University Revision Notes Series: Physics

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How to use this book

This book is designed for students studying a first year physics course at university. It was designed to be a companion to *University Physics* by Young; however, it is not rigidly tied to that book. Each section has keywords at the end, hence readers wishing to read further on the material of that section can simply look up the keywords in the index of their own textbooks, whether it be Young or another.

Each section has a brief description of the material, some worked examples, some problems for the reader to do with answers given, a list of keywords and important equations.

At the end of the book there are model exam papers with worked solutions and practice exam papers with the answers given. The book concludes with a glossary and an index.

Often the problem of understanding physics is one of being unable to see how the different topics in a chapter fit together. Hence at the end of each chapter is a concept map, which shows the relationship between the different topics. These have been developed by the editorial committee but are not necessarily unique. Hence it is suggested that on the completion of each chapter, readers draw their own concept maps and compare them with ours.

Acknowledgements

This book has been the work of a number of members of the teaching staff in the Applied Physics Department at the University of Technology, Sydney. Those directly involved in the writing of the chapters were Sue Hogg, Jim Franklin, Walter Kalceff, Ray Woolcott and myself. We have been assisted by Tony Moon, Geoff Anstis, David Blair, Bob Cheary and Alistair Thompson. In fact this book is really the next evolutionary stage of a program started in the early 1970s, when David Bailey, John Milledge and Ted Painter produced the first set of 'Physics Revision Notes' for the students at the then New South Wales Institute of Technology. The basic structure of each section has remained the same — a brief description, worked examples and problems for the reader. Most of the staff in the Applied Physics Department since that time have in some way contributed to these Revision Notes and we acknowledge their contribution, even though knowing who contributed what is lost in the annals of time.

I would like to especially thank the previous Head of Department, Professor Tony Moon, for his encouragement to transform our in-house 'Physics Revision Notes' into this book, Sue Hogg who looked after the project when I was away overseas on a study leave program and her son Trevor, himself a physics student, who typed the manuscript and made many insightful comments. I would like to thank Andrew Semmens of Addison–Wesley for his continued faith and patience that this book would eventually come together.

Contents

CHAPTER 1

Introduction

1.1 What is physics?

Physics is often defined as a study of force and energy and their interrelationship. It is a study of these quantities as well as matter and motion in an attempt to discover the laws that govern our universe. Physics is a quantitative science, which means it involves the process of measurement. Measurement is the process of *comparing* the quantity measured against a selected *reference.*

1.2 Physics and mathematics

Mathematics plays a vital role in physics. The following are the mathematical fundamentals that are important for physics.

1.2.1 Algebraic equations

The application of physical laws often leads to algebraic equations, which must be solved for the desired quantities. The basic rule in manipulating any algebraic equation is that both sides of the equation must be treated in the same way. Thus if a number is added to one side, or is multiplied by some factor, the same thing must be done to the other side.

Example: Consider the equation in x

$$2x - 6 = 10$$

To solve for x, start by adding 6 to both sides. This gives

$$2x = 16$$

Dividing by two gives the solution

$$x = 8$$

1.2.2 Quadratic equations

A quadratic equation of the form $ax^2 + bx + c = 0$ may be solved by using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1.2.3 Proportionality

If doubling quantity x always doubles quantity y, then we say that 'y is directly proportional to x' (often shortened to 'y is proportional to x'). This is expressed in symbols as

$$y \propto x$$

If y is proportional to x, then when y is plotted on a graph against x we get a straight line that passes through the origin (see Figure 1.1).

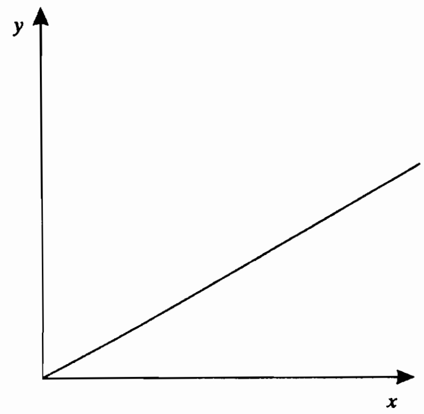


Figure 1.1

Now if y is proportional to x, there is a constant ratio between y and x. Thus

$$\frac{y}{x} = \text{a constant} = \text{k}$$

where 'k' is called the **constant of proportionality.** This proportionality can be written as the equality

$$y = \text{k}x$$

1.2.4 Exponents

Exponents play a very important role in physics. One key application is the system of scientific notation that is used to write both very large and very small numbers (more on this later). There are a few simple rules for exponents:

(1) In the expression 2^3 the number 2 is called the **base** and the number 3 is called the **index** or **exponent.**

(2) A positive exponent indicates the number of times the base must be multiplied by itself.

$$a^m = a \times a \times a \times \ldots \times a \quad (a \text{ multiplied by itself } m \text{ times})$$

$$\text{For example } 2^3 = 2 \times 2 \times 2 = 8$$

(3) A number having a negative exponent equals the reciprocal of that number with a positive exponent. That is,

$$a^{-m} = \frac{1}{a \times a \times a \times \ldots \times a} = \frac{1}{\text{a multiplied by itself } m \text{ times}}$$

$$\text{For example } 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

(4) Any number or quantity (except zero) raised to the exponent zero is always equal to one.

$$\text{For example } 3^0 = 1$$

(5) The **Index Rules** apply for all indices (exponents), whether positive, negative or zero. For all m and n:

	Rule	*Example*
(a)	$a^m \times a^n = a^{(m+n)}$	$2^3 \times 2^2 = 2^{(3+2)} = 2^5 = 32$
(b)	$\frac{a^m}{a^n} = a^{(m-n)}$	$\frac{3^5}{3^2} = 3^{(5-2)} = 3^3 = 27$
(c)	$(a^m)^n = a^{(m \times n)}$	$(2^2)^3 = 2^{(2 \times 3)} = 2^6 = 64$
(d)	$(a \times b)^m = a^m \times b^m$	$(7z)^2 = 7^2 \times z^2 = 49z^2$
(e)	$(\frac{a}{b})^m = \frac{a^m}{b^m}$	

1.2.5 Trigonometric functions

In any right-angled triangle, ABC, as shown in Figure 1.2, the trigonometric functions of angle θ are given by the following equations:

$$\sin\theta = \frac{a}{c}$$

$$\cos\theta = \frac{b}{c}$$

$$\tan\theta = \frac{a}{b}$$

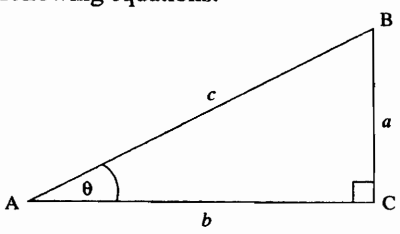


Figure 1.2

'Pythagoras' theorem

For the right-angled triangle ABC:

$$c^2 = a^2 + b^2$$

For a general triangle (see Figure 1.3):

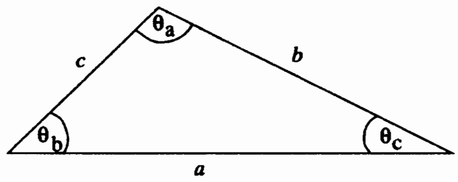


Figure 1.3

Sine rule:	$\dfrac{a}{\sin\theta_a} = \dfrac{b}{\sin\theta_b} = \dfrac{c}{\sin\theta_c}$
Cos rule:	$a^2 = b^2 + c^2 - 2bc\cos\theta_a$
Area of triangle	$\dfrac{1}{2}bc\sin\theta_a$

Trigonometrical relationships:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$$

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\sin C + \sin D = 2\sin\frac{C + D}{2}\cos\frac{C - D}{2}$$

1.2.6 Areas and volumes

In physics it is often necessary to calculate areas and volumes. The areas and volumes of important shapes are given below.

Areas:	Rectangle	Circle	Sphere
	$l \times b$	πr^2	$4\pi r^2$
Volumes:	Rectangular block	Cylinder	Sphere
	$l \times b \times h$	$\pi r^2 l$	$\dfrac{4}{3}\pi r^3$

1.2.7 Differentiation

Notation There are a variety of different notations for differentiation:

(1) Given a function $f(x)$, its derivative is

$$f'(x) \text{ or } \frac{d}{dx}(f(x))$$

(2) If the independent variable is called y and is written as $y = f(x)$, the derivative can be denoted by

$$y' \text{ or } \frac{dy}{dx}$$

1 Derivatives of polynomials

The four basic differentiation rules are:

(a) The derivative of x^n is nx^{n-1}.

(b) The derivative of a constant is zero.

(c) The derivative of cy *is* cy', where c is any constant.

(d) $\frac{d}{dx}(y \pm z) = \frac{dy}{dx} \pm \frac{dz}{dx}$

2 Derivatives of fractional and negative powers of *x*

Whether n be negative or a fraction (in fact when n is any real number)

$$\text{If } y = x^n \qquad \frac{dy}{dx} = nx^{n-1}$$

3 The product rule

$$\text{If } y = uv \qquad \frac{dy}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v$$

4 The quotient rule

This rule allows us to differentiate functions that are formed by dividing one expression by another:

$$\text{If } y = u/v \qquad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

5 The chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

6 Derivatives of trigonometric functions

There are two basic rules for differentiating trigonometric functions:

$$\text{If } y = \sin x \qquad \frac{dy}{dx} = \cos x$$

$$\text{If } y = \cos x \qquad \frac{dy}{dx} = -\sin x$$

Using these two results with other rules we already know, we can find:

$$\text{If } y = \tan x \qquad \frac{dy}{dx} = \sec^2 x$$

7 Exponential functions

The exponential function e^x has the special property that it is equal to its derivative. (The value of e is chosen so that this will be true.)

$$\text{If } y = e^x \qquad \frac{dy}{dx} = e^x$$

8 Logarithmic functions

The natural logarithm function $\ln x$ (or $\log_e x$) is defined by the property:

$$\text{If } x = e^y \qquad y = \ln x$$

$$\text{If } y = \ln x \qquad \frac{dy}{dx} = \frac{1}{x}$$

1.2.8 Integration

The indefinite integral: definition and notation

An 'indefinite integral' of a function f is a function whose derivative is f.

$\int f(x)\, dx$ stands for an indefinite integral of f.

Thus

$$\frac{d}{dx}\int f(x)\, dx = f(x)$$

Rules for integration:

Rule 1 For any constant, a

$$\int a\, dx = ax + C$$

Rule 2 For any rational number r other than $r = -1$

$$\int x^r dx = \frac{x^{r+1}}{r+1} + \mathrm{C}$$

Rule 3 For any constant, a

$$\int \mathrm{a}\, f(x)\, dx = a \int f(x)\, dx$$

Rule 4:
$$\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

Rule 5:
$$\int \mathrm{g(x)}^r\, \mathrm{g}'(x)\, dx = \frac{(\mathrm{g}(x))^{r+1}}{r+1} + C$$

Rule 6:
$$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$$

The definite integral

Sigma notation The Greek capital letter Σ is used in mathematics to indicate repeated addition.

Example

$$\sum_{i=1}^{9} i = 1 + 2 + 3 + \ldots + 9$$

i.e. the sum of the first nine positive integers

Area under a curve

Let f be a function such that $f(x) \geq 0$ for all x in the closed interval $[a,b]$. Then its graph is a curve lying on or above the x axis. We have an intuitive idea of the area A of the region lying under the curve, above the x axis and between the vertical lines $x = a$ and $x = b$.

If the area between a and b is divided into n strips of width Δx_i, then the area of the ith strip is approximately

$$f\left(x_i^*\right)\Delta x_i$$

where x_i^* is a value of x in the ith interval
and the total area is approximately

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

The approximation becomes better and better as the number of strips n in the interval $[a,b]$ becomes larger and larger. If the summations get arbitrarily close to a specific number as n approaches ∞, then this number is denoted

$$\int_a^b f(x)\,dx$$

and is called the **definite integral of *f* from *a* to *b*.**

1.2.9 Logarithms

The logarithm of a number to the base 10 is the power to which 10 must be raised to equal that number. Thus $100 = 10^2$, hence $\log_{10} 100 = 2$.

Use of logarithms in computations:

(1) Multiplication:

$$\log (m \times n) = \log m + \log n$$

(2) Division:

$$\log \frac{m}{n} = \log m - \log n$$

(3) Raising to a power

$$\log m^n = n \times \log m$$

(4) Extracting a root

$$\log \sqrt[n]{m} = \frac{1}{n} \times \log m$$

Natural logarithms, written as ln, use the exponential function e as their base. These are more basic and with the use of calculators more useful. Operations with the natural logarithms are the same as with logarithms to base 10.

1.2.10 Graphs

Graphs play an important role in physics. There are two main reasons for plotting experimental data on graphs. The first, and fairly obvious, reason is to communicate the results in a compact, easily comprehensible form. The second reason is as an aid in analysing the results.

When a point (x,y) is plotted on a graph as shown in Figure 1.4, the horizontal coordinate x is called the **abscissa** and the vertical coordinate y is called the **ordinate** of the point.

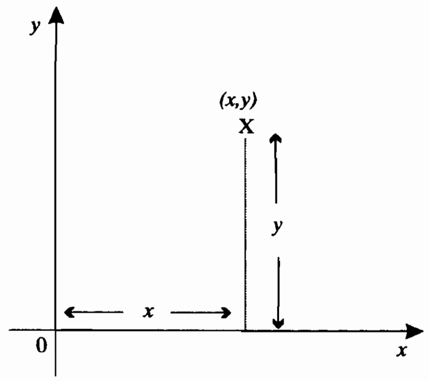


Figure 1.4

In this book most of the curves will be of simple relationships. Some of these and their equations are as follows:

Straight line (**Figure 1.5**)

$$y = mx + b$$

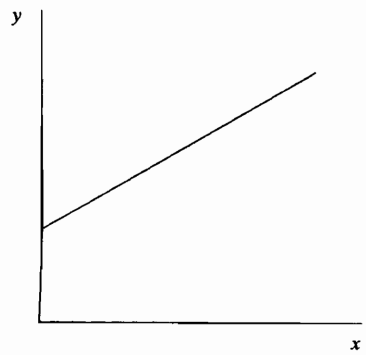


Figure 1.5

Circle (**Figure 1.6**)

$$x^2 + y^2 = r^2$$

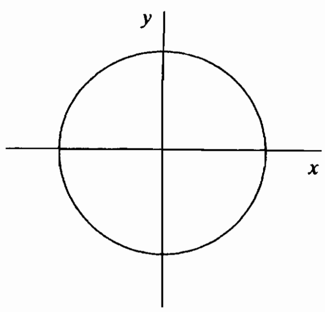


Figure 1.6

Parabola (**Figure 1.7**)

(*i*) $y = \pm ax^2$

(*ii*) $\pm y^2 = ax$

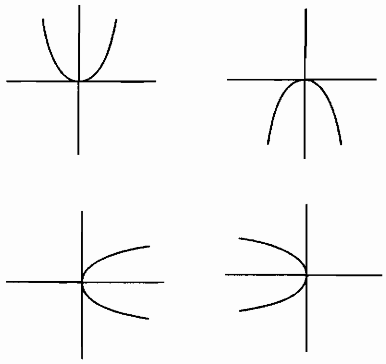


Figure 1.7

Hyperbola (**Figure 1.8**)

$xy = \text{constant}$

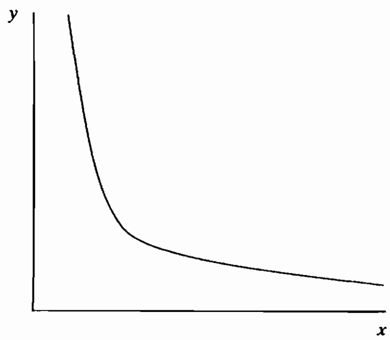


Figure 1.8

Exponential decay (**Figure 1.9**)

$y = y_0 \, e^{-ax}$

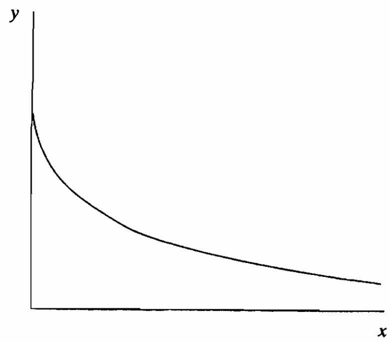


Figure 1.9

Exponential growth (**Figure 1.10**)

$$y = y_0\,(1 - e^{-ax})$$

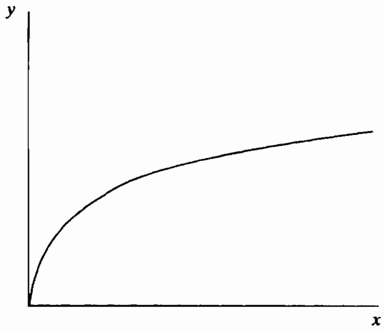


Figure 1.10

In physics, graphs are used for the following purposes:

- To record experimental values.
- To establish the relationships between two quantities.
- To interpolate (to obtain intermediate values).
- To extrapolate to values beyond the data.
- To calculate the gradient and intercept.
- To measure the area under the curve.
- To solve problems.

Graph calculations

Slope calculation:

At the point P (Figure 1.11), draw a tangent to the curve, extending the length to enable reasonable accuracy in the measurement of the 'rise', Δy, and the 'run' Δt.

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta t}$$

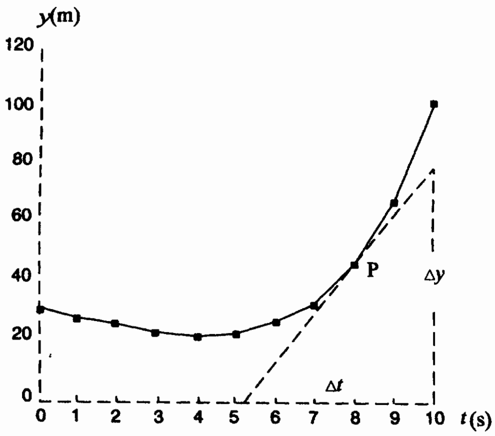


Figure 1.11

Area calculation (Figure 1.12):

To determine the integral, in this case of v with respect to t, between the times $t = 3$ s and $t = 8$ s, the area under the curve is divided into small area segments and then counted. Each segment has an area of 20 m/s × 1 s = 20 m. There are 19 whole segments + part segments totalling another 3 whole segments = 22

$$\int_{t=3}^{t=8} v(t)dt = \text{area} = 440\ m$$

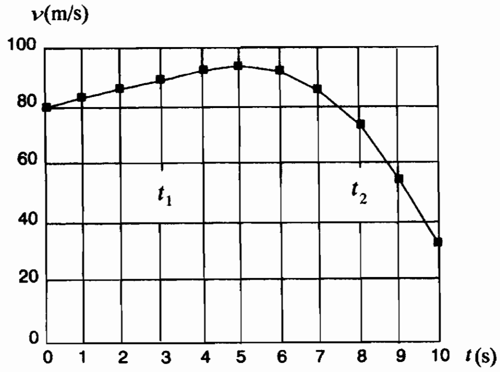


Figure 1.12

1.3 Scientific notation

In physics there is the need to manipulate both extremely large and extremely small numbers in order to express the values of various physical quantities. For example, in two millilitres of water there are approximately 66 870 000 000 000 000 000 000 water molecules and each molecule has a mass of about 0.000 000 000 000 000 000 000 000 059 82 kg. This method of writing numbers is not only awkward and cumbersome, it is also confusing and inconvenient. One is also very likely to make mistakes in writing so many zeros.

In order to overcome the difficulty of writing values with so many zeros, and to help make the computations involving these numbers simple, physicists use the method of **scientific notation.** In this method, any number N can be written as a number between 1 and 10 multiplied by a power of 10 corresponding to the number of decimal places the decimal point has been moved. When the decimal point is moved to the right, the power of 10 is given a negative sign. So in scientific notation the number N is written as

$$N = a \times 10^n$$

where a is a number between 1 and 10 and n is the power of 10 (an integer). The number 'a' is called the **mantissa** and the integer 'n' is known as the **index.**

By using scientific notation we can write the number of water molecules in two millilitres as 6.687×10^{22} and the mass of each molecule as 5.982×10^{-26} kg. See how much more compact this is than decimal notation (see Table 1.1).

Table 1.1 The first few powers of 10.

$1 = 10^0$	$1.0 = 10^0$
$10 = 10^1$	$0.1 = 10^{-1}$
$100 = 10^2$	$0.01 = 10^{-2}$
$1\ 000 = 10^3$	$0.001 = 10^{-3}$
$10\ 000 = 10^4$	$0.0001 = 10^{-4}$

Power of 10 notation is convenient to use in computation involving multiplication and division. The rules for scientific notation are those of the exponents in section 1.2.4.

1.4 Significant figures

Significant figures are those numbers in a measured value that give meaningful information about the quantity being measured. Although the accuracy of a physical measurement is always limited by the degree of refinement of the apparatus used and the skill of the observer, one and only one estimated figure (final digit) is retained in the reading, and is significant.

To determine the number of significant figures in a measured value, the following guidelines must be observed:

- All non-zero digits are significant.
- The location of a decimal point does not affect the number of significant figures in a reading.
- Zero is significant when it is between two significant digits.
- Zero is not significant if it merely indicates the location of a decimal point.

Significant figures in calculations

(i) In multiplication and division the answer should have as many significant figures as the least accurate of the factors.

(ii) When adding and subtracting, carry out the operation only as far as the first column that has an estimated figure.

1.5 The SI system

The system of measurement used in all scientific and medical work throughout the world is the *Système International d'Unites* or SI system. It is built up from the **base units** listed in Table 1.2.

Table 1.2 Base units of the SI system.

Dimension	*Unit*
Length	metre (m)
Mass	kilogram (kg)
Time	second (s)
Current	ampere (A)
Temperature	kelvin (K)
Luminous intensity	candela (cd)

Other units, known as **derived units,** are made up of multiples of these base units. For example, speed is distance travelled divided by time taken. So

$$\text{speed} = \frac{\text{distance (m)}}{\text{time (s)}} \quad \text{units } \frac{\text{m}}{\text{s}} = \text{m s}^{-1} \text{ or m/s}$$

A very important (and extremely useful) feature of the SI system is that it is both complete and self-consistent. This means that if you do a calculation with all the factors in SI units, then the answer is *always* in SI units. No extra calculations are required! However, if you use non-SI units in calculations then you may have to spend more time getting the units correct than you spend on the actual calculation. The rule of thumb is:

> 'Record all observations in the unit that the instrument reads in, but do all calculations in SI units.'

Prefixes

SI units are often too big or too small for the task at hand. So prefixes are used to indicate multiples and submultiples of units. Each prefix is given a symbol (see Table 1.3). It is vital that you become expert with prefixes.

Table 1.3 Decimal multiples and submultiples of SI units.

Prefix	*Symbol*	*Value*	
peta	P	1 000 000 000 000 000	$= 10^{15}$
tera	T	1 000 000 000 000	$= 10^{12}$
giga	G	1 000 000 000	$= 10^{9}$
mega	M	1 000 000	$= 10^{6}$
kilo	k	1 000	$= 10^{3}$
hecto	h	100	$= 10^{2}$
deca	da	10	$= 10^{1}$

		ONE	$= 10^0$
deci	d	0.1	$= 10^{-1}$
centi	c	0.01	$= 10^{-2}$
milli	m	0.001	$= 10^{-3}$
micro	μ	0.000 001	$= 10^{-6}$
nano	n	0.000 000 001	$= 10^{-9}$
pico	p	0.000 000 000 001	$= 10^{-12}$
femto	f	0.000 000 000 000 001	$= 10^{-15}$

Prefixes for submultiples are nearly all written in small letters from our alphabet, for example: m, n or p. The one exception is the Greek letter 'mu' written as μ. Prefixes for multiples between 1 and 1000 also use lower case letters. Capital letters are used by prefixes only for multiples greater than 1000.

1.6 Errors and accuracy

Whether a physical quantity has been determined by direct measurement, or deduced from measurements of related quantities, the value of that quantity is always inaccurate or uncertain to some degree. Consequently, whenever a numerical experimental result is quoted, some estimate of this uncertainty should be included. It is usual to refer to these uncertainties as errors.

These are three ways in which errors can arise:

- Systematic errors This is an error that can be minimised by improving the experimental technique, e.g. parallax error.
- Reading errors The accuracy of any instrument is always limited by the fineness of its scale graduations. The error due to the fineness of the scale is called the reading error.
- Random errors When a physicist makes a series of repeated measurements, with the greatest possible precision, the measurements often differ. These differences arise from various unknown and therefore uncontrollable sources of error. These are called random errors.

Types of errors

- Absolute error This is the uncertainty in measurement of some quantity. It is a measure of the range about our measured value in which the true value may be expected to lie.
- Fractional error A fractional error is the ratio of the absolute error to the measured value, that is,

$$\frac{\delta x}{x}$$

- Percentage error This is the absolute error expressed as a percentage of the measured value, that is,

$$\frac{\delta x}{x} \times 100\%$$

Compounding of errors

There are rules for estimating the error in a quantity that has been calculated from the measured values of other quantities.

If you have two quantities x and y with uncertainties Δx and Δy then

- when adding or substracting, add the *absolute* errors:

$$z = x \pm y \quad \Delta z = \Delta x + \Delta y$$

- when multiplying or dividing, add the *fractional errors* (or percentage errors) to give the combined *fractional error* (or percentage error):

$$z = x \times y \qquad \frac{\Delta z}{z} = \frac{\Delta x}{x} + \frac{\Delta y}{y}$$
$$z = x/y$$

- The fractional error in x^n is n times the *fractional error* in x;

$$z = x^n \qquad \frac{\Delta z}{z} = n \times \frac{\Delta x}{x}$$

The size of the absolute error determines the number of figures that are significant and should therefore be quoted in an answer, Conversely, to state a certain number of significant figures implies an associated degree of uncertainty.

Thus, a value quoted of $F = 2.68$ N implies an absolute error of ± 0.005 N. A value quoted of $\mu = 0.2$ implies an absolute error of ± 0.05.

Graphical errors

There is often a need to obtain a numerical result from the slope or intercept of a straight-line graph. The simplest way of estimating an absolute error in this case is to also draw graphs of maximum and minimum slope consistent with plotted points and to use these to obtain the range of error. Thus it is important, when plotting a graph, to indicate the error in each point plotted.

1.7 Dimensional analysis

The quantities used in physics may be separated into two types, fundamental and derived. The fundamental **dimensions** that occur in mechanics are **mass, length** and **time.** Electrical and magnetic quantities require **current** as an additional dimension. A given dimension can be expressed in a number of different **units** but in this book, only SI units will be used. For example, velocity is a measure of change of position with time. Thus, its dimensions are $\mathrm{L\,T^{-1}}$. It may be written as [velocity]

$= L\,T^{-1}$, where the square brackets mean 'dimensions of'. In SI units, velocity is measured in metres per second.

For an equation to be physically meaningful the dimensions of each term in the equation must be the same. That is, the powers of the fundamental dimension must be the same for each term. If this principle is not obeyed then an equation that is true for one set of units is not true for other equally valid sets of units.

The principle is useful in a number of ways. An equation can be shown to be incorrect. Another application is to derive a possible relationship between variables that are known to affect a particular process.

In order to use the method of dimensions effectively it is necessary to be aware of the dimensions of the most common derived quantities.

Some important quantities in mechanics are given in Table 1.4, together with their dimensions and SI units.

Table 1.4 The dimensions and SI units of some quantities.

Quantity	*SI unit*	*Dimension*
Area	m^2	L^2
Volume	m^3	L^3
Velocity	$m\,s^{-1}$	$L\,T^{-1}$
Acceleration	$m\,s^{-2}$	$L\,T^{-2}$
Density	$kg\,m^{-3}$	$M\,L^{-3}$
Momentum	$kg\,m\,s^{-1}$	$M\,L\,T^{-1}$
Force	$kg\,m\,s^{-2}$ (N)	$M\,L\,T^{-2}$
Energy	N m (J)	$M\,L^2\,T^{-2}$
Pressure	$N\,m^{-2}$ (Pa)	$M\,L^{-1}\,T^{-2}$
Power	$J\,s^{-1}$ (W)	$M\,L^2\,T^{-3}$
Angle	radian	L^0

Dimensions can be used in the following situations:

(1) **To check equations** If the dimensions of the left-hand and right-hand parts of an equation, and between additive individual terms on one side of the equation, are not the same, the equation cannot be a correct one

for example, the equation $$s = u + \frac{1}{2}at^2$$

cannot be correct since the LHS term 's', and the second term of the RHS both have the dimension of length (L), while the first term on the RHS has the dimension of a velocity ($L\,T^{-1}$).

Note: The correctness of dimensions does not guarantee that the equation is correct. Work and torque have the same dimensions but are quantities that are physically quite different.

(2) To obtain the form of an equation If it is known what physical quantities participate in a process being investigated, it is often possible to establish the nature of the relationship connecting the quantities by comparing the dimensions.

The method of dimensions is based on the requirement that the relationship between physical quantities must be independent of the selection of units, that is, LHS, RHS and all individual added terms must have identical dimensions in the system of units being used.

The method is most easily demonstrated by an example:

Example The acceleration, experienced by a body travelling in a circle towards the centre of that circle, is found to be dependent only on the tangential velocity 'v' at which the body is travelling and 'r', the distance that the body is from the centre of the circle.

Let us assume the acceleration is some type of power relationship of the two variables 'v', 'r', that is, let

$$a = \mathrm{k}v^{\alpha}r^{\beta}$$

where k is a dimensionless constant

Equating dimensions of both sides:

$$(\mathrm{L\,T^{-2}}) = (\mathrm{L\,T^{-1}})^{\alpha}\,(\mathrm{L})^{\beta}$$

The equality is satisfied if the powers of M, L and T on both sides are equal, that is,

M:	0	=	0
L:	1	=	$\alpha + \beta$
T:	−2	=	$-\alpha$

Hence we find that $\alpha = 2$, $\beta = -1$ and the equation must be

$$a = \frac{\mathrm{k}\,v^2}{r}$$

1.8 Vectors and scalars

Many of the quantities discussed in this book are completely specified by a statement of their magnitude; these are called **scalars.** Examples include mass, volume and time. However, there are some quantities that also require a statement of direction and these are called **vectors.** Two common examples are force and velocity. (For instance, there is a completely different response when a large upward force is applied to a book sitting on a table from the response to the same force applied to the book in a downwards direction.)

It is important to understand vectors because when dealing with vector quantities, such as a force, we need to know not only how big the force is, but also in which direction it is acting.

A **scalar** quantity has magnitude but no direction — it may be positive or negative (e.g. electric charge).

A **vector** quantitiy has both magnitude and direction — a negative vector points in a direction that is *opposite* to a positive vector.

1.8.1 Resolution of vectors into components

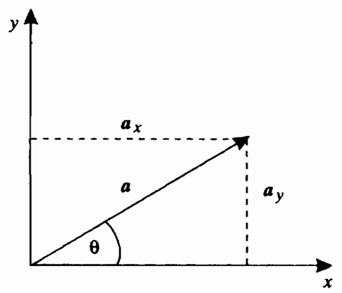


Figure 1.13

Vector a has components (a_x, a_y)

where $\quad a_x = |a| \cos\theta$

vector $\quad a_y = |a| \sin\theta$

$(|a| = \text{magnitude of})$

In terms of components (Figure 1.13):

the **magnitude** of a vector is given by

$$|a| = \sqrt{a_x^2 + a_y^2}$$

the **direction** is given by

$$\tan\theta = \frac{a_y}{a_x}$$

1.8.2 Addition and subtraction of vectors

One method involves the use of components. The procedure is to determine the components of each vector, then add the components:

that is, if
$$\boldsymbol{c} = \boldsymbol{a} \pm \boldsymbol{b}$$

then
$$c_x = a_x \pm b_x$$
$$c_y = a_y \pm b_y$$

and
$$c = \sqrt{c_x^2 + c_y^2}$$
$$\tan\theta = \frac{c_y}{c_x}$$

1.8.3 Multiplication of vectors

Vectors may be multiplied in two ways:

(1) To form a scalar (i.e. Scalar or Dot Product)

$$c = \boldsymbol{a} \cdot \boldsymbol{b}$$
$$c = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

(2) To form a vector (i.e. Vector or Cross Product)

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$
$$|\boldsymbol{c}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta$$

and the direction of $\boldsymbol{c}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ and is given by the right-hand screw rule. (The fingers on the right hand are in the direction from $\boldsymbol{a}$ to $\boldsymbol{b}$, see Figure 1.14.)

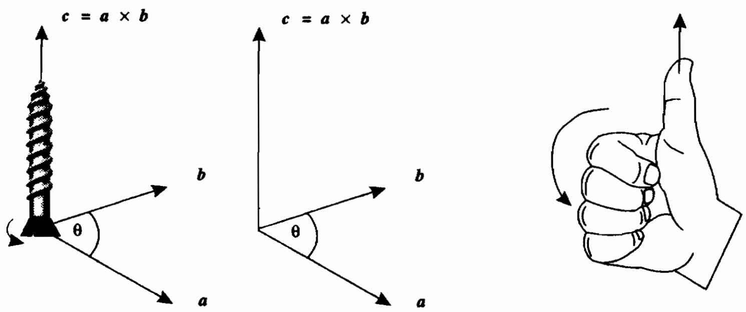


Figure 1.14

Worked examples

$|a| = 8.0$

$|b| = 12$

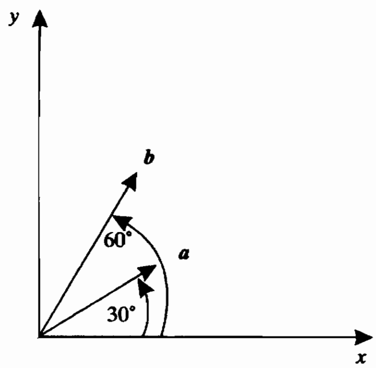


Figure 1.15

Find:

(a) $c = a + b$
(b) $d = a - b$
(c) $e = a \cdot b$
(d) $f = a \times b$

Solutions

(*a*)

$$a_x = |a| \cos \theta = 8 \cos 30^\circ = 8 \times 0.866 = 6.9$$

$$a_y = |a| \sin \theta = 8 \sin 30^\circ = 8 \times 0.5 = 4.0$$

$$b_x = |b| \cos \theta = 12 \cos 60^\circ = 12 \times 0.5 = 6.0$$

$$b_y = |b| \sin \theta = 12 \sin 60^\circ = 12 \times 0.866 = 10.4$$

$$c_x = a_x + b_x = 6.9 + 6.0 = 12.9$$

$$c_y = a_y + b_y = 4.0 + 10.4 = 14.4$$

$$\therefore \quad |c| = \sqrt{c_x{}^2 + c_y{}^2} = \sqrt{12.9^2 + 14.4^2} = 19$$

$$\tan \theta = \frac{c_y}{c_x} = \frac{14.4}{12.9} = 1.116$$

$$\theta = 48^\circ$$

(b) From (a):

$$d_x = a_x - b_x \qquad d_y = a_y - b_y$$

$$= 6.9 - 6.0 \qquad = 4.0 - 10.4$$

$$= 0.9 \qquad = -6.4$$

$$\therefore\ |\boldsymbol{d}| = \sqrt{d_x^{\ 2} + d_y^{\ 2}}$$

$$= \sqrt{0.9^2 + 6.4^2}$$

$$= 6.5$$

$$\tan\theta = \frac{d_y}{d_x}$$

$$= \frac{-6.4}{0.9}$$

$$= -7.11$$

$$\theta = -82^\circ$$

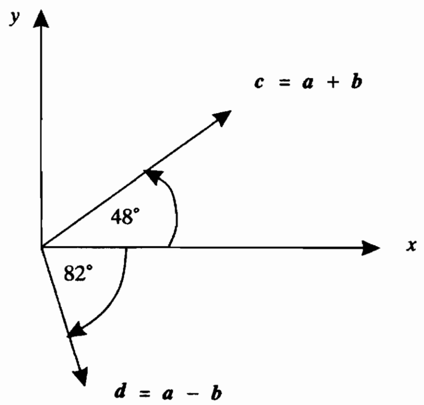


Figure 1.16

(c) $\boldsymbol{e} = \boldsymbol{a} \cdot \boldsymbol{b}$

$$e = |\boldsymbol{a}|\ |\boldsymbol{b}| \cos\theta$$

$$= 8.0 \times 12 \times \cos 30^\circ$$

$$= 8 \times 12 \times 0.866$$

$$e = 83$$

(d) $\boldsymbol{f} = \boldsymbol{a} \times \boldsymbol{b}$

$$|\boldsymbol{f}| = |\boldsymbol{a}|\ |\boldsymbol{b}| \sin\theta$$

$$= 8.0 \times 12 \times \sin 30^\circ$$

$$= 8 \times 12 \times 0.5$$

$$f = 48$$

direction is as illustrated

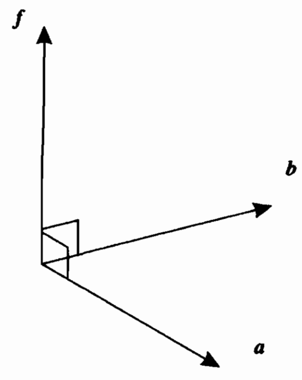


Figure 1.17

1.9 States of matter

To the physicist, 'matter' is anything that has mass and occupies space, and so is any material found on earth, with the exception of a vacuum (which is empty space).

Matter can exist in three physical states: either as a solid, as a liquid or as a gas. Water is a good exmple. It is found either as a solid (ice), liquid (water) or gas (steam). The state of matter can be altered by either heating or cooling.

The composition of matter has been closely investigated by physicists. It is found to be made up of tiny particles called **atoms,** which are usually bound to other atoms to form **molecules,** and are continually moving or vibrating.

1.9.1 Solids

In a solid the forces of molecular attraction are capable of locking molecules together in a rigid, ordered structure. The fact that the molecules are rigidly held in place gives solids a definite shape. In solids all the molecules 'touch' each other, so freezing almost always causes a slight decrease in volume (water is the only common exception to this rule).

The idea that molecules in solids are in contact suggests that all solids should be incompressible. Indeed most are (try compressing a block of iron). In fact the only solids that can be readily compressed are those that have a lot of empty space in their bulk structure. However, if they are compressed, there soon comes a point where there is no more empty space inside (the molecules are more or less touching). Elastic substances such as rubber lack the firmness of most solids because they are composed of very large molecules that are easily stretched.

1.9.2 Liquids

The molecules in liquids are moving rapidly enough so that individual molecules are not rigidly bound together. Consequently, liquids lack a definite shape and take the shape of the container holding them. Thus a water-filled balloon will alter its shape when the balloon is squeezed. Note that since the molecules of a liquid are in contact with each other, it is very difficult indeed to compress a liquid (they are almost incompressible).

1.9.3 Gases

Gases consist of widely separated, rapidly moving molecules that interact only when they collide with each other or with their container's walls. The molecules are moving rapidly and when they collide they bounce off each other (or from the container walls) without any loss of energy and continue moving just as fast as they did before the collision. This is just as well, since otherwise all the molecules would eventually slow down and settle at the bottom of the container! The gas molecules themselves only occupy a very small fraction of the total volume of a container of gas (this is why gases are so light). For example, air molecules take up less than 0.1% of the volume of a container filled with air; the other 99.9% of the volume is just empty space between the molecules.

The fact that gases mostly consist of empty space between the molecules means that one can greatly compress a gas and it will still remain as a gas. Another consequence of the fact that gases consist of widely separated, freely moving molecules is that a sample of gas will expand until it completely fills any container.

1.9.4 Fluids

Both liquids and gases can flow. Wind, for example, is a flow of air. It turns out that many properties of flowing matter are the same for both flowing liquids and gases. Thus scientists use the term **fluid** to describe any substance that will flow. Under ordinary conditions only gases and liquids are fluids. However, at high pressures some 'solids' flow very slowly.

CHAPTER 2

Mechanics

2.1 Motion along a straight line

Instantaneous displacement, '*s*', velocity, '*v*', acceleration, '*a*', are related by:

$$v = \frac{ds}{dt}, \quad a = \frac{dv}{dt}$$

Average velocity

$$v_{av} = \frac{\text{distance travelled}}{\text{time}}$$

2.1.1 Constant acceleration motion

Many problems involve constant acceleration, under which conditions the final velocity, v, and the displacement, s, are related to the initial velocity v_0, the time, t, and the acceleration, a, by:

$$v = v_0 + at$$

$$s = v_0 t + \frac{1}{2} at^2$$

$$v^2 = v_0^2 + 2as$$

2.1.2 Graphical methods: *s–t, v–t, a–t* curves

Particularly for cases of motion where several distinct sections occur, such as a car accelerating from rest, then travelling at constant velocity and then decelerating to rest, the property, that the integral of a variable with respect to time is the area under the curve of that variable plotted with respect to time, is useful:

$$\Delta s_{t_1 \to t_2} = \int_{t_1}^{t_2} v\,dt = \text{area under } v\text{–}t \text{ curve}$$

$$\Delta v_{t_1 \to t_2} = \int_{t_1}^{t_2} a\,dt = \text{area under } a\text{–}t \text{ curve}$$

$$v(t) = \text{slope of curve } s\text{–}t \text{ at time } t$$

$$a(t) = \text{slope of curve } v\text{–}t \text{ at time } t$$

2.1.3 Relative velocity

The motion of one body, A (see Figure 2.1), can be described in terms of its relative motion with respect to another body, B, whose motion is known:

$$\boldsymbol{v}_A = \boldsymbol{v}_{AB} + \boldsymbol{v}_B$$

where $\boldsymbol{v}_{AB}$ is the motion of A with respect to B

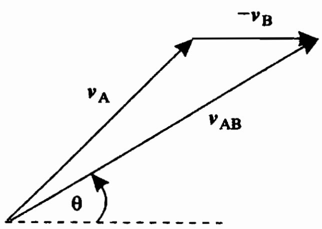


Figure 2.1

or, if B is also moving with respect to another body, C,

$$\boldsymbol{v}_{AC} = \boldsymbol{v}_{AB} + \boldsymbol{v}_{BC} = \boldsymbol{v}_{AB} - \boldsymbol{v}_{CB}$$

Worked example At the same time as one ball, A, is dropped from the top of a building 81.0 m high a second ball, B, is hit vertically with a starting velocity of 20.0 m s^{-1}. If gravity has the value g = 9.80 m s^{-2},

(a) How long does each ball take to hit the ground?

(b) With what velocity does each ball hit the ground?

Solution (vertically downwards taken as positive)

Ball A:
Data given:

$$v_0 = 0 \text{ m s}^{-1}$$
$$s = 81.0 \text{ m}$$
$$a = g = 9.80 \text{ m s}^{-2}$$

Calculations:

(a)
$$s = 81.0 = v_0 t + 0.5 \times a \times t^2$$
$$= 0 + 0.5 \times 9.80 \times t^2$$
$$t = \sqrt{\frac{2 \times 81.0}{9.80}} = 4.07 \text{ s}$$

(b)
$$v = v_0 + a\,t$$
$$= 0 + 9.80 \times 4.07$$
$$= 39.9 \text{ m s}^{-1} \text{ downwards}$$

Ball B:
Data given:

$$v_0 = -20.0 \text{ m s}^{-1}$$
$$s = 81.0 \text{ m}$$
$$a = g = 9.80 \text{ m s}^{-2}$$

Calculations:

(a)
$$s = 81.0 = v_0 t + 0.5 \times a \times t^2$$
$$= -20.0 \times t + 0.5 \times 9.80 \times t^2$$

i.e. quadratic equation:

$$4.90t^2 - 20.0t - 81.0 = 0$$

Solutions:

$$t = \frac{40.0 \pm \sqrt{400 + 4 \times 4.90 \times 81.0}}{9.80}$$
$$= 8.63 \text{ s or } -0.47 \text{ s}$$

The negative time is not physically possible, so

$$t = 8.63 \text{ s}$$

(b) $$v = v_0 + a\,t = -20.0 + 9.80 \times 8.63$$
$$= 64.6 \text{ m s}^{-1} \text{ downwards}$$

Problem A balloon has been ascending vertically from the ground at a constant speed for 10.0 s when a stone is allowed to fall from the side of the balloon. If it takes the stone a further 5.00 s to reach the ground, find

(a) the speed of the balloon; and

(b) its height at the instant that the stone is released.

(Ans: (a) 8.17 m s^{-1}; (b) 81.7 m)

Keywords

Displacement, speed, velocity (instantaneous), velocity (average), acceleration (instantaneous).

Important equations

$$\boxed{v = \frac{ds}{dt},\ a = \frac{dv}{dt}}$$

$$\boxed{\begin{gathered} v = v_0 + at \\ s = v_0 t + \frac{1}{2}at^2 \\ v^2 = v_0^2 + 2as \end{gathered}}$$

2.2 Motion in a plane

2.2.1 Vectorial extensions

The vectorial extensions of the one-dimensional equations are expressed as follows:

$$\mathbf{v} = \frac{d\mathbf{s}}{dt},\ \mathbf{a} = \frac{d\mathbf{v}}{dt}$$

$$\text{Average velocity} = \frac{\text{distance travelled}}{\text{time}}$$

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t$$

$$\mathbf{s} = \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$$

$$v^2 = v_0^2 + 2\mathbf{a}\cdot\mathbf{s}$$

2.2.2 Components

Motion in a plane can be analysed by resolving all vector displacements, velocity and acceleration into 'components' in two independent directions, for example horizontal and vertical, or parallel and perpendicular to an inclined plane, or tangential and radial to circular motion. Then simultaneously the one-dimensional kinematic equations are applied to the components of s, v_0, v, a:

For example, horizontally and vertically (x,y):

$$v_x = v_{0x} + a_x t\,,\ v_y = v_{0y} + a_y t$$

$$s_x = v_{0x}t + \frac{1}{2}a_x t^2\,,\ s_y = v_{0y}t + \frac{1}{2}a_y t^2$$

etc.

For example, along a plane, $\|$ and $\perp$ to the plane:

$$v_{\|} = v_{0\|} + a_{\|}t\,,\ v_{\perp} = v_{0\perp} + a_{\perp}t$$

$$s_{\|} = v_{0\|}t + \frac{1}{2}a_{\|}t^2\,,\ s_{\perp} = v_{0\perp}t + \frac{1}{2}a_{\perp}t^2$$

etc.

For example, circular motion:

radial,
$$s_N = R \text{ (constant)}$$
$$v_N = 0$$
$$a_N = \frac{v_T^2}{R}$$

and tangential,
$$s_T = R\,\theta$$
$$v_T = R\frac{d\theta}{dt} = R\omega$$
$$a_T = R\frac{d\omega}{dt} = R\,\alpha$$

Having calculated the individual components, the final answer is often given as a magnitude and direction (i.e. polar coordinates), for example in velocity:

$$|v| = \sqrt{v_x^2 + v_y^2}$$

$$\theta = \tan^{-1}\left[\frac{v_y}{v_x}\right]$$

Example: projectile motion (Figure 2.2)

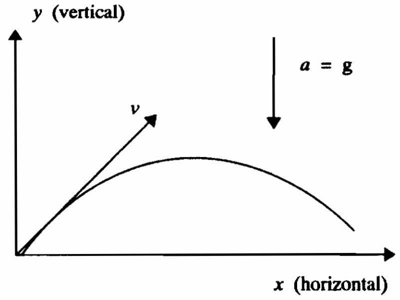


Figure 2.2

Horizontal motion

$$v_{x0} = v \cos \theta$$

$$a_x = 0$$

$$s_x = x$$

time limited by the vertical motion (see next column)

Distance travelled in x direction:

$$s_x = v_x t = v \cos \theta\, t = v \cos \theta \frac{2 \sin \theta}{g}$$

x component of the velocity with which it hits the ground is:

$$v \cos \theta$$

The instantaneous angle at any time is given by:

$$\theta' = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Vertical motion

Take s, v and a as positive upwards

$$v_{y0} = v \sin \theta$$

$$a_y = -g$$

$$s_y = y$$

$$y = v_{y0}t + \frac{1}{2} a_y t^2$$

If it started at ground level, $y = 0$

$$0 = v \sin \theta\, t - \frac{1}{2} g t^2$$

time to hit ground = 0 or $t = \dfrac{2v \sin \theta}{g}$

y component of the velocity with which it hits the ground is:

$$v_y = v \sin \theta - g\frac{2v \sin \theta}{g} = -v \sin \theta$$

Maximum height reached is obtained by observing that this height is reached when v_y instantaneously = 0

$$\text{i.e. } v_y = -v \sin \theta + gt = 0$$

Example: inclined plane (Figure 2.3)

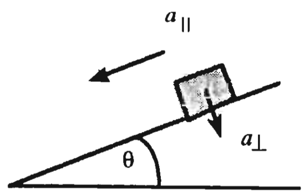


Figure 2.3

Motion perpendicular to the plane

None, that is: $v_{\perp} = 0$

$$a_{\perp} = 0$$

Motion parallel to the plane is the direction of the motion, that is:

$$v_{\parallel} = v_{0\parallel} + a_{\parallel}t$$

$$s_{\parallel} = v_{0\parallel}t + ½\, a_{\parallel}t^2$$

etc.

Example: circular motion (Figure 2.4)

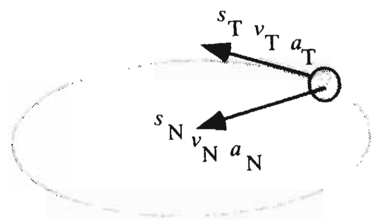


Figure 2.4

Tangential motion

substituting the relationships

$$s_T = r\,\theta\,,\ v_T = r\omega\,,\ a_T = r\alpha$$

in the equations

$$v_T = v_{0T} + a_T\,t \text{ etc.,}$$

obtain: $\omega = \omega_0 + \alpha\, t$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0{}^2 + 2\alpha\theta$$

for constant acceleration and in general.

$$\alpha = \frac{d\omega}{dt},\ \omega = \frac{d\theta}{dt}$$

Radial motion

$$s_N = R \text{ (constant)}$$

$$v_N = 0$$

$$a_N = \frac{v_T^2}{R}$$

2.2.3 Relative velocity (see 2.1.3)

When two bodies are each moving with different velocities it is sometimes important to calculate the relative motion of one body with regard to the other. In practice, when we refer to the velocity of body A, we are really referring to the velocity of body A relative to the velocity of the ground, which itself is in motion as the earth moves around its axis and around the sun etc. (see Figure 2.5). For example, using vectors:

$$v_{AB} = v_{AC} + v_{CB} = v_{AC} - v_{BC}$$

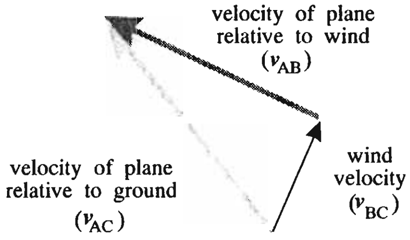


Figure 2.5

Calculations of the resulting motion may be made using either components, drawing the diagram closely to scale, or the 'cos rule' and the 'sine rule' 'triangle' relationships (see Introduction).

Worked examples

(a) A footballer kicks a football so that it leaves the ground at an angle of 30° with a velocity of 30 m s^{-1}. The ball is aimed directly at an opposing player 90 m away who, the moment the ball is kicked, starts to run forward to catch the ball (see Figure 2.6). At what speed must he run to just catch the ball?

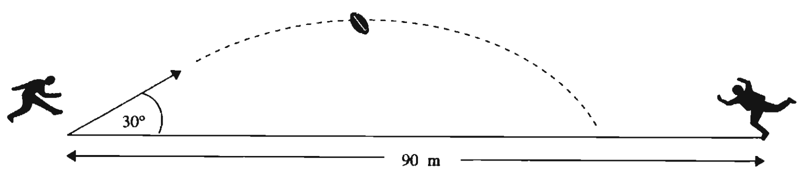


Figure 2.6

Solution

Data given:

$v = 30$ m s^{-1}
$\theta = 30^\circ$
a (vertical) $= g = 9.8$ m s^{-2}

Initial distance between kicker and receiver = 90 m

Calculations:

Vertically, if y is measured upwards:

$$y = v \sin\theta\, t - \frac{1}{2} g t^2$$

and the final value of y is zero

$$\therefore\ 0 = v \sin\theta\, t - \frac{1}{2} g t^2$$

$$t = \frac{2v \sin\theta}{g}$$

Horizontally,

$$x = v \cos\theta\, t = \frac{v \cos\theta\, 2v \sin\theta}{g} = \frac{v^2 \sin 2\theta}{g}$$

$$= \frac{30^2 \times 0.866}{9.8} = 79.5 \text{ m}$$

This is the distance before the ball lands.

The time of flight is:

$$t = \frac{x}{v \cos\theta} = \frac{79.5}{30 \times 0.866} = 3.06 \text{ s}$$

Thus the other player must run (90 – 79.5) = 10.5 m in 3.06 s in order to catch the ball.

His speed must be $^{10.5}/_{3.06} = 3.4$ m s^{-1}, i.e. 12.35 km h^{-1}

(b) Two boats A and B leave a marker buoy at the same time; boat A travels at a speed of 20 km h^{-1} in a direction 20° East of North, the other boat, B, heads due West at a speed of 15 km h^{-1}. At what speed does boat A seem to be travelling to an observer on boat B?

Solution With respect to the earth:

$$v_{AB} = v_{AE} - v_{BE}$$

The relative velocity vector diagram is shown in Figure 2.7.

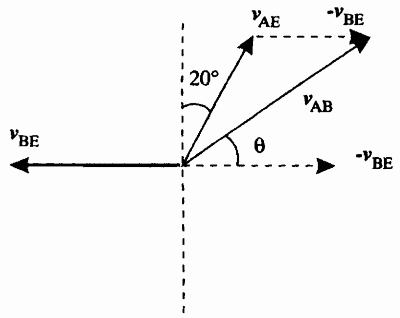


Figure 2.7

Using the method of components:

$R_x = 20 \sin 20 - (-15) = 21.84$

$R_y = 20 \cos 20 - 0 = 18.79$

$$\therefore \quad R = 28.8 \text{ km h}^{-1}$$

$$\theta = \tan^{-1}\frac{18.79}{21.84} = 40.7^{\circ}$$

Problem An object is thrown upwards at an angle of 37° to the horizontal with a speed of 10 m s^{-1} from the top of a building 20 m high.

(a) Find the time for the body to reach the ground.

(b) Find the horizontal distance that the body travels before reaching the ground.

(c) Find the velocity of the body when it reaches the ground.

(Ans: (a) 2.73 s; (b) 21.8 m; (c) 22.2 m s^{-1} directed 68.9° below the horizontal.)

Keywords

Projectile, projectile motion, range of projectile, normal component of acceleration, tangential component of acceleration, angular velocity, angular acceleration, relative velocity.

Important equations

$$\boxed{a_N = \frac{v_T^2}{R}}$$

$$\boxed{\alpha = \frac{d\omega}{dt}, \; \omega = \frac{d\theta}{dt}}$$

$$\boxed{\begin{aligned} s_T &= R\,\theta \\ v_T &= R\frac{d\theta}{dt} = R\,\omega \\ a_T &= R\frac{d\omega}{dt} = R\,\alpha \end{aligned}}$$

$$\boxed{v_{AB} = v_{AC} - v_{BC}}$$

2.3 Newton's laws of motion

First law

Every body remains at rest or continues to move in a straight line with a uniform speed if the resultant force acting on it is zero, that is,

$$\text{if} \sum F = 0 \text{ then } |v| = 0 \text{ or constant}$$

Second law

The rate change of momentum ($p = mv$) of a body is proportional to the resultant force acting on it and is in the direction of this force:

$$\boldsymbol{F} = \frac{d\,(mv)}{dt} = m\boldsymbol{a} \text{ if } m \text{ constant}$$

Third law

For every pair of bodies, the forces of action and reaction:

- are equal and opposite;
- are collinear;
- have the same point of action;
- act on different bodies.

Unit

The newton is the unbalanced force that will produce an acceleration of 1 m s^{-2} on a mass of 1 kg.

2.3.1 Solving problems involving forces: free-body method

A systematic approach to solving problems involving forces is important.

(1) Select a body to which Newton's laws are to be applied.

(2) Draw a diagram including all forces acting on a body. Do not include forces exerted by the body on some other body.

(3) Draw a set of coordinate axes.

(4) Resolve forces along each set of axes and apply Newton's Second Law in order to relate each component of the resultant force to each component of the acceleration.

Worked example A 5.0 kg block is placed on a smooth horizontal surface. A horizontal cord attached to the block passes over a light frictionless pulley and is attached to a 4.0 kg body. Find the acceleration and the tension in the cord when the system is released.

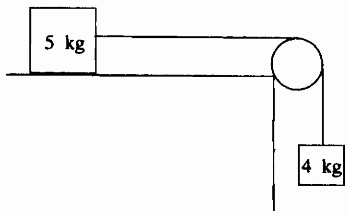


Figure 2.8

Considering the 5.0 kg mass and the 4.0 kg mass separately, the forces acting are as shown in Figure 2.9.

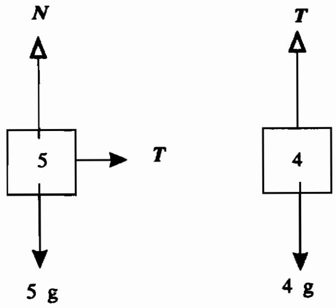


Figure 2.9

For the 5.0 kg block, Newton's Second Law gives: $T = 5.0a$

(The vertical forces on the 5.0 kg block are perpendicular to the direction of motion, are equal and in opposite directions, and do not cause any change in its motion.)

For the 4.0 kg body $4.0g - T = 4.0a$

Solving these two equations one obtains:

$$9.0a = 4.0g$$

$$\therefore\ a = 4.4\ \mathrm{m\ s^{-2}}$$

$$\text{and } T = 5.0a = 22\ \mathrm{N}$$

2.3.2 Friction

The frictional force F_f is proportional to the magnitude of the normal component of the contact force, N, and acts in a direction opposing the direction of motion or the direction of tendency to move.

If the body is stationary but tending to move, it experiences a force of static friction that can be larger than the force of kinetic friction that opposes its motion when it is moving. In each case, the size of the frictional force is expressed in terms of a coefficient of friction, μ, such that:

$$F_{\text{friction}} = \mu\, N$$

$$F_{\text{static}} = F_{\parallel\text{applied}}$$

$$(F_{\text{static}})_{\max} = \mu_s N$$

$$F_{\text{kinetic friction}} = \mu_k N$$

The values of both 'μ's depend on the nature of the two surfaces. In general μ_k has some dependence on the relative velocity of the objects, though at low speeds it may be regarded as a constant for two particular surfaces.

In rolling motion the frictional force is static, because there is no relative sliding movement between the material of the surface and the material of the rolling object.

Example: sliding motion on inclined plane

$$\sum F_{\perp} \, [= mg \cos \theta - N\,] = 0$$

$$\sum F_{\parallel} \, [= mg \sin \theta - \mu N\,] = ma_{\parallel}$$

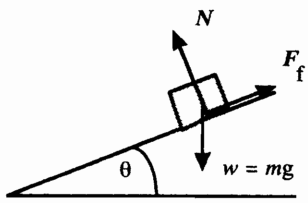


Figure 2.10

In the example illustrated in Figure 2.10, $\mu = \mu_s$ if the block is just on the point of moving, $\mu = \mu_k$ if the block is sliding down the plane. (Note: F_f would be acting down the plane if the block were being pushed up.)

2.3.3 'Weight' in accelerating frame of reference

Weight experienced by a body is the contact force w that the body exerts on whatever is supporting it.

According to Newton's Law of Gravitation:

$$F_{\text{gravitational}} = \mathrm{G}\frac{m_1 m_2}{r^2}$$

A freely falling body, by application of Newton's Second Law, experiences a force w equal to the gravitational force due to the earth:

$$w = mg$$

$$\text{i.e. } g = \mathrm{G}\frac{M_{\text{earth}}}{(r_{\text{earth}})^2}$$

In a frame of reference that is accelerating with acceleration **a**, such as an accelerating lift or an orbiting spacecraft, the weight experienced is given by:

$$\mathbf{w}_{\text{experienced}} = m\,(\mathbf{g} - \boldsymbol{a})$$

where $\boldsymbol{a}$ is positive if in the same direction as $\mathbf{g}$. The gravitational force on the object is still equal to mg.

2.3.4 Effect of the earth's rotation on g

A body at the North or South pole experiences the true 'g' as determined by Newton's Law of Gravitation. A body anywhere else on the earth experiences a 'g' value that is modified by the centripetal acceleration of the earth towards its axis. This has the greatest effect at the equator, where g is reduced by the term $R_E\,\omega^2$.

2.3.5 Force in circular motion

Since a body that is travelling in circular motion is constantly accelerating there must be one or more forces acting on it. The net force acting on the body towards the centre of the curved motion must (by Newton's Second Law) produce the centripetal acceleration:

$$\sum F_{\text{towards centre}} = ma_{\text{centripetal}} = m\frac{v_T^2}{R}$$

Example: horizontal circular motion A body travelling in a horizontal circle, for example a car on a flat surface, must have some force acting towards the centre of the circle. The car may have a rope attached to it as in Figure 2.11, or the friction in its tyres may be sufficient to provide this force as the frictional force will oppose the tendency of the car to move in a straight line (i.e. effectively outwards) as it would on a smooth surface.

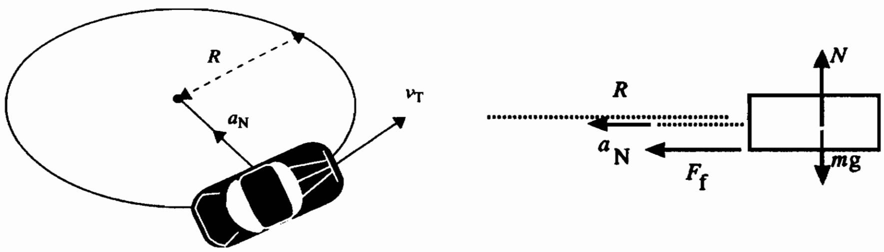


Figure 2.11

Vertical motion:

$$N - mg = 0$$

$$\therefore\ N = mg$$

Radial (horizontal) motion:

$$F_f = (\mu_s N = \mu_s mg) = ma_N\ (= m\frac{v^2}{R})$$

$$\therefore\ \mu = \frac{v^2}{Rg}$$

Example: critical banking for circular motion on a banked track If the banked track is smooth, no frictional force acts either up or down the plane. There is a critical speed at which the body may travel, maintaining circular motion as a result of the component $N \sin \theta$ of the track-supporting force acting towards the centre of the circle. If the track is rough and the body travels at the critical speed it will experience no tendency to slide up or down the track so that the same free-body diagram applies. If it goes faster than this critical speed there will tend to be a frictional force down the plane, which opposes the tendency to slide up the plane and at the same time adds to the force providing the centripetal acceleration. If it goes more slowly than this critical speed there will tend to be a frictional force acting up the plane, which opposes the tendency to slide down the plane and at the same time reduces the force providing the centripetal acceleration. The situation where the vehicle is moving at less than the critical speed is illustrated in Figure 2.12.

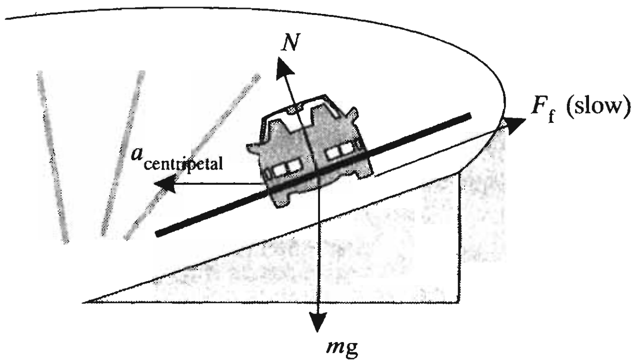


Figure 2.12

Critical case ($F_f = 0$)

Vertical (no motion, nor tendency to move):

$$N \cos \theta - mg = 0$$

$$\therefore\ N = \frac{mg}{\cos \theta}$$

Horizontal (centripetal motion):

$$F_N = N \sin \theta = ma_N = m\frac{v^2}{R}$$

i.e. $$\frac{mg}{\cos \theta} \sin \theta = m\frac{v^2}{R}$$

i.e. $$\tan \theta = \frac{v^2}{Rg}$$

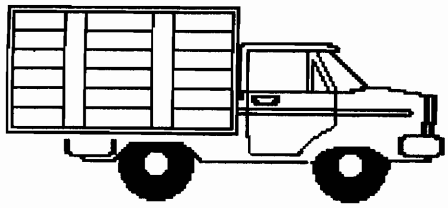


Figure 2.13

Worked example

A truck is loaded with a crate (see Figure 2.13). The coefficient of static friction between the crate and the floor of the truck is 0.25. If the truck is moving at 20 m s^{-1}, in how short a distance can the truck be stopped without the crate sliding?

Solution Since the crate is moving with the truck it has the same motion and the same opposing acceleration as the truck.

The only force acting on the crate in the direction of the acceleration is the force of static friction acting on the crate. The free-body diagram of the crate is given in Figure 2.14. Newton's Second Law applied to the crate:

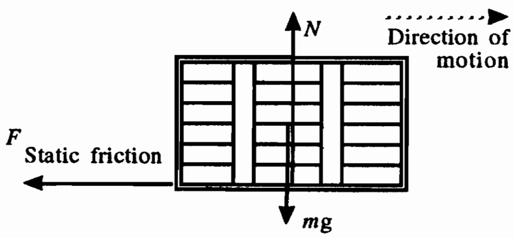


Figure 2.14

Vertically: $N - mg = 0$

Horizontally: $F_s = \mu_s N = ma$

Therefore:

$$a = \mu_s g = 0.25 \times 9.8 = 2.4 \text{ m s}^{-2}$$

Equation of motion for truck and crate:

$$v^2 = v_0^2 + 2as = 0$$

$$\therefore \quad s = \frac{20^2}{2 \times 2.4} = 83 \text{ m}$$

Therefore the shortest stopping distance without the crate sliding is 83 m.

