1)^t$, and t is the number $\left|\{\beta \in \Phi^+_\lambda \mid w_j\beta \notin \Phi^+\}\right|$. Also, $\Phi^+_\lambda = \{\alpha \in \Phi^+ \mid (\lambda, \alpha) = 0\}$, $\delta_\lambda = \frac{1}{2}\sum_{\alpha\in\Phi^+_\lambda}\alpha$, and $W/W_\lambda = \{w_j\}_{j=1}^m$.

If $\lambda = \lambda_1$, which lies on the wall $[0, \infty) \cdot \lambda_1$ of $\mathfrak{t}^*_+$, then

$$C_{\lambda_1} = \frac{(\alpha_1 + \alpha_2, \alpha_1)(\alpha_1 + \alpha_2, \alpha_1)(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2)}{(\lambda_1, \alpha)(\lambda_1, \alpha_1 + \alpha_2)(\frac{1}{2}\alpha_2, \alpha_2)} = 2, \tag{5.2.36}$$

where we let $(\alpha_j, \alpha_j) = 2, (\alpha_j, \alpha_i) = -1$, for $i \neq j$ and $i, j \in \{1, 2\}$, and $(\lambda_1, \alpha_1) = 1, (\lambda_1, \alpha_2) = 0$. Also, $sgn(\tau_{w_j}) = 1$ for all $w_j \in W/W_{\lambda_1}$. In addition, $\Phi^+_{e\cdot\lambda_1} = \{\alpha_2\}$, $\Phi^+_{\sigma_{\alpha_1}\cdot\lambda_1} = \{\alpha_1 + \alpha_2\}$ and $\Phi^+_{\sigma_2\sigma_{\alpha_1}\cdot\lambda_1} = \{\alpha_1\}$. Then, we have

$$\begin{aligned} \nu^{\vee_{\mathfrak{g}}}_{\lambda_1}|_{\mathfrak{t}}(H) = {} & 2\frac{e^{i\lambda_1(H)}}{i^2\alpha_1(H)(\alpha_1 + \alpha_2)(H)} + 2\frac{-e^{i(\lambda_1 - \alpha_1)(H)}}{i^2\alpha_1(H)\alpha_2(H)} \\ & + 2\frac{e^{i(\lambda_1 - \alpha_1 - \alpha_2)(H)}}{i^2\alpha_2(H)(\alpha_1 + \alpha_2)(H)}, \end{aligned} \tag{5.2.37}$$

which is exactly the same as $\left(\nu^p_{I_{\lambda_1}}\right)^{\vee_{\mathfrak{t}}}(H)$ that we calculated above.

Next, we determine the density function $\varphi(\lambda, \cdot)$, supported on $I_\lambda \cap \mathfrak{t}^*_+$.

Definition 5.2.6. Let $\lambda \in \Lambda^+ \subset \mathfrak{t}^*_+$ be a dominant highest weight, π_λ be the irreducible highest weight unitary representation of λ, and I_λ be the moment set of π_λ. By Proposition 5.2.1, the G-invariant probability measure of I_λ is defined by

$$\nu_{I_\lambda} = \int_{\mathfrak{t}^*_+} \varphi(\lambda, \lambda'')\nu_{\lambda''}\, d\lambda'', \tag{5.2.38}$$

where $\nu_{\lambda''}$ is the probability G-invariant measure on the coadjoint orbit $\mathcal{O}_{\lambda''}$, and $\varphi(\lambda, \lambda'')$ is a density function supported on $I_\lambda \cap \mathfrak{t}^*_+$, which relies on the choice of λ. Let $p : \mathfrak{g}^* \to \mathfrak{t}^*$ be the orthogonal projection map. The projection of ν_{I_λ}, denoted by $\nu^p_{I_\lambda}$, is defined by

$$\nu^p_{I_\lambda} = \int_{\mathfrak{t}^*_+} \varphi(\lambda, \lambda'')\nu^p_{\lambda''}\, d\lambda''. \tag{5.2.39}$$

Proposition 5.2.7. *Let $\mathfrak{t}{*}_0$ be the interior of the fundamental dual Weyl chamber $\mathfrak{t}^*_+$. Suppose the intersection $I_\lambda \cap \mathfrak{t}^*_0$ is non-empty. Then the density function*

$$\varphi(\lambda, \eta) = C^{-1}_\eta \left(\prod_{\alpha\in\Phi^+} \partial_\alpha\right) \nu^p_{I_\lambda}(\eta), \qquad \eta \in \mathfrak{t}^*_0, \tag{5.2.40}$$

where ∂_α is the directional derivative of the root vector α, and

$$C_\eta = \prod_{\alpha\in\Phi^+}(\delta,\alpha)\Big/\prod_{\alpha\in\Phi^+}(\eta,\alpha). \tag{5.2.41}$$

Proof. By the normalisation formula (4.1.9), let $\lambda \in \mathfrak{t}_0^*$,

$$\nu_\lambda^{\breve{\mathfrak{g}}}(H) = C_\lambda \frac{R(\lambda)(H)}{\prod_{\alpha\in\Phi^+} i\alpha(H)}, \qquad \forall H \in \mathfrak{t} \tag{5.2.42}$$

where $R(\lambda)(H) = \sum_{w\in W}\operatorname{sgn}(w)e^{iw\lambda(H)}$, and C_λ^{-1} is the continuous version of Weyl dimension formula in (4.1.9). Taking the $\mathfrak{t}$-Fourier transform of both sides of (5.2.39), we obtain

$$(\nu_{I_\lambda}^p)^{\vee_\mathfrak{t}}(H) = \int_{\mathfrak{t}_+^*}\varphi(\lambda,\lambda'')C_{\lambda''}\frac{R(\lambda'')(H)}{\prod_{\alpha\in\Phi^+} i\alpha(H)}d\lambda'', \tag{5.2.43}$$

where the Lebesgue measures on the walls of $\mathfrak{t}^*$ are zero. Also, if we multiply both sides of (5.2.43) by $\overline{R(\eta)(H)}$ and integrate over $\mathfrak{t}$, then by Lemma 4.2.1, we obtain

$$\begin{aligned}\varphi(\lambda,\eta) &= \frac{C_\eta^{-1}}{|W|}\sum_{w\in W}\operatorname{sgn}(w)\int_\mathfrak{t}\left(\prod_{\alpha\in\Phi^+} i\alpha(H)\right)\left(\nu_{I_\lambda}^p\right)^{\vee_\mathfrak{t}}(H)\,e^{-iw\eta(H)}\,dH \\ &= C_\eta^{-1}\int_\mathfrak{t}\left(\prod_{\alpha\in\Phi^+} i\alpha(H)\right)\left(\nu_{I_\lambda}^p\right)^{\vee_\mathfrak{t}}(H)\,e^{-i\eta(H)}\,dH\,.\end{aligned} \tag{5.2.44}$$

The last equality is obtained by the anti-symmetric properties of the product of all positive roots, that is

$$\prod_{\alpha\in\Phi^+} w^{-1}\alpha(H) = \operatorname{sgn}(w)\prod_{\alpha\in\Phi^+}\alpha(H) \tag{5.2.45}$$

by Lemma 4.2.2. Hence, the result follows from the fact that the Fourier transform of linear functionals are directional derivatives (Lemma 4.3.1). ☐

Example 5.2.7.1. Let $G = SU(3)$, and its Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$. The Cartan subalgebra $\mathfrak{t} \cong \mathbb{R}^2$ (and its dual $\mathfrak{t}^* \cong \mathbb{R}^2$). Let $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$ be the set of positive roots of $\mathfrak{g}$. Let W be the Weyl group (5.1.2) of $\mathfrak{t}, \mathfrak{t}^*$. Let $\delta = \alpha_1+\alpha_2 \in \Lambda^+$ be the half-sum of positive roots. Let π_δ be the irreducible unitary representation of δ, the set of the weights $\Pi(\delta) = \{\delta, \alpha_1, \alpha_2, 0, -\alpha_2, -\alpha_1, -\delta\}$ (only the '*zero*' weight has multiplicity 2, and others have multiplicity 1) and

I_δ be the moment set of π_δ. From Example 5.1.9.1 and Figure 5.1, we can see that $I_\delta \cap \mathfrak{t}^*_+$ is non-empty and is contained in the closure of $\mathfrak{t}^*_+$, which is spanned by the rays along the vectors λ_1 and λ_2 (walls of $\mathfrak{t}^*_+$), and $I_\delta \cap \mathfrak{t}^*_+$ is also not fully contained in any walls of $\mathfrak{t}^*_+$.

By equation (5.2.28), we have

$$\begin{aligned}
\nu^p_{I_\delta} &= e_\delta * F_{-\alpha_2} * F_{-\alpha_1} * (F_{-\delta})^{*2} * F_{-\alpha_2-\delta} * F_{-\alpha_1-\delta} * F_{-2\delta} \\
&+ e_{\alpha_1} * F_{\alpha_2} * F_{\alpha_2-\alpha_1} * (F_{-\alpha_1})^{*2} * F_{-\delta} * F_{-2\alpha_1} * F_{-\delta-\alpha_1} \\
&+ e_{\alpha_2} * F_{\alpha_1} * F_{\alpha_1-\alpha_2} * (F_{-\alpha_2})^{*2} * F_{-2\alpha_2} * F_{-\delta} * F_{-\delta-\alpha_2} \\
&+ 2e_0 * F_\delta * F_{\alpha_1} * F_{\alpha_2} * F_{-\alpha_2} * F_{-\alpha_1} * F_{-\delta} \\
&+ e_{-\alpha_2} * F_{\delta+\alpha_2} * F_\delta * F_{2\alpha_2} * (F_{\alpha_2})^{*2} * F_{-\alpha_1+\alpha_2} * F_{-\alpha_1} \\
&+ e_{-\alpha_1} * F_{\delta+\alpha_1} * F_{2\alpha_1} * F_\delta * (F_{\alpha_1})^{*2} * F_{-\alpha_2+\alpha_1} * F_{-\alpha_2} \\
&+ e_{-\delta} * F_{\alpha_1+\delta} * F_{\alpha_2+\delta} * (F_\delta)^{*2} * F_{\alpha_1} * F_{\alpha_2} * F_{2\delta},
\end{aligned} \tag{5.2.46}$$

where e_λ is a point mass at $\lambda \in \mathfrak{t}^*$, and F_λ is a uniform measure along the ray of the vector $\lambda \in \mathfrak{t}^*$. Thus, by equation (5.2.40), a density function $\varphi(\delta, \cdot)$ (we omit the constant coefficient) of the $I_\delta \cap \mathfrak{t}^*_+$ is given by

$$\begin{aligned}
\varphi(\lambda, \eta) &= \left(\prod_{\alpha \in \Phi^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}} \partial_\alpha \right) \nu^p_{I_\lambda}(\eta) \\
&= e_\delta * F_{-\delta} * F_{-\alpha_2-\delta} * F_{-\alpha_1-\delta} * F_{-2\delta} \\
&+ e_{\alpha_1} * F_{\alpha_2-\alpha_1} * F_{-\alpha_1} * F_{-2\alpha_1} * F_{-\delta-\alpha_1} \\
&+ e_{\alpha_2} * F_{\alpha_1-\alpha_2} * F_{-\alpha_2} * F_{-2\alpha_2} * F_{-\delta-\alpha_2} \\
&+ 2e_0 * F_{-\alpha_2} * F_{-\alpha_1} * F_{-\delta} \\
&+ e_{-\alpha_2} * F_{\delta+\alpha_2} * F_{2\alpha_2} * F_{\alpha_2} * F_{-\alpha_1+\alpha_2} \\
&+ e_{-\alpha_1} * F_{\delta+\alpha_1} * F_{2\alpha_1} * F_{\alpha_1} * F_{-\alpha_2+\alpha_1} \\
&+ e_{-\delta} * F_{\alpha_1+\delta} * F_{\alpha_2+\delta} * F_\delta * F_{2\delta},
\end{aligned} \tag{5.2.47}$$

for $\eta \in \mathfrak{t}^*_0$. Hence, we approximate and plot the density function $\varphi(\delta, \cdot)$ in Figure 5.3 and 5.4, and they show the density function is a linear piecewise polynomial with the highest density concentrated in the origin of $\mathfrak{t}^*$.

Also, the support of $\varphi(\delta, \cdot)$ coincides with the 2-dimensional non-convex region (a subset of $I_\delta \cap \mathfrak{t}^*$) in Figure 5.1. So, that non-convex region is not only a subset, but also the exact set for $I_\delta \cap \mathfrak{t}^*$.

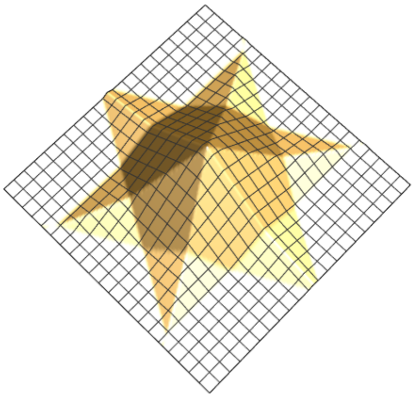

Figure 5.3: $I_\delta \cap \mathfrak{t}^*$ of A_2 (Top)

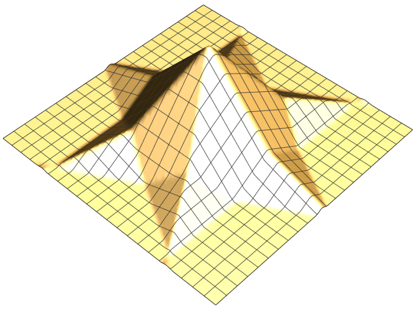

Figure 5.4: $I_\delta \cap \mathfrak{t}^*$ of A_2 (Side)

When $\lambda \in \Lambda^+$ lies on a 'wall or (edge of a wall)' of $\mathfrak{t}^*_+$, depending on the semisimple Lie algebra $\mathfrak{g}$, $I_\lambda \cap \mathfrak{t}^*_+$ could be fully contained in that wall. For example, every first fundamental weight λ_1 of the defining representation of $SU(n)$ lies on a minimal wall of $\mathfrak{t}^*_+$, and $I_{\lambda_1} \cap \mathfrak{t}^*_+$ is just the singular point λ_1, which is fully contained in a minimal wall (along the vector of λ_1). Hence, Proposition 5.2.7 no longer applies to these cases. Therefore, in the following part, we derive a new density function for $I_{\lambda_1} \cap \mathfrak{t}^*_+$ that is fully contained in a minimal wall.

Proposition 5.2.8. *Suppose $I_\lambda \cap \mathfrak{t}^*_+$ is fully contained in the closure of a minimal wall $\mathfrak{t}^*_\Xi$ of $\mathfrak{t}^*_+$, and let $\mathfrak{t}^*_{\Xi_0}$ denote the interior of $\mathfrak{t}^*_\Xi$, then the density function*

$$\varphi(\lambda, \eta) = C_\eta^{-1} \left(\prod_{\alpha \in \Phi^+ \setminus \Phi^+_{\Xi_0}} \partial_\alpha \right) \nu^p_{I_\lambda}(\eta), \qquad \eta \in \mathfrak{t}^*_{\Xi_0} \tag{5.2.48}$$

where $\Phi^+_{\Xi_0} = \{\alpha \in \Phi^+ : (\lambda, \alpha) = 0, \forall \lambda \in \mathfrak{t}^*_{\Xi_0}\}$, and

$$C_\eta = \frac{\prod_{\alpha \in \Phi^+} (\delta, \alpha)}{\prod_{\alpha \in \Phi^+ \setminus \Phi^+_{\Xi_0}} (\eta, \alpha) \cdot \prod_{\alpha \in \Phi^+_{\Xi_0}} (\delta_\eta, \alpha)}, \tag{5.2.49}$$

where $\delta_\eta = \frac{1}{2} \sum_{\alpha \in \Phi^+_{\Xi_0}} \alpha$.

Proof. Suppose that λ'' lies in the interior of a minimal wall $\mathfrak{t}^*_{\Xi_0}$, and define a subgroup of W, $W_{\Xi_0} = \{w \in W : w\lambda'' = \lambda'', \forall \lambda'' \in \Xi_0\}$, then W/W_{Ξ_0} is isomorphic to the Weyl orbit of λ''. Let $W/W_{\Xi_0} = \{w_j\}_{j=1}^m$. By Theorem 4.3.4, the measure $\nu^p_{\lambda''}$ is given by

$$\nu^p_{\lambda''} = C_{\lambda''} \sum_{j=1}^{m} \operatorname{sgn}(w_j) e_{w_j\lambda''} * \left(\prod_{\alpha \in \Phi^+_{\lambda''}} \partial_{w_j\alpha} \right) P\,, \tag{5.2.50}$$

where

$$\left(\prod_{\alpha \in \Phi^+_{\lambda''}} \partial_{w_j\alpha} \right) P = \operatorname{sgn}(\tau_{w_j}) \prod_{\alpha \in \Phi^+ \setminus \Phi^+_{w_j\lambda''}} * F_\alpha\,, \tag{5.2.51}$$

$\operatorname{sgn}(\tau_{w_j}) = (-1)^t$, and t is the number $\left|\{\beta \in \Phi^+_{\Xi_0} \mid w_j\beta \notin \Phi^+\}\right|$. Hence, the normalisation $\nu^{\vee\mathfrak{g}}_{\lambda''}|_{\mathfrak{t}}$ can be written as

$$\nu^{\vee\mathfrak{g}}_{\lambda''}|_{\mathfrak{t}}(H) = C_{\lambda''} \sum_{j=1}^{m} \frac{\operatorname{sgn}(w_j)}{\operatorname{sgn}(\tau_{w_j})} \frac{e^{iw_j\lambda''(H)}}{\prod_{\alpha \in \Phi^+ \setminus \Phi^+_{w_j\lambda''}} i\alpha(H)}, \qquad \forall H \in \mathfrak{t}. \tag{5.2.52}$$

Let $\mathfrak{t}^*_\Xi$ be a minimal wall of $\mathfrak{t}^*_+$ containing the support of $\varphi(\lambda, \cdot)$, $\mathfrak{t}^*_{\Xi_0}$ the interior of $\mathfrak{t}^*_\Xi$, so that the $\mathfrak{t}$-Fourier transform of both sides of (5.2.39) becomes

$$\left(\nu^p_{I_\lambda}\right)^{\vee\mathfrak{t}}(H) = \int_{\mathfrak{t}^*_\Xi} \varphi(\lambda, \lambda'')\, C_{\lambda''} \sum_{j=1}^{m} \frac{\operatorname{sgn}(w_j)}{\operatorname{sgn}(\tau_{w_j})} \frac{e^{iw_j\lambda''(H)}}{\prod_{\alpha \in \Phi^+ \setminus \Phi^+_{w_j\Xi_0}} i\alpha(H)}\, d\lambda'', \tag{5.2.53}$$

and rearrange and it becomes

$$\begin{aligned} &\left(\prod_{\alpha \in \Phi^+} i\alpha(H) \right) \left(\nu^p_{I_\lambda}\right)^{\vee\mathfrak{t}}(H) \\ &\qquad = \int_{\mathfrak{t}^*_\Xi} \varphi(\lambda, \lambda'')\, C_{\lambda''} \sum_{j=1}^{m} \frac{\operatorname{sgn}(w_j)}{\operatorname{sgn}(\tau_{w_j})} \left(\prod_{\beta \in \Phi^+_{w_j\Xi_0}} i\beta(H) \right) e^{iw_j\lambda''(H)}\, d\lambda''. \end{aligned} \tag{5.2.54}$$

We define a new function $\overline{R(\eta)(H)}$,

$$\overline{R(\eta)(H)} = \sum_{j=1}^{m} \frac{\operatorname{sgn}(w_j)}{\operatorname{sgn}(\tau_{w_j})} \frac{e^{-iw_j\eta(H)}}{\prod_{\beta \in \Phi^+_{w_j\Xi_0}} i\beta(H)}, \qquad \eta \in \mathfrak{t}^*_{\Xi_0}, \quad \forall H \in \mathfrak{t} \tag{5.2.55}$$

and in fact this function is an anti-symmetric function on $\mathfrak{t}$. Also, for $\alpha \neq \beta$, $f \in L^1(\mathfrak{t}^*_\Xi)$,

$$\partial_\alpha f * F_\beta(\eta) = \int_{\mathfrak{t}} \int_{\mathfrak{t}^*_\Xi} \frac{i\alpha(H)}{i\beta(H)} f(\lambda'')\, e^{i(\lambda'' - \eta)(H)}\, d\lambda''\, dH, \qquad \eta \in \mathfrak{t}^*_\Xi \tag{5.2.56}$$

Multiplying both sides of (5.2.54) by $\overline{R(\eta)(H)}$, and integrating over $\mathfrak{t}$, we obtain

$$\varphi(\lambda,\eta) = \frac{C_\eta^{-1}}{|W/W_{\Xi_0}|} \sum_{j=1}^{m} \frac{\operatorname{sgn}(w_j)}{\operatorname{sgn}(\tau_{w_j})} \int_{\mathfrak{t}} \left(\prod_{\alpha\in\Phi^+\setminus\Phi^+_{w_j\Xi_0}} i\alpha(H) \right) \left(\nu^p_{I_\lambda}\right)^{\vee_{\mathfrak{t}}}(H)\, e^{-iw_j\eta(H)}\, dH. \tag{5.2.57}$$

Also, notice that

$$\frac{\prod_{\alpha\in\Phi^+} iw_j^{-1}\alpha(H)}{\prod_{\beta\in\Phi^+_{w_j\Xi_0}} iw_j^{-1}\beta(H)} = \frac{\operatorname{sgn}(w_j^{-1})}{\operatorname{sgn}(\tau_{w_j^{-1}})} \frac{\prod_{\alpha\in\Phi^+} i\alpha(H)}{\prod_{\beta\in\Phi^{\pm}_{\Xi_0}} i\beta(H)}. \tag{5.2.58}$$

Now because $(w_j\eta,\beta) = (w_j^{-1}w_j\eta, w_j^{-1}\beta) = (\eta, w_j^{-1}\beta) = 0$, for $\beta \in \Phi^+_{w_j\Xi_0}$ and for all $\eta \in \mathfrak{t}^*_{\Xi_0}$, it follows that $w_j^{-1}\beta$ belongs to $\Phi^{\pm}_{\Xi_0}$. Therefore, equation (5.2.57) becomes

$$\varphi(\lambda,\eta) = C_\eta^{-1} \int_{\mathfrak{t}} \left(\prod_{\alpha\in\Phi^+\setminus\Phi^{\pm}_{\Xi_0}} i\alpha(H) \right) \left(\nu^p_{I_\lambda}\right)^{\vee_{\mathfrak{t}}}(H)\, e^{-iw_j\eta(H)}\, dH \tag{5.2.59}$$

Hence, the result follows. ☐

Remark. *We can use the Proposition 5.1.9 to determine a subset of* $I_\lambda \cap \mathfrak{t}^*$ *by identifying a set* $S(\lambda)$ *(Proposition 5.1.9) which contains the highest weight, so that we can determine whether or not it is fully contained in a wall of the* $\mathfrak{t}^*_+$*. Then we can choose the density functions of the* G*-invariant measures developed, (5.2.40) and (5.2.48), accordingly. The resulting measures calculated are* G*-invariant piece-wise polynomials with support contained in the convex hull of* $W\cdot\lambda$*. If the set* $S(\lambda)$ *is fully contained in the closure of a minimal wall, then we can compare the support of the version of the* G*-invariant measure of the wall with the* $S(\lambda)$ *to check the correspondence. Therefore, it is not straightforward to determine the density function for this case comparing to the root distinct highest weights. But our method provides a concrete way to calculate the density functions, thus the supports and singular supports of moment sets, especially when the moment set* I_λ *is not convex.*

Example 5.2.8.1. Let $G = SU(3)$, and its Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$. The Cartan subalgebra $\mathfrak{t} \cong \mathbb{R}^2$ (and its dual $\mathfrak{t}^* \cong \mathbb{R}^2$). Let $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$ be the set of positive roots of $\mathfrak{g}$. Let W be the Weyl group (5.1.2) of $\mathfrak{t}, \mathfrak{t}^*$. Let

$\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2 \in \Lambda^+$ be the first fundamental weight of $\mathfrak{t}^*$, and the Weyl orbit of λ_1 is the set $W \cdot \lambda_1 = \{\lambda_1, \lambda_1 - \alpha_1, \lambda_1 - \alpha_1 - \alpha_2\}$. Let π_{λ_1} be the irreducible unitary representation of λ_{λ_1}, which is also the defining representation of $SU(3)$, hence the set of weights $\Pi(\lambda_1) = W \cdot \lambda_1$. Each weight in $\Pi(\lambda_1)$ has multiplicity 1. By Example 5.1.2.2 , λ_1 is not root distinct, so the moment set I_{λ_1} is not convex, and I_{λ_1} is exactly the single coadjoint orbit $\mathcal{O}_{\lambda_1}$. The intersection $I_{\lambda_1} \cap \mathfrak{t}^*_+$ is $\{\lambda_1\}$, which is fully contained in the wall $[0, \infty) \cdot \lambda_1$ of $\mathfrak{t}^*_+$. Hence, the density function of $I_{\lambda_1} \cap \mathfrak{t}^*_+$ is

$$d\varphi(\lambda_1, \eta) = e_{\lambda_1}(\eta) d\eta, \quad \eta \in \mathfrak{t}^*_+, \tag{5.2.60}$$

where e_{λ_1} is the unit point mass at λ_1. To see this, notice that the Equation 5.2.33 that we calculated above, we have

$$\begin{aligned} \nu^p_{I_{\lambda_1}} &= e_{\lambda_1} * F_{-\alpha_1} * F_{\alpha_1+\alpha_2} \\ &\quad + e_{\lambda_1-\alpha_1} * F_{\alpha_1} * F_{-\alpha_2} \\ &\quad + e_{\lambda_1-\alpha_1-\alpha_2} * F_{\alpha_1+\alpha_2} * F_{\alpha_2}, \end{aligned} \tag{5.2.61}$$

and by Equation (5.2.48), the density function on $I_{\lambda_1} \cap \mathfrak{t}^*_+$ is given by

$$\begin{aligned} \varphi(\lambda_1, \cdot) &= \left(\prod_{\alpha \in \Phi^+ \setminus \Phi^+_{\lambda_1} = \{\alpha_1, \alpha_1+\alpha_2\}} \partial_\alpha \right) \nu^p_{I_{\lambda_1}} \\ &= e_{\lambda_1} + \partial_{\alpha_1+\alpha_2}(e_{\lambda_1-\alpha_1} * F_{-\alpha_2}) + \partial_{\alpha_1}(e_{\lambda_1-\delta} * F_{\alpha_2}). \end{aligned} \tag{5.2.62}$$

(We have omitted the constant coefficients.) Notice that if we let $f \in C^\infty_c(\mathfrak{t}^*)$, $\phi_{\alpha_1}, \phi_{\alpha_2}$ be the angles between the positive side of the x-axis of $\mathfrak{t}^* \cong \mathbb{R}^2$ and vectors α_1 and α_2 (for $\alpha_1 \neq \alpha_2$), respectively, then we have

$$\begin{aligned}
&\langle \frac{\partial}{\partial\alpha_1}F_{\alpha_2}, f\rangle = -\int_{\mathfrak{t}^*} F_{\alpha_2}(H)\frac{\partial}{\partial\alpha_1} f(H)\,dH \\
&= -\int_{\alpha_2\uparrow} \frac{\partial}{\partial\alpha_1} f(H)\,dH \\
&= -\int_{\alpha_2\uparrow} \left(\cos(\phi_{\alpha_1})\frac{\partial}{\partial x} + \sin(\phi_{\alpha_1})\frac{\partial}{\partial y}\right) f(x,y)\,d(x,y) \\
&= -\int_0^\infty \left(\cos(\phi_{\alpha_1})\frac{\partial}{\partial t}\frac{\partial t}{\partial x} + \sin(\phi_{\alpha_1})\frac{\partial}{\partial t}\frac{\partial t}{\partial y}\right) f(t\cos(\phi_{\alpha_2}), t\sin(\phi_{\alpha_2}))\,dt \\
&= -\int_0^\infty \left(\frac{\cos(\phi_{\alpha_1})}{\cos(\phi_{\alpha_2})}\frac{\partial}{\partial t} + \frac{\sin(\phi_{\alpha_1})}{\sin(\phi_{\alpha_2})}\frac{\partial}{\partial t}\right) f(t\cos(\phi_{\alpha_2}), t\sin(\phi_{\alpha_2}))\,dt \\
&= -\left(\frac{\cos(\phi_{\alpha_1})}{\cos(\phi_{\alpha_2})} + \frac{\sin(\phi_{\alpha_1})}{\sin(\phi_{\alpha_2})}\right)\int_0^\infty \frac{\partial}{\partial t} f(t\alpha_2)\,dt \\
&= \left(\frac{\cos(\phi_{\alpha_1})}{\cos(\phi_{\alpha_2})} + \frac{\sin(\phi_{\alpha_1})}{\sin(\phi_{\alpha_2})}\right) f(0) \\
&= C(\phi_{\alpha_1}, \phi_{\alpha_2})e_0 * f.
\end{aligned} \tag{5.2.63}$$

Hence, we conclude that $\varphi(\lambda_1, \cdot)|_{\mathfrak{t}^*_+} = e_{\lambda_1}$. So,

$$\nu_{I_{\lambda_1}} = \int_{\mathfrak{t}^*_+} e_{\lambda_1}(\lambda'')\nu_{\lambda''}\,d\lambda'' = \nu_{\lambda_1}, \tag{5.2.64}$$

which is exactly the probability G-invariant measure of the coadjoint orbit $\mathcal{O}_{\lambda_1}$.