Problem A 30 kg block is pushed from rest up a plane, which is inclined at 30° to the horizontal, by a force of 500 N parallel to the plane. The coefficient of sliding friction is 0.25.

Find

(a) the acceleration;

(b) the speed of the block after it has moved 10 m along the plane.

(Ans: (a) 9.64 m s^{-2}; (b) 13.9 m s^{-1})

Keywords

Force, resultant, action and reaction, mass, weight, friction, friction coefficients, statics of particle, dynamics problems, centripetal acceleration, banking of curves, gravitational field, gravitational mass, gravitational constant, gravity – effect of earth's rotation.

Important equations

$$\boxed{F = \frac{d(mv)}{dt} = ma \text{ if } m \text{ constant}}$$

$$\boxed{\begin{gathered} F_{\text{friction}} = \mu N \\ F_{\text{static}} = F_{\| \text{ applied}} \\ (F_{\text{static}})_{\max} = \mu_s N \\ F_{\text{kinetic friction}} = \mu_k N \end{gathered}}$$

$$\boxed{F_{\text{gravitational}} = G\frac{m_1 m_2}{r^2}}$$

$$\boxed{w_{\text{experienced}} = m(\mathbf{g} - \boldsymbol{a})}$$

$$\boxed{\sum F_{\text{towards centre}} = ma_{\text{centripetal}} = m\frac{v_T^2}{R}}$$

2.4 Work, energy and power

2.4.1 Work

The amount of work, W, done by a constant force, F, when the point of application of the force is displaced by s is defined by:

$$W = \boldsymbol{F} \cdot \boldsymbol{s} = F\, s \cos \theta$$

where θ is the angle between $\boldsymbol{F}$ and $\boldsymbol{s}$.

If the force is not constant but varies with displacement an integral expression is necessary:

$$W = \int_{s_1}^{s_2} \boldsymbol{F} \cdot d\boldsymbol{s}$$

The unit of work is the joule (N m).

2.4.2 Power

Power is the time rate of doing work. If W is the work done by a force on a body in a time 't' then

- the *average power P* delivered by that force is given by:

$$P = \frac{W}{t} = \frac{\boldsymbol{F} \cdot \boldsymbol{s}}{t} = \boldsymbol{F} \cdot \boldsymbol{v}$$

- the *instantaneous power P* delivered by that force is given by:

$$P = \frac{dW}{dt} = \frac{d\boldsymbol{F} \cdot \boldsymbol{s}}{dt} = \int \boldsymbol{F} \cdot d\boldsymbol{v}$$

The unit of power is the watt (J s^{-1}). A special power unit is the horsepower = 746 W.

2.4.3 Kinetic energy

The kinetic energy (KE or K) of a body is energy associated with its motion:

$$K = \frac{1}{2} mv^2$$

2.4.4 Work–energy principle

The work done by the resultant force on a body equals the change in kinetic energy of the body:

$$W = \Delta K$$

2.4.5 Conservative and dissipative forces

A force is called conservative if the total work it does on moving a body around any closed path is zero. The work done by a conservative force is independent of the path taken. Gravity is a conservative force. Friction is a non-conservative force.

2.4.6 Potential energy

The potential energy (U) of a body is its ability to do work because of its position or its internal configuration. It is calculated by determining the amount of work done by conservative forces in moving the body from rest at one point (or in one configuration) to rest at a new position (or to a new configuration).

Gravitational potential energy is gained when the body is shifted from one position to a higher position (further away from the centre of the earth) by supplying a force equal to those forces opposing its motion, that is, if no non-conservative forces are acting: $F_{grav} = mg$

Elastic potential energy is gained when, for example, a spring is compressed or extended by supplying a force equal to the opposing forces, that is, if no non-conservative forces are acting, the elastic force $F_{elastic} = kx$.

$$U_{grav} = mgh$$

$$U_{elastic} = \frac{1}{2}kx^2$$

The unit of energy is the same as the unit of work, the joule.

2.4.7 Internal work and energy

With a system that cannot be regarded as a single point, that is, extended bodies, one part of the system may exert forces and do work on another part, thus resulting in changes in the total kinetic energy of the composite system even though no work is done by forces applied by bodies outside the system. Such a change in the system may appear as a change in the temperature of the system, as discussed in Internal energy of a thermodynamic system (Section 3.7).

2.4.8 Conservation of mechanical energy

If the only forces that act on a system are those that are included in the potential energy term then the mechanical energy of a system ($K + U$) is constant:

$$\sum K + U = \text{constant}$$

i.e. loss in KE = gain in PE

2.4.9 Conservation of energy

The total amount of energy in an isolated system remains constant, although transformations from one form of energy to another may occur within the system (e.g. from mechanical energy to heat energy, energy of electromagnetic radiation and vice versa).

Worked example A girl pulls a 4.5 kg sled 12 m along a horizontal surface at constant speed. What work does she do on the sled if the coefficient of kinetic friction is 0.20 and her pull makes an angle of 45° with the horizontal?

Solution (see Figure 2.15):

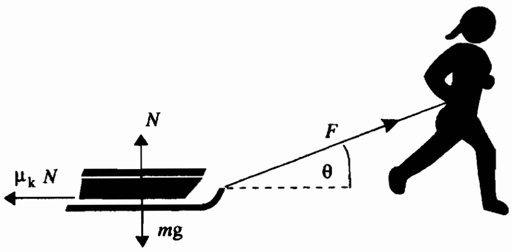


Figure 2.15

Data given:

v = constant

m = 4.5 kg

s = 12 m

μ_k = 0.20

θ = 45°

Vertical motion:

$$N + F \sin\theta - mg = 0$$

$$N = mg - F \cos\theta$$

Horizontal motion:

$$F \cos\theta - F_f = F\cos\theta - \mu_K N = 0$$

Combining these:

$$F \cos 45^\circ - 0.20 \times (mg - F \sin 45^\circ) = 0$$

$$\therefore \quad F = \frac{0.20 \times 4.5g}{\cos 45^\circ + 0.20 \sin 45^\circ} = \frac{8.82}{0.848} = 10 \text{ N}$$

Work done by F is:

$$F \times 12 \times \cos 45^\circ = 88 \text{ J}$$

Problem A pump is required to lift 1000 kg of water per minute from a well 8 m deep and eject it with a speed of 11 m s^{-1}.

(a) How much work is done per minute in lifting the water?

(b) What horsepower engine is needed to do this work?

(Ans: (a) 139 kJ; (b) 3.1 hp)

Keywords

Work, energy, kinetic energy, internal work, internal energy, conservation of mechanical energy, potential energy, gravitational potential energy, elastic potential energy, conservative force, dissipative force, power, joule, horsepower, conservation laws.

Important equations

$$\boxed{\begin{gathered} W = |F|\,|s| \cos\theta \\ W = \int_{path} F \cdot ds \end{gathered}}$$

$$\boxed{P = \frac{W}{t} = \frac{F \cdot s}{t} = F \cdot v}$$

$$\boxed{\begin{gathered} K = \frac{1}{2} mv^2 \\ W = \Delta K \end{gathered}}$$

$$\boxed{\begin{gathered} U_{\text{grav}} = mgh \\ U_{\text{elastic}} = \frac{1}{2} kx^2 \end{gathered}}$$

Conservation of mechanical energy:

$$\boxed{\begin{gathered} \sum K + U = \text{constant} \\ \text{i.e. loss in KE} = \text{gain in PE} \end{gathered}}$$

2.5 Impulse and momentum

2.5.1 Momentum

The linear momentum of a body, $p = mv$, was the term used in Newton's Second Law:

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt}$$

2.5.2 Impulse

Integrating Newton's Second Law results in an expression for the change in momentum resulting from an 'impulsive force', F, acting for a short time dt on a body:

$$\text{Impulse} \equiv J = F(t_2 - t_1) = m(v_2 - v_1)$$

$$\text{i.e. } J = \int_{t_1}^{t_2} F dt = \int_{v_1}^{v_2} m dv$$

2.5.3 Collisions

Collisions describe the interaction between two or more bodies.

2.5.4 Law of Conservation of Momentum

If no external force acts, momentum is conserved and the vector sum of the moments after impact is equal to that before impact.

If $\sum \boldsymbol{F}_{\text{ext}} = 0$ for a system of N particles

then $\sum_{i=1}^{N} m_i \boldsymbol{v}_i = \text{constant at any time}$

that is, $$\underbrace{\sum_{i=1}^{N} m_i \boldsymbol{v}_i}_{\textit{before}} = \underbrace{\sum_{i=1}^{N} m_i \boldsymbol{v}_i}_{\textit{after}}$$

In addition, mechanical energy may or may not be conserved:

- elastic collisions — momentum and energy are conserved.
- inelastic collisions — momentum is conserved, energy is not conserved.

Collision situations are solved by considering simultaneously:

- conservation of momentum in one, two, or three directions.
- energy conservation or known energy loss.

2.5.5 Centre of mass

When a body or a collection of particles is acted on by external forces, the centre of mass moves just as though all the mass was concentrated at that point and was acted on by a resultant force equal to the sum of the external forces on the system. The centre of mass $[\bar{x}, \bar{y}, \bar{z}]$ represents a weighted average position of the particles.

$$\bar{x} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i}, \quad \bar{y} = \frac{\sum_{i=1}^{N} m_i y_i}{\sum_{i=1}^{N} m_i}, \quad \bar{z} = \ldots \quad \text{for a system of } N \text{ particles}$$

$$\bar{x} = \frac{\int x dm}{M}, \bar{y} = \frac{\int y dm}{M}, \quad \bar{z} = \ldots \quad \text{for a continuous distribution}$$

2.5.6 Rocket propulsion

The thrust $m\frac{dv}{dt}$ is obtained by considering the conservation of momentum where both m and v are able to vary. For a rocket, the fuel burnt up and ejected from the rocket provides the necessary thrust to accelerate the rocket:

$$0 = \boldsymbol{F}_{\text{ext}} = \frac{dp}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

$$\therefore \quad m\frac{dv}{dt} \; (= ma\,) = v\,\frac{d(-m)}{dt}$$

This same equation applies to jet engines, which take in air (negative mass decrease) but then burn up fuel *and* the air that has been taken in, obtaining a thrust as they eject the burnt-up fuel and air.

2.5.7 Ballistic pendulum

The ballistic pendulum is a device for measuring the speed of a bullet. The bullet is allowed to make an inelastic collision (i.e. energy is not conserved in the collision) with a body of much greater mass. In many cases the bullet does not penetrate the target mass. The momentum of the system immediately after the collision equals the original momentum of the bullet, but since the velocity is very much smaller it can be determined much more easily, by the subsequent gain in potential energy by the pendulum.

Worked example A shell of mass 320 kg leaves the barrel of a gun of mass 20 000 kg with a horizontal velocity of 500 m s^{-1}.

(a) What impulse was given to the shell by the powder?

(b) With what velocity did the gun recoil?

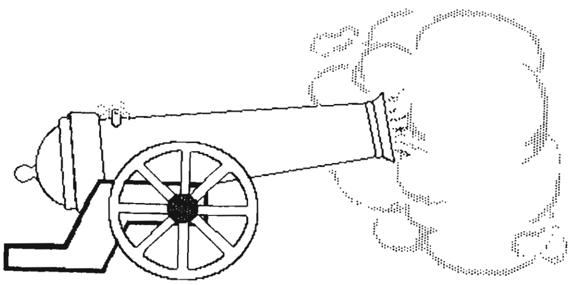


Figure 2.16

Solution Consider the motion in two parts — the first when the applied force from the gunpowder gives an impulse to the shell, and the second, following immediately, is that of the motion under no external force of the two-body system of shell and gun. Calculations are considered firstly for the second part of the motion, then for the first part:

Data given:

mass of shell = 320 kg

mass of gun = 20 000 kg

initial velocity (u) of shell = 0

final velocity (v) of shell = 500 m s^{-1}

initial velocity (u) of gun = 0

final velocity (v) of gun = ???

No external force, therefore momentum is conserved in a horizontal direction.

$$m_{\text{shell}}u_{\text{shell}} + m_{\text{gun}}u_{\text{gun}} = m_{\text{shell}}v_{\text{shell}} + m_{\text{gun}}v_{\text{gun}}$$

$$0 = 320 \times 500 + 2 \times 10^4\, v_{\text{gun}}$$

$$v_{\text{gun}} = \frac{-320 \times 500}{2 \times 10^4} = -8.0 \text{ m s}^{-1}$$

Impulse given to shell = change in momentum of shell

$= 320 \times 500 = 1.6 \times 10^5$ N s

Problem A bullet of mass 2.0 g travelling at 500 m s^{-1} is fired into a ballistic pendulum of mass 1.0 kg suspended from a cord 1.0 m long. The bullet penetrates the pendulum and emerges with a velocity of 100 m s^{-1}.

(a) What is the instantaneous velocity of the ballistic pendulum immediately after the collision?

(b) Through what vertical height will the pendulum then rise?

(Ans: (a) v = 0.8 m s^{-1}; (b) h = 33 m)

Keywords

Impulse, momentum, conservation of momentum, collision, elastic collision, inelastic collision, recoil, centre of mass, centre of gravity, rocket propulsion, ballistic pendulum.

Important equations

$$\boxed{p = mv}$$

$$\text{Impulse} \equiv J = F(t_2 - t_1) = m(v_2 - v_1)$$

$$\text{i.e. } J = \int_{t_1}^{t_2} F dt = \int_{v_1}^{v_2} m dv$$

$$\bar{x} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i}, \bar{y} = \frac{\sum_{i=1}^{N} m_i y_i}{\sum_{i=1}^{N} m_i}, \bar{z} = \ldots \text{ for a system of } N \text{ particles}$$

$$\bar{x} = \frac{\int x dm}{M}, \bar{y} = \frac{\int y dm}{M}, \bar{z} = \ldots \text{ for a continuous distribution}$$

2.6 Rotational motion

Rotational motion is not a new section of physics but a new application of the principles of linear motion. If we wish to examine the motions of a point going around in a circular path we can examine it as a rotation and consider its angular coordinates and, just as with linear motion, we can predict the position and velocity of the point if we know the acceleration.

As stated in Section 2.2, all the linear relationships in terms of s, v, a and t can be translated into equivalent rotational terms.

The normal relationships for the motion of such a point moving in a circular path are:

$$S_N = R \; ; v_N = 0 \; ; a_N = m\frac{v_T^2}{R} = mR\omega^2$$

Using the tangential relationships:

$$s_T = R\theta$$

$$v_T = R\frac{d\theta}{dt} = R\omega$$

$$a_T = R\frac{d\omega}{dt} = R\alpha$$

one can obtain a set of equivalent relationships in terms of angular displacement, θ, angular velocity, ω, angular acceleration, α, and time, t,

$$\alpha = \frac{d\omega}{dt}\ ;\ \omega = \frac{d\theta}{dt}$$

and, for the case of constant angular acceleration:

$$\omega = \omega_0 + \alpha t$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

2.6.1 Rotation of an extended, rigid body

In this case, all particles comprising the rigid, extended body are rotating with the same angular velocity and acceleration but the angular displacement varies depending on the location of the particles with respect to the axis of rotation. Consider the total rotational kinetic energy of such particles:

$$K_{\text{rotation}} = \sum_{i=1}^{N} m_i (v_{T_i})^2 = \left(\sum_{i=1}^{N} m_i R_i^2\right)\omega^2 \text{ since } \omega \text{ constant}$$

or, for a continuous distribution,

$$K_{\text{rotation}} = \int dm v_T^2 = \left(\int r^2 dm\right)\omega^2 \text{ since } \omega \text{ constant}$$

2.6.2 Moment of inertia

If we define the moment of inertia as the quantity in brackets in the expression given in Section 2.6.1, that is:

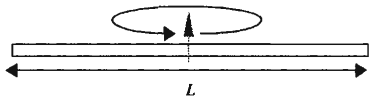
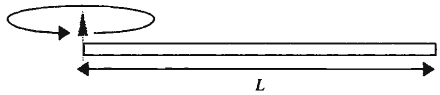
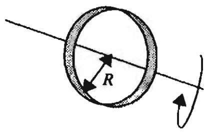
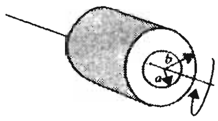
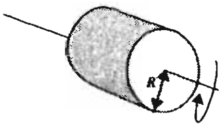
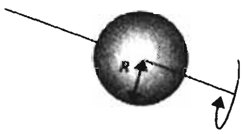
$$I = \sum_{i=1}^{N} m_i R_i^2 \text{ for a discrete set of } N \text{ particles}$$

or $$I = \int r^2 dm \text{ for a continuous distribution of mass}$$

then the moment of inertia about a rotational axis becomes the rotational 'mass-equivalent' of mass in linear motion. The kinetic energy of rotation now becomes:

$$K_{\text{rotation about axis}} = \frac{1}{2} I_{\text{axis}} \omega^2$$

Table 2.1

Object	Axis of rotation	Moment of inertia
Thin rod	about CM	$I = M\frac{L^2}{12}$; $k = \frac{L}{2\sqrt{3}}$
		
Thin rod	about end	$I = M\frac{L^2}{3}$; $k = \frac{L}{\sqrt{3}}$
		
Hoop	through centre	$I = MR^2$; $k = R$
		
Hollow cylinder	around principal axis	$I = M\frac{(a^2+b^2)}{2}$; $k = \sqrt{\frac{a^2+b^2}{2}}$
		
Solid cylinder	around principal axis	$I = M\frac{R^2}{2}$; $k = \frac{R}{\sqrt{2}}$
		
Solid sphere	around axis through centre	$I = M\frac{2R^2}{5}$; $k = \sqrt{\frac{2}{5}}R$
		

2.6.3 Radius of gyration

The radius of gyration, k, of a body is the distance from its axis of rotation to a point at which the total mass of the body might be concentrated without changing its moment of inertia about that axis:

$$I = Mk^2$$

2.6.4 Parallel axis theorem

The parallel axis theorem enables a calculation of the moment of inertia about one axis to be equated relative to the calculation about another parallel to that about which I is to be determined.

The minimum I is that about the centre of mass, and the parallel axis theorem is defined relative to the distance, d, of the particular axis to one parallel to it and acting through the centre of mass;

$$I_{x=d} = I_{CM} + Md^2$$

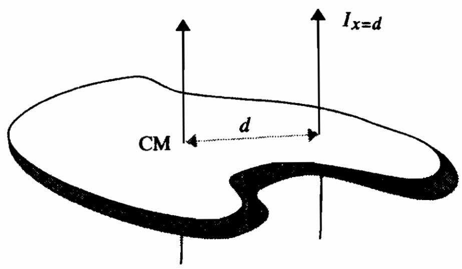


Figure 2.17

2.6.5 Torque (moment) of a force

A torque (or moment) of a force is the tendency of that force to cause a rotation of the body on which it acts. This turning tendency depends on the magnitude and direction of the force and on the location of the line of action of the force relative to the axis of rotation. A force that acts through the axis of rotation has no turning tendency about that axis.

The vector torque Γ of force $\boldsymbol{F}$ with respect to an axis is defined as:

$$\Gamma = \boldsymbol{r} \times \boldsymbol{F}$$

A useful working definition of torque or moment of a force gives the magnitude as:

$$\Gamma = F \times \text{perpendicular distance of line of action from the axis of rotation}$$

The direction of a torque will either be clockwise or counterclockwise about an axis.

2.6.6 Rotational dynamics about a fixed axis — Newton's Second Law equivalent

Analogous to the linear case, we may describe the overall rotational motion of the extended rigid body about a fixed axis by the rotational equivalent to Newton's Second Law:

$$\Gamma = I\,\alpha$$

with the opposing frictional torques needing to be accounted for in all dynamical situations, unless stated as negligible.

2.6.7 Rotational work and power

Work is done when a force torque causes a body to rotate through angle θ. The power is the rate at which that rotational work is done:

$$\text{Work}_{\theta_1 \to \theta_2} = \int_{\theta_1}^{\theta_2} \Gamma \cdot d\theta$$

$$\text{Power} = \frac{dW}{dt} = \int_{\omega_1}^{\omega_2} \Gamma \cdot d\omega$$

2.6.8 Angular momentum and angular impulse

The angular momentum of a body, L, is the vector quantity

$$L = I\omega$$

For a rotating point particle, the magnitude of L is:

$$L = I\omega = (mr^2)\frac{v}{r} = mvr$$

which is effectively the 'moment' of the linear momentum mv about the axis at distance r.

The angular impulse, J_θ, experienced by a body about an axis, is equal to the change of angular momentum of the body about the same axis:

$$J_\theta = I\omega_{\text{final}} - I\omega_{\text{initial}}$$

2.6.9 Conservation of angular momentum

This is used in rotational 'collisions' or interactions between two or more rotating bodies:

If $\sum \Gamma_{ext} = 0$ for a system of N particles then $\sum_{i=1}^{N} I_i\,\omega_i$ = constant at any time

i.e.
$$\underbrace{\sum_{i=1}^{N} I_i\,\omega_i}_{before} = \underbrace{\sum_{i=1}^{N} I_i\,\omega_i}_{after}$$

2.6.10 Rotational dynamics about a moving axis

If the axis about which a body is moving is also in motion (either linear or rotational) earlier techniques of relative velocity are used to regard the resulting motion as the combined motion of the centre of mass vectorially added to the rotation of the body around the centre of mass.

A point other than the centre of mass may be chosen for the relative motion. Thus the point of contact with the supporting surface for a rolling wheel is often chosen, as it is instantaneously at rest with respect to the surface, and the motion of all points on the rest of the wheel is one of pure rotation about the point of contact.

2.6.11 Rolling without slipping

Use a dynamical approach (using Newton's Second Law equivalents) *or* use an energy approach. In each case we have the choice of regarding the motion as:

- the centre of mass (CM) moving parallel to the direction of motion with linear velocity, $v = r\omega$, and the rest of the body rotating about the centre of mass with angular velocity ω;

or

- instantaneous pure rotation with angular velocity, ω, about the point of contact (P) between the rolling body and the surface.

Figure 2.18 is the free-body diagram for the above situation.

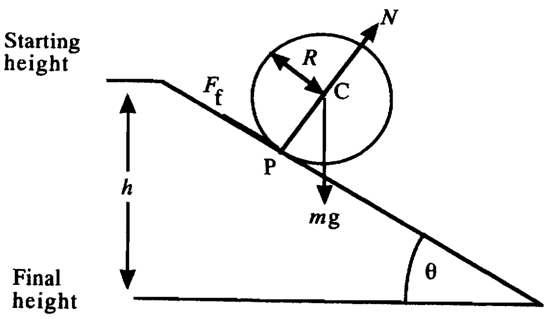


Figure 2.18

Dynamical approach

Considering the motion as a linear translation of the CM down the slope combined with rotation of the body around the CM, C.

(i) Translation perpendicular to plane

$N - mg\cos\theta = 0$ ...[1]

Translation parallel to plane

$mg\sin\theta - F_f = ma$ ...[2]

(ii) Rotation about C

Torque $= F_f R = I\alpha$ ...[3]

Using also the linear-angular relationship

$a = R\alpha$ ...[4]

and, by definition, $I = mk^2$ ...[5]

we eliminate (since we do not know its value unless the ball is just on the point of slipping) F_f from equations [2] and [3] (note that equation [1] is not needed in this case):

$$a = \frac{g\sin\theta}{(1 + k^2/R^2)}$$

Considering the motion as pure rotation about the point of contact P.

Torque $= mg\sin\theta\, R = I_P\alpha$ ...[1]

By the parallel-axis theorem

$I_P = I_C + mR^2 = m(k^2 + R^2)$ ...[2]

Using also the linear-angular relationship

$a = R\alpha$ ...[3]

obtaining from the three equations:

$$a = \frac{g\sin\theta}{(1 + k^2/R^2)}$$

Energy approach

Considering the motion as a linear translation of the CM down the slope, combined with rotation of the body around the CM, C.

$$I = mk^2 \ldots [1]$$

Conservation of mechanical energy:

Loss in PE = gain in KE

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

but $v = r\omega$

$$\therefore\ mgh = \frac{1}{2}mv^2 + \frac{1}{2}(mk^2)\frac{V^2}{R^2}$$

$$\text{i.e. } v = \sqrt{\frac{2gh}{(1 + k^2/R^2)}}$$

Considering the motion as pure rotation about the point of contact P.

By the parallel-axis theorem:

$$I_P = I_C + mR^2 = m(k^2 + R^2) \ldots [1]$$

Conservation of mechanical energy:

Loss in PE = gain in KE

$$mgh = \frac{1}{2}I_P\omega^2$$

but $v = r\omega$

$$\therefore\ mgh = \frac{1}{2}m\left(k^2 + R^2\right)\frac{v^2}{R^2}$$

$$\text{i.e. } v = \sqrt{\frac{2gh}{(1 + k^2/R^2)}}$$

Worked example A constant torque of 20 N m is exerted on a pivoted wheel for 10 s, during which time the angular velocity of the wheel increases from zero to 100 rpm. The external torque is then removed and the wheel is brought to rest by friction in its bearings, in 100 s.

Find

(a) the moment of inertia of the wheel;

(b) the friction torque;

(c) the total number of revolutions made by the wheel.

Solution (see Figure 2.19)

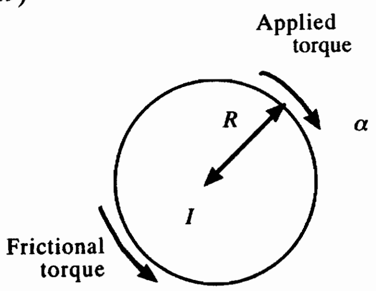


Figure 2.19

Data given:

Applied torque = 20 N m
Frictional torque = Γ_f

First part of motion $t = 10$ s

Starting angular velocity = 0 rpm

Final angular velocity = 100 rpm = $2\pi \times \dfrac{100}{60}$ rad s^{-1}

Second part of motion $t = 100$ s

Starting angular velocity = 100 rpm = $2\pi \times \dfrac{100}{60}$ rad s^{-1}

Final angular velocity = 0 rpm

Calculation:

First part of motion:

$$\omega = \omega_0 + \alpha t$$

$$\frac{2\pi \times 100}{60} = 0 + 10\alpha$$

$$\therefore \quad \alpha = \frac{\pi}{3} \text{ rad s}^{-2}$$

Newton's Second Law equivalent:

$$\Gamma - \Gamma_f = 20 - \Gamma_f = I\frac{\pi}{3} \ldots [1]$$

Second part of motion:

$$\omega = \omega_0 + \alpha t$$

$$0 = \frac{2\pi \times 100}{60} + 100\alpha$$

$$\therefore \quad \alpha = -\frac{\pi}{30}\ \text{rad s}^{-2}$$

$$\Gamma_f = I\frac{\pi}{30} \ldots [2]$$

Eliminating frictional torque from equations [1] and [2]:

$$20 = I\left(\frac{\pi}{3} - \frac{\pi}{30}\right) \therefore I = \frac{20}{(\pi/3 - \pi/30)} = 17\ \text{kg m}^2$$

Substituting for I in equation [2]; $\Gamma_f = 1.8$ N m

Angular displacement in first part of motion:

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}\frac{\pi}{3}10^2$$

$$\text{number of revolutions} = \frac{\theta}{2\pi} = \frac{50}{6}\ \text{rev}$$

Angular displacement in second part:

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{2\pi 100}{60}100 - \frac{1}{2}\frac{\pi}{30}100^2$$

$$\text{number of revolutions} = \frac{\theta}{2\pi} = \frac{500}{6}\ \text{rev}$$

$$\text{Total displacement} = 1/6\ (550) = 92\ \text{rev}$$

Problem A wheel of mass 3.0 kg and radius of gyration 150 mm is rotating on frictionless bearings at 1800 rpm. It is then coupled with another wheel of mass 21 kg and radius of gyration 300 mm, which was originally at rest. Calculate the final angular speed of the combination.
(Ans: 62 rpm)

Keywords

Rigid body, angular velocity, angular acceleration, kinetic energy in rotation, torque, moment of a force, moment of inertia, parallel-axis theorem, angular momentum, angular impulse, rolling friction, conservation of angular momentum.

Important equations

$$\alpha = \frac{d\omega}{dt};\ \omega = \frac{d\theta}{dt}$$

For constant acceleration:

$$\omega = \omega_0 + \alpha t$$
$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

$$I_{axis} = \int r^2 dm$$

Parallel Axis Theorem:

$$I_{\text{axis} \parallel \text{axis through CM}} = I_{\text{axis through CM}} + M\,(d_{\text{between axes}})^2$$

$$\text{Torque}_{\text{of force } F \text{ about axis}} \equiv \Gamma = \boldsymbol{r} \times \boldsymbol{F}$$

Newton's Second Law:

$$\Gamma = I\,\alpha$$

$$I = Mk^2$$

$$L = I\omega$$
$$J_\theta = \int_{t_1}^{t_2} \Gamma dt = \int_{\omega_1}^{\omega_2} I d\omega$$

$$K_{\text{rolling}} = \frac{1}{2}mv^2 + \frac{1}{2}I_{\text{CM}}\omega^2 = \frac{1}{2}I_{\text{point of contact}}\,\omega^2$$

2.7 Equilibrium of a rigid body

2.7.1 Dynamical equilibrium

For a rigid body to be in equilibrium under the net effect of a set of forces it must simultaneously be in translational equilibrium in any direction and in rotational equilibrium around any axis

$$\sum F_x = 0, \sum F_y = 0, \sum F_z = 0$$
$$\sum \Gamma_{\text{any axis}} = 0$$

Sufficient applications of these equations are made to obtain a solution. Generally, it will be necessary only to consider torques about one axis and this is chosen to minimise the number of torques acting.

Worked example Romeo places a uniform ladder 10.6 m long against a frictionless vertical wall with its lower end 6.4 m from the wall. The ladder has a mass of 36.3 kg. The coefficient of static friction between the foot of the ladder and the ground is 0.4. Romeo, of mass 68.1 kg, starts to climb up the ladder. How far up the ladder can he climb before the ladder starts to slip?

Solution (see Figure 2.20)

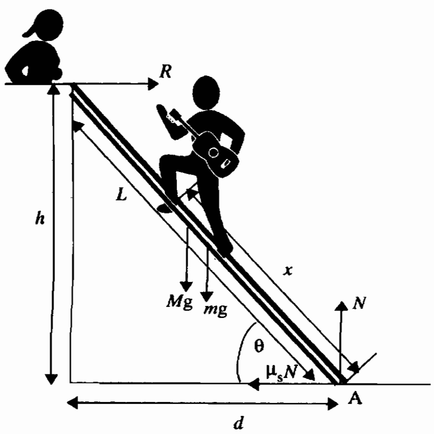


Figure 2.20

Data given:

$$M = 68.1 \text{ kg}$$
$$m = 36.3 \text{ kg}$$
$$\mu_s = 0.4$$
$$L_{\text{ladder}} = 10.6 \text{ m}$$
$$d = 6.4 \text{ m}$$

Let N and R be the normal components of the contact forces exerted by the ground and the wall, respectively, on the ladder.

Translational equilibrium:

$$\text{Vertically, } N - Mg - mg = 0 \ldots [1]$$
$$\text{Horizontally, } R - \mu_s N = 0 \ldots [2]$$

Rotational equilibrium about A:

$$\sum \text{clockwise moments} = \sum \text{anticlockwise moments}$$
$$R\,(10.6 \sin\theta) = mg\,(5.3 \cos\theta) + Mg\,x \cos\theta \ldots [3]$$

Geometrically:

$$\tan\theta = \frac{h}{L} \text{ and } h = \sqrt{L^2 - d^2}$$

Solving these equations gives:

$$x = 5.75 \text{ m}$$

That is, Romeo can climb the ladder 5.75 m before it starts to slip.

Problem A tall cylindrical block of mass 20 kg and radius 100 mm is on a horizontal surface so that its axis is vertical. The block will slide without overturning when a force ≥ 50 N is applied at a height of 200 mm above the base.

(a) Find the coefficient of static friction.

(b) If a force of 40 N is applied to the block it will not slide but will overturn, provided the force is applied above a certain point. Find the height of this point above the base.

(Ans: (a) 0.25; (b) 0.49 m)

Keywords

Equilibrium of rigid body, centre of mass, moment of a force.

Important equations

$$\boxed{\begin{gathered}\sum F_x = 0, \sum F_y = 0, \sum F_z = 0 \\ \sum \Gamma_{\text{any axis}} = 0\end{gathered}}$$

2.8 Fluid mechanics

2.8.1 Density

$$\text{Density } \rho = \frac{m}{V}$$

2.8.2 Fluid statics: hydrostatic pressure

The hydrostatic pressure at a point in a fluid is defined as the force per unit area on a small plane surface at that point. The direction of the force exerted by hydrostatic pressure is always normal to the plane, whatever its orientation. The pressure in an incompressible fluid at X, a depth h below the fluid surface, is equal to the pressure at the fluid surface, p_0, plus the pressure of the column of fluid above the point X:

$$\text{Pressure of fluid column: } \Delta p = \rho g h$$

$$\text{Pressure at a depth: } p = p_0 + \rho g h$$

$$\text{Absolute pressure (Pa)} = \frac{\text{pressure (mm Hg)} \times \rho\ \text{(Hg)} \times \text{g}}{1000}$$

$$\text{Gauge pressure} = \text{absolute pressure} - \text{atmospheric pressure}$$

$$1 \text{ atmosphere pressure} = 760 \text{ mm Hg} = 1000 \text{ mbars} \approx 10^5 \text{ Pa}$$

2.8.3 Units of pressure

Absolute unit of pressure is the pascal (Pa). However, because many pressure measurements involve the use of manometer readings of lengths of mercury, pressure measurements are often given in terms of 'mm Hg'.

2.8.4 Pascal's Law of Undiminished Pressures

Pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and the walls of the containing vessel.

2.8.5 Archimedes Principle and buoyancy

When a body is immersed in a fluid, the fluid exerts an upwards force on the body equal to the weight of the fluid displaced by the body. A body that is less dense than the fluid in which it is immersed will float, as it will experience an upthrust when submerged greater than its weight:

$$\text{Upthrust} = \rho\ \text{g}\ V$$

$$\text{where } V = \text{volume submerged}$$

$$\leq \text{volume of body}$$

$$\text{Flotation condition: weight} = \text{upthrust}$$

Worked example The left-hand side of an open-tube mercury manometer is connected to a gas tank. The mercury is 390 mm higher on the right-hand side than on the left when a barometer nearby reads 755 mm Hg. What is the pressure of the gas in Pa?

Solution (see Figure 2.21)

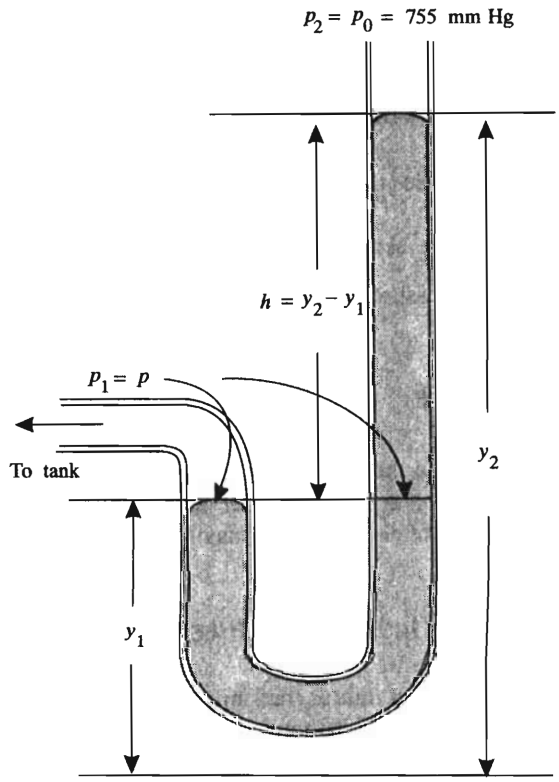


Figure 2.21

The gas pressure is the pressure at the top of the left-hand column of mercury. This is the same as the pressure at the same horizontal level in the right-hand column. The pressure at this level is the atmospheric pressure plus the pressure due to the column of fluid above this level. Therefore the absolute pressure of the gas is:

$$p = p_0 + \rho g h$$

$$p_0 = 1.36 \times 10^4 \times 9.8 \times 755 \times 10^{-3} = 1.01 \times 10^5 \text{ Pa}$$

$$p = 1.01 \times 10^5 + 1.36 \times 10^4 \times 9.8 \times 390 \times 10^{-3} = 1.53 \times 10^5 \text{ Pa}$$

Problem The escape hatch of a submarine is 0.40 m^2 in area and the submarine is submerged 30 m below the surface of the ocean. What force must be exerted on the hatch by a crew member wishing to escape from the submarine, in order to open the hatch?

Assume atmospheric pressure inside the submarine.

$\rho_{\text{sea water}} = 1030 \text{ kg m}^{-3}$

(Ans: $F = 1.2 \times 10^5$ N)

2.8.6 Surface tension

Any boundary surface of a liquid is under a state of tension due to the forces of attraction existing between molecules of the liquid (cohesive forces) and between molecules of the liquid and the walls of the container (adhesive forces).

The surface tension, γ, is measured by the force per unit length acting on a line in the surface (see Figure 2.22). The direction of the surface tension force is perpendicular to the line and in the plane of the surface.

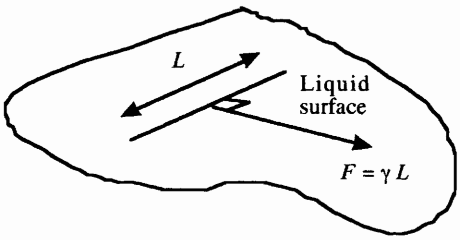


Figure 2.22

Surface tension can also be considered as the amount of energy required to increase the surface area of a material by unit area.

The curvature of a fluid surface can be related to the excess pressure across that surface and the surface tension of the liquid. In a capillary tube this excess pressure across the curved surface is balanced by the pressure of a column of liquid in the tube. This balance is also dependent on the nature of the liquid–solid 'contact angle' involved. Mercury on glass in air has a contact angle >90° therefore the mercury level is depressed in the capillary tube. Pure water–glass–air has a contact angle of 0°.

$$\text{Surface tension:} \quad \gamma = \frac{F}{\text{length}} \ \ or \ \ \gamma = \frac{\text{work}}{\Delta A}$$

$$\text{Excess pressure across spherical curved surface:} \quad \Delta p = \frac{2\gamma}{R}$$

$$\text{Excess pressure inside soap bubble (two surfaces):} \quad \Delta p = \frac{4\gamma}{R}$$

$$\text{Rise in capillary tube:} \quad \rho g h = \frac{2\gamma \cos \theta_{\text{contact}}}{R}$$

Worked example γ can be determined by dipping a microscope slide in liquid. When the bottom of the slide is at the same level as the surface of the liquid and in contact with the liquid, the force required to balance the slide is measured.

It is found that an additional mass of 6.0×10^{-5} kg is needed to balance a slide of length 8.0 cm and thickness 2 mm at this position. Determine the surface tension of the liquid.

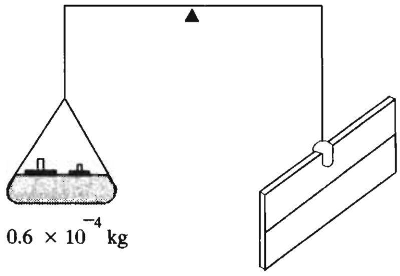


Figure 2.23

Solution Four surfaces of the slide are in contact with the liquid. Hence the total force on the slide due to the surface tension at the line of contact is (assuming contact angle is zero):

$$F = (2 \times 0.080 + 2 \times 0.002)\,\gamma$$

This force equals the weight of the additional balance mass so:

$$F = 6.0 \times 10^{-5} \times g$$

Therefore $\gamma = 3.6 \times 10^{-3}\ \mathrm{N\,m^{-1}}$

Problem A clean piece of glass tubing of internal radius 6.0 mm and external radius 7.0 mm is suspended from one arm of a beam balance so that its lower edge is just immersed in a beaker of water. What additional mass must be placed on the other side of the balance to compensate for the pull of surface tension on the tube? ($\gamma_{H_2O} = 7.2 \times 10^{-2}\ \mathrm{N\,m^{-1}}$)

(Ans: m = 600 mg)

2.8.7 Fluid dynamics: fluid flow

Fluid dynamics is concerned with the properties of a fluid in motion. An ideal fluid is a fluid that is incompressible and has no viscosity (see 2.8.8). If the ideal fluid is in a state of 'steady flow' so that every element passing through a given point follows the same line of flow as that of preceding elements, we have two important principles.

The **equation of continuity** must be observed since for incompressible flow the volume rate of flow ($Q = Av$) must be constant at all points along the flow:

$$A_1 v_1 = A_2 v_2$$

Bernoulli's equation is a consequence of the work–energy theorem applied to the fluid flow:

$$p_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 = p_2 + \rho g y_2 + \frac{1}{2}\rho v_2^2$$

Worked example Water is flowing through a pipe at a rate of 0.80 m^3 / min. The pipe narrows from an area of 1200 mm^2 at A to an area of 800 mm^2 at B, 2 m below A (Figure 2.24). If the pressure at A is 1.0×10^5 Pa, find

(a) the flow speed at A;

(b) the flow speed at B;

(c) the volume rate of flow at B;

(d) the pressure at B.

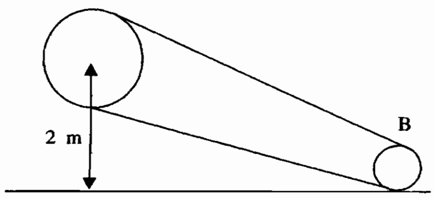


Figure 2.24

Solution

(a) At A:

$$Q = 0.80\ \text{m}^3\ \text{min}^{-1}$$
$$= 0.0133\ \text{m}^3\ \text{s}^{-1}$$
$$Q = A_1 V_1$$
$$0.0133 = \frac{1200}{10^6} \times v_1$$
$$v_1 = 11.1\ \text{m s}^{-1}$$

(b) At B:

$$A_1 V_1 = A_2 V_2$$
$$\frac{1200}{10^6} \times 11.1 = \frac{800}{10^6} \times v_2$$
$$v_2 = 16.7\ \text{m s}^{-1}$$

(c)

$$Q_A = Q_B = 0.013\ \text{m}^3\ \text{s}^{-1}$$

(d) $$p_A + \frac{1}{2}\rho v_A^2 + \rho g h_A = p_B + \frac{1}{2}\rho v_B^2 + \rho g h_B$$

$$10^5 + \frac{1}{2} \times 1000 \times (11.1)^2 + 1000 \times 9.8 \times 2 = p_B + \frac{1}{2} \times 1000 \times (16.7)^2 + 0$$

$$p_B = 4.2 \times 10^4 \text{ Pa}$$

Problem A Venturi meter when placed over the side of a boat shows a pressure difference of 0.10 m of water. What is the speed of the boat if the cross-sectional areas of the Venturi meter are in the ratio 3:1?

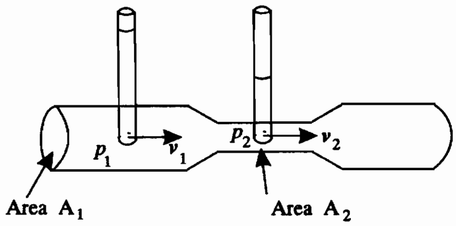


Figure 2.25

(Ans: 0.49 m s^{-1})

2.8.8 Viscosity

Viscosity is the internal friction of a fluid. It is described by a coefficient, η, being the ratio of shear stress to velocity gradient across the flowpath and increases rapidly as the temperature decreases for liquids, but increases with fall of temperature for gases.

$$\text{Viscosity } \eta = \frac{\text{shear stress}}{\text{rate of shear strain}} = \frac{\frac{F_{\parallel}}{A}}{\frac{dv}{dy}}$$

The viscosity causes an object moving through a viscous fluid to be acted on by a resistive force, which increases with velocity.

Stoke's Law: F_{drag} on spherical object $= 6\,\pi\,\eta\,r\,v$

Flow of a fluid through a tube is limited by its viscosity.

Poiseuille's Law: $$Q \equiv \frac{d(\text{volume})}{dt} = \frac{\pi}{8}\frac{R^4}{\eta}\frac{\Delta p}{L}$$

If the object is falling vertically the resistive (drag) force will add to the upthrust experienced by the object as a result of Archimedes' Principle. As a result, a 'terminal velocity' will be reached by the falling object.

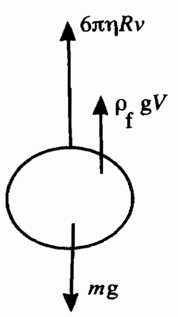
A newtonian fluid is one in which η does not change with changing shear stress. Reynold's number predicts the onset of turbulence for a non-ideal fluid.

$$\text{Reynold's number, } N_R = \frac{\rho v D}{\eta} \quad (< 2000 \text{ for steady streamline flow})$$

For values > 3000 the flow should be turbulent. Between 2000 and 3000 it may be either.

Worked example A sphere of radius 5.5 mm and density 2270 kg m^{-3} falls through a fluid of density 1000 kg m^{-3} and viscosity 0.10 Pa s. Find its terminal velocity.

Solution When the sphere has reached its terminal velocity there is zero resultant force acting on it. The forces to be considered are its weight, the force due to the liquid as determined by Archimedes' Principle and the resistive force as given by Stoke's Law (see Figure 2.26).



Data given:
- radius = 5.5 mm
- density of sphere $\rho_s = 2270$ kg m^{-3}
- density of fluid $\rho_f = 1000$ kg m^{-3}
- viscosity $\eta = 0.1$ Pa s

Figure 2.26

Taking downwards as positive:

$$mg - \rho_f g V - 6\,\pi\eta\, v_{\text{terminal}} R = 0$$

Since $\qquad V = \frac{4}{3}\pi R^3$ and $m = \rho V$

$$\rho_s g\left(\frac{4}{3}\pi R^3\right) - \rho_f g\left(\frac{4}{3}\pi R^3\right) - 6\pi\,\eta v_{\text{terminal}} R = 0$$

$$v_{\text{terminal}} = \left(\frac{2R^2 g}{9\eta}\right)(\rho_s - \rho_f)$$

Substituting the appropriate values gives:

$$v_t = \frac{2 \times (15.5 \times 10^{-3})^2 \times 9.8 \times (2270 - 1000)}{9 \times 0.1}$$

$$= 0.84 \text{ m s}^{-1}$$

Problem A hypodermic needle is used to inject 2 ml of liquid into a vein. If the needle is 20 mm long and has an internal diameter of 0.2 mm, calculate how long the injection will take if a force of 10 N is applied to the plunger, which has an area of cross-section of 100 mm^2. Assume streamline flow and take the coefficient of viscosity as 0.0005 Pa s.
(Ans: 5.1 s)

Keywords

Density, pressure, gauge pressure, atmospheric pressure, absolute pressure, barometer, surface tension, contact angle, ideal fluid, streamline, turbulence, Reynold's number, continuity equation, Bernoulli's equation, viscosity, Stoke's Law, newtonian fluid.

Important equations

$$\boxed{p = p_0 + \rho g h}$$

$$\boxed{\text{Surface tension: } \gamma = \frac{F}{\text{length}} \quad \textit{or} \quad \gamma = \frac{\text{Work}}{\Delta A}}$$

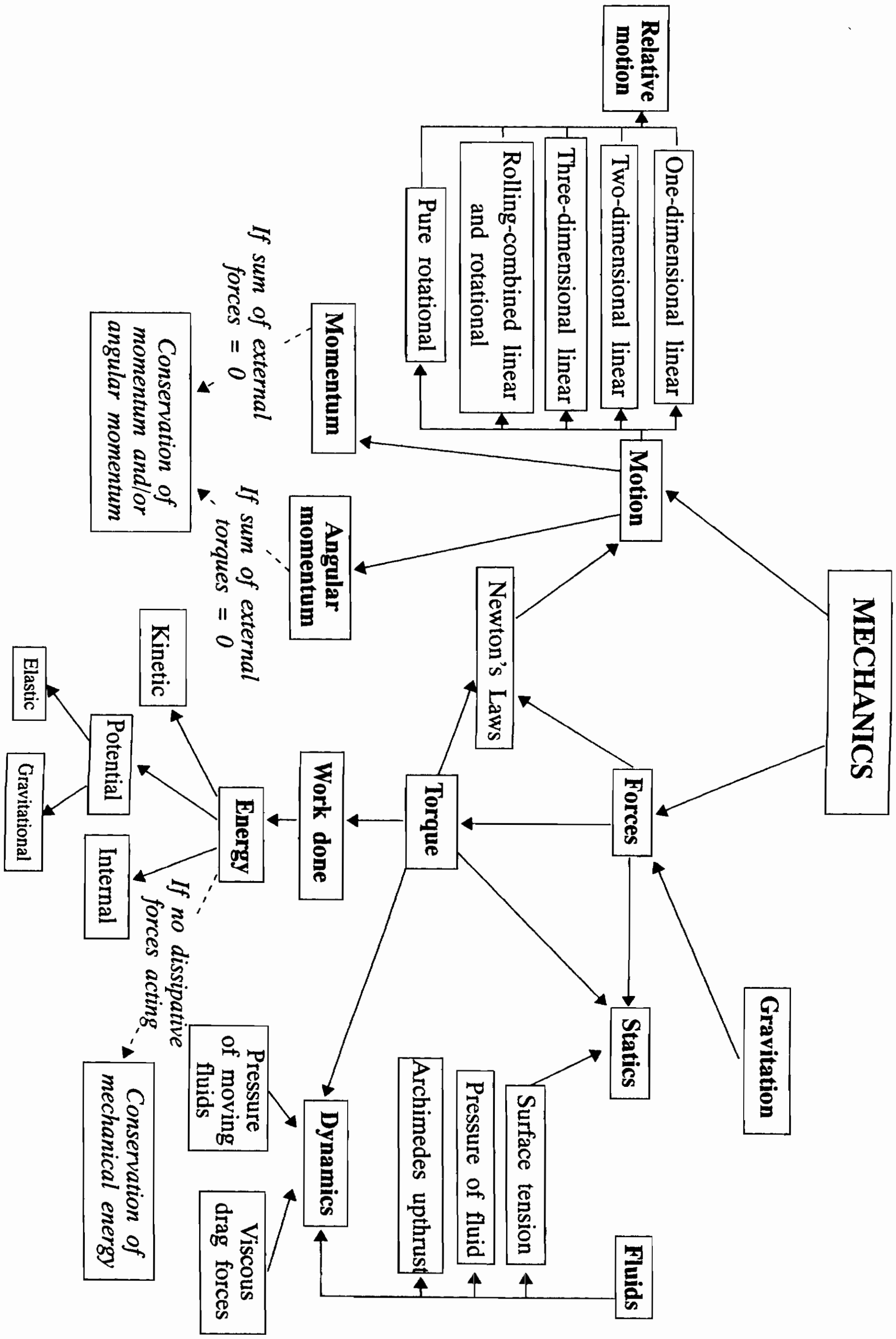
$$\boxed{A_1 v_1 = A_2 v_2}$$

$$\boxed{p_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 = p_2 + \rho g y_2 + \frac{1}{2}\rho v_2^2}$$

$$\text{Viscosity: } \eta = \frac{\text{shear stress}}{\text{rate of shear strain}} = \frac{\frac{F_{\parallel}}{A}}{\frac{dv}{dy}}$$

$$\text{Stoke's Law: } F_{\text{drag}} \text{ on spherical object} = 6\pi \eta r v$$

$$\text{Poiseuille's Law: } Q \equiv \frac{d\,(\text{volume})}{dt} = \frac{\pi}{8}\frac{R^4}{\eta}\frac{\Delta p}{L}$$


MECHANICS
Motion
Forces
Gravitation
Relative motion
One-dimensional linear
Two-dimensional linear
Three-dimensional linear
Rolling-combined linear and rotational
Pure rotational
Momentum
Angular momentum
If sum of external forces = 0
If sum of external torques = 0
Conservation of momentum and/or angular momentum
Newton's Laws
Torque
Work done
Energy
Kinetic
Potential
Elastic
Gravitational
Internal
If no dissipative forces acting
Conservation of mechanical energy
Statics
Dynamics
Surface tension
Pressure of fluid
Archimedes upthrust
Fluids
Pressure of moving fluids
Viscous drag forces

CHAPTER 3

Thermal physics

3.1 Temperature and thermometry

3.1.1 Thermal equilibrium

When the macroscopic properties of an isolated system become constant in time, the system is in thermal equilibrium.

The Zeroth Law of Thermodynamics

Two systems in thermal equilibrium with a third system are in thermal equilibrium with each other.

3.1.2 Temperature

Temperature is a measure of the 'hotness' or 'coldness' of a body. It is measured by a thermometer.

Thermometer

A thermometer is a device that can measure temperature by means of a change of a property such as length, volume or resistance.

Temperature scales

A temperature scale can be constructed as follows:

(1) Choose two fixed points, for example melting ice, boiling water.

(2) Divide the standard interval into sub-intervals called degrees.

(3) Assign a number to each degree.

This method was used before 1954. An example of such a scale is the Celsius scale. Since 1954 a single fixed point (the triple point of water) has been used in conjunction with absolute zero in the Kelvin scale.

The Celsius scale

The fixed points are melting ice, 0°C, and boiling water 100°C.

Absolute zero

The lowest possible temperature is –273.15°C, which is called absolute zero.

The Kelvin scale

A temperature scale can be started from absolute zero. The Kelvin scale is defined with reference to the Celsius scale:

$$T_{\text{kelvin}} = T_{\text{celsius}} + 273.15$$

A temperature *change* of one degree Celsius is identical to a change of one kelvin. The unit for a *change* in temperature is often written as C°.

Worked example What is 24.30°C expressed in kelvin?

Solution

$$\begin{aligned} T_K &= 273.15 + 24.30 \\ &= 297.45 \text{ K} \end{aligned}$$

Problem An object is heated from 20°C to 30°C. Determine the temperature difference in:

(a) °C

(b) K

(As the temperatures are only given to 2 significant figures, use –273°C for absolute zero.)

(Ans: (a) 10°C; (b) 10 K)

Keywords

Temperature, thermometer, absolute temperature.

Important equation

$$\boxed{T_{\text{kelvin}} = T_{\text{celsius}} + 273.15}$$

3.2 Thermal expansion

3.2.1 Change in length

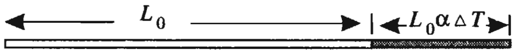


Figure 3.1

If a body of length L_0, made of material with coefficient of linear expansion α, is heated through a temperature range of ΔT (see Figure 3.1) the new length is:

$$L = L_0 (1 + \alpha \Delta T)$$

The change in length is:

$$\Delta L = L_0 \alpha \Delta T$$

3.2.2 Change in area

For an isotropic solid (one that expands equally in all directions), the coefficient of area expansion is equal to 2α (see Figure 3.2) for all practical purposes:

$$A = A_0 (1 + 2 \alpha \Delta T)$$

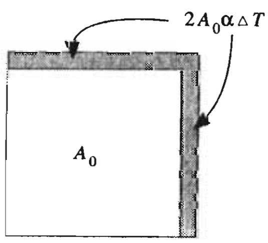


Figure 3.2

3.2.3 Change in volume

For an isotropic solid, the coefficient of cubic (or volume) expansion, β, is equal to 3α:

$$V = V_0 (1 + \beta \Delta T)$$
$$= V_0 (1 + 3\alpha \Delta T)$$
$$or\ \Delta V = V_0 \beta \Delta T$$

Expansion of a hole

A hole will expand when heated as if it were made of the material surrounding the hole.

Expansion of a liquid

For a liquid in a container the measured expansion is less than the true expansion because the container also expands.

Worked example A steel bridge is 70 m long at 0°C. The temperature has an annual variation from −25°C to 45°C. What is the difference in length of the bridge at these two temperature extremes?

Solution

Data given:

$$L_0 = 70 \text{ m}$$
$$\alpha_{\text{steel}} = 11 \times 10^{-6}\ (\text{C}^\circ)^{-1}$$
$$\Delta T = 70\ \text{C}^\circ$$

Asked for:

$$\Delta L = ???$$

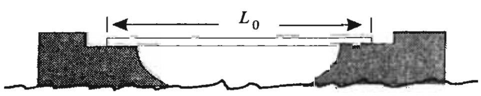


Figure 3.3

We assume that the value of α given for steel is the average value over this temperature range.

$$\therefore \text{ Change in length, } \Delta L = L_0\, \alpha\, \Delta T$$
$$= 70 \times 11 \times 10^{-6} \times 70$$
$$= 0.0539 \text{ m}$$
$$\Delta L = 54 \text{ mm}$$

Problem At 10°C a glass vessel can hold a volume of liquid equal to 200.00 ml.

(a) What volume of liquid can the vessel hold at 70°C, given that the coefficient of linear expansion of the glass is $0.8 \times 10^{-5}\ (\text{K})^{-1}$?

(b) What is the increase in volume of 200 ml of a liquid when its temperature is raised from 10°C to 70°C if the coefficient of volume expansion of the liquid is $7.0 \times 10^{-5}\ (\text{K})^{-1}$?

(c) If the liquid had been in the vessel at 10°C, what volume of liquid would spill out if the temperature was raised to 70°C?

(Ans: (a) 200.29 ml; (b) 0.84 ml; (c) 0.55 ml)

Keyword

Thermal expansion

Important equations

$$\boxed{\begin{aligned}\Delta L &= L_0\, \alpha\, \Delta T \\ \Delta V &= V_0\, \beta\, \Delta T\end{aligned}}$$

3.3 Heat capacity

3.3.1 Heat

Heat is a transfer of **energy**, which can flow from one body to another if a temperature difference exists between them.

Heat capacity

If a system receives an amount of energy ΔQ as heat flow, so that its temperature rises by an amount ΔT, then the heat capacity of the system is:

$$\frac{\Delta Q}{\Delta T}$$

Specific heat capacity

The specific heat c of a substance is its heat capacity per unit mass.

$$c = \frac{\Delta Q}{m\, \Delta T}$$

Thus, when there is a change in the temperature of a body, the heat flow to the body is:

$$\Delta Q = mc\, \Delta T$$

The specific heat of water = 4186 J kg^{-1} K^{-1}.

Molar heat capacity

The molar heat capacity C of a substance is the heat capacity of one mole (or one gram molecular weight) of the substance.

$$C = \frac{\Delta Q}{n\, \Delta T}$$

where n is the number of moles present,

and

$$n = \frac{m}{M}$$

where m is the mass in grams, and M is the molecular weight in grams, so C is in J mol^{-1} K^{-1}.

Method of mixtures

Heat transferred from a hot system = heat transferred to a cold system (see Figure 3.4).

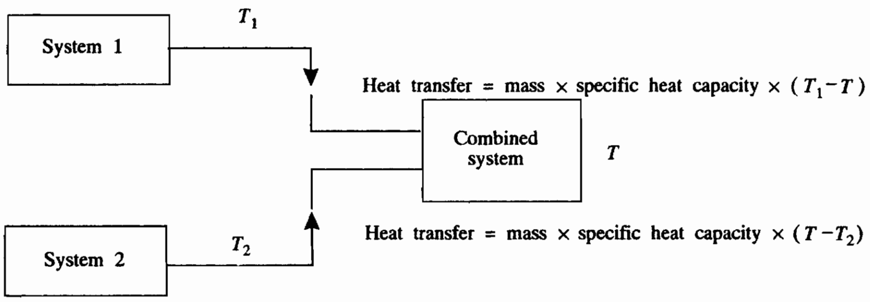


Figure 3.4

Worked example A glass beaker of mass 0.120 kg contains 0.160 kg of water at 27.0°C. Into it are dropped 0.090 kg of glass beads at 100°C. Calculate the final temperature given that:

$$c_{\text{water}} = 4186 \text{ J kg}^{-1} \text{ K}^{-1}$$
$$c_{\text{glass}} = 670 \text{ J kg}^{-1} \text{ K}^{-1}$$

Solution (to 3 significant figures)

Heat transfer from glass beads = heat transfer to water + heat transfer to beaker

$$(mc\,\Delta T)_g = (mc\,\Delta T)_w + (mc\,\Delta T)_b$$
$$0.090 \times 670 \times (100 - T) = 0.160 \times 4186 \times (T - 27) + 0.12 \times 670 \times (T - 27)$$
$$60.3 \times (100 - T) = 670 \times (T - 27) + 80.4 \times (T - 27)$$
$$6030 - 60.3T = 670T - 18100 + 80.4T - 2170$$
$$26300 = 810T$$
$$T = 32.5°C$$

Problem A 0.60 kg copper container ($c = 386.0$ J kg^{-1} K^{-1}) holds 1.50 kg of water at 20.0°C. A 1.00 kg iron ball ($c = 460.5$ J kg^{-1} K^{-1}) at 120°C is dropped into the water. What is the final temperature of the system?

($c_{\text{water}} = 4186$ J kg^{-1} k^{-1})

(Ans: 26.6°C)

Keywords
Specific heat capacity, molar heat capacity.

Important equations

$$\Delta Q = mc\,\Delta T$$
$$\Delta Q = nC\,\Delta T$$
Heat transferred from hot system = heat transferred to cold system

3.4 Change of state

3.4.1 Experiment

Consider an experiment where 0.5 kg of ice is heated from −20°C with a constant supply of heat energy. The temperature against time graph has the form shown in Figure 3.5.

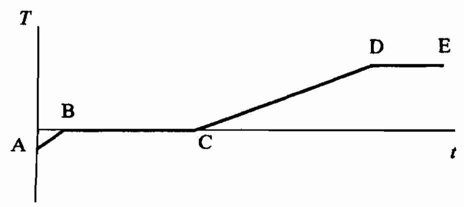


Figure 3.5

A ⟶ B ice heats up

B ⟶ C ice melts

C ⟶ D water heats up

D ⟶ E water boils and becomes steam

3.4.2 Heat of fusion

The heat of fusion (L_f) of a substance is the quantity of heat required to change unit mass of the substance from the solid to the liquid phase at the same temperature:

Heat of fusion of ice = 3.35×10^5 J kg^{-1} (at 0°C and 1 atm)

$$Q = mL_f$$

3.4.3 Heat of vaporization

The heat of vaporization (L_v) is the quantity of heat required to change unit mass of a substance from the liquid to the vapour phase at constant temperature:

Heat of vaporization of water = 22.56×10^5 J kg^{-1} (at 0°C and 1 atm)

$$Q = mL_v$$

Worked examples

(a) What quantity of heat is required to change 0.50 kg of ice at −20°C to steam at 100°C (specific heat capacity of ice = 2110 J kg^{-1} K^{-1})?

Solution Refer to Figure 3.5

AB
$$\Delta Q_{AB} = (mc\,\Delta T)_{ice} = 0.50 \times 2110 \times 20 = 21\,100 \text{ J}$$

BC
$$\Delta Q_{BC} = m\,L_f = 0.50 \times 3.35 \times 10^5 = 168\,000 \text{ J}$$

CD
$$\Delta Q_{CD} = (mc\,\Delta T)_{water} = 0.50 \times 4186 \times 100 = 209\,000 \text{ J}$$

DE
$$\Delta Q_{DE} = m\,L_v = 0.50 \times 2.26 \times 10^6 = 1\,130\,000 \text{ J}$$

Total heat energy = 1.53×10^6 J

(b) A 0.30 kg iron mug contains 0.90 kg of water at 75.0°C. When 0.10 kg of ice at 0°C is added, what is the final temperature of the water?

(Specific heats: iron = 470 J kg^{-1} K^{-1}
water = 4186 J kg^{-1} K^{-1}
ice = 2110 J kg^{-1} K^{-1}
Heat of fusion of ice = 3.35×10^5 J kg^{-1})

Solution Let T be the final temperature of the system.

$$\text{Heat transfer from hot objects} = \text{heat transfer to cold objects}$$

$$(mc\,\Delta T)_{\text{iron}} + (mc\,\Delta T)_{\text{hot water}} = (mL)_{\text{ice}} + (mc\,\Delta T)_{\text{melted ice}}$$

$$(0.30 \times 470 + 0.90 \times 4186) \times (75 - T) = 0.10 \times 3.35 \times 10^5 + 0.10 \times 4186 \times T$$

$$10\,600 - 141T + 283\,000 - 3770T = 33\,500 + 419T$$

$$260\,000 = 4330T$$

$$T = 60.0^\circ\text{C}$$

Problem A beaker of very small mass contains 0.500 kg of water at a temperature of 80.0°C. How many kilograms of ice at a temperature of −20.0°C must be dropped in the water so that the final temperature of the system will be 50.0°C?

(Specific heats: water = 4186 J kg^{-1} K^{-1}

ice = 2110 J kg^{-1} K^{-1}

Heat of fusion of ice = 3.35×10^5 J kg^{-1})

(Ans: 0.107 kg)

Keywords

Heat of fusion, heat of vaporization.

Important equation

$$\boxed{Q = mL}$$

3.5 Heat transfer

Heat can be transferred by conduction, convection and radiation:

- In conduction the energy is passed from particle to particle in the substance, the energy is transferred in the direction from high to low temperature.
- Convection is the process whereby fluids flow from one place to another because of density differences resulting from unequal temperatures.
- With radiation the energy is transferred as electromagnetic waves and does not require the presence of matter.

3.5.1 Thermal conduction

Consider a slab of material of cross-sectional area A and thickness L and having its faces differing in temperature by ΔT $(= T_2 - T_1)$.

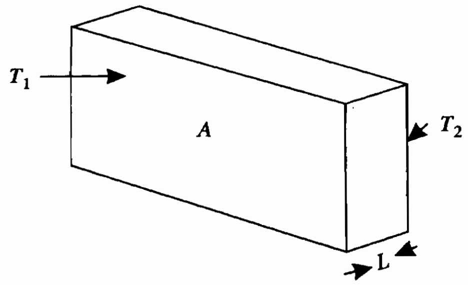


Figure 3.6

$$\frac{T_2 - T_1}{L}$$ is the temperature gradient.

The rate of flow of heat dQ/dt is proportional to the temperature gradient and the cross-sectional area.

$$\frac{dQ}{dt} = \mathrm{k}A\frac{dT}{dx} = \mathrm{k}A\frac{T_2 - T_1}{L}$$

Note that k depends only on the nature of the substance and is called the coefficient of thermal conductivity and has the units of J m^{-1} K^{-1} s^{-1}.

3.5.2 Heat transfer and loss of temperature

If heat is flowing from a body at a rate dQ/dt, then, if there is no change of state, the rate of loss of temperature is given by the equation:

$$\frac{dQ}{dt} = mc\frac{dT}{dt}$$

Worked example A boiler with a steel bottom 15 mm thick rests on a hot stove. The area of the bottom of the boiler is 0.18 m^2 and the conductivity for steel is 50 W m^{-1} K^{-1}.

(a) Determine the rate of heat flow by conduction through the bottom when the water inside the boiler is at 100°C and the lower surface of the boiler, in contact with the stove, is at 140°C.

(b) What mass of water evaporates in 5 minutes? (heat of vaporization = 2.26×10^6 J kg^{-1})

Solution

(a) Rate of heat transferred through the bottom:

$$\frac{dQ}{dt} = \frac{kA\,(T_2 - T_1)}{L}$$

$$= \frac{50 \times 0.18 \times (140 - 100)}{15 \times 10^{-3}}$$

$$= 2.4 \times 10^4 \text{ J s}^{-1}$$

(b) In 5 minutes:

$$\Delta Q = 2.4 \times 10^4 \times 5 \times 60$$

$$= 7.2 \times 10^6 \text{ J}$$

$$\therefore \quad as\ \Delta Q = m\,L_v$$

$$m = \frac{\Delta Q}{L_v} = \frac{7.2 \times 10^6}{2.26 \times 10^6} = 3.2 \text{ kg}$$

Problem A storage tank in the shape of a cube of edge 1.50 m is filled with water at 90°C and is lined on all six sides with felt insulation (thermal conductivity = 0.40 W m^{-1} K^{-1}) 8.0 mm thick.

(a) What is the rate of heat flow by conduction if the outside temperature is 15°C?

(b) If 1 m^3 of water has a mass of 1000 kg, and the specific heat of water is 4200 J kg^{-1} K^{-1}, how long would it take the temperature to drop by one Celsius degree?

(Ans: (a) 5.1×10^4 J s^{-1}; (b) 280 s)

3.5.3 Radiation

Radiation has the following characteristics:

- A blackened surface absorbs at a greater rate than a smooth polished one.
- A good reflector of radiation is a poor absorber and a poor emitter.
- A good absorber of radiation is a good emitter.

A perfectly **black body** is one that absorbs all the radiant energy falling on it.

Stefan–Boltzmann Law

The total radiation coming from a perfect radiator or black body is proportional to the fourth power of its **absolute** temperature T.

If H is the energy radiated per second from a **black body**, then:

$$H = A\,\sigma\,T^4$$

where σ is Stefan–Boltzmann's constant = 5.67×10^{-8} W m^{-2} K^{-4} and A is the surface area of the body

If the body is not a perfect radiator (black body), the formula becomes:

$$H = e A \sigma T^4$$

where e is the **coefficient of emissivity** or **relative emittance**.

The relative emittance is the ratio of the power radiated by a given area of surface to that radiated by the same area of a perfectly black body under the same conditions.

Where a body is both emitting and absorbing radiation the net rate of heat transfer from the body is:

$$H_{net} = eA \sigma (T_1^4 - T_2^4)$$

where T_1 is the temperature of the body and T_2 is the surrounding temperature.

Worked example A tungsten filament has dimensions 14 mm × 2.0 mm and has emissivity 0.35 and is of negligible thickness. It is heated to 2500°C by electricity. What is the rate of heat loss from the filament by radiation? (Surrounding temperature is 20°C)

Solution

Area of one side of the filament = $14 \times 10^{-3} \times 2.0 \times 10^{-3}$

$= 2.8 \times 10^{-5}$ m^2

$\therefore$ Total area = $2 \times 2.8 \times 10^{-5} = 5.6 \times 10^{-5}$ m^2

$$\begin{aligned} H_{net} &= e A \sigma (T_1^4 - T_2^4) \\ &= 0.35 \times 5.6 \times 10^{-5} \times 5.67 \times 10^{-8} \times \left((2500 + 273)^4 - (20 + 273)^4\right) \\ &= 66 \text{ J s}^{-1} \text{ (W)} \end{aligned}$$

Problem Calculate the rate of heat flow from a naked human body, given that its surface area is 1.40 m^2, its surface temperature is 34.0°C, and its emissivity is 0.60. The surrounding air temperature is 10.0°C. How much heat would flow in one hour?

(Ans: 118 W; 4.23×10^5 J)

Keywords

Thermal conductivity, radiant energy.

Important equations

$$\boxed{\frac{dQ}{dt} = \mathrm{k}A\,\frac{T_2 - T_1}{L}}$$

$$\boxed{H_{\text{net}} = \mathrm{e}\,A\,\sigma\,(T_1^4 - T_2^4)}$$

3.6 Equation of state

3.6.1 Ideal gas law

The following law applies for a fixed amount of gas changed from being in state 1 to being in state *2*.

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2} = \text{constant}$$

3.6.2 Equation of state for an ideal gas

If the left-hand side of the equation given in Section 3.6.1 is calculated in SI units for one mole, the constant is independent of the gas and is termed the universal gas constant, R:

For n moles:

$$pV = n\mathrm{R}T$$

where R = 8.31 J mol^{-1} K^{-1}

The number of molecules in a mole is called Avogadro's number, N_{A} (= 6.022×10^{23} molecules mole^{-1}). The equation of state can be written in terms of the number of molecules, N, as:

$$pV = N\mathrm{k}T$$

where k is Boltzmann's constant

$$\mathrm{k} = \frac{R}{N_{\mathrm{A}}} = 1.38 \times 10^{-23}\ \mathrm{J\ K^{-1}\ molecule^{-1}}$$

3.6.3 Partial pressures

Dalton's Law of Partial Pressure states that the total pressure of a gas equals the pressure due to each component gas as if it were acting alone occupying the entire volume:

$$p_T = \sum_{i=1}^{n} p_i$$

3.6.4 The triple point

At the triple point, gas, liquid and solid phases all exist together in equilibrium. It occurs at a characteristic temperature and pressure. It can be clearly seen on a p–T diagram, such as Figure 3.7.

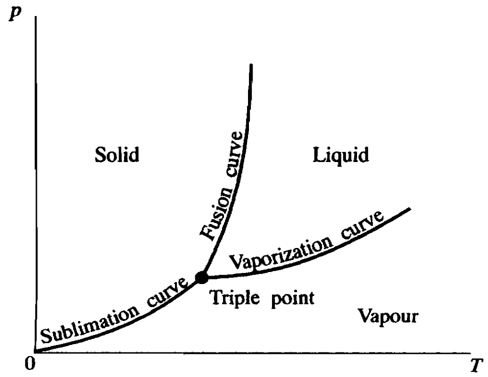


Figure 3.7

Worked examples

(a) A mass of oxygen occupies 2.00 litres at 5°C and 760 mm of mercury pressure. Determine its volume at 30°C and 800 mm pressure.

Solution

Data:

$p_1 = 760$ mm of Hg
$V_1 = 2.00$ L
$T_1 = 278$ K
$p_2 = 800$ mm of Hg
$T_2 = 303$ K

Asked for:

$V_2 = ???$

Using general gas law:

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$$

$$V_2 = \frac{p_1V_1T_2}{T_1p_2}$$

$$= \frac{760 \times 2 \times 303}{278 \times 800}$$

$$= 2.07 \text{ L}$$

Note: In the above example, it was not necessary to convert the pressures and volumes to SI units. This is because the equation can be rewritten as:

$$\frac{p_1}{p_2}\frac{V_1}{V_2} = \frac{T_1}{T_2}$$

and hence only involves a **ratio** of pressures and volumes and so, provided the units used are proportional to SI units, they may be used.

T has to be in kelvin rather than °C, because °C are not proportional to kelvin.

To determine the number of moles of oxygen present and hence the mass of oxygen, SI units must be used throughout.

Hence:

$$V = 2.00 \times 10^{-3} \text{ m}^3$$

$$T = 278 \text{ K}$$

$$p = 1.01 \times 10^5 \text{ Pa}$$

$$pV = nRT$$

$$\therefore \quad n = \frac{1.01 \times 10^5 \times 2.00 \times 10^{-3}}{8.31 \times 278} = 0.0874 \text{ moles}$$

$$m = nM = 0.0874 \times 32 = 2.80 \text{ g}$$

(b) A horizontal tube of uniform bore, which is closed at one end, has air imprisoned between the closed end and a short index of water (Figure 3.8). The other end of the tube is open to the atmosphere, the pressure of which is 750 mm Hg. At 12°C the length of the air column is 420 mm and at 50°C the length of the air column is 535 mm. The saturation vapour pressure (S.V.P.) of water at 12°C is 10.5 mm Hg. Find the S.V.P. of water at 50°C.

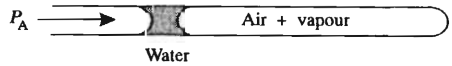


Figure 3.8

Solution For the trapped air

At 12°C $p_1 = 750 - 10.5$ by Dalton's Law

$= 739.5$ mm Hg

and $T_1 = 285$ K

If the cross-sectional area is A mm^2

$$V_1 = A \times 420 \text{ mm}^3$$

At 50°C $T = 323$ K

$$V_2 = A \times 535 \text{ mm}^3$$

∴ for trapped air:

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$$

$$p_2 = \frac{p_1V_1T_2}{T_1V_2}$$

$$= \frac{739.5 \times A \times 420 \times 323}{285 \times 535 \times A}$$

$$= 658 \text{ mm Hg}$$

∴ $p_{water} = 750 - 658 = 92$ mm Hg

Problem 0.10 m^3 of an ideal gas at a pressure of 1.0×10^5 Pa and a temperature of 300 K is heated at constant pressure until its volume is doubled. At this stage the gas is heated at constant volume until the pressure increases by 50%. Finally the gas is compressed at constant temperature to half its initial volume (i.e. 0.050 m^3).

(a) Construct a p–V graph to show the path taken by the gas.

(b) Determine:

(i) the number of moles of gas;

(ii) the pressure, volume and temperature after each stage of compression or expansion.

(Ans: (b)

(i) 4.0;

(ii) 1.0×10^5 Pa, 0.20 m^3, 600 K

1.5×10^5 Pa, 0.20 m^3, 900 K

6.0×10^5 Pa, 0.05 m^3, 900 K)

Keyword

Equation of state.

Important equations

$$\boxed{\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}}$$

$$\boxed{pV = nRT}$$

$$\boxed{pV = NkT}$$

3.7 The First Law of Thermodynamics

3.7.1 Internal energy

The internal energy U is the total energy of all the molecules. For an ideal gas it only depends on temperature.

3.7.2 First Law of Thermodynamics

The First Law of Thermodynamics states that

$$U_2 - U_1 = \Delta U = Q - W$$

where the heat transferred, Q, is positive if it *enters* the system and the work W is positive if it is done **by** the system.

The differential form of the First Law is

$$dU = dQ - dW$$

3.7.3 Work done in a thermodynamic process

$$\text{Work done} = \int_{V_1}^{V_2} p\,dV$$

(which equals the **area under a p–V diagram** between ordinates V_1 and V_2), as seen in Figure 3.9.

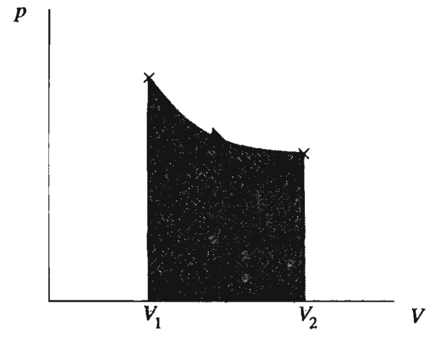


Figure 3.9

3.7.4 Different thermodynamic processes

- Isobaric process — constant pressure ($\Delta p = 0$)
- Isochoric process — constant volume ($\Delta V = 0$)
- Isothermal process — constant temperature ($\Delta T = 0$)
- Adiabatic process — no heat enters or leaves the system ($Q = 0$)

The four processes are shown on the p–V graph in Figure 3.10.

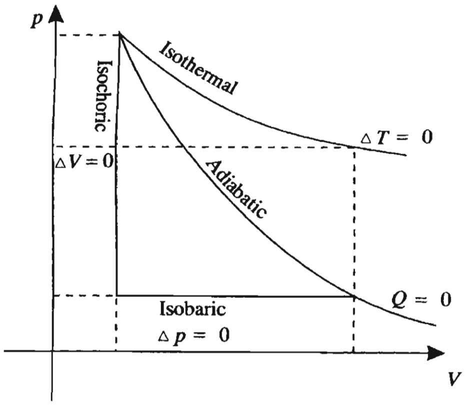


Figure 3.10

Heat capacities (specific heats) of an ideal gas

For an ideal gas

$$C_p - C_V = R$$

in which C_p and C_V are respectively the molar heat capacities at constant pressure and constant volume.

Isobaric process

$W = p\,(V_2 - V_1)$ $\quad Q = n\,C_p\,\Delta T$ $\quad \Delta U = n\,C_V\,\Delta T$

Isochoric process

$W = 0$ $\quad Q = n\,C_V\,\Delta T$ $\quad \Delta U = n\,C_V\,\Delta T$

(This particular value of ΔU will be the value for any process involving a change ΔT in temperature.)

Isothermal process

$W = n\,\mathrm{R}\,T \ln\left(\frac{V_2}{V_1}\right)$ $\quad Q = W$ $\quad \Delta U = 0$

Adiabatic process

Provided the adiabatic process is 'quasi-static' (i.e. the gas passes through a series of equilibrium states with the equation of state therefore applicable throughout), then it may be shown that:

$$pV^{\gamma} = \text{constant}$$

in which $\gamma = \frac{C_p}{C_V}$

$W = \frac{1}{\gamma - 1}(p_1V_1 - p_2V_2)$ $\quad Q = 0$ $\quad \Delta U = n\,C_V\,\Delta T$

Note: An alternative relation for W is: $W = -\Delta U = -n\,C_V\,\Delta T$

Worked example Consider 7.00 g of nitrogen (molecular weight = 28, $\gamma = \frac{7}{5}$) at 20° C and 1.01×10^5 Pa. This gas is taken around a cycle as shown in the p–V diagrams in Figure 3.11, where $V_B = 2V_A$ and $V_C = 3V_A$. The processes are A–B isobaric, B–C isothermal, C–D isochoric and D–A adiabatic.

Determine:

(a) the characteristics of nitrogen;

(b) the initial thermodynamic conditions (at A);

(c) the thermodynamic conditions at B and C;

(d) the thermodynamic conditions at D from the conditions at A;

(e) Q, ΔU and W for each process.

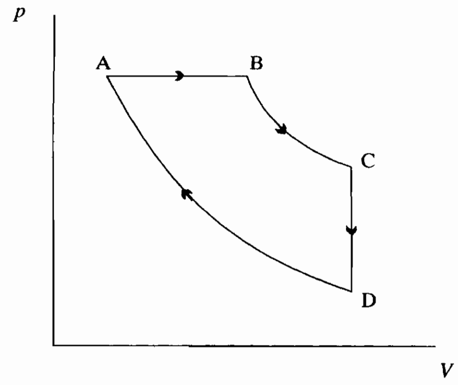


Figure 3.11

Solution

(a) $$\gamma = \frac{C_p}{C_V} = \frac{7}{5} \text{ and } C_p - C_V = R = 8.31 \text{ J mol}^{-1} \text{ K}$$

$$\therefore\ C_V = 20.8 \text{ J mol}^{-1} \text{ K}^{-1}$$

$$C_P = 29.1 \text{ J mol}^{-1} \text{ K}^{-1}$$

(b) $$p_A = 1.01 \times 10^5 \text{ Pa},\ T_A = 293 \text{ K},\ n = \frac{7.00}{28} = 0.25 \text{ mol}$$

$$\therefore\ V_A = \frac{nRT_A}{p_A} = 6.0 \times 10^{-3} \text{ m}^3$$

(c) $$p_B = p_A = 1.01 \times 10^5 \text{ Pa},\ V_B = 2V_A = 1.20 \times 10^{-2} \text{ m}^3$$

$$T_B = \frac{p_B V_B T_A}{p_A V_A} = 586 \text{ K}$$

$$V_C = 3V_A = 1.80 \times 10^{-2} \text{ m}^3,\ T_C = T_B = 586 \text{ K}$$

$$p_C = \frac{p_B V_B T_C}{T_B V_C} = 6.67 \times 10^4 \text{ Pa}$$

(d) $$V_D = 1.80 \times 10^{-2} \text{ m}^3, \quad p_A V_A{}^\gamma = p_D V_D{}^\gamma$$

$$p_D = 2.17 \times 10^4 \text{ Pa}$$

$$T_D = \frac{p_D V_D T_A}{p_A V_A} = 189 \text{ K}$$

(e) Isobaric (A–B) $Q = nC_p\Delta T = 2130$ J

$W = p\Delta V = 610$ J

$\Delta U = nC_V\Delta T = 1520$ J

Isothermal (B–C) $Q = W = n\mathrm{R}T \ln\left(\dfrac{V_2}{V_1}\right) = 490\ \mathrm{J}$
$\Delta U = 0\ \mathrm{J}$

Isochoric (C–D) $Q = nC_V\Delta T = -2060\ \mathrm{J}$
$W = 0\ \mathrm{J}$
$\Delta U = -2060\ \mathrm{J}$

Adiabatic (D–A) $Q = 0\ \mathrm{J}$
$W = -nC_V\Delta T = -540\ \mathrm{J}$
$\Delta U = 540\ \mathrm{J}$

Note: $\Delta U_{\mathrm{ABCD}} = 1520 + 0 - 2060 + 540 = 0\ \mathrm{J}$

Problem An engine operates by taking an ideal monatomic gas through the following cycle:

$$\left(C_p = \frac{5}{2}\mathrm{R},\ C_V = \frac{3}{2}\mathrm{R}\right)$$

(1) Start with n moles at p_0, V_0, T_0.

(2) Change to $2p_0$, V_0, at constant volume.

(3) Change to $2p_0$, $2V_0$, at constant pressure.

(4) Change to p_0, $2V_0$, at constant volume.

(5) Change to p_0, V_0, at constant pressure.

Show the cycle on a p–V diagram.
Calculate in terms of T_0:

(a) the temperature of the gas after each step has been performed;

(b) the heat absorbed or rejected in each step;

(c) the work done on or by the gas in each step;

(d) the change in internal energy of the gas in each step.

Confirm that, for the complete cycle, the net change in internal energy of the gas is zero.

(Ans:

(a) $2T_0$, $4T_0$, $2T_0$, T_0

(b) $+\frac{3}{2}n\mathrm{R}T_0$, $+5n\mathrm{R}T_0$, $-3n\mathrm{R}T_0$, $\frac{5}{2}n\mathrm{R}T_0$

(c) 0, $+2n\mathrm{R}T_0$, 0, $-n\mathrm{R}T_0$

(d) $+\frac{3}{2}n\mathrm{R}T_0$, $+3n\mathrm{R}T_0$, $-3n\mathrm{R}T_0$, $-\frac{3}{2}n\mathrm{R}T_0$)

Keywords

First law of thermodynamics, isothermal process, adiabatic process, isochoric process, isobaric process.

Important equations

$$\boxed{W = \int_{V_1}^{V_2} p\,dV}$$

$$\boxed{\Delta U = Q - W}$$

$$\boxed{C_p = C_V + R}$$

$$\boxed{\gamma = \frac{C_p}{C_V}}$$

$$\boxed{pV^{\gamma} = \text{constant}}$$

3.8 Kinetic theory of gases

3.8.1 Assumptions

The kinetic theory applies to an ideal gas and, as such, is based on a number of assumptions:

- That a gas is composed of a large number of molecules.
- The molecules are so small that they can be considered to be point masses.
- The molecules are in random motion.
- The molecules obey Newton's laws of motion.
- The molecules only exert forces on each other during a collision.
- Collisions between molecules or between the molecules and the walls of the container are perfectly elastic.
- The time of duration of any collision is small compared with the time between collisions.

3.8.2 Kinetic theory

Using these assumptions we can derive a number of equations that will describe the behaviour of a gas reasonably well.

The pressure p exerted by a gas is given by:

$$p = \frac{Nm\overline{v^2}}{3V} = \frac{1}{3}\rho\overline{v^2}$$

where: N = the number of molecules present
$\overline{v^2}$ = the mean square velocity of the molecules
V = the volume of the gas
m = the mass of a molecule
ρ = the density of the gas.

This may also be written:

$$p = \frac{1}{3}\rho\,(v_{rms})^2$$

(where $v_{rms} = \sqrt{\overline{v^2}}$ = root mean square velocity.)
Using the equation of state of an ideal gas it can be shown that:

$$\text{The average KE of translation per molecule} = \frac{1}{2}m\,(v_{rms})^2 = \frac{3}{2}kT$$

The previous equation indicates that the average kinetic energy per molecule is directly proportional to the absolute temperature, and the rms speed is proportional to the square root of the temperature. It also indicates that the molecules of all gases have the same average kinetic energy at the same temperature. It follows that at the same temperature, molecules with less mass have greater rms speed.

3.8.3 Equipartition of energy

The internal energy of an ideal gas depends only on the temperature and distributes itself in equal shares to each of the independent ways in which the molecule can absorb energy. Each independent mode of energy absorption is a **degree of freedom**.

To every degree of freedom of motion of a **molecule** there is attributed an average energy, at equilibrium, equal to $\frac{1}{2}kT$ (*or* $\frac{1}{2}RT$ *per mole*).

For ***monatomic ideal gases*** — 3 degrees of freedom (all translational):

Thus for 1 mole, internal energy $U = 3 \times \frac{1}{2}RT$ and $\frac{dU}{dT} = \frac{3}{2}R$

Now $\Delta U = n\,C_V\,\Delta T$, $\therefore\ C_V = \frac{3}{2}R$, $C_p - C_V = R$, $C_p = \frac{5}{2}R$

Hence $$\gamma = \frac{C_p}{C_V} = \frac{5}{3} = 1.67$$

For ***diatomic ideal gases*** — 5 degrees of freedom (3 translational, 2 rotational):

Thus for 1 mole, internal energy $U = 5 \times \frac{1}{2}\,RT$ and $\frac{dU}{dT} = \frac{5}{2}\,R$

Now $C_V = \frac{5}{2}\,R$, $C_p = \frac{7}{2}\,R$

Hence $$\gamma = \frac{C_p}{C_V} = \frac{7}{5} = 1.40$$

For ***polyatomic ideal gases*** — the behaviour varies but for some, for example CH_4, there are 6 degrees of freedom (3 translational, 3 rotational):

Thus for 1 mole, internal energy $U = 6 \times \frac{1}{2}\,RT$ and $\frac{dU}{dT} = 3R$
Now $C_V = 3R$, $C_p = 4R$

Hence $$\gamma = \frac{C_p}{C_V} = \frac{4}{3} = 1.33$$

Worked example Calculate:

(a) the average translational kinetic energy of an oxygen molecule at 27.0°C;

(b) the rms velocity of the molecules at this temperature;

(c) the average total kinetic energy of an oxygen molecule at this temperature, given that oxygen is diatomic;

(d) the internal energy of 1 mole of oxygen at this temperature (molecular weight of $O_2 = 32$).

Solution

(a)
$$\begin{aligned} KE_{\text{translational}} &= \frac{3}{2}\,kT \\ &= \frac{3}{2} \times 1.38 \times 10^{-23} \times 300 \\ &= 6.21 \times 10^{-21}\ \text{J} \end{aligned}$$

(b) Mass of 1 mole of oxygen = 32 g

$$\text{mass of 1 molecule of oxygen} = \frac{32 \times 10^{-3}}{6.02 \times 10^{23}}\ \text{kg}$$

$$\frac{1}{2}\,m\,(v_{\text{r m s}})^2 = 6.21 \times 10^{-21}$$

$$(v_{\mathrm{r\,m\,s}})^2 = \frac{6.21 \times 10^{-21} \times 6.02 \times 10^{23}}{\frac{1}{2} \times 32 \times 10^{-3}}$$

$$= 2.33 \times 10^5$$

$$v_{\mathrm{r\,m\,s}} = 480 \text{ m s}^{-1}$$

(c) The average kinetic energy attributed to each of the 5 degrees of freedom is 1/2 kT, and the translational kinetic energy accounts for 3 degrees of freedom. That is, the translational KE is 3/5 of the total KE.
Therefore:

$$K_{\mathrm{tot}} = \frac{5}{3} \times 6.21 \times 10^{-21} \text{ J}$$
$$= 1.04 \times 10^{-20} \text{ J}$$

(d) The internal energy for one molecule is 1.04×10^{-20} J.
Therefore, for Avogadro's number of molecules (for one mole):

$$\text{internal energy} = 1.04 \times 10^{-20} \times 6.02 \times 10^{23}$$
$$= 6.26 \times 10^3 \text{ J}$$

Problem What is the average translational kinetic energy of a molecule of nitrogen at 127°C? What is the average value of the square of its speed? What is the rms speed? What is the momentum of a nitrogen molecule travelling at this speed?

Suppose a molecule travelling at this speed bounces back and forth between opposite sides of a cubical vessel 100 mm on a side, moving at right angles to those sides (x-direction). What is the average force it exerts on *each* of these two walls of the container?

(*Hint:* use Newton's Second Law.)

What is the average force per unit area?

How many molecules travelling at this speed are needed to produce an average pressure of 1 atmosphere (1.01×10^5 Pa)?

Compare the above number with the number of nitrogen molecules actually contained in a vessel of this size at 1 atm pressure and 127°C.

(*Hint:* $n = \frac{p\mathrm{V}}{\mathrm{R}T}$ and $N = nN_A$)

(Molecular weight $N_2 = 28$)

(Ans: 8.21×10^{-21} J; 35.4×10^4 m^2 s^{-2}; 595 m s^{-1}; 2.78×10^{-23} kg m s^{-1}; 1.65×10^{-19} N; 1.65×10^{-17} Pa; 6.15×10^{21} molecules; 18.5×10^{21} molecules)

Keywords

Kinetic theory, equipartition of energy.

Important equation

$$\boxed{\frac{1}{2} m (v_{\mathrm{r\,m\,s}})^2 = \frac{3}{2} \mathrm{k}T}$$

3.9 Carnot cycle and the Second Law of Thermodynamics

3.9.1 Carnot cycle

A Carnot cycle consists of a series of four *reversible* processes, alternately isothermal and adiabatic. The heat engine operates in a cycle between two reservoirs at temperatures T_H (hot reservoir) and T_C (cold reservoir) absorbing heat Q_H and rejecting heat Q_C respectively, each cycle. The cycle is shown on the p–V diagram in Figure 3.12.

1–2 Isothermal expansion (at T_H)
2–3 Adiabatic expansion
3–4 Isothermal compression (at T_C)
4–1 Adiabatic compression

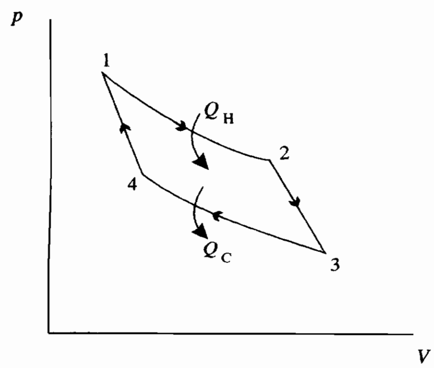


Figure 3.12

The work done by the engine (which equals the area of the 'cycle' on a p–V diagram) is given by:

$$W = Q_H + Q_C$$

(According to the sign convention used, Q_C is negative as the heat energy Q_C leaves the system.)

The **thermal efficiency**, e, of a heat engine is:

$$e = \frac{W}{Q_H}$$

So,
$$e = \frac{Q_H + Q_C}{Q_H} = 1 + \frac{Q_C}{Q_H}$$

For a Carnot engine it can be shown that:

$$\frac{Q_H}{Q_C} = -\frac{T_H}{T_C}$$

$$e = 1 - \frac{T_C}{T_H}$$

This is true for any working substance.

A **Carnot refrigerator** involves the Carnot cycle being reversed so that heat Q_C is absorbed from the cold reservoir (T_C), heat Q_H delivered to the hot reservoir (T_H), and work W done *on* the working substance:

Again
$$W = Q_H + Q_C \text{ and } \frac{Q_H}{Q_C} = -\frac{T_H}{T_C}$$

(According to the sign convention used, W and Q_H are negative for the refrigerator.) The coefficient of performance for a Carnot refrigerator is:

$$K = \frac{Q_C}{|W|} = \frac{T_C}{T_H - T_C}$$

3.9.2 The Second Law of Thermodynamics

Different, equally valid, statements of the Second Law of Thermodynamics are as follows:

(1) Heat flows spontaneously from a hot object to a cold object; heat will not flow spontaneously from a cold object to a hot object.

(2) No cyclic process is possible in which the net result is to convert a quantity of heat, Q_H, from a source at a single temperature totally into work W (so $W = Q_H$).

(3) No cyclic process is possible the only result of which is a heat flow out of a system at one temperature and an equal magnitude heat flow into a second system at a higher temperature.

(4) All reversible engines operating between the same two temperatures have the same efficiency; no irreversible engine operating between the same two temperatures can have an efficiency greater than a Carnot engine.

(5) In any real (irreversible) process, some energy becomes unavailable to do useful work.

(6) Real processes tend to move systems towards a state of greater disorder.

3.9.3 Entropy

The Second Law of Thermodynamics can be phrased in several other ways. The quantity **entropy** tends to increase in all real processes. At the microscopic level, entropy is closely related to the randomness or disorder of the constituents; as noted in the sixth statement of the second law, all systems tend toward states of greater disorder.

In terms of entropy there are two more statements of the Second Law of Thermodynamics:

(7) The entropy of an isolated thermodynamic system never decreases. It either stays constant (reversible processes) or increases (irreversible spontaneous processes).

(8) The total entropy of any system and its environment increases as a result of any spontaneous process.

Worked example If a Carnot engine takes 6300 J of heat each cycle from the high-temperature reservoir at 500 K and gives out 3780 J to the low-temperature reservoir, calculate the temperature of the latter. What is the thermal efficiency of the cycle, and how much external work is done in each cycle?

Solution

Now, $$\frac{Q_H}{Q_C} = -\frac{T_H}{T_C} \text{ for a Carnot cycle.}$$

i.e. $$T_C = -\frac{Q_C}{Q_H} \cdot T_H$$

$$= -\frac{-3780}{6300} \times 500$$

$$T_C = 300 \text{ K}$$

Thermal efficiency: $$e = 1 - \frac{T_C}{T_H} = 1 - \frac{300}{500} = \frac{2}{5}$$

$$e = 40\%$$

External work done, each cycle: $$W = Q_H + Q_C = 6300 + (-3780)$$

$$= 2520 \text{ J}$$

Problem A Carnot engine is operated between two heat reservoirs at temperatures of 400 K and 300 K.

(a) If the engine receives 5000 J from the reservoir at 400 K in each cycle, how many joules does it reject to the reservoir at 300 K?

(b) If the engine is operated in reverse, as a refrigerator, and received 5000 J from the reservoir at 300 K, how many joules does it deliver to the reservoir at 400 K?

(c) How many joules would be produced if the mechanical work required to operate the refrigerator in part (b) were converted directly to heat?

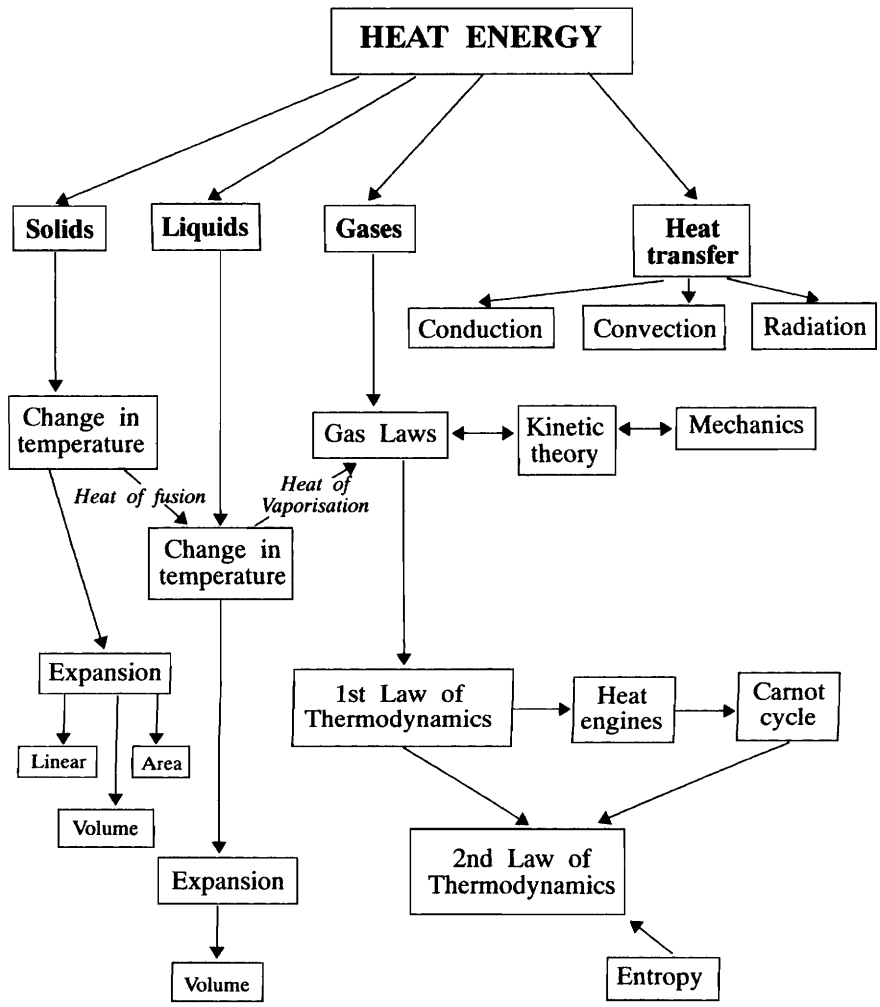
(Ans: (a) 3750 J; (b) 6670 J; (c) 1670 J)

Keywords

Carnot cycle, refrigerator, entropy.

Important equation

$$\boxed{e = \frac{W}{Q_H} = 1 - \frac{T_C}{T_H}}$$


HEAT ENERGY
Solids
Liquids
Gases
Heat transfer
Conduction
Convection
Radiation
Change in temperature
Gas Laws
Kinetic theory
Mechanics
Heat of fusion
Heat of Vaporisation
Change in temperature
Expansion
Linear
Area
Volume
1st Law of Thermodynamics
Heat engines
Carnot cycle
2nd Law of Thermodynamics
Expansion
Volume
Entropy

CHAPTER 4

Waves

4.1 Elasticity

A material is **elastic** when, after having forces exerted on it, it returns to its original shape.

Stress is the force per unit area. The stress in a bar is **tensile** when the bar is pulled at opposite ends with equal forces. The stress is **compressive** if the bar is pushed at opposite ends with equal forces:

$$\text{stress} = \frac{\text{force}}{\text{area}}$$

Strain is the ratio of the change in some dimension of a body, when it is subject to forces, to the original dimension of the body. **Tensile strain** of a bar is the ratio of the elongation to the original length. **Compressive strain** is the ratio of the decrease in the length to the original length:

$$\text{strain} = \frac{\Delta l}{l_0}$$

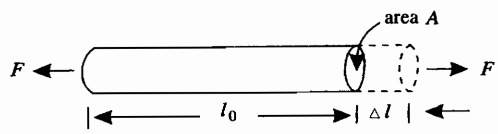


Figure 4.1

A typical stress–strain graph for a ductile metal under tension is shown in Figure 4.2.

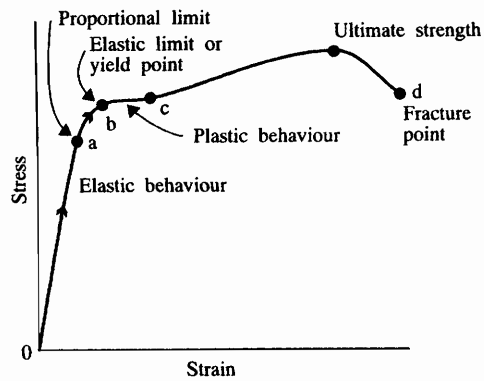


Figure 4.2

4.1.1 Hooke's Law

A material follows **Hooke's Law** if stress is proportional to strain:

$$k = \frac{\text{stress}}{\text{strain}}$$

k is a constant called the **modulus of elasticity.**

4.1.2 Young's modulus

In the case of longitudinal stress and strain, the constant of proportionality is called **Young's modulus**, Y:

$$Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{F/A}{\Delta l/l_0}$$

When a material stretches under stress the dimension perpendicular to the direction of the stress becomes smaller (see Figure 4.3). **Poisson's ratio** σ relates the fractional change in width to the fractional change in length:

$$\frac{\Delta w}{w_0} = -\sigma\frac{\Delta l}{l_0}$$

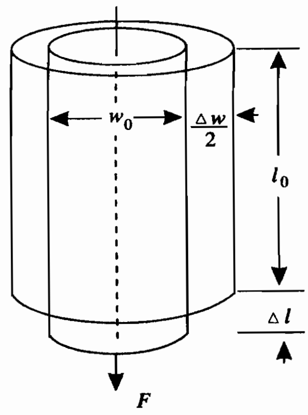


Figure 4.3

4.1.3 Bulk stress and strain

When a solid or fluid is subject to a uniform pressure over its whole surface the deformation is measured in terms of the change in volume:

$$\text{volume strain} = \frac{\Delta V}{V_0}$$

The **bulk modulus** is defined as:

$$B = \frac{\text{change in pressure}}{\text{volume strain}} = -\frac{\Delta p}{\Delta V/V_0}$$

4.1.4 Shear stress and strain

Shear stress is the force tangent to a surface divided by the area of the surface (see Figure 4.4).

Shear strain: $\varphi = x/h$

Shear modulus: $S = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F/A}{x/h}$

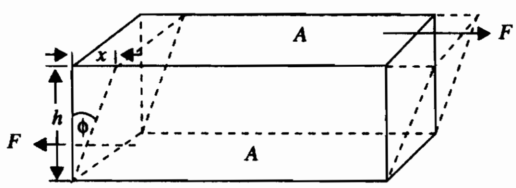


Figure 4.4

4.1.5 Units

Strains are dimensionless quantities.

Stresses have the dimensions of force per unit area and so in the SI system are measured in N m^{-2} or pascals, symbol Pa. 1 Pa = 1 N m^{-2}.

Elastic constants, which are defined as the ratio of stresses to strains, have the same dimensions as stress and so are measured in N m^{-2} in the SI system.

Worked examples

(a) An elastic rod 5 m long and 0.03 m^2 in cross-section stretches 0.15 m when a 270 N weight is hung on it (Figure 4.5). Find the stress, strain and Young's modulus for the material of the rod.

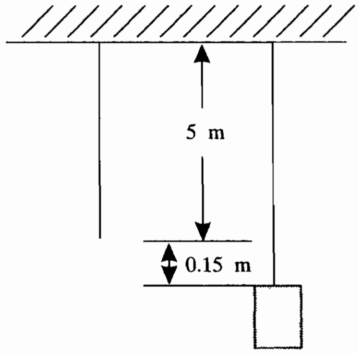


Figure 4.5

Solution

$$\text{stress} = \frac{F}{A} = \frac{270\ \text{N}}{0.03\ \text{m}^2} = 9 \times 10^3\ \text{N m}^{-2}$$

$$\text{strain} = \frac{\Delta l}{l_0} = \frac{0.15\ \text{m}}{5\ \text{m}} = 0.03$$

$$\text{Y} = \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

$$= \frac{9 \times 10^3}{3 \times 10^{-2}}\ \text{N m}^{-2}$$

$$= 3.0 \times 10^5\ \text{N m}^{-2}$$

(b) A cube of metal 100 mm on a side (Figure 4.6) is subjected to a shearing force of 1520 N. The top face of the cube is displaced 0.25 mm with respect to the bottom. Calculate the shearing stress, the shearing strain, and the shear modulus.

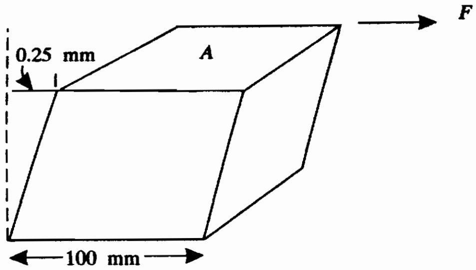


Figure 4.6

Solution

$$\text{Shearing stress} = \frac{\text{tangential force}}{\text{area of face}}$$

$$= \frac{1520\ \text{N}}{(0.1\ \text{m})^2}$$

$$= 1.52 \times 10^5\ \text{N m}^{-2}$$

$$\text{Shearing strain} = \frac{\text{displacement}}{\text{altitude}}$$

$$= \frac{2.5 \times 10^{-4}\ \text{m}}{0.10\ \text{m}}$$

$$= 2.5 \times 10^{-3}\ \text{rad}$$

$$\text{Shear modulus} = \frac{\text{shearing stress}}{\text{shearing strain}}$$

$$= \frac{1.52 \times 10^5\ \text{N m}^{-2}}{2.5 \times 10^{-3}}$$

$$= 6.1 \times 10^7\ \text{N m}^{-2}$$

(c) A wire of 1.0 mm diameter and length 850 mm is stretched by a force to a length of 1.0 m. Calculate the new diameter of the wire if Poisson's ratio for the metal is 0.5.

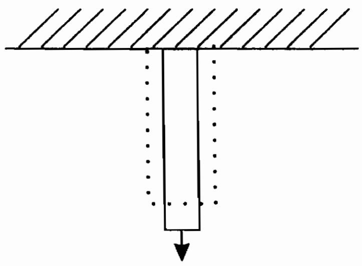


Figure 4.7

Solution

$$\Delta W = -\frac{\sigma\, W \Delta l}{l_0}$$

$$= -\frac{0.5 \times 1 \times 10^{-3}\ \text{m} \times 0.15\ \text{m}}{0.85\ \text{m}}$$

$$= -8.8 \times 10^{-5}\ \text{m} = -0.088\ \text{mm}$$

$\therefore$ new diameter = 0.91 mm

Problem A copper wire 1.0 mm in diameter and 2.0 m long is used to support a mass of 5.0 kg. By how much does it stretch under the load? If the 'ultimate strength' of copper is 3.4×10^8 N m^{-2}, what is the maximum mass the wire can support without breaking? (Ultimate strength means the greatest stress a material can withstand.) (Young's modulus for copper = 1.17×10^{11} N m^{-2})

(Ans: 1.1×10^{-3} m, 27 kg)

Keywords

Elasticity, stress, strain, Young's modulus, Poisson's ratio, bulk modulus, shear modulus.

Important equations

$$\boxed{Y = \frac{F/A}{\Delta l/l_0}}$$

$$\boxed{B = -\frac{\Delta p}{\Delta V/V_0}}$$

$$\boxed{S = \frac{F/A}{x/h}}$$

$$\boxed{\sigma = -\frac{\Delta w/w_0}{\Delta l/l_0}}$$

4.2 Simple harmonic motion (SHM)

4.2.1 Definition

If a body moves in a straight line in such a way that its acceleration (a) is proportional to its displacement (x) from some equilibrium position but always directed oppositely to its displacement, then the body is said to execute SHM, that is,

$$a = -\omega^2 x$$

where the constant of proportionality is written as ω^2 for future convenience.

Alternatively:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

This differential equation has as its solution:

$$x = A \sin(\omega t + C)$$

in which A and C are constants. A is called the **amplitude**.

From this solution, the velocity v and acceleration a of the body are found to be:

$$v = \omega A \cos(\omega t + C)$$
$$a = -\omega^2 A \sin(\omega t + C)$$

By eliminating $(\omega t + C)$ between the equations for x and v we can show that:

$$v = \pm\, \omega \sqrt{A^2 - x^2}$$

The quantity $(\omega t + C)$ is called the **phase angle** (or the phase) and C is called the **initial phase angle.**
[*Note:* A set of equations equivalent to the above equations may be based on the alternative solution, $x = A \cos(\omega t + D)$]

4.2.2 Force constant

Since the total force on a body is $F = ma$, the force, like the acceleration, is proportional to x. That is:

$$F = -\mathrm{k}x$$

where k, called the **force constant,** is the restoring force per unit displacement. Clearly we have the connecting formula:

$$\omega^2 = \frac{\mathrm{k}}{m}$$

4.2.3 Period and frequency

The **period**, T, is the time for one complete oscillation. The **frequency**, f, is the number of complete oscillations per second. The SI unit of frequency is the hertz (Hz).

These quantities are connected to the **angular frequency**, ω, and to each other by:

$$f = \frac{1}{T}$$

$$\omega = 2\pi f$$

Hence the period is: $$T = \frac{2\pi}{\omega}$$

4.2.4 Spiral spring

If the only force acting on the body is due to a spring obeying Hooke's Law, then $F = -\mathrm{k}x$. It will execute SHM with a period:

$$T = 2\pi\sqrt{\frac{m}{\mathrm{k}}}$$

where k now becomes the **stiffness factor** of the spring.

Typically, the body hangs vertically from a spiral spring and the weight of the spring must also be considered. However, it is only a small correction and for the discussion here we will assume a spring of negligible mass.

4.2.5 Simple pendulum

For a pendulum of length L oscillating through a small angle, that is $\theta \approx \sin\theta$, the restoring force is $mg \sin\theta$,

$$\therefore \quad ma = -mg\sin\theta \approx -mg\,\theta \approx -mg\frac{x}{L}$$

where x is the displacement from the equilibrium position.

Hence the period is: $$T = 2\pi\sqrt{\frac{L}{\mathrm{g}}}$$

4.2.6 Energy of SHM

Kinetic energy $$K = \frac{1}{2}mv^2$$

$$= \frac{1}{2} m\omega^2 A^2 \cos^2 (\omega t + C)$$

Potential energy
$$U = \frac{1}{2} kx^2$$
$$= \frac{1}{2} m\omega^2 A^2 \sin^2 (\omega t + C)$$

Total energy
$$E = K + U$$
$$= \frac{1}{2} m\omega^2 A^2$$

The fact that the total mechanical energy is constant is expressed by the equation:

$$\frac{1}{2} kx^2 + \frac{1}{2} mv^2 = \frac{1}{2} kA^2$$

Worked examples

(a) A particle vibrates with an SHM of amplitude 10.0 mm and period 0.50 s. At time $t = 0$ it is at the equilibrium position and moving in the negative direction. Determine:

(i) the angular frequency;

(ii) the initial phase;

(iii) the time at which the particle first has a displacement of –3.0 mm.

Solution

(i) Now
$$T = \frac{2\pi}{\omega}$$

so
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.50}$$
$$\omega = 12.57 \text{ rad s}^{-1}$$
$$= 13 \text{ rad s}^{-1}$$

(ii)

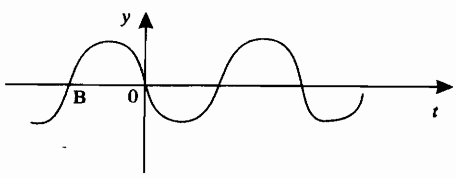


Figure 4.8

Now
$$y = A \sin (\omega t + C)$$

substituting
$$t = 0, y = 0$$

gives
$$\sin C = 0$$

$$\therefore \; C = 0 \text{ rad } or \; C = \pi \text{ rad}$$
$$\text{(assuming that } 0 \le C \le 2\pi)$$

Now $$\frac{dy}{dt} = A\,\omega \cos(\omega t + C)$$

which is negative when $t = 0$ and $C = 0$.

Hence the only solution is $C = \pi$ radians.

(iii) Now, $$y = A \sin(\omega t + C)$$

Substituting $y = -3.0$ mm gives $\sin(12.57t + \pi) = -0.30$.

We must now solve this equation for t given that $t > 0$. Simply using a calculator to find the inverse sine of both sides and solving for t gives $t = -0.274$ s, which is clearly wrong. The problem is that the inverse sine function has infinitely many possible solutions and the calculator always gives the solution to this function in the range $-\pi$ to π. However, in this case the phase $(12.57t + \pi)$ is more than π.

The simplest way to deal with this problem is to recognise that:

$$\sin(\theta + \pi) = -\sin(\theta)$$

so $$\sin(12.57t + \pi) = -\sin(12.57t) = -0.30$$

and $$\sin(12.57t) = 0.30$$

Thus $$12.57t = \sin^{-1}(0.30) = 0.305^*$$

and $$t = \frac{0.305}{12.57}$$
$$= 0.024 \text{ s}$$

* *Note:* The calculator must be switched to radians mode before the angle is calculated.

(b) When a 2.0 kg mass is suspended from a spiral spring, the spring stretches by 300 mm. What will be the period of the oscillation if the mass is drawn down a further 60 mm and then released? What will its maximum velocity and the total energy of the system be? (See Figure 4.9.)

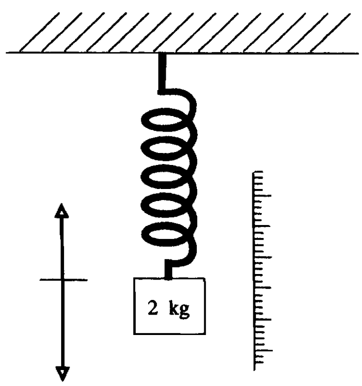


Figure 4.9

Solution The first set of data tells us about the stiffness factor of the spring:

$$F = mg = ky$$

where $m = 2.0$ kg
$y = 0.300$ m

Hence

$$\begin{aligned} k &= \frac{mg}{y} \\ &= \frac{2 \times 9.8}{0.3} \\ &= 65.3 \text{ N m}^{-1} \end{aligned}$$

For a spiral spring, the period of oscillation is given by:

$$\begin{aligned} T &= 2\pi \sqrt{\frac{m}{k}} \\ &= 2\pi \sqrt{\frac{2}{65.3}} \\ &= 1.1 \text{ s} \end{aligned}$$

The maximum velocity will be attained when the displacement is zero. Hence we substitute $y = 0$ in the equation:

$$\frac{1}{2} ky^2 + \frac{1}{2} mv^2 = \frac{1}{2} kA^2$$

to obtain

$$v_{max} = \sqrt{\frac{k}{m}} A = \sqrt{\frac{65.3}{2}} \times 0.06$$

$$= 0.34 \text{ m s}^{-1}$$

The total energy of the system is equal to the maximum kinetic *or* the maximum potential energy:

$$\begin{aligned} K_{max} &= \frac{1}{2} mv_{max}^2 \\ &= \frac{1}{2} \times 2 \times (0.34)^2 \\ &= 0.12 \text{ J} \end{aligned}$$

or

$$\begin{aligned} U_{max} &= \frac{1}{2} ky_{max}^2 \\ &= \frac{1}{2} kA^2 \end{aligned}$$

$$= \frac{1}{2} \times 65.3 \times (0.06)^2$$
$$= 0.12 \text{ J}$$

$\therefore$ Total energy of the system:

$$E = 0.12 \text{ J}$$

Problem A particle is executing a SHM with amplitude 0.120 m and frequency 3.00 Hz. At time $t = 0$ the displacement of the particle from the equilibrium position is +0.120 m.

(a) What is the angular frequency?

(b) What is the initial phase?
(Advice: for b, d, e draw a freehand graph of y versus t).

(c) Write down the formula giving the displacement as a function of t only.

(d) Where is the particle at time $t = 0.020$ s ?

(e) Where is the particle at time $t = 1.500$ s ?

(f) Given that the particle has mass 0.200 kg, calculate the force constant.

(Ans: (a) 18.85 rad s^{-1}; (b) $\frac{\pi}{2}$ rad or 90°; (c) $y = 0.120 \sin (18.85t + \frac{\pi}{2})$;
(d) + 0.112 m; (e) –0.120 m; (f) 71.1 N m^{-1})

Keywords

Simple harmonic motion, period, frequency, angular frequency, phase angle.

Important equations

$$\boxed{a = -\omega^2 x}$$

$$\boxed{x = A \sin (\omega t + C)}$$

$$\boxed{U = \frac{1}{2} kx^2}$$

4.3 Wave motion

4.3.1 Mechanical waves

A **wave** is a propagating (i.e. travelling) disturbance that transports energy from one region to another. A key feature of a wave is that **the disturbance propagates while the medium only undergoes small fluctuations about its equilibrium position.**

In a **longitudinal** mechanical wave, the direction of the disturbance is along the direction of propagation, for example a compression wave in a spring, sound in a gas. In a **transverse** mechanical wave, the direction of the disturbance is perpendicular to the direction of propagation, for example a wave in a stretched string. Some waves, such as water waves, combine both transverse and longitudinal motion.

A **periodic wave** is a propagating pattern of disturbance that repeats itself over a distance λ called the **wavelength**.

The number of complete waves that go past a fixed point in one second is called the **frequency**, f. The SI unit of frequency is the hertz, Hz. The time for one wave to go past a fixed point is called the **period**, T. The period is related to the frequency by equation:

$$f = \frac{1}{T}$$

Since the waveform advances one wavelength in one period, the **speed of propagation**, c, is given by: speed = distance/time or $c = \lambda/T$ and so

$$c = f\lambda$$

4.3.2 Mathematical description of a mechanical wave

Any wave can be described by a **wave function**, $y\,(x, t)$. The wave function describes the position of an arbitrary particle in the medium at any time.

The most important type of periodic wave is the sinusoidal wave. (In fact, any periodic wave can be considered to be the sum of sinusoidal waves.) A sinusoidal wave travelling in the positive direction along the x axis has the wave function:

$$y\ (x, t) = A \sin\,(\omega t - kx + \varphi)$$

where:

$y\,(x, t)$	is the displacement at position x at time t
$\omega = 2\,\pi f$	is the angular frequency
$k = 2\pi/\lambda$	is the wave number
φ	is the initial phase, and
A	is the amplitude.

The term in brackets is called the **phase** of the wave and should be calculated in radians. (The calculator must be switched to radians mode before the sine of the phase is calculated.) The amplitude, A, is the maximum value of the displacement from equilibrium.

Notes

(1) The wave function can be written in terms of cosines by changing the initial phase:

$$y\,(x, t) = A \sin(\omega t - \mathrm{k}x + \varphi) = A \cos(\omega t - \mathrm{k}x + [\varphi - \pi])$$

(2) A wave travelling in the negative x direction has the wave function:

$$y\,(x, t) = A \sin(\omega t + \mathrm{k}x + \varphi)$$

(3) If the wave is transverse (e.g. a stretched string) then the displacement y is perpendicular to the direction of wave propagation. In longitudinal waves (e.g. sound) the displacement y is along the direction of wave propagation.

(4) The speed of the propagating wave is given by:

$$c = f\lambda = \omega/\mathrm{k}$$

(5) The velocity of a particle at position x at time t is found by differentiating the displacement with respect to time while holding x fixed:

$$v\,(x, t) = \frac{\partial y}{\partial t} = \omega A \cos(\mathrm{k}x - \omega t + \varphi)$$

Do not confuse the particle speed, *v*, with the speed of the wave, *c*.

(6) The acceleration of a particle at position x at time t is the derivative of its velocity:

$$a\,(x, t) = \frac{\partial v}{\partial t} = -\,\omega^2 A \sin(\mathrm{k}x - \omega t + \varphi) = -\,\omega^2 y\,(x, t)$$

Thus the passage of the wave causes the particles to undergo simple harmonic motion.

(7) The angular frequency, ω, is the rate at which the phase at a given place changes with time. Thus for a fixed value of x, the phase change from time t_1 to time t_2 is:

$$\omega(t_2 - t_1)$$

(8) The wave number, k, is the rate at which the phase at a given time changes with position. Thus for a fixed value of t, the phase change from position x_1 to position x_2 is:

$$\mathrm{k}\,(x_2 - x_1)$$

(9) You can check by substitution that the **wave equation** given below holds:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Worked example In a certain wave motion $y = A \sin(\mathrm{k}x - \omega t)$, the amplitude is 80.0 mm, the period 2.00 seconds and the velocity of propagation 200 mm s^{-1}.

Find at any given instant:

(a) the wavelength;

(b) the displacement of a particle, P, whose phase is $\pi/3$ radians;

(c) the direction of motion of P;

(d) the phase angle of a particle, Q, 100 mm to the right of P; and

(e) the distances from P of three particles with phase angle of zero.

Solution

(a) $$f = 1/T = 0.5 \text{ Hz}$$

So $$\lambda = \frac{c}{f} = \frac{200}{0.5} = 400 \text{ mm}$$

(b)

Now $$y = A \sin(kx - \omega t)$$

where $$kx - \omega t = \pi/3 \text{ radians for particle P}$$

Thus $$y = 80 \sin(\pi/3)$$
$$= 69.3 \text{ mm}$$

(c) $$v = \frac{dy}{dt} = -A\,\omega \cos(kx - \omega t) = -A\,\omega \cos\left(\frac{\pi}{3}\right)$$

Thus y is +ve and v is –ve

Therefore P must be moving downward towards the equilibrium position.

(d) Now, k $(x_1 - x_2)$ gives the phase change from position x_1 to position x_2 for a fixed value of t.

So $$k(x_1 - x_2) = \frac{2\pi}{\lambda} \times 100 = \frac{2\pi}{400} \times 100 = \frac{\pi}{2}$$

Therefore $$\text{phase angle of } Q = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6} \text{ radians.}$$

(e) For a particle to have zero phase, it must be lagging $\pi/3$ radians behind P. Thus if δx is the distance between the particle and particle P, then

$$\frac{2\pi}{400} \times \delta x = \frac{-\pi}{3}$$

and $$\delta x = -66.7 \text{ mm}$$

Other particles distant from this one by integral multiples of the wavelength must be in phase with it, and therefore, also satisfy the requirements of zero phase, for example at –466.7, –866.7 mm etc. from P.

[There are further solutions for particles on the other side of P at 333 mm etc.]

Problems

(a) Write the equation for a progressive wave moving along the negative x-axis and having amplitude 0.020 m, frequency 550 Hz, velocity 330 m s^{-1}.

(Ans: $y = 0.020 \sin \frac{\pi}{0.30} (x + 330t)$ [Note this is only one form])

(b) When a train of plane waves of wavelength 2.40 m traverses a medium, individual particles execute a periodic motion given by the relation:

$$y = 40.0 \sin \frac{2\pi t}{6} \text{ where the displacement is in mm.}$$

(i) Find the velocity of the wave disturbance.

(ii) Find the difference in phase for two positions of the same particles that are occupied at time intervals 1.00 second apart.

(iii) What is the difference in phase, at any given instant, of two particles 2.10 m apart?

(iv) If the displacement of a certain particle at a certain time is 30 mm, find where it will be 2 seconds later.

(Ans:(a) 400 mm s^{-1}; (b) 60°; (c) 315°; (d) 7.9 or –39 mm)

(c) A sinusoidal transverse progressive wave of amplitude 50.0 mm, wavelength 200 mm, is travelling along the positive direction of the x-axis with a velocity of 4.0 m s^{-1}. At a given instant there is a crest of the wave at the origin. Determine the displacement, velocity and direction of motion of a particle 120 mm to the right of the origin:

(i) at the given instant;

(ii) 1/ 50 second later.

(Ans: (a) – 40.5 mm, 3.69 m s^{-1} upward; (b) +15.5 mm, 5.98 m s^{-1} downward)

Keywords

Transverse wave, longitudinal wave, wave speed, wave pulse, periodic motion, period, frequency, wavelength, amplitude, compression, rarefaction, wave function, phase angle, propagation constant, wave number, angular frequency, wave equation.

Important equations

$$\boxed{c = f\lambda}$$

$$\boxed{y(x, t) = A \sin(\omega t - kx + \varphi)}$$

4.4 Interference of waves

Interference occurs when two waves act on the same point at the same time. The resulting displacement is the **algebraic sum** of the displacements that each wave would cause if acting alone. (This is called the **principle of superposition.**) Algebraically, if y_1 is the disturbance due to wave one and y_2 is the disturbance due to wave two, the resulting disturbance is:

$$y = y_1 + y_2$$

If the two waves are travelling in (almost) the same direction and have the same frequency then they will interfere **constructively** and **reinforce** each other to give a maximum displacement if they have a phase difference of 0, $\pm 2\pi$, $\pm 4\pi$, etc. If the phase difference is π, $\pm 3\pi$, etc. then they will **interfere destructively** and partially or wholly **annul** each other.

Thus if the waves are initially in phase:

- constructive interference occurs when the path difference is 0, λ, 2λ ...
- destructive interference occurs when the path difference is $\lambda/2$, $3\lambda/2$, $5\lambda/2$ ...

Standing waves are an important example of interference between waves travelling in opposite directions (see the sections on stretched strings and organ pipes).

If two waves with similar frequencies travelling in the same direction interfere, then the resultant has an amplitude that varies sinusoidally with a **beat frequency** of:

$$\text{beat frequency} = | f_1 - f_2 |$$

Worked example

The equation

$$y = 2.00 \sin [2\pi (1000t - 0.0100x)]$$

represents a transverse plane wave travelling in the x-direction. Write down the equations of waves that, when combined with the above, will produce annulment.

Solution For complete destruction or annulment the second wave must be identical with the first, but lead (or lag) by π radians, thus its equation will be:

$$y = 2.00 \sin [2\pi (1000t - 0.0100x) \pm \pi]$$

Keywords

Interference, constructive interference, destructive interference, cancellation, reinforcement.

Important equation

$$\boxed{\text{Beat frequency} = | f_1 - f_2 |}$$

4.5 Mechanical properties of waves

4.5.1 The speed of waves

Stretched string (transverse wave)

$$c = \sqrt{\frac{S}{\mu}}$$

S = tension in string
μ = mass per unit length of the string

Fluid (longitudinal wave)

$$c = \sqrt{\frac{B}{\rho}}$$

B = bulk modulus
ρ = density of the fluid

Solid (longitudinal wave)

$$c = \sqrt{\frac{Y}{\rho}}$$

Y = Young's modulus
ρ = density of the fluid

Speed of sound (ideal gas)

$$c = \sqrt{\frac{\gamma p}{\rho}}$$

$$c = \sqrt{\frac{\gamma RT}{M}}$$

γ = ratio of specific heats
p = pressure
ρ = density of the gas
R = universal gas constant
T = absolute temp
M = molecular mass

4.5.2 Rate of energy transfer

Power in a stretched string

The power of a wave on a stretched string is the rate at which energy is transferred by the wave. A wave of amplitude A and angular frequency ω in a string of mass per unit length μ has a power of:

$$P = 2\,\pi^2 A^2 f^2 \mu\, c$$

Intensity

The power transmitted per unit area is called the **intensity**:

$$I = \frac{P}{A}$$

Bulk medium

For a wave (e.g. sound), in a bulk medium of density ρ the power transmitted per unit area is given by:

$$I = 2\,\pi^2 A^2 f^2 \rho\, c$$

Note that for *all* types of waves (including electromagnetic waves such as light and radio waves) the intensity is proportional to $A^2 f^2 c$.

Worked example What is the speed of sound in air (mean molecular mass 28.5 g) at 20°C?

$$c = \sqrt{\frac{\gamma R T}{M}}$$

$$= \sqrt{\frac{1.40 \times 8.31 \times (273 + 20)}{28.5 \times 10^{-3}}}$$

$$= 346 \text{ m s}^{-1}$$

(*Note:* SI units must be used for quantities if the answer is to be in SI units. So T should be in kelvin and M in kilograms.)

Problems

(a) One end of a long, horizontal rope is moved up and down exactly twice per second with a maximum displacement of 30 mm above and 30 mm below its equilibrium position. The string weighs 0.200 kg/m and is under a tension of 50.0 N. Find the amplitude, velocity, frequency and wavelength of the resulting waves. How much energy per second must be supplied to the rope?

(Ans: 30 mm, 15.8 m s^{-1}, 2.00 Hz, 7.9 m, 0.22 W)

(b) Calculate the intensity of a plane sound wave of frequency 500 Hz and amplitude 2.0×10^{-5} m in air of density 1.3 g per litre and at a pressure of 1.00×10^5 Pa ($\gamma = 1.4$ for air).

(Ans: 0.84 W m^{-2})

Keywords

Bulk modulus, intensity, wave speed.