Example 5.2.8.2. Let $G = Spin(5, \mathbb{R})$, a 5-dimensional real spin group, which is also the double cover group of $SO(5, \mathbb{R})$. They share the same Lie algebra $\mathfrak{g}$, which can be represented by the compact real form of B_2 (in the B_n family), and the Cartan subalgebra $\mathfrak{t} \cong \mathbb{R}^2$. The root system of B_2 has positive roots: $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$. The first fundamental weight λ_1 coincides with a positive root, which is $\lambda_1 = \alpha_1 + \alpha_2$. Let π_{λ_1} be the irreducible unitary representation of λ_1. The set of weights of π_{λ_1} is

$$\Pi(\lambda_1) = \{\lambda_1, \lambda_1 - \alpha_1, 0, \lambda_1 - \alpha_1 - 2\alpha_2, \lambda_1 - 2\alpha_1 - 2\alpha_2\}, \tag{5.2.65}$$

and each weight has multiplicity 1. Thus we have

$$\begin{aligned}\nu^p_{I_{\lambda_1}} &= e_{\lambda_1} * (F_{\alpha_1} * F_{\alpha_1+\alpha_2} * F_{\alpha_1+2\alpha_2} * F_{2\alpha_1+2\alpha_2})\\ &+ e_{\lambda_1-\alpha_1} * (F_{\alpha_1} * F_{\alpha_2} * F_{-2\alpha_2} * F_{\alpha_1+2\alpha_2})\\ &+ e_0 * (F_{\alpha_1+\alpha_2} * F_{\alpha_2} * F_{\alpha_2} * F_{\alpha_1+\alpha_2})\\ &+ e_{\lambda_1-\alpha_1-\alpha_2} * (F_{\alpha_1+2\alpha_2} * F_{-2\alpha_2} * F_{\alpha_2} * F_{\alpha_1})\\ &+ e_{\lambda_1-2\alpha_1-2\alpha_2} * (F_{2\alpha_1+2\alpha_2} * F_{\alpha_1+2\alpha_2} * F_{\alpha_1+\alpha_2} * F_{\alpha_1}).\end{aligned} \tag{5.2.66}$$

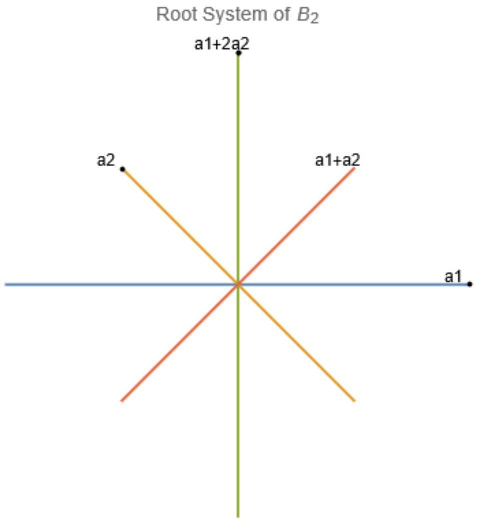

Figure 5.5: Root System of B_2

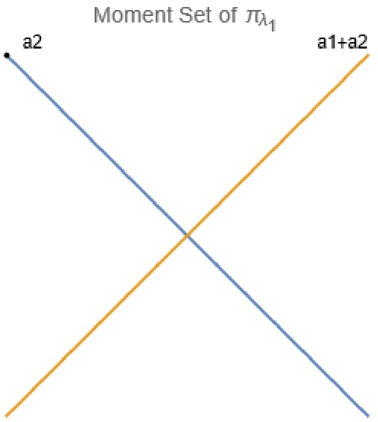

Figure 5.6: I_{λ_1} of B_2

Since λ_1 is not root distinct, the moment set I_{λ_1} is not a convex set. Let $S(\lambda_1)$ be the largest possible set of weights containing λ_1 such that the pairwise difference of any pair of weights in the set is not a root. We can see that $S(\lambda_1) = \{\lambda_1, \lambda_1 - 2\alpha_1 - 2\alpha_2\}$, because $2\alpha_1 + 2\alpha_2 \notin \Phi$. Hence the Weyl orbit $W \cdot S(\lambda_1)$ is also contained in $I_{\lambda_1} \cap \mathfrak{t}^*$, which is shown in Figure 5.6. Hence, we assume $I_\lambda \cap \mathfrak{t}^*_+$ is fully contained in the wall $[0, \infty) \cdot \lambda_1$. Thus we can apply the density function (5.2.48). We have $\Phi^+ \backslash \Phi^+_{\lambda_1} = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$, and we obtain the density function $\varphi(\lambda_1, \cdot)$ of $I_{\lambda_1} \cap \mathfrak{t}_+^*$

$$\begin{aligned}\varphi(\lambda_1, \cdot) &= \left(\prod_{\alpha \in \{\alpha_1, \alpha_1+\alpha_2, \alpha_1+2\alpha_2\}} \partial_\alpha \right) \nu_{I_{\lambda_1}}^p \\ &= e_{\lambda_1} * (F_{2\alpha_1+2\alpha_2}) \\ &+ C(\alpha_1+\alpha_2, -\alpha_2)\, e_{\lambda_1-\alpha_1} * (F_{\alpha_2}) \\ &+ C(\alpha_1, \alpha_2) \partial_{\alpha_1+2\alpha_2} e_0 \\ &+ C(\alpha_1+\alpha_2, -\alpha_2)\, e_{\lambda_1-\alpha_1} * (F_{\alpha_2}) \\ &+ e_{\lambda_1-2\alpha_1-2\alpha_2} * (F_{2\alpha_1+2\alpha_2})\end{aligned} \tag{5.2.67}$$

where $C(\alpha, \beta)$ is a constant coefficient determined by vectors α, β. Thus we obtain a distribution along the wall spanned by λ_1, and it is given by $d\varphi(\lambda_1, \eta) = F_{\lambda_1}(\lambda_1 - \eta) + F_{\lambda_1}(-\lambda_1 - \eta)\, d\eta$, for $\eta = t\lambda_1, t \in (0, \infty)$. This formula can be reduced to

$$d\varphi(\lambda_1, \eta) = F_{\lambda_1}(\lambda_1 - \eta)\, d\eta \tag{5.2.68}$$

for $\eta = t\lambda_1, t \in (0, \infty)$. This is an arc-length measure on $(0, 1) \cdot \lambda_1$. Its support coincides with the 'possible' moment set of I_{λ_1}, which is shown in Figure 5.6.

Could the moment set be bigger than this? In other words, is the intersection $I_\lambda \cap \mathfrak{t}_0^*$ non-empty? We can determine it by the density function (5.2.40), so we have

$$\varphi(\lambda_1, \cdot) = \left(\prod_{\alpha \in \Phi+} \partial_\alpha \right) \nu_{I_{\lambda_1}}^p. \tag{5.2.69}$$

Because we differentiate the projection $\nu_{I_{\lambda_1}}^p$ with respect to all the positive roots, we obtain a new distribution

$$\varphi(\lambda_1, \cdot) = C(\alpha, \beta) e_{\lambda_1}, \tag{5.2.70}$$

which is a point mass at e_{λ_1}, and λ_1 is fully contained in the wall $[0, \infty) \cdot \lambda_1$. This does not match our assumption that $I_\lambda \cap \mathfrak{t}_0^*$ is non-empty. Hence, it is a contradiction, and the set in Figure 5.6 is exactly the support of $I_{\lambda_1} \cap \mathfrak{t}_+^*$.

Chapter 6

A Non-Commutative Kirillov Method

We introduce a non-commutative Kirillov method by combining an updated version of Nelson's formula [40] for finite-dimensional non-commutative self-adjoint operators and a family of G-invariant measures on moment sets of irreducible unitary representations developed in Chapter 5 to explicitly calculate the exponential of skew-Hermitian irreducible highest weight representations of Lie algebras of compact simply connected semisimple Lie groups. We also demonstrate how the differential operators induced by the action of G interact with the lift of matrix coefficients that were studied in Chapter 3. Furthermore, we extend our method to non-skew-Hermitian irreducible highest weight representations.

In this chapter, we let G be a compact connected semisimple Lie group, $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$, $\mathfrak{t}, \mathfrak{t}^*$ be the (dual) Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{g}^*$, respectively, and W be the Weyl group of $\mathfrak{t}$, $\mathfrak{t}^*$. In addition, we let $\mathfrak{t}^*_+$ be the fundamental dual Weyl chamber of $\mathfrak{t}^*$, $\mathfrak{t}^*_0$ be the interior of $\mathfrak{t}^*_+$, Φ be the set of roots of $\mathfrak{g}$, Δ be a subset of simple roots of Φ, Φ^+ be the subset of positive roots of Φ, Λ be the set of the weights of $\mathfrak{t}^*$ and Λ^+ be the set of dominant weights of Λ in $\mathfrak{t}^*_+$. In addition, we let $p : \mathfrak{g}^* \to \mathfrak{t}^*$ be the projection map with respect to the Killing form of $\mathfrak{g}^*$.

In Section 6.1, we review the definitions of Weyl calculus, E. Nelson's formula for Weyl calculus in a finite setting, and R. Raffoul's results of support and singular support of Weyl calculus for general compact Lie groups.

In Section 6.2, we discuss some interesting properties of the non-constant-coefficient differential operator $\zeta\frac{\partial}{\partial\zeta}$ in $\mathfrak{g}$ (which generalises $r\frac{\partial}{\partial r}$ for $\mathfrak{su}(2)$). In

Proposition 6.2.4, we derive generalised Bessel functions for compact connected semisimple Lie groups. In Proposition 6.2.5, we use this Bessel function to prove that $\zeta\frac{\partial}{\partial\zeta}$ is a G-invariant differential operator in $\mathfrak{g}$.

In Section 6.3, we derive a Kirillov-type non-commutative formula for an irreducible unitary highest weight representation π_λ. In Proposition 6.3.3, we derive an updated version of Nelson's formula by introducing a recursive function ψ of the differential operator $\zeta\frac{\partial}{\partial\zeta}$. In Proposition 6.3.5, we show that the $\mathfrak{g}$-Fourier transform of $\pi_\lambda \circ \exp$ is a polynomial of differential operators in fundamental weight directions, root directions and $\zeta\frac{\partial}{\partial\zeta}$, acting on the G-invariant measure ν_{I_λ} of the moment set I_λ of π_λ.

In Section 6.4, in Proposition 6.4.1, we show that the Kirillov type non-commutative formula can be extended to non-unitary highest weight representations.

We use examples of $SU(2)$ and $SU(3)$ to illustrate these propositions.

6.1 Weyl Calculus and Coadjoint Orbits

The Weyl calculus was introduced by Hermann Weyl [53] to model the quantisation of a function $f(P, Q)$ for which P, Q are the moment and position operators. It was studied in the general case of self-adjoint operators by Robert Anderson [5]. Edward Nelson [40] introduced the theory of operants as a framework for studying Weyl calculus, and proved an explicit formula for the Weyl calculus in the finite-dimensional setting with an ingenious application of recursion relations and induction. Brian Jefferies [31] provided an alternative proof of Nelson's formula with an application of the Cayley-Hamilton theorem and binomial expressions. Raed Raffoul [44] applied the Weyl calculus and operants to study the unitary representations of compact Lie groups.

In this section, we review Nelson's explicit formula of the Weyl calculus for finite-dimensional non-commutative self-adjoint operators and Raffoul's results for the support and singular support of the Weyl calculus $\mathcal{W}_A$ when A is the infinitesimal version of an irreducible unitary representation of a compact simply connected semisimple Lie group.

Definition 6.1.1. Let $\mathcal{H} \cong \mathbb{C}^n$ be a finite and n-dimensional complex Hilbert space. Let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator on $\mathcal{H}$, and let $\mathcal{H}$ be endowed with a uniform operator norm

$$||A|| = sup\{||Av|| \,:\, ||v|| \neq 0, ||v|| = 1, ||v|| \in \mathcal{H}\}\,. \tag{6.1.1}$$

Define $\mathcal{A} = (A_1, ..., A_d)$ as a d-tuple of n-dimensional Hermitian matrices acting on $\mathcal{H}$. For $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$, the matrix $\xi \cdot \mathcal{A} = \sum_{j=1}^{d} \xi_j A_j$ is again Hermitian.

Theorem 6.1.2 ([40], Theorem 6, [44], Theorem 2.1.5)**.** *Let $\mathcal{H} \cong \mathbb{C}^n$ be a complex Hilbert space. Let $\mathcal{A} = (A_1, ..., A_d)$ be a d-tuple of finite-dimensional Hermitian matrices on $\mathcal{H}$, where $A_1, ...A_d$ do not necessarily commute. Let $\mathcal{L}(\mathcal{H})$ denote the space of all operators on $\mathcal{H}$ endowed with the uniform operator norm $|| \cdot ||$. Then, for $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$, $e^{i\xi \cdot \mathcal{A}} \in \mathcal{L}(\mathcal{H})$ and*

$$||e^{i\xi \cdot \mathcal{A}}|| = 1. \tag{6.1.2}$$

Let $f : \mathbb{R}^d \to \mathbb{C}$ be any function with integrable Fourier transform $\widehat{f}$. Then the $\mathcal{L}(\mathcal{H})$-valued integral

$$f(\mathcal{A}) = \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{i\xi \cdot \mathcal{A}} \, d\xi, \tag{6.1.3}$$

is convergent.

Definition 6.1.3. Let $\mathcal{H} \cong \mathbb{C}^n$ be a complex Hilbert space. Let $\mathcal{A} = (A_1, ..., A_d)$ be a d-tuple of finite-dimensional Hermitian matrices on $\mathcal{H}$, and let $\mathcal{L}(\mathcal{H})$ be the space of all operators on $\mathcal{H}$. Let $f \in \mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing functions (It is known that $\widehat{f} \in \mathcal{S}(\mathbb{R}^d)$). The **Weyl calculus** of $\mathcal{A}$ is a $\mathcal{L}(\mathcal{H})$-valued tempered distribution $\mathcal{W}_{\mathcal{A}}$, which is given by

$$\mathcal{W}_{\mathcal{A}}(f) = \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{i\xi \cdot \mathcal{A}} \, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d)\,. \tag{6.1.4}$$

Next, we state E. Nelson's explicit formula for the Weyl calculus of finite-dimensional self-adjoint operators.

Theorem 6.1.4 ([40], Theorem 9, [44], Theorem 4.1.1)**.** *(E. Nelson) Let $\mathcal{H} \cong \mathbb{C}^n$ be an finite-dimensional Hilbert space, and $\mathcal{L}(\mathcal{H})$ be the Banach algebra of bounded operators on $\mathcal{H}$. Let $\Sigma \subset \mathcal{L}(\mathcal{H})^*$ be a subset of the dual space of all linear functionals of the form $A \mapsto (Au, u)$, for $A \in \mathcal{L}(\mathcal{H})$, where u lies on the unit sphere Ω in $\mathcal{H}$ with a unitarily invariant probability measure*

$\nu(\Omega) = 1$, and μ be the unitarily invariant measure on Σ such that $\mu(\Sigma) = 1$. Let V_0 be the real vector space of all self-adjoint elements of $\mathcal{L}(\mathcal{H})$. Suppose $A = (A_1, ..., A_d)$ is a d-tuple of self-adjoint operators in V_0. Let $x \in \mathbb{R}^d, \xi \in \left(\mathbb{R}^d\right)^*$ and $x \cdot A = \sum_{j=1}^d x_j A_j$, and let $\mathcal{W}_{x \cdot A}$ be the Weyl calculus of $x \cdot A$, which is also the Fourier transform of the $\mathcal{L}(\mathcal{H})$-valued function $x \cdot A \mapsto e^{ix \cdot A}$. Then, the explicit form of $\mathcal{W}_{x \cdot A}$ is given by

$$\begin{aligned}\mathcal{W}_{x \cdot A} = \frac{1}{(n-1)!} \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \sum_{m=0}^{j} (-1)^{n-k-j-1} \frac{\phi_{n-k-j-1}\left(-i\frac{\partial}{\partial \xi} \cdot A\right)}{(n-j+m-1)!} \\ \cdot \left(-i\frac{\partial}{\partial \xi} \cdot A\right)^k \left(-\xi\frac{\partial}{\partial \xi} - dI\right)^m \cdot \mu,\end{aligned} \tag{6.1.5}$$

where $\phi_k(A)$ denotes the principal minor of the matrix A with degree k, and the inverse Fourier transform of μ is given by

$$\check{\mu}(x \cdot A) = \int_{\Sigma} e^{i\xi \cdot x}\, d\mu(\xi) = \int_{\Omega} e^{i(x \cdot Au, u)} d\nu(u). \tag{6.1.6}$$

Remark. *By the Paley-Wiener-Schwartz theorem, the inverse Fourier transform of $\mathcal{W}_{x \cdot A}$ is a matrix of analytic functions on $\mathbb{R}^d$.*

Example 6.1.4.1 ([5], Theorem 4.1)**.** Let $\mathcal{H} \cong \mathbb{C}^2$ be a Hilbert space of dimension 2 and $\mathcal{L}(\mathcal{H})$ be the Banach algebra of bounded operators on $\mathcal{H}$. Let A_1, A_2, A_3 be Pauli spin matrices:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.1.7}$$

Together with the identity matrix, these form a basis for $\mathcal{L}(\mathcal{H})$. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, and $x \cdot A = x_1 A_1 + x_2 A2 + x_3 A_3$, which is

$$x \cdot A = \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix}. \tag{6.1.8}$$

Then, for $\xi = (\xi_1, \xi_2, \xi_3) \in (\mathbb{R}^3)^*$, the $\mathcal{L}(\mathcal{H})$-valued distribution $\mathcal{W}_{x \cdot A}$, also the

Fourier transform of $e^{ix\cdot A}$ is given by

$$\begin{aligned}\mathcal{W}_{x\cdot A} &= \sum_{k=0}^{1}\sum_{j=0}^{1-k}\sum_{m=0}^{j}(-1)^{1-k-j}\frac{\phi_{1-k-j}\left(-i\frac{\partial}{\partial\xi}\cdot A\right)}{(1-j+m)!}\\ &\cdot\left(-i\frac{\partial}{\partial\xi}\cdot A\right)^{k}\left(-\xi\frac{\partial}{\partial\xi}-3I\right)^{m}\cdot\mu\\ &= -\phi_1\left(-i\frac{\partial}{\partial\xi}\cdot A\right)I\mu+\phi_0\left(-i\frac{\partial}{\partial\xi}\cdot A\right)I\mu\\ &+\phi_0\left(-i\frac{\partial}{\partial\xi}\cdot A\right)I\left(-\xi\frac{\partial}{\partial\xi}-3I\right)\mu+\phi_0\left(-i\frac{\partial}{\partial\xi}\cdot A\right)\left(-i\frac{\partial}{\partial\xi}\cdot A\right)\mu\\ &= \left(I+\left(-i\frac{\partial}{\partial\xi}\cdot A\right)+\left(-\xi\frac{\partial}{\partial\xi}-3\right)I\right)\mu\end{aligned} \tag{6.1.9}$$

where $\phi_1(\cdot)=0, \phi_0(\cdot)=1$. Hence,

$$\mathcal{W}_{x\cdot A}=\begin{pmatrix}-r\frac{\partial}{\partial r}-2+\frac{\partial}{\partial\xi_3} & -i\frac{\partial}{\partial\xi_1}+\frac{\partial}{\partial\xi_2}\\ -i\frac{\partial}{\partial\xi_1}-\frac{\partial}{\partial\xi_2} & -r\frac{\partial}{\partial r}-2-\frac{\partial}{\partial\xi_3}\end{pmatrix}\mu \tag{6.1.10}$$

where $r^2=\xi_1^2+\xi_2^2+\xi_3^2$ on $(\mathbb{R}^3)^*$ and μ is the probability surface measure on the unit sphere Σ of $\mathbb{R}^3$.

Remark. *In fact, this result was first proposed by R. Anderson* [5]*, and it is also identical to Cazzaniga's result for the Fourier transform of the defining representation* $e^{d\pi_1}$ *in* $\mathfrak{su}(2)$ *in Proposition* 2.5.9*.*

Now, we state R. Raffoul's results for the support and singular support of the Weyl calculus $\mathcal{W}_{d\pi_\lambda}$ of an irreducible highest weight unitary representation π_λ of a compact simply connected semisimple Lie groups.

Theorem 6.1.5 ([44], Theorem 6.1.3)**.** *Let G be a compact simply connected semisimple Lie group, $\mathfrak{g}$ be the Lie algebra of G. Let T be a maximal torus of G, $\mathfrak{t}$ be the Lie algebra of T, and $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. Let $\lambda \in \Lambda^+ \subset \mathfrak{t}^*$ be a dominant highest weight, π_λ be the irreducible unitary representation of λ, $d\pi_\lambda$ be the skew-Hermitian and infinitesimal version of π_λ in $\mathfrak{g}$, and I_λ be the moment set of π_λ. Suppose dim$(\mathfrak{g})=d$, and let $\{X_1,...,X_d\}$ be a basis for the Lie algebra $\mathfrak{g}$, and $x\in\mathbb{R}^d$. Then the support of the Weyl calculus $\mathcal{W}_{-i\,x\cdot d\pi_\lambda(X)}$ satisfies*

$$supp(\mathcal{W}_{-i\,x\cdot d\pi_\lambda(X)})\subseteq I_\lambda. \tag{6.1.11}$$

Proof. By Theorem 6.1.4 and Proposition 5.2.2, we conclude that the measure ν of $\mathcal{W}_{-i\,x\cdot d\pi_\lambda(X)}$ is exactly the probability G-invariant measure ν_{I_λ} supported on the moment set I_λ. We can also write $\mathcal{W}_{-i\,x\cdot d\pi_\lambda(X)} = D_\lambda \cdot \nu_{I_\lambda}$, where D_λ is a matrix of differential operators of the form $\xi\frac{\partial}{\partial\xi}$ and $\frac{\partial}{\partial\xi_1}, \dots \frac{\partial}{\partial\xi_d}$, for $\xi \in \mathbb{R}^d$. Since a differential operator, in general, reduces the support of a distribution ([48], Ch.3), we have

$$supp(D_\lambda \cdot \nu_{I_\lambda}) \subseteq supp(\nu_{I_\lambda}). \tag{6.1.12}$$

Hence, the theorem follows. □

Theorem 6.1.6 ([44], Theorem 6.28)**.** *Let G be a compact simply connected semisimple Lie group, $\mathfrak{g}$ be the Lie algebra of G. Let T be a maximal torus of G, $\mathfrak{t}$ be the Lie algebra of T, and $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. Let $\lambda \in \Lambda^+ \subset \mathfrak{t}^*$ be a dominant highest weight, π_λ be the irreducible unitary representation of λ, $d\pi_\lambda$ the skew-Hermitian and infinitesimal version of π_λ in $\mathfrak{g}$, and I_λ be the moment set of π_λ. Let Φ be the root system of $\mathfrak{t}^*$, δ be the half sum of all the positive roots in Φ. Let $dim(\mathfrak{g}) = d$, $X \in \mathfrak{g}, x \in \mathbb{R}^d$. Then, the convex hull of the δ-shift of $\mathcal{W}_{-i\,x\cdot d\pi_\lambda(X)}$ is equal to the convex hull of the coadjoint orbit $\mathcal{O}_{\lambda+\delta}$, that is,*

$$conv\ supp\left(\nu_\delta * \mathcal{W}_{-i\,x\cdot d\pi_\lambda(X)}\right) = conv(\mathcal{O}_{\lambda+\delta}), \tag{6.1.13}$$

where ν_δ is the Liouville measure of the coadjoint orbit $\mathcal{O}_\delta$.

Example 6.1.6.1. Let $G = SU(2)$, $\mathfrak{su}(2) \cong \mathbb{R}^3$ the Lie algebra of $SU(2)$, π_n be the irreducible unitary representation of a positive integer n. The δ element in $\mathfrak{t}^* \cong \mathbb{R}$ is equal to 1. Let I_n be the moment set of π_n, ν_1 be the Liouville measure of $\mathcal{O}_1$, and ν_{I_n} be the probability G-invariant measure of I_n. According to Proposition 5.1.8 and Proposition 5.2.4, we have

$$\nu_1 * \nu_{I_n} = \nu_1 * (\nu_1)^{*n} = \nu_1 * (\nu_1)^{*(n+1)} = \nu_{I_{n+1}}, \tag{6.1.14}$$

where the support $\nu_{I_{n+1}}$ is the ball in $\mathbb{R}^3$ with radius $n+1$, and it is also the convex hull of the sphere $\mathcal{O}_{n+1}$ in $\mathbb{R}^3$ with radius $n+1$.

In addition, R. Raffoul also proved the following result about the singular support of $\nu_\delta * \mathcal{W}_{-i\,x\cdot d\pi_\lambda}$.

Theorem 6.1.7 ([44], Theorem 6.25)**.** *Let G be a compact simply connected semisimple Lie group, $\mathfrak{g}$ be the Lie algebra of G. Let T be a maximal torus of G, $\mathfrak{t}$ be the Lie algebra of T, and $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. Let $\lambda \in \Lambda^+ \subset \mathfrak{t}^*$ be a dominant highest weight, π_λ be the irreducible unitary representation of λ, $d\pi_\lambda$ the skew-Hermitian and infinitesimal version of π_λ in $\mathfrak{g}$, and I_λ be the moment set of π_λ. Let Φ be the root system of $\mathfrak{t}^*$, δ be the half sum of all positive roots in Φ, and W be the Weyl group of $\mathfrak{t}^*$. Let $dim(\mathfrak{g}) = d$, $X \in \mathfrak{g}, x \in \mathbb{R}^d$. Then, the singular support of $\nu_\delta * \mathcal{W}_{-i\,x\cdot d\pi_\lambda(X)}$ is given by*

$$singsupp\left(\nu_\delta * \mathcal{W}_{x\cdot d\pi_\lambda(X)}\right) = \bigcup_{w\in W} \mathcal{O}_{\lambda+w\,\delta}, \tag{6.1.15}$$

where $w \cdot \delta$ lies on the Weyl orbit of δ.

Remark. *This theorem is a generalisation of Raffoul's own result for $SU(2)$ in Theorem 2.5.11. However, we disagree with his result. Because in Proposition 5.2.4, we have shown $\nu_1 * \nu_{I_n} = (\nu_1)^{*(n+1)}$, which is the $n+1$-fold convolution of the Liouville measure of $\mathcal{O}_1$ for $SU(2)$. Also, in Proposition 4.4.1, we have shown*

$$singsupp\left((\nu_1)^{*(n+1)}\right) = \bigcup_{j=0}^{\lfloor\frac{n+1}{2}\rfloor} \mathcal{O}_{n+1-2j}, \tag{6.1.16}$$

which is different from Raffoul's result. In our opinion, in general the singular support of $\nu_\delta * \mathcal{W}_{-i\,x\cdot d\pi_\lambda}$ is fairly complex to describe.