Important equations

$$\boxed{c = \sqrt{\frac{S}{\mu}}}$$

$$\boxed{c = \sqrt{\frac{\gamma p}{\rho}}}$$

4.6 Sound waves

Sound is a longitudinal wave in a bulk medium. It consists of a propagating pattern of **compression** and **rarefaction**. The pressure at point x at time t is given by:

$$p\,(x,\,t) = B\,\mathrm{k}\,A\cos\,(\omega\,t - \mathrm{k}x)$$

where B is the bulk modulus of the medium and A the particle's amplitude.
In an ideal gas:

$$B = \gamma p$$

where γ is the ratio of specific heats and p is the average pressure.
The **pressure amplitude** is:

$$p_{\max} = B\,\mathrm{k}A$$

Sound intensity, I, is the time average rate at which energy is transported across a unit area. Its SI unit is W m^{-2}

$$I = \frac{p_{\max}^2}{2\rho c} = \frac{p_{\max}^2}{2\sqrt{\rho B}}$$

The intensity of sound of frequency f may also be expressed as:

$$I = 2\pi^2 A^2 f^2 \rho\, c$$

In fact, the intensity of all types of waves is proportional to the amplitude squared, the square of the frequency, and directly proportional to the wave speed.

The ear can detect sounds that vary in intensity over many orders of magnitude. So the **sound intensity level,** β, is given in **decibels** by the equation:

$$\beta = 10\log\frac{I}{I_0}\ \mathrm{dB}$$

where I is the sound's intensity and I_0 is a reference level of $I_0 = 10^{-12}$ W m^{-2}. At a frequency of 1000 Hz, I_0 is approximately the threshold of audibility for people with very good hearing. (The threshold of pain is about 120 dB). Prolonged exposure to sound louder than 90 dB will destroy hearing permanently.

Worked example The sound level intensity of a rock band is 105 dB 10 m away from the band. What will be the sound level intensity 60 m away? (Assume that the sound is coming from a point source and is emitted equally in all directions.)

Solution

$$I = \frac{P}{A} = \frac{P}{4\pi r^2}$$

Hence

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}$$

As

$$\beta = 10 \log \frac{I}{I_0}$$

we have

$$105 = 10 \log \frac{I_1}{10^{-12}}$$

therefore

$$I_1 = 3.16 \times 10^{-2}$$

$$I_2 = \frac{3.16 \times 10^{-2} \times 10^2}{60^2} = 8.78 \times 10^{-4}$$

and

$$\beta_2 = 10 \log \frac{8.78 \times 10^{-4}}{10^{-12}} = 89 \text{ dB}$$

Problems

(a) A plane horizontal wave travels down a cylindrical tube of diameter 20 mm, transferring energy at a rate of 1.0 mW. The tube contains hydrogen at STP, which has a molecular weight of 2.00 and $\gamma = 1.40$.

(i) What is the sound intensity?

(ii) What is the sound's intensity level?

(iii) What is the pressure amplitude as a fraction of the total pressure?

(Ans: (i) 3.2 W m^{-2}; (ii) 125 dB; (iii) 2.6×10^{-4})

(b) A small source radiates sound of acoustic energy 1.5 W uniformly in all directions. Find the intensity at a point 25.0 m from the source if there is 10% absorption over that distance.

(Ans: 1.7×10^{-4} W m^{-2})

(c) If two sounds differ in their loudness by 23 dB, what is the ratio of their intensities?

(Ans: 200)

Keywords

Sound, pressure amplitude, sound intensity, sound intensity level, decibel, threshold of audibility, ultrasonic, pitch.

Important equation

$$\boxed{\beta = 10 \log \frac{I}{I_0}}$$

4.7 Waves in stretched strings and pipes

4.7.1 Motion of a stretched string

Boundary conditions

When a wave in a stretched string strikes a discontinuity, part of the wave is reflected and part is transmitted. The phase of the transmitted wave is the same as that of the original wave. If the mass per unit length increases across the boundary then the reflected wave has a phase shift of 180 degrees. If the mass per unit length decreases across the boundary then the reflected wave has no phase shift.

Principle of superposition

If two waves are acting on a point then the displacement is the **algebraic sum** of the displacements that would occur if each wave were acting alone.

Standing waves in strings

If a string of length L is fixed at both ends, then the waves travelling in each direction are reflected at the ends and combine to set up a standing wave (Figure 4.10).

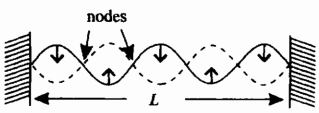


Figure 4.10

The two waves in the string are:

$$y_1 = A \sin (kx - \omega t) \text{ (travelling to the right), and}$$
$$y_2 = A \sin (kx + \omega t) \text{ (travelling to the left)}$$

and the sum is

$$y = y_1 + y_2 = 2A \sin (kx) \cos (\omega t)$$

This **standing wave** has **nodes** every half wavelength where no motion occurs ($kx = 0, \pi, 2\pi, \ldots$) separated by **antinodes** where the displacement is a maximum ($kx = \pi/2, 3\pi/2, \ldots$) (see Figure 4.10).

If there are n antinodes in a string of length L (each separated by $\lambda/2$), then $L = n\lambda/2$ and $\lambda = 2L/n$ and so the frequency of the wave is given by:

$$F_n = \frac{c}{\lambda} = \frac{n}{2L}\sqrt{\frac{S}{\mu}}$$

The lowest possible frequency of vibration (with $n = 1$) is called the **fundamental**, f_1. The other possible frequencies of vibration are called **overtones**. An overtone that has m times the frequency of the fundamental is called the mth **harmonic**:

$n = 1$	f_1	fundamental	
$n = 2$	$f_2 = 2f_1$	1st overtone	2nd harmonic
$n = 3$	$f_3 = 3f_1$	2nd overtone	3rd harmonic

These are the **normal modes of vibration**. They are the only types of vibration in which all the particles of the system oscillate with the same frequency. If you try to excite any other type of motion then it will die away very rapidly.

Worked examples

(a) The equation:

$$y = 2.00 \sin [2\pi (1000t - 0.0100x)]$$

represents a transverse plane wave travelling in the x-direction. Write down the equations of a wave that, when combined with the above, will produce a standing wave.

Solution To produce a standing or stationary wave requires a second wave that is identical with the first but in the opposite direction. This involves changing the '–' to a '+'

$$\therefore \quad y = 2.00 \sin [2\pi (1000t + 0.0100x)]$$

(b) A 4.30 m long piece of thread with a weight of 100 g at one end is passed over a pulley and the other end attached to the prong of a tuning fork (Figure 4.11). When the fork vibrates the string divides into six complete segments. If the weight of the thread is 0.200 g, calculate the frequency of the tuning fork.

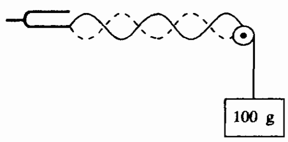


Figure 4.11

Solution Evidently, standing waves have formed on the thread, with each segment being of length 4.30/ 6 m. Therefore the wavelength of component waves is:

$$\lambda = 2 \times \frac{4.30}{6}$$

(since $\lambda/2$ = distance between two successive nodes). Now the tension in the thread equals the weight of the mass at the end of the string. Thus the velocity of waves on the thread is:

$$c = \sqrt{\frac{S}{\mu}} = \sqrt{\frac{0.100 \times 9.8}{2.00 \times 10^{-4}/4.30}}$$

$$= 145.2 \text{ m s}^{-1}$$

Therefore frequency of component waves is:

$$f = \frac{c}{\lambda} = \frac{145.2 \times 3}{4.30} = 101 \text{ Hz}$$

This must be the frequency of the tuning fork that generates the waves.

Problems

(a) Two plane waves of frequency 50 Hz, velocity 200 m s^{-1} and amplitude 0.50 m are travelling in the same direction and have a phase difference of $\pi/4$ radians. Write down the equation of the resultant wave when they are compounded.

(Ans: $y = \sin \pi \left(\frac{x}{2} - 100t + \frac{1}{8}\right) \cos \frac{\pi}{8}$)

(b) A string vibrates according to the equation:

$$y = 50 \sin \frac{\pi x}{3} \cos 40\pi t$$

where x and y are in mm and t is in seconds.

(i) What are the amplitude and velocity of the component waves the superposition of which can give rise to this vibration?

(ii) What is the distance between nodes?

(iii) What is the velocity of a particle of the string at position $x = 15$ mm when $t = 9/8$ s?

(Ans: (i) 25 mm, 0.12 m s^{-1}; (ii) 3.0 mm; (iii) 0 m s^{-1})

4.7.2 Longitudinal standing waves in an organ pipe

The air in contact with a closed end of an organ pipe cannot move, so that end must form a pressure node. Conversely, an open end will form a pressure antinode, a place of maximum variation in pressure.

Closed at one end

If an organ pipe of length L is closed at one end and open at the other end then the standing wave pattern may be schematically represented as in Figure 4.12,

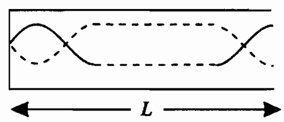


Figure 4.12

so that $L = n\lambda / 4$ with $n = 1, 3, 5, \ldots$ Thus for the normal modes of vibration:

$$\lambda = \frac{4L}{n} \text{ and } f_1 = \frac{nc}{4L}$$

For $n = 1$ we have the fundamental, f_1. The other values of n represent all the *odd* harmonics. Even harmonics are not possible.

Open at both ends

If an organ pipe of length L is open at both ends then the standing wave pattern may be schematically represented in Figure 4.13,

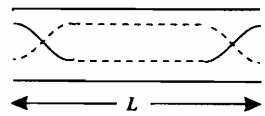


Figure 4.13

so that $L = n\lambda / 2$ with $n = 1, 2, 3, \ldots$ Thus for the normal modes of vibration:

$$\lambda = \frac{2L}{n} \text{ and } f_1 = \frac{nc}{2L}$$

For $n = 1$ we have the fundamental, f_1. The other values of n represent all the harmonics. Note that both even and odd harmonics are possible.

Problems

(a) Find the fundamental frequency and the first four overtones in air of a 0.15 m organ pipe:

(i) if the pipe is open at both ends;

(ii) if the pipe is closed at one end.

(Assume speed of sound is 345 m s^{-1}.)

(Ans:(i) 1150 Hz, 2300 Hz, 3400 Hz, 4600 Hz, 5800 Hz; (ii) 580 Hz, 1700 Hz, 2900 Hz, 4000 Hz, 5200 Hz)

(b) Two sources, 0.20 m apart, send out identical travelling waves having a wavelength of 5.0 mm. Find the shortest perpendicular distance PA where we would expect to get annulment of the waves at P (see Figure 4.14).

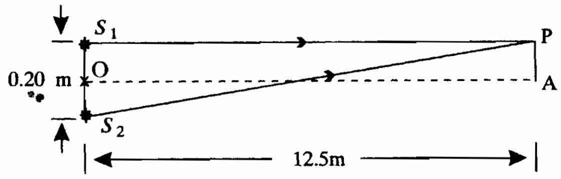


Figure 4.14

OA is the perpendicular bisector of S_1S_2 and equal to 12.5 m.

[*Hint:* Angle POA is small, so you may make some trigonometrical approximations.]

(Ans: 0.16 m)

4.7.3 Resonance

If you make a system undergo **forced oscillation** then the response of the system (i.e. the amplitude of oscillation) will vary dramatically with driving frequency. The frequencies at which the response is a maximum are called the **resonant frequencies**. The normal-mode frequencies are resonant frequencies. They are called the **natural frequencies** of oscillation since if the system is no longer forced, then it soon ends up vibrating in some combination of the normal modes.

Problem A retired opera singer wants to rebuild his shower cubicle so that it will resonate at low C (132 Hz) when he sings in the shower. If the speed of sound in the hot air in the shower is 355 m s^{-1}, how far apart should the walls of the shower cubicle be built?

(Ans: 1.34 m)

Keywords

Boundary conditions, superposition, node, antinode, normal mode, fundamental frequency, overtone, harmonic, pressure node, pressure antinode, resonance, forced oscillator.

Important equation

$$\boxed{y = y_1 + y_2 = 2A \sin(kx)\, cos(\omega t)}$$

4.8 The doppler effect

If a moving source emits a sound with a frequency of f_S, then the frequency of the sound perceived by a listener, f_L, will usually be different. This is called the **doppler effect**. The simplest case to analyse is where the source and listener are heading directly towards each other or directly away from each other. Let the direction from the listener to the source be taken as the positive direction. Then if the velocity of the source with respect to the medium is v_S and the velocity of the listener with respect to the medium is v_L, the emitted and perceived frequencies are related by the formula below. The speed of sound, c, is always taken as positive:

$$\frac{f_L}{c + v_L} = \frac{f_S}{c + v_S}$$

Worked example The whistle of a steam train has a frequency of 1000 Hz. If the steam train, travelling North at 30 m s^{-1}, emits a blast from its whistle, what frequency will be heard on a diesel train heading South at 45 m s^{-1} while the diesel is still North of the steam train? The speed of sound is 335 m s^{-1}.

Solution Rearranging the above equation gives:

$$f_L = \frac{f_S\,(c + v_L)}{c + v_S}$$

The positive direction is from the listener to the source, that is, South in this problem. Thus the diesel's velocity is positive while that of the steam train is negative. Hence:

$$f_L = \frac{1000\,(335 + 45)}{335 - 30} = 1246 \text{ Hz}$$

The sign convention used in deriving in the above formula is that **the positive direction is from the listener to the source.** So this sign convention *must* be adopted in order to use the formula.

Problems The whistle of a steam train has a frequency of 1000 Hz. The steam train is travelling North at 30 m s^{-1} and emits a blast from its whistle. The speed of sound is 335 m/s.

(a) What frequency will be heard on a diesel train heading North at 45 m s^{-1} while the diesel is still South of the steam train?

(b) What frequency will be heard on the diesel after it passes the steam train?

(c) What frequency will be heard by a stationary listener due South of the steam train?

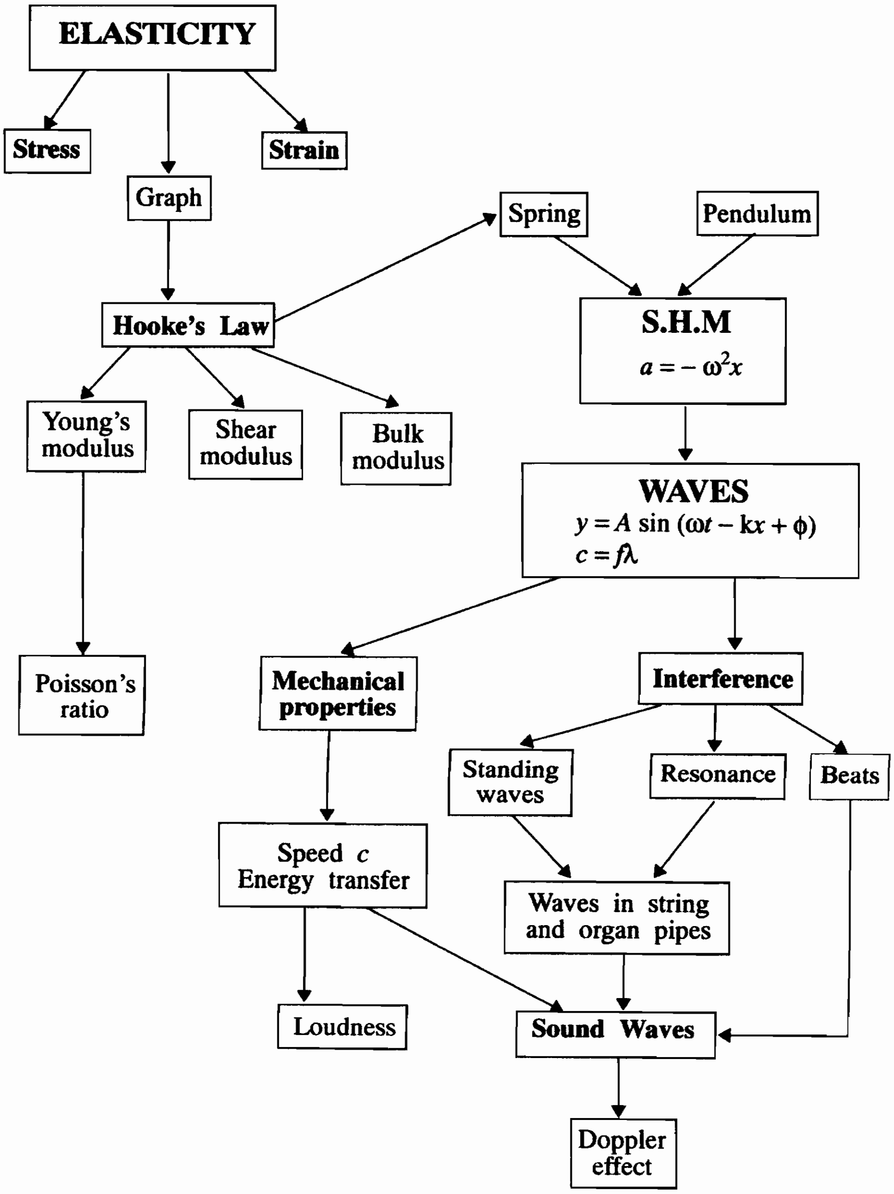
(Ans: (a) 1041 Hz; (b) 951 Hz; (c) 918 Hz)

Keyword

Doppler effect.

Important equation

$$\boxed{f_L = \frac{f_S\,(c + v_L)}{c + v_S}}$$


ELASTICITY
Stress
Strain
Graph
Hooke's Law
Young's modulus
Shear modulus
Bulk modulus
Poisson's ratio
Spring
Pendulum
S.H.M
$a = -\omega^2 x$
WAVES
$y = A \sin(\omega t - kx + \phi)$
$c = f\lambda$
Mechanical properties
Interference
Standing waves
Resonance
Beats
Speed c
Energy transfer
Waves in string and organ pipes
Loudness
Sound Waves
Doppler effect

CHAPTER 5

Electricity

5.1 Electric charge

5.1.1 Electrification of objects – types of charge and the forces between them

It was known by the ancient Greeks that amber ('electron' in Greek), when rubbed with wool or fur, acquires the property of attracting light objects.

By the end of the 17th century, it had been established that there were 'two distinct electricities: vitreous and resinous... and the characteristics of these are such that each repels all of the same kind and attracts the other'.

Today we know these 'electricities' as being due to charge imbalance within atoms, and refer to the two types of charge as positive and negative.

Consistent with the Rutherford and Bohr models, we now know that each atom consists of a very small ($\approx 10^{-14}$ m), relatively massive, positively charged nucleus containing protons and neutrons, with one or more very much lighter negatively charged electrons in orbits ($\approx 10^{-10}$ m diameter) about the nucleus. If the atom is undisturbed (no electron has been added to or removed from the space around the nucleus), the number of electrons is equal to the number of positive charges (protons) in the nucleus.

If some electrons are removed from or added to an atom, the atom acquires an overall charge imbalance (becomes an ion) and is said to be electrically charged.

The basic unit of charge is that of the electron and is equal to 1.6×10^{-19} coulomb (C).

Conservation of charge

Electrifying (charging) an object involves the **transfer** of charge from one object to another; there can be no creation or destruction of charge.

5.1.2 Conductors and insulators

Some materials, called conductors (e.g. gold, copper, aluminium and other metals) permit electric charge to move from one region to another; others, called insulators (e.g. glass, rubber, most plastics) do not.

The attribute that distinguishes conductors and insulators is the nature of the outermost electrons in their atoms: in conductors they are relatively free to move about, while in insulators, they are more tightly bound to their respective nuclei.

Worked example A glass rod is charged by rubbing with a silk cloth. If the total positive charge given to the rod is 10^{-10} C, and there are 10^{24} electrons in the rod, what fraction of the electrons are rubbed off from the glass to the cloth?

Solution

Number of electrons rubbed off $= \dfrac{10^{-10}\ (\text{C})}{1.6 \times 10^{-19}\ (\text{C})} = 6.3 \times 10^{8}$

Therefore fraction rubbed off $= \dfrac{6.3 \times 10^{8}}{10^{24}} = 6.3 \times 10^{-16}$

Problems

(a) What is the total positive charge of all the nuclei in 1 mole of silver? (A silver atom contains 79 electrons.)

(b) What is the total negative charge of all the electrons in 10 g of gold? (A gold atom contains 70 electrons and has an atomic weight of 197.)

(Ans: (a) 7.6×10^{6} C; (b) 3.4×10^{5} C)

Keywords

Electric charge, electron, proton, neutron, ion, coulomb, conductor, insulator.

5.2 Coulomb's Law

The force exerted on a charged body (charge q_1) by another charged body of charge q_2 is given by Coulomb's Law:

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$$

where q_1, q_2 are the charges

r is the distance between the charges

ε_0 is the permittivity of free space = 8.854×10^{-12} C^2 N^{-1} m^{-2}

The constant $\dfrac{1}{4\pi\varepsilon_0} = 9.0 \times 10^9$ N m^2 C^{-2}

Worked example Three charges, q_1, q_2, and q_3 are arranged as shown in Figure 5.1.

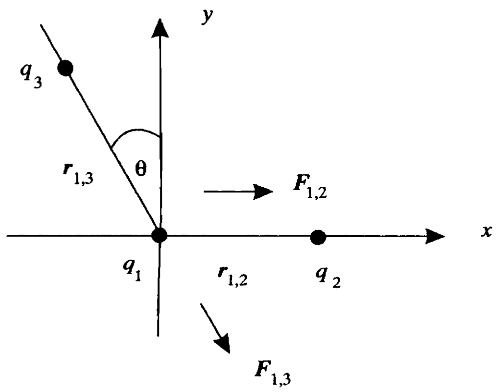


Figure 5.1

$$q_1 = -1.0 \times 10^{-6}\ \text{C}$$

$$q_2 = +3.0 \times 10^{-6}\ \text{C}$$

$$q_3 = -2.0 \times 10^{-6}\ \text{C}$$

$$r_{1,2} = 0.15\ \text{m}$$

$$r_{1,3} = 0.10\ \text{m}$$

$$\theta = 30^\circ$$

What is the force on q_1 due to the other two charges?

Solution The force on q_1 due to q_2 is:

$$F_{1,2} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{(r_{1,2})^2}$$

$$= 9.0 \times 10^9 \times \frac{1.0 \times 10^{-6} \times 3.0 \times 10^{-6}}{(0.15)^2}$$

$$= 1.2\ \text{N}$$

The force on q_1 due to q_3 is:

$$F_{1,3} = 9.0 \times 10^9 \times \frac{1.0 \times 10^{-6} \times 2.0 \times 10^{-6}}{(0.10)^2}$$

$$= 1.8\ \text{N}$$

and their directions are as indicated in Figure 5.2.

The resultant force will be given by resolving into x and y components.

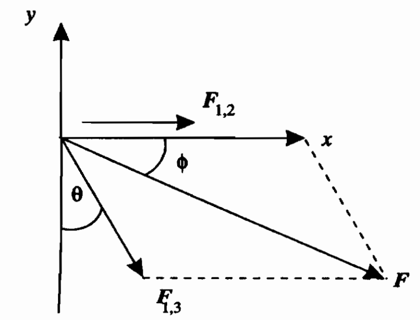


Figure 5.2

Then:

$$F_x = F_{12} + F_{13} \sin\theta$$

$$F_y = -F_{13} \cos\theta$$

$$\therefore\ F_x = 1.2 + 1.8 \sin 30^\circ$$

$$= 2.1\ \text{N}$$

and

$$F_y = -1.8 \cos 30^\circ$$

$$= -1.6\ \text{N}$$

Now

$$F = \left[F_x^2 + F_y^2\right]^{\frac{1}{2}}$$

$$= \left[(2.1)^2 + (1.6)^2\right]^{\frac{1}{2}}$$

$$= 2.6\ \text{N}$$

The direction of F is given by:

$$\tan \varphi = \frac{F_y}{F_x}$$

$$= \frac{-1.6}{2.1}$$

$$= -0.76$$

$$\therefore \quad \varphi = -37^o$$

that is, 37° below the x axis, as defined in Figure 5.2.

Problems

(a) Point charges of 2.0×10^{-9} coulomb are situated at each of three corners of a square of side 0.20 m. What would be the magnitude and direction of the resultant force on a point charge of -1.0×10^{-9} coulomb if it were placed

(i) at the centre of the square;

(ii) at the vacant corner of the square?

(Ans: (i) 9×10^{-7} N towards the middle charge; (ii) 8.6×10^{-7} N towards the diagonally opposite corner)

(b) Two very small spheres each weighing 3.0×10^{-5} N are attached to silk fibres 5.0×10^{-2} m long and hung from a common point.
When the spheres are given equal quantities of negative charge each supporting fibre makes an angle of 30° with the vertical.
Find the magnitude of the charges.

(Ans: 2.2×10^{-9} coulomb)

Keywords

Electric force, Coulomb's Law.

Important equation

$$\boxed{F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}}$$

5.3 Electric field

The electric field at any point, $\boldsymbol{E}$, is the force experienced by a unit charge placed there:

$$\boldsymbol{E} = \frac{\boldsymbol{F}}{q}$$

The units of electric field are N C^{-1} or V m^{-1}.

The field due to a point charge q_1 is:

$$E = \frac{q_1}{4\pi\varepsilon_0 r^2}$$

The field due to a collection of charges is the vector sum of the fields due to each charge.

Note: $\boldsymbol{F} = m\boldsymbol{a} = \boldsymbol{E}q$ enables the acceleration of a charged particle of mass m to be found if it is in a field $\boldsymbol{E}$. From $\boldsymbol{a}$ the velocity and displacement can be found using the equations of rectilinear motion:

$$v = u + at\,,\ s = x_2 - x_1 = ut + \frac{1}{2}at^2 \text{ and } v^2 = v_0^2 + 2as$$

Worked example Two charges of +1.00 μC and −3.00 μC are located on the x-axis as shown in Figure 5.3. The positive charge is at the origin, while the negative charge is at x = 1.00 m. What is the magnitude and direction of the electric field due to these charges at:

(a) $x = 0.50$ m

(b) $x = 1.50$ m?

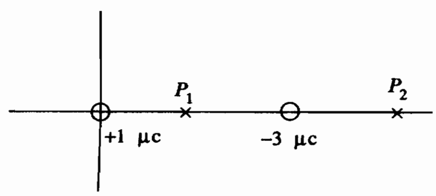


Figure 5.3

Solution Electric field is positive **away from** +1 μC and positive **towards** −3 μC. Thus field directions will be as shown in Figure 5.4,

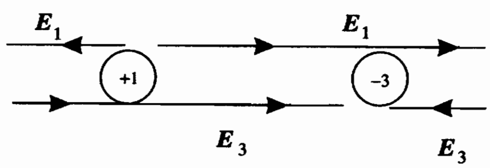


Figure 5.4

where E_1, E_3 are fields due to +1 μC and −3 μC respectively.

Thus, in region $x < 0$: E_1 and E_3 are opposed.

Resultant field, $\quad E = E_3 - E_1$

In region $0 < x < 1$: E_1 and E_3 are in same direction.

Resultant field, $$E = E_1 + E_3$$

In region $1 < x$: E_1 and E_3 are opposed.

Resultant field, $$E = E_1 - E_3$$

(a) at $x = 0.5$ m

$$E = \frac{1}{4\pi\varepsilon_0} \times \frac{1.00 \times 10^{-6}}{(0.5)^2} + \frac{1}{4\pi\varepsilon_0} \times \frac{3.00 \times 10^{-6}}{(0.5)^2}$$

$$= \frac{0.9 \times 10^9 \times 1.00 \times 10^{-6}}{0.25} + \frac{9.00 \times 10^9 \times 3.00 \times 10^{-6}}{0.25}$$

$$= \frac{9.00 \times 10^9 \times 4.00 \times 10^{-6}}{0.25}$$

$$= 1.44 \times 10^5 \text{ N C}^{-1}$$

(b) at $x = 1.5$ m

$$E = \frac{1}{4\pi\varepsilon_0} \times \frac{1 \times 10^{-6}}{(1.50)^2} - \frac{1}{4\pi\varepsilon_0} \times \frac{3 \times 10^{-6}}{(0.50)^2}$$

$$= 9.0 \times 10^9 \times 10^{-6} \left[\frac{1}{(1.50)^2} - \frac{3}{(0.50)^2}\right]$$

$$= 9.0 \times 10^3 \,[0.45 - 12]$$

$$= -1.04 \times 10^5 \text{ N C}^{-1}$$

Problems

(a) Calculate the electric field a perpendicular distance of 3.0 m from the midpoint of a line joining two charges, one +1.0 μC, the other −1.0 μC, which are 10 cm apart.

(Ans: 33 N C^{-1} parallel to axis)

(b) A uniform electric field exists in the region between two oppositely charged plane parallel plates. An electron is released from rest at the surface of the negatively charged plate and strikes the surface of the opposite plate, 20 mm distant from the first, in a time interval of 1.5×10^{-8} s.

(i) Find the electric field intensity. (Mass of electron = 9.11×10^{-31} kg, charge of electron = 1.6×10^{-19} coulomb).

(ii) Find the velocity of the electron when it strikes the second plate.

(Ans: (i) 1.0 kN C^{-1}; (ii) 2.7×10^6 m s^{-1})

Keyword

Electric field

Important equations

$$\boxed{E = \frac{F}{q}}$$

$$\boxed{E = \frac{q_1}{4\pi\varepsilon_0 r^2}}$$

5.4 Gauss' Law

Gauss' Law provides a way to determine the electric field for certain symmetric charge distributions. This law is applied to any closed surface, real or hypothetical (called a Gaussian surface).

The law states that: 'the total electrical flux passing through a closed surface is proportional to the charge enclosed by that surface':

$$\oint E \cdot dA = \frac{Q}{\varepsilon_0}$$

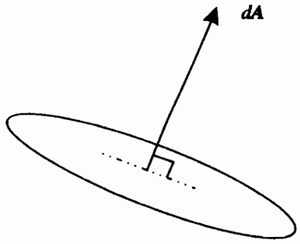


Figure 5.5

Notes:

(1) In Figure 5.5 dA is an element of surface and its direction is perpendicular to that surface.

(2) A common convention used in electrostatics is to denote individual charges by lower case 'q', while a capital 'Q' represents the total amount of charge.

Worked example Use Gauss' Law to find the electric field at a distance 'r' from an infinite plane sheet of charge of surface charge density σ C m^{-2} (Figure 5.6).

Solution Consider a Gaussian surface that is a closed circular cylinder of cross-sectional area A and length h on each side of the sheet.

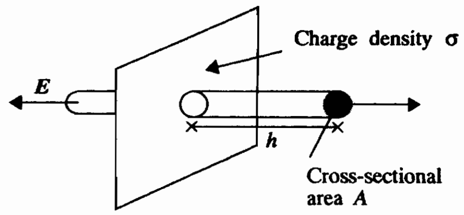


Figure 5.6

At all points $\boldsymbol{E}$ is perpendicular to the surface of the sheet and thus parallel to $d\boldsymbol{A}$.

$$\therefore \ \boldsymbol{E} \cdot d\boldsymbol{A} = E \, dA$$

Also $\boldsymbol{E}$ passes through the end of the cylinder **only**.

Thus
$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q}{\varepsilon_0}$$

becomes
$$E \cdot \oint dA = \frac{Q}{\varepsilon_0}$$

$$\therefore \ E \cdot 2A = \frac{Q}{\varepsilon_0}$$

but charge enclosed
$$Q = \sigma A$$

$$\therefore \ 2EA = \frac{\sigma A}{\varepsilon_0}$$

$$\therefore \ E = \frac{\sigma}{2\varepsilon_0}$$

Problems

(a) Use Gauss' Law to calculate the electric field produced by the following:
   (i) A point charge of $+q$,
   (ii) An infinite sheet of charge of density σ.
   (iii) Two infinite parallel plane sheets of charge separated by a distance d. Both sheets have charge density σ but are of opposite sign.

(Ans: (i) $\frac{q}{4\pi\varepsilon_0 r^2}$; (ii) $\frac{\sigma}{2\varepsilon_0}$; (iii) inside $\frac{\sigma}{2\varepsilon_0}$, outside 0)

(b) A non-conducting sphere of radius R contains a uniform charge density throughout its volume. Using Gauss' Law, find the electric field at a distance r from the centre of the sphere

(i) for $r < R$;

(ii) for $r > R$.

(Ans: (i) $\frac{\rho r}{3\varepsilon_0}$; (ii) $\frac{\rho R^3}{3\varepsilon_0 r^2}$)

Keywords

Electric field, electric flux, Gauss' Law.

Important equations

$$\boxed{E = \frac{F}{q}}$$

$$\boxed{\text{Flux} = \oint E \cdot dA = \frac{Q}{\varepsilon_0}}$$

5.5 Electric potential

5.5.1 Definition

The electric potential difference between two points A and B is:

$$V_{AB} = V_A - V_B = \frac{W_{AB}}{q}$$

which is equal to the work per unit charge required to move a test charge at constant speed from B to A.

There is a strong connection between electric potential and electric field.

We have:

$$F = qE \text{ and } W = \int F \cdot dl$$

$$\therefore \quad W = q\int E \cdot dl$$

$$\therefore \quad V_A - V_B = \int_a^b E \cdot dl$$

The potential V due to a point charge is the work done moving a unit charge from infinity to the point:

$$V = \int_r^\infty E \cdot dl = \int_r^\infty \frac{q\,dr}{4\pi\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0 r}$$

Potential is a scalar quantity. Thus, the potential at a point due to a number of point charges is given by:

$$V = \frac{1}{4\pi\varepsilon_0}\sum \frac{q_i}{r_i}$$

5.5.2 Potential gradient

$$E_x = -\frac{dV}{dx} \text{ (units of V m}^{-1}\text{)}$$

5.5.3 Electron volt

A special unit of energy for electrostatics is the electron volt (eV). This is the energy acquired by an electron when it is accelerated through a potential difference of 1 V:

$$1 \text{ eV} = 1.6 \times 10^{-19} \text{ joules}$$
$$\text{energy} = qv$$
$$1 \text{ eV} = 1.6 \times 10^{-19} \text{ C} \times 1 \text{ V} = 1.6 \times 10^{-19} \text{ J}$$

Worked example Find the work done in assembling the charges as illustrated in Figure 5.7.

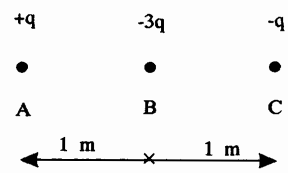


Figure 5.7

Solution Consider no charges in place:

(a) Work done in moving charge $+q$ from ∞ to A is zero (since no electric field exists).

(b) Work done in moving charge $-3q$ from ∞ to B is:

$$W = (-3q) \times \text{potential at B (due to } +q)$$
$$= -3q \times \frac{1}{4\pi\varepsilon_0} \times \frac{q}{d}$$
$$= -\frac{3q^2}{4\pi\varepsilon_0 d}$$

(c) Work done in moving charge $-q$ from ∞ to C is:

$$W = (-q) \times \text{ potential at C (due to } +q \text{ and } -3q)$$
$$= -q \times \left[\frac{1}{4\pi\varepsilon_0} \times \frac{q}{2d} + \frac{1}{4\pi\varepsilon_0} \times \frac{-3q}{d}\right]$$
$$= \frac{1}{4\pi\varepsilon_0} \times \left[\frac{-q^2}{2d} + \frac{3q^2}{d}\right]$$

Thus, total work done to assemble charges is:

$$W = \frac{1}{4\pi\varepsilon_0 d} \times \left[-3q^2 - \frac{q^2}{2} + 3q^2\right]$$
$$= \frac{1}{4\pi\varepsilon_0 d}\left[\frac{-q^2}{2}\right]$$
$$= -\frac{q^2}{8\pi\varepsilon_0 d}$$

Problems

(a) (i) What is the difference in potential between two points 20 and 40 cm distant from a point charge of -60 μC?

(ii) How much work must be done on a +2.0 μC charge to move it from the point of lower potential to that of higher potential?

(Ans: (i) 1.35×10^6 V; (ii) 2.7 J)

(b) A beam of electrons of energy 2000 eV is injected at right angles into an electric field of 5000 V m^{-1}. Calculate the linear deviation of the beam when it has travelled a distance of 10 cm in the field.

(Ans: 0.63 cm)

(c) A small sphere of mass 0.20 g hangs by a thread between two parallel vertical plates 5.0 cm apart. The sphere holds a charge of 6.0×10^{-9} C.
What difference in potential between the plates will cause the thread to assume an angle of 30° with the vertical?

(Ans: 9.4 kV)

Keywords
Potential, electric potential, potential difference, potential gradient, electron volt.

Important equations

$$\boxed{V_{AB} = V_A - V_B = \frac{W_{AB}}{q}}$$

$$\boxed{V_{AB} = \int_A^B E \cdot dl}$$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0} \sum \frac{q_i}{r_i}}$$

$$\boxed{E_x = -\frac{dV}{dx}}$$

5.6 Capacitance

Capacitance is the ratio of the charge Q to the applied potential:

$$C = \frac{Q}{V}$$

The unit of capacitance is the farad, F. Since this is a large unit it is more usual to work with either μF (microfarads, 10^{-6} F) or pF (picofarads, 10^{-12} F).

When connected in parallel, the effect of capacitors is additive (Figure 5.8),

that is $$C_T = \sum_{i=1}^{n} C_i = C_1 + C_2 + C_3 + \ldots$$

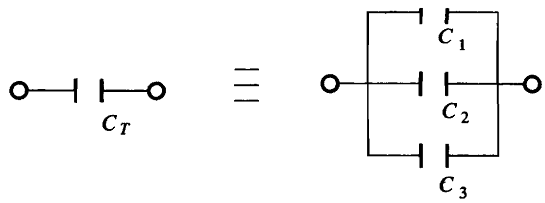


Figure 5.8

Note: ≡ means equivalent to.

When connected in series (Figure 5.9):

$$\frac{1}{C_T} = \sum_{i=1}^{n} \frac{1}{C_i} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots$$

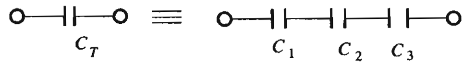


Figure 5.9

5.6.1 Energy storage

The energy stored in a capacitor is given by:

$$U = \int V\,dq = \int_0^Q \frac{q\,dq}{C} = \frac{Q^2}{2C} = \frac{QV}{2} = \frac{1}{2}V^2C$$

Parallel plate capacitor

Two parallel plates of area A separated by a distance d make up a parallel plate capacitor (see Figure 5.10):

$$C = \frac{\varepsilon_0 A}{d}$$

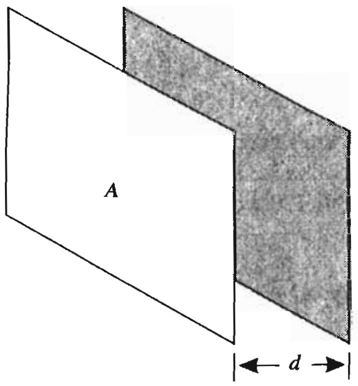


Figure 5.10

Worked example

A potential difference of 300 V is applied to a 2.0 μF and an 8.0 μF capacitor in series.

(a) Find the charge and the potential difference for each capacitor.

(b) If the capacitors are now reconnected with their positive plates together, no external voltage being applied, what will be the charge and potential difference for each capacitor?

Solution (see Figures 5.11 and 5.12)

(a)

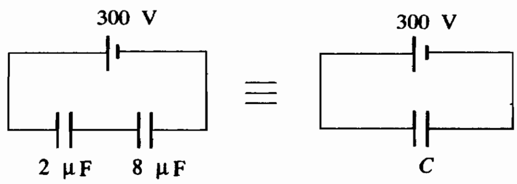


Figure 5.11

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$= \frac{1}{2.0} + \frac{1}{8.0}$$

$$= \frac{5.0}{8.0}$$

$$\therefore \quad C = 1.6\ \mu\text{F}$$

$$Q = CV$$

$$= 1.6 \times 10^{-6} \times 300$$

$$= 4.8 \times 10^{-4}\ \text{C}$$

This charge will reside on *each* capacitor.

Thus for each capacitor:

$$V = \frac{Q}{C}$$

$$\therefore \quad V_2 = \frac{4.8 \times 10^{-4}}{2.0 \times 10^{-6}}$$

$$= 2.4 \times 10^2$$

$$= 240\ \text{V}$$

and

$$V_8 = \frac{4.8 \times 10^{-4}}{8.0 \times 10^{-6}}$$

$$= 0.6 \times 10^2$$

$$= 60\ \text{V}$$

(b)

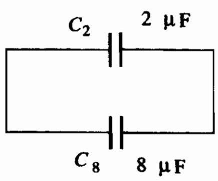


Figure 5.12

The charge on each capacitor in part (a) is 4.8×10^{-4} C.

Thus total charge in circuit is 9.6×10^{-4} C.

This will be redistributed between the two capacitors (now in parallel) under the condition that the voltage across each must be the same.

$$V = \frac{Q}{C}$$

$$\frac{Q_2}{C_2} = \frac{Q_8}{C_8}$$

So

$$Q_2 = \frac{C_2}{C_8} Q_8$$

and

$$Q_2 + Q_8 = 9.6 \times 10^{-4}\ \text{C}$$

$$\therefore \quad Q_8 \left(\frac{C_2}{C_8} + 1 \right) = 9.6 \times 10^{-4}\ \text{C}$$

$$\therefore \quad Q_8 = 7.7 \times 10^{-4}\ \text{C}$$

and

$$Q_2 = 9.6 \times 10^{-4} - 7.7 \times 10^{-4}$$

$$= 1.9 \times 10^{-4}\ \text{C}$$

Also

$$V_{2\,\mu\text{F}} = \frac{Q_2}{C_2} = V_{8\,\mu\text{F}} = \frac{Q_8}{C_8}$$

$$= \frac{1.9 \times 10^{-4}}{2.0 \times 10^{-6}}$$

$$= 95\ \text{V}$$

5.6.2 Dielectrics

Most capacitors are made with some insulating material, called a dielectric, between the conducting plates. This material serves to:

- physically hold the plates apart;
- allow a larger electric field to be formed between the plates before the onset of breakdown, thereby increasing the capacitance.

With air between the plates,

$$C_0 = \frac{A\varepsilon_0}{d}$$

With some other dielectric of permittivity ε,

$$C = \frac{A\varepsilon}{d}$$

$$\therefore \quad \frac{C}{C_0} = \frac{\varepsilon}{\varepsilon_0} = K$$

K is the relative permittivity or dielectric constant. Some typical values are:

air $K = 1.00059$
water $K = 80.4$

Problems

(a) Three capacitors having capacitances of 8.0, 8.0 and 4.0 μF are connected in series across a 12 volt line.

(i) What is the charge on the 4.0 μF capacitor?

(ii) What is the total energy of all three capacitors?

(iii) The capacitors are disconnected from the line and reconnected in parallel with the positively charged plates connected together. What is the voltage of the new combination?

(iv) What is the energy of the combination?

(Ans: (i) 24 μC; (ii) 1.4×10^{-4} joule; (iii) 3.6 V; (iv) 1.3×10^{-4} joule)

(b) A 1.0 μF capacitor and a 2.0 μF capacitor are connected in parallel across a 1200 volt supply line.

(i) Find the charge on each capacitor and the voltage across each.

(ii) The charged capacitors are then disconnected from the line and from each other, and reconnected with terminals of unlike sign together.
Find the final charge on each and the voltage across each.

(Ans: (i)1.2×10^{-3}C, 2.4×10^{-3} C, 1200 V; (ii) 4.0×10^{-4} C, 8.0×10^{-4} C, 400 V)

(c) A capacitor consists of two parallel plates separated by a layer of air 0.40 cm thick, the area of each plate being 202 cm^2.

(i) Compute its capacitance.

(ii) If the capacitor is connected across a 500 volt source, find the charge on each plate, the energy stored in the capacitor, and the electric field strength between the plates.

(Ans: (i) 45 pF; (ii) 22×10^{-9} C, 5.6×10^{-6} J, 1.3×10^{5} V m^{-1})

Keywords

Capacitance, capacitor, farad, parallel-plate capacitor, dielectric, dielectric constant.

Important equations

$$\boxed{C = \frac{Q}{V}}$$

$$\boxed{C_T = \sum_{i=1}^{n} C_i = C_1 + C_2 + C_3 + \ldots}$$

$$\boxed{\frac{1}{C_T} = \sum_{i=1}^{n} \frac{1}{C_i} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots}$$

$$\boxed{U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2}$$

$$\boxed{C = \frac{\varepsilon_0 A}{d}}$$

$$\boxed{K = \frac{\varepsilon}{\varepsilon_0}}$$

5.7 Electric current

5.7.1 Current, I

A current of electricity exists in a conductor whenever electric charge q is being transferred from one point to another in that conductor. If charge is transferred at the uniform rate of 1 coulomb per second, then the constant current existing in the conductor is 1 ampere:

$$I \text{ (current)} = \frac{q \text{ (charge transferred)}}{t \text{ (time taken to transfer this charge)}}$$

This is sometimes written in differential form, $I = \dfrac{dq}{dt}$

5.7.2 Current density, j

$$j = \frac{I}{A} = n_+ q_+ v_+ + n_- q_- v_-$$

v = drift velocity of charge carrier

C = cross -sectional area

n = number of charge carriers per unit volume

q = charge per carrier

5.7.3 Potential difference, *V*

The potential difference V between two points in a conductor is measured by the work W required to transfer unit charge from one point to another. The **volt** is the potential difference (abbreviated pd) between two points in a conductor when 1 joule of work is required to transfer 1 coulomb of positive charge from one point to the other:

$$V \text{ (potential difference)} = \frac{W \text{ (work to transfer charge}}{q \text{ (charge transferred)}}$$

If two points of an external circuit have a potential difference V, then, when a charge q moves from the higher to the lower potential point, the electric field does an amount of work $W = qV$ on the charge.

5.7.4 Electromotive force, ε

An agent such as a battery or generator has an electromotive force (emf) if it does work on the charge moving through it, the charge receiving electrical energy as it moves from the lower to the higher potential side. Emf is measured by the pd (potential difference) between the terminals when the battery or generator is not delivering current. The units of emf are the same as the units of pd, since both are measured by work per unit charge. The SI unit of emf is the volt.

5.7.5 Resistance, *R*

The resistance of a conductor is a property that depends on its dimensions, material and temperature. It determines the current produced in the conductor when a given potential difference is established across it. The **ohm** (Ω) is the resistance of an ohmic conductor in which there is a current of 1 ampere when the potential difference between its ends is 1 volt:

$$R \text{ (resistance)} = \frac{V \text{ (potential difference)}}{I \text{ (current)}}$$

5.7.6 Resistivity, ρ

Resistivity is a characteristic of the material, and is independent of its dimensions:

$$\rho = \frac{E}{j} \text{ (units are } \Omega \text{ m)}$$

Hence, for a cylindrical ohmic conductor of length l and cross-section A, since: $E = \frac{V}{l}$, $j = \frac{I}{A}$ and $R = \frac{V}{I}$

$$\rho = \frac{RA}{l}$$

or

$$R = \rho \frac{l}{A}$$

Conductivity, σ

Conductivity is defined as the reciprocal of resistivity.

$$\sigma = \frac{1}{\rho}$$

Temperature coefficient of resistivity, α

If a resistor is heated, its resistance and resistivity change according to:

$$R_T = R_0 (1 + \alpha \Delta t)$$

$$\rho_T = \rho_0 (1 + \alpha \Delta t)$$

where ΔT is the temperature change from 0°C (where R_0 is the resistance at 0°C) and α is the temperature coefficient of resistivity.

Ohm's Law

From the definition of resistance the value of the steady electrical current I in an ohmic conductor (such as a metal) at a constant temperature is equal to the potential difference V between the ends of the conductor divided by the resistance R of the conductor:

$$I \text{ (current)} = \frac{V \text{ (potential difference)}}{R \text{ (resistance)}}$$

Ohm's Law may be applied to any part of a circuit or to the entire circuit. Thus the potential difference, or voltage drop, across any part of a conductor is equal to the current I in the conductor multiplied by the resistance R of that part, or $V = IR$.

As applied to the entire circuit (containing a source or sources of emf), Ohm's Law states that:

$$\text{Total current } I \text{ in circuit} = \frac{\text{total emf } \varepsilon \text{ in circuit}}{\text{total resistance } R \text{ of circuit}}$$

$$or\ I = \frac{\varepsilon}{R}$$

Measurement of resistance by ammeter and voltmeter

The **current** is measured by inserting in series a (low-resistance) ammeter into the circuit. The **potential difference** is measured by connecting the terminals of a (high-resistance) voltmeter across the resistance being measured, that is, in parallel. The **resistance** is computed by dividing the voltmeter reading by the ammeter reading according to Ohm's Law, $R = V / I$. (If an *exact* value of the resistance is required, the resistances of the voltmeter and ammeter must be considered parts of the circuit.) (See Figure 5.13.)

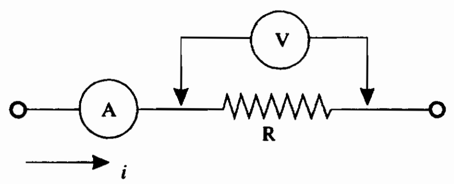


Figure 5.13

5.7.7 Terminal voltage

The terminal voltage of a battery or generator when it delivers a current I is equal to the total electromotive force minus the potential drop (or voltage drop) in its internal resistance r.

- **When delivering current** (on discharge) — terminal voltage = emf – voltage drop in internal resistance = $\varepsilon - Ir$.
- **When receiving current** (on charge) — terminal voltage = emf + voltage drop in internal resistance = $\varepsilon + Ir$.
- **When no current exists** — terminal voltage = emf of battery or generator.

5.7.8 Electrical energy, heat and power

In a circuit, the work W done in transferring a charge q between two terminals having a potential difference V is:

$$W = qV = (It)V = IVt$$

By Ohm's Law, $V = IR$ so $IV = I^2R$. Thus the electrical energy converted into heat in an ohmic conductor of resistance R carrying a current I is:

$$W = I^2 Rt$$

which is called Joule's Law of Heating.

Since average power $P = \frac{W}{t} = \frac{IVt}{t} = IV$, using Ohm's Law we obtain

$$P = IV = I^2R = \frac{V^2}{R}$$

The unit of power is the watt (W).

Worked examples

(a) A steady current of 6.0 A is maintained in a metallic conductor. What charge q is transferred through it in 1.0 minute?

Solution

$$q = It = 6.0 \text{ A} \times 60 \text{ s} = 360 \text{ C}$$

(b) A current of 6.0 A exists in a copper wire of cross-section 0.050 cm^2. Calculate the average drift speed of the active electrons in the wire, assuming each atom of the metal contributes one electron to the conduction process. Copper has density 8.92 g cm^{-3} and atomic weight 63.5.

Solution

Mass of 1 cm length of wire = volume × density

$$= 0.050 \text{ cm}^3 \times 8.92 \text{ g cm}^{-3} = 0.446 \text{ g}$$

A mass of 63.5 g Cu contains 6.02×10^{23} atoms and 6.02×10^{23} free electrons.

$$\text{Free electrons / cm of wire} = \frac{0.446 \text{ g / cm}}{63.5 \text{ g}} \times 6.02 \times 10^{23} \text{ electrons}$$

$$= 4.23 \times 10^{21} \text{ electrons}$$

$6 \text{ A} = 6 \text{ C s}^{-2} = 6 \times (6.24 \times 10^8 \text{ electrons}) \text{ s}^{-1} = 0.375 \times 10^{20} \text{ electrons s}^{-1}$ passing through a given section of wire.

$$\text{Drift speed of active electrons} = \frac{0.375 \times 10^{20} \text{ electrons s}^{-1}}{4.23 \times 10^{21} \text{ electrons cm}^{-1}}$$

$$= 8.9 \times 10^{-3} \text{ cm s}^{-1} = 89\ \mu\text{m s}^{-1}$$

(c) Compute the resistance of a hardened copper rod 12 m long and 8.0 mm in diameter if the resistivity of the material is $1.756 \times 10^{-8}\ \Omega$ m.

Solution

$$R = \rho\frac{l}{A} = 1.756 \times 10^{-8}\ \Omega\ \text{m} \times \frac{12\ \text{m}}{0.25\pi\ (8.0 \times 10^{-3})\ \text{m}^2} = 4.2 \times 10^{-3}\ \Omega$$

(d) The resistance of a platinum thermometer is 6.00 Ω at 30°C. Determine its resistance at 100°C. The temperature coefficient of resistance of platinum is 0.00392 $(°\text{C})^{-1}$.

Solution

To find R_0 (R at 0°C): $R_T = R_0\,(1 + \alpha\Delta T)$

$$6 = R_0\,(1 + 0.00392 \times 30)$$

$$R_0 = 5.37\ \Omega$$

$$R_{100} = R_0(1 + \sigma\Delta T) = 5.37\ \Omega \times (1 + 0.00392°\text{C}^{-1} \times 100°\text{C}) = 7.48\ \Omega$$

Note that it is incorrect to use R_{30} for initial resistance and ΔT for the temperature change; doing so would lead to an answer of:

$$R_T = 6.00\ \Omega \times (1 + 0.00392°\text{C}^{-1} \times 70°\text{C}) = 7.65\ \Omega\ \text{(incorrect)}$$

(e) What is the potential drop across an electric hotplate which draws 5.0 amperes from the line when its hot resistance is 24 ohms?

Solution

$$V = IR = 5.0\ \text{A} \times 24\ \Omega = 120\ \text{V}$$

(f) The ammeter–voltmeter method is used to measure an unknown resistance R (see Figure 5.14). An ammeter (A) connected in series with the resistance reads 0.8 A. A voltmeter (V) placed across the ends of the resistance reads 1.5 V. Compute the value of the resistance R. Neglect any errors caused by the instruments.

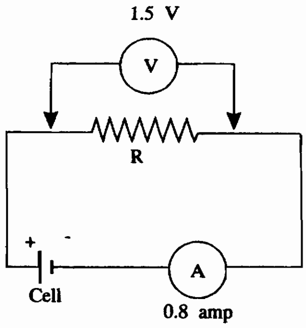


Figure 5.14

Solution

$$R = \frac{V}{I} = \frac{1.5\ \text{V}}{0.8\ \text{A}} = 1.87\ \Omega$$

(g) Compute the work and the average power required to transfer 96 000 C in one hour through a potential difference of 50 V.

Solution

$$\text{Work } W = qV = 96\ 000\ \text{C} \times 50\ \text{V} = 4.8 \times 10^8\ \text{J}$$

$$\text{Power } P = \frac{W}{T} = \frac{4.8 \times 10^6\ \text{J}}{3.6 \times 10^3\ \text{s}} = 1.33 \times 10^3\ \text{W} = 1.33\ \text{kW}$$

Problems

(a) Compute the time required to pass 36 000 C through an electroplating bath using a current of 5 A.

(Ans: 2 h)

(b) Determine the potential difference between the ends of a wire of resistance 5 Ω if 720 C pass through it per minute.

(Ans: 60 V)

(c) A length of wire has resistance 12.64 Ω at 30°C and 11.22 Ω at 0°C. Compute:

(i) the temperature coefficient of resistance;
(ii) its resistance if heated to 300°C.

(Ans: (i) 4.2×10^{-3} per °C; (ii) 25.42 Ω)

(d) How much resistance will a wire 150 m long have if its cross-sectional area is 1.5 mm^2 and its resistivity is $4.6 \times 10^{-4}\ \Omega$ m?

(Ans: $R = 4.6 \times 10^4\ \Omega$)

(e) If a current of 100 A is flowing through a silver conductor of 1 cm^2 cross-sectional area, calculate the drift velocity that has been imposed on the free electrons in the silver.

(Atomic weight of silver = 108, valency = 1, density = 10 500 kg m^{-3})

(Ans: $V_d = 10^{-4}$ m s^{-1})

(f) A nichrome heater dissipates 500 W when the applied potential difference is 110 V and the wire temperature is 800°C. How much power would it dissipate if the wire temperature were held at 200°C by immersion in a bath of cooling oil? (The applied potential difference remains the same and the mean temperature coefficient of resistivity for nichrome is 4×10^{-4} °C^{-1}.)

(Ans: 611 W)

(g) A cell has emf 1.54 V. When it is in series with a 1.0 Ω resistance, the reading of a voltmeter connected across the cell terminals is 1.40 V. Determine its internal resistance.

(Ans: 0.10 Ω)

Keywords

Current, current density, potential difference, electromotive force, resistance, resistivity, Ohm's Law, temperature coefficient of resistivity, ammeter, voltmeter, internal resistance, energy, power.

Important equations

$$\boxed{I = \frac{Q}{t}}$$

$$\boxed{J = \frac{I}{A}}$$

$$\boxed{R = \frac{V}{I}}$$

$$\boxed{R = \frac{\rho l}{A}}$$

$$\boxed{R_T = R_0\,(1 + \alpha\Delta T)}$$

$$\boxed{\rho_T = \rho_0\,(1 + \alpha\Delta T)}$$

$$\boxed{V = \varepsilon - Ir}$$

$$\boxed{W = IVt = I^2Rt}$$

$$\boxed{P = IV = I^2R = \frac{V^2}{R}}$$

5.8 Circuit theory

5.8.1 Sign convention

We are concerned with the change in potential as one moves around a loop in a specified direction. Hence in terms of the diagram in Figure 5.15, the potential change is positive ($+\varepsilon$) if one moves from A to B, and negative ($-\varepsilon$) if one moves from B to A:

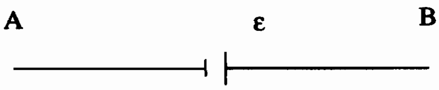


Figure 5.15

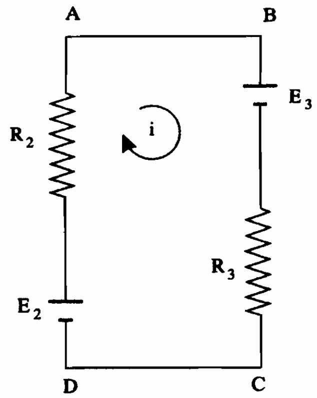


Figure 5.16

As current flows from higher to lower potential, the potential change across a resistor is negative if moving in the direction of the current.

A normal circuit consists of a collection of resistances and sources of emf connected by conductors of negligible resistance (see Figure 5.16). The analysis of these circuits will enable us to determine the current passing through any component of the circuit or to determine the potential across any two points in the circuit.

In a series circuit as shown in Figure 5.17:

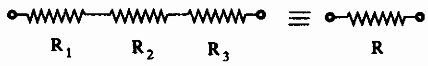


Figure 5.17

- **Resistance,** $R = R_1 + R_2 + R_3 + \ldots$
  where R is the equivalent resistance of a series combination of conductors having resistances $R_1, R_2, R_3, \ldots$
- **Potential difference** The total potential difference across several resistors connected in series is equal to the sum of the potential differences across the separate resistors.
- **Current** Current is the same in every part of the series circuit.

In a parallel circuit as in Figure 5.18:

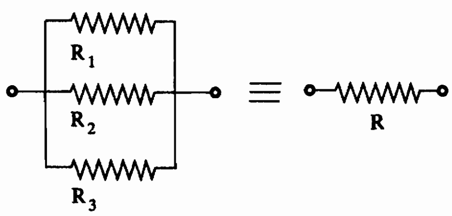


Figure 5.18

- **Resistance,** $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots$
  where R is the equivalent resistance of a parallel combination of conductors having resistances R_1, R_2, R_3, … R is always less than the smallest of the individual resistances. Connecting additional resistors in parallel decreases the joint resistance of the combination.
- **Potential difference** The potential difference across several resistors in parallel is the same as that across each of the resistors. The potential difference is the same across all branches.
- **Current** The sum of the currents in the branches is equal to the value of the current flowing in. Current values in the different branches vary inversely as the resistances of the branches.

When a circuit consists of only one source of emf then the formulae above can be simply applied to give a complete solution to the circuit.

Worked examples

(a) For the circuit shown in Figure 5.19, determine the current I through the battery.

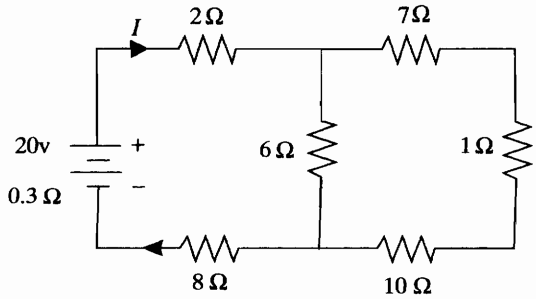


Figure 5.19

Solution

The 7.0, 1.0 and 10.0 Ω resistors are in series so their combined resistance is 18 ohms. Then 18.0 Ω is in parallel with 6.0 Ω so their combined resistance is:

$$\frac{1}{R_1} = \frac{1}{18} + \frac{1}{6}$$
$$R_1 = 4.5\ \Omega$$

Hence the equivalent resistance of the entire circuit is

$$4.5 + 2.0 + 8.0 + 0.3 = 14.8\ \Omega$$

and the battery current is

$$I = \frac{\varepsilon}{R} = \frac{20}{14.8} = 1.35\ \text{A}$$

(b) **For the circuit shown in Figure 5.20, determine the voltage across and current through every resistance.**

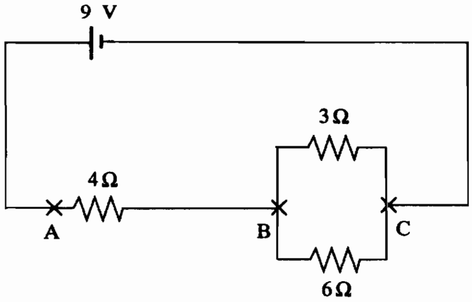


Figure 5.20

Solution

$$\frac{1}{R_{BC}} = \frac{1}{3.0} + \frac{1}{6.0}$$

$$\therefore \; R_{BC} = 2\,\Omega$$

$$R_{total} = R_T = R_{AB} + R_{BC} = 6.0\,\Omega$$

$$I_T = \frac{V_T}{R_T} = 1.5\text{ A } \;(= I_{4\,\Omega})$$

$$V_{AB} = I_T R_{AB} = 1.5 \times 4.0 = 6.0\text{ V } (V_{4\,\Omega})$$

$$V_{BC} = I_T R_{BC} = 1.5 \times 2.0 = 3.0\text{ V}$$

$$(\therefore \; V_{3\,\Omega} = 3.0\text{ V and } V_{6\,\Omega} = 3.0\text{ V})$$

$$I_{3\,\Omega} = \frac{V_{BC}}{R_{3\,\Omega}} = 1.0\text{ A}$$

$$I_{6\,\Omega} = \frac{V_{BC}}{R_{6\,\Omega}} = 0.5\text{ A}$$

Problems

(a) **How much current will pass through the 10 Ω resistance in the circuit shown in Figure 5.21?**

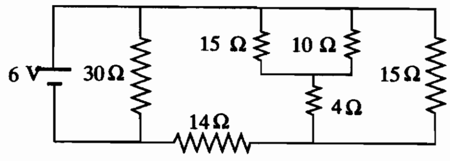


Figure 5.21

(Ans: 0.11 A)

(b) Three resistors of 40, 60 and 120 Ω are connected in parallel and this parallel group is connected in series with 15 and 25 Ω. The whole system is then connected to a 120 volt source. Determine

(i) the current in the 25 Ω resistor;

(ii) the potential drop in the parallel group;

(iii) the potential drop in the 25 Ω resistor;

(iv) the current in the 60 Ω resistor;

(v) the current in the 40 Ω resistor;

(Ans: (i) 2.0 A; (ii) 40 V; (iii) 50 V; (iv) 0.67 A; (v)1.0 A)

Keywords

Series resistors, parallel resistors, circuit.

Important equations

$$\boxed{R = \frac{V}{I}}$$

$$\boxed{\begin{aligned} &R = R_1 + R_2 + R_3 + \dots \\ &\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots \end{aligned}}$$

5.9 Kirchhoff's laws

5.9.1 Multi-loop, multi-source electrical circuits

When there is **more than one source of emf** within a circuit, particularly when the circuit is a 'multiloop' circuit, then the complete solution can be a lot more complicated and tedious. To solve this sort of circuit we may use Kirchhoff's laws.

5.9.2 Kirchhoff's First Law

The algebraic sum of all the currents entering any junction of conductors in the circuit is equal to zero:

$$\sum I = 0$$

(If this were not so then there would be a net gain or loss of charge at some points in the circuit.)

Often the positive direction of current in each branch is chosen arbitrarily and indicated by arrows as in Figure 5.22.

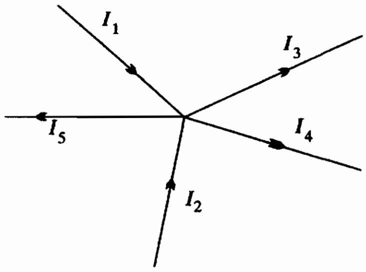


Figure 5.22

In this case, Kirchhoff's First Law becomes

$$I_1 + I_2 = I_3 + I_4 + I_5$$

5.9.3 Kirchhoff's Second Law

The net potential **change** around any closed loop is zero.

Consider a current as shown in Figure 5.23.

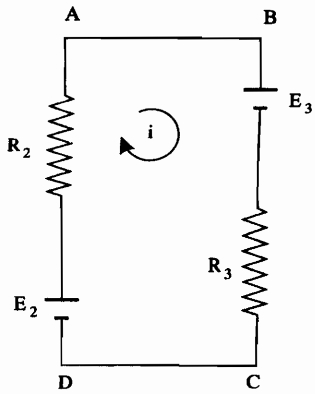


Figure 5.23

We arbitrarily assume that the current flows in a clockwise direction. (If this is the 'wrong' decision, a negative value of I will come out of the arithmetic; this is interpreted as a counter-clockwise current.)

Going around the loop A B C D A we obtain:

$$-E_3 - IR_3 + E_2 - IR_2 = 0$$

More generally, we say $$\Sigma\,\varepsilon - \Sigma\, IR = 0$$
or $$\Sigma\,\varepsilon = \Sigma\, IR$$
where the right-hand side means 'the sum of the potential drops', for example in Figure 5.24, where the directions of the current have been arbitrarily assigned:

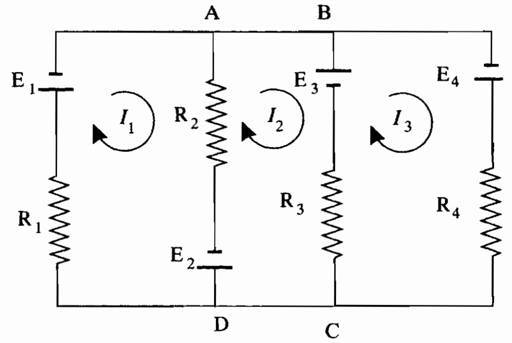


Figure 5.24

Around loop ABCDA

$$-E_2 - E_3 = (I_2 - I_3)\, R_3 + (I_2 - I_1)\, R_2$$

5.9.4 Solving multi-loop circuits

Kirchhoff's laws can be used in one of the following ways to solve multiloop circuits:

- **Loop currents** This method uses Kirchhoff's First Law indirectly and his Second Law directly.
- Direct use of Kirchhoff's two laws.

Circuit solution by loop currents

Method:

(1) Divide circuit into 'loops'.

(2) Assign one current to pass around each loop.

(3) Sum the potential drops around each loop.

(4) Solve the simultaneous equations for all the unknown currents.

Worked examples

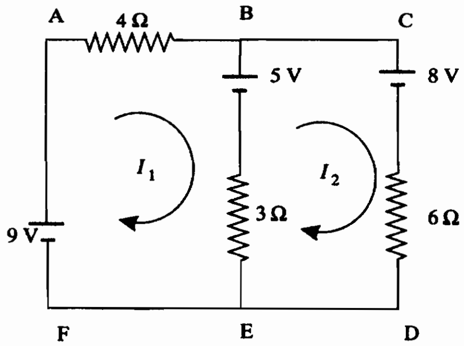


Figure 5.25

For loop ABEFA
$$-5 + 9 = 4I_1 + 3I_1 - 3I_2$$
$$4 = 7I_1 - 3I_2 \qquad [1]$$

For loop BCDEB
$$-8 + 5 = 6I_2 + 3I_2 - 3I_1$$
$$-3 = 9I_2 - 3I_1 \qquad [2]$$

Multiply equation [1] by 3
$$12 = 21I_1 - 9I_2 \qquad [3]$$

Current through 4 Ω resistor:

Add [2] and [3]
$$18I_1 = 9$$
$$I_1 = 0.5\text{A}$$

Current through 6 Ω resistor:

Substitute $I_1 = 0.5$ A in equation [1]

$$4 = 3.5 - 3I_2$$

$$-3I_2 = 0.5$$
$$I_2 = -\frac{1}{6}\text{ A} = 0.2\text{ A}$$

Current through 3 Ω resistor:
$$= I_1 - I_2$$
$$= I_1 - \frac{-1}{6}$$
$$= \frac{3+1}{6}$$
$$= \frac{4}{6}$$
$$= \frac{2}{3}\text{ A} = 0.7\text{ A}$$

Direct use of Kirchhoff's two laws

When there is an unknown resistance or emf, this method must be used. (It can also be used quite generally.) The method is shown in the following worked example.

Worked example

Given that the current through R is 0.50 A and other currents and emfs are as shown in Figure 5.26, determine the resistance R and the emf ε.

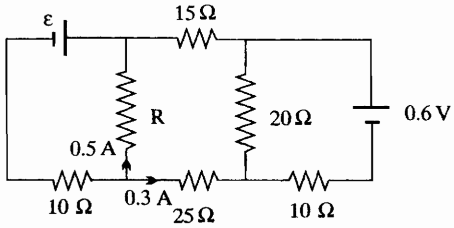


Figure 5.26

Solution

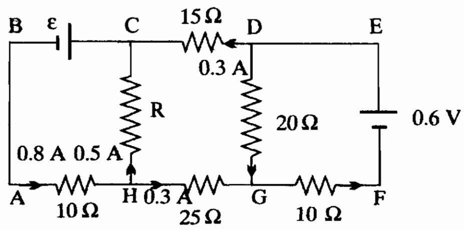


Figure 5.27

Current from A to H = 0.50 + 0.30 = 0.80 A.

Let I = current (DG).
Hence current (GF) = I + 0.30 A (at junction G)
and current (DC) = 0.30 A (at junction D)

In loop DGFE:

$$\Sigma\varepsilon = \Sigma IR$$

$$0.6 = 20I + 10(I + 0.30)$$

$$I = -\frac{2.4}{30} = -0.80 \text{ A}$$

(so current flows from G to D)

In loop CHGD

$$\Sigma\varepsilon = \Sigma IR$$

$$0 = R(-0.50) + 25(0.30) + 20(0.80) + 15(0.30)$$

$$R = 56\ \Omega$$

In loop BAHC

$$\Sigma\varepsilon = \Sigma IR$$
$$-\varepsilon = 10 \times 0.80 + 56 \times 0.50$$
$$\varepsilon = -36 \text{ V}$$

Problems

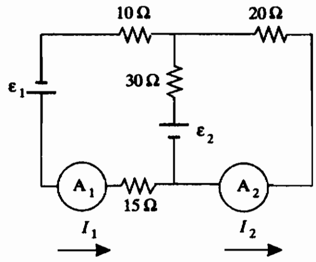


Figure 5.28

(a) Refer to figure 5.28. The ammeters A_1, A_2 (which have negligible resistance) show currents $I_1 = 0.30$ A, and $I_2 = 0.20$ A respectively. Determine the unknown emfs ε_1, ε_2 and interpret the sign of the answers.

(Ans: +11.5 V; –1.0 V)

(b) Calculate the current I_1 passing through the 30 Ω resistance in the circuit shown in Figure 5.29.

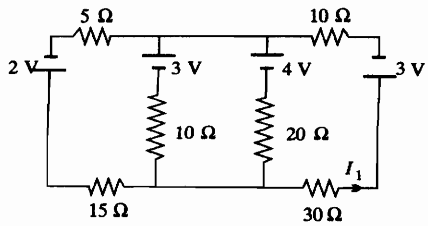


Figure 5.29

(Ans: –0.11 A)

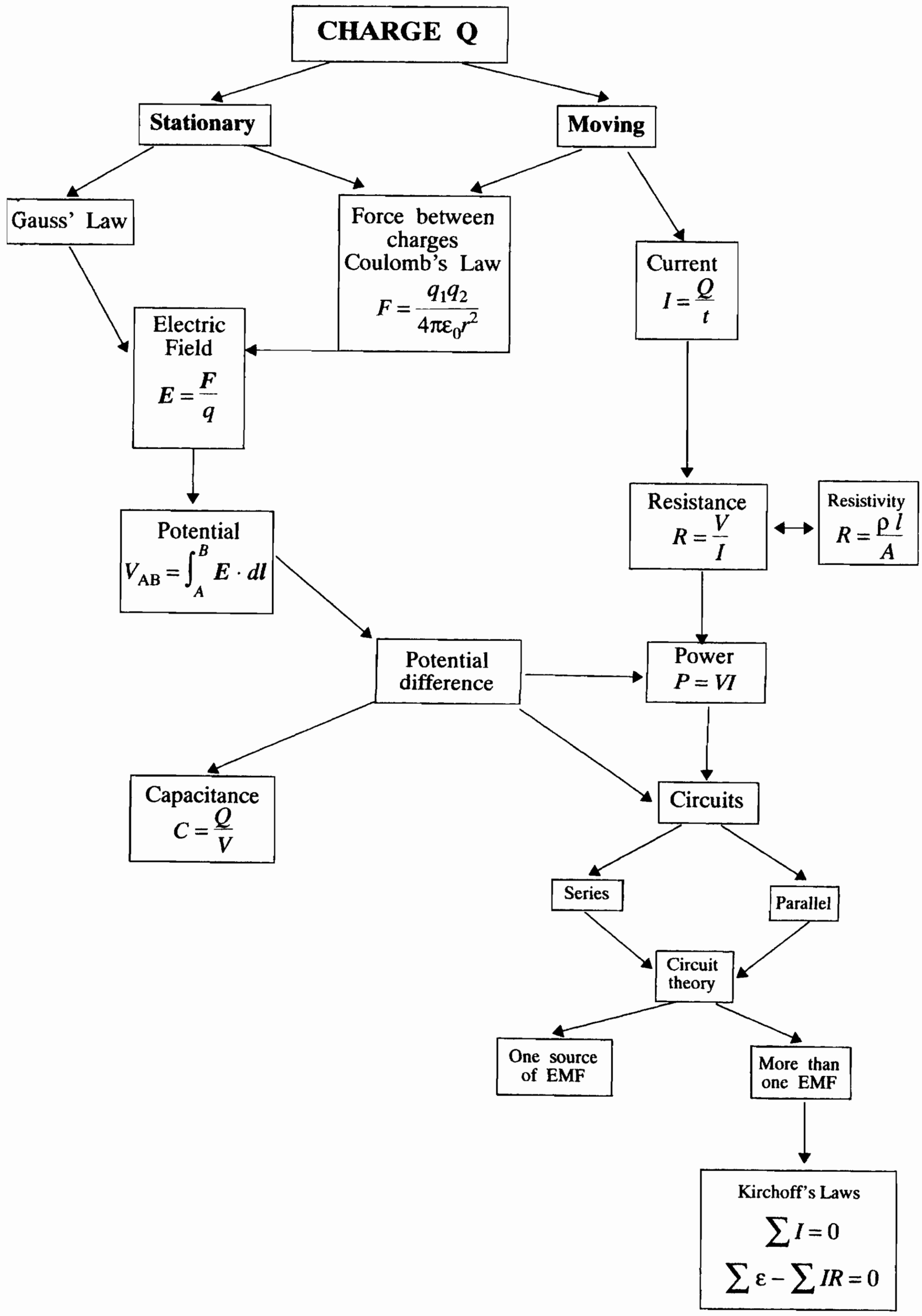
Keywords

Kirchhoff's loop rule, Kirchhoff's point rule.

Important equations

$$\boxed{\Sigma I = 0}$$

$$\boxed{\Sigma\varepsilon - \Sigma IR = 0}$$


CHARGE Q
Stationary
Moving
Gauss' Law
Force between charges Coulomb's Law
$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$
Current
$I = \frac{Q}{t}$
Electric Field
$E = \frac{F}{q}$
Potential
$V_{AB} = \int_A^B E \cdot dl$
Resistance
$R = \frac{V}{I}$
Resistivity
$R = \frac{\rho l}{A}$
Potential difference
Power
$P = VI$
Capacitance
$C = \frac{Q}{V}$
Circuits
Series
Parallel
Circuit theory
One source of EMF
More than one EMF
Kirchoff's Laws
$\sum I = 0$
$\sum \varepsilon - \sum IR = 0$

CHAPTER 6

Electromagnetism

6.1 Magnetic field and magnetic forces

6.1.1 Magnetic field

A moving charge or a current sets up or creates a magnetic field in the space surrounding it. However, magnetic properties were observed at least 2000 years ago in pieces of iron ore. Such 'permanent magnets' were found to orient themselves in the earth's 'magnetic field' where the earth's South geographic pole is close to the earth's magnetic North pole and the North geographic pole is close to the magnetic South pole (see Figure 6.1).

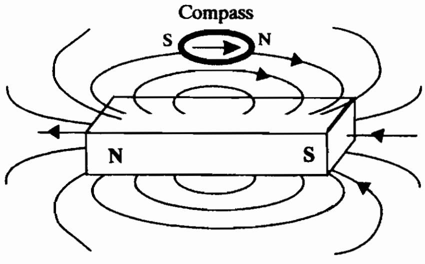


Figure 6.1

6.1.2 Magnetic field lines

The lines shown in Figure 6.1 (a bar magnet) indicate the direction of the magnetic field vector, B, and are described as 'magnetic field lines'. They show the direction that a unit north pole (monopole) would move if such a monopole could exist.

The unit of B is the **tesla**.

6.1.3 Magnetic flux

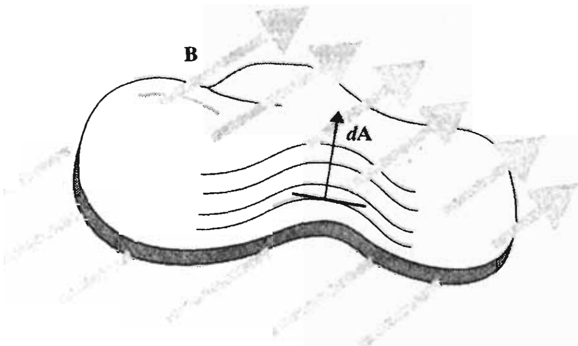


Figure 6.2

The magnetic flux, Φ, through a surface, of area A, is as shown in Figure 6.2.

$\boldsymbol{dA}$ is a vector of length dA and in a direction perpendicular to the surface dA at that point.

$$\Phi = \int \boldsymbol{B} \cdot \boldsymbol{dA}$$

for B uniform over plane surface A

$$\Phi = B_{\perp}A = BA \cos \theta$$

The unit of Φ is the **weber**.

6.1.4 Motion of charged particles in a magnetic field

The magnetic field lines do not show the direction that a unit positive charge would move (unlike the electric field lines) because the force experienced by charge q moving at velocity $\boldsymbol{v}$ in magnetic field $\boldsymbol{B}$ is given by the cross-product:

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$$

In other words, the force experienced by such a moving charge is at right angles to the directions of both $\boldsymbol{v}$ and $\boldsymbol{B}$.

If both a magnetic field, $\boldsymbol{B}$, and an electric field, $\boldsymbol{E}$, are present, the force on the charged particle is the vector sum:

$$\boldsymbol{F} = q\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

For a charged particle moving in a magnetic field, the nature of the cross-product determines that the force, and hence acceleration, of the particle is always at right angles to the velocity — a property of circular motion. The path of the charged particle q, of mass m, moving with constant velocity $\boldsymbol{v}$ in a constant magnetic field, $\boldsymbol{B}$, perpendicular to $\boldsymbol{v}$ is a circle of radius R such that:

$$F = qvB = m\left(\frac{v^2}{R}\right)$$

$$\text{Hence } R = \frac{mv}{Bq}, \quad T = \frac{2\pi m}{qB}$$

where T is the period of the motion.

6.1.5 Magnetic force on a conductor

A current flowing through a wire is equivalent to a flow of positive charge in the direction of the current. Therefore the force experienced by a current element $\boldsymbol{dl}$ in a magnetic field $\boldsymbol{B}$, is:

$$\boldsymbol{F} = i\boldsymbol{dl} \times \boldsymbol{B}$$

6.1.6 Force and torque on a current loop

Consider a rectangular current loop in a uniform magnetic field, as in Figure 6.3.

Let the current loop abcd lie with its plane twisted so that its normal N lies at angle θ to B.

The net force on the loop is zero and forces on sides ad and bc act through the same line of action, thus having no turning effect. The forces on ab and cd, however, have lines of action offset by distance h and therefore contribute a resultant torque on the loop.

This torque may be calculated as:

$$|\Gamma| = 2|\boldsymbol{F}_{ab}|\,h = 2|\boldsymbol{F}_{ab}|\,\frac{1}{2}\,w \sin\theta$$

$$= I\,l\,B\,w \sin\theta = I \text{ Area B} \sin\theta$$

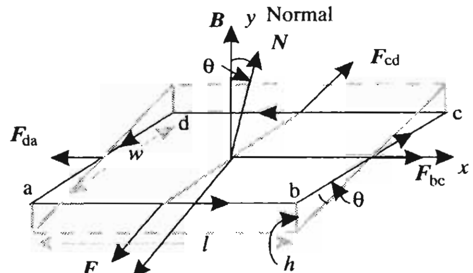


Figure 6.3

For an N-turn loop the torque will be N times the magnitude of the torque on a single-turn loop.

If we define the ***magnetic moment*** m of the single loop as a vector with direction normal to the plane of the loop and of magnitude:

$$m = IA$$

then magnetic moment $\boldsymbol{m} = NiA$

and the torque may be calculated as: $\Gamma = \boldsymbol{m} \times \boldsymbol{B} = NiAB \sin \theta$

Such a current loop is termed a 'magnetic dipole' of magnetic moment $\boldsymbol{m}$.

The torque developed when a current is sent through a coil in a magnetic field is the basis of the operation of the direct current motor.

It may be determined that the potential energy when a magnetic dipole of magnetic moment $\boldsymbol{m}$ is inclined to the magnetic field vector $\boldsymbol{B}$ is:

$$U_B = -\boldsymbol{m} \cdot \boldsymbol{B}$$

6.1.7 The Hall effect

Because of the force experienced by moving charges, the nature of the charges involved in carrying current through a conductor can be determined. The force experienced by positive charges moving in one direction is the same as that experienced by negative charges moving in the opposite direction. Hence, there will be a charge build-up of the charge carriers on one side of the conductor and there will be a net negative charge if the negative carriers are dominant, a net positive charge if the positive carriers are dominant, and zero if both carry charge equally. It is found that a definite potential exists across any metal in such a direction as to conclude that the negative carriers are the dominant carriers in electric current through a metal (see Figure 6.4).

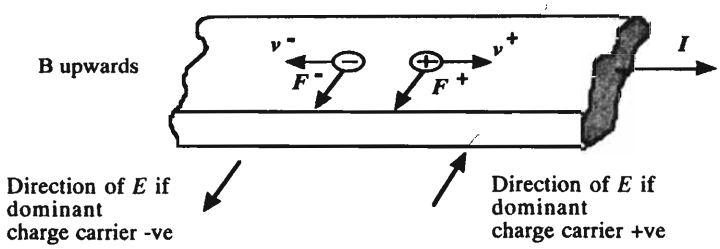


Figure 6.4

The measurement of the Hall effect enables a direct measurement of the density of current-carrying charges in the material.

Worked example What is the speed of a beam of electrons when the simultaneous influence of an electric field of 35×10^4 N C^{-1} and a magnetic field of 2.0×10^{-2} T, both fields being normal to the beam and to each other, produces no deflection of the electrons?

Solution

Data given:

Magnitude of $E = 35 \times 10^4$ N C^{-1}
Magnetic field $B = 2.0 \times 10^{-2}$ T
Resultant force = 0
Electric force $F_E = qE$
Magnetic force $F_B = qvB$ $(\sin\theta = 1)$
(These forces must be equal and opposite.)

$$qE = qvB$$

$$v = \frac{E}{B} = \frac{35 \times 10^4}{2.0 \times 10^{-2}} = 17 \times 10^6 \text{ m s}^{-1}$$

Problem Calculate the energy required to rotate a square, 100-turn coil with sides of 0.1 m, through an angle of 60° as shown in Figure 6.5, when the field is 0.3 T and the current in the coil is 5 A.
(Ans: 1.3 J)

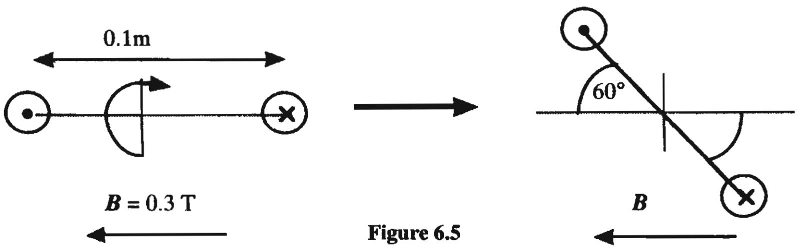


Figure 6.5

Keywords

Magnetism, magnetic pole, magnetic field, tesla, magnetic-field line, magnetic flux, weber, magnetic moment, torque on current loop, magnetic dipole, potential energy of magnetic dipole, Hall effect.

Important equations

$$\boxed{\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}}$$

$$\boxed{\boldsymbol{F} = q\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})}$$

$$\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{A}$$

for B uniform over plane surface A

$$\Phi = B_{\perp} A = BA \cos\theta$$

$$\boxed{\boldsymbol{F} = i d\boldsymbol{l} \times \boldsymbol{B}}$$

magnetic moment $\boldsymbol{m} = NiA$

$$\Gamma = \boldsymbol{m} \times \boldsymbol{B} = NiAB \sin\theta$$

$$\boxed{U_B = -\boldsymbol{m} \cdot \boldsymbol{B}}$$

6.2 Sources of magnetic field

6.2.1 Magnetic field of a moving charge

Similar to the electric field E created by a stationary charge, a moving charge q moving with velocity $\boldsymbol{v}$ creates a magnetic field at vector displacement $\boldsymbol{r}$ from it proportional to the size of the charge and inversely proportional to the square of the displacement.

$$\boldsymbol{B} = \frac{\mu_0}{4\pi} \frac{q\boldsymbol{v} \times \hat{\boldsymbol{r}}_r}{r^2}$$

The constant μ_0 is termed the permeability of free space and has the value of $4\pi \times 10^{-7}$ N A^{-2}.

6.2.2 Biot-Savart's Law: magnetic field of a current element

The total magnetic field caused by several moving charges is the vector sum of the fields caused by the individual charges. Thus the magnetic field contribution dB of a current element of vector length dl to the total magnetic field at location P (see Figure 6.6) is given by Biot-Savart's Law:

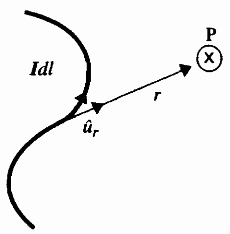


Figure 6.6

$$d\boldsymbol{B} = \frac{\mu_0}{4\pi} I d\boldsymbol{l} \times \frac{\hat{\boldsymbol{r}}_r}{r^2}$$

the magnitude of which is:

$$dB = \frac{\mu_0}{4\pi} \frac{Idl \sin\theta}{r^2}$$

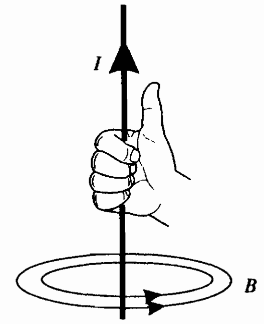


Figure 6.7

and the direction is given by the right-hand grasp rule (see Figure 6.7)

6.2.3 Magnetic field of a straight conductor

Using the Biot-Savart Law, the magnitude of the magnetic field at perpendicular distance r from a long current-carrying conductor is:

$$B = \frac{\mu_0 I}{2\pi r}$$

the direction being obtained by the right-hand grasp rule.

6.2.4 Force between parallel conductors

Two parallel conductors exert a force on each other due to the magnetic field each creates at the location of the other. The force between each is equal and opposite. If the currents are in the same direction the forces will be attractive, if in the opposite direction they will be repulsive (see Figure 6.8).

Force per unit length:

$$\frac{F_{12}}{l} = \frac{\mu_0 I_1 I_2}{2\pi a}$$

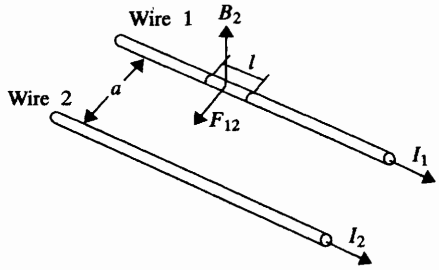


Figure 6.8

6.2.5 Magnetic field of a circular loop

Using the Biot-Savart Law, the magnetic field at distance x along the axis of a circular loop of radius a (see Figure 6.9) is given by:

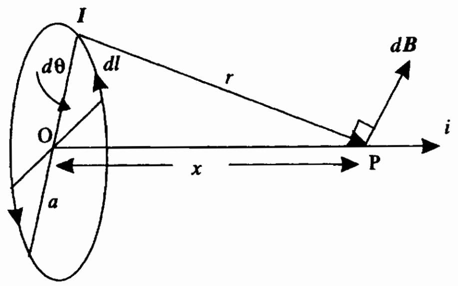


Figure 6.9

$$B_x = \frac{\mu_0 I a^2}{2\,(x^2 + a^2)^{3/2}}$$

$$= \frac{\mu_0 I}{2a} \text{ at } x = 0$$

6.2.6 Ampere's Law

For cases of symmetry, B may be calculated by the use of Ampere's Law:

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I$$

The path of the integral is any closed loop, and I is the net current piercing any surface whose boundary is the closed loop. The path is chosen so that B is constant or zero over the path — or perpendicular to the path so that the dot product is zero. The path does not have to be an actual physical boundary.

6.2.7 Magnetic field inside a solenoid

Using Ampere's Law the magnetic field may be calculated along the axis of a long solenoid (see Figure 6.10).

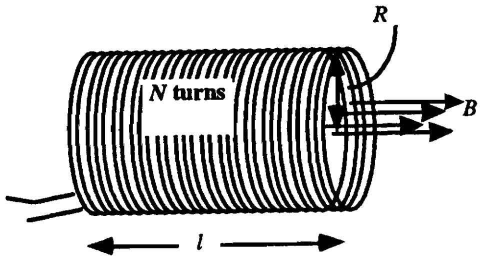


Figure 6.10

$$B = \mu_0 In = \mu_0 I \frac{N}{l}$$

where n is the number of turns per unit length.

For a short solenoid ($R > l$), the magnetic field is reduced and tapers off from a maximum value in the centre of the solenoid.

6.2.8 Magnetic materials

If the centres of solenoids are filled with a 'magnetic' material, the value of B is altered.

Materials fall into three main categories:

- Paramagnetic materials increase B slightly.
- Diamagnetic materials decrease B slightly.
- Ferromagnetic materials increase B greatly.

The alignment of atomic current loops within the material provides the source of extra magnetic fields. For ferromagnetic materials this increase is non-linear and reaches saturation with large applied currents. Such magnetic behaviour may be described by replacing μ_0 in the previous equations by the term μ, where:

$$\mu = \mu_r \mu_0$$

and μ_r is the relative magnetic permeability.

Alternatively, it may be described by χ_m the magnetic susceptibility, where:

$$\chi_m = \mu_r - 1$$

6.2.9 Magnetic fields and displacement current

In situations involving time-varying fields, such as an AC current 'passing through' a capacitor, the current in Ampere's Law must include the conduction current I_C and the displacement current I_D. This displacement current contributes to the calculation of the total magnetic field, with the modified Ampere's Law being:

$$\oint \boldsymbol{B} \cdot dl = \mu_0 (I_C + I_D)$$

In empty space, where there is no conduction current, Ampere's Law may be written in terms of the electric flux Ψ

$$I_D = \varepsilon_0 \frac{d\Psi}{dt}$$

$$\oint \boldsymbol{B} \cdot dl = \mu_0 \varepsilon_0 \frac{d\Psi}{dt}$$

Worked example A long wire carrying a current of 15 A is parallel to the plane of a rectangular coil with 25 turns and a current of 2.0 A. From the dimensions shown on Figure 6.11, find the total force on the coil.

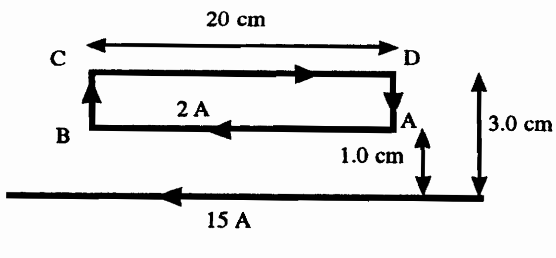


Figure 6.11

Solution Forces on arms BC and AD are parallel to the long wire and cancel each other. The field due to the long wire at distance d is:

$$B = \frac{\mu_0 I}{2\pi d}$$

At wire AB this is:

$$B_{AB} = \frac{4\pi \times 10^{-7} \times 15}{2\pi \times 0.01} = 3.0 \times 10^{-4}\ \mathrm{T}$$

At wire CD it is:

$$B_{\mathrm{CD}} = \frac{4\pi \times 10^{-7} \times 15}{2\pi \times 0.03} = 1.0 \times 10^{-4}\ \mathrm{T}$$

Force on wire AB is towards the long wire = $3.0 \times 0.2 \times 3.0 \times 10^{-4}$ N
Force on wire DC is away from the long wire = $1.0 \times 0.2 \times 3.0 \times 10^{-4}$ N

Net force on loop = $F_{\mathrm{AB}} + F_{\mathrm{CD}}$ in a direction towards the long wire.

$$= 2.0 \times 0.2 \times 3.0 \times 10^{-4}$$

$$= 1.2 \times 10^{-4}\ \mathrm{N}$$

Problem A long solenoid of diameter 0.15 m with 100 turns/m carries a current of 500 mA. Calculate the magnetic flux through the coil;

(a) with air only inside the coil;

(b) with a rod of magnetic material of diameter 0.02 m and relative permeability 1200.

(Ans: (a) 1.1 μWb; (b) 25 μWb)

Keywords

Biot and Savart, magnetic field of current element, permeability, Ampere's Law, solenoid, paramagnetism, ferromagnetism, diamagnetism, magnetic susceptibility, displacement current and magnetic field.

Important equations

Biot-Savart Law:

$$\boxed{d\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{I d\boldsymbol{l} \times \hat{\boldsymbol{r}}_r}{r^2}}$$

Ampere's Law:

$$\boxed{\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I}$$

Straight line conductor:

$$\boxed{B = \frac{\mu_0 I}{2\pi r}}$$

Solenoid:

$$\boxed{B = \mu_0 I n}$$

Circular loop:

$$\boxed{\begin{aligned} B_x &= \frac{\mu_0 I a^2}{2\,(x^2 + a^2)^{3/2}} \\ &= \frac{\mu_0 I}{2a} \text{ at } x = 0 \end{aligned}}$$

6.3 Electromagnetic induction

6.3.1 Induced currents and emfs

The phenomenon of 'induced currents' and 'induced emfs' results from:

- moving a conductor through a magnetic field, so that the field lines are being cut by the conductor as the conductor moves through (motional emf);
- changing the magnitude of magnetic flux by:
  - varying $|\boldsymbol{B}|$
  - varying $|\boldsymbol{A}|$
  - varying the orientation of $|\boldsymbol{B}|$ to $|\boldsymbol{A}|$, such as rotating A or rotating B.

6.3.2 Motional electromotive force

Because of the magnetic forces built up on the charges in a conductor moving through a magnetic field, B, there is a migration of the charges to opposite ends of the conductor, resulting in an 'induced emf' between the ends of the conductor. The electric field resulting from this potential between the ends of the conductor produces electric forces on the charges that act in the opposite direction to the magnetic forces and an equilibrium situation is reached where:

the electric field $E = emf/l$

Equating the magnetic and electric forces: $q\,(emf/l) = qvB$

and the induced emf is $emf = vBl$

and, more generally, $emf = \int \boldsymbol{v} \times \boldsymbol{B} \cdot \boldsymbol{dl}$

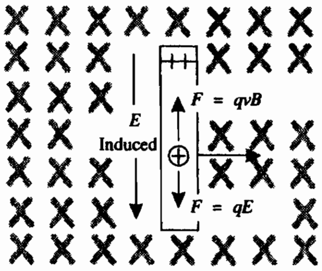


Figure 6.12

6.3.3 Faraday's Law of Induction

The induced emf in a circuit resulting from the change of magnetic flux through the circuit is equal to the rate of change of the magnetic flux through it.

6.3.4 Lenz's Law

The direction of an induced emf is such that it opposes the change that produced it. Faraday's Law and Lenz's Law are combined in the equation:

$$emf = -\frac{d\phi}{dt}$$

where the minus sign describes the 'opposing' direction of the emf as stated by Lenz.

In some cases both the motional emf equation and Faraday's Law are effectively different methods of making the same calculation. However, Faraday's Law has a broader application, since it does not require any part of the circuit to be in motion.

6.3.5 Eddy currents

Metals moving in magnetic fields or located in varying magnetic fields experience induced emfs and induced currents, the path of such induced currents being completed through the metal. Because of Lenz's Law these currents always oppose the change which causes them and can be a considerable nuisance. To reduce the effect of the eddy currents in transformers the core is laminated, thus limiting the area of the possible metal loops and hence the effect of the changing flux.

6.3.6 Maxwell's equations

Maxwell's equations are the four equations derived in electric, magnetic and electromagnetic theory, relating to the surface and line integrals of ***E*** and ***B***. The second of these has a value of 0 since no magnetic monopole exists. The fourth equation is equivalent to Faraday's Law, using the relationship between electric potential and electric field:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q}{\varepsilon_0}$$

$$\oint \boldsymbol{B} \cdot d\boldsymbol{A} = 0$$

$$\oint B \cdot dl = \mu_0 \left(I_C + \varepsilon_0 \frac{d\Psi}{dt} \right)$$

$$\oint E \cdot dl = -\frac{d\Phi}{dt}$$

6.3.7 Inductance

The relationship of induced emf to the rate of change with time of the original current creating that induced emf in a loop or a system of loops is termed the 'inductance' of that system. This inductance relates to the capacity of the system to 'store' magnetic energy.

Unit of inductance is the **henry.**

Mutual inductance

If we have two linked coils, each in the flux created by the other, the flux in coil 2 due to a current i_1 flowing in *each* loop of coil 1 is:

$$\phi_{2,1} = \int_2 B_1 \cdot dA_2$$

The total emf induced in each as a result of a changing flux in the other is (since $B \propto i$ hence $\phi \propto i$):

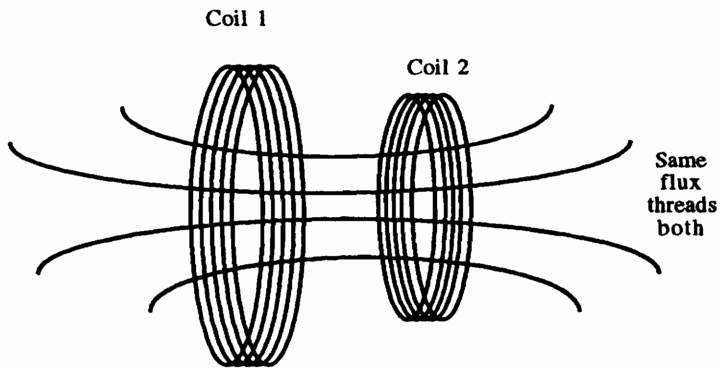


Figure 6.13

$$emf_2 = -N_1 N_2 \frac{d\phi_{2,1}}{dt} = -M_{2,1} \frac{di_1}{dt}$$

$$emf_1 = -N_1 N_2 \frac{d\phi_{1,2}}{dt} = -M_{1,2} \frac{di_2}{dt}$$

It can be shown that $M_{1,2} = M_{2,1} \equiv M$ is a constant depending only on the geometry of the system of coils.

Thus we have:

$$emf_2 = -M\frac{dI_2}{dt} = -N_2\frac{d\phi_2}{dt}$$

and

$$M = \frac{N_2\phi_2}{i_1} \text{ or } = \frac{N_1\phi_1}{i_2}$$

6.3.8 Self inductance

If a change occurs in the current flowing in a coil this will also cause a change in the magnetic flux threading through the coil itself and an emf will be induced in such a direction as to oppose the change (hence an increasing current would experience an induced current in the opposite direction, a decreasing current would experience an induced current in the same direction). With a similar derivation to that for mutual inductance, the self inductance of a system, L, is defined as a constant (dependent only on the geometry of the system) such that:

$$emf = -L\frac{di}{dt} = -N\frac{d\phi}{dt}$$

and

$$L = \frac{N\phi}{i}$$

6.3.9 Energy storage in inductor

The instantaneous power P resulting from a changing current is the rate of energy supplied:

$$P = \frac{dW}{dt} = emf\ i = Li\frac{di}{dt}$$

$$\therefore\ \Delta\,\text{energy} = L\int_{I_i}^{I_f} i\ di = \frac{1}{2}L(I_f^2 - I_i^2)$$

This is then stored as 'magnetic energy'

$$U_B = \frac{1}{2}LI^2$$

Worked example A long straight wire carries a current $I = I_0 \sin(\omega t + \alpha)$ where α = constant. The wire lies in the plane of a rectangular loop of N turns of wire.

Determine the emf induced in the loop by the magnetic field due to the changing current in the straight wire.

Data: $I_0 = 50$ A, $\omega = 100\pi$ rads^{-1}, $N = 80$ turns, $r_1 = 5.0$ cm, $r_2 = 15.0$ cm, $h = 12$ cm.

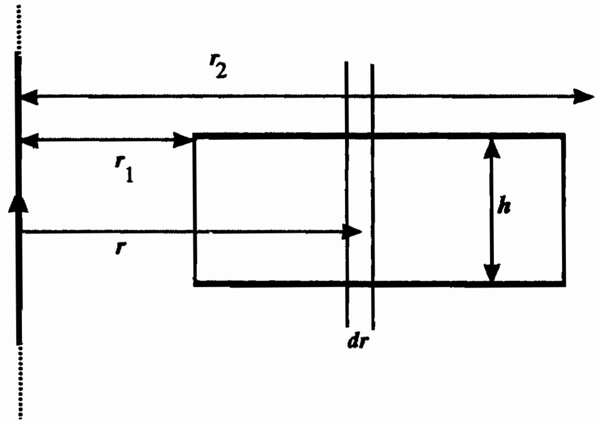


Figure 6.14

Solution Consider the element dr at distance r from the current-carrying wire. The flux $d\phi$ through this element is:

$$d\phi = \frac{\mu_0 I}{2\pi r} dA = \frac{\mu_0 I}{2\pi r} h dr$$

Total flux through the coil is:

$$\phi = N \frac{\mu_0 I}{2\pi} h \int_{r_1}^{r_2} \frac{1}{r} dr = N \frac{\mu_0 I_0 \sin(\omega t + \alpha)}{2\pi} h \ln \frac{r_2}{r_1}$$

and the induced emf is:

$$emf = -\frac{d\phi}{dt} = -N \frac{\mu_0 I_0 \, \omega \cos(\omega t + \alpha)}{2\pi} h \ln \frac{r_2}{r_1}$$

$$= -(2 \times 10^{-7}) \times 80 \times 0.12 \times 50 \times 100\pi \times \ln \frac{15}{5} \times \cos(100\pi t + \alpha)$$

$$= -33 \cos(100\pi t + \alpha) \text{ mV}$$

Problem A metal disc of radius 0.15 m rotating at 100 rpm is situated in a uniform magnetic field of 1.0 T parallel to the axis of rotation. Calculate the emf induced between the rim and the centre of the disc.

(Ans: 0.12 V)

Keywords

Motional electromotive force, electromagnetic induction, Faraday's Law, Lenz's Law, induced electric field, induced emf, Maxwell's equations, inductor, inductance, mutual inductance, self inductance, solenoid, energy in inductor.

Important equations

Motional emf

$$\boxed{\begin{aligned} emf &= \int_a^b \boldsymbol{v} \times \boldsymbol{B} \cdot d\boldsymbol{l} \\ &= vBl \end{aligned}}$$

Faraday's Law

$$\boxed{emf = -\frac{d\phi}{dt}}$$

Maxwell's equations

$$\boxed{\begin{gathered} \oint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q}{\varepsilon_0} \\ \oint \boldsymbol{B} \cdot d\boldsymbol{A} = 0 \\ \oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \left(I_C + \varepsilon_0 \frac{d\Psi}{dt} \right) \\ \oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi}{dt} \end{gathered}}$$

$$\boxed{\begin{aligned} emf &= -M\frac{di}{dt} = -N\frac{d\phi}{dt} \\ emf &= -L\frac{di}{dt} = -N\frac{d\phi}{dt} \end{aligned}}$$

$$\boxed{U_B = \frac{1}{2}LI^2}$$

6.4 R–C and R–L circuits

6.4.1 Transients

Transient or temporary currents flow in electric circuits when a change is made in the steady state conditions, such as switching the circuit on or switching the circuit off or increasing the current or decreasing the current. Because of the properties of inductors and capacitors the existence of these elements in electric circuits causes a delayed response to such changes.

R–C circuits

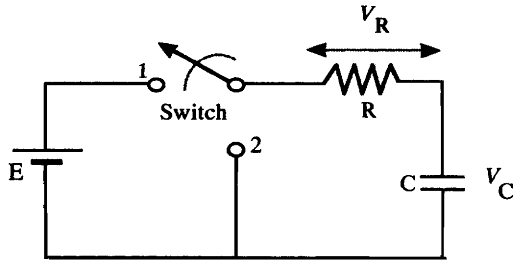


Figure 6.15

When the switch in Figure 6.15 is closed to position 1 positive charge flows from the battery to the top plate, thus repelling the positive charge from the bottom plate, which therefore becomes negatively charged. The transfer of electrons is very rapid at first, slowing down as the potential across the capacitor approaches the applied voltage of the battery. When the voltage across the capacitor equals the battery voltage, the transfer of electrons will cease and the plates will have a net charge determined by $Q = CE$.

When the switch is thrown to position 2, the capacitor will begin to discharge. The established voltage across the capacitor will create a flow of charge in the closed path that will eventually discharge the capacitor completely.

Mathematics:

A first-order differential equation in charge q results from analysis of the circuit:

$$E = iR + V_C = \frac{dq}{dt}R + \frac{q}{C}$$

On discharging, the same equation applies except with $E = 0$.

The solutions are obtained for the charge q, and from this the current flowing and the resultant voltage are obtained. These are summarized as follows:

Charge:

Charging:

$$q = EC\,(1 - e^{-t/RC}) = q_0\,(1 - e^{-t/RC})$$

Discharging:

$$q = EC\,e^{-t/RC} = q_0\,e^{-t/RC}$$

Current flowing in the circuit:

Charging:

$$i = \frac{E}{R} e^{-t/RC} = i_0 e^{-t/RC}$$

Discharging:

$$i = -\frac{E}{R} e^{-t/RC} = -i_0 e^{-t/RC}$$

Voltage across R and C:

Charging:

$$V_C = E(1 - e^{-t/RC}) ; V_R = E e^{-t/RC}$$

Discharging:

$$V_C = E e^{-t/RC} ; V_R = -E e^{-t/RC}$$

These results are illustrated in Figure 6.16.

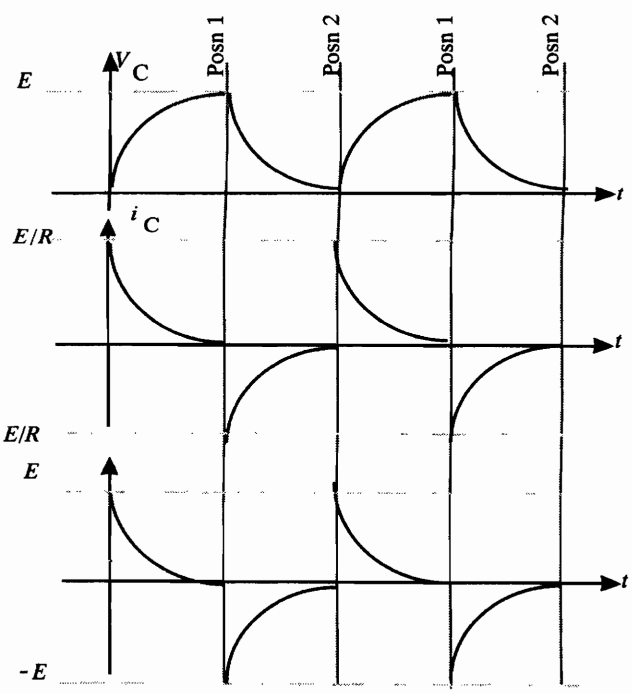


Figure 6.16

R–L circuits

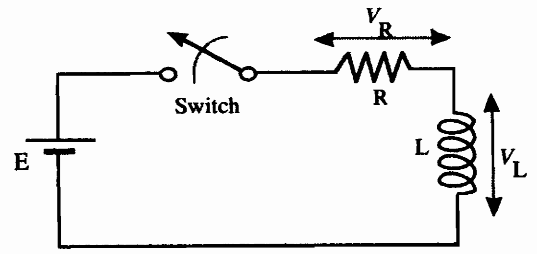


Figure 6.17

At the instant the switch in Figure 6.17 is closed the inductance of the coil will prevent an instantaneous change in current through the coil. The potential drop across the coil V_L, will equal the impressed voltage E. The current i_L will then build up from zero, establishing a voltage drop across the resistor and a corresponding drop in V_L. The current will continue to increase until the voltage across the inductor drops to zero volts and the full impressed voltage appears across the resistor.

When the switch is opened the induced current will oppose the drop to zero, possibly accompanied by a spark due to the rapid change in current through the inductor.

Mathematics

In the circuit shown in Figure 6.17 the first-order differential equation is in i:

Charging: $$E = iR - L\frac{di}{dt}$$

Discharging: the same equation with $E = 0$.

The solutions are obtained directly for the current flowing and the resultant voltage obtained. These are summarised as follows:

Current flowing in the circuit:

Charging: $$i = \frac{E}{R}(1 - e^{-R/Lt}) = i_0(1 - e^{-R/Lt})$$

Discharging: $$i = \frac{E}{R}e^{-R/Lt} = i_0 e^{-R/Lt}$$

Voltage across R and L:

Charging: $$V_L = E\,e^{-R/Lt};\ V_R = E(1 - e^{-R/Lt})$$

Discharging: $$V_L = -E\,e^{-R/Lt};\ V_R = E\,e^{-R/Lt}$$

These results are illustrated in Figure 6.18.

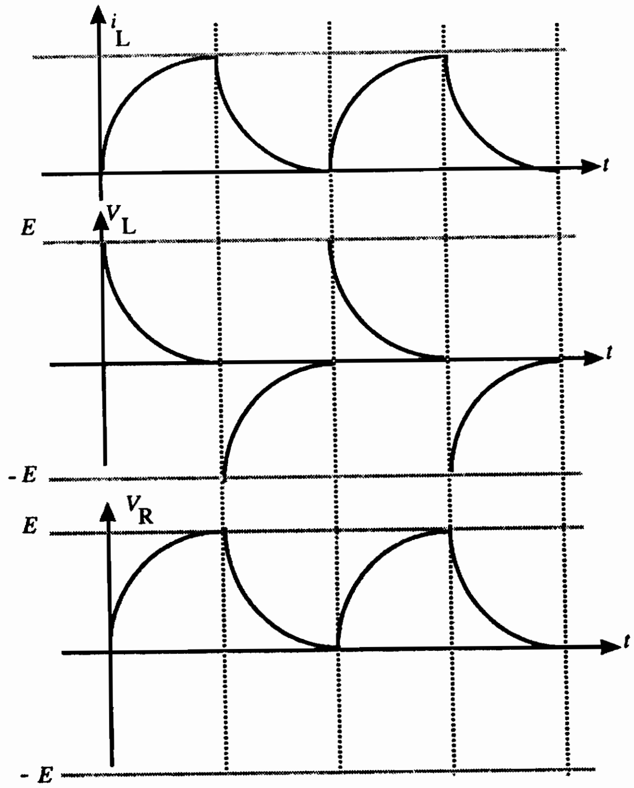


Figure 6.18

6.4.2 Time constants

The exponential term in the solutions of the transient equations is most usefully described in terms of a 'time constant', τ, such that when $t = 1\tau$ a decaying current/potential has decayed to $\frac{1}{e}$ of its maximum value or the rising current/potential has risen to $(1 - \frac{1}{e})$ of its maximum value.

For R–L circuits, the time constant is:

$$\tau_L = \frac{L}{R}\ \mathrm{s}$$

For R–C circuits, the time constant is:

$$\tau_C = RC\ \mathrm{s}$$

A rising potential is then of the form:

$$V = V_0\, e\,(1 - e^{-t/\tau})\ \text{etc.}$$

L–C circuits

When a circuit contains an inductance L and capacitance C, oscillations will occur as the energy is transferred from electrical energy in the capacitor to magnetic energy in the oscillator and back again. In practice, it is not possible to have an inductance without a resistance so the effect is studied in AC circuits, under resonance.

Worked example For the circuit shown in Figure 6.19,

(a) Determine the time constant of the circuit.

(b) Write the mathematical equation for the voltage V_C after the closing of the switch.

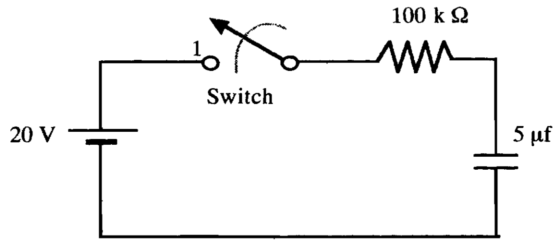


Figure 6.19

(c) Determine the voltage V_C after one, three and five time constants.

(d) Write the equations for the current i_C and the voltage V_R.

(e) Sketch the waveforms for V_C and I_C.

Solution (see Figure 6.20)

(a) $\tau_C = RC = 100 \times 10^3 \times 5 \times 10^{-6} = 0.5$ s

(b) Charging equation for V_C: $V_C = E\,(1 - e^{-t/\tau}) = 20\,(1 - e^{-2t})$

(c) $t = 1\tau$, $V_C = 12.64$ V; $t = 3$, $V_C = 19.00$ V; $t = 5\tau$, $V_C = 19.87$ V

(d) $i_C = \dfrac{E}{R} e^{-t/\tau} = 0.2 \times 10^{-3} e^{-2t}$ $V_R = 20\, e^{-2t}$

(e)

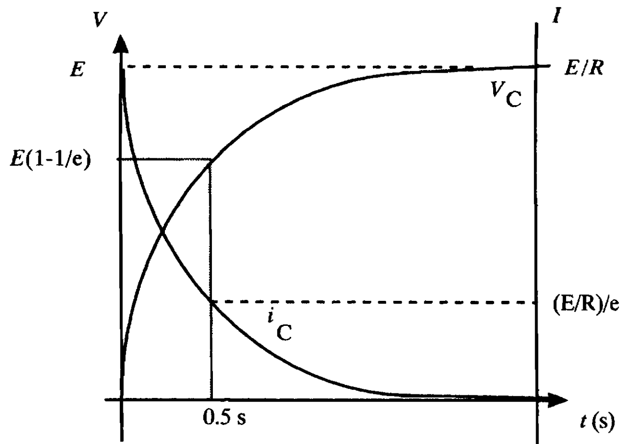


Figure 6.20

Problem A 10 000 Ω resistor and an inductor are connected in series and a 10 V potential is suddenly applied. If the potential across the inductor is 5.0 V after 1.0 μs, find:

(a) the inductance of the inductor;

(b) the time for the current in the circuit to reach 95% of its steady state value.

(Ans: 14 mH; 4.3 μs)

Keywords

R–L circuit, R–C series circuit, time constant, transient current.

Important equations

$$\boxed{\tau_L = \frac{L}{R}}$$

$$\boxed{\tau_C = RC}$$

6.5 AC circuit theory

6.5.1 Source of alternating current – the AC generator

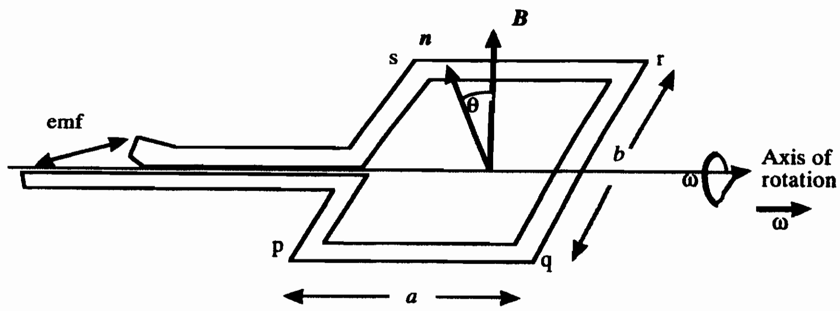


Figure 6.21

When a coil plane is inclined at angle θ as shown in Figure 6.21 then the flux through the coil

$$\phi = \text{area} \times B \cos\theta = abB\cos\theta$$

But $\theta = \omega t$ at time t. Therefore the induced emf is:

$$emf = -\frac{d}{dt}\phi = -\frac{d}{dt}\,\text{area} \times B\cos\omega t$$

that is, $$emf = \omega AB \sin\omega t$$

or, using $$V \equiv emf$$

or $$V = V_{max}\sin\omega t$$

6.5.2 AC terminology

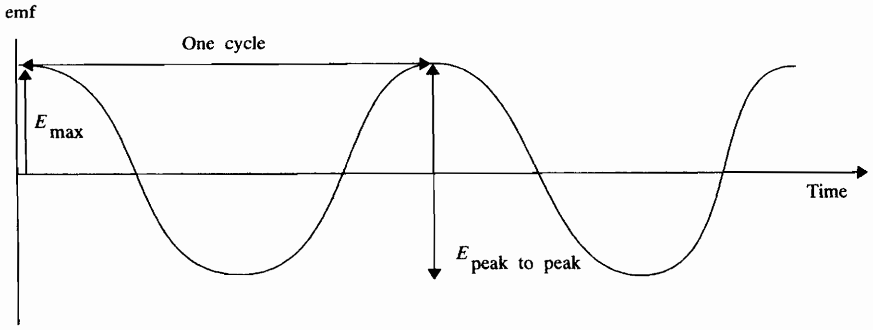


Figure 6.22

Phase

If a voltage such as that produced by the AC alternator,

$$V = V_{max} \cos \omega t$$

is applied across an AC circuit element, R, L or C or a combination of these, the current flowing will be of the form:

$$\text{Current: } I = I_{max} \cos (\omega t + \phi)$$

where

$$\begin{aligned} \text{Phase } \phi &= 0^\circ = 0 \text{ radians for a resistor} \\ &= -90^\circ = -\frac{\pi}{2} \text{ radians for an inductor} \\ &= 90^\circ = \frac{\pi}{2} \text{ radians for a capacitor} \end{aligned}$$

in other words

voltage is in phase with i for a resistor
voltage is 90° ahead of i for an inductor
voltage is 90° behind i for a capacitor.

Reactance and impedance

If current $I = I_{max} \cos \omega t$ flows through a resistor R the voltage $|V_R| = I_{max} R$
If current $I = I_{max} \cos \omega t$ flows through an inductor L the voltage $|V_L| = I_{max} X_L$
If current $I = I_{max} \cos \omega t$ flows through a capacitor C the voltage $|V_C| = I_{max} X_C$

where:

Reactance of inductance L: $X_L = \omega L = 2\pi f L$

of capacitance C: $X_C = \dfrac{1}{\omega C} = \dfrac{1}{2\pi f C}$

If current $I = I_{max} \cos \omega t$ flows through a general AC circuit element of impedance Z:

$$|V| = I_{max} |Z|$$

If the phase of impedance Z is ϕ, then

$$V = IZ = I_{max} |Z| \cos(\omega t + \phi)$$

$$\text{Phase angle } \phi\text{: } \tan\phi = \frac{X}{R}$$

Phasor diagram

A phasor is a rotating vector used to represent the magnitude and phase of AC quantities such as I, V, Z. The vector rotates counterclockwise with constant angular velocity ω.

The instantaneous value of a quantity is then represented by the projection onto a horizontal axis of the phasor.

In such a circuit the current I is always in the same direction as V_R.

If the phasor diagram represents a series circuit then the same current I vector applies to all components and the voltages have various phases with respect to I. If the phasor diagram represents a parallel circuit element then the same applied voltage V vector applies to all components and the currents have various phases with respect to V.

R–L–C series circuit

The phasor diagram for such a circuit is shown in Figure 6.23,

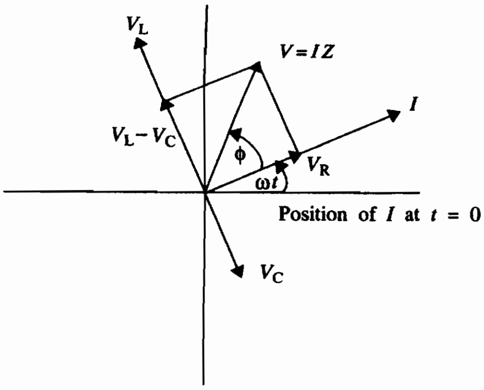


Figure 6.23

from which it may be seen that Z for such a circuit consisting only of R, L and C in series is given by:

$$Z = \sqrt{R^2 + X^2} = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

which is frequency dependent.

6.5.3 Resonance in AC circuits

In the above expression for Z in a series R–L–C circuit it can be seen that Z will reach a minimum value when $X_L = X_C$, that is, the current will peak at this point.

The frequency at which this peak occurs is called the resonant frequency f_R or resonant angular frequency ω_R. By equating $X_L = X_C$ we obtain:

$$\omega_R = \sqrt{\frac{1}{LC}}$$

$$f_R = \frac{1}{2\pi\sqrt{LC}}$$

The sharpness of the peak depends on the value of R in the circuit. Small R produces a very sharp peak, large R a broader, flatter peak.

Resonance also occurs in parallel circuits — in this case the current reaches a minimum at the resonant frequency. The resonant frequency is close to the series circuit value but the resonance peak is not as regularly defined as in the series case.

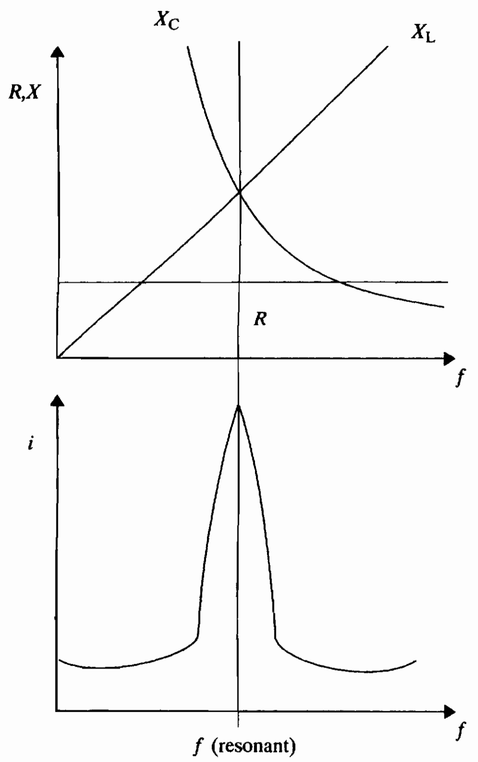


Figure 6.24

6.5.4 Power and RMS values in AC circuits

The instantaneous power absorbed by a load is:

$$p(t) = i(t)\,v(t) = I_{max} \cos \omega t V_{max} \cos(\omega t - \phi)$$

Averaging this over a whole cycle ($T = 2\pi/w$) we obtain the average power

$$P_{av} = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{2} I_{max} V_{max} \cos\phi = I_{rms} V_{rms} \cos\phi$$

where I_{rms} and V_{rms} are the 'root-mean-square' values of I and V.
For sinusoidal signals:

$$I_{rms} = \frac{I_{max}}{\sqrt{2}}\,;\; V_{rms} = \frac{V_{max}}{\sqrt{2}}$$

Laboratory analogue and digital meters normally read rms values. Oscilloscopes display the 'peak-to-peak' value, that is, twice the maximum value of the amplitude of the sinusoidal signal.

6.5.5 Transformers

A transformer consists of two coils, electrically insulated from each other but wound on the same core, so that the flux produced by one threads through the other. Power is supplied to the input (primary) coil and obtained from the output (secondary) coil (see Figure 6.25).

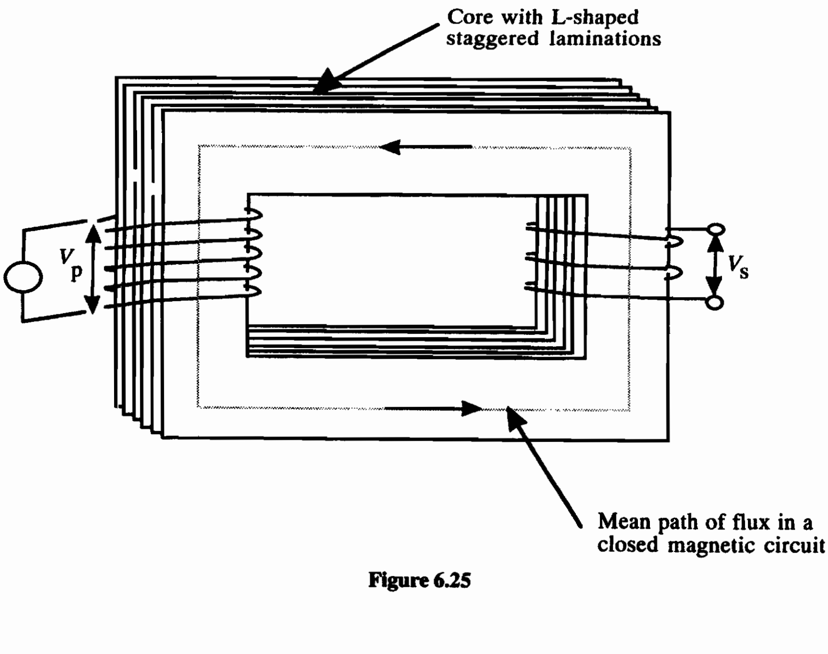


Figure 6.25

Transformers may be used to transform voltage and current levels in AC circuits, stepping up or stepping down the voltage/current according to the ratio of the turns in the primary and secondary (see Figure 6.26).

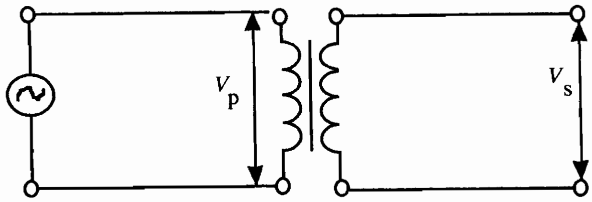


Figure 6.26

For an ideal transformer both coils are pure inductors and there are no power losses. The ideal transformer relationships are:

$$\frac{V_s}{V_p} = \frac{n_s}{n_p} = \frac{I_p}{I_s}$$

Worked example An alternating emf ($V = 50$ V, $f = 1.0$ kHz) drives a series R–L–C circuit ($L = 2.0$ mH, $R = 10\ \Omega$, $C = 80\ \mu$F) as shown in Figure 6.27. Determine the potential difference across each of the circuit components at an instant when the potential difference across the emf is equal to 50 V.

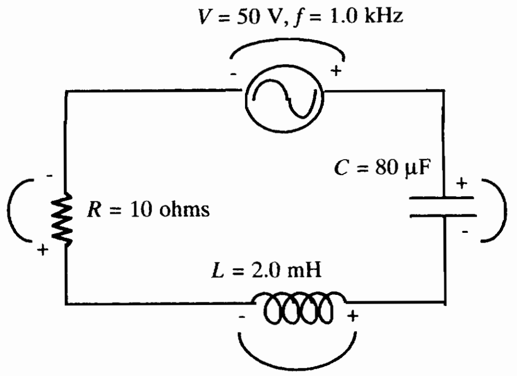


Figure 6.27a

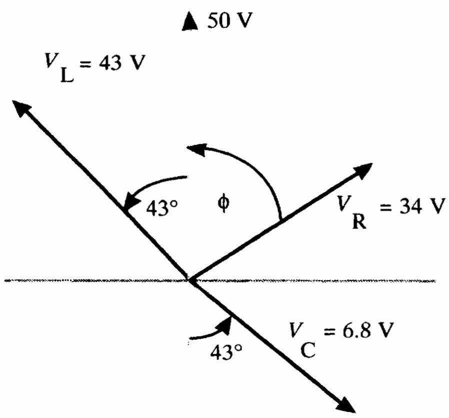


Figure 6.27b

Solution

$$X_L = 2\pi fL = 2\pi \times 1000 \times 0.002 = 12.6\ \Omega$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 1000 \times 8.0 \times 10^{-5}} = 2.0\ \Omega$$

$$X = X_L - X_C = 10.6\ \Omega$$

$$Z = \sqrt{R^2 + X^2} = 14.6\ \Omega$$

$$I = \frac{V}{Z} = \frac{50}{14.6} = 3.42 \text{ A}$$

$$\phi = \tan^{-1}\frac{X}{R} = \tan^{-1}\frac{10.6}{10} = 46.7° = 0.815 \text{ rad}$$

$$V_L = IX_L = 3.42 \times 12.6 = 43.1 \text{ V}$$

$$V_C = IX_C = 3.42 \times 2.0 = 6.8 \text{ V}$$

$$V_R = IR = 3.42 \times 10 = 34.2 \text{ V}$$

From the phasor diagram:

$$\Delta V_L = V_L \cos 43.3° = 31 \text{ V}$$

$$\Delta V_C = -V_C \cos 43.3° = -5.0 \text{ V}$$

$$\Delta V_R = V_R \cos 46.7° = 24 \text{ V}$$

Note: $\Delta V_L + \Delta V_C + \Delta V_R = 31 - 5.0 + 24 = 50 \text{ V}$

Problem A series circuit consists of $R = 25\ \Omega$, $C = 5.0\ \mu$F and an AC supply of 50 V_{rms} with a frequency of 1 kHz. Determine:

(a) the magnitude of the current (i.e. the measured value);

(b) the measured voltages across R and C;

(c) the phase angle of the current with respect to the applied voltage.

(Ans: (a) 1.2 A; (b) 31 V; (c) 39 V; (d) 52°)

Keywords

Sinusoidal emf, AC circuits, reactance inductive, reactance capacitive, impedance, phase angle in L–R–C circuit, phasor, phasor diagram, series resonance, resonance parallel, average value in AC circuits, power in AC circuits, root-mean-square value, transformer.