Example 6.1.7.1. We demonstrate another example. Let $G = SU(3)$, and its Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$. The Cartan subalgebra $\mathfrak{t} \cong \mathbb{R}^2$ (and its dual $\mathfrak{t}^* \cong \mathbb{R}^2$). Let $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$ be the set of positive roots of $\mathfrak{t}$. Let W be the Weyl group (5.1.2) of $\mathfrak{t}, \mathfrak{t}^*$. Also, let $W_{\lambda_1} = \{e, \sigma_{\alpha_2}\}$ be the stabiliser subgroup. Let $\lambda \in \Lambda^+$ be a dominant weight. If $\lambda = \lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$, the first fundamental weight of $\mathfrak{t}^*$ (in Figure 5.2), then the Weyl orbit of λ_1 is the set

$$W \cdot \lambda_1 = \{\lambda_1, \lambda_1 - \alpha_1, \lambda_1 - \alpha_1 - \alpha_2\} \cong W/W_{\lambda_1} = \{e, \sigma_{\alpha_1}, \sigma_2\sigma_1\}. \tag{6.1.17}$$

Let π_{λ_1} be the irreducible unitary representation of λ_1, $\Pi(\lambda_1)$ be the set of weights of π_{λ_1} which coincides with $W \cdot \lambda_1$, and I_{λ_1} be the moment set of π_{λ_1}. Since π_{λ_1} is the defining representation of $SU(3)$, I_{λ_1} is exactly the single coadjoint orbit $\mathcal{O}_{\lambda_1}$, and the measure $\nu_{I_{\lambda_1}} = \nu_{\lambda_1}$. Let δ be the half sum of all positive roots in Φ^+. We wish to find the support and singular support of the

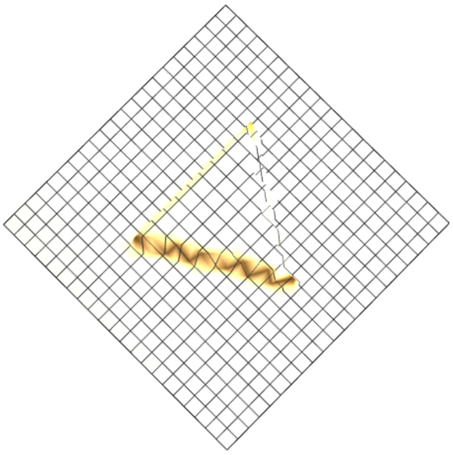

Figure 6.1: $\nu^p_{\lambda_1}$ of A_2

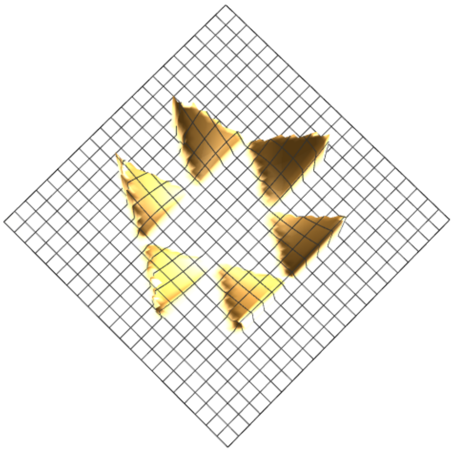

Figure 6.2: $\nu_\delta * \nu_{\lambda_1}|_{\mathfrak{t}^*}$ of A_2

convolution $\nu_\delta * \nu_{I_{\lambda_1}} = \nu_\delta * \nu_{\lambda_1}$. We recall from Chapter 4, the explicit formula (4.3.18) for the convolution of the regular coadjoint orbit $\mathcal{O}_\delta$ and the singular coadjoint orbit $\mathcal{O}_{\lambda_1}$ is given by

$$\nu_\delta * \nu_{\lambda_1} = \int_{\mathfrak{t}^*_+} \sum_{w\in W} \operatorname{sgn}(w) e_{w\delta} * \nu^p_{\lambda_1}(\lambda'')\mu_{\lambda''}\, d\lambda'' . \tag{6.1.18}$$

So, the density function for the support of $(\nu_\delta * \nu_{\lambda_1}) \cap \mathfrak{t}^*_+$ is

$$d\varphi(\delta, \lambda_1, \lambda'') = \sum_{w\in W} \operatorname{sgn}(w) e_{w\delta} * \nu^p_{\lambda_1}(\lambda'')\, d\lambda'', \quad \lambda'' \in \mathfrak{t}^*_+ \tag{6.1.19}$$

Hence, we first approximate and draw $\nu^p_{\lambda_1}$ in Figure 6.1, and we notice that this projection measure is a uniform measure supported on an equilateral triangle, whose edges are in the set $W \cdot \lambda_1$, which is centred at the origin of $\mathfrak{t}^*$. The final form of the density function $\varphi(\delta, \lambda_1, \cdot)$ is the anti-symmetric summation of $\nu^p_{\lambda_1}$ shifted by the Weyl orbit of δ, which can be seen in Figure 6.2. Hence, the support of $\nu_\delta * \nu_{\lambda_1}|_{\mathfrak{t}^*}$ is the set of six equilateral triangles, and the singular support is the set of all edges of these six triangles. In addition, the support of $\nu_\delta * \nu_{\lambda_1}|_{\mathfrak{t}^*_+}$ is the triangle with edges $\{\lambda_1 + \delta, \lambda_1 + \alpha_2, \lambda_1\} = \{2\lambda_1 + \lambda_2, 2\lambda_2, \lambda_1\}$, which is fully contained in $\mathfrak{t}^*_+$.

6.2 The Theta Differential Operators

In this section, we study the properties of the non-constant-coefficient differential operator $x\frac{\partial}{\partial x}$ that appeared in the explicit formula for the Weyl calculus.

It is a transversal differential operator in $\mathfrak{g}$(Chapter 3). We show that it is a G-invariant differential operator in $\mathfrak{g}$.

Definition 6.2.1. Let $\mathfrak{g}$ be a real semisimple Lie algebra of dimension d, $x \in \mathbb{R}^d$, and $\{X_1, ..., X_d\}$ be a basis for $\mathfrak{g}$. We let $x \cdot X = x_1 X_1 + \cdots + x_d X_d \in \mathfrak{g}$. Then the **theta differential operator** is defined by

$$x\,\frac{\partial}{\partial x} = x_1\,\frac{\partial}{\partial x_1} + \cdots + x_d\,\frac{\partial}{\partial x_d}. \tag{6.2.1}$$

Remark. *The name 'theta' differential operator is used in [52] to study Bessel functions. This is also a Cauchy-Euler operator on $\mathbb{R}^d$, and it is equal to $r\frac{\partial}{\partial r}$ in polar coordinates. It is also called a homogeneous operator, since the eigenspaces of this operator are homogeneous polynomials in $\mathbb{R}^d$.*

Proposition 6.2.2. *Let G be a compact connected semisimple Lie group, $\mathfrak{g}$ be the Lie algebra of G. Let T be a maximal torus in G, $\mathbb{T}$ be a 1-torus in T. Define $\varphi : \mathbb{R} \to \mathbb{T}, t \mapsto e^{it}$ as a one-parameter subgroup of $\mathbb{T}$. Let $f \in C^\infty(\mathfrak{g})$, if we differentiate f in the direction of the 'rotation' defined by φ and evaluate it at $t = 0$, then we obtain*

$$\frac{d}{dt} f(\varphi(t)\, x \cdot X)\bigg|_{t=0} = i\, x\, \frac{\partial}{\partial x}\, f(x \cdot X), \tag{6.2.2}$$

for $t \in \mathbb{R}, x \in \mathbb{R}^d, X \in \mathfrak{g}$.

Proof. Since the value of f depends only on $x \in \mathbb{R}^d$, we can define $\tilde{\varphi}(t) = \varphi(t)x$, and by the chain rule, we have

$$\frac{d}{dt} f(\varphi(t)\, x)\bigg|_{t=0} = \frac{d\tilde{\varphi}(t)}{dt}\,\frac{d}{d\tilde{\varphi}(t)} f(\tilde{\varphi}(t))\bigg|_{t=0} = i\, x \frac{\partial}{\partial x} f(x), \tag{6.2.3}$$

so the proposition follows. ☐

We can also show that the theta differential operator is G-invariant. Firstly, we derive the Bessel function for $\mathfrak{g}$.

Theorem 6.2.3 ([37], Theorem 8.60)**.** *(**Weyl Integration Formula**) Let G be a compact connected semisimple Lie group, T be a maximal torus of G. Let dg, dt be the normalised Haar measure on G and T such that $\int_G dg = 1, \int_T dt =$*

1, *respectively. Let* $\mathfrak{g}$ *be the Lie algebra of* G, *and* Φ *be a chosen root system of* $\mathfrak{g}$. *Suppose* $f \in C(G)$. *We have*

$$\int_G f(g)\, dg = \int_T \left(\int_G f(gtg^{-1})\, dg\, u(t)\, dt \right), \tag{6.2.4}$$

where

$$u(t) = \left| \prod_{\alpha \in \Phi^+} \left(e^{\frac{i\alpha(t)}{2}} - e^{\frac{-i\alpha(t)}{2}} \right) \right|^2 = \left| \sum_{w \in W} sgn(w) e^{iw\delta(t)} \right|^2, \quad t \in T, \tag{6.2.5}$$

and Φ^+ *is the subset of all positive roots in* Φ, δ *is the half sum of all positive roots in* Φ^+, W *is the Weyl group of the Lie algebra* $\mathfrak{t}$ *of* T.

Corollary 6.2.3.1. *Let* G *be a compact connected semisimple Lie group,* T *be a maximal torus of* G. *Let* $\mathfrak{g}$, $\mathfrak{t}$ *be the Lie algebras of* G, T, *respectively. If* dX *is a Lebesgue measure on* $\mathfrak{g}$ *and* $\varphi \in L^1(\mathfrak{g})$, *then the Weyl integration formula on* $\mathfrak{g}$ *is given by*

$$\int_{\mathfrak{g}} \varphi(X)\, dX = \int_{\mathfrak{t}_+} \prod_{\alpha \in \Phi^+} \alpha(H)^2 \int_G \varphi(Ad(g)H)\, dg\, dH \tag{6.2.6}$$

where dH *is a Lebesgue measure on the positive Weyl chamber* $\mathfrak{t}^+$, *and* $Ad(\cdot)$ *is the adjoint representation of* G.

Proposition 6.2.4. *(Bessel function) Let* G *be a compact connected semisimple Lie group,* T *be a maximal torus of* G. *Let* $\mathfrak{g}$, $\mathfrak{t}$ *be the Lie algebras of* G, T, *respectively. Also, let* $\mathfrak{t}^*$ *be the dual of* $\mathfrak{t}$. *If* $f \in L^1(\mathfrak{g})$ *is adjoint invariant, then for* $\beta \in \mathfrak{t}^*$, *the Fourier transform of* f *is given by*

$$\int_{\mathfrak{g}} f(X) e^{i\beta(X)}\, dX = \int_{\mathfrak{t}^+} f(H) \prod_{\alpha \in \Phi^+} \alpha(H)^2 \frac{1}{d(\beta)} j \cdot \chi_{\beta-\delta}(H)\, dH, \tag{6.2.7}$$

where $d(\beta)$ *is the continuous version of Weyl dimension formula (Theorem 4.1.6), and* $j \cdot \chi_{\beta-\delta}(H)$ *is the continuous version of the function in the Kirillov character formula, (4.1.9). In fact,* $\frac{1}{d(\beta)} j \cdot \chi_{\beta-\delta}$ *is a generalised Bessel function for* G, *which is a special case of the version for symmetric spaces in* [10].

Proof. We can obtain (6.2.7) using the Weyl integration formula (6.2.4) and our continuous version of the Kirillov character formula (4.1.9). Let $\beta \in \mathfrak{t}^*$, dg be the normalised Haar measure on G. Then,

$$\begin{aligned}
\widehat{f}(\beta) &= \int_{\mathfrak{g}} f(X)e^{i\beta(X)}dX \\
&= \int_{\mathfrak{t}_+} \prod_{\alpha\in\Phi^+} \alpha(H)^2 \int_G f(Ad(g)H)e^{iAd^*(g^{-1})\beta(H)}\, dg\, dH \\
&= \int_{\mathfrak{t}_+} \prod_{\alpha\in\Phi^+} \alpha(H)^2\, f(H)\, \frac{\prod_{\alpha\in\Phi^+}(\delta\,,\,\alpha)}{\prod_{\alpha\in\Phi^+}(\beta\,,\,\alpha)} \int_{\mathcal{O}_\beta} e^{i\lambda(H)}\, d\mu_\beta(\lambda)\, dH \\
&= \int_{\mathfrak{t}_+} f(H) \prod_{\alpha\in\Phi^+} \alpha(H)^2\, \frac{\prod_{\alpha\in\Phi^+}(\delta\,,\,\alpha)}{\prod_{\alpha\in\Phi^+}(\beta\,,\,\alpha)}\, \frac{R(\beta)(H)}{\prod_{\alpha\in\Phi^+} i\alpha(H)}\, dH
\end{aligned} \tag{6.2.8}$$

where $R(\beta)(H) = \sum_{w\in W} sgn(w)e^{iw\beta(H)}$. But if β_0 lies on a wall of a Weyl chamber in $\mathfrak{t}^*$, by the dominated convergence theorem, we may approximate β_0 by a regular element, that is,

$$\widehat{f}(\beta_0) = \int_{\mathfrak{t}_+} f(H) \prod_{\alpha\in\Phi^+} \alpha(H)^2\, C_{\beta_0} \sum_{j=1}^{m} \frac{\mathrm{sgn}(w_j)e^{iw_j\beta_0(H)}}{\mathrm{sgn}(\tau_{w_j}) \prod_{\alpha\in\Phi^+\backslash\Phi^+_{w_j\beta_0}} i\alpha(H)}\, dH \tag{6.2.9}$$

which corresponds to the normalisation formula in (5.2.52). □

Example 6.2.4.1. Let $G = SU(2)$, its Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$, and $\mathfrak{t} \cong \mathbb{R}$ be the Lie algebra of a maximal torus T (1-torus) in $SU(2)$. If $f \in L^1(\mathfrak{g})$ is adjoint (rotational) invariant, then the Fourier transform of f is given by

$$\widehat{f}(\beta) = \int_0^\infty f(H)\,(H)^2 \frac{\sin\beta(H)}{\beta(H)}\, dH, \tag{6.2.10}$$

for $\beta \in \mathfrak{t}^* \cong \mathbb{R}$. This formula coincides with and generalise the radial Fourier transform in $\mathbb{R}^3$ ([49], Sec 7.7).

Next, we show that the theta differential operator $x\frac{\partial}{\partial x}$ in $\mathfrak{g}$ is G-invariant.

Proposition 6.2.5. *Let G be a compact connected semisimple Lie group, T be a maximal torus of G. Let $\mathfrak{g}$, $\mathfrak{t}$ be the Lie algebras of G, T, respectively. Let $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. Suppose $dim(\mathfrak{g}) = d$, and we let $x \in \mathbb{R}^d$, and $\xi \in (\mathbb{R}^d)^*$. If $f \in \mathcal{S}(\mathfrak{g})$ is an adjoint invariant Schwartz function, then*

$$\widehat{\left(x\frac{\partial}{\partial x} f\right)} = -\left(\xi\frac{\partial}{\partial \xi} + dI\right)\widehat{f}, \tag{6.2.11}$$

and $x\frac{\partial}{\partial x}$ and $\xi\frac{\partial}{\partial \xi}$ are G-invariant differential operators.

Proof. The equity in (6.2.11) follows by taking the Fourier transform with respect to the Killing form. To show that $x\frac{\partial}{\partial x}$ is a G-invariant differential operator, we assume that $\left(x\frac{\partial}{\partial x}\right) f$ is also adjoint invariant. So, for $\beta \in \mathfrak{t}^*$, $g \in G$, we can write

$$\begin{aligned}\left(x\frac{\partial}{\partial x} f\right)^{\widehat{}}(\mathrm{Ad}^*(g)\beta) &= -\mathrm{Ad}^*(g)\beta \,\frac{\partial}{\partial \mathrm{Ad}^*(g)\beta}\widehat{f}(\mathrm{Ad}^*(g)\beta) - d\,\widehat{f}(\mathrm{Ad}^*(g)\beta)\\ &= -\beta\frac{\partial}{\partial\beta}\widehat{f}(\beta) - d\,\widehat{f}(\beta)\\ &= \left(x\frac{\partial}{\partial x} f\right)^{\widehat{}}(\beta).\end{aligned} \tag{6.2.12}$$

Now, we evaluate $\left(x\frac{\partial}{\partial x} f\right)^{\widehat{}}|_{\mathfrak{t}^*}$ with respect to the Bessel function in (6.2.7). Because we have assumed $\left(x\frac{\partial}{\partial x} f\right)$ is adjoint invariant, so we can use the Weyl integration formula (6.2.4), to show that $\left(x\frac{\partial}{\partial x} f\right)^{\widehat{}}(Ad^*(h)\beta)$, for $h \in G$, is

$$\begin{aligned}&= \int_{\mathfrak{t}_+} \prod_{\alpha\in\Phi^+} \alpha(H)^2 \int_G Ad(g)H\,\frac{\partial}{\partial Ad(g)H} f(Ad(g)H)\, e^{iAd^*(g^{-1}h)\beta(H)}\,dg\,dH\\ &= \int_{\mathfrak{t}_+} H\,\frac{\partial}{\partial H} f(H) \prod_{\alpha\in\Phi^+}\alpha(H)^2 \int_G e^{iAd^*(g)\beta(H)}\,dg\,dH\\ &= \int_{\mathfrak{t}_+} H\frac{\partial}{\partial H}\, f(H)\, \frac{\prod_{\alpha\in\Phi^+}(\delta\,,\,\alpha)}{\prod_{\alpha\in\Phi^+}(\beta\,,\,\alpha)} \prod_{\alpha\in\Phi^+}\frac{1}{i}\alpha(H) \sum_{w\in W}\mathrm{sgn}(w)e^{iw\beta(H)}\,dH\\ &= -\int_{\mathfrak{t}_+} f(H)\left(\frac{\partial}{\partial H}H\right)\frac{\prod_{\alpha\in\Phi^+}(\delta\,,\,\alpha)}{\prod_{\alpha\in\Phi^+}(\beta\,,\,\alpha)} \prod_{\alpha\in\Phi^+}\frac{1}{i}\alpha(H) \sum_{w\in W}\mathrm{sgn}(w)e^{iw\beta(H)}\,dH\\ &= -\left(\beta\frac{\partial}{\partial\beta} + (|\Delta| + |\Phi|)\; I\right)\int_{\mathfrak{t}_+} f(H)\,\frac{\prod_{\alpha\in\Phi^+}(\delta\,,\,\alpha)}{\prod_{\alpha\in\Phi^+}(\beta\,,\,\alpha)} \prod_{\alpha\in\Phi^+}\frac{1}{i}\alpha(H)\\ &\cdot\sum_{w\in W}\mathrm{sgn}(w)e^{iw\beta(H)}\,dH\\ &= -\left(\beta\frac{\partial}{\partial\beta} + d\,I\right)\widehat{f}(\beta),\end{aligned} \tag{6.2.13}$$

where the quotient rule is applied in the second-last equality. We have $d = |\Delta| + |\Phi|$, where $|\Delta|$ is the cardinality of the set of roots Φ, Δ is the cardinality of the set of simple roots Δ. Hence, we get an output that is the same as the second equality of (6.2.12); thus the proposition follows. ☐

Example 6.2.5.1. Let $G = SU(2)$, its Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$, and $\mathfrak{t} \cong \mathbb{R}$ be the

Lie algebra of a maximal torus T (1-torus) in $SU(2)$. Recall that by Theorem 2.5.7 and Proposition 2.5.9, and for $(x_1, x_2, x_3) \in \mathbb{R}^3$, $r\frac{\partial}{\partial r} = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}$, and $|X| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, we have

$$\left(r\frac{\partial}{\partial r}\frac{\sin(|X|)}{|X|}\right)^{\vee} = -\left(r^*\frac{\partial}{\partial r^*} + 3\right)\nu_1, \tag{6.2.14}$$

where ν_1 is the Liouville measure on the coadjoint orbit $\mathcal{O}_1$. Notice that $\frac{\sin|X|}{|X|}$ is both differentiable and G-invariant, and $r\frac{\partial}{\partial r}$ preserves the G-invariance since

$$r\frac{\partial}{\partial r}\frac{\sin(|X|)}{|X|} = \cos(|X|) - \frac{\sin(|X|)}{|X|}, \tag{6.2.15}$$

which is G-invariant. Consequently, we have

$$r\frac{\partial}{\partial r}\cos(|X|) = -|X|\sin(|X|), \tag{6.2.16}$$

which is also G-invariant.

6.3 A Kirillov-Type Non-Commutative Formula

In this section, we derive an updated version of Nelson's formula (6.1.5) for irreducible unitary highest weight representations of compact simply connected semisimple Lie groups, which includes a novel recursive formula ψ for the invariant theta differential operator $\zeta\frac{\partial}{\partial \zeta}$.

Firstly, we derive this ψ function. Let $r \in \mathbb{R}$ and $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$, given that the ζ_k's are **distinct**, and we consider the following linear system

$$\begin{aligned} e^{r\zeta_1} &= g_0 + g_1\zeta_1 + g_2\zeta_1^2 + \cdots + g_{n-1}\zeta_1^{n-1} \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ e^{r\zeta_n} &= g_0 + g_1\zeta_n + g_2\zeta_n^2 + \cdots + g_{n-1}\zeta_n^{n-1} \end{aligned} \tag{6.3.1}$$

which uniquely determines the functions $g_0, \ldots, g_{n-1}$ in real variables (r, ζ). Then, we can determine $g_k(r, \zeta)$ as follows.

Lemma 6.3.1. *Let $\gamma_k(\zeta) = g_k(1,\zeta)$. Then $\gamma_k(\zeta)$ can be determined by*

$$\gamma_k(\zeta) = \sum_{j=0}^{n-k-1} \sum_{m=0}^{j} (-1)^{n-k-j-1} \phi_{n-k-j-1}(\zeta) \binom{j}{m} \frac{(n-1)!}{(n-j+m-1)!} \psi(m-1) \cdot \gamma_{n-1}(\zeta), \tag{6.3.2}$$

where $\phi_j(\zeta)$ is the j-th elementary symmetric function of ζ,

$$\gamma_{n-1}(\zeta) = \frac{\begin{vmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-2} & e^{\lambda_1} \\ \vdots & & & & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^{n-2} & e^{\lambda_n} \end{vmatrix}}{D}, \tag{6.3.3}$$

where D is the Vandermonde determinant

$$D = \begin{vmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ \vdots & & & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^{n-1} \end{vmatrix} = \prod_{i<j} (\lambda_j - \lambda_i). \tag{6.3.4}$$

Then, ψ can be defined recursively as

$$\psi(p) = \sum_{q=0}^{p} (-1)^{p-q} \frac{p!}{q!} \left(\zeta \cdot \frac{\partial}{\partial \zeta} \right) \cdot \psi(q-1), \tag{6.3.5}$$

with $\psi(-1) = 1$.

Proof. Following Nelson's proof (which is fundamental but elegant) in ([40], Theorem 9]), we obtain the relation

$$r^k \gamma_k(r\zeta) = \sum_{j=0}^{n-k-1} \sum_{m=0}^{j} (-1)^{n-k-j-1} \phi_{n-k-j-1}(\zeta) \binom{j}{m} \frac{(n-1)!}{(n-j+m-1)!} \frac{\partial^m}{\partial r^m} \gamma_{n-1}(r\zeta), \tag{6.3.6}$$

where

$$\gamma_{n-1}(r\zeta) = \sum_{k=1}^{n} (-1)^{n-1} e^{r\zeta_k} \prod_{j \neq k} (r\zeta_j - r\zeta_k)^{-1}, \tag{6.3.7}$$

by expanding the determinant function in denominator of (6.3.3) from the last column. That is,

$$\begin{aligned}
\gamma_{n-1}(r\zeta) &= \sum_{k=1}^{n}(-1)^{n+k}e^{r\zeta_k}\prod_{i<j,\, i,j\neq k}(r\zeta_j - r\zeta_i)\prod_{i<j}(r\zeta_j - r\zeta_i)^{-1} \\
&= \sum_{k=1}^{n}(-1)^{n+k}e^{r\zeta_k}\prod_{n\geqslant j>k}(r\zeta_j - r\zeta_k)^{-1}\prod_{1\leqslant i<k}(r\zeta_k - r\zeta_i)^{-1} \\
&= \sum_{k=1}^{n}(-1)^{n+2k-1}e^{r\zeta_k}\prod_{j\neq k}(r\zeta_j - r\zeta_k)^{-1} \\
&= \sum_{k=1}^{n}(-1)^{n-1}e^{r\zeta_k}\prod_{j\neq k}(r\zeta_j - r\zeta_k)^{-1}.
\end{aligned} \tag{6.3.8}$$

for $j \in \{1,...,n\}$ and $j \neq k$. If we differentiate $\gamma_{n-1}(r\zeta)$ with respect to r, we obtain

$$\frac{\partial}{\partial r}\gamma_{n-1}(r\zeta) = \frac{1}{r}\left(\zeta\cdot\frac{\partial}{\partial\zeta}\right)\gamma_{n-1}(r\zeta), \tag{6.3.9}$$

or

$$r\frac{\partial}{\partial r}\gamma_{n-1}(r\zeta) = \left(\zeta\cdot\frac{\partial}{\partial\zeta}\right)\gamma_{n-1}(r\zeta). \tag{6.3.10}$$

To see this, for each term $e^{r\zeta_k}\prod_{j\neq k}(r\zeta_j - r\zeta_k)^{-1}$, we have

$$r\frac{\partial}{\partial r}\frac{e^{r\zeta_k}}{r^{n-1}\prod_{j\neq k}(\zeta_j-\zeta_k)} = \frac{e^{r\zeta_k}(r\zeta_k-(n-1))}{r^{n-1}\prod_{j\neq k}(\zeta_j-\zeta_k)}. \tag{6.3.11}$$

Let $\zeta\frac{\partial}{\partial\zeta} = \sum_{j=1}^{n}\zeta_j\frac{\partial}{\partial\zeta_j}$, we have

$$\begin{aligned}
&\zeta_k\frac{\partial}{\partial\zeta_k}\frac{e^{r\zeta_k}}{r^{n-1}\prod_{j\neq k}(\zeta_j-\zeta_k)} = \\
&\qquad\sum_{i\neq k}\frac{\zeta_k\, e^{r\zeta_k}}{r^{n-1}(\zeta_i-\zeta_k)^2\prod_{j\neq k, j\neq i}(\zeta_j-\zeta_k)} + \frac{r\zeta_k e^{r\zeta_k}}{r^{n-1}\prod_{j\neq k}(\zeta_j-\zeta_k)}
\end{aligned} \tag{6.3.12}$$

and for each $i \neq k$, we have

$$\zeta_i\frac{\partial}{\partial\zeta_i}\frac{e^{r\zeta_k}}{r^{n-1}\prod_{j\neq k}(\zeta_j-\zeta_k)} = \frac{-\zeta_i\, e^{r\zeta_k}}{r^{n-1}(\zeta_i-\zeta_k)^2\prod_{j\neq k, j\neq i}(\zeta_j-\zeta_k)}. \tag{6.3.13}$$

Hence,

$$\left(\zeta_k\frac{\partial}{\partial\zeta_k} + \sum_{i\neq k}\zeta_i\frac{\partial}{\partial\zeta_i}\right)\frac{e^{r\zeta_k}}{r^{n-1}\prod_{j\neq k}(\zeta_j-\zeta_k)} = \frac{e^{r\zeta_k}(r\zeta_k-(n-1))}{r^{n-1}\prod_{j\neq k}(\zeta_j-\zeta_k)}. \tag{6.3.14}$$

Lastly, if we combine (6.3.6), (6.3.9), and evaluate $r^k\gamma_k(r\zeta)\big|_{r=1}$, then the lemma follows by induction.

Inductive proof for the ψ function:

We first let $P(m)$ be the equality

$$\frac{\partial^m}{\partial r^m}\gamma_{n-1}(r\zeta) = \frac{1}{r^m}\psi(m-1)\gamma_{n-1}(r\zeta). \tag{6.3.15}$$

This holds for $m = 0, 1$. We assume $P(m)$ is true for $m \geqslant 1$, and wish to show $P(m+1)$ is also true. That is,

$$\frac{\partial^{m+1}}{\partial r^{m+1}}\gamma_{n-1}(r\zeta) = \frac{1}{r^{m+1}}\psi(m)\gamma_{n-1}(r\zeta). \tag{6.3.16}$$

So we have

$$\begin{aligned}\frac{\partial}{\partial r}\left(\frac{\partial^m}{\partial r^m}\gamma_{n-1}(r\zeta)\right) &= \frac{\partial}{\partial r}\left(\frac{1}{r^m}\psi(m-1)\gamma_{n-1}(r\zeta)\right)\\ &= \frac{1}{r^{m+1}}\sum_{q=0}^{m-1}(-1)^{m-q}\,\frac{m!}{q!}\left(\zeta\cdot\frac{\partial}{\partial\zeta}\right)\cdot\psi(q-1)\gamma_{n-1}(r\zeta)\\ &+\left(\zeta\cdot\frac{\partial}{\partial\zeta}\right)\left(\frac{1}{r^{m+1}}\sum_{q=0}^{m-1}(-1)^{m-1-q}\,\frac{(m-1)!}{q!}\left(\zeta\cdot\frac{\partial}{\partial\zeta}\right)\cdot\psi(q-1)\gamma_{n-1}(r\zeta)\right).\end{aligned} \tag{6.3.17}$$

When $q = m$, the term $\psi_{q=m}(m)$ in $\psi(m)$ is equal to $\left(\zeta\cdot\frac{\partial}{\partial\zeta}\right)\psi(m-1)$, and the second summand above is equal to $\frac{1}{r^{m+1}}\,\psi_{q=m}(m)\,\gamma_{n-1}(r\zeta)$. Hence the proof is complete. □

Remark. *In Nelson's original proof [40], he proposed*

$$\frac{\partial}{\partial r}\gamma_{n-1}(r\zeta) = \left(\zeta\cdot\frac{\partial}{\partial\zeta}\right)\gamma_{n-1}(r\zeta), \tag{6.3.18}$$

which is missing $1/r$ on the right-hand side, so the ψ function was originally not included in Nelson's formula (6.1.5).