Important equations

$$\boxed{\begin{aligned} X_L &= \omega L = 2\pi f L \\ X_C &= \frac{1}{\omega C} = \frac{1}{2\pi f C} \end{aligned}}$$

$$\boxed{V = IZ = I_{max}\,|Z| \cos(\omega t + \phi)}$$

$$\boxed{\tan\phi = \frac{X}{R}}$$

$$\text{R–L–C Series:}\quad Z=\sqrt{R^2+X^2}=\sqrt{R^2+(X_L-X_C)^2}=\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}$$

$$\omega_R=\sqrt{\frac{1}{LC}}$$

$$f_R=\frac{1}{2\pi\sqrt{LC}}$$

$$P_{av}=I_{rms}V_{rms}\cos\phi$$

$$\frac{V_s}{V_p}=\frac{n_s}{n_p}=\frac{I_p}{I_s}$$

6.6 Electromagnetic waves

6.6.1 Features of electromagnetic waves

An electromagnetic wave consists of time-varying electric and magnetic fields that travel through space with a definite speed (see Figure 6.28).

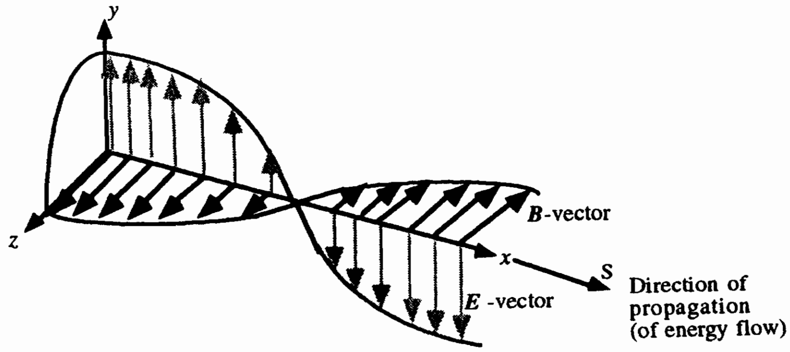


Figure 6.28

6.6.2 Maxwell's equations

Maxwell's equations apply to electromagnetic waves (see Section 6.3.6).

The equations tell us that the electric and magnetic fields are linked in such a way that any variation in one affects the other. It can be shown from the equations that:

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

where ε_0 is the permittivity in free space = 8.842×10^{-12} C^2 N^{-1} m^{-2}, and μ_0 is the permeability of free space = $4\pi \times 10^{-7}$ Wb A^{-1} m^{-1}.

The wave is transverse — both E and B are perpendicular to the direction of propagation of the wave and to each other (i.e. there is no vibration in the direction of propagation).

There is a definite ratio between the magnitudes of E and B

$$B = \frac{E}{c} = \sqrt{\varepsilon_0 \mu_0}\, E$$

In a vacuum the wave travels with a definite and unchanging speed.

In a medium other than vacuum, the ε_0 and μ_0 values are replaced by ε and μ for that medium.

6.6.3 The electromagnetic spectrum

As for all wave motion, the equation of wave propagation is:

$$c = f\lambda$$

where c = velocity, f = frequency, λ = wavelength.

Each of the E, B vectors propagate sinusoidally, according to the equations:

$$E = E_{\max} \sin(kx - \omega t)\ ;\ B = B_{\max} \sin(kx - \omega t)$$

describing waves of frequency $\omega/2\pi$, wavelength $2\pi/k$ in the +ve x–direction.

The particular features (colour, etc.) of a wave depend on its frequency, which is set at the source of the wave and does not vary as the wave moves through different media. Thus we have the electromagnetic spectrum with the corresponding frequency ranges, though the corresponding wavelengths in vacuum are also quoted, being useful as a memorable value, for example 400 nm for the high frequency end of the visible spectrum, 700 nm for the low frequency end etc. (see Figure 6.29).

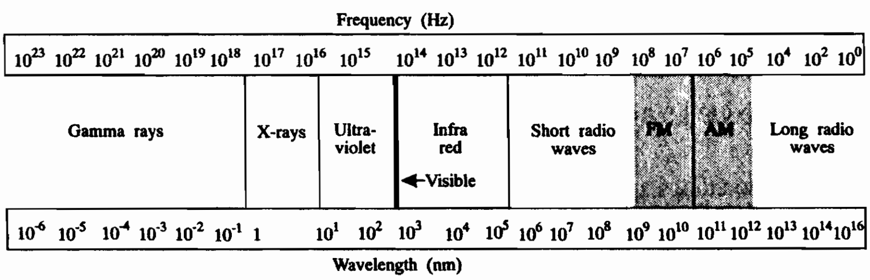


Figure 6.29

6.6.4 Energy in electromagnetic wave

The total energy density where electric and magnetic fields are present is given by

$$u = \frac{U}{\text{unit volume}} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\mu_0 B^2 = \varepsilon_0 E^2 \;(= \mu_0 B^2)$$

The electric energy density and the magnetic energy density are equal.
The energy flow per unit time, per unit area, is, from this

$$S = \varepsilon_0 \, c \, E^2 = \frac{EB}{\mu_0}$$

The Poynting vector S is a vector of this magnitude in the direction of propagation of the wave. It gives the flow of energy through a cross-section perpendicular to the propagation direction, per unit area and per unit time. The intensity of the radiation at any point may be described by the average value of the magnitude of the Poynting vector. The total energy flow per unit time through any surface may be calculated by:

$$P = \int S \cdot dA$$

Radiation pressure may be regarded as being due to the transfer of momentum by the waves. When an electromagnetic wave is completely absorbed by a surface perpendicular to the propagation direction the force on the surface equals the time rate of change of momentum. In this case the radiation pressure would equal the time rate of change of momentum per unit area, which can be shown to be S/c. If the wave were elastically reflected the radiation pressure would equal $2S/c$.

6.6.5 Standing waves (electromagnetic)

Conducting surfaces or interfaces between two insulating materials with different dielectric or magnetic properties will reflect electromagnetic waves. In general when an electromagnetic wave meets such a boundary there will be partial reflection and partial transmission.

By placement of conducting boundaries at appropriate intervals it is possible to set up standing waves in electromagnetic waves similar to those set up in the resonance of sound waves in open/closed pipes or in stretched strings.

Radiation from an antenna

The simplest example of an oscillating charge producing an electromagnetic wave is that of the oscillating dipole — a pair of equal, oppositely charged particles whose charge magnitude varies sinusoidally with time. The Poynting vector (energy flow) from such a radiating dipole is radially outwards.

Worked example The average intensity of radiation from sunlight incident on the earth is 1.4 kW m^{-2}.

Assuming that all the light from earth is totally absorbed, what force does this exert on the earth (radius of the earth = 6400 km)?

Solution

$S = 1400$ Wm^{-2}.

Area of cross-section of the earth (i.e. area in the path of the sunlight) = $\pi\, r_E{}^2$.

Power absorbed = SA since S is perpendicular to A:

$$= 1400 \times \pi \times (6.4 \times 10^6)^2 = 1.8 \times 10^{17}\ \text{W}$$

$$F = \frac{d}{dt}(\text{momentum}) = \frac{d\frac{S}{c}A}{dt} = \frac{1}{c}P = \frac{1.8 \times 10^{17}}{3.0 \times 10^8} = 6.0 \times 10^5\ \text{kN}$$

Problem The average intensity of radiation from sunlight incident on the earth is 1.4 kW m^{-2}. What are the values of E_{max} and B_{max} for this radiation?

(Ans: 0.73 kV m^{-1}; 2.4 μT)

Keywords

Electromagnetic wave, Maxwell's equations, energy density in electric field, energy density in magnetic field, energy density in electromagnetic wave, Poynting vector, intensity of electromagnetic wave, radiation pressure, electromagnetic spectrum, dipole oscillating.

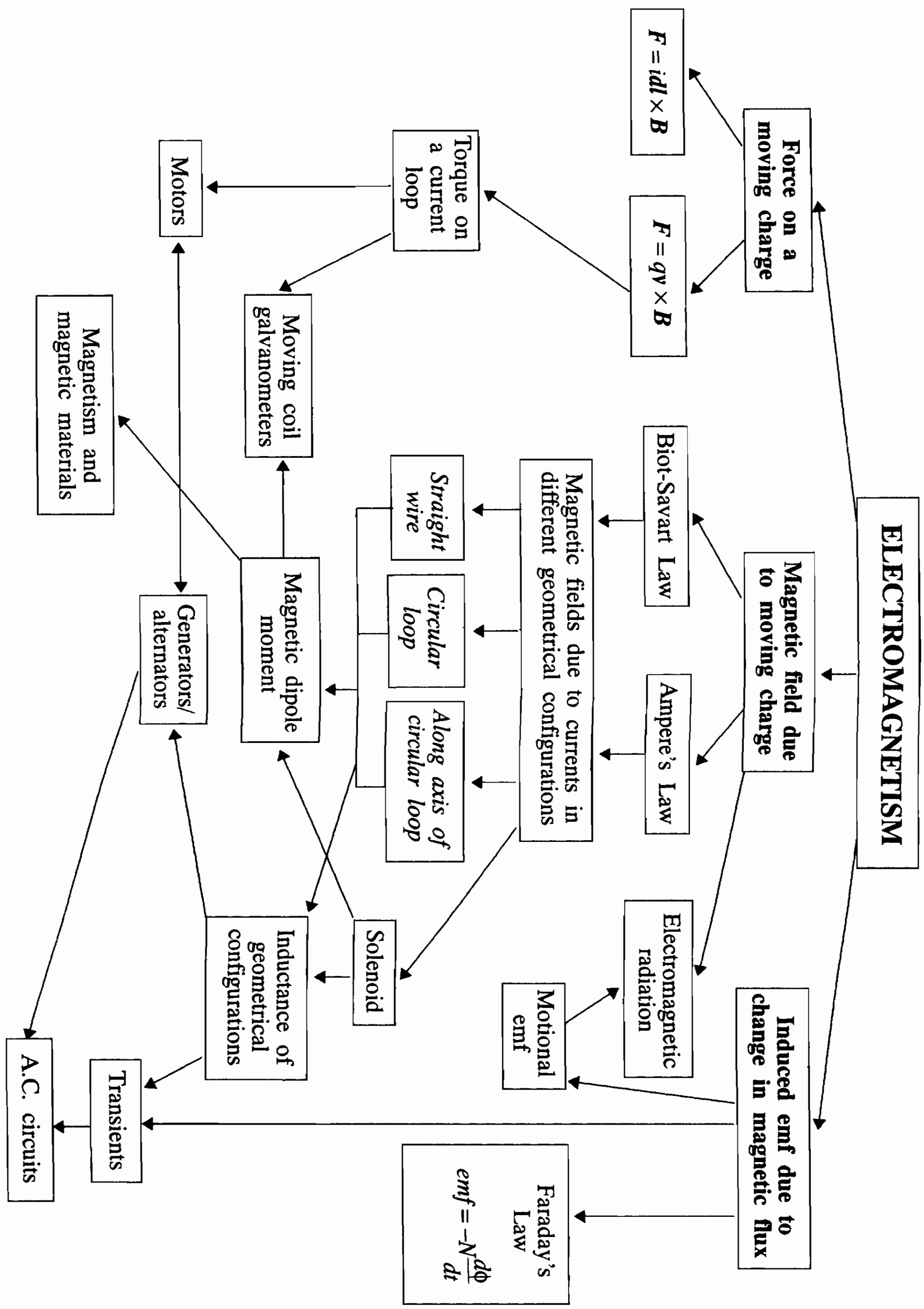
Important equations

$$\boxed{c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}}$$

$$\boxed{B = \frac{E}{c} = \sqrt{\varepsilon_0 \mu_0}\, E}$$

$$\boxed{S = \varepsilon_0 c E^2 = \frac{EB}{\mu_0}}$$

$$\boxed{P = \int S \cdot dA}$$



ELECTROMAGNETISM
Force on a moving charge
$F = idl \times B$
$F = qv \times B$
Torque on a current loop
Motors
Moving coil galvanometers
Magnetism and magnetic materials
Magnetic field due to moving charge
Biot-Savart Law
Ampere's Law
Magnetic fields due to currents in different geometrical configurations
Straight wire
Circular loop
Along axis of circular loop
Magnetic dipole moment
Generators/ alternators
Solenoid
Inductance of geometrical configurations
Electromagnetic radiation
Motional emf
Induced emf due to change in magnetic flux
Faraday's Law
$emf = -N\frac{d\phi}{dt}$
Transients
A.C. circuits

CHAPTER 7

Optics

7.1 Electromagnetic radiation

7.1.1 Electromagnetic waves

Light is an electromagnetic wave with a wavelength in the range 400 nm to 700 nm (1 nm = 10^{-9} m). Electromagnetic waves were discussed in the chapter on Electromagnetism (Chapter 6). An electromagnetic wave consists of time-varying electric and magnetic fields that travel through space with a definite speed, c.

Important properties:

- The wave is transverse — both E and B are perpendicular to the direction of propagation of the wave and to each other (i.e. there is no vibration in the direction of propagation).
- The wave travels in a vacuum with a definite and unchanging speed, namely:

$$c = c_{vac} = 3.00 \times 10^{8}\ \text{m s}^{-1}$$

- The speed of wave propagation, c, is given by

$$c = f\lambda$$

  where f = frequency and λ = wavelength.

7.2 Reflection and refraction of light

At the interface between two optical materials (see Figure 7.1) the following are observed:

(1) The incident, reflected, and refracted rays, and the normal to the surface, all lie in the same plane.

(2) The angle of reflection, ϕ_r, is equal to the angle of incidence, ϕ_a, for all wavelengths and any pair of substances. Thus:

$$\phi_r = \phi_a$$

(3) The ratio of sine of the angle of incidence (ϕ_a) and sine of the angle of refraction (ϕ_b) is a constant (this is called Snell's Law).

$$\frac{\sin \phi_a}{\sin \phi_b} = \frac{n_b}{n_a}$$

Thus if $n_b > n_a$ then $\phi_b < \phi_a$ and vice versa.

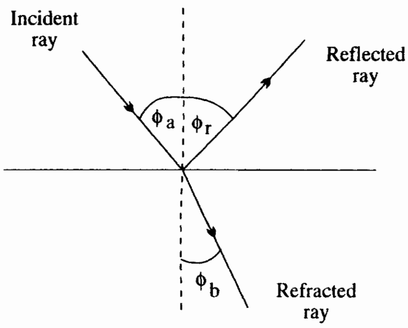


Figure 7.1

7.2.1 Refractive index

The refractive index (n_a) of medium (a) is related to the speed of light in the medium (c_a) by the equation

$$n_a = \frac{c}{c_a}$$

where c is the speed of light in a vacuum ($n = 1$ exactly).

Between two media

$$\frac{n_a}{n_b} = \frac{c_b}{c_a}$$

7.2.2 Wave speed, frequency and wavelength

$$c = f\lambda$$

f = frequency, λ = wavelength of the light wave.

Frequency does not change on passing from one medium to another, but *velocity* and wavelength both *decrease* on passing from an optically 'less dense' medium to a 'more dense' one.

7.2.3 Critical angle and total internal reflection

If a ray in a medium strikes an interface with a less optically dense medium at such an angle that the refracted ray emerges parallel to the interface, that angle of incidence is called the critical angle (ϕ_{crit}) (Figure 7.2). Rays incident at angles greater than the critical angle will be totally internally reflected.

Snell's Law:

$$\frac{n_b}{n_a} = \frac{\sin \phi_a}{\sin \phi_b}$$

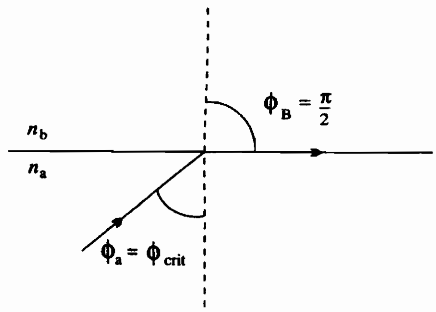


Figure 7.2

In this case:

$$\frac{n_b}{n_a} = \frac{\sin \phi_{crit}}{\sin \frac{\pi}{2}} = \frac{\sin \phi_{crit}}{1}$$

If medium (b) is a vacuum and medium (a) has refractive index n, then

$$\frac{1}{n} = \sin \phi_{crit}$$

7.2.4 Optical path length

If t is the thickness of a medium of refractive index n, the **optical path length** is given by nt. This is sometimes called the **effective optical path**.

Worked example Light of wavelength 600 nm passes through air (refractive index = 1.00) and enters a block of glass (n = 1.50) of thickness 10.0 mm.

Determine

(a) the frequency of the light in air;

(b) the speed of the light in glass;

(c) the frequency of the light in glass;

(d) the wavelength of the light in glass;

(e) the angle of refraction if the angle of incidence was 30.0°;

(f) the critical angle for light passing from glass to air;

(g) the optical path length for light passing through the block.

Solution

(a)
$$f_a = \frac{c}{\lambda} = \frac{3.0 \times 10^8}{600 \times 10^{-9}} = 5.0 \times 10^{14}\ \text{Hz}$$

(b)
$$\frac{n_a}{n_b} = \frac{c_b}{c_a}$$

$$\frac{1.0}{1.5} = \frac{c_b}{3.00 \times 10^8}$$

$$c_b = \frac{3.0 \times 10^8}{1.5} = 2.0 \times 10^8\ \text{m s}^{-1}$$

(c)
$$f_b = f_a = 5.0 \times 10^{14}\ \text{Hz}$$

(d)
$$\lambda_b = \frac{c_b}{f_b} = \frac{2.0 \times 10^8}{5.0 \times 10^{14}} = 4.0 \times 10^{-7}\ \text{m (400 nm)}$$

(e)
$$n_a \sin \phi_a = n_b \sin \phi_b$$

$$1 \times \sin 30 = 1.5 \times \sin \phi_b$$

$$\sin \phi_b = \frac{0.5}{1.5}$$

$$\phi_b = 19.5^\circ$$

(f)
$$\sin i_{crit} = \frac{1}{n}$$
$$= \frac{1}{1.5}$$
$$i_{crit} = 41.8^{o}$$

(g)
$$nt = 1.5 \times 10 = 15 \text{ mm}$$

Problem Light, in going from carbon disulphide (n = 1.63) into glass (n = 1.50), has an angle of incidence of 35.0° (see Figure 7.3).

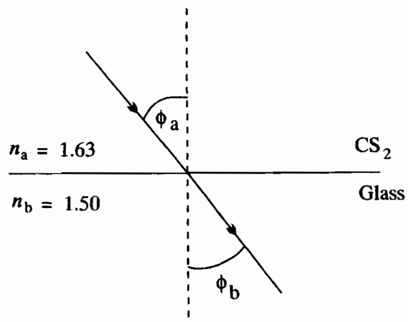


Figure 7.3

(a) Find the angle of refraction in the glass.

(b) At what angle would the ray have been totally reflected?

(Ans: (a) 38.6°; (b) 67.0°)

Keywords

Reflection, refraction, Snell's Law, total internal reflection.

Important equations

$\phi_a = \phi_r$	(reflection)
$n_a \sin \phi_a = n_b \sin \phi_b$	(refraction)

7.3 Mirrors

Curved surfaces, either cylindrical or spherical, form the basic optical components of many systems, for example shaving mirrors or make-up mirrors, vehicle headlight reflectors, rear vision mirrors, astronomical telescopes. This section investigates the images formed by **spherical** mirrors.

When light is reflected by a plane mirror its angle of reflection equals its angle of incidence. This is also true for spherical mirrors. Images formed by spherical mirrors can be located by drawing in two of the three principal rays described below.

(1) All rays passing through the centre of curvature (C) of a mirror strike the mirror normally and hence are reflected back in the opposite direction along the same path to pass through C again (Figure 7.4a and 7.4b). If the rays do not actually pass through C, but travel towards it (e.g. for the case of a convex mirror), then the reflected rays retrace the paths of the incident rays as above (Figure 7.4c).

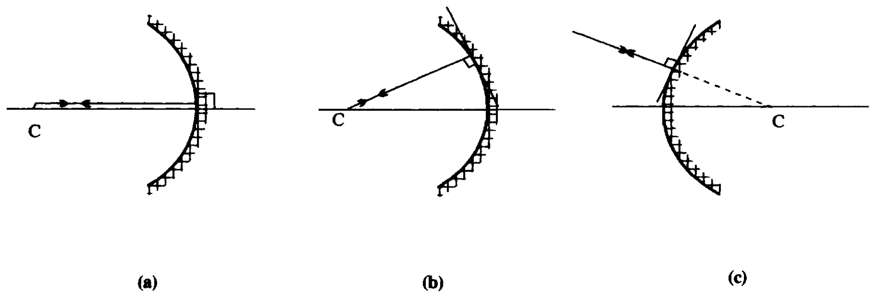


Figure 7.4

(2) When reflected, all incident rays parallel to the principal axis either pass through or appear to come from a point half way between the mirror and its centre of curvature. This point is called the focal point, F (Figure 7.5).

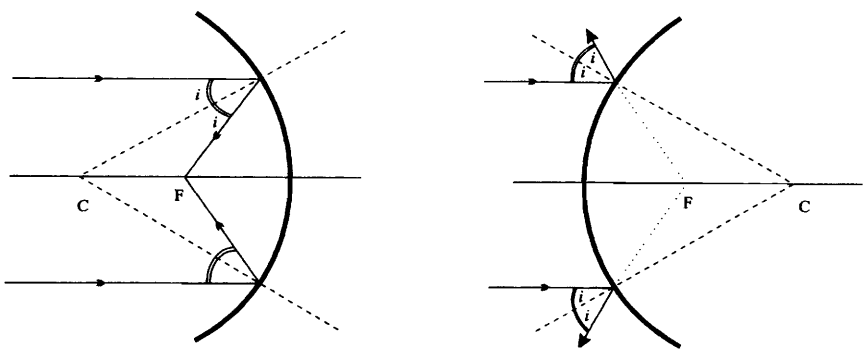


Figure 7.5

(3) **All rays passing through, or travelling towards, the focal point of a mirror will be reflected parallel to the axis.**

The three examples in Figure 7.6 show the positions (and sizes) of the images formed by spherical mirrors. The distance between the **object** and the mirror is s and the distance between the **image** and the mirror is s'.

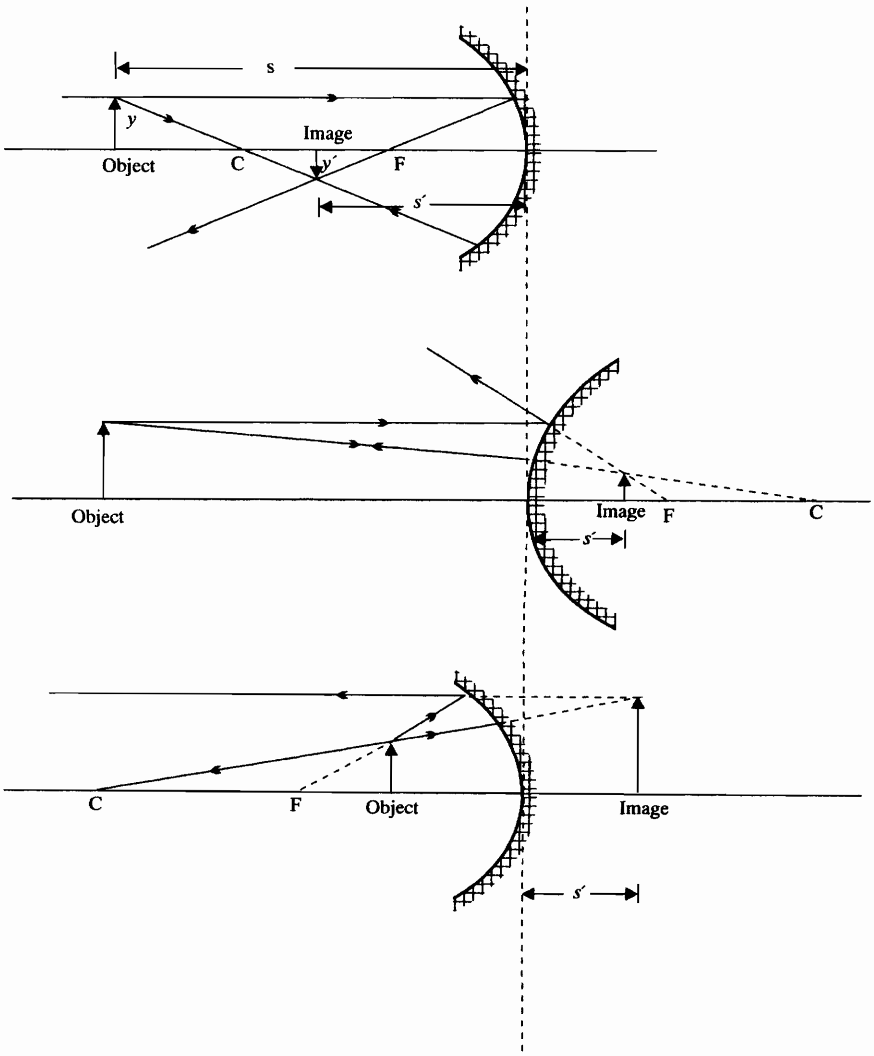


Figure 7.6

7.3.1 Sign convention

The sign convention used in this book is the 'real is positive'. The R-side (or real side) of a mirror is the side from which incident light comes; the V-side (or virtual side) is the back of the mirror, where no light rays can ever be present.

Objects

The object distance, s, is positive if the object is **real** or the rays **diverge** from the object. The object distance is negative if the object is **virtual**, that is, the rays **converge** towards a point but meet the mirror before reaching the point.

Images

The image distance is positive and the image is **real** if it lies on the **R-side**. The image distance is negative and the image is **virtual** if it lies on the **V-side**.

Radii of curvature

The radius of curvature (r) is positive if the centre of curvature is on the **R-side** (i.e. if the mirror is **concave**.) The radius of curvature (r) is negative if the centre of curvature is on the **V-side** (i.e. if the mirror is **convex**).

Focal lengths

The focal length (f) is positive if rays are made to converge (i.e. if the mirror is concave.) The focal length (f) is negative if rays are made to diverge (i.e. if the mirror is convex).

7.3.2 Spherical mirror formulae

The distance of the object to the mirror (s) and the distance of the image to the mirror (s') are related to the radius of curvature (R) and the focal length of the mirror (f) by

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$$

where

$$f=\frac{R}{2}$$

Note:

(1) When applying the above formulae a ray diagram should *always* be drawn to check the correctness of your answers. (See next worked example for an illustration.)

(2) The correct sign conventions must be used.

(3) The formulae are sufficiently accurate only when the rays make small angles ($< 15°$) with the principal axis. Hence they are known as 'paraxial' or 'near to the axis' ray equations.

7.3.3 Linear magnification

The linear magnification (m) is the ratio of the image height (y') to the object height (y), or, the image distance to the object distance.

$$m = \frac{y'}{y} = -\frac{s'}{s}$$

If **m is positive**, image is **upright**.
If **m is negative**, image is **inverted**.
If **$m > 1$**, image is **magnified**.
If **$m < 1$**, image is **diminished**.

Worked example Find the nature of the image formed of an object that is 1.00 m in front of a convex mirror having a radius of curvature of 0.50 m.

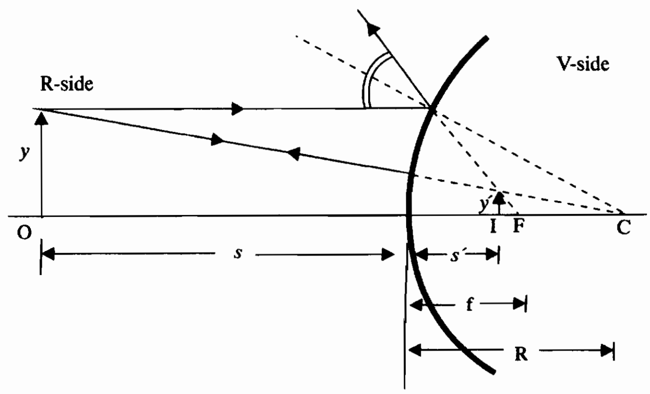


Figure 7.7

Solution Always draw a ray diagram that is approximately to scale.
Since the object is on the R-side of the mirror (see Figure 7.7), s is positive and since the centre of curvature C is on the V-side, r and f are negative.

Data given:
$s = 1.00$ m
$R = -0.50$ m
Asked for:
$s' = ?$
$m = ?$

The mirror formulae we use are accurate only when the light rays make small angles with the axis. Hence, the object height y must be appropriately small:

Now, $$f = \frac{R}{2}$$

$$= -0.25 \text{ m}$$

and, $$\frac{2}{R} = \frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

i.e. $$\frac{1}{s'} = \frac{1}{f} - \frac{1}{s}$$

$$= \frac{-1}{0.25} - \frac{1}{1.00}$$

$$= \frac{-5}{1.00} \text{ m}^{-1}$$

$$\therefore \; s' = -0.20 \text{ m}$$

Also, magnification $$m = -\frac{s'}{s}$$

$$= -\frac{-0.20}{1.00}$$

$$\therefore \; m = 0.20$$

Therefore, the image is virtual (s' is negative), upright (m is positive), diminished (0.20) and it appears 0.20 m behind the mirror.

Problem A motor car driver sees an image of a following car in his convex driving mirror of radius of curvature 1.20 m. Assuming the following car to be 1.50 m high and to be 13.5 m in front of the mirror, find the position and size of the image.
(Ans: –0.574 m; 63.3 mm)

Keyword
Mirror.

Important equations

$$\boxed{\frac{1}{s} + \frac{1}{s'} = \frac{2}{R} = \frac{1}{f}}$$

$$\boxed{m = -\frac{s'}{s}}$$

7.4 Lenses

7.4.1 Thin lenses

A lens is an optical system that includes two refracting surfaces. This book only considers the case where the two spherical surfaces are sufficiently close together that the distance between them can be neglected. This is called a **thin lens.**

Sign convention

The sign convention used here for s, s' and R is the 'real is positive' convention as met in most textbooks. It is not the only sign convention in use and other texts may give you a choice of conventions. The pragmatic approach is that you may use any convention you like as long as it will enable you to obtain the correct answer *every time.* You should always ensure that you make all steps in your answers clear so that it is possible to follow your analysis step by step from the initial data through to the final answer. The rules are basically the same as for mirrors with the real and virtual sides reversed. The real side of a lens is behind the lens.

7.4.2 Rules for drawing ray diagrams involving thin lenses

There are three **principal rays** that can be used to relate object and image positions.

- One ray used is the same for both converging and diverging lenses.

  Any ray that passes through the centre of a lens continues in a straight line (Figure 7.8).

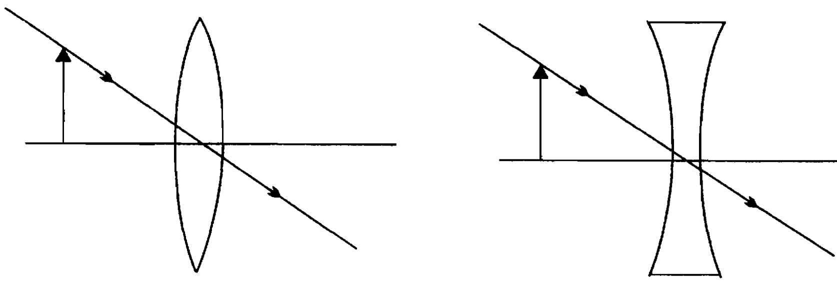


Figure 7.8

- For converging lenses, the other two rays used are as follows:

  Any ray that travels parallel to the principal axis of a converging lens will emerge from the lens through the focus. Any ray that approaches a converging lens through its focus will emerge parallel to its principal axis. (See Figure 7.9.) (The principal axis is the line that passes through both surfaces of the lens and is perpendicular to both lens surfaces.)

Note: Lenses have focal points on both sides of the lens.

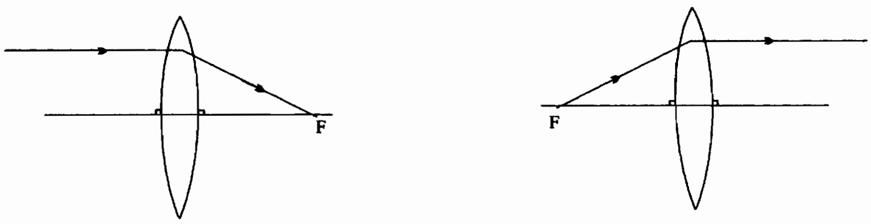


Figure 7.9

- For diverging lenses, the other two rays are as follows:

  Any ray that approaches a diverging lens parallel to its principal axis will emerge from the lens as though it came from the focus on the approach side of the lens. Any light that approaches a diverging lens as though it were converging to the focus on the exit side of the lens will emerge from the lens travelling parallel to the principal axis of the lens. (See Figure 7.10.)

Figure 7.10

7.4.3 Lens formulae

The thin lens equation

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$$

where s, s' and f are the object distance, the image distance and the focal length respectively.

The lens maker's equation

The focal length can be calculated from the characteristics of the lens. For a lens in air, f can be obtained from the Lens maker's equation, which is

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

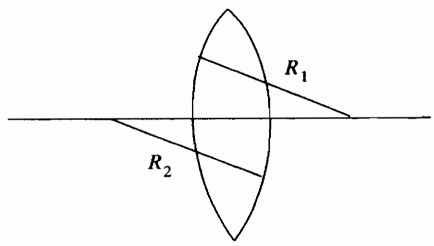


Figure 7.11

7.4.4 Linear magnification

The linear magnification m is the ratio of the image height (y') to the object height (y). This can be shown to equal the ratio of the image distance to the object distance:

$$m = \frac{y'}{y} = -\frac{s'}{s}$$

the sign convention for linear magnification is the same as for mirrors.

7.4.5 Lens power

The **power** (P) in **dioptres** (D) of a thin lens of focal length f in air is given by

$$P = \frac{1}{f} \quad \text{where } f \text{ is in metres.}$$

Worked examples

(a) An equi-convex lens is made of glass of refractive index 1.55 and has a focal length in air of 200 mm. Determine the radius of curvature of each side.

Solution Equi-convex means that the radius of curvature is the same from each side (say R) (but by the sign convention $R_2 = -R_1 = -R$) (see Figure 7.11).

Hence from the Lens maker's equation:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{200} = (1.55-1)\left(\frac{1}{R} - \frac{1}{-R}\right)$$

$$\frac{2}{R} = \frac{1}{200 \times 0.55}$$

$$\therefore \quad R = 220 \text{ mm}$$

(b) An object is 300 mm from the biconvex lens of focal length 200 mm. Where will the image appear?

Solution

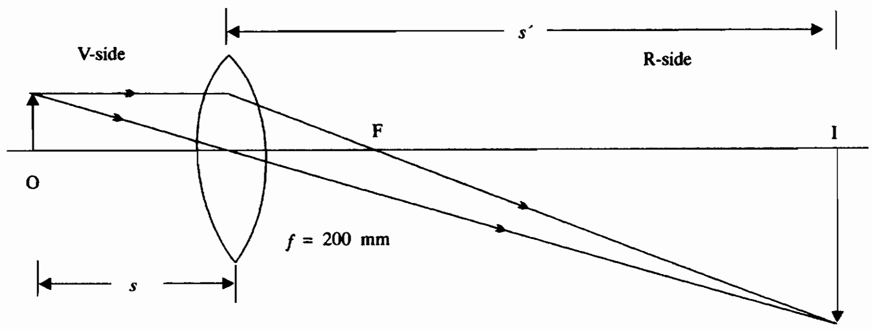


Figure 7.12

Since the lens is biconvex (converging) and is situated in air, the focal length is positive. And since the object is real, the object distance is positive.

Data given:

$f = +200$ mm

$s = +300$ mm

Asked for:

$s' = ?$

The appropriate equation for a thin lens in air is:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

rearranging

$$\frac{1}{s'} = \frac{1}{f} - \frac{1}{s}$$

$$= \frac{1}{200} - \frac{1}{300}$$

$$= \frac{1}{600}$$

$$\therefore \; s' = 600 \text{ mm}$$

Problem

(a) An illuminated object is 1 m in front of a convex lens of 150 mm focal length. Where will the image appear? If the object is 80 mm high how high will the image be?

(b)Where would the image have been if the lens were concave instead of convex and with the same focal length?

(Ans: (a) 176.4 mm, 14.1 mm; (b) –130 mm)

Keywords

Lenses, thin lenses.

Important equations

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

$$\boxed{m = \frac{y'}{y} = -\frac{s'}{s}}$$

$$\boxed{\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

7.5 Optical instruments

7.5.1 Thin lenses in contact

If two or more thin lenses are in contact the power of the combination is the sum of the powers of each lens:

$$P = P_1 + P_2 + P_3 + \ldots$$

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \ldots$$

7.5.2 Thin lenses not in contact

An image formed by one lens can serve as the object for a second lens.

Worked example An object is placed 30 mm in front of a combination of lenses ($f_a = 10.0$ mm, $f_b = 12.5$ mm), which are placed 35 mm apart. Determine the nature and position of the image,

(a) by ray tracing;

(b) by using the lens formulae.

Solution See Figure 7.13.

(a)

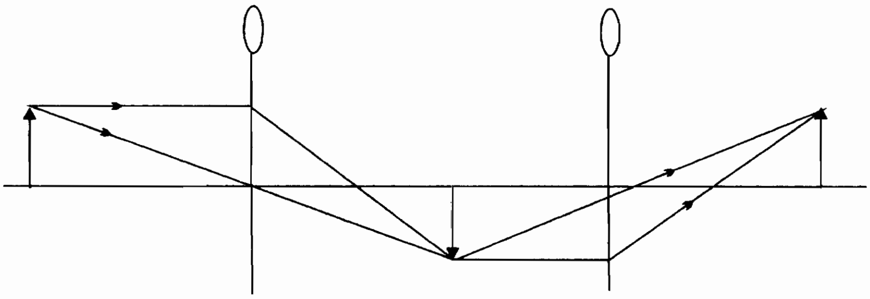


Figure 7.13

(b) First lens: $s_1 = 30$ mm, $f_1 = 10$ mm

$$\frac{1}{s_1} + \frac{1}{s_1'} = \frac{1}{f}$$

$$\frac{1}{30} + \frac{1}{s_1'} = \frac{1}{10}$$

$$\therefore \quad s_1' = 15 \text{ mm}$$

Use this as the object for the second lens

$$s_2 = 35 - 15 = 20 \text{ mm}, f_2 = 12.5 \text{ mm}$$

$$\frac{1}{s_2} + \frac{1}{s_2'} = \frac{1}{f_2}$$

$$\frac{1}{20} + \frac{1}{s_2'} = \frac{1}{12.5}$$

$$\therefore \quad s_2' = 33 \text{ mm}$$

$$m_1 = -\frac{s_1'}{s_1} = -\frac{15}{30} = -0.5$$

$$m_2 = -\frac{s_2'}{s_2} = -\frac{33}{20} = -1.67$$

$$m_{total} = m_1 m_2 = -0.5 \times -1.67 = +0.83$$

Therefore the image will be real, upright, diminished and 33 mm behind the second lens.

Problem

(a) An image of an object 1600 mm away is formed on a screen by placing a convex lens between the object and the screen. What is the focal length of the lens if the magnification of the image is 3.00? (*Hint*: $m = -3.00$)

(b) By placing a concave lens in contact with the lens, a sharp image of the object on the screen is formed when the lens is placed precisely at the midpoint between the object and the screen. What is the focal length of the concave lens?

(Ans: (a) 300 mm; (b) 1200 mm)

7.5.3 Simple magnifier

When a single converging lens is used as a magnifier, the object to be examined is placed a little closer than the principal focus. An enlarged, erect, virtual image of the object is then seen. The image should be at the distance of **most distinct vision** (or 'least distance of distinct vision'), which is about 250 mm from the eye (Figure 7.14).

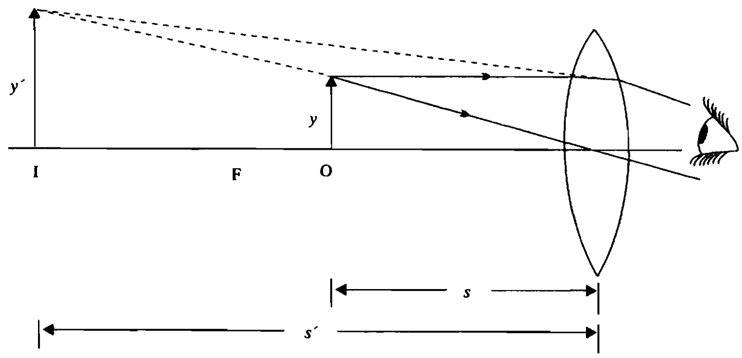


Figure 7.14

The eye is placed as close to the lens as possible, so the image distance s' is ≈ 250 mm.

Now the thin lens equation gives:

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$$

$$\frac{s'}{s}+1=\frac{s'}{f}$$

Now, $s' = -250$ mm

Therefore, $$\frac{s'}{s}=\frac{-250}{f}-1$$

Linear magnification where f is in mm

$$m=\frac{-s'}{s}=\frac{250}{f}+1$$

A magnifier is often used as an eyepiece in an optical instrument in combination with other image-forming lenses (e.g. a microscope). In such a situation, experienced users of optical instruments frequently place the second-last image formed by a multi-lens instrument at the principal focus of the eyepiece so that parallel rays enter the eye and the final image is at infinity (relaxed vision) rather than at 250 mm. Muscles that adjust the eye are then relaxed, for least eye-strain.

7.5.4 The eye and the camera

The eye and a simple camera consists of a single lens that forms real images. The essential elements of the camera are seen in Figure 7.15.

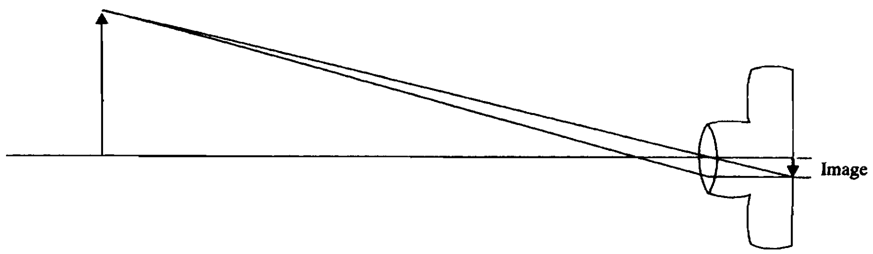


Figure 7.15

7.5.5 The slide projector

In a slide projector, the concave mirror and condenser lenses gather and direct the light from the lamp so it will enter the projection lens after passing through the slide, as shown in Figure 7.16.

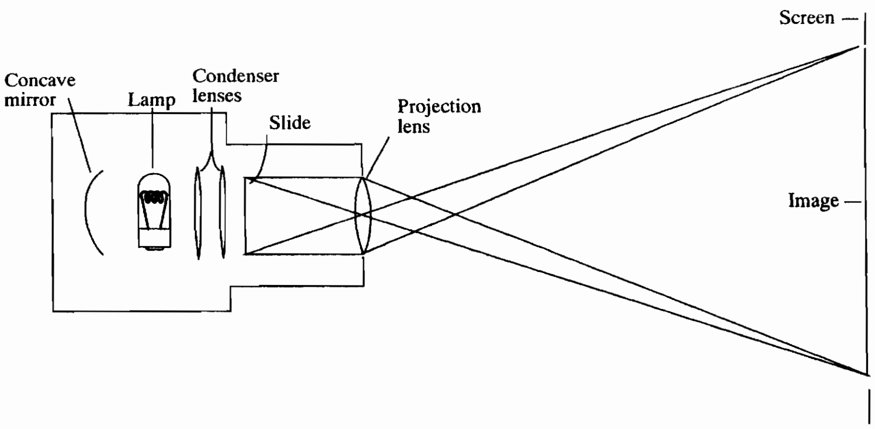


Figure 7.16

7.5.6 Telescopes

The astronomical telescope

Rays that enter the system parallel leave it parallel, as shown in Figure 7.17.

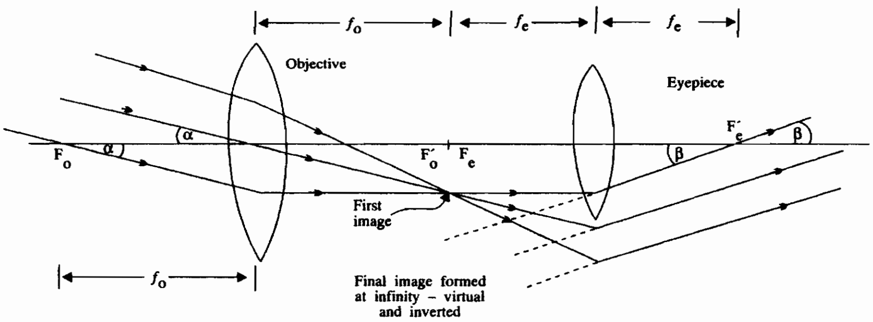


Figure 7.17

The final image is formed at infinity — virtual and inverted.

The distance, D, between the lens is given by $D = f_o + f_e$

The **angular magnification, *M*,** is the ratio of the angle β subtended at the eye by the image, to the angle α, which the object subtends at the objective lens.

$$M = \frac{\beta}{\alpha} = -\frac{f_o}{f_e}$$

Magnifications greater than about 1000 are rarely used in astronomy.

The Galilean telescope

To obtain an upright image, the eyepiece lens needs to be concave. This is called a Galilean telescope and is shown in Figure 7.18.

The same equations apply but it must be remembered that f_e is now negative.

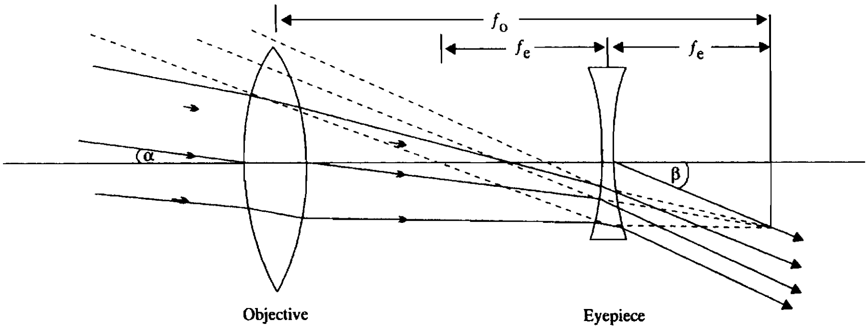


Figure 7.18

7.5.7 The compound microscope

When an angular magnification larger than that attainable with a simple magnifier is desired, it is necessary to use a **compound microscope**, usually called simply a **microscope**. The essential elements of a microscope are illustrated in Figure 7.19. The object, O, to be examined is placed just beyond the first focal point, F_1, of the **objective** lens, which forms a real and enlarged image, I_1. This image lies just within the first focal point F_e of the **eyepiece**, which forms a final virtual image of I_1 at I_2.

For the image to be at the least distance of distinct vision, s_e' must equal 250 mm, and the angular magnification M is given by:

$$M = \frac{250 \times s_o'}{f_o \times f_e}$$

(where all quantities are in mm).

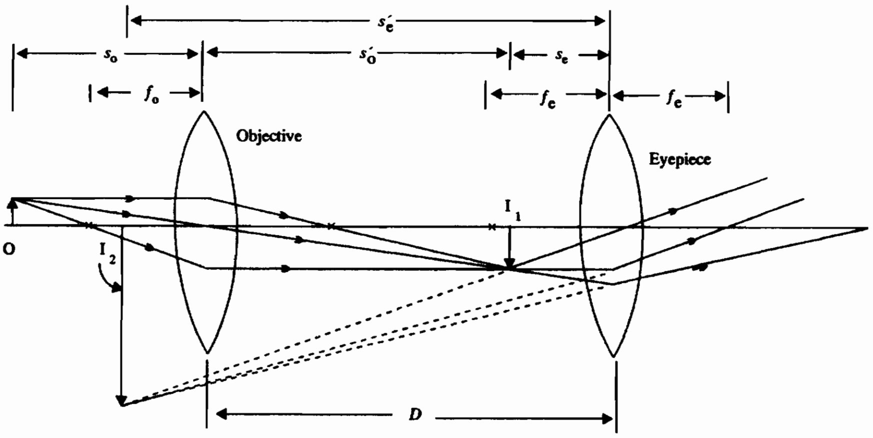


Figure 7.19

Worked examples

(a) How could an astronomical telescope be made from two lenses with powers +2.00 and +8.00 dioptres?

Solution

$$f_1 = \frac{1}{D_1} = \frac{1}{2} = 0.5 \text{ m} = 500 \text{ mm}$$

$$f_2 = \frac{1}{D_2} = \frac{1}{8} = 0.125 \text{ m} = 125 \text{ mm}$$

Long focal length objective $f_o = 500$ mm

Short focal length eyepiece $f_e = 125$ mm

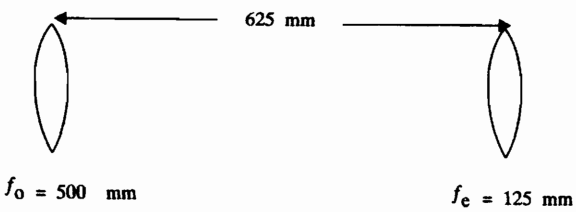


Figure 7.20

$$M = -\frac{f_o}{f_e} = -\frac{500}{125} = -4.00$$

distance between lenses = 500 + 125 = 625 mm

(b) A compound microscope has an objective of focal length 5.0 mm. The eyepiece has a focal length of 40 mm. The lenses are separated by a distance of 245 mm.

(i) Where must the object be placed in order that the final image be formed at the distance of most distinct vision?

(ii) What is the linear magnification produced by

- the objective;
- eyepiece;
- the complete system?

Solution

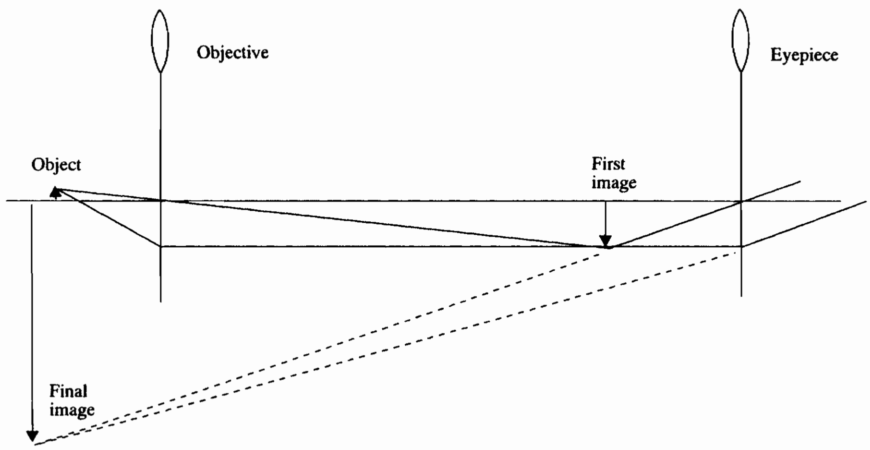


Figure 7.21

(i) For the eyepiece:

Object distance $s_e = ?$

$$\frac{1}{f} = \frac{1}{s_e} + \frac{1}{s_e'}$$

Image $s_e' = -250$ mm $\qquad f_e = 40.0$ mm

$$\frac{1}{40} = \frac{1}{s_e} + \frac{1}{-250}$$

$$\therefore\ s_e = 34.5 \text{ mm}$$

For the objective: $s_o = ?$

$s_o' = 245 - 34.5 = 210.5$ mm $\qquad f_o = 5.0$ mm

$$\frac{1}{f} = \frac{1}{s_o} + \frac{1}{s_o'}$$

$$\frac{1}{5} = \frac{1}{s_o} + \frac{1}{210.5}$$

$$\therefore \; s_o = 5.12 \text{ mm}$$

(ii) Objective:

$$m_o = -\frac{s_o'}{s_o} = -\frac{210.5}{5.12} = -41.1$$

Eyepiece:

$$m_e = -\frac{s_e'}{s_e} = -\frac{-250}{34.5} = +7.25$$

Complete system, the total magnification is:

$$m_{\text{total}} = m_o \times m_e$$

$$m_t = -41.1 \times 7.25$$

$$m_t = -298$$

Note: The angular magnification:

$$M = \frac{250 \times 210.5}{5 \times 40} = 263$$

Problem Convex lenses of focal lengths 30.0 mm and 50.0 mm are used respectively as the objective and eyepiece of a microscope. If the object is 35.0 mm from the objective and the final image is 250 mm from the eyepiece, what is the distance between the centres of the lenses?
(Ans: 252 mm)

Keywords
Magnifier, eye, camera, telescope, microscope.

Important equation

$$\boxed{M = -\frac{f_o}{f_e}}$$

7.6 Polarization

7.6.1 Polarization of electromagnetic waves

The normal random orientation of the planes of vibration of transverse waves such as an electromagnetic radiation may be altered by transmission or reflection:

Transmission:

- through natural polarizing crystals;
- through commercial polaroid filters;
- through special 'double refracting' crystals.

Reflection:

- at a mirror or reflecting surface (e.g. off the sea at the beach);
- by scattering off suspended particles (e.g. dust in the air or colloidal particles in suspension).

Light that is not polarized has vibrations in any direction perpendicular to the direction of propagation.

Light that is polarized linearly, say along the y axis, has vibrations only in the positive and negative y directions while it is moving along the x axis. For example, incident unpolarized light may have been passed through a polarizing filter, with axis along the y axis, so that it allows only vibrations in the y direction to pass through. (See Figure 7.22.)

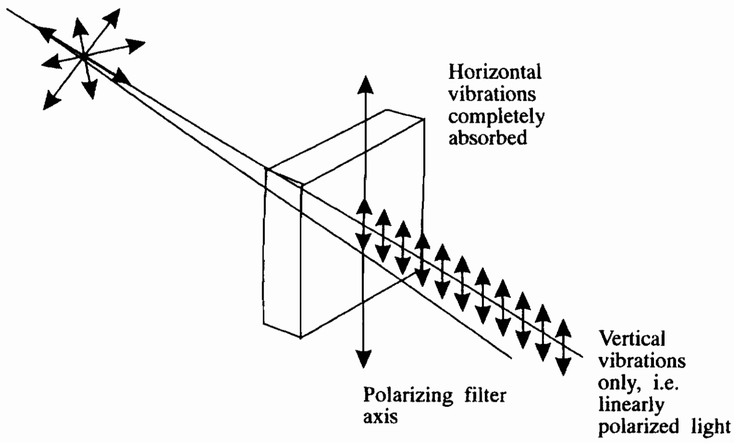


Figure 7.22

7.6.2 Malus' Law (polarization due to transmission)

When incident electromagnetic radiation is polarized at a direction θ to the polarizing axis of the material (filter) through which it is passing, only the component of the vector parallel to the direction of the axis passes through — the remainder is obstructed, as shown in Figure 7.23.

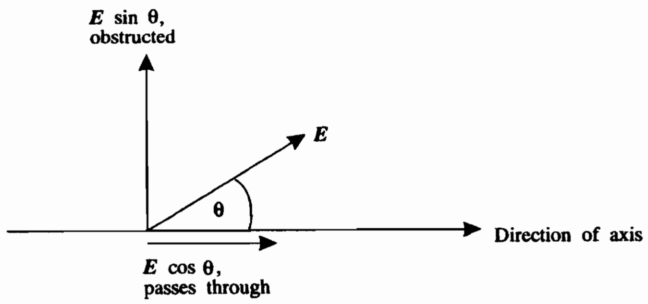


Figure 7.23

Since the intensity of the radiation is proportional to the square of the amplitude, it follows that the intensity is reduced by a factor of $(\cos\theta)^2$, as stated in Malus' Law:

$$I = I_{max}(\cos\theta)^2$$

For non-polarized incident light, the intensity will be reduced by half since, on the average of the infinite directions of E, half will pass through and half will be obstructed, or, in formal mathematics:

$$I_{av} = \frac{1}{2\pi}\int_{-\pi}^{\pi} I_{max}(\cos\theta)^2\,d\theta = \frac{1}{2}\frac{I_{max}}{2\pi}\int_{-\pi}^{\pi}(1+\cos 2\theta)\,d\theta = \frac{1}{2}I_{max}$$

7.6.3 Brewster's Law (limiting case of polarization due to reflection)

When electromagnetic radiation is forced to change direction, such as in the case of reflection, more limitations are now placed on its possible directions of vibration since:

- the wave can have no vibrations parallel to the original direction of propagation; and
- the wave can have no vibrations parallel to the new direction of propagation.

Brewster's Law describes the critical case where the incident angle is equal to the **polarizing angle**, so that no light whatever is reflected except a portion of the light in which the electric vector is perpendicular to the plane of incidence. In that case Brewster observed that the reflected ray and the refracted ray were at 90° to each other, as seen in Figure 7.24.

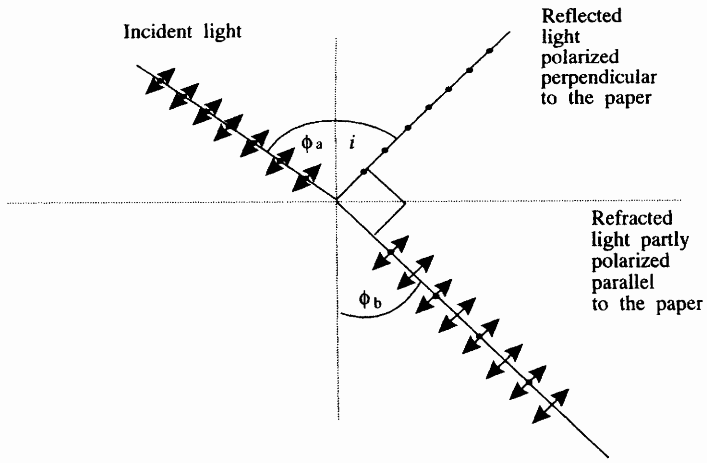


Figure 7.24

Using Snell's Law, we can find the polarizing angle, ϕ_p

$$n_a \sin \phi_p = n_b \sin \phi_b$$

$$= n_b \sin\left(\frac{\pi}{2} - \phi_p\right) = n_b \cos \phi_p$$

$$\therefore \quad \tan \phi_p = \frac{n_b}{n_a} \quad \text{(Brewster's Law)}$$

Worked example How much light, originally unpolarized, passes through two sheets of polaroid material inclined at an angle of 30° to each other?

Solution Since the incident light is unpolarized, the intensity after the first sheet is

$$\frac{1}{2} I_{max}$$

For the intensity after the second sheet, apply Malus' Law:

$$I = I_{incident} (\cos \theta)^2 = \frac{1}{2} I_{max} (\cos \theta)^2 = \frac{1}{2} (\cos 30)^2 I_{max} = 0.375\, I_{max}$$

Problem Unpolarized light falls on two polarizing sheets placed on top of one another. What must be the angle between the characteristic directions of the sheets if the intensity of the transmitted light is:

(a) 1/3 of the intensity of the beam transmitted by the first sheet;

(b) 1/3 of the intensity of the incident beam?

(Assume that the polarizing sheet is ideal, i.e. it reduces the intensity of unpolarized light by exactly 50%.)

(Ans: (a) 54.7°; (b) 35.3°)

Keywords

Polarization, Malus' Law, Brewster's Law.

Important equations

$$\boxed{I = I_{max}\,(\cos\theta)^2}$$

$$\boxed{\tan\phi_p = \frac{n_b}{n_a}}$$

7.7 Physical optics

7.7.1 Interference and diffraction

Interference refers to any situation in which two or more waves overlap in space.

Diffraction effects occur when incident waves bend around obstacles or spread out from apertures in the familiar manner of water waves. Although many effects involving light can be analysed in terms of rectilinear propagation, careful investigation reveals that light waves also diffract into regions that would not be illuminated if only straight line propagation were possible.

It is convenient to separate the phenomena associated with the diffraction of light into two categories. **Fresnel diffraction** refers to cases in which the viewing screen or the observer's eye is relatively close to the diffracting obstacle or aperture. **Fraunhofer diffraction** refers to cases in which both the source and the eye or screen are at very large distances from the diffracting obstacle or aperture.

Electromagnetic waves combine or interfere according to the superposition principle previously discussed regarding mechanical waves. In place of net particle displacements, one finds a resultant electric or magnetic field vector. Thus for two electromagnetic waves:

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$$

For light polarized in one direction, the vector notation can be dropped and the problem examined as for mechanical waves in one dimension. E depends on the amplitudes of E_1 and E_2 and on their **relative amplitudes** at the point of interest.

Interference is only observed if the phase differences between the waves remains constant in time. Sources of such waves are said to be **coherent.**

Interference leads to alternate regions of light and dark on a screen. This occurs because light intensity depends on resultant amplitude squared, which in turn depends on the *path difference from each source to the point of interest* on the screen. Such patterns occur because of the wavelike nature of light.

7.7.2 Young's double slit interference

When coherent light is shone onto two slits a distance d apart, a pattern of alternate light and dark fringes is formed on a screen a distance D away. (Figure 7.25 is exaggerated in the y direction and in the following discussion θ is considered to be small.)

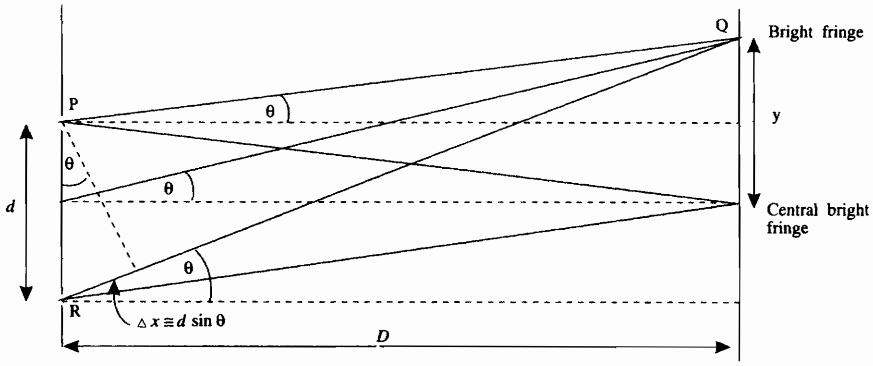


Figure 7.25

The path difference between PQ and RQ $= \Delta x$

$$= d \sin \theta$$

This is an approximation, which works because $D >> \lambda$

Constructive interference (i.e. bright fringes) occurs when

$$d \sin \theta = n\lambda \qquad n = 0, \pm 1, \pm 2$$

Destructive interference (i.e. dark fringes) occurs when

$$d \sin \theta = \frac{(2n+1)\,\lambda}{2} \qquad n = 0, \pm 1, \pm 2$$

$$\sin \theta = \frac{(2n+1)\,\lambda}{2d} \qquad n = 0, \pm 1, \pm 2$$

Fringe separation

For small θ, in radians, $\sin\theta \approx \theta \approx \tan\theta$

$$\therefore\ \sin\theta = \frac{n\lambda}{d} \approx \theta \approx \tan\theta = \frac{y}{D}$$

$$y = \frac{n\lambda D}{d}$$

So for the nth bright fringe $\qquad y_n = \dfrac{n\lambda D}{d}$

for the $(n+1)$th bright fringe $\qquad y_{n+1} = \dfrac{(n+1)\lambda D}{d}$

separation of fringes $\qquad = y_{n+1} - y_n$

$$= \frac{\lambda D}{d}$$

$$\text{Fringe separation} = \frac{\lambda D}{d}$$

and $\qquad \text{Angular fringe separation} = \dfrac{\lambda}{d}$

Note: The equations in this section all show that the spreading of fringes is proportional to λ.

7.7.3 Double slit intensity

If two waves (each of amplitude E_o and intensity I_o) arriving at a screen are out of phase by an angle $\Delta\phi$, the resultant wave has an amplitude

$$E = 2E_o \cos\left(\frac{\Delta\phi}{2}\right)$$

and intensity $\qquad I = 4\,E_o^2 \cos^2\left(\dfrac{\Delta\phi}{2}\right)$

$$= 4\,I_o \cos^2\left(\frac{\Delta\phi}{2}\right)$$

where $\qquad \Delta\phi = \dfrac{2\pi}{\lambda} \times d\sin\theta$

7.7.4 Thin film interference

A situation is shown in Figure 7.26 in which part of the light is reflected from the top surface and part from the bottom surface. (This diagram is exaggerated, the incident angle should be close to 0.)

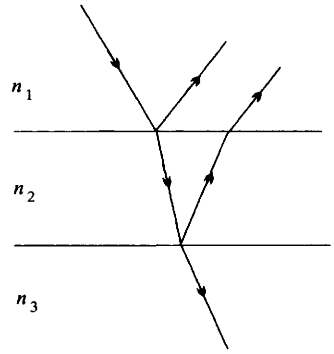


Figure 7.26

Note: On reflection, a wave experiences a phase change of π if $n_1 < n_2$, but no phase change occurs if $n_1 > n_2$.

The light reflected from the top and bottom surfaces combine.

If $n_2 > n_1$ and $n_3 > n_2$ then, dark fringes will be observed when the difference in phase angle $\Delta\phi$ is given by:

$$\Delta\phi = (2m + 1)\,\pi$$

where $m = 0, 1, 2, \ldots$

From the diagram $$\Delta\phi = 2\pi\left(\frac{2n_2 d}{\lambda}\right)$$

where d is the thickness of the film.

Worked example A thin film of refractive index 1.54 is coated onto a material with higher refractive index. The unit is placed in sunlight and designed to have its weakest reflection at a wavelength of 600 nm where the solar spectrum peaks. What film thickness is required?

Solution Since $n_{air} < n_{film} < n_{substrate}$ a phase shift of π occurs at both surfaces.

$$\therefore \quad \Delta\phi \equiv 2\pi\left(\frac{2\nu\tau}{\lambda}\right)$$

$$= (2m + 1)\,\pi \qquad \text{For annulment } m = 0, 1, 2, \ldots$$

where $n = n_{film}$

$$\therefore \quad 2nt = (2m + 1)\frac{\lambda}{2}$$

with $m = 0$ $$t = \frac{\lambda}{4} \times \frac{1}{n}$$

(This is the principle of the quarter-wave plate.)

$$t = \frac{1}{1.54} \times \frac{600 \times 10^{-9}}{4} = 97 \text{ nm}$$

(Films about 100 nm thick are commonly used for this purpose.)

7.7.5 Single slit diffraction

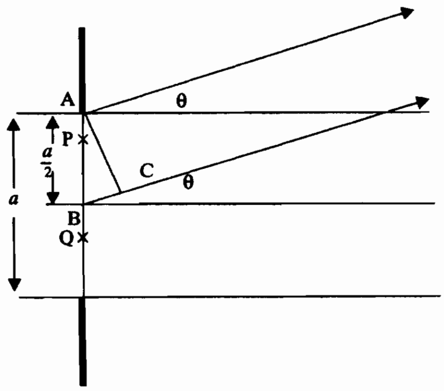


Figure 7.27

For a single slit of finite width a, we consider waves coming from point A at the side of the slit and from the midpoint B as shown in Figure 7.27. These two waves travelling at an angle θ to the horizontal will have a path difference $\frac{a}{2}\sin\theta$. The same situation occurs for the pair of waves coming from any point in the top half of the slit and the corresponding point half a slit width lower in the bottom half (e.g. P and Q in Figure 7.27). For any of these the path difference between the two waves is the same, $\frac{a}{2}\sin\theta$. When this path difference is $\frac{\lambda}{2}$, all the waves will cancel precisely, giving an extinction. This is the position of the first minimum. This occurs when:

$$\frac{a}{2}\sin\theta = \frac{\lambda}{2}$$

$$a\sin\theta = \lambda$$

The general equation for all minima caused by single slit diffraction is:

$$\sin\theta = \frac{n\lambda}{a} \qquad n = \pm 1, \pm 2$$

Note: $n = 0$, is a maximum not a minimum.

7.7.6 Single slit intensity

To determine the intensity we have to recognize that each point within the slit acts as a source of waves and that each will be progressively more out of phase with the top wave as we move down the slit. The wave disturbances reaching an arbitrary point can be represented by amplitude and phase on a phase vector diagram (Figure 7.28).

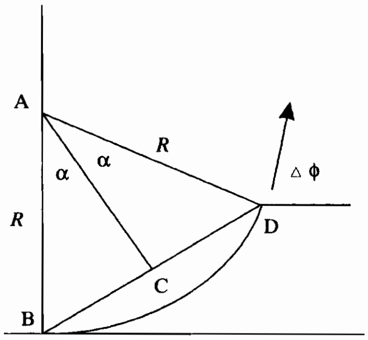


Figure 7.28

The resultant intensity I is:

$$I = I_{max} \times \frac{(\sin \alpha)^2}{\alpha^2}$$

where $$\alpha = \frac{\pi a \sin \theta}{\lambda}$$

For the case of Young's double slit experiment the above intensity is modulated due to interference effects, so that the overall intensity at an angle θ is given by:

$$I = I_{max} \times \frac{(\sin \alpha)^2}{\alpha^2} \times (\cos \beta)^2$$

where $$\alpha = \frac{\pi a \sin \theta}{\lambda} \text{ and } \beta = \frac{\pi d \sin \theta}{\lambda}$$

The resultant pattern looks as shown in Figure 7.29.

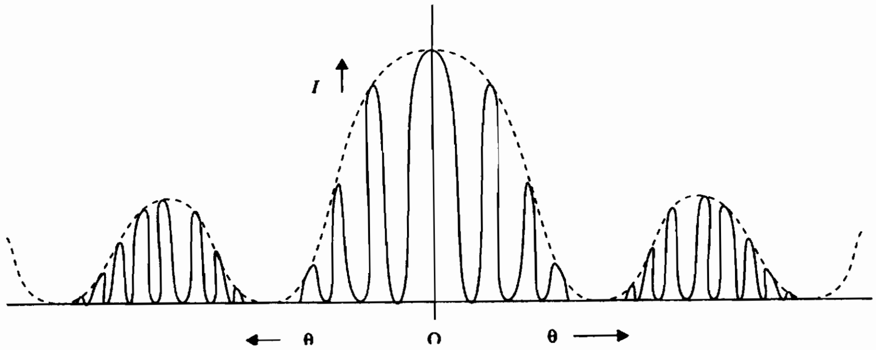


Figure 7.29

7.7.7 Circular hole diffraction

If diffraction originates from a circular hole instead of a thin slit then the effects of a number of thin slits of varying length must be summed (Figure 7.30), which results in a modification to the equation for a single slit,

The first minima occur when:

$$a \sin \theta = \pm 1.22 \lambda$$

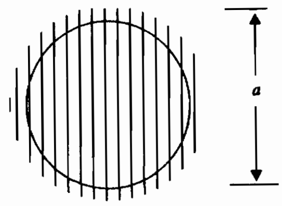


Figure 7.30

where a is the diameter of the hole.

The image of a circular hole is a series of circular fringes around the central maximum, as shown in Figure 7.31.

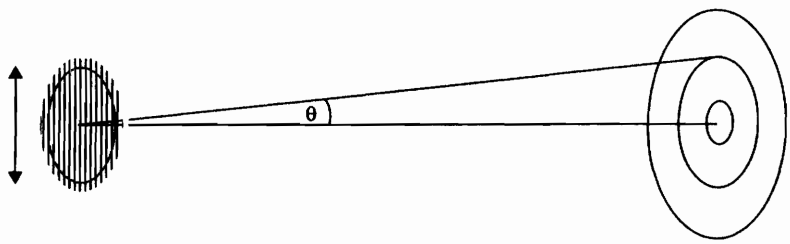


Figure 7.31

7.7.8 Resolution of (circular) sources

Two sources are considered to be resolved when the central diffraction maxima of one image coincides with the first minima of the second — this is called the

Rayleigh Criterion. When this happens it can be shown that the intensity between the central maxima falls to 81% of the peak intensity (Figure 7.32).

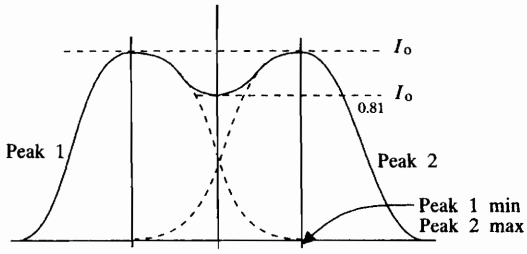


Figure 7.32

Note: This definition arises from the experimental observation that an intensity decrease to about 80% of I_0 can be just detected by the eyes and hence the two images can be seen as separate.

This theory can be applied to the use of a telescope to resolve two distant stars, or of a microscope to resolve two close microscope objects, or of the pupil of the eye in looking at two close objects. In each case, 'a' is the aperture of the objective lens. Radio telescopes have the same problem resolving radio sources.

Worked example An example of the above concept is the resolution of a distant set of headlights (1.50 m apart) at night into two distinct sources. For the eye at night, the pupil of the eye may be as large as 5 mm and the main wavelength of the headlights may be taken to be about 6×10^{-7}m (Figure 7.33).

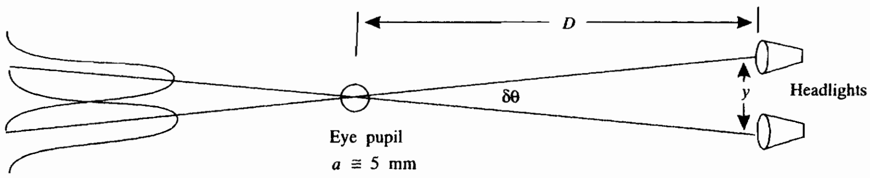


Figure 7.33

Solution

For resolution
$$a \sin \delta\theta = 1.22\,\lambda \approx a\delta\theta \approx \frac{ay}{D}$$

$$\therefore \; D = \frac{ay}{1.22\,\lambda}$$

$$\therefore \; D = \frac{1.5 \times 5 \times 10^{-3}}{1.22 \times 6 \times 10^{-7}}$$

$$= 10.3 \times 10^{3}\text{ m}$$

$$\cong 10\text{ km}$$

7.7.9 Multiple slit diffraction

As the number of slits increases from two towards infinity, changes as shown in Figure 7.34 are observed. The positions of the principal maxima are always given by

$$d \sin \theta = m\lambda$$

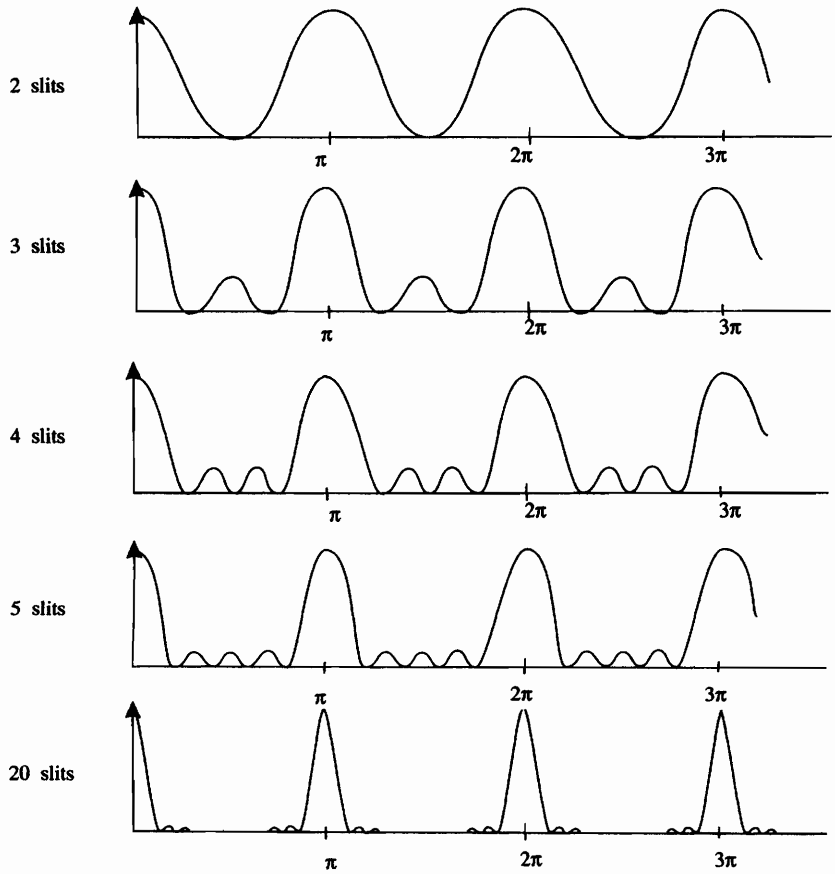


Figure 7.34

7.7.10 Diffraction gratings

A transmission diffraction grating is a piece of material that has had many narrow parallel grooves cut into it. It diffracts light in the same way as double or multiple slits. If parallel light is shone onto a diffraction grating maxima will occur when

$$d \sin \theta = m\lambda$$

where m is the order of the diffraction. Hence a single spectral line of wavelength λ will occur at different angles given by this equation, with $m = \pm 1, \pm 2, \pm 3$ etc. until θ reaches ± 90. This is shown in Figure 7.35.

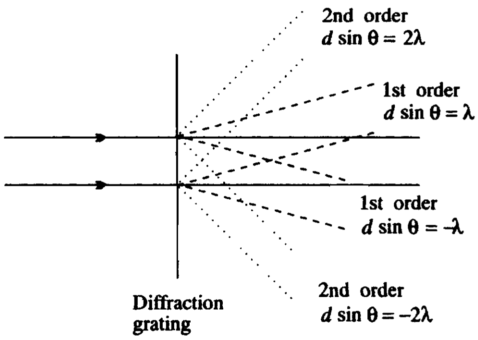


Figure 7.35

Resolution of diffraction gratings or multiple slits

To distinguish light waves the wavelengths of which are close together, the maxima (bright bands) should be as narrow as possible, that is, the grating should have high resolving power.

The resolving power is given by

$$R = \frac{\lambda}{\Delta\lambda}$$

where λ is the mean wavelength of the two close wavelengths to be resolved, and $\Delta\lambda$ is the wavelength difference between them.

From this definition it is possible to derive an alternative expression for the resolution of a grating:

$$R = mN$$

where N = total number of slits, and
m = order of spectral line used.

Thus the more slits or the higher the order, the better the resolving power.

Worked examples

(a) A grating has 600 lines mm^{-1} and is 60 mm wide.

(i) What is the smallest wavelength interval that can be resolved in the third order at $\lambda = 500$ nm?

(ii) For this wavelength and this grating, can the resolution be increased? How?

Solution

(i) Grating has 600 lines mm^{-1} and is 60 mm wide
Therefore total number of lines = 36 000
Resolving power of a grating in the third order:

$$R = \frac{\lambda}{\Delta\lambda} = mN$$

$$\therefore \quad \Delta\lambda = \frac{\lambda}{\mu N} = \frac{500 \times 10^{-9}}{36\,000 \times 3}$$

$$= 4.6 \times 10^{-12} \text{ m}$$

(ii) Resolution of grating depends on mN. Here N is fixed, as we are limited to the same grating. The other possibility is to use a higher order, say $m = 4$, but is this possible?

Here:

$$d = \frac{1}{600} \text{ mm} = 1.67 \times 10^{-6} \text{ m}$$

$$\lambda = 500 \times 10^{-9}$$

$$m = 4$$

$$d \sin\theta = m\lambda$$

$$\therefore \quad \sin\theta = \frac{4\lambda}{d} = \frac{4 \times 5 \times 10^{-7}}{1.67 \times 10^{-6}} = 1.2$$

This is undefined for θ.

∴ there is no 4th order in the spectrum

∴ resolution cannot be increased.

(b) A thin flake of mica (n = 1.58) is used to cover one slit of a double-slit arrangement (Figure 7.36). The central point on the screen is occupied by what used to be the 7th bright fringe. If λ = 550 nm, what is the thickness of the mica?

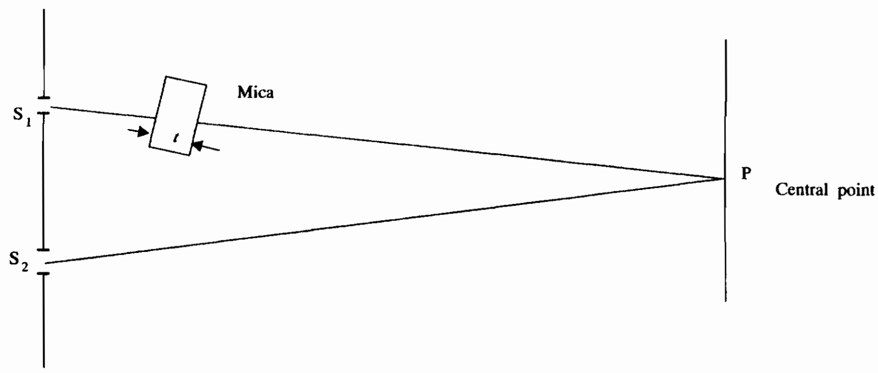


Figure 7.36

Solution Central point is now occupied by what was the 7th fringe (max), that is, the optical path difference between the waves from S_1 and S_2 must be 7λ. This is caused by the mica flake.

The paths S_1P and S_2P are identical except that for:

S_1P light traverses a mica flake of optical path = nt
S_2P light traverses air of same thickness with optical path = t

Therefore path difference $= nt - t = t(n-1) = 7\lambda$

$$t = \frac{7\lambda}{n-1} = \frac{7 \times 5.5 \times 10^{-7}\ \text{m}}{1.58 - 1}$$

$$= 6.6 \times 10^{-6}\ \text{m}$$

This principle is used in interferometry.

Problem

(a) Parallel light of wavelength 550 nm is shone onto two identical slits that are 3.3 mm apart. Bright fringes are observed on a screen 1.2 m away. How far apart are the bright fringes?

(b) If one slit is covered up, the distance from the central maximum to the 1st minimum is 4.00 mm. How wide is the slit?

(Ans: (a) 0.20 mm; (b) 0.165 mm)

Keywords

Diffraction, interference, Rayleigh Criterion, diffraction grating.

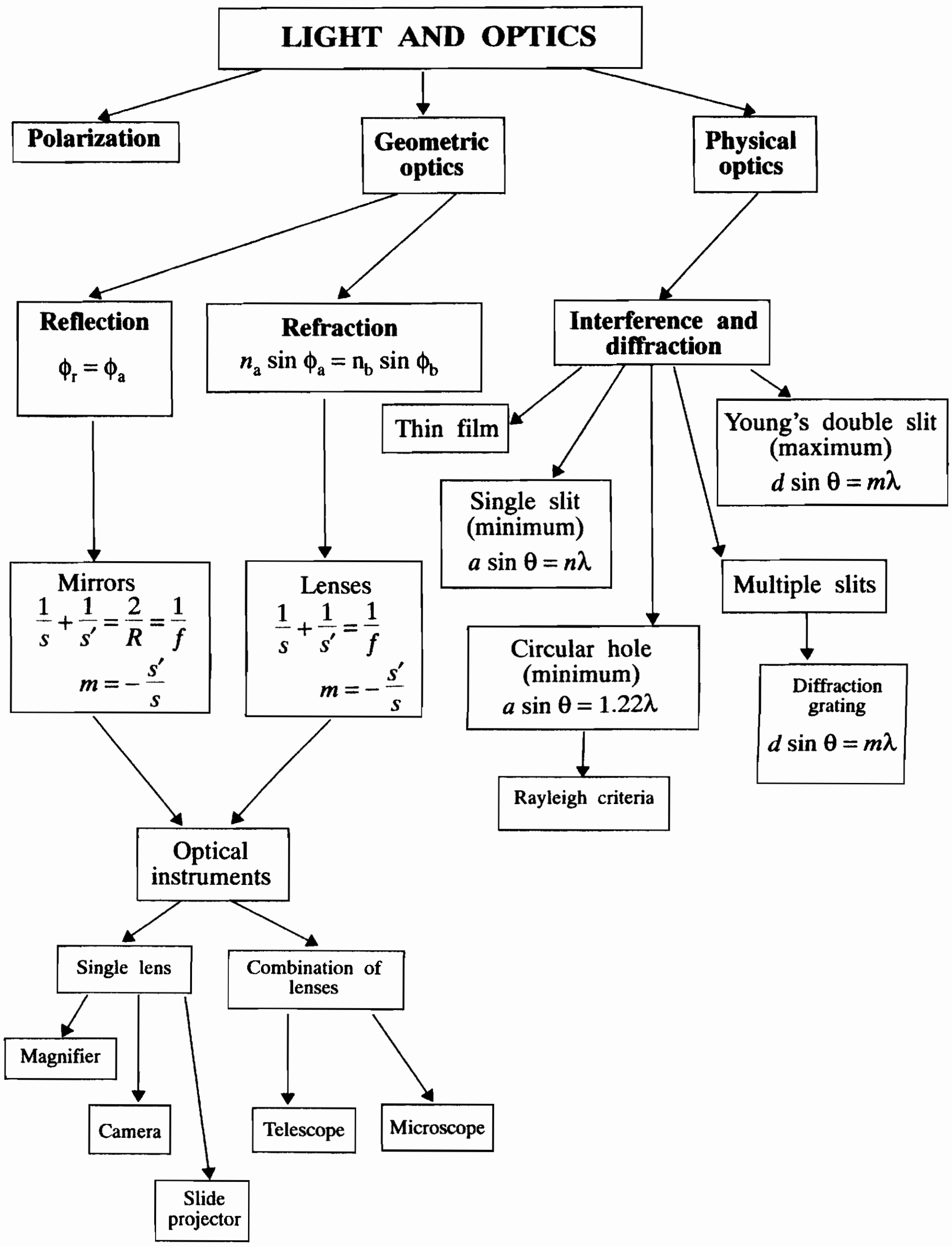
Important equations

$$\boxed{d \sin\theta = m\lambda \text{ (constructive) (double slit)}}$$

$$\boxed{a \sin\theta = n\,\lambda \text{ (dark fringes) (single slit)}}$$

$$\boxed{a \sin\theta = \pm 1.22\,\lambda \text{ (circular hole)}}$$

$$\boxed{d \sin\theta = m\lambda \text{ (diffraction grating)}}$$


LIGHT AND OPTICS
Polarization
Geometric optics
Physical optics
Reflection
$\phi_r = \phi_a$
Refraction
$n_a \sin \phi_a = n_b \sin \phi_b$
Interference and diffraction
Thin film
Young's double slit (maximum)
$d \sin \theta = m\lambda$
Single slit (minimum)
$a \sin \theta = n\lambda$
Mirrors
$\frac{1}{s} + \frac{1}{s'} = \frac{2}{R} = \frac{1}{f}$
$m = -\frac{s'}{s}$
Lenses
$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$
$m = -\frac{s'}{s}$
Multiple slits
Circular hole (minimum)
$a \sin \theta = 1.22\lambda$
Diffraction grating
$d \sin \theta = m\lambda$
Rayleigh criteria
Optical instruments
Single lens
Combination of lenses
Magnifier
Camera
Slide projector
Telescope
Microscope

CHAPTER 8

Atomic and nuclear physics

8.1 Waves and particles

8.1.1 Planck's quantum hypothesis

When any electromagnetic radiation is emitted by a body the values of energy are not continuous but are quantized such that the energy values are an integer times a minimum amount of energy, E:

$$E = \mathrm{h}f$$

where h is Planck's constant and equals 6.63×10^{-34} J s, f is the frequency of the radiation.

8.1.2 The photoelectric effect

When light is shone on a metal surface, electrons can be emitted. A curious aspect of this phenomenon is that electrons are only emitted if the frequency of the light is greater than a particular frequency that is characteristic of a particular metal.

To explain the photoelectric effect (see Figure 8.1), Einstein extended Planck's hypothesis by saying that the energy of radiation is emitted as bundles of energy called **quanta** or **photons**. Each photon carries energy

$$E = \mathrm{h}f$$

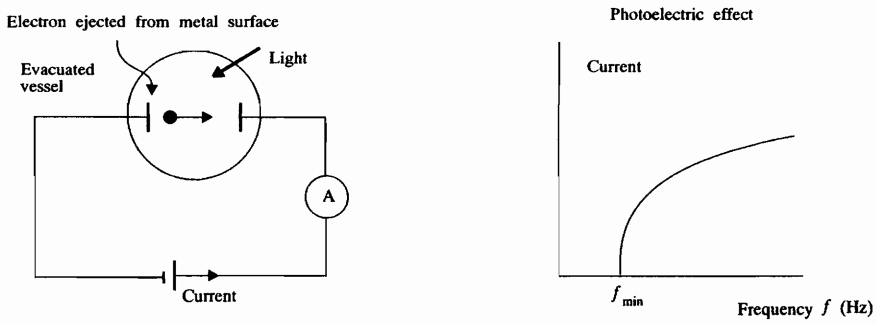


Figure 8.1

By this theory, when light is shone on a metal surface no electrons are emitted unless the energy imparted by the incident photon to individual electrons in the metal is sufficient to overcome the energy barrier, Φ, of the metal surface. When $hf < \Phi$, no electrons escape and no current flows in the circuit shown. When the energy imparted by the photon exceeds the energy, Φ, electrons escape and a current is observed. The minimum frequency at which current flows is known as the **threshold frequency**, f_0. The associated wavelength, $\lambda_0 = \frac{c}{f_0}$, is known as the **threshold wavelength**. The energy barrier Φ of a metal surface is known as the **work function.**

At the threshold function:

$$\Phi = hf_0$$

When $hf > \Phi$ the kinetic energy of the emitted (K) will be greater than 0. If all the energy of an incident photon is imparted to an electron, the kinetic energy of the escaping electron will be given by:

$$K = hf - \Phi$$

This is the maximum possible value for the kinetic energy of an electron (K_{max}).

Worked example The work function of nickel is 4.9 eV. When UV light of frequency 2.0×10^{15} Hz is shone on a nickel surface, what will be the maximum velocity of the emitted electrons?

Solution

$$\begin{aligned} K_{max} &= hf - \Phi \\ &= 6.63 \times 10^{-34} \times 2 \times 10^{15} - 4.9 \times 1.602 \times 10^{-19} \\ &= 5.4 \times 10^{-19}\ \text{J} \end{aligned}$$

Put $K_{max} = \frac{1}{2}mv^2$ with $m = 9.11 \times 10^{-31}$ kg

to give $v = 1.1 \times 10^6$ m s^{-1}

Thus was introduced the idea that all electromagnetic radiation has both a wave and a particle nature.

8.1.3 Particles as waves — the de Broglie hypothesis

Just as electromagnetic radiation can act as a particle, entities like electrons and neutrons, which are normally thought of as particles, can act as waves. In 1924, de Broglie proposed a relation between momentum, mv, of a particle and the wavelength, where

$$\lambda = \frac{h}{mv}$$

Worked example What is the wavelength of an electron with a velocity of 5×10^6 m s^{-1}?

Solution

$$\lambda = \frac{h}{mv}$$

$$= \frac{6.63 \times 10^{-34}}{9.11 \times 10^{-31} \times 5 \times 10^6}$$

$$= 1.46 \times 20^{-10} \text{ m}$$

Problem

(a) The work function of sodium is 3.68×10^{-19} J. If electrons are to be ejected with a minimum kinetic energy of 2.00×10^{-19} J, determine the wavelength and frequency of the incident light beam.

(Ans: 3.51×10^{-7} m, 8.57×10^{14} Hz)

(b) Calculate the wavelength for the electrons emitted with maximum kinetic energy in problem (a) above.

(Ans: 1.55×10^{-9} m)

Keywords

Quantum, photoelectric effect, photon, Planck's constant, de Broglie wavelength.

Important equations

$$\boxed{E = \mathrm{h}f}$$

$$\boxed{K_{max} = \mathrm{h}f - \Phi}$$

$$\boxed{\lambda = \frac{\mathrm{h}}{mv}}$$

8.2 Theory of the atom

By 1910 it was known that a number of spectra of hydrogen satisfied the equation

$$\frac{1}{\lambda} = \mathrm{R}\left[\frac{1}{m^2} - \frac{1}{n^2}\right]$$

where R is Rydberg's constant = $1.097 \times 10^7\ \mathrm{m}^{-1}$ and m and n are integers.

For each spectral line there is a particular value of m and n ($n > m$).

8.2.1 Bohr theory of the atom (1913)

Bohr postulates

(1) Only certain electron orbits are *allowed*, and as long as an electron remains in its particular orbit no light is emitted or absorbed. Each orbit has a definite energy.

(2) Electrons in allowed orbits have an angular momentum of $n\mathrm{h} / 2\pi$ where n is an integer and h is Planck's constant.

(3) When an electron jumps from one orbit to another, it emits or absorbs a photon. If an electron changes from an orbit with an energy E_i to one with a lower energy E_f, the frequency f of the emitted photon is given by:

$$\mathrm{h}_f = E_i - E_f$$

From the postulates Bohr showed:

$$E_i = -\frac{m\,e^4\,Z^2}{8\varepsilon_0^2\,h^2} \cdot \frac{1}{n_i^2}$$

So

$$\mathrm{h}f = E_i - E_f$$

$$= \frac{me^4 Z^2}{8\,\varepsilon_0^{\,2} h^2}\left[\frac{1}{n_f^{\,2}} - \frac{1}{n_i^{\,2}}\right]$$

as
$$c = f\lambda$$

$$\frac{1}{\lambda} = \frac{me^4 Z^2}{8\,\varepsilon_0^{\,2} h^3 c}\left[\frac{1}{n_f^{\,2}} - \frac{1}{n_i^{\,2}}\right]$$

that is,
$$\frac{1}{\lambda} = R\left[\frac{1}{m^2} - \frac{1}{n^2}\right]$$

Worked example **A hydrogen atom emits a photon with an energy of 12.75 eV. Assuming the final state of the atom is its ground state, determine the quantum number of the orbit from which the electron dropped.**

Solution

The energy of the photon in joules is:

$$\begin{aligned} E &= 12.75 \times 1.60 \times 10^{-19} \\ &= 2.043 \times 10^{-18} \\ &= \frac{\mathrm{hc}}{\lambda} \\ \lambda &= \frac{\mathrm{hc}}{E} \\ &= \frac{6.63 \times 10^{-34} \times 3.00 \times 10^{8}}{2.043 \times 10^{-18}} \\ &= 9.74 \times 10^{-8}\ \mathrm{m} \end{aligned}$$

From Bohr theory:

$$\begin{aligned} \frac{1}{\lambda} &= \mathrm{R}\left[\frac{1}{m^2} - \frac{1}{n^2}\right] \\ \frac{1}{9.74 \times 10^{-8}} &= 1.097 \times 10^{7}\left[\frac{1}{1^2} - \frac{1}{n^2}\right] \\ \frac{1}{n^2} &= 1 - \frac{1}{9.74 \times 10^{-8} \times 1.097 \times 10^{7}} \\ &= 6.41 \times 10^{-2} \\ \therefore\ n &= 4 \end{aligned}$$

Problem What is the photon energy involved in each of the following transitions in a hydrogen atom?

(a) The electron jumps from the 5th energy level to the 2nd.

(b) The electron jumps from the 20th energy level to the lowest energy level.

(c) In each case, is the radiation emitted or absorbed?

(Ans: (a) 4.581×10^{-19} J; (b) 21.77×10^{-19} J; (c) emitted, emitted)

Keywords

Bohr model, Rydberg constant, energy level.

Important equations

$$\boxed{\frac{1}{\lambda} = \mathrm{R}\left[\frac{1}{m^2} - \frac{1}{n^2}\right]}$$

$$\boxed{\mathrm{h}f = E_\mathrm{i} - E_\mathrm{f}}$$

8.3 Lasers

A **laser** (*L*ight *A*mplification by *S*timulated *E*mission of *R*adiation) is a source of intense, coherent, monochromatic radiation in the ultraviolet, visible and infrared regions of the electromagnetic spectrum. They are sometimes known as optical quantum generators, and are the optical version of the **maser**, which, using micro-waves, was developed in 1953. Laser action has been achieved in gaseous, solid and liquid media. The first laser device was built in the USA by T. Meiman in 1960.

8.3.1 Mechanism

Stimulated emission

An incoming photon with energy $E = \mathrm{h}f$ stimulates an electron in a high energy state, E_1, to jump to a lower energy state, E_2, where

$$E_1 - \mathrm{E}_2 = \mathrm{h}f$$

The released photon is travelling in the same direction as the incoming photon and therefore joins the stimulating photon beam, the intensity of which therefore builds up exponentially.

Population inversion

The laser beam requires a large number of electrons to be in a high energy metastable state — a situation known as population inversion.

Consider a three level atomic system, as shown in Figure 8.2

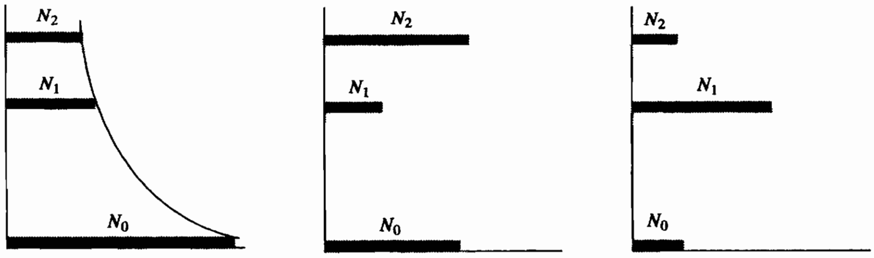


Figure 8.2

Electrons are excited into a higher energy state, then fall rather rapidly into an intermediate, metastable state, from which they are unlikely to fall unless stimulated.

The method of changing the populations of the energy levels is called **optical pumping**.

Resonance cavity

Various ways are used to keep a large number of the photons in the laser system, for example mirrors at either end of the tube, with partial silvering at the emitting end, so that a certain number can be emitted. Because of the cavity resonance, various models will be emitted from the laser.

Keywords

Laser, stimulated emission, population inversion, resonance cavity.

8.4 Nuclear physics

8.4.1 The nucleus—composition

In an atom (diameter $\approx 10^{-10}$ m, the nucleus (diameter $\approx 10^{-15}$ m) occupies very little of the volume of the atom, but contains nearly all the mass and all the positive charge. The **nucleus** consists of protons (positively charged) and **neutrons** (electrically neutral), collectively referred to as **nucleons**.

The number of protons in an atom is its **atomic number** and is given the symbol Z. If the number of neutrons in the atom is N then the **mass number** of the atom is $A = Z + N$.

Nuclei with the same Z but different N are called **isotopes** of an element.

8.4.2 Mass of the nucleus

Nuclear masses are expressed in (atomic) mass units, 'u', where:

$$1\ \text{u} = \frac{1}{12}\ \text{of a}\ C^{12}\ \text{atom}$$
$$= 1.660566 \times 10^{-27}\ \text{kg}$$

Clearly, we have:

mass of atom = mass of nucleus + mass of Z electrons

Note: $m_p = 1.00728$ u, $m_n = 1.00866$ u, $m_e = 0.00055$ u

8.4.3 Mass defect and binding energy

Similarly, the energy of the whole nucleus is less than the energy of the separate (constituent) particles. The difference is known as the **'Binding Energy'** — actually an **energy defect**. Binding energy represents the work needed to pull the nucleus apart. The way the binding energy for a nucleus divided by the number of nucleons in the nucleus (i.e. binding energy per nucleon) varies as a function of mass number is shown in Figure 8.3.

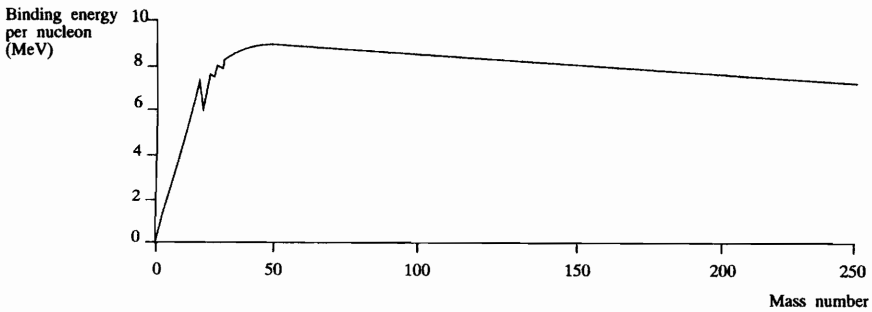


Figure 8.3

8.4.4 Equivalence of mass and energy

All energy has mass and all mass has energy. One of the consequences of Einstein's Special Theory of Relativity is a relationship between the energy, E, and the mass, m.

A body of mass m possesses an energy known as its 'rest energy', E_0, given by

$$E_0 = mc^2$$

(called 'rest energy', because the body possesses this energy even when at rest, i.e. even when the kinetic energy has dropped to zero).

8.4.5 Radioactivity

Radioactivity is the spontaneous breaking up of a nucleus and the emission of a particle. This is referred to as **nuclear decay**. There are three types, depending on the nature of the particle emitted:

- α decay — the emission of an α particle (He^{++} or ${}^{4}_{2}He$) generally:

$$ {}^{A}_{Z}X - {}^{A-4}_{Z-2}Y + {}^{4}_{2}He $$

  for example:

$$ \underset{\text{(uranium)}}{{}^{238}_{92}U} \rightarrow \underset{\text{(thorium)}}{{}^{234}_{90}Th} + {}^{4}_{2}He $$

  This is a common mode of decay because the ${}^{4}_{2}He$ nucleus is particularly stable.

- β decay — the emission of a β particle (e^-, an electron), generally:

$$ {}^{A}_{Z}X \rightarrow {}^{A}_{Z+1}Y + {}^{0}_{-1}e + \bar{\nu} \quad \text{where } \bar{\nu} \text{ is an antineutrino,} $$

  for example:

$$ \underset{\text{(artificially produced)}}{{}^{28}_{13}Al} \rightarrow {}^{28}_{14}Si + {}^{0}_{-1}e + \bar{\nu} $$

- γ decay — the emission of a γ particle (quantum), generally:

$$ \underset{\text{(excited)}}{{}^{A}_{Z}X^{*}} \rightarrow \underset{\text{(unexcited)}}{{}^{A}_{Z}X} + \gamma $$

A γ ray photon is a very high-energy photon; its wavelength is less than that of X-rays, and is typically $\approx 10^{-14}$ m.

For example:

The β decay above gives ${}^{28}_{14}Si^{*}$, which is in an excited state (similar to an atom in which the electron is in an excited state). Then:

$$ \underset{\text{(excited)}}{{}^{28}_{14}Si^{*}} \rightarrow \underset{\text{(unexcited)}}{{}^{28}_{14}Si} + \gamma $$

Half-life

The half-life ($t_{1/2}$) of a decay process is the time it takes for half of the atoms originally present to decay.

Now,

$$ N = N_0 e^{-\lambda t} $$

so $$\frac{1}{2}N_0 = N_0 e^{-\lambda t_{1/2}}$$

$$\frac{1}{2} = e^{-\lambda t_{1/2}}$$

where $$e^{\lambda t_{1/2}} = 2$$

$$\lambda t_{1/2} = \ln 2$$

$$t_{1/2} = \frac{\ln 2}{\lambda}$$

Activity

The activity of a sample is defined as the number of decays per unit time.

$$\text{Activity} = A = \frac{dN}{dt}$$

Now, $$N = N_0 e^{-\lambda t}$$

$$\therefore \quad \frac{dN}{dt} = -\lambda N_0 e^{-\lambda t}$$

$$A = \lambda N_0 e^{-\lambda t}$$

$$A = \lambda N$$

A is usually calculated using this expression; also, if A_0 is the activity at $t = 0$ then

$$A_0 = \lambda N_0$$

and $$A = A_0 e^{-\lambda t}$$

Units of activity:

1 becquerel (Bq) = 1 disintegration/second
1 curie (Ci) = 3.70×10^{10} disintegrations/second

8.4.6 Nuclear energy

Nuclear reactions produce energy if the final nuclei are more strongly bound than the initial nuclei. To achieve this stronger binding, mass is lost, thus energy ($E_0 = mc^2$) is produced. There are two ways in which energy may be produced in useful amounts.

Nuclear fission

Bombardment of ^{235}U with neutrons splits the nucleus into fragments:

$$^{1}_{0}n + {}^{235}_{92}U \rightarrow {}^{140}_{54}Xe + {}^{94}_{38}Sr + 2\,^{1}_{0}n + 200\ \text{MeV}$$
(xenon) (strontium)

This reaction produces two neutrons, which can react again with ^{235}U and so on. This is a self-sustaining reaction, called a **chain reaction**. Thus, if a sufficient amount of ^{235}U (called the 'critical mass') is present so that a large fraction of the neutrons react with a uranium nucleus before reaching the surface and escaping from the ^{235}U material, the reaction builds to a very large release of energy in an instant. This is the process that occurs in the atomic bomb.

To use this reaction to generate energy in a controlled situation it must be slowed down. This is done in a nuclear reactor using material (moderators) to absorb neutrons so that only one from each nucleus reacts with another nucleus.

Nuclear fusion

Heavy hydrogen nuclei (deuterium—each deuterium nucleus is called a **deuteron**) are fused to form helium. In essence, the equation is:

$$^2_1H + ^2_1H \rightarrow ^4_2He + \text{energy}$$

To fuse the two nuclei they need to be driven together very quickly to overcome the coulomb repulsion force. For this, the nuclei need a high kinetic energy, equivalent to a temperature of $\approx 10^8$K. Thus, to achieve this reaction a gas (called a plasma) of deuterons plus free electrons at this temperature are needed. Containing such a high-temperature plasma is very difficult. The problem of producing fusion energy is mainly an engineering one: to contain a sufficient number of nuclei at a high enough temperature to obtain more energy output than the energy put in to achieve this state.

Worked examples

(a) How much energy would be required to separate the ^9_4Be atom, of mass 9.01219 u into its component nucleons?

Solution

$A = 9, Z = 4, \therefore N = A - Z = 9 - 4 = 5$

$$\begin{aligned}
\text{mass of nucleus} &= 9.01219 - 5 \times (\text{mass of electron}) \\
&= 9.01219 - 5 \times 0.00055 \\
&= 9.00944\ \text{u} \\
\text{mass of nucleons} &= 4 \times (\text{mass of proton}) + 5 \times (\text{mass of neutron}) \\
&= 4 \times 1.00728 + 5 \times 1.00866 \\
&= 9.07242\ \text{u} \\
\text{mass defect} &= 9.07242 - 9.00944 = 0.06298\ \text{u} \\
\text{energy required} &= 0.06298 \times 1.661 \times 10^{-27} \times (3 \times 10^8)^2 \\
&= 9.41 \times 10^{-12}\ \text{J}
\end{aligned}$$

(b) A Geiger counter records 9.00 counts per second when placed near a radioactive sample. After 3000 s the counting rate has dropped to 0.20 counts per second (after applying a correction to subtract the background radiation). Calculate the decay constant.

Solution

$$A = A_0 e^{-\lambda t}$$

$$0.20 = 9.00\, e^{-\lambda 3000}$$

$$e^{-3000\lambda} = \frac{0.20}{9.00}$$

$$e^{3000\lambda} = \frac{9.00}{0.20} = 45$$

$$3000\,\lambda = \ln 45$$

$$\lambda = \frac{\ln 45}{3000} = 1.3 \times 10^{-3}\ \text{s}^{-1}$$

Problems

(a) What is the binding energy of $^{7}_{3}\text{Li}$ (7.00160 u)?

(Ans: 52.7 MeV)

(b) The unstable isotope ^{40}K is used for dating rock samples. Its half-life is 2.4×10^{8} years.

(i) How many decays occur per second in a sample containing 2×10^{-6}g of ^{40}K?

(ii) What is the activity of the sample, in curies?

(Ans: (i) 2.76 decays; (ii) 7.45×10^{-11} Ci)

Keywords

Nucleus, atomic number, mass number, atomic mass unit, isotope, mass defect, proton, neutron, radioactivity, alpha particle, beta particle, gamma ray, activity, half-life, decay constant, fission, fusion, chain reaction.

Important equations

$$\boxed{A = Z + N,\ ^{A}_{Z}\text{X}}$$

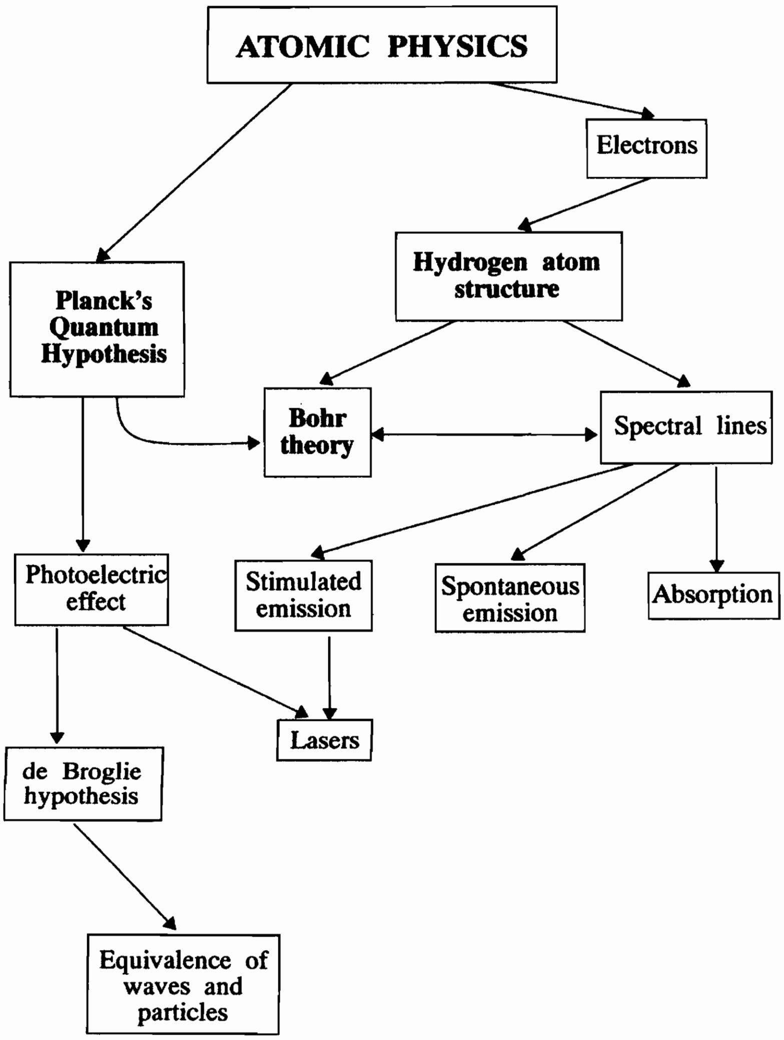
$$\boxed{E_0 = mc^2}$$

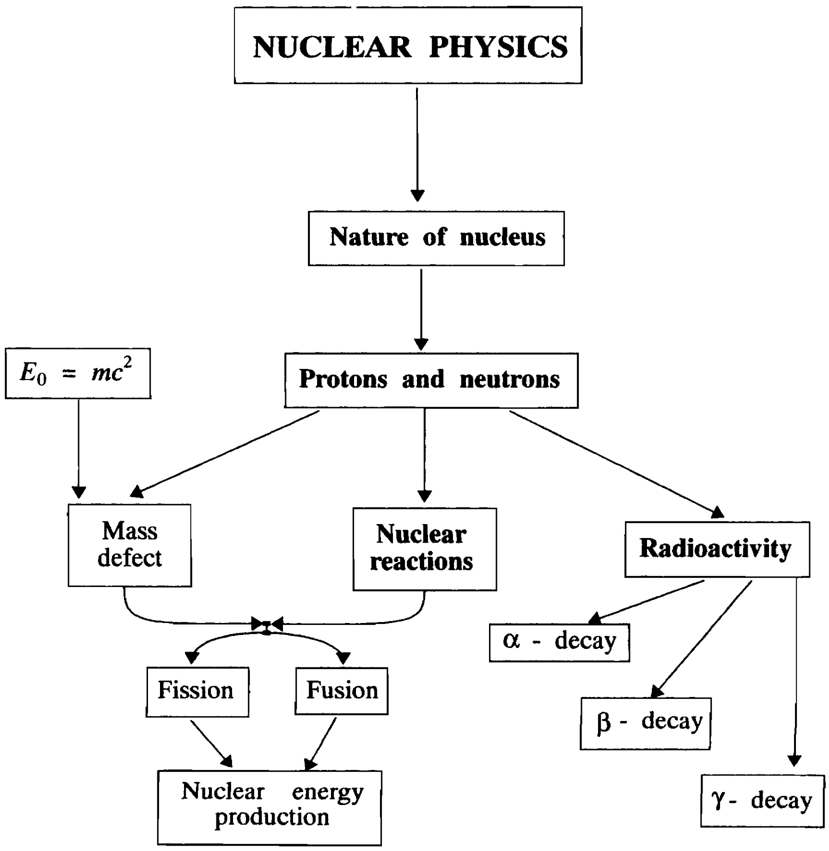
$$\boxed{E = (\Delta m)\, c^2}$$

$$\boxed{N = N_0 e^{-\lambda t}}$$

$$\boxed{t_{1/2} = \frac{\ln 2}{\lambda}}$$

$$\boxed{A = \lambda N}$$


ATOMIC PHYSICS
Electrons
Hydrogen atom structure
Planck's Quantum Hypothesis
Bohr theory
Spectral lines
Photoelectric effect
Stimulated emission
Spontaneous emission
Absorption
Lasers
de Broglie hypothesis
Equivalence of waves and particles


NUCLEAR PHYSICS
Nature of nucleus
$E_0 = mc^2$
Protons and neutrons
Mass defect
Nuclear reactions
Radioactivity
α - decay
Fission
Fusion
β - decay
Nuclear energy production
γ- decay

CHAPTER 9

Model examination papers

The following are two typical examination papers for the material covered in this book. The material would normally be part of a two-semester 1st-year physics course. Hence Examination paper 1 could be considered as the 1st semester examination paper and paper 2 the 2nd semester paper. Each paper is intended to be done in 3 hours and is divided into sections A and B. Students are given a choice of 3 out of 4 questions in each section.

The material covered in each question is given in the table below. Because of the nature of an examination paper, a number of the questions cover material from different parts of the course (especially D3).

Structure of model examination papers

A Mechanics

1 Units, dimensions, vectors, rectilinear motion, projectiles
2 Newton's laws, friction, circular motion
3 Energy and momentum
4 Equilibrium and rotation

B Thermal and waves

1 Temperature, specific heat capacity, change of state, heat transfer
2 Ideal gas equation, 1st law, kinetic theory, heat engines
3 SHM and wave motion
4 Elasticity, mechanical properties of waves, sound, reflection, refraction

C Electricity and magnetism

1 Electrostatics
2 Circuits and Kirchhoff's law
3 Magnetic fields and Faraday's law
4 Inductance, R–C, R–L and AC circuits

D Optics, fluids, atomic and nuclear

1 Mirrors, lenses, optical instruments
2 Fluid statics, surface tension, fluid dynamics
3 Viscosity, physical optics, photoelectric effect
4 Bohr atom, the nucleus, radioactivity

Full worked solutions for these papers are included. There are two further examination papers based on the same breakdown of material in Chapter 10 for practice.

Autumn Semester Examination

Physics 1A (F/T, P/T)

Full-time: Start 9.30 a.m. Finish 12.40 p.m.
Time allowed: 3 hours + 10 minutes

This paper is designed to be completed in 3 hours. An extra 10 minutes has been added to the time allowed, and it is recommended that you use these 10 minutes to read the paper before starting to answer the questions.

Part-time: Start 9.30 a.m. Finish 11.10 a.m.
Time allowed: $1\frac{1}{2}$ hours + 10 minutes

This paper is designed to be completed in $1\frac{1}{2}$ hours. An extra 10 minutes has been added to the time allowed and it is recommended that you use these 10 minutes to read the paper before starting to answer the questions.

Full-time: Answer **3** questions from Section **A**, and **3** questions from Section **B.**
Part-time: Answer **3** questions from Section **A** only and leave the room quietly at the time stated.

Calculators may be used.

Answer each question in a *separate* booklet

Clearly *mark the question number on the front* of each booklet.

Physics 1 data sheet

$g = 9.8\ \text{m s}^{-2}$

$\sigma = 5.670 \times 10^{-8}\ \text{W m}^{-2}\text{K}^{-4}$

$R = 8.314\ \text{J mol}^{-1}\ \text{K}^{-1}$

$N_0 = 6.022 \times 10^{23}$ molecules mole^{-1}

$k = 1.381 \times 10^{-23}\ \text{J K}^{-1}$ molecule^{-1}

$c = 2.998 \times 10^{8}\ \text{m s}^{-1}$

$h = 6.626 \times 10^{-34}\ \text{J s}$

$\varepsilon_0 = 8.854 \times 10^{-12}\ \text{F m}^{-1}$

$\mu_0 = 4\pi \times 10^{-7}\ \text{H m}^{-1}$

$e = 1.602 \times 10^{-19}\ \text{C}$

$m_e = 9.110 \times 10^{-31}\ \text{kg} = 0.00055\ \text{u}$

$m_p = 1.673 \times 10^{-27}\ \text{kg} = 1.00728\ \text{u}$

$m_n = 1.675 \times 10^{-27}\ \text{kg} = 1.00866\ \text{u}$

$G = 6.673 \times 10^{-11}\ \text{m}^2\ \text{N kg}^{-2}$

$1\ \text{u} = 1.661 \times 10^{-27}\ \text{kg}$

Rydberg's constant = $1.097 \times 10^{7}\ \text{m}^{-1}$

Temperature of ice point = 273.15 K

1 atmosphere = 1.013×10^{5} Pa

Mass of earth = 5.974×10^{24} kg

Radius of earth = 6.37×10^{6} m

Section A

(Attempt any *three* questions in this section)

Question 1 *Marks*

(a) A bicycle of mass m, travels at a speed v around a circle of radius R on a flat surface (coefficient of static friction between the tyre and the surface is μ_s). A rope attached from the centre of the circle to the bike experiences a tension T. A student finds that :

$$T = \frac{mv^2}{R} - \mu_s mg$$

(i) What are the dimensions of T and μ_s?

(ii) Is the equation dimensionally correct?

3

(b) The Indian pacific train is heading due west across the Nullarbor Plain at 90 km h^{-1}. A light plane is travelling at 130 km h^{-1} in a direction 20° East of North.

(i) Draw a vector diagram showing the velocity of the train, the velocity of the plane and the relative velocity of the plane as seen by a passenger on the train.

(ii) What is the relative velocity (magnitude and direction) of the plane as seen by a passenger on the train?

5

(c) A rocket sled is used by the army to test the stamina of their soldiers under stress. The rocket travels on frictionless rails in a straight line away from the base under the following conditions:

time 0–5 seconds, acceleration = 37.0 m s^{-2}
time 5–15 seconds, acceleration = 0.0 m s^{-2}
time 15–? seconds, acceleration = –6.0 m s^{-2}

(i) At what time does the sled come to rest?

(ii) Calculate the total distance moved by the sled.

6

(d) A cricketer hits a ball at waist height (0.9 m) with a speed of 40 m s^{-1} at an angle of 35° above the horizontal. Will the ball clear the top of a 7.0 m sight-screen located 150 m from the batsman? If not, at what height will it hit the screen? (Show all working.)

6

20

Question 2 *Marks*

(a) Explain why Newton's First Law could be regarded as a particular example of the Second Law.

3

(b) A tugboat exerts a force of 3300 N. It pulls a boat of mass 25 000 kg by means of a strong steel cable of mass 200 kg. The cable can be considered to be stretched horizontally under these conditions.
If there is no frictional force:

(i) Draw force diagrams showing all the forces relevant to the motion.

(ii) Determine the acceleration achieved by the tug.

(iii) What is the tension in the cable at the end closer to the tug?

(iv) What is the tension in the cable at the end closer to the boat?

6

(c) A car travels at a steady 90 km h^{-1} along a road leading over a small hill. The top of the hill is rounded so that, in the vertical plane, the road approximately follows an arc of circle of radius 85 m.

(i) What is the centripetal acceleration of the car at the top of the hill?

(ii) Consider that at the top of the hill the speed of the car is 90 km h^{-1} but is increasing at a rate of 2 m s^{-2}. Draw a diagram showing the component of the acceleration vector of the car in the direction towards the centre of the circle and the component tangential to the arc of the circle.

(iii) Determine the angle the acceleration vector makes with respect to the horizontal at the top of the hill and the magnitude of the acceleration vector of the car.

6

(d) A cyclist is trying to negotiate a tight curve on an unbanked track. The radius of curvature of the curve is 30 m and the static coefficient of friction of the tyres on the surface is 0.65.

(i) Include a diagram showing all forces acting on the cyclist/bicycle.

(ii) What is the maximum constant speed (in km h^{-1}) with which the cyclist may go around the corner without sliding?

5

20

Question 3 *Marks*

(a) An object of mass m starts at the bottom of a slope with speed u. It is pulled up the slope, a distance s, with a constant force F. A frictional force, f, acts on the object and when it has gained a vertical height, h, its speed is v.

(i) In a diagram show all the forces acting on the body.

(ii) Write an expression for the initial energy.

(iii) Write an expression for the final energy.

(iv) Write an expression for the work done by friction.

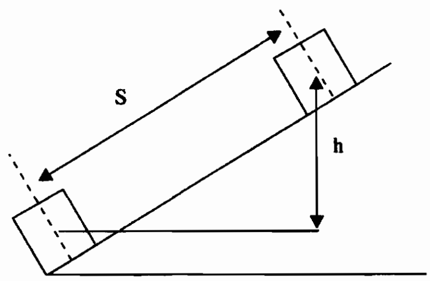


Figure 9.0

4

(b) The nucleus of an atom of radioactive copper, at rest, undergoes a beta decay, and simultaneously emits an electron and a neutrino. The momentum of the electron is 2.64×10^{-22} kg m s^{-1} and that of the neutrino is 1.97×10^{-22} kg m s^{-1}. The angle between their directions of motion is 30.0°. If the mass of the residual nucleus is 1.06×10^{-25} kg what is the recoil velocity of the nucleus after the decay?

9

(c) A skier of mass 70 kg skis from rest, down a jump ramp (see diagram, which is not to scale).

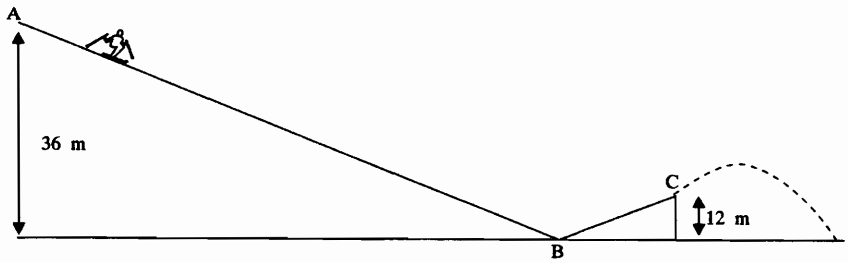


Figure 9.1

The distance of AB is 60 m and BC is 20 m.

(i) Find the work done by friction as the skier travels from A to B, if the skier's speed at B is 25.5 m s^{-1}.

(ii) Find the frictional force on the skis as the skier travels from A to B, assuming a constant frictional force.

(iii) Assuming a constant frictional force over the entire ramp distance ABC, find the speed of the skier at C.

7

20

Question 4 *Marks*

(a) State the conditions under which a body is in a state of mechanical equilibrium.

2

(b) It is possible for two sheets of wood to be placed so that they lean against each other, at an angle to the horizontal as shown in the diagram, without either falling down. Each sheet has a mass of 37.00 kg, a length of 2.85 m and the coefficient of friction between wood and floor is 0.75. (Assume that the force of interaction between the two pieces of wood at their tops are horizontal.)

(i) Draw a diagram showing all the forces acting on one of the sheets.

(ii) What is the smallest angle θ where this is possible.

7

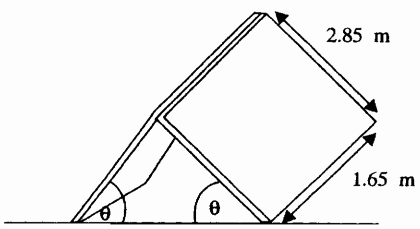


Figure 9.2

(c) A uniform solid cylinder of mass 15 kg and radius 0.15 m is free to rotate about a horizontal axis through its centre (see diagram). A rope, wrapped around the curved surface of the cylinder, carries at its end a block of mass 8.0 kg. Determine:

(i) the angular acceleration of the cylinder;

(ii) the tension in the rope;

(iii) the angle (in radians) through which the cylinder turns in 5.0 s starting from rest;

(iv) what force is exerted on the cylinder at its axis of rotation.

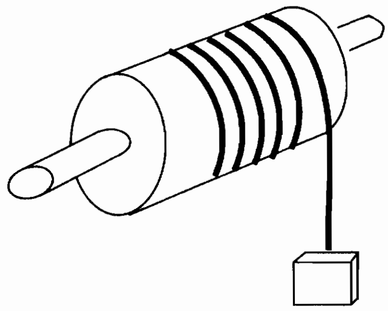


Figure 9.3

11

20

Section B

(Attempt any *three* questions in this section)

Question 5 *Marks*

(a) (i) State the Zeroth Law of Thermodynamics.

(ii) What is meant by the heat of vaporization of a substance? (If you use an equation define all symbols.)

4

(b) (i) A hole of diameter 20.000 mm is drilled in a steel plate when it is at a temperature of 25°C. What is the diameter of the hole when the temperature of the plate is 500°C?
(Coefficient of linear expansion of steel: 1.2×10^{-5} $(°C)^{-1}$)

(ii) At 7°C the density of water is 0.9999×10^3 kg m^{-3} and at 10°C it is 0.9997×10^3 kg m^{-3}. Determine the average coefficient of volume expansion of water in the range of temperature 7°C to 10°C.

6

(c) Determine the quantity of heat that must be transferred from 5 kg of steam at 300°C and 0.1 MPa so that it condenses into liquid water at 70°C and 0.1 MPa.

(Heat of vaporization of water 2.256×10^6 J kg^{-1}; specific heat capacities at constant pressure: steam 1970 J kg^{-1} $(°C)^{-1}$, water 4186 J kg^{-1}$(°C)^{-1}$)

4

(d) A flat sheet of insulator of area 0.50 m^2 and thickness 20 mm is kept at a temperature of 100°C on one side while the other side radiates heat to and absorbs heat from the surroundings, which are at 20°C. Assuming the radiating surface has an emissivity of 0.2, show that the temperature of the radiating surface is approximately 42°C.
(Thermal conductivity of the insulator: 0.01 W m^{-1} $(°C)^{-1}$)

6

20

Question 6 *Marks*

(a) (i) Use a fully labelled P–V diagram to explain the essential features of a Carnot cycle.

(ii) What is the efficiency of a Carnot engine operating between 180°C and 20°C?

4

(b) A closed vessel contains 0.30 m^3 of nitrogen at 1.0×10^5 Pa and 25°C. Calculate the number of moles in the container. (Molecular weight of nitrogen = 28)

2

(c) According to the kinetic theory, what would be the root mean square speed of the nitrogen molecules in part (b).

3

(d) The nitrogen gas is taken around a cycle as shown in the P–V diagram (where BC is an isothermal expansion, AB and DC are horizontal lines and AD is a vertical line). The volume at B is twice the volume at A and the volume at C is three times the volume at A. ($V_B = 2V_A$; $V_C = 3V_A$)

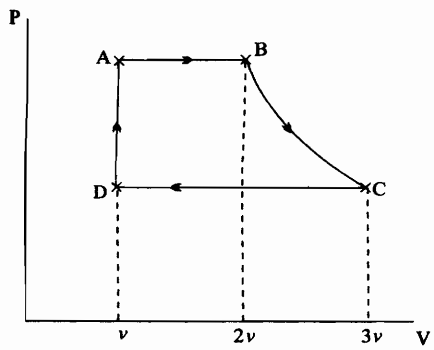


Figure 9.4

Copy the following two tables into your answer booklet and complete them:

(i) The thermodynamic conditions at A, B, C and D:

	P	T	V
A	1.0×10^5 Pa	25°C	0.30 m^3
B	?	?	?
C	?	?	?
D	?	?	?

(ii) The names of the thermodynamic processes and the work done in each process:

	Description	W
A–B	?	?
B–C	Isothermal	?
C–D	?	?
D–A	?	?

11

20

Question 7 *Marks*

(a) The equation representing linear simple harmonic motion is given by:

$$y = A\sin(\omega t + \alpha)$$

State what the symbols A, ω and α represent. 3

(b) A spring with a mass of 0.30 kg attached to its free end hangs vertically from a fixed support. A further 0.20 kg is added to the free end, causing the length of the spring to increase by 50 mm. The combined mass is then set in oscillation with an amplitude of 25 mm.

(i) Determine the period of the oscillation.

(ii) Calculate the maximum kinetic energy of the system.

(iii) Find the magnitude and the direction of the acceleration of the combined mass when it is 10 mm below the equilibrium position. 7

(c) A wave travelling on a length of string under tension is represented by the equation:

$$y = 20\sin(\frac{\pi}{6}x - 20\,\pi\, t)$$

where x and y are in mm.

(i) Determine the wave's amplitude, wavelength, frequency and speed of propagation.

(ii) In what direction is the wave travelling?

(iii) Determine the velocity of a particle in the string at position $x = 9.0$ mm and $t = 0.31$ s. 7

(d) Show that the equation:

$$y = 40 \sin\left(\frac{\pi}{6}x\right) \cos(20\,\pi t)$$

represents the standing wave produced when the wave described in part (c) combines with an identical wave travelling in the opposite direction.

3

20

Question 8 *Marks*

(a) Define the following, explaining all symbols used:

(i) Poisson's ratio.

(ii) Shear (or rigidity) modulus.

3

(b) A copper rod of length 2.10 m and cross-sectional area 205 mm^2 is fastened end-to-end to a steel rod of length L and cross-sectional area 103 mm^2, as shown in the diagram. The compound rod is subjected to equal and opposite pulls of magnitude 3.00×10^4 N at its ends.

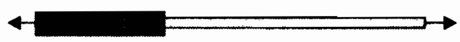


Figure 9.5

(i) Find the original length L of the steel rod if the elongations of the two rods are equal.

(ii) What is the stress in each rod?

(iii) What is the strain in each rod?

($Y_{steel} = 2.0 \times 10^{11}$ Pa; $Y_{copper} = 1.1 \times 10^{11}$ Pa)

6

(c) At an open-air concert of the rock band, 'Disaster Area', it is found that the sound intensity level 3 m from the loudspeaker (considered to be a point source) is 120 dB. How far from the speaker does one need to go for the sound level to drop to 85 dB?

5

(d) A ray of light of wavelength 500 nm is passing from air (refractive index = 1.00) into a 60° glass prism (refractive index = 1.60) at an angle of incidence of 40° (see diagram). Calculate:

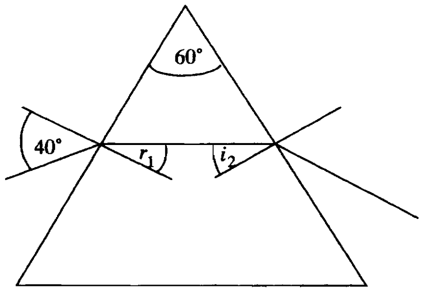


Figure 9.6

(i) the frequency of light in air;

(ii) the wavelength of light in the glass;

(iii) the angle r_1 in Figure 9.6, and

(iv) the angle i_2.