Lemma 6.3.2. *Let G be a compact simply connected semisimple Lie group, T be a maximal torus of G, and $\mathfrak{g}, \mathfrak{t}$ be the Lie algebras of G, T, respectively. Let $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$, and $\mathfrak{t}^*_+$ be a chosen positive Weyl chamber of $\mathfrak{t}^*$. Let $\lambda \in \Lambda^+ \subset \mathfrak{t}^*_+$*

be a dominant highest weight, π_λ *be the irreducible unitary representation of the highest weight* λ *acting on a Hilbert space* $\mathcal{H}$*,* n *be the dimension of* π_λ*,* $d\pi_\lambda$ *be the infinitesimal version of* π_λ*,* I_λ *be the moment set of* π_λ*, and* ν_{I_λ} *be the probability* G*-invariant measure on* I_λ *(pushforward of the unitarily probability measure of the unit sphere of* $\mathcal{H}_\lambda$*). Then each function* γ_k *in (6.1.5), for* $k \in \{0, 1, ..., n-1\}$ *can be extended to* $d\pi_\lambda$ *on* $\mathfrak{g}$*, and*

$$(n-1)!\,\gamma_{n-1}(d\pi_\lambda(X)) = \int_{I_\lambda} e^{i\beta(X)}\,d\nu_{I_\lambda}(\beta) \tag{6.3.19}$$

for all $X \in \mathfrak{g}$*.*

Proof. If we replace $\zeta \in \mathbb{R}^n$ in $\gamma_{n-1}(\zeta)$ (6.3.2) by a diagonal matrix A with distinct entries $\zeta_1, ...\zeta_n$, then (6.3.1) is equivalent to

$$e^A = \gamma_0(A)I + \gamma_1(A)A + \cdots + \gamma_{n-1}(A)A^{n-1}. \tag{6.3.20}$$

If we let U be a unitary matrix, and denote $\widetilde{A} = UAU^*$, then

$$\begin{aligned} e^{\widetilde{A}} = Ue^AU^* &= \gamma_0(A)UIU^* + \gamma_1(A)UAU^* + \cdots + \gamma_{n-1}(A)(UAU^*)^{n-1} \\ &= \gamma_0(A)I + \gamma_1(A)\widetilde{A} + \cdots + \gamma_{n-1}(A)\widetilde{A}^{n-1} \\ &= \gamma_0(\widetilde{A})I + \gamma_1(\widetilde{A})\widetilde{A} + \cdots + \gamma_{n-1}(\widetilde{A})\widetilde{A}^{n-1}. \end{aligned} \tag{6.3.21}$$

So, γ_k for $k \in \{0, 1, ..., n-1\}$ each depends only on the diagonal matrix A, and is unitarily invariant. In addition, the $\phi_j(\zeta)$ function in (6.3.2) is the j-th elementary symmetric function of ζ. If we replace ζ by $\widetilde{A}$, then $\phi_j(\widetilde{A})$ is equal to the j-th elementary symmetric functions of the eigenvalues for $\widetilde{A}$, which is also equal to the sum of the principal minors of $\widetilde{A}$ of degree j, so that $\phi_0(\widetilde{A}) = 1, \phi_1(\widetilde{A}) = Tr(\widetilde{A})$ and $\phi_n(\widetilde{A}) = det(\widetilde{A})$.

Let $\Pi(\lambda) = \{\lambda_1, ..., \lambda_n\}$ be the set of weights of π_λ such that the multiplicity of each weight $multi(\lambda_j) = 1$. Since $d\pi_\lambda$ is skew-Hermitian and G-invariant, so its value only depends on $\mathfrak{t}$. Hence we have

$$\gamma_{n-1}(d\pi_\lambda(H)) = (-1)^{n-1}\sum_{k=1}^{n} \frac{e^{i\lambda_k(H)}}{\prod_{\substack{j\neq k \\ 1\leqslant j\leqslant n}} i(\lambda_j - \lambda_k)(H)}, \tag{6.3.22}$$

for $H \in \mathfrak{t}$. Now, by Proposition 5.2.2 and Proposition 5.2.3, we have

$$(n-1)!\gamma_{n-1}(d\pi_\lambda(H)) = \int_{\mathfrak{t}^*} e^{i\beta(H)}\, d\nu^p_{I_\lambda}(\beta) = \int_{\mathfrak{g}^*} e^{i\beta(H)}\, d\nu_{I_\lambda}(\beta), \tag{6.3.23}$$

for $H \in \mathfrak{t}$. Also, if $\Pi(\lambda) = \{\lambda_1, ..., \lambda_n\}$ is the set of weights of π_λ such that there exists a weight λ_j with $multi(\lambda_j) > 1$, then the relation in (6.3.23) still holds, by Proposition 5.2.5. Hence, the lemma follows. □

Now, we propose a *Kirillov-type non-commutative formula* for compact simply connected semisimple Lie groups.

Proposition 6.3.3. *Let G be a compact simply connected semisimple Lie group, T be a maximal torus of G, and $\mathfrak{g}, \mathfrak{t}$ be the Lie algebras of G, T, respectively. Let $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$, and $\mathfrak{t}^*_+$ be a chosen positive Weyl chamber of $\mathfrak{t}^*$. Let $\lambda \in \Lambda^+ \subset \mathfrak{t}^*_+$ be a dominant highest weight, π_λ be the irreducible unitary representation of the highest weight λ acting on a Hilbert space $\mathcal{H}$, n be the dimension of π_λ, $d\pi_\lambda$ be the infinitesimal version of π_λ, I_λ be the moment set of π_λ, and ν_{I_λ} be the probability G-invariant measure on I_λ (pushforward of the unitarily probability measure of the unit sphere of $\mathcal{H}_\lambda$). Let $\{X_1, ..., X_d\}$ be a basis of $\mathfrak{g}$, and denote $x{\cdot}X = \sum_{k=1}^d x_j X_j$, for $x \in \mathbb{R}^d$. Then, the closed form of the exponential mapping $d\pi_\lambda \mapsto e^{d\pi_\lambda}$ is given by*

$$\begin{aligned} e^{d\pi_\lambda(x\cdot X)} = \frac{1}{(n-1)!} \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \sum_{m=0}^{j} (-1)^{n-k-j-1} \frac{\phi_{n-k-j-1}\left(x \cdot d\pi_\lambda(X)\right)}{(n-j+m-1)!} \\ \cdot \left(x \cdot d\pi_\lambda(X)\right)^k\, \psi(m-1)\, \check{\nu}_{I_\lambda}(x \cdot X), \end{aligned} \tag{6.3.24}$$

where $\phi_j(A)$ is the sum of the principal minors of A of degree j,

$$\check{\nu}_{I_\lambda}(x \cdot X) = \int_{\mathfrak{g}^*} e^{i\beta(x\cdot X)}\, d\nu_{I_\lambda}(\beta), \tag{6.3.25}$$

and

$$\psi(p) = \sum_{q=0}^{p} (-1)^{p-q} \frac{p!}{q!} \left(x \cdot \frac{\partial}{\partial x}\right) \cdot \psi(q-1), \tag{6.3.26}$$

with $\psi(-1) = 1$.

Example 6.3.3.1. Let $G = SU(2)$, $\mathfrak{g} = \mathfrak{su}(2) \cong \mathbb{R}^3$ be the Lie algebra of $SU(2)$, $n \in \Lambda^+ \cong \mathbb{Z}^+$ be an integer highest weight, π_n be the irreducible highest weight unitary representation of n acting on a Hilbert space $\mathcal{H}_n$. Suppose if we let $n = 2$, then by (2.5.18), a basis of $\mathcal{H}_2$ is $\{\sqrt{3}z^2, \sqrt{6}zw, \sqrt{3}w^2\}$, for $(z, w) \in \mathbb{C}^2$. Recall that a basis of $\mathfrak{su}(2)$, $\{X_1, X_2, X_3\}$, can be found in (2.5.3), and a basis of the complexification $\mathfrak{su}(2)^{\mathbb{C}}$, $\{H, X, Y\}$ is given in (2.3.1). Also, by (3.1.17), we have

$$\begin{aligned} d\pi(H)^R &= z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}, \\ d\pi(X)^R &= z\frac{\partial}{\partial w}, \\ d\pi(Y)^R &= w\frac{\partial}{\partial z}. \end{aligned} \tag{6.3.27}$$

Hence, the actions of infinitesimal representation $d\pi$ are given by:

$$\begin{gathered} d\pi(H)^R\sqrt{3}z^2 = 2(\sqrt{3}z^2), \quad d\pi(H)^R\sqrt{6}zw = 0, \quad d\pi(H)^R\sqrt{3}w^2 = -2(\sqrt{3}w^2). \\ d\pi(X)^R\sqrt{3}z^2 = 0, \quad d\pi(X)^R\sqrt{6}zw = \sqrt{2}(\sqrt{3}z^2), \quad d\pi(X)^R\sqrt{3}w^2 = \sqrt{2}(\sqrt{6}zw). \\ d\pi(Y)^R\sqrt{3}z^2 = \sqrt{2}(\sqrt{6}zw), \quad d\pi(Y)^R\sqrt{6}zw = \sqrt{2}(\sqrt{3}w^2), \quad d\pi(Y)^R\sqrt{3}w^2 = 0. \end{gathered} \tag{6.3.28}$$

We take the compact real forms of $\mathfrak{su}(2)^{\mathbb{C}}$ and obtain

$$d\pi(X_1) = d\pi(X - Y), \quad d\pi(X_2) = id\pi(X + Y), \quad d\pi(X_3) = id\pi(H). \tag{6.3.29}$$

If we let $(x_1, x_2, x_3) \in \mathbb{R}^3$, and denote $x \cdot X = \sum_{j=1}^{3} = x_j X_j$, then the infinitesimal skew-Hermitian highest weight representation $d\pi_2$ is given by

$$d\pi_2(x \cdot X) = \begin{pmatrix} i2x_3 & \sqrt{2}(x_1 + ix_2) & 0 \\ -\sqrt{2}(x_1 - ix_2) & 0 & \sqrt{2}(x_1 + ix_2) \\ 0 & -\sqrt{2}(x_1 - ix_2) & -i2x_3. \end{pmatrix}, \tag{6.3.30}$$

By Proposition 5.2.4, we have the (rotationally) G-invariant analytic function

$$\check{\nu}_{I_2}(r) = \left(\frac{\sin r}{r}\right)^2, \tag{6.3.31}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Now, we substitute $d\pi_2$ into (6.3.24), and for $r\frac{\partial}{\partial r} =$

$x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}$, we have

$$\begin{aligned}
e^{d\pi_2} &= \frac{1}{2!}\sum_{k=0}^{2}\sum_{j=0}^{2-k}\sum_{m=0}^{j}(-1)^{2-k-j}\frac{\phi_{2-k-j}\,(d\pi_2)}{(2-j+m)!}\cdot(d\pi_2)^k\;\psi(m-1)\,\check{\nu}_{I_2}\\
&= \phi_0(d\pi_2)\psi(-1)I+\phi_0(d\pi_2)\psi(-1)d\pi_2+\frac{1}{2}\phi_0(d\pi_2)\psi(-1)(d\pi_2)^2\\
&-\frac{1}{2}\phi_1(d\pi_2)\psi(-1)I+\phi_1(d\pi_2)\psi(-1)d\pi_2+\frac{1}{2}\phi_2(d\pi_2)\psi(-1)I\\
&+2\phi_0(d\pi_2)\psi(0)I+\frac{1}{2}\phi_0(d\pi_2)\psi(0)d\pi_2-\frac{1}{2}\phi_1(d\pi_2)\psi(0)I+\frac{1}{2}\phi_0(d\pi_2)\psi(1)I\\
&= I+d\pi_2+\frac{1}{2}(d\pi_2)^2+2\left(x_1^2+x_2^2+x_3^2\right)I+2r\frac{\partial}{\partial r}I+\frac{1}{2}r\frac{\partial}{\partial r}d\pi_2\\
&+\frac{1}{2}\left(-r\frac{\partial}{\partial r}+\left(r\frac{\partial}{\partial r}\right)^2\right)I,
\end{aligned} \tag{6.3.32}$$

multiplied by $\check{\nu}_{I_2}(r)$. Therefore, we denote $e^{d\pi_2}_{i,j}$ the (i,j)-th element of the matrix $e^{d\pi_2}$, we have

$$\begin{aligned}
e^{d\pi_2}_{1,1} &= \left(1+\frac{3}{2}r\frac{\partial}{\partial r}+\frac{1}{2}\left(r\frac{\partial}{\partial r}\right)^2+(r^2-x_3^2)+i2x_3+ix_3r\frac{\partial}{\partial r}\right)\check{\nu}_{I_2}(r)\\
&= \left(\frac{\sin r}{r}\right)^2+\frac{3(r\cos r-\sin r)\sin r}{r^2}+\frac{2+2(-1+r^2)\cos 2r-3r\sin 2r}{2r^2}\\
&+(r^2-x_3^2)\left(\frac{\sin r}{r}\right)^2+i2x_3\left(\frac{\sin r}{r}\right)^2+ix_3\left(\frac{2(r\cos r-\sin r)\sin r}{r^2}\right)\\
&= \frac{(r\cos r+ix_3\sin r)^2}{r^2}=\left(\cos r+ix_3\frac{\sin r}{r}\right)^2,
\end{aligned} \tag{6.3.33}$$

which is exactly the $\alpha^2(\exp x\cdot X)$ by (2.5.25), and is also the lift of the $(1,1)$-th matrix coefficient of π_2, by Example 2.5.5.1. The remaining elements of $e^{d\pi_2}$

are:

$$\begin{aligned}
e^{d\pi_2}_{1,2} &= \left(\sqrt{2}x_1 + \frac{1}{\sqrt{2}}x_1\, r\frac{\partial}{\partial r} + i\sqrt{2}x_2 + i\frac{1}{\sqrt{2}}x_2\, r\frac{\partial}{\partial r} + i\sqrt{2}x_1x_3 - \sqrt{2}x_2x_3\right)\check{\nu}_{I_2}(r)\\
&= \frac{1}{\sqrt{2}}(x_1+ix_2)\left(2 + r\frac{\partial}{\partial r} + i2x_3\right)\left(\frac{\sin r}{r}\right)^2\\
&= \frac{\sqrt{2}\sin r(x_1+ix_2)(r\cos r + ix_3\sin r)}{r^2}\\
&= \sqrt{2}\,\alpha(\exp x\cdot X)\beta(\exp x\cdot X).\\
e^{d\pi_2}_{1,3} &= \left(x_1^2 + i2x_1x_2 - x_2^2\right)\left(\frac{\sin r}{r}\right)^2\\
&= (x_1+ix_2)^2\left(\frac{\sin r}{r}\right)^2\\
&= \beta^2(\exp x\cdot X).
\end{aligned} \tag{6.3.34}$$

$$\begin{aligned}
e^{d\pi_2}_{2,1} &= \left(-\sqrt{2}x_1 - \frac{1}{\sqrt{2}}x_1\, r\frac{\partial}{\partial r} + i\sqrt{2}x_2 + i\frac{1}{\sqrt{2}}x_2\, r\frac{\partial}{\partial r} - i\sqrt{2}x_1x_3 - \sqrt{2}x_2x_3\right)\check{\nu}_{I_2}(r)\\
&= -\sqrt{2}\,\alpha(\exp x\cdot X)\overline{\beta}(\exp x\cdot X).\\
e^{d\pi_2}_{2,2} &= \left(1 + \frac{3}{2}r\frac{\partial}{\partial r} + \frac{1}{2}\left(r\frac{\partial}{\partial r}\right)^2 + 2x_3^2\right)\check{\nu}_{I_2}(r)\\
&= \cos 2r + 2x_3^2\left(\frac{\sin r}{r}\right)^2 = (\cos r)^2 - \frac{(\sin r)^2(x_1^2+x_2^2-x_3^2)}{r^2}\\
&= \alpha(\exp x\cdot X)\overline{\alpha}(\exp x\cdot X) - \beta(\exp x\cdot X)\overline{\beta}(\exp x\cdot X).\\
e^{d\pi_2}_{2,3} &= \left(\sqrt{2}x_1 + \frac{1}{\sqrt{2}}x_1\, r\frac{\partial}{\partial r} + i\sqrt{2}x_2 + i\frac{1}{\sqrt{2}}x_2\, r\frac{\partial}{\partial r} - i\sqrt{2}x_1x_3 + \sqrt{2}x_2x_3\right)\check{\nu}_{I_2}(r)\\
&= \sqrt{2}\,\overline{\alpha}(\exp x\cdot X)\beta(\exp x\cdot X).
\end{aligned} \tag{6.3.35}$$

$$\begin{aligned}
e_{3,1}^{d\pi_2} &= \left(x_1^2 - i2x_1x_2 - x_2^2\right)\left(\frac{\sin r}{r}\right)^2\\
&= (-\overline{\beta})^2(\exp x\cdot X).\\
e_{3,2}^{d\pi_2} &= \left(-\sqrt{2}x_1 - \frac{1}{\sqrt{2}}x_1\, r\frac{\partial}{\partial r} + i\sqrt{2}x_2 + i\frac{1}{\sqrt{2}}x_2\, r\frac{\partial}{\partial r} + i\sqrt{2}x_1x_3 + \sqrt{2}x_2x_3\right)\check{\nu}_{I_2}(r)\\
&= -\sqrt{2}\,\overline{\alpha}(\exp x\cdot X)\overline{\beta}(\exp x\cdot X).\\
e_{3,3}^{d\pi_2} &= \left(1 + \frac{3}{2}r\frac{\partial}{\partial r} + \frac{1}{2}\left(r\frac{\partial}{\partial r}\right)^2 + (r^2 - x_3^2) - i2x_3 - ix_3r\frac{\partial}{\partial r}\right)\check{\nu}_{I_2}(r)\\
&= \overline{\alpha}^2(\exp x\cdot X).
\end{aligned} \tag{6.3.36}$$

where we notice that $x_1^2+x_2^2 = r^2-x_3^2$. Hence, we have completed the calculation for $e^{d\pi_2}$. Also, recall that by Proposition 3.2.5, every matrix coefficient $T_{i,j}$ of the irreducible unitary representation of G satisfies

$$T_{i,j}\circ\exp = D_{i,j}^G\cdot T_{1,1}\circ\exp \tag{6.3.37}$$

where $D_{i,j}^G$ is a polynomials of differential operators in the Lie algebra $\mathfrak{g}$ induced by the G action. By (3.2.19), we have

$$\begin{aligned}
H_\alpha^G &= \frac{1}{i}X_3^G: && \frac{1}{i}\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right),\\
X_\alpha^G &= \frac{1}{2}(X_1^G - iX_2^G): && (ix_1+x_2)\frac{\partial}{\partial x_3} - x_3(i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}),\\
Y_\alpha^G &= -\frac{1}{2}(X_1^G + iX_2^G): && (ix_1-x_2)\frac{\partial}{\partial x_3} - x_3(i\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}).
\end{aligned} \tag{6.3.38}$$

By Example 3.2.5.1, we have

$$\begin{aligned}
Y_\alpha^G\cdot\alpha^2(\exp x\cdot X) &= -\frac{1}{2}\left(X_1^G + iX_2^G\right)\cdot\alpha^2(\exp x\cdot X)\\
&= \left((ix_1-x_2)\frac{\partial}{\partial x_3} - x_3(i\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2})\right)\\
&\cdot\left(1 + \frac{3}{2}r\frac{\partial}{\partial r} + \frac{1}{2}\left(r\frac{\partial}{\partial r}\right)^2 + (r^2-x_3^2) + i2x_3 + ix_3r\frac{\partial}{\partial r}\right)\left(\frac{\sin r}{r}\right)^2.
\end{aligned} \tag{6.3.39}$$

Since

$$\left(r\frac{\partial}{\partial r}\right)^p\left(\frac{\sin r}{r}\right)^q, \tag{6.3.40}$$

is G-invariant for any integers p, q (Proposition 6.2.5), the actions of the differential operators Z^G induced by the action of G,

$$Z^G \cdot \left(r\frac{\partial}{\partial r}\right)^p \left(\frac{\sin r}{r}\right)^q = 0, \quad \forall Z \in \mathfrak{su}(2), \tag{6.3.41}$$

by the definition of the induced differential operators by action of G (Definition 3.2.1). Hence, a quick calculation shows

$$\begin{aligned} Y_\alpha^G \cdot \alpha^2(\exp x \cdot X) &= \left((ix_1 - x_2)\frac{\partial}{\partial x_3} - x_3(i\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2})\right) \\ &\quad \cdot \left(x_1^2 + x_2^2 + i2x_3 + ix_3 r\frac{\partial}{\partial r}\right)\left(\frac{\sin r}{r}\right)^2 \\ &= -(x_1 + ix_2)\left(2 + r\frac{\partial}{\partial r} + i2x_3\right)\left(\frac{\sin r}{r}\right)^2 \\ &= -2\alpha(\exp x \cdot X)\beta(\exp x \cdot X). \end{aligned} \tag{6.3.42}$$

Also, we have

$$Y_\alpha^G \cdot \alpha(\exp x \cdot X)\beta(\exp x \cdot X) = -\beta^2(\exp x \cdot X). \tag{6.3.43}$$

Therefore, we have $(-Y_\alpha^G) \cdot T_{1,1} \circ \exp = \sqrt{2}\, T_{1,2} \circ \exp$, $(-Y_\alpha^G) \cdot T_{1,2} \circ \exp = \sqrt{2}\, T_{1,3} \circ \exp$. Similarly, we have $X_\alpha^G \cdot T_{1,1} \circ \exp = \sqrt{2}\, T_{2,1} \circ \exp$, and by further applying polynomials of X_α^G and Y_α^G, we can obtain every other matrix coefficient of $\pi_2 \circ \exp$. We put these polynomials in a matrix:

$$\begin{pmatrix} 1 & -Y_\alpha^G & (-Y_\alpha^G)^2 \\ X_\alpha^G & -(1 + Y_\alpha^G X^G) & 2Y_\alpha^G + (Y_\alpha^G)^2 X_\alpha^G \\ (X_\alpha^G)^2 & -(X_\alpha^G + Y_\alpha^G (X_\alpha^G)^2) & 1 + 2Y_\alpha^G X_\alpha^G + (Y_\alpha^G)^2 (X_\alpha^G)^2 \end{pmatrix} \tag{6.3.44}$$

This show our result satisfies Prop 3.1.7.

We may also write down the polynomial of $P_{1,1}^3$ of $e_{1,1}^{d\pi_3}$ ($d\pi_3$ in (3.1.19)), that is,

$$\begin{aligned} P_{1,1}^3 &= \frac{1}{6}\left(r\frac{\partial}{\partial r}\right)^3 + \left(r\frac{\partial}{\partial r}\right)^2 + \frac{11}{6}\left(r\frac{\partial}{\partial r}\right) + 1 \\ &\quad + ix_3\left(\frac{1}{2}\left(r\frac{\partial}{\partial r}\right)^2 + \frac{5}{2}\left(r\frac{\partial}{\partial r}\right) + 3\right) + x_3^2\left(\frac{1}{6}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{2}\right) \\ &\quad + \frac{7}{6}(r^2 - x_3^2)\left(r\frac{\partial}{\partial r}\right) + \frac{3}{2}(r^2 - x_3^2)ix_3 + \frac{7}{2}(r^2 - x_3^2) + \frac{1}{2}ix_3^3. \end{aligned} \tag{6.3.45}$$

Hence, a short calculation shows that

$$P^3_{1,1} \cdot \left(\frac{\sin r}{r}\right)^3 = \left(\cos r + ix_3 \frac{\sin r}{r}\right)^3 = \alpha^3(\exp x \cdot X). \tag{6.3.46}$$

Remark.

- *This example was the starting point for this project. When we first applied Nelson's formula to calculate* $e^{d\pi_2}$*, the result did not match* $\pi_2 \circ \exp$ *in Example 2.5.5.1. This led to a review of Nelson's formula, and later we found the missing* ψ *recursive function. Nelson's formula works for the 2-dimensional case in Example 6.1.4.1 because* $\psi(-1) = 1$*, which therefore does not affect the final solution.*

- *Also, initially we let*
$$\breve{\nu}_{I_2}(r) = \left(\frac{\sin r}{r}\right)^2,$$
just based on our assumption that the moment set I_n *is the* n*-fold convolution of coadjoint orbit* $\mathcal{O}_1$*, from Cazzaniga's result for* $SU(2)$*. Later, we derived an explicit formula for the measure* ν_{I_λ} *(in Chapter 5). Then, we discovered that this assumption is indeed true for* $SU(2)$*, by Proposition 5.2.4.*

- *In addition, since each row and column of the irreducible unitary representation of* G *form a basis for the underlying Hilbert space, we can then combine the theorem of the highest weight and differential operators induced by the action of* G *to further check that our formula (6.3.24) indeed produces the correct result.*

Example 6.3.3.2. We calculate the example of $SU(3)$. Let $\mathfrak{su}(3)$ be the Lie algebra of $SU(3)$. Let $\gamma = \left(\{h_i\}_{i=1}^2, \{x_j\}_{j=1}^3, \{y_k\}_{k=1}^3\right) \in \mathbb{R}^8$. A standard basis for $\mathfrak{su}(3)$ (8-dimensional) is given in (2.2.16), and the matrix form for $X \in \mathfrak{su}(3)$ can be written as

$$X = \begin{pmatrix} ih_1 & x_1 + iy_1 & x_3 + iy_3 \\ -x_1 + iy_1 & ih_2 - ih_1 & x_2 + iy_2 \\ -x_3 + iy_3 & -x_2 + iy_2 & -ih_2 \end{pmatrix} \tag{6.3.47}$$

This matrix is also the irreducible skew-Hermitian highest weight representation with respect to the first fundamental weight λ_1 of $\mathfrak{su}(3)$. We denote it by $d\pi_{\lambda_1}$.

The Cartan subalgebra of $\mathfrak{su}(3)$, is $\mathfrak{t} \cong \mathbb{R}^2$ (and its dual $\mathfrak{t}^* \cong \mathbb{R}^2$). Let $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ be the set of positive roots of $\mathfrak{t}^*$. Let W be the Weyl group (5.1.2) of $\mathfrak{t}, \mathfrak{t}^*$. Also, let $W_{\lambda_1} = \{e, \sigma_{\alpha_2}\}$ be the stabiliser subgroup. Let $\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$ be the first fundamental weight of $\mathfrak{t}^*$, then the Weyl orbit of λ_1 is the set

$$W \cdot \lambda_1 = \{\lambda_1, \lambda_1 - \alpha_1, \lambda_1 - \alpha_1 - \alpha_2\} \cong W/W_{\lambda_1} = \{e, \sigma_{\alpha_1}, \sigma_2\sigma_1\}. \tag{6.3.48}$$

Let π_{λ_1} be the irreducible unitary representation of λ_1, $\Pi(\lambda_1)$ be the set of weights of π_{λ_1} which coincides with $W \cdot \lambda_1$, and I_{λ_1} be the moment set of π_{λ_1}. Since $d\pi_{\lambda_1}$ is also the defining representation of $\mathfrak{su}(3)$, so the moment set of I_{λ_1} is a single coadjoint orbit $\mathcal{O}_{\lambda_1}$. So, we recall that the Fourier transform of the G-invariant measure $\nu_{I_{\lambda_1}}$ in (5.2.33), is given by

$$\begin{aligned}
\left(\nu_{I_{\lambda_1}}\right)^{\vee_{\mathfrak{g}}}|_{\mathfrak{t}}(H) &= (\nu_{\lambda_1})^{\vee_{\mathfrak{g}}}|_{\mathfrak{t}}(H) \\
&= 2\frac{e^{i\lambda_1(H)}}{i^2(-\alpha_1)(H)(-\alpha_1-\alpha_2)(H)} + 2\frac{e^{i(\lambda_1-\alpha_1)(H)}}{i^2(\alpha_1)(H)(-\alpha_2)(H)} \\
&+ 2\frac{e^{i(\lambda_1-\alpha_1-\alpha_2)(H)}}{i^2(\alpha_1+\alpha_2)(H)(\alpha_2)(H)},
\end{aligned} \tag{6.3.49}$$

for $H \in \mathfrak{t}$. Now, by (6.3.24), the exponential mapping $e^{d\pi_{\lambda_1}}$ is given by

$$e^{d\pi_{\lambda_1}} = \frac{1}{2!}\sum_{k=0}^{2}\sum_{j=0}^{2-k}\sum_{m=0}^{j}(-1)^{2-k-j}\frac{\phi_{2-k-j}\left(d\pi_{\lambda_1}\right)}{(2-j+m)!}\cdot\left(d\pi_{\lambda_1}\right)^k\,\psi(m-1)\,\check{\nu}_{I_{\lambda_1}}. \tag{6.3.50}$$

We denote $P^{\lambda_1}_{i,j}$ the (i,j)-th element of the polynomials of $e^{d\pi_{\lambda_1}}$, and let $\gamma\frac{\partial}{\partial\gamma} = \sum_{i=1}^{2} h_i\frac{\partial}{\partial h_i} + \sum_{j=1}^{3} x_j\frac{\partial}{\partial x_j} + \sum_{k=1}^{3} y_k\frac{\partial}{\partial y_k}$. Then we have,

$$\begin{aligned}
P^{\lambda_1}_{1,1} &= 2 + 3\gamma\frac{\partial}{\partial\gamma} + \left(\gamma\frac{\partial}{\partial\gamma}\right)^2 + ih_1\left(2 + \gamma\frac{\partial}{\partial\gamma} + ih_2\right) + h_2^2 + x_2^2 + y_2^2, \\
P^{\lambda_1}_{1,2} &= (x_1 + iy_1)\left(2 + \gamma\frac{\partial}{\partial\gamma} + ih_2\right) + (-x_2 + iy_2)(x_3 + iy_3), \\
P^{\lambda_1}_{1,3} &= (x_3 + iy_3)\left(2 + \gamma\frac{\partial}{\partial\gamma} + ih_1 - ih_2\right) + (x_1 + iy_1)(x_2 + iy_2),
\end{aligned} \tag{6.3.51}$$

$$\begin{aligned}
P^{\lambda_1}_{2,1} &= (-x_1 + iy_1)\left(2 + \gamma\frac{\partial}{\partial\gamma} + ih_2\right) + (x_2 + iy_2)(-x_3 + iy_3),\\
P^{\lambda_1}_{2,2} &= 2 + 3\gamma\frac{\partial}{\partial\gamma} + \left(\gamma\frac{\partial}{\partial\gamma}\right)^2 + (ih_2 - ih_1)\left(2 + \gamma\frac{\partial}{\partial\gamma}\right) + h_1h_2 + x_3^2 + y_3^2,\\
P^{\lambda_1}_{2,3} &= (x_2 + iy_2)\left(2 + \gamma\frac{\partial}{\partial\gamma} - ih_1\right) + (-x_1 + iy_1)(x_3 + iy_3),
\end{aligned} \tag{6.3.52}$$

$$\begin{aligned}
P^{\lambda_1}_{3,1} &= (-x_3 + iy_3)\left(2 + \gamma\frac{\partial}{\partial\gamma} + ih_1 - ih_2\right) + (-x_1 + iy_1)(-x_2 + iy_2),\\
P^{\lambda_1}_{3,2} &= (-x_2 + iy_2)\left(2 + \gamma\frac{\partial}{\partial\gamma} - ih_1\right) + (x_1 + iy_1)(-x_3 + iy_3),\\
P^{\lambda_1}_{3,3} &= 2 + 3\gamma\frac{\partial}{\partial\gamma} + \left(\gamma\frac{\partial}{\partial\gamma}\right)^2 + ih_2\left(-2 - \gamma\frac{\partial}{\partial\gamma} + ih_1\right) + h_1^2 + x_1^2 + y_1^2.
\end{aligned} \tag{6.3.53}$$

We also have differential operators induced by the action of G in the root directions given in (3.2.22). Let $\Pi(\lambda_1) = \{\lambda_1, \lambda_1 - \alpha_1, \lambda_1 - \alpha_1 - \alpha_2\}$, and $\{v_1, v_2, v_3\}$ be the corresponding set of weight vectors. The actions of the lowering operators on v_1 are given by

$$d\pi(Y_{\alpha_1})v_1 = v_2, \quad d\pi(Y_{\alpha_2})v_1 = 0, \quad d\pi(Y_{\alpha_3})v_1 = v_3. \tag{6.3.54}$$

Hence, we can show that $-Y^G_{\alpha_1}\cdot \overset{d\pi_{\lambda_1}}{e_{1,1}} = \overset{d\pi_{\lambda_1}}{e_{1,2}}$ and $-Y^G_{\alpha_3}\cdot \overset{d\pi_{\lambda_1}}{e_{1,1}} = \overset{d\pi_{\lambda_1}}{e_{1,3}}$. First, we have

$$\begin{aligned}
Y^G_{\alpha_1} &= (ix_1 - y_1)\frac{\partial}{\partial h_1} + \frac{1}{2}(2h_1 - h_2)\frac{\partial}{\partial y_1} - i\frac{1}{2}(2h_1 - h_2)\frac{\partial}{\partial x_1} - \frac{1}{2}(x_2 - iy_2)\frac{\partial}{\partial x_3}\\
&\quad - \frac{1}{2}(y_2 + ix_2)\frac{\partial}{\partial y_3} + \frac{1}{2}(x_3 + iy_3)\frac{\partial}{\partial x_2} - \frac{1}{2}(ix_3 - y_3)\frac{\partial}{\partial y_2},\\
Y^G_{\alpha_3} &= (ix_3 - y_3)\left(\frac{\partial}{\partial h_1} + \frac{\partial}{\partial h_2}\right) + \frac{1}{2}(h_1 + h_2)\frac{\partial}{\partial y_3} - i\frac{1}{2}(h_1 + h_2)\frac{\partial}{\partial x_3}\\
&\quad + \frac{1}{2}(x_2 + iy_2)\frac{\partial}{\partial x_1} + \frac{1}{2}(ix_2 - y_2)\frac{\partial}{\partial y_1} - \frac{1}{2}(x_1 + iy_1)\frac{\partial}{\partial x_2} - \frac{1}{2}(ix_1 - y_1)\frac{\partial}{\partial y_2}.
\end{aligned} \tag{6.3.55}$$

Also, because $X^G \cdot \left(\gamma\frac{\partial}{\partial\gamma}\right)^p \check{\nu}_{I_{\lambda_1}}(\gamma) = 0$, for all $X \in \mathfrak{su}(3)$, so we can reduce $P^{\lambda_1}_{1,1}$ to

$$\tilde{P}^{\lambda_1}_{1,1} = ih_1\left(2 + \gamma\frac{\partial}{\partial\gamma} + ih_2\right) + h_2^2 + x_2^2 + y_2^2. \tag{6.3.56}$$

Hence, quick calculations show that

$$-Y^G_{\alpha_1} \cdot \tilde{P}^{\lambda_1}_{1,1} = P^{\lambda_1}_{1,2}, \quad -Y^G_{\alpha_3} \cdot \tilde{P}^{\lambda_1}_{1,1} = P^{\lambda_1}_{1,3}. \tag{6.3.57}$$

Furthermore, since every row and column of $e^{d\pi_{\lambda_1}}$ is a copy of $\{v_1, v_2, v_3\}$, we can also determine a matrix of polynomials of induced differential operators by action of G, acting on $e^{d\pi_1}_{1,1}$. That is,

$$D_{\lambda_1} = \begin{pmatrix} 1 & -Y_{\alpha_1} & -Y_{\alpha_3} \\ X_{\alpha_1} & 1 - Y_{\alpha_1} X_{\alpha_1} & -Y_{\alpha_3} X_{\alpha_1} \\ X_{\alpha_3} & -Y_{\alpha_1} X_{\alpha_3} & 1 - Y_{\alpha_3} X_{\alpha_3} \end{pmatrix}. \tag{6.3.58}$$

In addition, we can obtain a global version of $\nu^{\vee\mathfrak{g}}_{\lambda_1}|_{\mathfrak{su}(3)}$. If we take the eigenvalues of $d\pi_{\lambda_1}$ in (6.3.47), and use the *Cardano formula* for the cubic root, then we have

$$\begin{aligned} &\nu^{\vee\mathfrak{g}}_{\lambda_1}(\gamma) = \\ &\quad -\frac{e^{ia(\gamma)}(a(\gamma) - 2b(\gamma)) + e^{-ib(\gamma)}(-2a(\gamma) + b(\gamma)) + e^{-i(a(\gamma)-b(\gamma))}(a(\gamma) + b(\gamma))}{(a(\gamma) - 2b(\gamma))(2a(\gamma) - b(\gamma))(a(\gamma) + b(\gamma))}, \end{aligned} \tag{6.3.59}$$

where

$$\begin{aligned} a(\gamma) &= \sqrt[3]{-\frac{1}{2}d(\gamma) + \sqrt[2]{\left(\frac{1}{2}d(\gamma)\right)^2 + \left(\frac{1}{3}c(\gamma)\right)^3}} \\ &\quad + \sqrt[3]{-\frac{1}{2}d(\gamma) - \sqrt[2]{\left(\frac{1}{2}d(\gamma)\right)^2 + \left(\frac{1}{3}c(\gamma)\right)^3}}, \\ b(\gamma) &= \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \sqrt[3]{-\frac{1}{2}d(\gamma) + \sqrt[2]{\left(\frac{1}{2}d(\gamma)\right)^2 + \left(\frac{1}{3}c(\gamma)\right)^3}} \\ &\quad + \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \sqrt[3]{-\frac{1}{2}d(\gamma) - \sqrt[2]{\left(\frac{1}{2}d(\gamma)\right)^2 + \left(\frac{1}{3}c(\gamma)\right)^3}}, \end{aligned} \tag{6.3.60}$$

and

$$\begin{aligned} c(\gamma) &= -(h_1^2 - h_1h_2 + h_2^2 + x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2), \\ d(\gamma) &= -(h_1^2h_2 - h_1h_2^2 + h_2x_1^2 - h_1x_2^2 + h_1x_3^2 - h_2x_3^2 + 2x_2x_3y_1 + h_2y_1^2 \\ &\quad + 2x_1x_3y_2 - h_1y_2^2 - 2x_1x_2y_3 + 2y_1y_2y_3 + h_1y_3^2 - h_2y_3^2). \end{aligned} \tag{6.3.61}$$

Notice that $c(\gamma)$ is the Killing form of $\mathfrak{su}(3)$. A representation of $SO(3)$ can be obtained by restricting to (x_1, x_2, x_3), and this can be compared with the spherical coordinate representation of $SO(3)$.

We can also recover the character of π_{λ_1} by taking the sum of the diagonal elements, that is, $Tr(P^{\lambda_1})\,\breve{\nu}_{\lambda_1}$. The half-sum of the positive roots is given by $\delta = \alpha_1 + \alpha_2$. Hence, the character formula of λ_1 is given by

$$\chi_{\lambda_1}(e^H) = \frac{\sum_{w\in W} sgn(w)e^{iw(\lambda_1+\delta)(H)}}{\sum_{w\in W} sgn(w)e^{iw(\delta)(H)}}, \quad H \in \mathfrak{t}. \tag{6.3.62}$$

To perform a concrete calculation, let $\mathfrak{t} \cong \mathbb{R}^2$. We shall choose a basis $\{H_1, H_2\}$ of $\mathbb{R}^2$ such that

$$\alpha_1(H_1) = 2, \quad \alpha_1(H_2) = -1, \quad \alpha_2(H_1) = -1, \quad \alpha_2(H_2) = 2. \tag{6.3.63}$$

Therefore, H_1, H_2 can be realised as $H_1 = (1, 1)$ and $H_2 = (-1, 0)$, and $\alpha_1 = (1, 1)$ and $\alpha_2 = (-2, 1)$. The Weyl orbit of $\lambda_1 + \delta$ is

$$W \cdot (\lambda_1 + \delta) = \{\lambda_1 \underset{+}{+} \delta, \lambda_1 - \underset{-}{\alpha_1} + \alpha_2, \lambda_1 - \underset{+}{\alpha_1} - 2\alpha_2, \lambda_1 \underset{-}{-} 2\delta, \lambda_1 \underset{+}{-} 2\alpha_1, \lambda_1 \underset{-}{+} \alpha\}, \tag{6.3.64}$$

and the plus-minus signs are with respect to the Weyl group. The Weyl orbit of δ is

$$W \cdot (\delta) = \{\underset{+}{\delta}, \underset{-}{\alpha_2}, \underset{+}{-}\alpha_2, \underset{-}{-}\delta, \underset{+}{-}\alpha_1, \underset{-}{\alpha_1}\}, \tag{6.3.65}$$

Let $(h_1 - h_2, h_1) \in \mathbb{R}^2$, then the character formula (6.3.62) on $\mathbb{R}^2$ is given by

$$\chi_{\lambda_1}(h_1 - h_2, h_1) = e^{ih_1} + e^{-i(h_1-h_2)} + e^{-ih_2}. \tag{6.3.66}$$

On the other hand, by taking the trace of $e^{d\pi_{\lambda_1}}$, the character formula of λ_1 can be calculated as

$$\chi_{\lambda_1}(h_1-h_2,h_1) = \left(6 + 9\,\gamma\frac{\partial}{\partial\gamma} + 3\left(\gamma\frac{\partial}{\partial\gamma}\right)^2 + h_1^2 - h_1h_2 + h_2^2\right)\check{\nu}_{\lambda_1}(h_1-h_2,h_1), \tag{6.3.67}$$

where $\gamma\frac{\partial}{\partial\gamma} = h_1\frac{\partial}{\partial h_1} + h_2\frac{\partial}{\partial h_2}$ and

$$\check{\nu}_{\lambda_1}(h_1-h_2,h_1) = \frac{e^{-i(h_1+h_2)\left(-e^{i(2h_1+h_2)}(h_1-2h_2)+e^{ih_1}(2h_1-h_2)-e^{i2h_2}(h_1+h_2)\right)}}{(h_1-2h_2)(2h_1-h_2)(h_1+h_2)}, \tag{6.3.68}$$

which is also equal to $e^{ih_1} + e^{-i(h_1-h_2)} + e^{-ih_2}$.

We are also interested in working out the Fourier transform of the Kirillov-type non-commutative formula (6.3.24). Firstly, we examine the Fourier transform of polynomials in $\mathfrak{g}$.

Let $\mathfrak{g}^{\mathbb{C}}$ be a complex semisimple Lie algebra. Let $|\Phi^+| = r$ be the cardinality of the set of all positive roots of $\mathfrak{g}^{\mathbb{C}}$, and $|\Delta| = l$ be the cardinality of the set of simple roots, and $d = l + 2r$. Let $\mathfrak{g}$ be the compact real form of $\mathfrak{g}^{\mathbb{C}}$. Recall that by Definition 4.1.4, if we let $h = (h_1, ..., h_l) \in \mathbb{R}^l, x = (x_1, ..., x_r) \in \mathbb{R}^r, y = (y_1, ..., y_r) \in \mathbb{R}^r$, then the decomposition of $\mathfrak{g}$ satisfies

$$\mathfrak{g} = \underbrace{\sum_{\alpha\in\Delta}\mathbb{R}(iH_\alpha)}_{h\cdot H} + \underbrace{\sum_{\alpha\in\Phi^+}\mathbb{R}(X_\alpha - X_{-\alpha})}_{x\cdot X} + \underbrace{\sum_{\alpha\in\Phi^+}\mathbb{R}i(X_\alpha + X_{-\alpha})}_{y\cdot Y}. \tag{6.3.69}$$

where each X_α ($X_{-\alpha}$) belongs to the 1-dimensional root space $\mathfrak{g}_\alpha$ ($\mathfrak{g}_{-\alpha}$) for every $\alpha \in \Phi^+$.

Lemma 6.3.4. *Let $\beta = (\beta_1, ...\beta_l) \in \left(\mathbb{R}^l\right)^*$, $(\xi_1, ..., \xi_r) \in (\mathbb{R}^r)^*$, $(\eta_1, ..., \eta_r) \in (\mathbb{R}^r)^*$. Let $c_{i,j}$ be (i,j)-th element of the* ***Cartan matrix*** *C of $\mathfrak{g}$. Let $f \in L^1(\mathfrak{g})$. With respect to Killing form $(\cdot,\cdot)$ of $\mathfrak{g}$, we have*

$$(h_jf)\hat{} = -i\left(c_j^{-1}\cdot\frac{\partial}{\partial\alpha}\right)\hat{f}, \qquad \textit{and} \qquad ((x_j+iy_j)f)\hat{} = -i\left(\frac{\partial}{\partial\xi_j} + i\frac{\partial}{\partial\eta_j}\right)\hat{f}, \tag{6.3.70}$$

where $c_j^{-1} = (c_{j1}^{-1}, ...c_{jl}^{-1})$ is the j-th row of the inverse C^{-1} and is also the set of coefficients of the fundamental weight associated with the set of simple roots Δ.

Proof. Let $H = (H_1, ..., H_l)$ be the tuple of a basis of Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$.

We have

$$(\beta \cdot H, h \cdot H) = (\beta_1 H_{\alpha_1} + ... + \beta_l H_{\alpha_l}\,,\; h_1 H_{\alpha_1} + ... + h_l H_{\alpha_l}). \tag{6.3.71}$$

We normalise the Killing form to be $(H_{\alpha_i}, H_{\alpha_j}) = c_{i,j}$, and since the Killing form is non-degenerate on $\mathfrak{t}$, that is, $(H, H) \neq 0$ for $H \in \mathfrak{t}$ and $(H, X) = 0$ for $X \in \mathfrak{g}/\mathfrak{t}$. Then we only need to calculate

$$\langle \beta \cdot H, h \cdot H \rangle = (c_1 \cdot h)\beta_1 + ... + (c_l \cdot h)\beta_l\,, \tag{6.3.72}$$

where c_j is the j-th row of Cartan matrix C. If we apply the differential operator $c_j^{-1} \cdot \frac{\partial}{\partial \beta}$ to $e^{i\langle \beta \cdot H, h \cdot H \rangle}$, we obtain

$$\begin{pmatrix} c_{j1}^{-1} & \dots & c_{jl}^{-1} \end{pmatrix} \cdot \begin{pmatrix} c_{11} & \dots & c_{1l} \\ \vdots & \ddots & \vdots \\ c_{l1} & \dots & c_{ll} \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_l \end{pmatrix} = h_j. \tag{6.3.73}$$

Also, the Killing form satisfies $(X_\alpha, X_{-\alpha}) \neq 0$ for $\alpha \in \Phi^+$, so we have

$$(x_j f)\widehat{} = -i \frac{\partial}{\partial \xi_j} \widehat{f} \qquad \text{and} \qquad (y_j f)\widehat{} = -i \frac{\partial}{\partial \eta_j} \widehat{f}. \tag{6.3.74}$$

Hence the lemma follows. □

Remark. *Notice that* $\left(c_j^{-1} \cdot \frac{\partial}{\partial \beta}\right)$ *is the directional derivative in a fundamental weight* λ_j *direction, and* $\left(\frac{\partial}{\partial \xi_j} - i \frac{\partial}{\partial \eta_j}\right)$ *is the directional derivative in the positive root* α_j *direction (negative root direction if there is a plus sign within the parentheses).*

Now, we propose the Fourier transform of the Kirillov-type non-commutative formula for compact simply connected semisimple Lie groups.

Proposition 6.3.5. *Let* G *be a compact simply connected semisimple Lie group,* T *be a maximal torus of* G*, and* $\mathfrak{g}, \mathfrak{t}$ *be the Lie algebras of* G, T *respectively. Let* $\mathfrak{t}^*$ *be the dual of* $\mathfrak{t}$*, and* $\mathfrak{t}^*_+$ *be a chosen positive Weyl chamber of* $\mathfrak{t}^*$*. Let* $\lambda \in \Lambda^+ \subset \mathfrak{t}^*_+$ *be a dominant highest weight,* π_λ *be the irreducible highest weight unitary representation of* λ *acting on a Hilbert space* $\mathcal{H}$*,* n *be the dimension of* π_λ*,* $d\pi_\lambda$ *be the infinitesimal version of* π_λ*,* I_λ *be the moment set of* π_λ*, and* ν_{I_λ} *be the probability* G*-invariant measure on* I_λ *(pushforward of the unitarily probability measure of the unit sphere of* $\mathcal{H}_\lambda$*). Let* $\{X_1, ..., X_d\}$ *be a basis of* $\mathfrak{g}$*,* *and denote* $x \cdot X = \sum_{k=1}^{d} x_j X_j$, *for* $x \in \mathbb{R}^d$. *Let* C *be the Cartan matrix of* $\mathfrak{g}^{\mathbb{C}}$, *and* $\zeta = (\beta, \xi, \eta) \in \left(\mathbb{R}^d\right)^*$. *Then the* $\mathfrak{g}$*-Fourier transform of* $e^{d\pi_\lambda(x \cdot X)}$, *denoted by* $\left(e^{d\pi_\lambda(x \cdot X)}\right)^{\wedge_{\mathfrak{g}}}$, *can be written as*

$$\begin{aligned} \left(e^{d\pi_\lambda(x \cdot X)}\right)^{\wedge_{\mathfrak{g}}} = & \frac{1}{(n-1)!} \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \sum_{m=0}^{j} (-1)^{n-k-j-1} \frac{\phi_{n-k-j-1}\left(d\pi_\lambda(X) \cdot -i\frac{\partial}{\partial \zeta}\right)}{(n-j+m-1)!} \\ & \cdot \left(d\pi_\lambda(X) \cdot -i\frac{\partial}{\partial \zeta}\right)^k \psi(m-1) \cdot \nu_{I_\lambda} \\ = & \; D^\lambda \cdot \nu_{I_\lambda}, \end{aligned} \tag{6.3.75}$$

where ϕ_k *is the sum of principal minors of order k, and* ψ *is a recursive function of the theta differential operator* $\zeta\frac{\partial}{\partial \zeta}$, *given by*

$$\psi(p) = \sum_{q=0}^{p} (-1)^{p-q} \frac{p!}{q!} \cdot \left(-\zeta\frac{\partial}{\partial \zeta} - dI\right) \cdot \psi(q-1)$$

where $\psi(-1) = 1$, *and*

$$-i\frac{\partial}{\partial \zeta} = -i\left(C^{-1} \cdot \frac{\partial}{\partial \beta}, \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}\right).$$

Also,

$$supp\left(D^\lambda \cdot \nu_{I_\lambda}\right) \subseteq conv(\mathcal{O}_\lambda) \quad and \quad supp\left(D^\lambda \cdot \nu_{I_\lambda}\right) \cap \mathfrak{t}^* \subseteq conv(W \cdot \lambda).$$

The matrix of differential operators D^λ *consists of polynomial of Euclidean differential operator in some fundamental weight directions* $\frac{\partial}{\partial \lambda_i}$, *root directions* $\frac{\partial}{\partial \alpha_j^+}$, $\frac{\partial}{\partial \alpha_k^-}$, *and theta differential operator* $\zeta\frac{\partial}{\partial \zeta}$. *We have*

$$\frac{\partial}{\partial \lambda_i} = C_i^{-1} \cdot \frac{\partial}{\partial \beta},$$

where C_i^{-1} *is the i-th row of the inverse of the Cartan matrix* C, $\beta = (\beta_1, ..., \beta_l)$ *is the tuple of simple roots* $\Delta \subset \Phi$. *Also, we have*

$$\frac{\partial}{\partial \alpha_j^+} = \frac{\partial}{\partial \xi_j} - i\frac{\partial}{\partial \eta_j}, \qquad and \qquad \frac{\partial}{\partial \alpha_k^-} = \frac{\partial}{\partial \xi_k} + i\frac{\partial}{\partial \eta_k},$$

where α_j^+ *is a positive root in* Φ, *and* α_k^- *is a negative root in* Φ.

Remark. *We can also discuss the significance of the existence of the theta differential operator in the non-commutative formulas* (6.3.24), (6.3.75)*:*

- *The G-invariant distribution $\left(\zeta\frac{\partial}{\partial\zeta}\right)^k(\nu_{I_\lambda})$ is annihilated by the differential operators induced by the action of G, by Proposition* 3.2.5*. So, these induced differential operators only act on the polynomials given by the non-commutative formula.*

- *It is closely related to the Clebsch-Gordan theory (Theorem* 5.1.6*). Let $\lambda, \lambda' \in \Lambda^+$ be the highest dominant weights, we can take the tensor product of irreducible representations π_λ and $\pi_{\lambda'}$ to obtain exactly one copy of the irreducible representation $\pi_{\lambda+\lambda'}$, where the highest weight matrix coefficient is given by $\pi_{1,1}^{\lambda+\lambda'} = \pi_{1,1}^{\lambda}\pi_{1,1}^{\lambda'}$. The Euclidean Fourier transform of $\pi_{1,1}^{\lambda+\lambda'}\circ\exp$ contains terms in the form of*

$$\left(\zeta\frac{\partial}{\partial\zeta}\right)^p \nu_{I_\lambda} * \left(\zeta\frac{\partial}{\partial\zeta}\right)^q \nu_{I_{\lambda'}}, \tag{6.3.76}$$

  for some integers p, q. In general, the sumset of moment sets $I_\lambda + I_{\lambda'}$ is not equal to $I_{\lambda+\lambda'}$, by Lemma 5.1.7*. Also, because the $\zeta\frac{\partial}{\partial\zeta}$ is a non-constant-coefficient and it does not commute with convolution operation, the existence of $\zeta\frac{\partial}{\partial\zeta}$ ensures that the support of the measure $(\pi_{1,1}^{\lambda+\lambda'}\circ\exp)^{\wedge\mathfrak{g}}$ is not equal to the support of the convolution of measures $\nu_{I_\lambda} * \nu_{I_{\lambda'}}$.*

Example 6.3.5.1. We use the previous calculations for $SU(3)$ in Example 6.3.3.2. The set of all positive roots $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$. The Cartan matrix C for its Lie algebra $\mathfrak{su}(3)$ is given by

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}. \tag{6.3.77}$$

So, the two fundamental weights are: $\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \lambda_1 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$. Let $P_{i,j}^{\lambda_1}$ denote the polynomials of the matrix of $e^{d\pi_{\lambda_1}}$. For examples, we have

$$\begin{aligned} P_{1,1}^{\lambda_1} &= 2 + 3\gamma\frac{\partial}{\partial\gamma} + \left(\gamma\frac{\partial}{\partial\gamma}\right)^2 + ih_1\left(2 + \gamma\frac{\partial}{\partial\gamma} + ih_2\right) + h_2^2 + x_2^2 + y_2^2, \\ P_{1,2}^{\lambda_1} &= (x_1 + iy_1)\left(2 + \gamma\frac{\partial}{\partial\gamma} + ih_2\right) + (-x_2 + iy_2)(x_3 + iy_3), \\ P_{1,3}^{\lambda_1} &= (x_3 + iy_3)\left(2 + \gamma\frac{\partial}{\partial\gamma} + ih_1 - ih_2\right) + (x_1 + iy_1)(x_2 + iy_2), \end{aligned} \tag{6.3.78}$$

If we take the $\mathfrak{g}$-Fourier transform of these $P^{\lambda_1}_{1,j}, j \in \{1,2,3\}$, we obtain

$$\begin{aligned}
(P^{\lambda_1}_{1,1})^{\wedge_{\mathfrak{g}}} &= 2 - 3\left(\zeta\frac{\partial}{\partial\zeta} + 8\right) + \left(\zeta\frac{\partial}{\partial\zeta} + 8\right)^2 + \left(\frac{2}{3}\frac{\partial}{\partial\beta_1} + \frac{1}{3}\frac{\partial}{\partial\beta_2}\right)\left(-\zeta\frac{\partial}{\partial\zeta} - 6 + \cdots\right. \\
&\quad \left.\left(\frac{1}{3}\frac{\partial}{\partial\beta_1} + \frac{2}{3}\frac{\partial}{\partial\beta_2}\right)\right) - \left(\frac{1}{3}\frac{\partial}{\partial\beta_1} + \frac{2}{3}\frac{\partial}{\partial\beta_2}\right)^2 - \left(\frac{\partial}{\partial\xi_2} + i\frac{\partial}{\partial\eta_2}\right)\left(\frac{\partial}{\partial\xi_2} - i\frac{\partial}{\partial\eta_2}\right), \\
(P^{\lambda_1}_{1,2})^{\wedge_{\mathfrak{g}}} &= -i\left(\frac{\partial}{\partial\xi_1} + i\frac{\partial}{\partial\eta_1}\right)\left(-\zeta\frac{\partial}{\partial\zeta} - 6 + \left(\frac{1}{3}\frac{\partial}{\partial\beta_1} + \frac{2}{3}\frac{\partial}{\partial\beta_2}\right)\right) \\
&\quad + \left(\frac{\partial}{\partial\xi_2} - i\frac{\partial}{\partial\eta_2}\right)\left(\frac{\partial}{\partial\xi_3} + i\frac{\partial}{\partial\eta_3}\right), \\
(P^{\lambda_1}_{1,3})^{\wedge_{\mathfrak{g}}} &= -i\left(\frac{\partial}{\partial\xi_3} + i\frac{\partial}{\partial\eta_3}\right)\left(-\zeta\frac{\partial}{\partial\zeta} - 6 + \left(\frac{1}{3}\frac{\partial}{\partial\beta_1} + \frac{2}{3}\frac{\partial}{\partial\beta_2}\right) - \left(\frac{2}{3}\frac{\partial}{\partial\beta_1} + \frac{1}{3}\frac{\partial}{\partial\beta_2}\right)\right) \\
&\quad - \left(\frac{\partial}{\partial\xi_1} + i\frac{\partial}{\partial\eta_1}\right)\left(\frac{\partial}{\partial\xi_2} + i\frac{\partial}{\partial\eta_2}\right),
\end{aligned} \tag{6.3.79}$$

where

$$\frac{\partial}{\partial\lambda_1} = \frac{2}{3}\frac{\partial}{\partial\beta_1} + \frac{1}{3}\frac{\partial}{\partial\beta_2}, \quad \frac{\partial}{\partial\lambda_2} = \frac{1}{3}\frac{\partial}{\partial\beta_1} + \frac{2}{3}\frac{\partial}{\partial\beta_2}, \tag{6.3.80}$$

are directional derivatives in fundamental weights directions, and

$$\frac{\partial}{\partial\alpha_j^+} = \left(\frac{\partial}{\partial\xi_j} - i\frac{\partial}{\partial\eta_j}\right), \quad \frac{\partial}{\partial\alpha_j^-} = \left(\frac{\partial}{\partial\xi_j} + i\frac{\partial}{\partial\eta_j}\right), \quad j \in \{1,2,3\} \tag{6.3.81}$$

are directional derivatives in a positive root α_j^+ direction and a negative root α_j^- direction.

Corollary 6.3.5.1. *The Fourier transform of the Kirillov character formula is given by*

$$\mu_{\lambda+\delta} = d(\lambda+\delta)\ Tr\left(D^G_\lambda\right)\ \nu_\delta * \left(D^\lambda_{v_\lambda,v_\lambda}\ \nu_{I_\lambda}\right) \tag{6.3.82}$$

where $D^\lambda_{v_\lambda,v_\lambda} = D^\lambda_{v_\lambda,v_\lambda}\left(C^{-1}\cdot\frac{\partial}{\partial\lambda_j}, \frac{\partial}{\partial\alpha_j^\pm}, \zeta\frac{\partial}{\partial\zeta}\right)$, *and* $Tr\left(D^G_\lambda\right)$ *is the trace of the matrix of induced differential operators by* G *and it annihilates adjoint invariant functions. In addition,* $Tr\left(D^G_\lambda\right)\ D^\lambda_{v_\lambda,v_\lambda}$ *is a polynomial of differential operators of the theta differential operator* $\zeta\frac{\partial}{\partial\zeta}$ *and constant coefficient* G*-invariant differential operators.*

Remark. *We provide a new perspective for determining the support and singular supports of* $\nu_\delta * \mathcal{W}_{-i\,d\pi_\lambda(\cdot)}$ *and Kirillov character formula:*

- *If we closely observe the non-commutative formula (6.3.75), we notice there always exists terms of constants or constant coefficient differential operators that commute with convolutions (whereas the non-constant coefficient operator $\zeta\frac{\partial}{\partial\zeta}$ does not), which act on the measure ν_{I_λ}. So it suffices to consider the support and singular support of the convolution $\nu_\delta * \nu_{I_\lambda}$. Notice that by formulas (5.2.38), (4.2.9), we have*

$$\nu_\delta * \nu_{I_\lambda} = \int_{\mathfrak{t}^*_+} \varphi(\lambda, \lambda'')\nu_\delta * \nu_{\lambda''}\, d\lambda'' \tag{6.3.83}$$

  where the resulting density function is again a piecewise polynomial with support intersecting $\mathfrak{t}^*$ and contained in the convex hull of $W\cdot(\lambda+\delta)$, and the singular support depends on the transitions between piece-wise polynomials of the measures (Example 6.1.7.1). This is a lot more difficult to describe for general compact simply connected and semisimple Lie groups, than it is for $SU(2)$.

- *Let D_{χ_λ} be the trace of the non-commutative formula (6.3.75) which consists of polynomials of G-invariant constant coefficient differential operators and the theta differential operator, acting on ν_{I_λ}. Then the restriction to $\mathfrak{t}^*$ is a distribution*

$$D_{\chi_\lambda}\nu_{I_\lambda}|_{\mathfrak{t}^*} = \sum_{\mu\in\Pi(\lambda)} m(\mu)e_\mu, \tag{6.3.84}$$

  where each μ is a weight in the set of weights $\Pi(\lambda)$ of representation π_λ, and e_μ is the unit point mass at the weight μ. Note that the support and singular support of the measure above are exactly the weights of the irreducible representation π_λ. This is also the $\mathfrak{t}$-Fourier transform of $\chi_\lambda(e^H)$, for $H \in \mathfrak{t}$. Also, the Kirillov character formula (2.4.2) can be interpreted *as*

$$\begin{aligned}\big(j(H)\chi_\lambda(e^H)\big)^{\vee_{\mathfrak{t}}} &= \sum_{w\in W} sgn(w)e_{w\delta} * \prod_{\alpha\in\Phi^+} *F_\alpha * \sum_{\mu\in\Pi(\lambda)} m(\mu)e_\mu \\ &= \left(\sum_{w\in W} sgn(w)e_{w\delta}\right) * \left(\sum_{\mu\in\Pi(\lambda)} m(\mu)e_\mu\right) * \prod_{\alpha\in\Phi^+} *F_\alpha \\ &= \sum_{w\in W} sgn(w)e_{w(\lambda+\delta)} * \prod_{\alpha\in\Phi^+} *F_\alpha \\ &= \mu^p_{\lambda+\delta},\end{aligned} \tag{6.3.85}$$

by the Weyl integration formula, the Weyl character formula, and (4.1.9)*. Hence each character χ_λ acts as a shift operator that shifts the coadjoint orbit $\mathcal{O}_\delta$ to $\mathcal{O}_{\lambda+\delta}$. This also explains the importance of Kirillov character formula in deriving the wrapping map in* [19]*.*

6.4 Highest Weight Representations

The formula we proposed in Proposition 6.3.3 is based on Nelson's formula [40] and Raffoul's results [44], and only applies to unitary representations. However, it can be difficult to explicitly calculate the skew-Hermitian infinitesimal irreducible representation $d\pi_\lambda$ for a highest dominant weight λ for any compact simply connected semisimple Lie groups. But we have observed that if we apply *Weyl's unitarian trick*, the non-commutative formula (6.3.24) can be directly applied to an infinitesimal irreducible highest weight representation $d\tilde{\pi}_\lambda$.

Proposition 6.4.1. *Let G be a compact simply connected semisimple Lie group, T be a maximal torus of G, and $\mathfrak{g}, \mathfrak{t}$ be their Lie algebras, respectively. Let $\tilde{\pi}_\lambda$ be an irreducible finite-dimensional highest weight representation (not necessarily unitary) of G, and $d\tilde{\pi}_\lambda$ be the infinitesimal version of $\tilde{\pi}_\lambda$ in $\mathfrak{g}$ (not necessarily skew-Hermitian). Then, for every $X \in \mathfrak{g}$, there exists a $H \in \mathfrak{t}$ and a $g \in G$ such that the skew-Hermitian representation $d\tilde{\pi}_\lambda(H)$ satisfies $d\tilde{\pi}_\lambda(H) = \tilde{\pi}(g)d\tilde{\pi}(X)\tilde{\pi}(g^{-1})$, and the Kirillov type non-commutative formula* (6.3.24) *in Proposition* 6.3.3 *can be extended to the highest weight representation $d\tilde{\pi}_\lambda$.*

Proof. If G is simply connected, then the integration of $d\tilde{\pi}$ is equal to $\tilde{\pi}$ of G (Theorem 3.1.6), and $d\tilde{\pi}$ satisfies $d\tilde{\pi}(gXg^{-1}) = \tilde{\pi}(g)d\tilde{\pi}(X)\tilde{\pi}(g^{-1})$, for all $X \in \mathfrak{g}$

and $g \in G$. Because G is also compact, every $X \in \mathfrak{g}$ is also conjugate to some $H \in \mathfrak{t}$, which means for every $X \in \mathfrak{g}$, there exists $H \in \mathfrak{t}$ and $g \in G$ such that $d\tilde{\pi}_\lambda(H)$ satisfies $d\tilde{\pi}_\lambda(H) = \tilde{\pi}(g)d\tilde{\pi}(X)\tilde{\pi}(g^{-1})$.

Since $d\tilde{\pi}(H)$ is a diagonal matrix, then by Lemma 6.3.2, we have

$$e^{d\tilde{\pi}(H)} = \gamma_0(d\tilde{\pi}(H))I + \gamma_1(d\tilde{\pi}(H))d\tilde{\pi}(H) + \cdots + \gamma_{n-1}(d\tilde{\pi}(H))d\tilde{\pi}(H)^{n-1}, \quad H \in \mathfrak{t}. \tag{6.4.1}$$

This leads to

$$e^{d\tilde{\pi}(gHg^{-1})} = \gamma_0(d\tilde{\pi}(H))I+\gamma_1(d\tilde{\pi}(H))d\tilde{\pi}(gHg^{-1})+\cdots+\gamma_{n-1}(d\tilde{\pi}(H))d\tilde{\pi}(gHg^{-1})^{n-1}. \tag{6.4.2}$$

Also, by Lemma 6.3.1, each γ_k depends on γ_{n-1}, and γ_{n-1} is adjoint invariant, that is, $\gamma_{n-1}(d\tilde{\pi}(gHg^{-1})) = \gamma_{n-1}(d\tilde{\pi}(H))$. Now, because π_λ is not necessarily unitary, we can apply Weyl's unitarian trick to make $\tilde{\pi}$ unitary. Let $\mathcal{H}_{\tilde{\pi}}$ be a Hilbert space spanned by the weight spaces of $\tilde{\pi}$. Let $(\cdot,\cdot)$ be a given norm on $\mathcal{H}_{\tilde{\pi}}$. Then, we have a new norm

$$\langle u, v\rangle = \int_G (\tilde{\pi}(g)u, \tilde{\pi}(g)v)\, dg, \quad \text{for } u, v \in \mathcal{H} \tag{6.4.3}$$

on $\mathcal{H}_{\tilde{\pi}}$, so that $\tilde{\pi}$ is unitary on $\mathcal{H}_{\tilde{\pi}}$. The moment set of the unitary representation $\tilde{\pi}$, denoted by $I_{\tilde{\pi}}$, is given by $I_{\tilde{\pi}} = \{\langle d\tilde{\pi}(\cdot)u, u\rangle : ||u|| = 1\}$. Hence, we have the G-invariant and continuous function

$$\gamma_{n-1}(d\tilde{\pi}(X)) = \int_\Omega e^{\langle d\tilde{\pi}(X)u,u\rangle}\, d\nu(u) = \int_{I_{\tilde{\pi}}} e^{i\beta(X)}\, d\nu_{I_{\tilde{\pi}}}(\beta), \quad X \in \mathfrak{g} \tag{6.4.4}$$

where Ω is the unit sphere in $\mathcal{H}_{\tilde{\pi}}$ with respect to $\langle\cdot,\cdot\rangle$, and $\nu_{I_{\tilde{\pi}}}$ is the G-invariant measure of the moment set $I_{\tilde{\pi}}$. Therefore the proposition follows. □

Example 6.4.1.1. Let $G = SU(2), \mathfrak{g} = \mathfrak{su}(2) \cong \mathbb{R}^3$. Let $n \in \Lambda^+ \cong \mathbb{Z}^+$ be a highest integer weight. Suppose we do not have an explicit basis for the Hilbert space $\mathcal{H}_n$ in (2.5.18) to construct the representations $d\pi_n$ for $\mathfrak{su}(2)$. Instead, we start with the irreducible highest weight representations for the complexification $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{su}(2)^{\mathbb{C}}$. Let $\{H, X, Y\}$ in (2.3.1) be a basis for $\mathfrak{sl}_2(\mathbb{C})$. Let $n = 2$, and suppose that the set of weight vectors $\{v_0, v_1, v_2\}$ span the $\mathfrak{sl}_2$-module. If we follow the constructions in Example 2.3.2.1, then we have

$$\begin{aligned} &d\pi(H)v_0 = 2v_0, \quad d\pi(H)v_1 = 0, \quad d\pi(H)v_2 = -2v_2, \\ &d\pi(Y)v_0 = 2v_1, \quad d\pi(Y)v_1 = v_2, \quad d\pi(Y)v_2 = 0, \\ &d\pi(X)v_0 = 0, \quad d\pi(X)v_1 = v_0, \quad d\pi(X)v_2 = 2v_1. \end{aligned} \tag{6.4.5}$$

Let $\{X_1, X_2, X_3\}$ be the basis for $\mathfrak{su}(2)$ in (2.5.3), and take the compact real forms of $\mathfrak{su}(2)^{\mathbb{C}}$, we obtain

$$d\pi(X_1) = d\pi(X - Y), \quad d\pi(X_2) = id\pi(X + Y), \quad d\pi(X_3) = id\pi(H). \tag{6.4.6}$$

If we let $(x_1, x_2, x_3) \in \mathbb{R}^3$, and $x \cdot X = \sum_{j=1}^{3} x_j X_j$, then we have

$$d\tilde{\pi}_2(x \cdot X) = \begin{pmatrix} i2x_3 & 2(x_1 + ix_2) & 0 \\ -(x_1 - ix_2) & 0 & (x_1 + ix_2) \\ 0 & -2(x_1 - ix_2) & -i2x_3. \end{pmatrix} \tag{6.4.7}$$

which is <u>not</u> skew-Hermitian. If we substitute $d\tilde{\pi}_2$ into the non-commutative formula (6.3.24), and we denote $e^{d\pi_2}_{i,j}$ the (i, j)-th element of the matrix $e^{d\pi_2}$, then we have

$$\begin{aligned} e^{d\tilde{\pi}_2}_{1,1} &= \left(1 + \frac{3}{2} r \frac{\partial}{\partial r} + \frac{1}{2}\left(r\frac{\partial}{\partial r}\right)^2 + (r^2 - x_3^2) + i2x_3 + ix_3 r\frac{\partial}{\partial r}\right) \check{\nu}_{I_2}(r) \\ &= \frac{(r\cos r + ix_3 \sin r)^2}{r^2} = \left(\cos r + ix_3 \frac{\sin r}{r}\right)^2 \\ &= \alpha^2(\exp x \cdot X). \end{aligned} \tag{6.4.8}$$

The rest of the elements of $e^{d\pi_2}$ are:

$$\begin{aligned}
e^{d\tilde{\pi}_2}_{1,2} &= \left(2x_1 + x_1\, r\frac{\partial}{\partial r} + i2x_2 + ix_2\, r\frac{\partial}{\partial r} + i2x_1x_3 - 2x_2x_3\right)\check{\nu}_{I_2}(r)\\
&= (x_1 + ix_2)\left(2 + r\frac{\partial}{\partial r} + i2x_3\right)\left(\frac{\sin r}{r}\right)^2\\
&= 2\frac{\sin r(x_1 + ix_2)(r\cos r + ix_3\sin r)}{r^2}\\
&= 2\,\alpha(\exp x\cdot X)\beta(\exp x\cdot X). \qquad (6.4.9)\\
e^{d\tilde{\pi}_2}_{1,3} &= \left(x_1^2 + i2x_1x_2 - x_2^2\right)\left(\frac{\sin r}{r}\right)^2\\
&= (x_1 + ix_2)^2\left(\frac{\sin r}{r}\right)^2\\
&= \beta^2(\exp x\cdot X).
\end{aligned}$$

$$\begin{aligned}
e^{d\tilde{\pi}_2}_{2,1} &= \left(-x_1 - \frac{1}{2}x_1\, r\frac{\partial}{\partial r} + ix_2 + i\frac{1}{2}x_2\, r\frac{\partial}{\partial r} - ix_1x_3 - x_2x_3\right)\check{\nu}_{I_2}(r)\\
&= -\,\alpha(\exp x\cdot X)\overline{\beta}(\exp x\cdot X).\\
e^{d\tilde{\pi}_2}_{2,2} &= \left(1 + \frac{3}{2}r\frac{\partial}{\partial r} + \frac{1}{2}\left(r\frac{\partial}{\partial r}\right)^2 + 2x_3^2\right)\check{\nu}_{I_2}(r)\\
&= \alpha(\exp x\cdot X)\overline{\alpha}(\exp x\cdot X) - \beta(\exp x\cdot X)\overline{\beta}(\exp x\cdot X).
\end{aligned} \tag{6.4.10}$$

$$\begin{aligned}
e^{d\tilde{\pi}_2}_{3,1} &= \left(x_1^2 - i2x_1x_2 - x_2^2\right)\left(\frac{\sin r}{r}\right)^2\\
&= (-\overline{\beta})^2(\exp x\cdot X).\\
e^{d\tilde{\pi}_2}_{3,2} &= \left(-2x_1 - x_1\, r\frac{\partial}{\partial r} + i2x_2 + ix_2\, r\frac{\partial}{\partial r} + i2x_1x_3 + 2x_2x_3\right)\check{\nu}_{I_2}(r)\\
&= -2\,\overline{\alpha}(\exp x\cdot X)\overline{\beta}(\exp x\cdot X).\\
e^{d\tilde{\pi}_2}_{3,3} &= \left(1 + \frac{3}{2}r\frac{\partial}{\partial r} + \frac{1}{2}\left(r\frac{\partial}{\partial r}\right)^2 + (r^2 - x_3^2) - i2x_3 - ix_3r\frac{\partial}{\partial r}\right)\check{\nu}_{I_2}(r)\\
&= \overline{\alpha}^2(\exp x\cdot X).
\end{aligned} \tag{6.4.11}$$

Hence, we have completed the calculation for $e^{d\tilde{\pi}_2}$. If we compare this calculation for $e^{d\tilde{\pi}_2}$ with the previous $e^{d\pi_2}$, then they are exactly the same except having different constant coefficients. Also, note that $\{\alpha^2, 2\alpha\beta, \beta^2\}$ is a copy of the

$\mathfrak{sl}_2$-module $\{v_0, v_1, v_2\}$. Because

$$\begin{aligned} &d\pi(H)^R\alpha^2 = 2\alpha^2, \quad d\pi(X)^R\alpha\beta = 0, \quad d\pi(X)^R\beta^2 = -2\beta^2, \\ &d\pi(X)^R\alpha^2 = 0, \quad d\pi(X)^R\alpha\beta = \alpha^2, \quad d\pi(X)^R\beta^2 = 2\alpha\beta, \\ &d\pi(Y)^R\alpha^2 = 2\alpha\beta, \quad d\pi(Y)^R\alpha\beta = \beta^2, \quad d\pi(Y)^R\beta^2 = 0, \end{aligned} \tag{6.4.12}$$

where the differential operators

$$d\pi(H)^R = z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}, \quad d\pi(X)^R = z\frac{\partial}{\partial w}, \qquad d\pi(Y)^R = w\frac{\partial}{\partial z}, \tag{6.4.13}$$

are in (3.1.17). This observation also satisfies the case for non-unitary representations mentioned in Proposition 3.1.7.

We can also extend the Kirillov-type non-commutative formula to general compact connected semisimple Lie groups.

Theorem 6.4.2 ([43], J. Price)**.** *(Structure Theorem) Let G be a compact connected semisimple Lie group. Then, G can be decomposed as*

$$G \cong (T \times \prod_{i\in I} G_i)\,/\,Z \tag{6.4.14}$$

where T is a k-dimensional torus $\mathbb{T}^k$, each G_i is a compact and simply connected semisimple Lie group, and Z is a finite subgroup of the centre of $T \times \prod_{i\in I} G_i$.

Proposition 6.4.3. *Let G be a compact connected semisimple Lie group. Then every irreducible highest weight representation of a compact connected Lie group G is associated with a measure in the form of*

$$\left(e_{n_1} * \cdots * e_{n_p}\right) \otimes \left(D_{\lambda_{m_1}}\mu_{I_{\lambda_{m_1}}} \otimes \cdots \otimes D_{\lambda_{m_q}}\mu_{I_{\lambda_{m_q}}}\right), \tag{6.4.15}$$

where each e_{n_i} is a unit point mass at an integer n_j in $\mathbb{R}$, and each $D_{\lambda_{m_j}}\mu_{I_{\lambda_{m_j}}}$ is a distribution in a dual Lie algebra $\mathfrak{g}_j^*$ of a compact simply connected semisimple Lie group G_j that consists of an operator of differential operators acting on the (unique) G-invariant measure of the moment set of a highest dominant weight λ_{m_j}.

Proof. Recall that every irreducible representation π_T of T is one-dimensional and they are also called characters of T, and it can be written as

$$\pi_T = e^{in_1\theta_1} \cdots e^{in_k\theta_k}. \tag{6.4.16}$$

Taking the Fourier transform of π_T in $\mathbb{R}^n$, we obtain the point mass distributions at integer points $n_1, ..., n_k$. For each G_j, we obtain the measure $D_{\lambda_{m_j}} \mu_{I_{\lambda_{m_j}}}$ with respect to the highest weight representation $\pi_{\lambda_{m_j}}$. Now, we can pick out all pairs of integers $(n_1, ..., n_k)$ and weights $(\lambda_{m_1}, ..., \lambda_{m_q})$ such that the representation

$$\pi_T \otimes (\pi_{\lambda_{m_1}} \otimes \cdots \otimes \pi_{\lambda_{m_q}}) \tag{6.4.17}$$

is trivial on the finite group Z. In these cases, they are the highest weight representations of the compact connected Lie group G. Hence, the proposition follows. □

Example 6.4.3.1 ([21], Example 5.4)**.** Let $T = \mathbb{T}$ be the 1-torus, $G = SU(2)$. Let $\{(-1, I), (1, I)\}$ be the finite centre for $\mathbb{T} \times SU(2)$. Then, the unitary group of dimension 2 can be represented by

$$U(2) \cong (\mathbb{T} \times SU(2)) \,/\, \{(-1, -I), (1, I)\}. \tag{6.4.18}$$

Let $n \in \mathbb{Z}^+$, the function $\mathbb{R} \to \mathbb{T}$, $x \mapsto e^{inx}$ be the character of $\mathbb{T}$, π_n be an irreducible unitary representation of $SU(2)$ with dimension $n + 1$. Hence, the irreducible representation $\tilde{\pi}_{m,n}$ of $U(2)$ is given by

$$\tilde{\pi}_{m,n} \cong e^{imx} \times \pi_n, \tag{6.4.19}$$

where $m \in \mathbb{Z}, n \in \mathbb{Z}^+$ are both odd or both even.

Chapter 7

Riemannian Symmetric Spaces of The Compact Type

Let G be a compact connected Lie group, $G \times G$ be the direct product of G with itself, ΔG the diagonal of $G \times G$. Then the pair $(G \times G, \Delta G)$ is a symmetric space. The quotient group $G \times G/\Delta G$ is isomorphic to G. Therefore, G is a symmetric space of compact type.

Hence, it is natural to extend the non-commutative Kirillov method developed in Chapter 6 to symmetric spaces of compact type to calculate zonal and non-zonal spherical functions.

In Section 7.1, we review the definitions and properties of Riemannian symmetric spaces.

In Section 7.2, we review the structure of the L^2 functions of the symmetric pair (G, K) of compact type.

In Section 7.3, we state the Cartan-Helgason theorem and show how to calculate zonal and non-zonal spherical functions through the Kirillov-type non-commutative formula in Chapter 6, and we demonstrate this by calculating a simple example of $SU(2)$.

In Section 7.4, we look at the wrapping map and e-function, and show how we can combine these with the non-commutative Kirillov method to transfer the spherical Fourier analysis on G/K of the symmetric pair (G, K) of compact type to the K-invariant Fourier analysis on the tangent space $\mathfrak{p}$ of G/K.

7.1 Symmetric Spaces and Cartan Decomposition

Definition 7.1.1. Let G be a connected semisimple Lie group. Let K be a maximal compact subgroup of G such that K is a compact fixed point set of an involution σ in $\mathrm{Aut}(G)$, such that $K = \{g \in G \mid \sigma(g) = g\}$, and σ satisfies $\sigma^2 = I$. We call the homogeneous space G/K **symmetric space of the Riemannian symmetric pair** (G, K).

Definition 7.1.2. Let G be a connected semisimple Lie group, $\mathfrak{g}$ be its Lie algebra, K be a maximal compact subgroup of G with respect to an involution σ, where σ induces a Lie algebra automorphism on $\mathfrak{g}$ by differentiating σ at identity, i.e., $d\sigma$, such that $d\sigma^2 = I$. It follows that the eigenvalues of $d\sigma$ are ± 1. The $+1$ eigenspace is the Lie algebra $\mathfrak{k}$ of K, and the -1 eigenspace is the subspace of $\mathfrak{g}$ denoted by $\mathfrak{p}$, such that

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \tag{7.1.1}$$

and let $X, Y \in \mathfrak{g}$, then $d\sigma([X, Y]) = [d\sigma(X), d\sigma(Y)]$. Hence,

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{m}, \tag{7.1.2}$$

Notice that $\mathfrak{p}$ is K-invariant, since, let $X \in \mathfrak{p}, k \in K$, then

$$\frac{d}{dt}\sigma(\exp Ad(k)tX)\bigg|_{t=0} = -Ad(k)X, \tag{7.1.3}$$

assuming $\sigma(g) = s_p \circ g \circ s_p$, where s_p is an involutive isometry of the fixed point $p \in G/K$, i.e., $s_p(p) = p$.

There is a theorem for general properties of Riemannian symmetric spaces.

Theorem 7.1.3 ([28], Theorem 3.3)**.**

- *Let S be a Riemannian globally symmetric space and p a point in S, G be the group of isometries of S, and K be a maximal compact subgroup of G that fixes p, then K is a maximal compact subgroup of the connected group G and G/K is analytically diffeomorphic to S under mapping $gK \mapsto g \cdot p$, for $g \in G$.*

- *The mapping* $\sigma : g \mapsto s_p g s_p$, $(s_p^2 = 1)$ *is an involutive automorphism of* G *such that* K *lies between the closed group* K_σ *of all fixed points of* σ *and the identity components of* K_σ, *i.e.,* $K_\sigma \subset K \subset K_o$. *The group* K *contains no normal subgroup of* G *other than* $\{e\}$.

- *Let* $\mathfrak{g}$ *and* $\mathfrak{k}$ *denote the Lie algebras of* G *and* K, *respectively. Then* $\mathfrak{k} = \{X \in \mathfrak{g} \,|\, (d\sigma)_e X = X\}$, *and if* $\mathfrak{p} = \{X \in \mathfrak{g} \,|\, (d\sigma)_e X = -X\}$, *we have* $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ *(as a direct sum).*

- *Let* f *denote the natural mapping* $f : g \mapsto g \cdot p$ *of* G *onto* S. *Then* $(df)_e$ *maps* $\mathfrak{k}$ *onto* $\{0\}$, *and* $\mathfrak{p}$ *isomorphically onto the tangent space at* p, *denoted by* TS_p. *If* $X \in \mathfrak{p}$, *then the geodesic emanating from* p *with tangent vector* $(d\pi)_e X$ *is given by*

  $$\gamma_{(d\pi)_e X}(t) = \exp(tX) \cdot p, \tag{7.1.4}$$

  If $Y \in TS_p$, *then* $(de^{tX})_p(Y)$ *is the parallel transport of* Y *along the geodesic.*

Lemma 7.1.4 ([28], Lemma 3.6)**.** *Let* $\mathfrak{g}$ *be a complex semi-simple Lie algebra and* Φ *be the root system. The root space decomposition says* $\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Phi^+} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$. *Let* $d\sigma = \theta$, $P_+ = \{\alpha \,|\, \alpha \in \Phi^+, \alpha \neq \alpha^\theta\}$ *and* $P_- = \{\alpha \,|\, \alpha \in \Phi^+, \alpha = \alpha^\theta\}$, *where* $\alpha^\theta(H) = \alpha(\theta H)$, *for* $H \in \mathfrak{t}$. *The* ***Cartan decomposition*** *of* $\mathfrak{g}$ *with respect to* σ *is:*

$$\mathfrak{g} = \underbrace{\mathfrak{m} + \mathfrak{l}}_{\mathfrak{k}} + \underbrace{\mathfrak{h}_\mathfrak{p} + \mathfrak{q}}_{\mathfrak{p}}, \tag{7.1.5}$$

where $\mathfrak{h}_\mathfrak{p}$ *is a maximal abelian algebra in* $\mathfrak{p}$, *and*

$$\begin{aligned} \mathfrak{m} &= \mathfrak{h}_\mathfrak{k} + \sum_{\beta \in P_-} (\mathfrak{g}_\beta + \mathfrak{g}_{-\beta}), \\ \mathfrak{l} &= \sum_{\beta \in P_+} \mathbb{C}(X_\beta + \theta X_\beta), \\ \mathfrak{q} &= \sum_{\beta \in P_+} \mathbb{C}(X_\beta - \theta X_\beta). \end{aligned} \tag{7.1.6}$$

Definition 7.1.5. Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$, and $\mathfrak{g}_0$ be the split real form of $\mathfrak{g}$ dual to $(\mathfrak{u}, \theta)$, $\mathfrak{k}$ the Lie algebra of K. Then θ gives the decomposition:

$$\mathfrak{u} = \mathfrak{k}_0 + \mathfrak{p}_*, \qquad \mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0\,, \tag{7.1.7}$$

where $\mathfrak{p}_* = i\mathfrak{p}_0$. Let $\mathfrak{h}_{\mathfrak{p}_*}$ denote a maximal abelian subspace of $\mathfrak{p}_*$. Then the space $\mathfrak{h}_{\mathfrak{p}_0} = i\mathfrak{h}_{\mathfrak{p}_*}$ is a maximal abelian subspace of $\mathfrak{p}_0$.

Definition 7.1.6. Let (G, K) be a Riemannian symmetric pair of the compact type. Let M and M' be the centraliser and normaliser of the maximal abelian subspace $\mathfrak{h}_{\mathfrak{p}}$ in $\mathfrak{p}$. That is,

$$\begin{aligned} M &= \{k \in K \mid \mathrm{Ad}(k)H = H \ \text{ for each } \ H \in \mathfrak{h}_{\mathfrak{p}}\}, \\ M' &= \{k \in K \mid \mathrm{Ad}(k)\mathfrak{h}_{\mathfrak{p}} \subset \mathfrak{h}_{\mathfrak{p}} \ \text{ for each } \ H \in \mathfrak{h}_{\mathfrak{p}}\}. \end{aligned} \tag{7.1.8}$$

Then, the quotient M'/M is called the **restricted Weyl group**. Let $\Phi_{\mathfrak{p}}$ denote the set of roots that do not vanish on $\mathfrak{h}_{\mathfrak{p}}$ (maximal abelian subalgebra of $\mathfrak{p}$), and $\Phi_{\mathfrak{p}}$ is called the **restricted root system**. The restricted Weyl group is generated by the reflections s_α, $\alpha \in \Phi_{\mathfrak{p}}$, and

$$s_\alpha(H) = H - 2\frac{\alpha(H)}{\alpha(H_\alpha)}H_\alpha, \quad H \in \mathfrak{h}_{\mathfrak{p}}, \tag{7.1.9}$$

for $\alpha \in \Phi_{\mathfrak{p}}$.

7.2 K-invariant functions

Definition 7.2.1. Let (G, K) be a Riemannian symmetric pair of the compact type, and $q : G \to G/K$ be the natural quotient map. We can define a G-invariant measure ν on G/K by $\nu(E) = \mu(EK) = \mu(q^{-1}(E))$, for $E \subseteq G/K$, given that μ is a left invariant Haar measure on G.

Definition 7.2.2. A function f on G/K can be given by a right K-invariant function $\tilde{f}$ on G defined by $\tilde{f}(g) = f(gK)$. In other words, we can realise $L^2(G/K, \nu)$ as the set of right K-invariant functions in $L^2(G, \mu)$. We also have the set of bi-K-invariant functions in $L^2(G, \mu)$, such that $f(k_1 g k_2) = f(g)$, for all $k_1, k_2 \in K$ and $g \in G$. These correspond to the left K-invariant functions in $L^2(G/K, \nu)$.

Definition 7.2.3. Suppose an irreducible unitary representation π of the compact Lie group G has a K-invariant vector, that is, there exists a vector $v \in \mathcal{H}_\pi$ such that $\pi(k)v = v$ for all $k \in K$, then the matrix coefficient $\pi_{u,v}(g) = \langle \pi(g)v, u \rangle$ for $u \in \mathcal{H}_\pi$ is K-invariant, and $\langle \pi(g)v, v \rangle$ is bi-K-invariant

Theorem 7.2.4 ([27], Theorem 3.5)**.** *Let G be a compact Lie group, $(\pi, \mathcal{H}_\pi)$ be an irreducible unitary representation of G. Define the subspace $\mathcal{H}_\pi^K = \{\xi \in \mathcal{H}_\pi : \pi(k)\xi = \xi, \forall k \in K\}$. Let $d(\pi), l(\pi)$ be the dimension of $\mathcal{H}_\pi$ and $\mathcal{H}_\pi^K$, respectively. For $\pi \in \widehat{G}$, let $C_\pi(G/K)$ denote the subspace of continuous functions $C(G/K)$ spanned by right K-invariant matrix coefficients of π. Then the functions*

$$gK \to \langle \pi(g)v_j, v_i \rangle = \pi_{i,j}(g), \quad 1 \leqslant i \leqslant d(\pi), 1 \leqslant j \leqslant l(\pi), \tag{7.2.1}$$

form a basis of $C_\pi(G/K)$, and the orthogonal Hilbert space decomposition is given by

$$L^2(G/K) = \oplus_{\pi \in \widehat{G}_K} C_\pi(G/K), \tag{7.2.2}$$

where $\widehat{G}_K$ is the set of π that has K-invariant vectors. Let $C_\pi^K(G,K)$ denote the subspace of $C_\pi(G/K)$ that consists of bi-K-invariant matrix coefficients of π. Then,

$$C_\pi^K(G,K) = \sum_{1 \leqslant i,j \leqslant l(\pi)} \mathbb{C}\pi_{i,j}, \quad \pi \in \widehat{G}_K, \tag{7.2.3}$$

and we have the orthogonal direct sum

$$L_K^2(G/K) = \oplus_{\pi \in \widehat{G}_K} C_\pi^K(G/K). \tag{7.2.4}$$

Remark. *If $l(\pi) \leqslant 1$ for all π, then a symmetric pair (G,K) is called a* ***Gelfand pair****.*

7.3 Spherical Functions

Let G be a non-compact real semi-simple Lie group with the Iwasawa decomposition $G = KAN$, M the centraliser of A in K. Let $G^{\mathbb{C}}$ be the complexification of G. Also, let U be the compact real form of $G^{\mathbb{C}}$ dual to G.

Theorem 7.3.1 ([27], Theorem 5.4.1)**.** *(Cartan-Helgason Theorem) Let σ be an irreducible representation of G on a finite-dimensional vector space V.*

1. *$\sigma(K)$ has a non-zero vector fixed by K unique up to scalar multiplication if and only if $\sigma(M)$ fixes the highest weight vector of σ. Also, σ is called an irreducible* ***spherical representation*** *if σ has a non-zero K-invariant vector.*

2. *Let λ be a linear form on the real form of the Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$. Then λ is the highest weight of an irreducible finite-dimensional spherical rep-*

resentation of G if and only if

$$\lambda(i(\mathfrak{h}\cap\mathfrak{k}))=0 \quad \textit{and} \quad \frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}^+, \quad \forall\alpha\in\Phi^+, \tag{7.3.1}$$

where $\mathfrak{h}$ is the Cartan subalgebra of $G^{\mathbb{C}}$, and $\mathfrak{k}$ is the Lie algebra of $K^{\mathbb{C}}$, respectively.

Corollary 7.3.1.1. *Let U be a compact simply connected semisimple Lie group, K be the fixed point subgroup of an involutive automorphism of U. Then condition 2 above characterises the highest weights of the irreducible spherical representations of U.*

Theorem 7.3.2 ([23], Theorem 12.3.13)**.** *Let π_λ be a spherical representation of a highest weight λ of U, then the K-invariant vector v_K is given by*

$$v_K=c\int_K \pi_\lambda(k)v_\lambda\,dk, \tag{7.3.2}$$

for some scalar c, and v_K is the projection of v_λ by $P=\int_K\pi_\lambda(k)\,dk$.

Definition 7.3.3. The **zonal spherical function** of an irreducible spherical representation π_λ of U is given by

$$\phi_\lambda(g)=\langle\pi_\lambda(g)v_K,v_K\rangle,\ g\in G. \tag{7.3.3}$$

Theorem 7.3.4 ([27], Theorem 4.4.2)**.** *The spherical function of the irreducible spherical representation π_λ of a compact Lie group G is given by*

$$\phi_\lambda(g)=\int_K \chi_\lambda(g^{-1}k)\,dK, \tag{7.3.4}$$

where χ_λ is the character for π_λ.

Proposition 7.3.5. *Let π_λ be an irreducible unitary and spherical representation of U of highest weight λ, given by the inverse Fourier transform of the non-commutative Kirillov formula, $\mathfrak{s}$ a neighborhood of 0 in $\mathfrak{u}$ such that* $\exp:\mathfrak{s}\to K$ *is a diffeomorphism, and dH a suitable measure on $\mathfrak{s}$, then*

$$v_K=\int_{\mathfrak{s}} J(H)\pi_\lambda(\exp H)P^{\lambda}_{1,1}\check{\nu}_{I_\lambda}(H)\,dH, \tag{7.3.5}$$

where P_{11}^{λ} is the polynomial of the lift of the highest weight matrix coefficient, $\check{\nu}_{I_\lambda}$ is the inverse Fourier transform of the Liouville measure of the moment set I_λ, and J is the Jacobian of the exponential map.

Remark. *We can work out the lift of spherical functions through the spherical representations given by the non-commutative Kirillov formula, and when we restrict to the tangent subspace $\mathfrak{p}$, we recover a K-invariant measure in $\mathfrak{p}$. Similarly,we can work out the non-K-invariant measures for the non-zonal spherical functions.*

It is interesting to work out an explicit version of the non-commutative Kirillov formula for K-invariant and non-K-invariant measures of spherical functions. This will be a subject of future work.

Example 7.3.5.1. The simplest symmetric space of compact type is the pair $(SU(2), SO(2))$, the Cartan decomposition of $\mathfrak{su}(2)$ is given by

$$\mathfrak{su}(2) = \underbrace{x_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\mathfrak{k}_0=\mathfrak{l}_0} + \underbrace{\underbrace{x_2 \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}}_{\mathfrak{a}_*} + x_3 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}_{\mathfrak{p}_*}, \tag{7.3.6}$$

for $(x_1, x_2, x_3) \in \mathbb{R}^3$. The condition (2) in Theorem 7.3.1 is satisfied because $i(\mathfrak{h} \cap \mathfrak{k})$ is zero, it follows that $\lambda(i(\mathfrak{h} \cap \mathfrak{k})) = 0$. The condition

$$\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+, \quad \forall \alpha \in \Phi^+, \tag{7.3.7}$$

is satisfied only if $\lambda = k\alpha$ for a positive integer k, since $\alpha = 2$ for $SU(2)$. Hence only the even integer highest weights λ are associated to a spherical representation. Assume $\lambda = 2$. The spherical representation $\pi^{(2)}_{SO(2)}$ of $SU(2)$ restricted to $SO(2)$ is

$$\pi^{(2)}_{SO(2)}(\exp x_1 \cdot X_1) = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\cos 2x_1 & \frac{1}{\sqrt{2}}\sin 2x_1 & \frac{1}{2} - \frac{1}{2}\cos 2x_1 \\ -\frac{1}{\sqrt{2}}\sin 2x_1 & \cos 2x_1 & \frac{1}{\sqrt{2}}\sin 2x_1 \\ \frac{1}{2} - \frac{1}{2}\cos 2x_1 & -\frac{1}{\sqrt{2}}\sin 2x_1 & \frac{1}{2} + \frac{1}{2}\cos 2x_1 \end{pmatrix} \tag{7.3.8}$$

where $x \in (-\pi, \pi]$, and $\frac{1}{2\pi}\mu((-\pi, \pi]) = 1$ is a natural Haar measure on $SO(2)$. The first row of $\pi^{(2)}$ is $\{\alpha^2, \sqrt{2}\alpha\beta, \beta^2\}$ (Example 2.5.5.1), so by (7.3.5), the unique $SO(2)$-fixed vector is

$$v_{SO(2)} = \frac{1}{2}(\alpha^2 + \beta^2). \tag{7.3.9}$$

In fact, every homogeneous polynomial of the form $(\alpha^2 + \beta^2)^k$ of degree $2k$ is $SO(2)$-invariant. Therefore, we can calculate the spherical function of $\lambda = 2$. Recall the calculations for $\pi^{(2)} \circ \exp$ in Example 6.3.3.1, we have

$$\begin{aligned}\phi_\lambda(\exp x \cdot X) &= \left\langle \pi_\lambda(\exp x \cdot X)(\alpha^2 + \beta^2), (\alpha^2 + \beta^2) \right\rangle \\ &= (\alpha^2 + \overline{\alpha}^2 + \beta^2 + \overline{\beta}^2)(\exp x \cdot X) \\ &= \left(2 + 3r\frac{\partial}{\partial r} + \left(r\frac{\partial}{\partial r}\right)^2 + 4(r^2 - x_2^2 - x_3^2)\right)\left(\frac{\sin\sqrt{x_1^2 + x_2^2 + x_3^2}}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right)^2\end{aligned} \tag{7.3.10}$$

for $X \in \mathfrak{su}(2), r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. When we restrict to $\mathfrak{a}_*$, the spherical function is

$$\phi_\lambda(\exp x_3 X_3) = 2\cos 2x_3, \quad x_3 X_3 \in \mathfrak{a}_*. \tag{7.3.11}$$

This is identical to the results of the calculation given in ([27], p.23). Thus, we can define a new basis for $\mathcal{H}_2$, which is

$$\left\{\sqrt{3}(\alpha^2 + \beta^2), \sqrt{6}\alpha\beta, \sqrt{3}(\alpha^2 - \beta^2)\right\}. \tag{7.3.12}$$

The non-zonal spherical functions of π_2 with respect to this basis are ϕ', ϕ'', where

$$\begin{aligned}\phi'_\lambda(\exp x \cdot X) &= \left\langle \pi_\lambda(\exp x \cdot X)\sqrt{3}(\alpha^2 + \beta^2), (\sqrt{6}\alpha\beta) \right\rangle \\ &= \sqrt{2}(\overline{\alpha}\beta + \alpha\overline{\beta})(\exp x \cdot X) \\ &= i\sqrt{2}\left(x_2\left(r\frac{\partial}{\partial r} + 2\right) - 2x_3x_1\right)\left(\frac{\sin\sqrt{x_1^2 + x_2^2 + x_3^2}}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right)^2.\end{aligned} \tag{7.3.13}$$

The restriction to $\mathfrak{p}_*$ is

$$\begin{aligned}\phi'_\lambda(\exp(x_2X_2+x_3X_3)) &= i\sqrt{2}x_2\left(r\frac{\partial}{\partial r}+2\right)\left(\frac{\sin\sqrt{x_2^2+x_3^2}}{\sqrt{x_2^2+x_3^2}}\right)^2 \\ &= i\sqrt{2}x_2\frac{\sin 2\sqrt{x_2^2+x_3^2}}{\sqrt{x_2^2+x_3^2}}.\end{aligned}\tag{7.3.14}$$

The second one ϕ'', is given by

$$\begin{aligned}\phi''_\lambda(\exp x\cdot X) &= \left\langle \pi_\lambda(\exp x\cdot X)\sqrt{3}(\alpha^2+\beta^2), (\sqrt{6}\alpha\beta)\right\rangle \\ &= \sqrt{2}(\alpha^2-\overline{\alpha}^2+\beta^2-\overline{\beta}^2)(\exp x\cdot X) \\ &= i2\left(x_3\left(r\frac{\partial}{\partial r}+2\right)+2x_1x_2\right)\left(\frac{\sin\sqrt{x_1^2+x_2^2+x_3^2}}{\sqrt{x_1^2+x_2^2+x_3^2}}\right)^2,\end{aligned}\tag{7.3.15}$$

and the restriction to $\mathfrak{p}_*$ is

$$\begin{aligned}\phi''_\lambda(\exp(x_2X_2+x_3X_3)) &= i2x_3\left(r\frac{\partial}{\partial r}+2\right)\left(\frac{\sin\sqrt{x_2^2+x_3^2}}{\sqrt{x_2^2+x_3^2}}\right)^2 \\ &= i2x_3\frac{\sin 2\sqrt{x_2^2+x_3^2}}{\sqrt{x_2^2+x_3^2}}.\end{aligned}\tag{7.3.16}$$

Notice that we have

$$\begin{aligned}i\frac{\partial}{\partial x_2}\phi_\lambda(\exp(x_2X_2+x_3X_3)) &= \phi'_\lambda(\exp(x_2X_2+x_3X_3)), \\ i\frac{\partial}{\partial x_3}\phi_\lambda(\exp(x_2X_2+x_3X_3)) &= \phi''_\lambda(\exp(x_2X_2+x_3X_3)).\end{aligned}\tag{7.3.17}$$

7.4 Wrapping Map and e-function

We first look at the **wrapping map** for compact connected semisimple Lie groups, which was discovered and proved by A.Dooley and N.Wildberger [19].

Definition 7.4.1. Let G be a compact connected semisimple Lie group, and let $\mathfrak{g}$ be its Lie algebra. Let ν be a distribution with compact support on $\mathfrak{g}$. Let $f \in C^\infty(G)$, and define $\tilde{f} = f\circ\exp$. Suppose j is the square root of the Jacobian determinant of exp. We define a linear functional $\Phi(\nu)$ on G to be

$$(\Phi(\nu), f)_G = (\nu, j\tilde{f})_{\mathfrak{g}}\tag{7.4.1}$$

and $\Phi(\nu)$ is called the **wrap** of ν. It is a distribution on G.

Theorem 7.4.2 ([19], Theorem 2.1)**.** *Let μ and ν be G-invariant distributions of compact support on $\mathfrak{g}$, or two G-invariant integrable functions. Then,*

$$\Phi(\mu *_{\mathfrak{g}} \nu) = \Phi(\mu) *_G \Phi(\nu) \tag{7.4.2}$$

where the convolution on the left is on $\mathfrak{g}$ and the convolution on the right is on G.

Remark. *Actually, the wrapping map Φ is a homomorphism between the Banach algebra of G-invariant distributions of compact support on $\mathfrak{g}$ and the Banach algebra of central distributions on G. The Kirillov character formula is a consequence of this homomorphism [14]. In addition, the convolution structures studied in Chapter 4 can also be wrapped to G by Φ. In fact, the idea of the wrapping map can be extended to symmetric spaces of compact type.*

Let $S = G/K$ denote a simply connected symmetric (coset) space; G is a connected Lie group, σ is an involution of G, and K is the connected component of the identity e of G in the fixed point subgroup of G under σ. Let $p : G \to G/K$, $p(g) = gK$ be the canonical projection, and $o = p(e) = K$ be the origin of S. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of G with respect to $d\sigma$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the eigenspaces of $d\sigma$ with eigenvalues 1 and -1, respectively, and $\mathfrak{p}$ is identified as the tangent space to S at the origin o. Also, let exp and $\mathrm{Exp} = p \circ \exp$ denote the exponential mapping of G and S, defined on $\mathfrak{g}$ and $\mathfrak{p}$, respectively.

For $X \in \mathfrak{p}$, let $j(X)$ be the square root of the Jacobian determinant of Exp, and $j(X)$ is K-invariant. Let $\mathfrak{s}$ be a K-invariant neighborhood of o in $\mathfrak{p}$ so that $\mathrm{Exp} : \mathfrak{s} \to S$ is a diffeomorphism. We can extend the idea of wrapping to symmetric spaces: let η be a K-invariant distribution of compact support on $\mathfrak{p}$, and for $f \in C_c^\infty(G/K)$, $\tilde{f} = f \circ \mathrm{Exp}$, we define the linear functional

$$(\Phi(\eta), f)_{G/K} = (\nu, j\tilde{f})_{\mathfrak{p}} \tag{7.4.3}$$

Suppose η, ξ are K-invariant distributions supported in $\mathfrak{s}$, and $\eta *_{\mathfrak{p}} \xi$ is also supported in $\mathfrak{s}$. Now, the interesting question is: Is there a wrapping map Φ for a symmetric space of compact type G/K? That is,

$$\Phi(\eta *_{\mathfrak{p}} \xi) = \Phi(\eta) *_{G/H} \Phi(\xi) \tag{7.4.4}$$

F.Rouvière [46] developed a so called 'twisted' convolution of 7.4.4 by introducing an analytical function $\mathfrak{s} \times \mathfrak{s} \to \mathbb{R}$, which he called the e-**function**.

Theorem 7.4.3 ([46], Theorem 2.2, Proposition 3.14)**.** *Let η, ξ be a K-invariant distribution which is locally integrable on $\mathfrak{s}$. Then,*

$$\Phi(\eta *_{\mathfrak{p},e} \xi) = \Phi(\eta) *_{G/K} \Phi(\xi) \tag{7.4.5}$$

where

$$(\eta *_{\mathfrak{p},e} \xi)(X) = \int_{\mathfrak{p}} \eta(Y)\xi(X-Y)e(X,Y)\,dY, \tag{7.4.6}$$

and the $e(X,Y)$ is a function of variables $(X,Y) \in \mathfrak{s} \times \mathfrak{s}$. This e-function can be expressed as

$$e(X,Y) = \frac{j(X)j(Y)}{j(X+Y)}\psi(X,Y) \tag{7.4.7}$$

where $\psi(X,Y)$ is the Jacobian determinant of the change of variables

$$(Ad(k_1)X, Ad(k_2)Y) \mapsto (X,Y), \text{ for } k_1, k_2 \in K. \tag{7.4.8}$$

Rouvière [46] has calculated the e-function as an infinite series that converges in a neighborhood of o in $\mathfrak{s}$. A.Dooley [16] has derived an explicit formula of the e-function for symmetric spaces of compact type, which generalises the wrapping map for a compact Lie group.

Let (G,K) be a Riemannian symmetric pair of compact type. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition with respect to the Cartan involution $d\sigma$. Let $\mathfrak{a}$ be the maximal abelian subalgebra of $\mathfrak{p}$, and let $\mathfrak{a}_+$ be the restricted positive Weyl chamber with respect to a set of restricted roots Φ_r^+. Let $A_+ = \exp(\mathfrak{a})$, then $G = KA_+K$ is the Cartan decomposition of G.

Theorem 7.4.4 ([16])**.** *Let (G,K) be a Riemannian symmetric pair of compact type. Let $S = G/K$ be the corresponding symmetric space and let $\mathfrak{p}$ be the tangent space of S at eK. For $X, Y \in \mathfrak{p}$, $H_1, H_2 \in \mathfrak{a}_+$, $k_1, k_2 \in K$, let $X = Ad(k_1)H_1$, $Y = Ad(k_2)H_2$, and $X + Y = Ad(k_3)H_3$. Then the e-**function** is given by*

$$e(X,Y) = \frac{j(H_1)j(H_2)}{j(H_3)} \prod_{\alpha \in \Phi_r^+} \prod_{w,w' \in W_r} \left[\frac{(\alpha(H_1) + \alpha^w(H_2) + \alpha^w(H_3))}{\cos\frac{1}{2}\left(\alpha(H_1) + \alpha^w(H_2) + \alpha^w(H_3)\right)}\right]^{\frac{m_\alpha}{2}} \tag{7.4.9}$$

where m_α *is the multiplicity of the restricted root* α*,* W_r *is the restricted Weyl group of* $\mathfrak{a}$*, and* $\alpha^w(H)$ *is the image of the root* α *under the* W_r*-action. The* j*-function is* K*-invariant and is the square root of the Jacobian of Exp, given by*

$$j(H) = \prod_{\alpha \in \Phi_r^+} \left[\frac{\sin \alpha(H)}{\alpha(H)} \right]^{\frac{m_\alpha}{2}}, \text{ for } H \in \mathfrak{a}^+. \tag{7.4.10}$$

Remark. *Note that when the symmetric pair is* $(G \times G, \Delta G)$*, the e-function is equal to* 1*. Thus, the convolution is described by the wrapping map. Another interpretation of this e-function is the sectional curvature of* S*.*

Example 7.4.4.1 ([14], Section. 4)**.** Let $(G, K) = (SO(3), SO(2))$ be the symmetric pair, and the symmetric space $S = SO(3)/SO(2)$ is the 2-sphere. The K-orbit in the tangent space $\mathfrak{p}$ is a circle. If we consider the convolution of the probability measures on two circles centred at the origin in the plane $\mathfrak{p}$ with radius r_1, r_2. We can observe that if for any X be on circle r_1 and Y be on circle r_2, then the distance from any point $X + Y$ to origin is given by

$$|X + Y| = (r_1^2 + r_2^2 + 2r_1 r_2 \cos\theta)^{\frac{1}{2}} \tag{7.4.11}$$

where θ is the angle between X and Y. Let μ_{r_1}, μ_{r_2} denote the probability measures on the circles, and let Ω be the support of $\mu_{r_1} * \mu_{r_2}$. Then

$$\begin{aligned} \mu_{r_1} * \mu_{r_2}(\Omega) &= \int_\Omega 1(|X+Y|) d\mu_{r_1}\, d\mu_{r_2} \\ &= \frac{1}{\pi} \int_0^\pi 1((r_1^2 + r_2^2 + 2r_1 r_2 \cos\theta)^{\frac{1}{2}})\, d\theta \\ &= \frac{1}{\pi} \int_{|r_1 - r_2|}^{r_1 + r_2} \frac{r}{r_1 r_2 \sin\theta}\, dr, \end{aligned}$$

by changing variable from angle θ to radius r. Also, we notice that $2r_1 r_2 \sin\theta$ is the area of the triangle of $o, Y, X + Y$. Hence, we can use Heron's formula to rewrite the the probability distribution f of the convolution of two circles r_1, r_2 as

$$f_{r_1, r_2}(r) = \frac{1}{\pi} \frac{2r}{\prod_\pm (r \pm r_1 \pm r_2)^{\frac{1}{2}}} \chi_{[|r_1 - r_2|, r_1 + r_2]}(r) \tag{7.4.12}$$

where χ is the uniform distribution in the radial direction. If we now calculate the area of this triangle when we 'wrap' it on the surface of the 2-sphere, we can obtain the probability distribution of the convolution of two circles r_1, r_2

on the surface of the sphere as

$$g_{r_1,r_2}(r) = \frac{1}{\pi}\frac{\sin r}{\prod_{\pm}\cos\frac{1}{2}(r \pm r_1 \pm r_2)^{\frac{1}{2}}}\chi_{[|r_1-r_2|,r_1+r_2]}(r). \tag{7.4.13}$$

If we compare these two distributions f and g, we obtain the e-function for the 2-sphere. That is,

$$e(X,Y) = \frac{r}{\sin r}\prod_{\pm}\left[\frac{\cos\frac{1}{2}(r \pm r_1 \pm r_2)}{\frac{1}{2}(r \pm r_1 \pm r_2)}\right]^{\frac{1}{2}}, \tag{7.4.14}$$

where $\pm$ denotes the all possible combinations of plus and minus, $r_1 = |X|, r_2 = |Y|$ and $r = |X+Y|$. Notice that $\sin r/r$ is the j-function of Exp on $\mathfrak{p}$. This idea can be extended to the symmetric spaces of $SO(n)/SO(n-1)$, and the e-function is given by

$$e(X,Y) = \left(\frac{\sin r_1 \sin r_2}{r_1 r_2}\right)^{n-2}\frac{2r}{\sin r}\prod_{\pm}\left[\frac{(r \pm r_1 \pm r_2)}{\cos\frac{1}{2}(r \pm r_1 \pm r_2)}\right]^{\frac{n-3}{2}}. \tag{7.4.15}$$

So the general idea is to restrict the calculations to two-dimensional plane, and compare the density functions of the geodesic triangle and the flat triangle.

Theorem 7.4.5 ([16]). *Let (G,K) be a Riemannian symmetric pair of the compact type. Let $S = G/K$ be the corresponding symmetric space, and let $\mathfrak{p}$ be the tangent space of S at eK. Then, e-function is defined on the entire tangent space $\mathfrak{p}$. Also, let μ and ν be K-invariant distributions of compact support on $\mathfrak{p}$, we have*

$$\Phi(\mu *_{\mathfrak{p},e} \nu) = \Phi(\mu) *_S \Phi(\nu) \tag{7.4.16}$$

Remark. *This 'twisted' convolution described by the e-function in (7.4.16) establishes a new correspondence between zonal spherical harmonic analysis on G/K and K-invariant harmonic analysis on $\mathfrak{p}$. We can combine this correspondence with the Kirillov-type non-commutative formula for K-invariant distributions on $\mathfrak{p}^*$ to transfer the Fourier analysis of zonal spherical distributions on G/K to Fourier analysis of K-invariant distributions with compact support on its tangent space $\mathfrak{p}$.*

Proposition 7.4.6. *Let ϕ_λ be a spherical function (defined in (7.3.3)) of the symmetric pair (G,K), where G is a compact simply connected semi-simple Lie* *group, and K is the fixed point group of an involutive automorphism of G. Then, the Fourier transform of the convolution zonal spherical distributions $\Phi(\mu)$ and $\Phi(\nu)$, with respect to the 'twisted' convolution of K-invariant distributions μ and ν, of compact support on $\mathfrak{p}$, is given by*

$$(\Phi(\mu) *_S \Phi(\nu), \phi_\lambda) = \int_p \int_p \mu(X)\nu(Y)e(X,Y)j(X+Y)\phi_\lambda(Exp(X+Y))\, dXdY, \tag{7.4.17}$$

and the Euclidean Fourier transform of $\phi_\lambda \circ Exp$ in $\mathfrak{p}$ is equal to $D_\lambda^K \nu_\lambda^K$, which is a K-invariant measure of $\mathfrak{g}^*$, restricted to $\mathfrak{p}^*$.

Remark. *In the calculation, the '$1/j(X+Y)$' function inside the e-function can be canceled with the j-function in front of the ϕ_λ. The K-invariant measure $D_\lambda^K \nu_\lambda^K$ can be determined by the Kirillov-type non-commutative formula, which is described in Section 7.3.*

Example 7.4.6.1. We can combine the e-function of the 2-sphere (7.4.14) and the spherical function of the 2-sphere when $\lambda = 1$ (7.3.11) to explicitly write down the Fourier transform of spherical distribution $\Phi(\mu) * \Phi(\nu)$ with compact support on the sphere. That is,

$$\begin{aligned}(\Phi(\mu) *_S \Phi(\nu), \phi_1) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} j(X)\mu(X)j(Y)\nu(Y) \prod_{\pm} \left[\frac{\cos \frac{1}{2}(|X+Y| \pm |X| \pm |Y|)}{\frac{1}{2}(|X+Y| \pm |X| \pm |Y|)} \right]^{\frac{1}{2}} \\ \cdot \cos 2|X+Y|\, dXdY,\end{aligned} \tag{7.4.18}$$

where the j-function is $j(X) = \sin|X|/|X|$, $X \in \mathfrak{p}$.

It is also interesting and motivating to derive the K-invariant measures for spherical functions with respect to the **restricted root system** of symmetric spaces. This is a subject of future work.

Appendix A

Symbolic Calculations

A.1 Example

A.1.1 Fundamental Representation of $Spin(5, \mathbb{R})$

The Cartan matrix of B_2 is

$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \tag{A.1.1}$$

We can choose a basis in $\mathbb{R}^2$ such that

$$\alpha_1(H_1) = 2, \quad \alpha_1(H_2) = -2, \quad \alpha_2(H_1) = -1, \quad \alpha_2(H_2) = 2 \tag{A.1.2}$$

and we can let $\alpha_1 = (2, 0)$, $\alpha_2 = (0, 2)$, and $H_1 = (1, -\frac{1}{2})$, $H_2 = (-1, 1)$, and

$$\begin{aligned} (\nu_{I_{\lambda_1}})^{\vee \mathfrak{t}}|_{\mathfrak{t}} \left(h_1 - h_2, -\frac{h_1}{2} + h_2 \right) = {} & \big[e^{-h(h_1+2h_2)} \big(-h_1^2 e^{i2h_1} - h_1^2 e^{i4h_1} + \\ & (h_1 - 2h_2)^2 e^{i2h_2} + (h_1 - 2h_2)^2 e^{i2(h_1+h_2)} + 8(h_1 - h_2) h_2 e^{i(h_1+2h_2)} \big) \big] \\ & \div \big(8h_1^2 (h_1 - 2h_2)^2 h_2 (h_1 - h_2) \big) \end{aligned} \tag{A.1.3}$$

The polynomial corresponding to the trace of Kirillov-type non-commutative formula of $d\pi_\lambda|_{\mathfrak{t}}$ is given by

$$\begin{aligned} Tr(P_{\lambda_1}|_{\mathfrak{t}}) = 120 &+ 250\left(\gamma\frac{\partial}{\partial\gamma}\right) + 175\left(\gamma\frac{\partial}{\partial\gamma}\right)^2 + 50\left(\gamma\frac{\partial}{\partial\gamma}\right)^3 + 5\left(\gamma\frac{\partial}{\partial\gamma}\right)^4 \\ &+ 72h_1^2 + 42h_1^2\left(\gamma\frac{\partial}{\partial\gamma}\right) + 6h_1^2\left(\gamma\frac{\partial}{\partial\gamma}\right)^2 + h_1^4 - 144h_1h_2 \\ &- 84h_1h_2\left(\gamma\frac{\partial}{\partial\gamma}\right) - 12h_1h_2\left(\gamma\frac{\partial}{\partial\gamma}\right)^2 - 4h_1^3h_2 + 144h_2^2 \\ &+ 84h_2^2\left(\gamma\frac{\partial}{\partial\gamma}\right) + 12\left(\gamma\frac{\partial}{\partial\gamma}\right)^2 + 4h_1^2h_2^2, \end{aligned} \tag{A.1.4}$$

and a calculation shows

$$Tr(P_{\lambda_1}|_{\mathfrak{t}}) \cdot (\nu_{I_{\lambda_1}})^{\vee\mathfrak{t}}|_{\mathfrak{t}} = 1 + e^{ih_1} + e^{-ih_1} + e^{i(h_1-2h_2)} + e^{-i(h_1-2h_2)}, \tag{A.1.5}$$

which is the character of π_{λ_1}. We can also calculate the global version formula. The irreducible skew-Hermitian matrix $d\pi_{\lambda_1}$ is equal to

$$\begin{pmatrix} ih_1 & x_1+iy_1 & \sqrt{2}(x_3+iy_3) & -\sqrt{2}(x_4+iy_4) & 0 \\ -x_1+iy_1 & -ih_1+i2h_2 & \sqrt{2}(x_2+iy_2) & 0 & -\sqrt{2}(x_4+iy_4) \\ -\sqrt{2}(x_3-iy_3) & -\sqrt{2}(x_2-iy_2) & 0 & \sqrt{2}(x_2+iy_2) & -\sqrt{2}(x_3+iy_3) \\ \sqrt{2}(x_4-iy_4) & 0 & -\sqrt{2}(x_2-iy_2) & ih_1-i2h_2 & x_1+iy_1 \\ 0 & \sqrt{2}(x_4-iy_4) & \sqrt{2}(x_3-iy_3) & -x_1+iy_1 & -ih_1 \end{pmatrix} \tag{A.1.6}$$

for $X \in \mathfrak{spin}(5)$. This is obtained with respect to the Chevalley basis of B_2 ([30], Exercise 25.6). Thus, we can calculate the $P^\lambda_{1,1}$ polynomial, that is,

$$\begin{aligned}
&\mathrm{d}\gamma^4 + ih_1\mathrm{d}\gamma^3 + 10\mathrm{d}\gamma^3 + h_1^2\mathrm{d}\gamma^2 + 4h_2^2\mathrm{d}\gamma^2 + x_1^2\mathrm{d}\gamma^2 + 4x_2^2\mathrm{d}\gamma^2 + 2x_3^2\mathrm{d}\gamma^2 \\
&+ 2x_4^2\mathrm{d}\gamma^2 + y_1^2\mathrm{d}\gamma^2 + 4y_2^2\mathrm{d}\gamma^2 + 2y_3^2\mathrm{d}\gamma^2 + 2y_4^2\mathrm{d}\gamma^2 + 9ih_1\mathrm{d}\gamma^2 - 4h_1h_2\mathrm{d}\gamma^2 \\
&+ 35\mathrm{d}\gamma^2 + ih_1^3\mathrm{d}\gamma + 7h_1^2\mathrm{d}\gamma + 4ih_1h_2^2\mathrm{d}\gamma + 28h_2^2\mathrm{d}\gamma + ih_1x_1^2\mathrm{d}\gamma - 2ih_2x_1^2\mathrm{d}\gamma + 7x_1^2\mathrm{d}\gamma \\
&+ 4ih_1x_2^2\mathrm{d}\gamma + 28x_2^2\mathrm{d}\gamma + 14x_3^2\mathrm{d}\gamma - 2ih_1x_4^2\mathrm{d}\gamma + 4ih_2x_4^2\mathrm{d}\gamma + 14x_4^2\mathrm{d}\gamma + ih_1y_1^2\mathrm{d}\gamma \\
&- 2ih_2y_1^2\mathrm{d}\gamma + 7y_1^2\mathrm{d}\gamma + 4ih_1y_2^2\mathrm{d}\gamma + 28y_2^2\mathrm{d}\gamma + 14y_3^2\mathrm{d}\gamma - 2ih_1y_4^2\mathrm{d}\gamma + 4ih_2y_4^2\mathrm{d}\gamma \\
&+ 14y_4^2\mathrm{d}\gamma + 26ih_1\mathrm{d}\gamma - 4ih_1^2h_2\mathrm{d}\gamma - 28h_1h_2\mathrm{d}\gamma - 4ix_2x_3y_1\mathrm{d}\gamma - 4ix_1x_3y_2\mathrm{d}\gamma \\
&+ 4i\sqrt{2}x_3x_4y_2\mathrm{d}\gamma + 4ix_1x_2y_3\mathrm{d}\gamma + 4i\sqrt{2}x_2x_4y_3\mathrm{d}\gamma - 4iy_1y_2y_3\mathrm{d}\gamma - 4i\sqrt{2}x_2x_3y_4\mathrm{d}\gamma \\
&+ 4i\sqrt{2}y_2y_3y_4\mathrm{d}\gamma + 50\mathrm{d}\gamma + 4ih_1^3 + 12h_1^2 + 16ih_1h_2^2 + 48h_2^2 + 4ih_1x_1^2 - 8ih_2x_1^2 \\
&+ 12x_1^2 + 2x_1^2x_2^2 + 16ih_1x_2^2 + 48x_2^2 + 2h_1^2x_3^2 + 8h_2^2x_3^2 - 8h_1h_2x_3^2 + 24x_3^2 + 4x_2^2x_4^2 \\
&- 8ih_1x_4^2 + 16ih_2x_4^2 + 24x_4^2 + 2x_2^2y_1^2 + 4ih_1y_1^2 - 8ih_2y_1^2 + 12y_1^2 + 2x_1^2y_2^2 + 4x_4^2y_2^2 \\
&+ 2y_1^2y_2^2 + 16ih_1y_2^2 + 4\sqrt{2}x_1x_4y_2^2 + 48y_2^2 + 2h_1^2y_3^2 + 8h_2^2y_3^2 - 8h_1h_2y_3^2 + 24y_3^2 \\
&+ 4x_2^2y_4^2 + 4y_2^2y_4^2 - 8ih_1y_4^2 + 16ih_2y_4^2 + 24y_4^2 + 24ih_1 - 16ih_1^2h_2 - 48h_1h_2 \\
&- 4\sqrt{2}x_1x_2^2x_4 - 16ix_2x_3y_1 + 4h_1x_2x_3y_1 - 8h_2x_2x_3y_1 - 16ix_1x_3y_2 + 4h_1x_1x_3y_2 \\
&- 8h_2x_1x_3y_2 + 16i\sqrt{2}x_3x_4y_2 + 4\sqrt{2}h_1x_3x_4y_2 - 8\sqrt{2}h_2x_3x_4y_2 + 8\sqrt{2}x_2x_4y_1y_2 \\
&+ 16ix_1x_2y_3 - 4h_1x_1x_2y_3 + 8h_2x_1x_2y_3 + 16i\sqrt{2}x_2x_4y_3 + 4\sqrt{2}h_1x_2x_4y_3 \\
&- 8\sqrt{2}h_2x_2x_4y_3 - 16iy_1y_2y_3 + 4h_1y_1y_2y_3 - 8h_2y_1y_2y_3 + 4\sqrt{2}y_1y_2^2y_4 \\
&- 16i\sqrt{2}x_2x_3y_4 - 4\sqrt{2}h_1x_2x_3y_4 + 8\sqrt{2}h_2x_2x_3y_4 - 4\sqrt{2}x_2^2y_1y_4 \\
&- 8\sqrt{2}x_1x_2y_2y_4 + 16i\sqrt{2}y_2y_3y_4 + 4\sqrt{2}h_1y_2y_3y_4 - 8\sqrt{2}h_2y_2y_3y_4 + 24
\end{aligned} \tag{A.1.7}$$

where $d\gamma$ represents $\gamma\frac{\partial}{\partial\gamma}$. To show this indeed is the correct output, we show that $Y_\alpha.\, v_{\lambda_1} = v_{\lambda_1-\alpha}$, and that is exactly $Y_{\alpha_1}.\, P^{\lambda}_{1,1}\left(\nu_{I_{\lambda_1}}\right)^{\vee_{\mathfrak{g}}} = P^{\lambda}_{1,2}\left(\nu_{I_{\lambda_1}}\right)^{\vee_{\mathfrak{g}}}$, where Y_{α_1} is the G-induced non-constant coefficient differential operator given by

$$\begin{aligned}
Y_{\alpha_1} = {}& (y_1 - ix_1)\frac{\partial}{\partial h_1} + (h_2 - h_1)\frac{\partial}{\partial y_1} - i(h_2 - h_1)\frac{\partial}{\partial x_1} + \frac{1}{2}(x_2 - iy_2)\frac{\partial}{\partial x_3} \\
&+ \frac{1}{2}(y_2 + ix_2)\frac{\partial}{\partial y_3} - \frac{1}{2}(x_3 + iy_3)\frac{\partial}{\partial x_2} - \frac{1}{2}(y_3 - ix_3)\frac{\partial}{\partial y_2}
\end{aligned} \tag{A.1.8}$$

Since the induced differential operators induced by G annihilate $(\nu_{I_{\lambda_1}})^{\vee\mathfrak{g}}$, we have $P^{\lambda}_{1,1}\,(\nu_{I_{\lambda_1}})^{\vee\mathfrak{g}}$ produces a polynomial, which is

$$
\begin{aligned}
& x_1\mathrm{d}\gamma^3 + iy_1\mathrm{d}\gamma^3 + 2ih_2x_1\mathrm{d}\gamma^2 + 9x_1\mathrm{d}\gamma^2 - 2x_2x_3\mathrm{d}\gamma^2 + 9iy_1\mathrm{d}\gamma^2 - 2h_2y_1\mathrm{d}\gamma^2 \\
& + 2ix_3y_2\mathrm{d}\gamma^2 - 2ix_2y_3\mathrm{d}\gamma^2 - 2y_2y_3\mathrm{d}\gamma^2 + x_1^3\mathrm{d}\gamma + iy_1^3\mathrm{d}\gamma + 2x_1x_2^2\mathrm{d}\gamma + 2x_1x_3^2\mathrm{d}\gamma \\
& - 2x_1x_4^2\mathrm{d}\gamma + x_1y_1^2\mathrm{d}\gamma + 2x_1y_2^2\mathrm{d}\gamma + 2\sqrt{2}x_4y_2^2\mathrm{d}\gamma + 2iy_1y_2^2\mathrm{d}\gamma + 2x_1y_3^2\mathrm{d}\gamma \\
& + 2\sqrt{2}x_4y_3^2\mathrm{d}\gamma + 2iy_1y_3^2\mathrm{d}\gamma - 2x_1y_4^2\mathrm{d}\gamma - 2iy_1y_4^2\mathrm{d}\gamma + h_1^2x_1\mathrm{d}\gamma + 14ih_2x_1\mathrm{d}\gamma \\
& - 2h_1h_2x_1\mathrm{d}\gamma + 26x_1\mathrm{d}\gamma - 4ih_2x_2x_3\mathrm{d}\gamma - 14x_2x_3\mathrm{d}\gamma - 2\sqrt{2}x_2^2x_4\mathrm{d}\gamma - 2\sqrt{2}x_3^2x_4\mathrm{d}\gamma \\
& + ih_1^2y_1\mathrm{d}\gamma + ix_1^2y_1\mathrm{d}\gamma + 2ix_2^2y_1\mathrm{d}\gamma + 2ix_3^2y_1\mathrm{d}\gamma - 2ix_4^2y_1\mathrm{d}\gamma + 26iy_1\mathrm{d}\gamma - 2ih_1h_2y_1\mathrm{d}\gamma \\
& - 14h_2y_1\mathrm{d}\gamma + 14ix_3y_2\mathrm{d}\gamma - 4h_2x_3y_2\mathrm{d}\gamma + 4i\sqrt{2}x_2x_4y_2\mathrm{d}\gamma - 14ix_2y_3\mathrm{d}\gamma + 4h_2x_2y_3\mathrm{d}\gamma \\
& - 4i\sqrt{2}x_3x_4y_3\mathrm{d}\gamma - 4ih_2y_2y_3\mathrm{d}\gamma - 14y_2y_3\mathrm{d}\gamma - 2i\sqrt{2}x_2^2y_4\mathrm{d}\gamma + 2i\sqrt{2}x_3^2y_4\mathrm{d}\gamma \\
& + 2i\sqrt{2}y_2^2y_4\mathrm{d}\gamma - 2i\sqrt{2}y_3^2y_4\mathrm{d}\gamma - 4\sqrt{2}x_2y_2y_4\mathrm{d}\gamma - 4\sqrt{2}x_3y_3y_4\mathrm{d}\gamma + 4x_1^3 + 4iy_1^3 \\
& + 2ih_1x_1x_2^2 + 8x_1x_2^2 - 2ih_1x_1x_3^2 + 4ih_2x_1x_3^2 + 8x_1x_3^2 - 8x_1x_4^2 - 4x_2x_3x_4^2 + 4x_1y_1^2 \\
& + 2x_2x_3y_1^2 + 2ih_1x_1y_2^2 + 8x_1y_2^2 + 8\sqrt{2}x_4y_2^2 + 2i\sqrt{2}h_1x_4y_2^2 + 8iy_1y_2^2 - 2h_1y_1y_2^2 \\
& - 2ih_1x_1y_3^2 + 4ih_2x_1y_3^2 + 8x_1y_3^2 + 8\sqrt{2}x_4y_3^2 - 2i\sqrt{2}h_1x_4y_3^2 + 4i\sqrt{2}h_2x_4y_3^2 + 8iy_1y_3^2 \\
& + 2h_1y_1y_3^2 - 4h_2y_1y_3^2 - 8x_1y_4^2 - 4x_2x_3y_4^2 - 8iy_1y_4^2 + 4ix_3y_2y_4^2 - 4ix_2y_3y_4^2 - 4y_2y_3y_4^2 \\
& + 4h_1^2x_1 + 24ih_2x_1 - 8h_1h_2x_1 + 24x_1 - 2h_1^2x_2x_3 - 2x_1^2x_2x_3 - 16ih_2x_2x_3 \\
& + 4h_1h_2x_2x_3 - 24x_2x_3 - 8\sqrt{2}x_2^2x_4 - 2i\sqrt{2}h_1x_2^2x_4 - 8\sqrt{2}x_3^2x_4 + 2i\sqrt{2}h_1x_3^2x_4 \\
& - 4i\sqrt{2}h_2x_3^2x_4 + 4\sqrt{2}x_1x_2x_3x_4 + 4ih_1^2y_1 + 4ix_1^2y_1 + 8ix_2^2y_1 - 2h_1x_2^2y_1 + 8ix_3^2y_1 \\
& + 2h_1x_3^2y_1 - 4h_2x_3^2y_1 - 8ix_4^2y_1 + 24iy_1 - 8ih_1h_2y_1 - 24h_2y_1 - 4ix_1x_2x_3y_1 + 4i\sqrt{2}x_2x_3x_4y_1 \\
& + 4ix_3x_4^2y_2 + 2ix_3y_1^2y_2 + 2ih_1^2x_3y_2 - 2ix_1^2x_3y_2 + 24ix_3y_2 - 4ih_1h_2x_3y_2 - 16h_2x_3y_2 \\
& + 16i\sqrt{2}x_2x_4y_2 - 4\sqrt{2}h_1x_2x_4y_2 + 4x_1x_3y_1y_2 - 4ix_2x_4^2y_3 - 2ix_2y_1^2y_3 - 2ih_1^2x_2y_3 \\
& + 2ix_1^2x_2y_3 - 24ix_2y_3 + 4ih_1h_2x_2y_3 + 16h_2x_2y_3 - 16i\sqrt{2}x_3x_4y_3 - 4\sqrt{2}h_1x_3x_4y_3 \\
& + 8\sqrt{2}h_2x_3x_4y_3 - 4x_1x_2y_1y_3 - 2h_1^2y_2y_3 - 2x_1^2y_2y_3 - 4x_4^2y_2y_3 + 2y_1^2y_2y_3 - 16ih_2y_2y_3 \\
& + 4h_1h_2y_2y_3 - 4\sqrt{2}x_1x_4y_2y_3 - 4ix_1y_1y_2y_3 - 4i\sqrt{2}x_4y_1y_2y_3 - 24y_2y_3 - 8i\sqrt{2}x_2^2y_4 \\
& + 2\sqrt{2}h_1x_2^2y_4 + 8i\sqrt{2}x_3^2y_4 + 2\sqrt{2}h_1x_3^2y_4 - 4\sqrt{2}h_2x_3^2y_4 + 8i\sqrt{2}y_2^2y_4 \\
& - 2\sqrt{2}h_1y_2^2y_4 - 8i\sqrt{2}y_3^2y_4 - 2\sqrt{2}h_1y_3^2y_4 + 4\sqrt{2}h_2y_3^2y_4 - 16\sqrt{2}x_2y_2y_4 \\
& - 4i\sqrt{2}h_1x_2y_2y_4 + 4\sqrt{2}x_1x_3y_2y_4 + 4i\sqrt{2}x_3y_1y_2y_4 + 4\sqrt{2}x_1x_2y_3y_4 - 16\sqrt{2}x_3y_3y_4 \\
& + 4i\sqrt{2}h_1x_3y_3y_4 - 8i\sqrt{2}h_2x_3y_3y_4 + 4i\sqrt{2}x_2y_1y_3y_4
\end{aligned}
\tag{A.1.9}
$$

and this is exactly the same as $P^{\lambda}_{1,2}$. Similarly, we can apply other ‘lowering’ operators to generate the first row of the matrix of polynomials. Due to the complexity of the computation, we omit the calculation. However, it can be produced using the programs referenced in Appendix A.2.

A.2 Computing Programs

We look at the technical aspects of how to implement the Kirillov-type non-commutative formula for arbitrary compact real forms of semisimple Lie algebras, and how to approximate the G-invariant measures of coadjoint orbits and moment sets of unitary representations by the techniques of 'box splines'.

Implement the Kirillov-type non-commutative formula:

Let G be a compact simply connected semisimple Lie group, T be a maximal torus of G. Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{t}$ be the Lie algebra of T. We describe an algorithm to calculate the highest weight representation of G, using the Kirillov-type non-commutative formula on the highest weight representation of $\mathfrak{g}$.

i. For each complexification of $\mathfrak{g}$, there is a **Cartan matrix** C that describes the Killing form among all simple roots. Taking the inverse of C, we can obtain the fundamental weights $\{\lambda_j\}$ in $\mathfrak{t}^*$.

ii. Every dominant weight can be written as a positive integral linear combination of all fundamental weights: $\lambda = \sum_{j=1}^{l} k_j \lambda_j$, each k_j is non-negative, and l is the rank of $\mathfrak{t}$. For each dominant highest weight λ, we can apply the lowering operators of the negative root spaces to the highest weight vector v_λ to generate all the weights in the saturated set $\Pi(\lambda)$ of the highest weight λ. In addition, we can apply the Frendenthal formula ([30], Theorem 22.3) to calculate the multiplicity of each weight.

iii. Let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$, we can then work out the highest weight representation $d\pi_\lambda$ of $\mathfrak{g}^{\mathbb{C}}$, with respect to the root decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\alpha \in \Phi^+} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}). \tag{A.2.1}$$

Then, we can convert $d\pi_\lambda(\mathfrak{g}^{\mathbb{C}})$ to the compact real form $d\pi_\lambda(\mathfrak{g})$.

iv. Let n be the dimension of $d\pi_\lambda(\mathfrak{g})$. We can calculate the matrix of the polynomials of the inverse Fourier transform of the Kirillov-type non-commutative formula for $d\pi_\lambda$ by (6.3.75). We can also calculate the inverse Fourier transform of the G-invariant measure of the moment set I_λ, denoted by $(\nu_{I_\lambda})^\vee(H)$, for $H \in \mathfrak{t}$. To obtain the global version, we can directly calculate the eigenvalues of $d\pi_\lambda$ and insert them into the G-invariant function $(\nu_{I_\lambda})^\vee$.

v. In this way, we obtain the highest weight representation $\pi(\exp \mathfrak{g})$ of G. We can then apply Weyl's unitarian trick to make π unitary.

We have done a few examples in Wolfram Mathematica.

1. The irreducible representations of $SU(2)$: π_1, π_2 and π_3.

   https://tgou1055.github.io/phd/Non-comm-Kirillov-formula-of-SU(2).pdf

2. The fundamental representation of $SU(3)$.

   https://tgou1055.github.io/phd/Non-comm-Kirillov-formula-of-SU(3)-fundamental.pdf

3. The highest weight δ representation of $SU(3)$.

   https://tgou1055.github.io/phd/Representations-of-delta-A2.pdf

4. The fundamental representation of $Spin(5)$ (double cover group of $SO(5)$).

   https://tgou1055.github.io/phd/B2-fundamental-representation.pdf

Approximate Measures:

Let F_α be the arc-length measure along the ray of a vector α in the real Cartan subalgebra $\mathfrak{t}$. Let $E = \prod_j * F_{\alpha_j}$ be a measure in $\mathfrak{t}$. Also, let $E = \lim_{n\to\infty} \prod_j * F_{n\alpha_j}$ in the sense of a distribution, where $n\alpha = [0,n]\cdot\alpha$. We can use the techniques of 'box splines' [11] to approximate E. That is,

i. Pick two arc-length measures $F_{n\alpha_1}, F_{n\alpha_2}$, and the convolution $F_{n\alpha_1} * F_{n\alpha_2}$ can be represented by the uniform distribution f on the parallelogram spanned by $\{\alpha_1, \alpha_2\}$.

ii. The convolution $F_{\alpha_3} * f$ can be calculated by

$$F_{\alpha_3} * f(\beta) = \int_0^n f(\beta - n\alpha_3)\, dn, \tag{A.2.2}$$

and the rest of the convolutions can also be calculated this way.

iii. Thus, the measure $E = \lim_{n\to\infty}\prod_j *F_{n\alpha_j}$ is a piecewise polynomial distribution on the Euclidean space $\mathfrak{t}$ (or $\mathfrak{t}^*$).

We have constructed some examples in Wolfram Mathematica.

1. The convolutions of unit spheres for $SU(2)$.

   `https://tgou1055.github.io/phd/Convolutions-of-unit-spheres-SU(2).pdf`

2. The projection Liouville measures of regular and singular coadjoint orbits of $SU(3)$.

   `https://tgou1055.github.io/phd/Projection-Measure-of-A2.pdf`

3. The projection Liouville measures of the coadjoint orbit of delta of $Spin(5)$.

   `https://tgou1055.github.io/phd/Projection-Measure-of-B2.pdf`

4. The intersection between the moment set of delta element and dual Cartan subalgebra of $SU(3)$.

   `https://tgou1055.github.io/phd/Moment-set-of-delta-of-SU(3).pdf`

Remark. *The symbolic calculation for multiple-fold convolutions of the arc-length measures of root and weight vectors will become very complicated and Mathematica on my laptop was unable to produce any output when $n \geqslant 5$ (when $\mathfrak{t}$ is two-dimensional). But our approach is more calculable and provides a more direct approach to calculate the piecewise polynomials of the density functions of convolution structures of coadjoint orbits and moment sets of irreducible unitary representations.*

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