$\frac{6}{20}$

A Answers

Question I

(a) (i) T has dimensions $\mathrm{M\,L\,T^{-2}}$

μ_s has no dimensions $\mathrm{M^0\,L^0\,T^0}$

(ii) $\dfrac{mv^2}{R}$ has dimensions $\mathrm{M\,L^2\,T^{-2}\,L^{-1} = M\,L\,T^{-2}}$

$\mu_s\,mg$ has dimensions $\mathrm{M\,L\,T^{-2}}$

As all terms have the same dimensions, the equation is dimensionally correct.

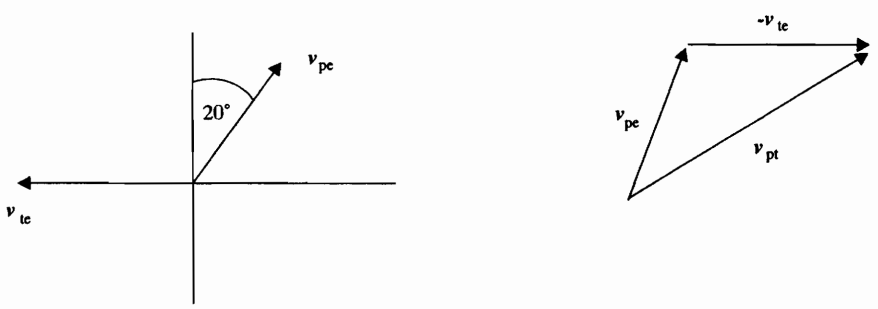


Figure 9.7

(b) (i) p – plane; t – train; e – earth

(ii) $v_{\text{p rel t}} = v_{\text{p rel e}} - v_{\text{t rel e}}$

Components:

$R_x = 90 + 130 \cos 70 = 90 + 44.5 = 134.5\ \mathrm{km\ h^{-1}}$

$R_y = 130 \sin 70 = 122.2\ \mathrm{km\ h^{-1}}$

$R = \sqrt{R_x^2 + R_y^2}$

$= \sqrt{134.5^2 + 122.2^2}$

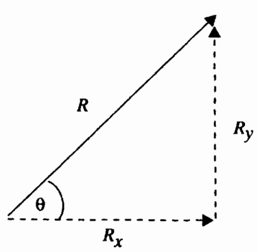


Figure 9.8

$= 182 \text{ km h}^{-1}$

$\theta = \tan^{-1}\left(\frac{122.2}{134.5}\right) = 42.6^{\circ}$

$\therefore$ 43° N of E

(c)

	$v = u + at$	$s = ut + \frac{1}{2}at^2$
0–5 seconds	$v = 185 \text{ m s}^{-1}$	$s = 463$ m
5–15 seconds	$v = 185 \text{ m s}^{-1}$	$s = 1850$ m
15+ seconds	$0 = 185 - 6t$	$s = 185 \times 30.8 + \frac{1}{2} \times -6 \times 30.8^2$
	$t = 30.85$ s	$= 5700 - 2850$
		$= 2850$ m

(i) The sled will come to rest at $t = 30.8$ s; (ii) total distance moved = 463 + 1850 + 2850 = 5160 m

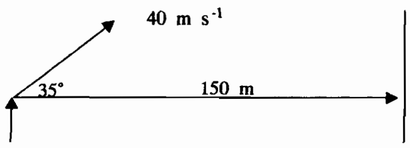


Figure 9.9

(d)

$u_x = 40 \cos 35$ $\quad$ $u_y = 40 \sin 35$

$v = u_x t$ $\quad$ $S_y = ut + \frac{1}{2}at^2$

$t = \frac{150}{u_x}$ $\quad$ $= 40 \sin 35 \times 4.58 - \frac{1}{2}9.8 \times 4.58^2$

$= 4.58$ s $\quad$ $= 105.1 - 102.8$

$= 2.32$ m

Therefore at sightscreen ball has height = 0.9 + 2.32 = 3.22 m, therefore doesn't clear the screen.

Question 2

(a) Newton's Second Law states that: $F = \dfrac{d(mv)}{dt}$. Newton's First Law is the special case where F is zero, assuming m is constant.

(b) (i)

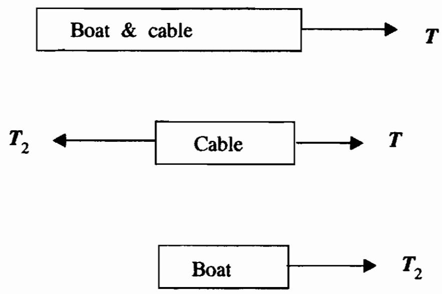


Figure 9.10

(ii) $F = ma$

$3300 = 25\ 200a$

$a = 0.13 \text{ m s}^{-2}$

(iii) $T = 3300$ N; (iv) $T_2 = 25\ 000a = 3250$ N

(c) (i) $v = \dfrac{90 \times 1000}{60 \times 60} = 25.0 \text{ m s}^{-1}$

$a_c = \dfrac{v^2}{R} = \dfrac{(25.0)^2}{85} = 7.4 \text{m s}^{-2}$

(ii)

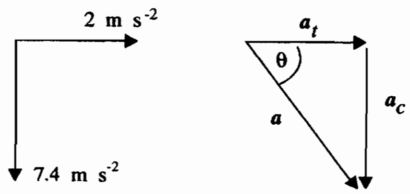


Figure 9.11

(iii) $a_{\text{total}} = \sqrt{a_c{}^2 + a_t{}^2}$

$= \sqrt{(7.4)^2 + (2)^2} = 7.6 \text{ m s}^{-1}$

$\tan\theta = \dfrac{7.4}{2.0} = 3.68$

$\theta = 75°$

(d) (i)

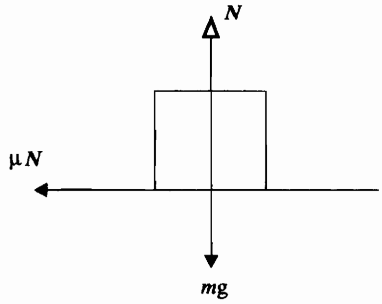


Figure 9.12

(ii) $\mu N = \mu mg = \dfrac{mv^2}{r}$

$v = \sqrt{\mu g R} = \sqrt{0.65 \times 9.8 \times 30}$

$= 13.8 \text{ m s}^{-1}$

$= 50 \text{ km h}^{-1}$

Question 3

(a) (i)

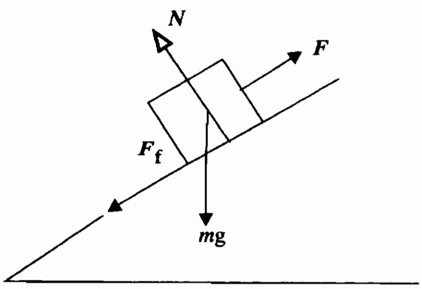


Figure 9.13

(ii) $E_0 = \dfrac{1}{2}mu^2$; (iii) $E_t = \dfrac{1}{2}mv^2 + mgh$ (iv) $W_f = F_f\, s$

(b)

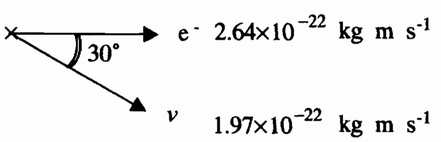


Figure 9.14

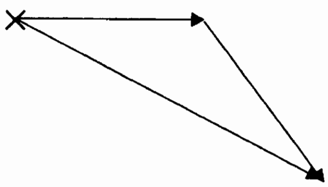


Figure 9.15

Add these momenta

By cosine rule:

$$R = \sqrt{2.64^2 + 1.97^2 - 2 \times 2.64 \times 1.97 \times \cos 150 \times 10^{-22}}$$
$$= 4.46 \times 10^{-22} \text{ kg m s}^{-1}$$

By sine rule:

$$\frac{\sin\theta}{1.97 \times 10^{-22}} = \frac{\sin 150}{4.46 \times 10^{-22}}$$
$$\theta = 12.8^{\circ}$$

Nucleus will recoil in the direction shown in the diagram, with a momentum of magnitude $= 4.46 \times 10^{-22}$ kg m s^{-1}

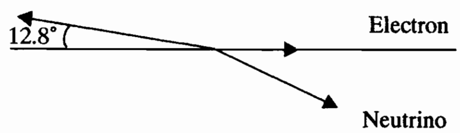


Figure 9.16

$$\therefore\ v = \frac{p}{m}$$
$$= \frac{4.46 \times 10^{-22}}{1.06 \times 10^{-25}}$$
$$= 4.21 \times 10^{3} \text{ m s}^{-1}$$

(c) (i) At A, $P = mgh = 70 \times 9.8 \times 36 = 2.47 \times 10^4$ J

At B, $K = \frac{1}{2}mv^2 = \frac{1}{2} \times 70 \times (25.5)^2 = 2.28 \times 10^4$ J

$\therefore$ Work done by friction $= 2.47 \times 10^4 - 2.28 \times 10^4$
$= 1.90 \times 10^3$ J

(ii) $W_f = F_f d$

$1.90 \times 10^3 = F_f \times 60$

$F_f = 32$ N

(iii) At C, $P = mgh = 70 \times 9.8 \times 12 = 8.23 \times 10^3$ J

$K = \frac{1}{2}mv^2 = \frac{1}{2} \times 70 \times v^2 = 35v^2$

$\therefore$ Total work done by friction $= F_f\, d = 32 \times 80$
$= 2.54 \times 10^3$ J

Energy at A = energy at C + work done by friction

$$2.47 \times 10^4 = 8.23 \times 10^3 + 35v^2 + 2.54 \times 10^3$$
$$35v^2 = 1.39 \times 10^4$$
$$v = 20 \text{ m s}^{-1}$$

Question 4

(a) A body is in a state of mechanical equilibrium when:

- the sums of the components of the forces in x and y directions are zero;
- the algebraic sum of the moments about any one point (or about every point) is zero.

(b) (i)

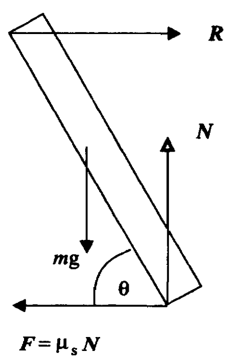


Figure 9.17

(ii) Consider one sheet. The sum of the vertical forces acting on the sheet gives:

$N - mg = 0 \quad \therefore N = mg = 37 \times 9.8$ N

The sum of the horizontal forces acting on the sheets gives:

$R = \mu_s N = 0.75 \times 37 \times 9.8$ N

Moments about the bottom of the sheet gives:

$mg \frac{l}{2} \cos\theta = Rl \sin\theta$, where l is the length of the sheet.

$$\tan\theta = \frac{mg}{2R} = \frac{mg}{2\mu\, mg}$$

$$\tan\theta = \frac{1}{1.5} \quad \therefore \theta = 34^\circ$$

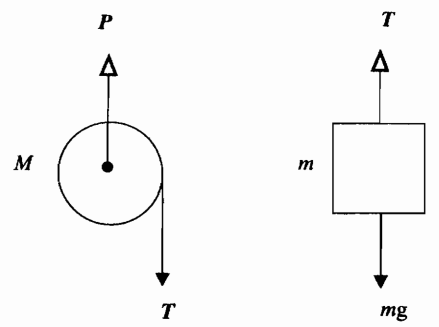


Figure 9.18

(c) (i) $mg - T = ma$ [1] (forces acting on block)

$a = R\,\alpha$ [2] (for the drum)

$\Gamma = I\,\alpha$ [3] (for the drum)

$\therefore\ TR = \frac{1}{2}MR^2\,\alpha$

So, $[1] \rightarrow mg - T = mR\,\alpha$ [4]

$[3] \rightarrow T = \frac{1}{2}MR\,\alpha$ [5]

$[4] + [5]\ mg = (m + \frac{1}{2}M)\,R\,\alpha$

$8 \times 9.8 = (8 + \frac{1}{2} \times 15) \times 0.15\,\alpha$

$\alpha = 33.7 \text{ rad s}^{-2}$

(ii) Substitute in [5] $T = \frac{1}{2} \times 15 \times 0.15 \times 33.7$

$= 37.9 \text{ N}$

(iii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$= 0 + \frac{1}{2} \times 33.7 \times 5^2$

$= 421 \text{ rad}$

(iv) $P = T = 37.9 \text{ N}$

Question 5

(a) (i) Two systems in thermal equilibrium with a third system are in thermal equilibrium with each other.

(ii) The quantity of heat required to cause unit mass of the substance to change from the liquid phase to the vapour.

(b) (i) $\Delta L = \alpha L_0 \Delta T$

$= 1.2 \times 10^{-5} \times 20 \times 475$

$= 0.114$

$\therefore \; L = 20.114$ mm

(ii) $\beta = \dfrac{\Delta V}{V_0 \Delta T}$ and $\rho = \dfrac{m}{V} \;\; \therefore \; V = \dfrac{m}{\rho}$

$$V_0 = \frac{m}{0.9999 \times 10^3}$$

$$\Delta V = \frac{m}{0.9997 \times 10^3} - \frac{m}{0.9999 \times 10^3}$$

$\Delta T = 3$

$\therefore \; \beta = 6.67 \times 10^{-5}\ (^\circ\text{C})^{-1}$

(c) (i) $Q = m \times c_{st} \times \Delta T_{st} + m \times L_v + m \times c_w \times \Delta T_w$

$= 5.00 \times 1970 \times (300 - 100) + 5.00 \times 2.256 \times 10^6 + 5.00 \times 4186 \times (100 - 70)$

$= 1.97 \times 10^6 + 1.128 \times 10^7 + 6.279 \times 10^5$

$= 1.39 \times 10^7$ J

(d)

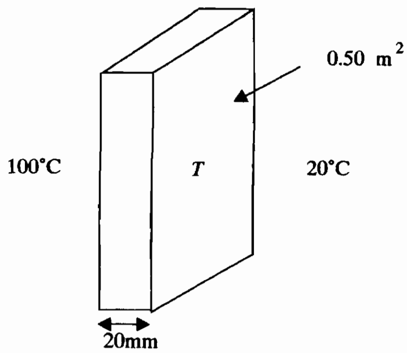


Figure 9.19

Thermal conductivity through sheet:

$$\frac{dQ}{dt} = \frac{kA\,\Delta T}{\Delta x} = \frac{0.01 \times 0.5 \times (100 - T)}{20 \times 10^{-3}}$$

Radiation loss from surface:

$$\frac{dQ}{dt} = e\,\sigma A\,(T^4 - T_0^{\,4})$$

$$= 0.2 \times 5.67 \times 10^{-8} \times 0.5 \times [(273 + T)^4 - (273 + 20)^4]$$

Try $T = 42°C$,

$$\left(\frac{dQ}{dt}\right)_{cond} = 14.5\ \mathrm{J\,s^{-1}},\ \left(\frac{dQ}{dt}\right)_{rad} = 14.0\ \mathrm{J\,s^{-1}}$$

Note: at $T = 45°C$,

$$\left(\frac{dQ}{dt}\right)_{cond} = 13.8\ \mathrm{J\,s^{-1}},\ \left(\frac{dQ}{dt}\right)_{rad} = 16.2\ \mathrm{J\,s^{-1}}$$

Question 6

(a) (i) Carnot cycle

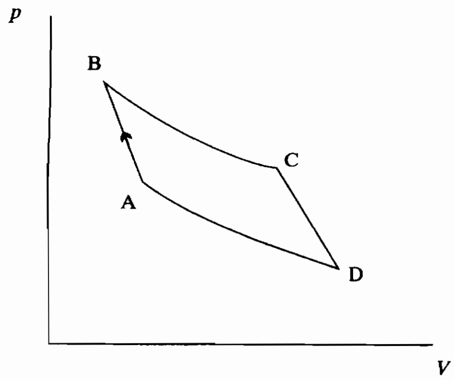


Figure 9.20

A–B and C–D are adiabatic; B–C and D–A are isothermal; between B and C heat energy, Q_1, taken in (+); between D and A heat energy, Q_2, given out (–).

Work done: $W = Q_1 + Q_2$

Efficiency: $e = \dfrac{W}{Q_1} = 1 - \dfrac{T_2}{T_1}$

(ii)
$$e = 1 - \frac{T_2}{T_1}$$
$$= 1 - \frac{273 + 20}{273 + 180}$$
$$= 0.35$$

(b)
$$pV = n\text{R}T$$
$$1 \times 10^5 \times 0.3 = n \times 8.314 \times (273 + 25)$$
$$n = 12 \text{ moles}$$

(c)
$$\frac{1}{2}m\overline{v^2} = \frac{3}{2}\text{k}T$$
$$\therefore \quad \frac{1}{2} \times \frac{28 \times 10^{-3}}{6.022 \times 10^{23}} \times \overline{v^2} = \frac{3}{2} \times 1.381 \times 10^{-23} \times (273 + 25)$$
$$\overline{v^2} = 2.66 \times 10^5$$
$$v_{\text{r m s}} = 520 \text{ m s}^{-1}$$

(d) (i) $p_B = p_A = 1.0 \times 10^5$ Pa

$V_B = 2V_A = 0.60 \text{ m}^3$

$$\frac{p_A V_A}{T_A} = \frac{p_B V_B}{T_B}$$

$T_B = 2T_A = 2 \times (273 + 25) = 596$ K

$T_C = T_B = 596$ K

$V_C = 3V_A = 0.90 \text{ m}^3$

$$p_C = \frac{p_B V_B}{V_C} = 6.67 \times 10^4 \text{ Pa}$$

$V_D = V_A = 0.30 \text{ m}^3$

$p_D = p_C = 6.67 \times 10^4$ Pa

$$\frac{p_C V_C}{T_C} = \frac{p_D V_D}{T_D}$$

$$\therefore \quad T_D = \frac{p_D V_D}{p_C V_C} \times T_C$$
$$= \frac{0.3}{0.9} \times 596$$
$$= 199 \text{ K}$$

(ii) A–B isobaric

$$W = p\,\Delta V = 1.0 \times 10^5 \times 0.3$$
$$= 3.0 \times 10^4 \text{ J}$$

B–C isothermal

$$W = p_B V_B \ln \frac{V_C}{V_B}$$
$$= 1.0 \times 10^5 \times 0.6 \ln \frac{0.9}{0.6}$$
$$= 2.4 \times 10^4 \text{ J}$$

C–D isobaric

$$W = p\,\Delta V = -6.67 \times 10^4 \times 0.6$$
$$= -4.0 \times 10^4 \text{ J}$$

D–A isochoric

$$W = 0$$

Question 7

(a) A is the amplitude of SHM; ω is the angular frequency; α is the initial phase.

(b) (i)

$$kx = mg$$
$$k = \frac{0.2 \times 9.8}{0.05} = 39.2 \text{ N m}^{-1}$$
$$T = 2\pi\sqrt{\frac{m}{k}}$$
$$= 2\pi\sqrt{\frac{0.5}{39.2}}$$
$$= 0.71 \text{ s}$$

(ii)

$$K_{max} = E_{total} = P_{max} = \frac{1}{2} kA^2$$
$$= \frac{1}{2} \times 39.2 \times (0.025)^2$$
$$= 1.23 \times 10^{-2} \text{ J}$$

(iii)

$$a = -\omega^2 y$$
$$\omega = \frac{2\pi}{T} = 8.8$$

$\therefore \quad a = -(8.8)^2 \times 0.010$

$= -0.78 \text{ m s}^{-2}$

the direction is upwards (towards the equilibrium position).

(c) (i) $A = 20$ mm (amplitude)

$\lambda = \text{wavelength} = \dfrac{2\pi}{k,}$ but $k = \dfrac{\pi}{6}$, so $\lambda = 2 \times 6 = 12$ mm

$\omega = 2\pi f = 20\pi$, so f = frequency = 10 Hz

$c = f\lambda$

$= 10 \times 12 \times 10^{-3}$

$= 0.12 \text{ m s}^{-1}$

(ii) In the positive direction, i.e. from left to right.

(iii)
$$y = 20 \sin\left(\frac{\pi}{6}x - 20\pi t\right)$$

$$v = \frac{dy}{dt} \text{ (i.e. } x \text{ is fixed)}$$

$$= -400\,\pi \cos\left(\frac{\pi}{6}x - 20\pi t\right)$$

at $x = 9.0$ mm $\quad t = 0.31$ s

$$v = -400\,\pi \cos\left(\frac{3}{2}\pi - 0.31 \times 20\,\pi\right)$$

$$= 740 \text{ mm s}^{-1}$$

i.e. $v = 740 \text{ mm s}^{-1}$ upwards.

(d)
$$y = y_1 + y_2$$

$$y_1 = 20 \sin\left(\frac{\pi}{6}x + 20\pi t\right)$$

$$y_2 = 20 \sin\left(\frac{\pi}{6}x + 20\pi t\right)$$

$$y = 20 \sin\left(\frac{\pi}{6}x - 20\pi t\right) + 20 \sin\left(\frac{\pi}{6}x + 20\pi t\right)$$

Now $\sin A + \sin B = 2 \sin\left(\dfrac{A+B}{2}\right) \cos\left(\dfrac{A-B}{2}\right)$

$$y = 40 \sin\left(\frac{\pi}{6}x\right) \cos(20\pi t)$$

QED

Question 8

(a) (i) $\sigma = -\dfrac{\Delta w/w_0}{\Delta l/l_0}$

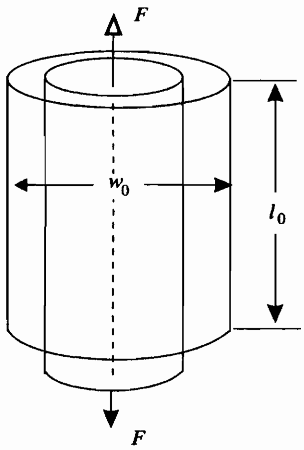


Figure 9.21

(ii)

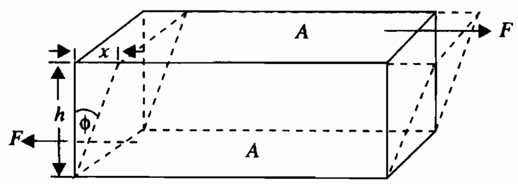


Figure 9.22

$$S = \frac{F/A}{x/h}$$

(b) (i) $Y = \dfrac{F\,l_0}{\Delta l A}$

For copper: $1.1 \times 10^{11} = \dfrac{3. \times 10^4 \times 2.10}{\Delta l \times 205 \times 10^{-6}}$

$$\therefore \quad \Delta l = \frac{3 \times 10^4 \times 2.10}{1.1 \times 10^{11} \times 205 \times 10^{-6}} = 2.79 \times 10^{-3}\ \text{m}$$

For iron: $2.0 \times 10^{11} = \dfrac{3 \times 10^4 \times L_0}{2.79 \times 10^{-3} \times 1.03 \times 10^{-6}}$

$$\therefore \quad L_0 = 1.92\ \text{m}$$

(ii) Stress $= \dfrac{F}{A}$

For copper: stress $= \dfrac{30\,000}{205 \times 10^{-6}} = 1.46 \times 10^8\ \text{N m}^{-2}$

For iron: stress $= \dfrac{30\,000}{103 \times 10^{-6}} = 2.91 \times 10^{8}$ N m^{-2}

(iii) Strain $= \dfrac{\Delta l}{l_0}$

For copper: strain $= \dfrac{2.79 \times 10^{-3}}{2.10} = 1.33 \times 10^{-3}$

For iron: strain $= \dfrac{2.79 \times 10^{-3}}{1.92} = 1.46 \times 10^{-3}$

(c) $\beta_1 - \beta_2 = 10 \log \dfrac{I_1}{I_2}$

$120 - 85 = 10 \log \dfrac{I_1}{I_2}$

$\dfrac{I_1}{I_2} = 3160 = \left(\dfrac{r_2}{r_1}\right)^2$

$r_2 = r_1 \times \sqrt{3160} = 169$ m

(d) (i) $c = f\lambda$

$f = \dfrac{c}{\lambda} = \dfrac{3.0 \times 10^{8}}{500 \times 10^{-9}} = 6.0 \times 10^{14}$ Hz

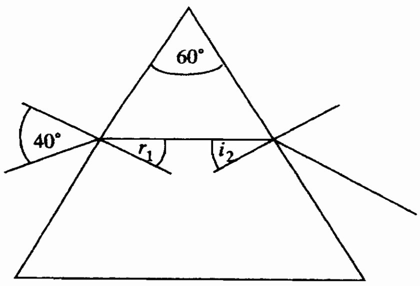


Figure 9.23

(ii) $n_1 c_1 = n_2 c_2$

$c_2 = \dfrac{1.0 \times 3.0 \times 10^{8}}{1.60} = 1.88 \times 10^{8}$ m s^{-1}

f remains the same

$\lambda_2 = \dfrac{c}{f} = \dfrac{1.88 \times 10^{8}}{6.0 \times 10^{14}} = 3.13 \times 10^{-7}$ m

(iii) $n_1 \sin i_1 = n_2 \sin r_1$

$1.0 \sin 40 = 1.6 \sin r_1$

$r_1 = 23.7^{\circ}$

(iv) Third angle = 180 – (60 + 66.3) = 53.7°

Therefore $i_2 = 90 - 53.7 = 36.3^{\circ}$

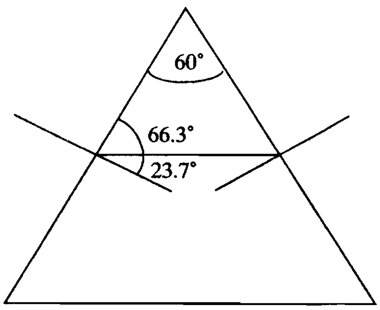


Figure 9.24

Spring Semester Examination

Physics 1S (F/T, P/T)

Full-time: Start 9.30 a.m. Finish 12.40 p.m.
Time allowed: 3 hours + 10 minutes

This paper is designed to be completed in 3 hours. An extra 10 minutes has been added to the time allowed and it is recommended that you use these 10 minutes to read the paper before starting to answer the questions.

Part-time: Start 9.30 a.m. Finish 11.10 a.m.
Time allowed: $1\frac{1}{2}$ hours + 10 minutes

This paper is designed to be completed in $1\frac{1}{2}$ hours. An extra 10 minutes has been added to the time allowed and it is recommended that you use these 10 minutes to read the paper before starting to answer the questions.

Full-time: Answer **3** questions from Section **C** and **3** questions from Section **D**.
Part-time: Answer **3** questions from Section **C** only and leave the room quietly at the time stated.

Calculators may be used.

Answer *each* question in a *separate* booklet.

Clearly *mark the question number on the front* of each booklet.

Section A

(Attempt any *three* questions in this section)

Question 1 *Marks*

(a) Explain what is meant by the electric potential difference, V_{ab}, of a point a with respect to point b.

Give both a 'word explanation' and an equation.

3

(b) A particle of mass 2.0×10^{-27} kg carries a charge $q = +3.0\ \mu C$. It moves a distance $d = 0.50$ m from point a to point b along a straight line. The electric field is uniform along this line, in the direction from a to b, its magnitude being $E = 200\ N\ C^{-1}$.

Find:

(i) the force on q;

(ii) the work done on it by the field;

(iii) the potential difference, $V_a - V_b$.

7

(c) Charges of $+ 10.0 \times 10^{-9}$ C and -10×10^{-9} C are placed on the two electrodes of a capacitor formed by two parallel metal plates of area 40 mm^2 separated by a distance of 5.0 mm.

(i) Determine the capacitance.

(ii) What is the energy stored by the capacitor?

(iii) With what speed will an electron released from rest from the negative plate strike the positive plate?

10

20

Question 2 *Marks*

(a) (i) Explain what is meant by the term 'electromotive force'.

(ii) Write an expression for the heat generated when a current I passes through a resistor R for a time t.

3

(b) A copper wire of length 1200 m has a cross-sectional area of 1.20 mm^2. The resistivity of copper is $1.725 \times 10^{-8}\ \Omega$ m at 20°C and the temperature coefficient of resistance is $3.92 \times 10^{-3}\ K^{-1}$. Determine the resistance at 100°C.

5

(c) If a current of 2.0 A flows through the 20 Ω resistance in the circuit shown in the diagram determine:

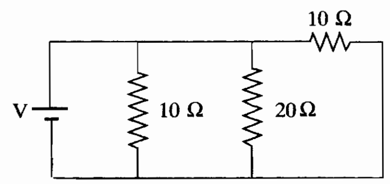


Figure 9.25

(i) the emf of the (ideal) battery V;

(ii) the power dissipated by the 20 Ω resistance.

5

(d) Determine the current passing through the 6.25 Ω resistance in the circuit shown in the diagram. (Use Kirchhoff's laws.)

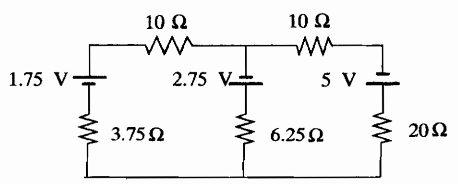


Figure 9.26

7

20

Question 3 *Marks*

(a) State Lenz's Law.

2

(b) Use Lenz's Law to answer each of the following questions:

(i) When the N pole of a bar magnet is moved towards a large metal plate, a current is induced in the plate. Describe the nature of the induced current and indicate, with justification, its direction.

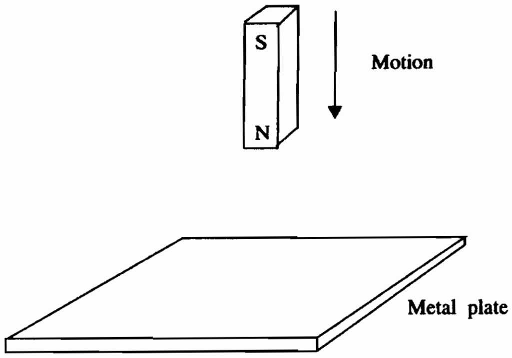


Figure 9.27

(ii) A conducting rod, in contact with a pair of rails, moves at right angles to a magnetic field directed into the page. In what direction does the current flow? Explain your answer.

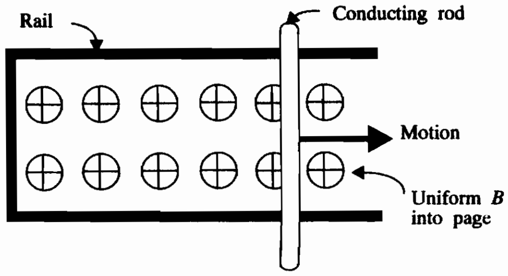


Figure 9.28

5

(c) A 20 cm length of wire is suspended from a spring balance. The wire is in a uniform magnetic field, which acts in a horizontal plane perpendicular to the wire as shown on the diagram. When a 20 A current is passed through the wire the balance reads 8.7 g. When the current is reversed the balance reads 5.3 g. Determine the magnetic field acting on the wire. 5

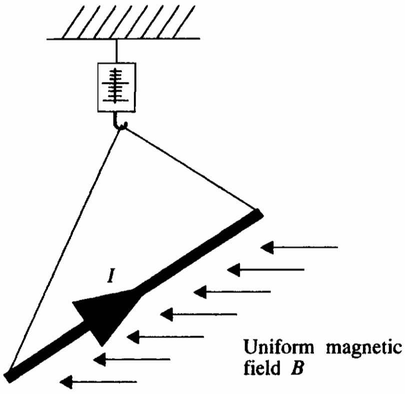


Figure 9.29

(d) How strong a magnetic field is needed to cause an electron moving at 1.0×10^7 m s^{-1} to change its direction of motion by 90° within the dimensions of a square of side 200 mm?

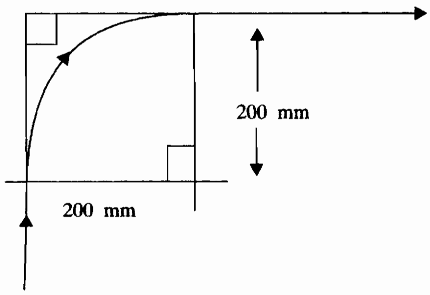


Figure 9.30

4

(e) Use Ampere's circuital law to obtain the magnetic induction at a distance of 0.080 m from a long straight conductor carrying a current of 17.0 A.

4

20

Question 4 Marks

(a) An AC voltmeter is used to measure the (AC) source emf and the voltages across the inductor, the capacitor and the resistor in a series LCR circuit.

(i) Draw a circuit diagram showing this circuit.

(ii) Draw a phasor diagram for the voltage across each component and explain why the emf is larger than the voltage across the resistor but smaller than the algebraic sum of voltages across the inductor, capacitor and resistor.

5

(b) An LCR circuit consists of an inductor of 15.0 mH, a 2.8 μF capacitor and a resistor of 5.0 Ω connected in series to a source of alternating emf $E = E_0 \sin\omega t$ where $E_0 = 0.60$ V and $\omega = 4.7 \times 10^3$ rad s^{-1}.

(i) What is the impedance of this circuit?

(ii) What is the maximum current in the circuit?

(iii) What is the phase angle of the current with respect to the applied emf?

(iv) If the frequency was allowed to vary, at what value of ω would resonance occur?

8

(c) The voltage between A and B, in the RC circuit shown in the diagram, is held at 20 V until the capacitor is fully charged. At this time the potential difference between A and B instantaneously dropped to 0 V by connecting a short circuit between these points.
Determine:

(i) the time constant of the circuit;

(ii) the voltage across, and the current through, each component at the instant the short circuit is made;

(iii) the voltage across the capacitor and the current through it, 1 second after the short circuit is applied.

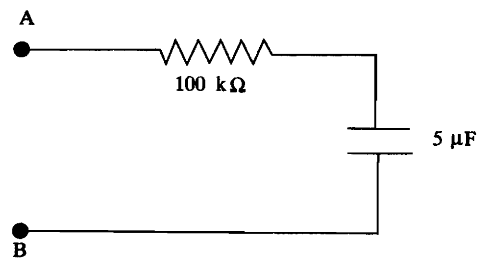


Figure 9.31

7

20

Section B

(Attempt any *three* questions in this section)

Question 5 Marks

(Note: When drawing ray diagrams in this question, different scales can be used in the horizontal and vertical directions.)

(a) State the lens maker's formula. With the aid of a diagram, clearly define each symbol, including the conditions under which it is positive. 3

(b) A pin, 12 mm high, is placed at a point O, on the axis, 150 mm from the pole, P, of a concave spherical mirror of focal length 500 mm.
  (i) Determine, by calculation, the nature, position and size of the image.
  (ii) Draw a ray diagram to determine the position of the image. 6

(c) If a 2.0 mm high object is placed 0.50 m to the left of a biconvex lens of focal length 0.20 m, determine the position, nature and size of the image by:
  (i) calculation;
  (ii) a ray diagram. 6

(d) A biconcave lens with a focal length of 0.10 m is placed 0.50 m to the right of the biconvex lens of focal length 0.20 m (i.e. the lens in part (c)). Determine by calculation the position, nature and size of the image formed by the combination of lenses.

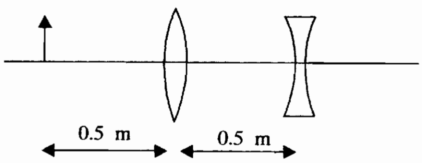


Figure 9.32

5
20

Question 6 *Marks*

(a) A tank containing a liquid of density, ρ, has a small hole in the side at a depth below the surface, which is open to the atmosphere. Use Bernoulli's equation to derive a formula for the speed with which the liquid leaves the hole.

4

(b) A child insists on taking a large cylindrical kitchen pot into the bath with her and floating it among the other toys. Her mother notices that the 0.97 kg pot floats with only 37 mm submerged in the water (density 1000 kg m^3), but when her favourite doll is placed in the pot, the pot sinks until its base is 97 mm below the surface. What is the mass of the doll?

6

(c) A glass U-tube of radius 0.10 mm is placed upside down so that one end is immersed in a container of water (density = 1000 kg m^{-3} and surface tension = 72×10^{-3} N m^{-1}) and the other in a container of alcohol (density = 830 kg m^{-3}).

The gas pressure in the U-tube is reduced to a value lower than atmospheric and it is found that the height of the water column is 0.250 m while that of the alcohol is 0.230 m (see diagram).

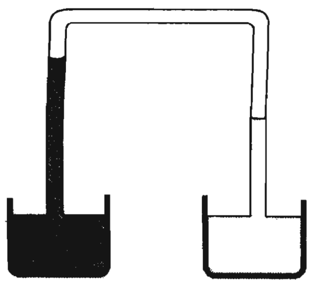


Figure 9.33

(i) Find an expression for the pressure in the U-tube in terms of the atmospheric pressure.

(ii) Find the surface tension of the alcohol. (Assume all angles of contact are zero.)

10

20

Question 7 *Marks*

(a) (i) Show that the terminal speed for a sphere falling through a medium is given by:

$$v_t = 2R^2 \frac{g}{9\eta} (\rho_s - \rho_L)$$

State the meaning of all the symbols used.

(ii) A glass marble of radius 4.0 mm falls through the liquid in a large tank and reaches terminal velocity before it hits the bottom. Just before impact with the bottom the marble is found to be moving at 24 m s^{-1}. How fast would a marble of radius 3.1 mm, dropped from the same height, be moving at the bottom if the glass of the first marble has a density five times that of the liquid while the second marble's density is three times that of the liquid.

6

(b) Light of wavelength 550 nm is shone onto two identical slits set a distance *d* apart. Bright fringes 0.20 mm apart are observed on a screen 1.20 m away.

(i) How far apart are the slits?

If one slit is covered up the width of the central maximum is observed to be 0.62 mm.

(ii) How wide is the slit?

6

(c) The pupil of the eye has a diameter of about 5 mm at night. Two lights (wavelength 550 nm) are 0.80 m apart, and are just resolved by the naked eye. How far from the eye are the lights?

3

(d) The photoelectric work function for sodium is 2.3 eV.

(i) What is the threshold wavelength?

(ii) Will light with a wavelength of 580 nm cause the emission of a photoelectron? (You must include an explanation in your solution.)

(iii) If light with a wavelength of 350 nm falls on sodium, what is the maximum kinetic energy of the photoelectrons?

5

20

Question 8 *Marks*

(a) State the postulates on which the Bohr theory of the atom is based. 2

(b) Find the frequency of radiation emitted when an electron jumps from level 3 to level 2 in the Bohr model of the hydrogen atom. 4

(c) If 30% of the atoms of a radioactive element decay in 12 days, what is the half-life of the element? 6

(d) A He–Ne laser emits 2 pulses per second, each pulse lasting for 10 ms. If the laser is rated at 5 mW at a wavelength of 634 nm:

(i) How much energy is emitted during each pulse?

(ii) What is the energy of each photon emitted?

(iii) How many photons are released each pulse? 5

(e) How much energy is released during the following reaction? (Mass of nucleus is quoted.)

$$\underset{(3.0155\ \text{u})}{{}^{3}_{1}\text{H}} + \underset{(2.0136\ \text{u})}{{}^{2}_{1}\text{H}} \longrightarrow \underset{(4.00150\ \text{u})}{{}^{4}_{2}\text{He}} + \underset{(1.00866\ \text{u})}{{}^{1}_{0}\text{n}}$$

3

20

A Answers

Question 1

(a) Words: V_{ab} is the work done by the electric field per unit charge during the motion of a charge from a to b.

Equation: In general, $V_{ab} = \int_a^b \boldsymbol{E} \cdot d\boldsymbol{l}$

$$= \frac{W}{q}$$

(b) (i) $F = qE = 3.0 \times 10^{-6} \times 200 = 600\ \mu\text{N}$ in the direction of E

(ii) $W = Fs = 600 \times 10^{-6} \times 0.50 = 300\ \mu\text{J}$

(iii) $V_a - V_b = \frac{W}{q} = \frac{300\ \mu\text{J}}{3.0\ \mu\text{C}} = 100\ \text{V}$

(c) (i) $C = \frac{A\varepsilon_0}{d} = \frac{40 \times 10^{-6} \times 8.85 \times 10^{-12}}{0.005} = 0.071\ \text{pF}$

(ii) $U = \frac{1}{2} CV^2 = \frac{1}{2} C \frac{q^2}{C^2}$

$$= \frac{1}{2} \times \frac{(10^{-8})^2}{0.071 \times 10^{-12}} = 7.0 \times 10^{-4}\ \text{J}$$

(iii) KE gained by electron $= qV = \frac{q^2}{C} = \frac{1}{2} mv^2$

$$\therefore \quad v = \sqrt{\frac{2q^2}{mC}}$$

$$= \sqrt{\frac{2 \times (1.6 \times 10^{-19})^2}{9.11 \times 10^{-31} \times 0.071 \times 10^{-12}}}$$

$$= 890\ \text{m s}^{-1}$$

Question 2

(a) (i) emf — the influence that makes charge move from a lower to a higher potential.

(ii) Energy = I^2Rt

(b) $R = \frac{\rho\, l}{A} = \frac{1.725 \times 10^{-8} \times 1200}{1.20 \times 10^{-6}}$

$$= 17.25\ \Omega$$

$$R = R_0(1 + \alpha \Delta t)$$
$$17.25 = R_0 (1 + 3.92 \times 10^{-3} \times 20)$$
$$R_0 = 16.00\ \Omega$$
$$\therefore\ R_{100} = 16.00 (1 + 3.92 \times 10^{-3} \times 100)$$
$$= 22.7\ \Omega$$

(c) (i) All three resistors are in parallel
$\therefore$ voltage across each resistor $V = IR$
$= 2 \times 20$
$= 40$ V

$\therefore$ emf of battery = 40

(ii) $P = I^2 R = 2^2 \times 20 = 80$ W

(d)

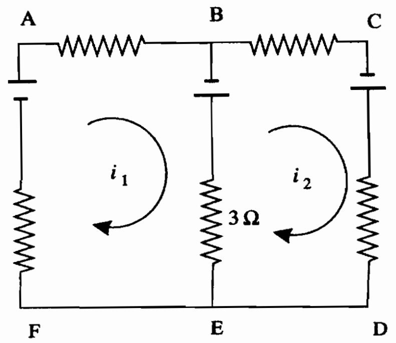


Figure 9.34

In loop ABEFA: $-10i_1 + 2.75 - 6.25i_1 + 6.25i_2 - 3.75i_1 + 1.75 = 0$ [1]

In loop BCDEB: $-10i_2 + 5 - 20i_2 - 6.25i_2 + 6.25i_1 - 2.75 = 0$ [2]

Equation [1] becomes: $20i_1 - 6.25i_2 = 4.50$ [3]

Equation [2] becomes: $36.25i_2 - 6.25i_1 = 2.25$ [4]

Solving [3] and [4] gives: $i_1 = 0.258$ A; and $i_2 = 0.107$ A

hence current through 6.25 Ω resistor = $i_1 - i_2 = 0.152$ A

Question 3

(a) Lenz's Law states that 'the direction of an induced current is such as to oppose the cause producing it'.

(b) (i) As the magnet approaches the plate it will produce an increasing flux into the plate. By Lenz's Law, the induced current will itself

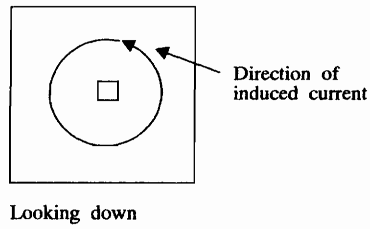


Figure 9.35

produce a field opposing the increasing applied flux, that is, a flux out of the plate. So the induced current will flow in anticlockwise loops around the plate.

(ii) When the conducting rod is moved as shown, the magnetic flux into the page in the circuit is increased. The induced current will oppose this — current will be anticlockwise producing magnetic flux out of the page

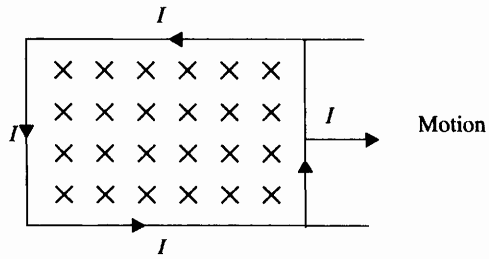


Figure 9.36

(c)

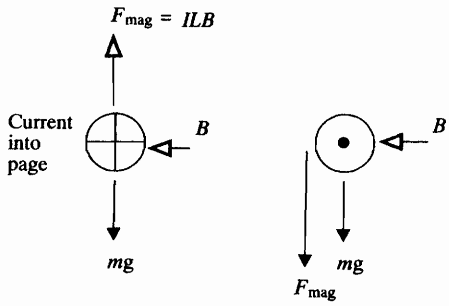


Figure 9.37

The difference in the forces = $2ILB = (8.7 - 5.3) \times 10^{-3} \times g$

$$3.4 \times 10^{-3} \times 9.8 = 2 \times 20 \times 0.2 \times B$$

$$B = 4.2 \times 10^{-3}\,\mathrm{T}$$

(d) $F = qvB = \dfrac{mv^2}{R}$

$$B = \frac{mv}{qR} = \frac{9.31 \times 10^{-31} \times 1.0 \times 10^{7}}{1.6 \times 10^{-19} \times 0.2}$$

$= 2.91 \times 10^{-4}$ T

(e) $$\oint B \cdot dl = \mu_0 I$$

for a long wire:

$B \times 2\,\pi\,a = \mu_0 I$

$$\therefore\ B = \frac{\mu_0 I}{2\pi a}$$

$$= \frac{4\pi \times 10^{-7} \times 17.0}{2\pi \times 0.080}$$

$= 4.25 \times 10^{-5}$ T

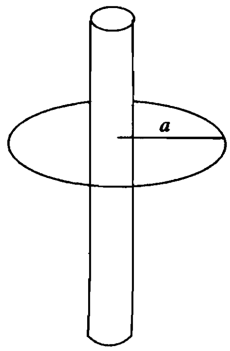


Figure 9.38

Question 4

(a) (i)

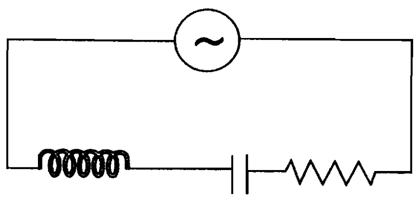


Figure 9.39

(ii) **Voltages across the resistor, capacitor and inductor are out of phase with each other, as shown in the phase diagram:**

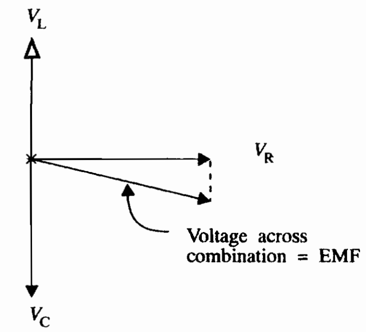


Figure 9.40

Hence V_L and V_C could both be quite large in magnitude but the resultant of V_R, V_L and V_C, which is equal to the applied emf, is not as large since V_L and V_C always oppose each other in direction. In general:

$$|V_{emf}| < |V_R| + |V_L| + |V_C|$$

(b) $X_L = \omega \times 15.0 \times 10^{-3} = 4.7 \times 10^3 \times 15.0 \times 10^{-3} = 70.5\ \Omega$

$$X_C = \frac{1}{4.7 \times 10^3 \times 2.8 \times 10^{-6}} = 76.0\ \Omega$$

(i) $Z = \sqrt{R^2 + (X_L - X_C)^2} = 7.43\ \Omega$

(ii) $I_{max} = \frac{V}{|Z|} = \frac{0.60}{7.43} = 81$ mA

(iii) $\varphi = \tan^{-1}\frac{X_L - X_C}{R} = -48°$

(iv) $\omega L = \frac{1}{\omega C}$ gives resonance at $\omega = \frac{1}{\sqrt{LC}} = 4.88 \times 10^3$ rad s^{-1}

(c) (i) Time constant $\tau_C = RC = 100 \times 10^3 \times 5 \times 10^{-6}$
$= 0.5$ s

(ii) At time $t = 0$, $V_C = 20$ V and $V_C + V_R = 0$

$\therefore V_R = -20$ V

also $V_R = IR$

$$I = -\frac{20}{100 \times 10^3}$$

$= -200\ \mu$A

(iii) $V_C = V_0(e)^{-t/RC} = 20(e)^{-1/0.5}$

$= 2.71$ V

$V_R = IR = 2.71$

$I = 27\ \mu$A

Question 5

(a)

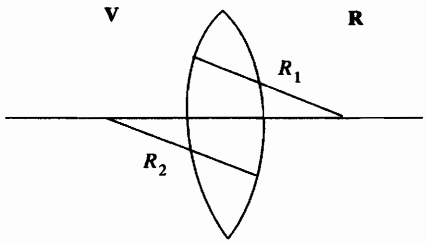


Figure 9.41

For a lens in air: $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

where f = focal length, positive if parallel beam is made convergent; R_1, R_2 are the radii, positive if centre is on R-side (shown); n is the refractive index of the lens.

(b) (i)

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

$$\frac{1}{150} + \frac{1}{s'} = \frac{1}{500}$$

$$s' = -214 \text{ mm}$$

$$m = -\frac{s'}{s} = -\frac{-214}{150} = 1.43$$

$$\therefore \quad |\,\text{size}\,| = 12 \times 1.43 = 17 \text{ mm}$$

Therefore image is virtual, 17 mm high and 214 mm behind the mirror.

(ii)

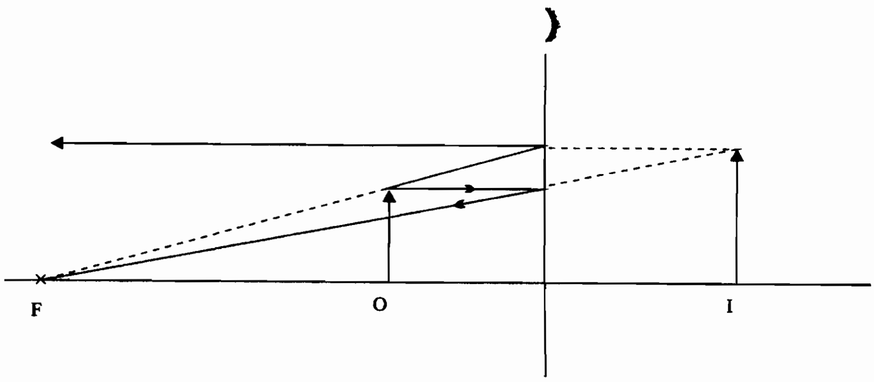


Figure 9.42

(c) (i)

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$$

$$\frac{1}{0.50}+\frac{1}{s'}=\frac{1}{0.20}$$

$$s'=0.33\text{ m}$$

$$m_1=-\frac{s'}{s}=-\frac{0.33}{0.50}=-0.66$$

$$\therefore\ |\text{size}|=2\times 0.66\ =1.3\text{ mm}$$

Therefore image is real, inverted, 1.3 mm high and 0.33 m behind the lens.

(ii)

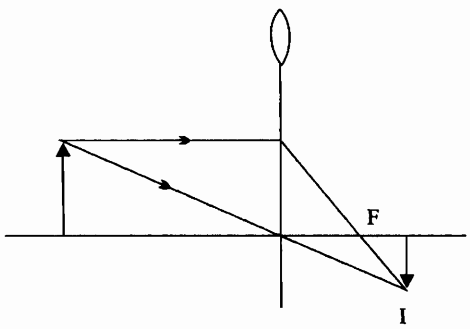


Figure 9.43

(d)

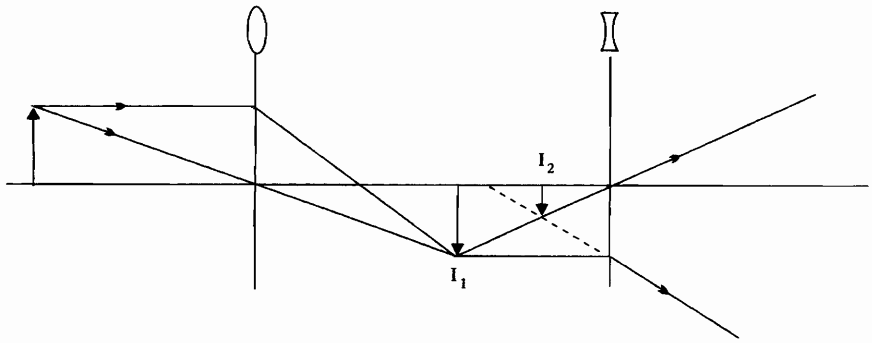


Figure 9.44

$$s = 0.50 - 0.33 = 0.17\ \text{m}$$

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

$$\frac{1}{0.17} + \frac{1}{s'} = \frac{1}{-0.10}$$

$$s' = -0.063\ \text{m}$$

$$m_2 = -\frac{s'}{s} = -\frac{0.063}{0.17} = 0.37$$

$$\therefore\quad |\,\text{size}\,| = 0.37 \times 1.33 = 0.49\ \text{mm}$$

Therefore image is virtual, inverted, 0.49 mm high and 0.063 m to the left of the biconcave lens.

Question 6

(a)

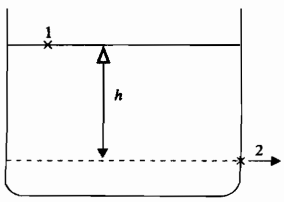


Figure 9.45

Bernoulli's equation:

$$p_1 + \rho g h_1 + \frac{1}{2}\rho v_1 = p_2 + \rho g h_2 + \frac{1}{2}\rho v_2{}^2$$

at 1: $p_1 = p_{\text{at}},\ h_1 = h,\ v_1 = 0$

at 2: $p_2 = p_{\text{at}},\ h_2 = 0,\ v_2 = ?$

$$\therefore\quad p_{\text{at}} + \rho g h + 0 = p_{\text{at}} + 0 + \frac{1}{2}\rho v_2{}^2$$

$$\frac{1}{2}\rho v_2{}^2 = \rho g h$$

$$v_2 = \sqrt{2gh}$$

(b) By Archimedes' Principle — for an object floating:
$mg = \rho V_s g$, where V_s is submerged volume

case 1: $m_p g = \rho V_{s_1} g = \rho A h_1 g$

$$0.97 = 1000 \times A \times 0.37$$

$$A = 2.62 \times 10^{-3}\ \text{m}^2$$

case 2: $(m_p + m_d)g = \rho V_{s_2} g = \rho A h_2 g$

$$(m_d + 0.97) = 1000 \times 2.62 \times 10^{-3} \times 0.97$$

$$m_d = 2.54 - 0.97 = 1.57\ \text{kg}$$

(c) Let the gas pressure inside the U-tube be P_0.

For the water: $P_{at} = P_0 - \dfrac{2\gamma_w}{r}\cos\theta_w + \rho_w g h_w$

For the alcohol: $P_{at} = P_0 - \dfrac{2\gamma_a}{r}\cos\theta_a + \rho_a g h_a$

$$\therefore P_0 - \frac{2\gamma_w}{r}\cos\theta_w + \rho_w g h_w = P_0 - \frac{2\gamma_a}{r}\cos\theta_a + \rho_a g h_a$$

if $\theta_w = \theta_a = 0$

$$\frac{2\gamma_a}{r} = \frac{2\gamma_w}{r} + \rho_a g h_a - \rho_w g h_w$$

$$\gamma_a = 72 \times 10^{-3} + \frac{0.1 \times 10^{-3}}{2} \times 9.8\,(830 \times 0.23 - 1000 \times 0.25)$$

$$= 72 \times 10^{-3} - 29 \times 10^{-3}$$

$$= 43 \times 10^{-3}\ \text{N m}^{-1}$$

Question 7

(a) (i) R radius; g acceleration due to gravity; η coefficient of viscosity; ρ_s density of solid; ρ_l density of liquid.

At terminal speed, v_t, there is no net force on the sphere.

Weight = upthrust + retarding force

$$\frac{4}{3}\pi R^3 \rho_s g = \frac{4}{3}\pi R^3 \rho_l g + 6\pi\eta R v_t$$

$$\therefore\ v_t = \frac{\frac{4}{3}\pi R^3 g[\rho_s - \rho_l]}{6\pi\eta R}$$

$$= \frac{2R^2 g}{9\eta}[\rho_s - \rho_l]$$

(ii) $\frac{v_1}{v_2} = \frac{R_1{}^2[\rho_1 - \rho_l]}{R_2{}^2[\rho_2 - \rho_l]}$

$$\frac{24}{v_2} = \frac{4^2[5-1]}{3.1^2[3-1]}$$

$$v_2 = \frac{24 \times 3.1^2}{4^2 \times 2} = 7.2 \text{ m s}^{-1}$$

(b) (i) Distance between slits d

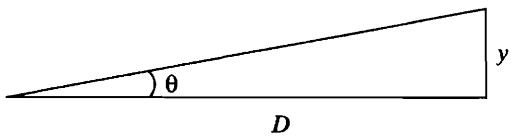


Figure 9.46

$d \sin\theta = n\,\lambda$

$\therefore$ first bright fringe: $\sin\theta = \frac{\lambda}{d}$

For triangle: $\sin\theta \approx \tan\theta \approx \frac{y}{D}$

$$\therefore\ d = \frac{\lambda D}{y} = \frac{550 \times 10^{-9} \times 1.2}{0.20 \times 10^{-3}} = 3.30 \times 10^{-3} \text{ m}$$

(ii) Slit width a

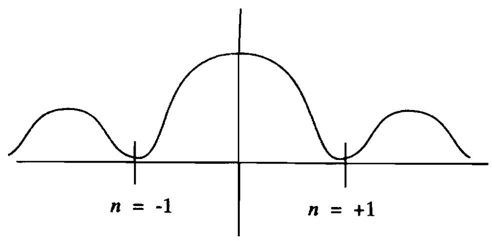


Figure 9.47

$a \sin\theta = n\lambda$

For half central maximum : $\sin\theta \approx \frac{0.31 \times 10^{-3}}{1.2}$

$$\therefore\ a = \frac{1 \times 550 \times 10^{-9} \times 1.2}{0.31 \times 10^{-3}}$$

$$= 2.13 \times 10^{-3} \text{ m}$$

(c) $a \sin \Delta\theta = 1.22\lambda$ is the Rayleigh criterion for resolution of images.

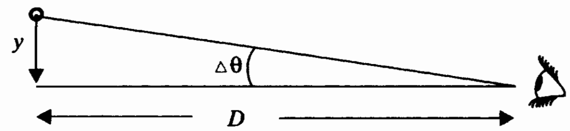


Figure 9.48

$$\sin \Delta\theta = \frac{y}{D}$$

$$\therefore \frac{ay}{D} = 1.22\lambda$$

$$\therefore D = \frac{ay}{1.22\lambda} = \frac{5.00 \times 10^{-3} \times 0.80}{1.22 \times 550 \times 10^{-9}} = 5960 \text{ m}$$

(d) (i) $\varphi = hf_c = \dfrac{hc}{\lambda_c}$

$\varphi = 2.3 \times 1.60 \times 10^{-19} = 3.68 \times 10^{-19}$ J

$$\lambda_c = \frac{hc}{\varphi} = \frac{6.63 \times 10^{-34} \times 3.00 \times 10^{8}}{3.68 \times 10^{-19}} = 5.40 \times 10^{-7} \text{ m}$$

(ii) No, because $\lambda > \lambda_c$, the energy is not sufficient for the emission of a photon.

(iii) $E = \varphi + K_{max} = hf$

$$K_{max} = \frac{hc}{\lambda} - \varphi$$

$$= \frac{6.63 \times 10^{-34} - 3.00 \times 10^{8}}{350 \times 10^{-9}} - 3.68 \times 10^{-19}$$

$$= 5.68 \times 10^{-19} - 3.68 \times 10^{-19}$$

$$= 2.00 \times 10^{-19} \text{ J}$$

Question 8

(a) (i) No light is emitted from an electron in a fixed orbit;

(ii) $mvr = nh / 2\pi$;

(iii) $hf = E_i - E_f$

(b) $$\frac{1}{\lambda} = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

Now $c = f\lambda$

$$f = cR\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$
$$= 3 \times 10^8 \times 1.097 \times 10^7\left(\frac{1}{2^2} - \frac{1}{3^2}\right)$$
$$= 3 \times 10^8 \times 1.097 \times 10^7(0.14)$$
$$= 4.6 \times 10^{14}\ \text{Hz}$$

(c) $N = N_0\, e^{-\lambda t}$

at time $t = 12$, $N = 0.7\, N_0$

$$0.7\, N_0 = N_0\, e^{-12\lambda}$$
$$12\,\lambda = \ln\left(\frac{1}{0.7}\right)$$
$$\lambda = \frac{1}{12}\ln\left(\frac{1}{0.7}\right)$$
$$= 0.030\ \text{d}^{-1}$$
$$t_{½} = \frac{\ln 2}{\lambda}$$
$$= \frac{\ln 2}{0.03}$$
$$= 23\ \text{days}$$

(d) (i) Half the energy is emitted during each pulse.

$$E = \frac{Pt}{2} = \frac{5.0 \times 10^{-3} \times 10 \times 10^{-3}}{2} = 2.5 \times 10^{-5}\ \text{J}$$

(ii) $E = \text{h}f$
$$= \frac{\text{h}c}{\lambda}$$
$$= \frac{6.63 \times 10^{-34} \times 3 \times 10^8}{6.34 \times 10^{-9}}$$
$$E = 3.1 \times 10^{-19}\ \text{J}$$

(iii) Energy for pulse $= Pt$

$$\therefore \text{ number of photons per pulse} = \frac{2.5 \times 10^{-5}}{3.1 \times 10^{-19}}$$
$$= 8.1 \times 10^{13}$$

(e) $\Delta m = (3.0155 + 2.0136) - (4.0015 + 1.00866)$
$= 5.0291 - 5.0106$
$= 0.01894 \text{ u}$
$= 0.01894 \times 1.6604 \times 10^{-27} \text{ kg}$
$= 3.1 \times 10^{-29} \text{ kg}$

$\Delta E = (\Delta m)\, c^2$
$= 3.1 \times 10^{-29} \times (3 \times 10^8)^2$
$= 2.8 \times 10^{-12} \text{ J}$

CHAPTER 10

Practice examination paper

Physics Examination

Section A

(Attempt any three questions in this section)

Question 1 *Marks*

(a) Show that Bernoulli's equation:

$$p_1 + \frac{1}{2}\rho v_1{}^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2{}^2 + \rho g h_2 \text{ is dimensionally correct.}$$

3

(b) A river flows due north at a speed of 0.25 m s^{-1}. A man can row a boat at a speed of 0.5 m s^{-1} relative to the water.

(i) If the man starts from the west bank, in what direction should the boat be headed in order to reach a point on the opposite bank directly east from the starting point?
(ii) What is the velocity of the boat relative to the earth?
(iii) If the river is 100 m wide, how much time is required to cross the river?

5

(c) A police car is waiting at the beginning of an expressway where the speed limit is 110 km h^{-1}. A car travelling at a constant speed of 150 km h^{-1} passes the police car, which immediately gives chase. If the police car can accelerate at a uniform rate of 2 m s^{-2} up to a maximum speed of 180 km h^{-1}, determine:

(i) how long it takes the police car starting from rest to catch up with the speeding car;
(ii) how far the police car travels before it catches the other car.

8

(d) A projectile shot at an angle of 60° above the horizontal strikes a building 30 m away at a point 15 m above the point of projection. Find the speed of projection.

4

20

Question 2 *Marks*

(a) Explain why velodromes (for bicycle racing) have banked tracks. (Include appropriate diagrams in your answer.)

5

(b) A 10 kg block placed on a rough plane inclined at an angle of 25° to the horizontal is observed to slide down the plane with an acceleration of 1.92 m s^{-2}. If a force F is now applied to the 10 kg in a horizontal direction as shown in the diagram, the block is able to be moved up the plane at constant velocity:

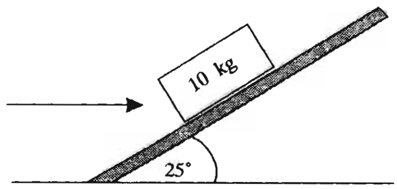


Figure 10.1

What is the magnitude of force F?

9

(c) A passenger lift consists of a lift cage of 900 kg and a counterweight of 990 kg connected by a cable runner over a pair of pulleys as shown. Neglecting the mass of cable and pulleys:

(i) What is the upward acceleration of the lift cage if the pulleys are allowed to run freely?
(ii) What is the tension in the cable?
(iii) What are the tensions in the cable if the pulleys are locked by a braking mechanism so that the lift remains stationary?

(iv) What supporting force would the lift floor be exerting on a 50 kg man standing in the lift which was running freely solely under the lift of the counterweight?

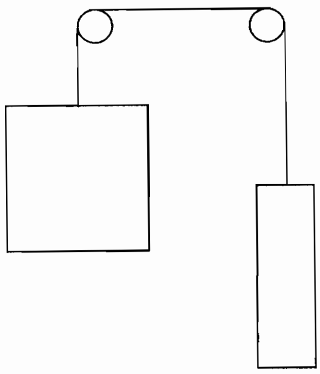


Figure 10.2

6

20

Question 3 *Marks*

(a) Explain, using the Law of Conservation of Momentum, why it may need more than one fire-brigade officer to hold a fire hose delivering 1000 litres of water per minute at a speed of 25 m s^{-1}. (Density of water ρ = 1000 kg m^{-3}) 4

(b) In a wrecking yard, a crash test is being made on a car of mass 2000 kg. When the car is dropped from a height of 9.25 m onto a target at ground level, it is found to rebound to a height of 0.3 m.

(i) How much work has been done by non-conservative forces in the impact?

(ii) In what way is the work done by these forces likely to be observed? 5

(c) A ball moving at a speed of 3 m s^{-1} along the x-axis strikes a stationary ball of the same mass. After the collision the balls move as shown in the diagram:

(i) Determine the speed of each ball after the collision.

(ii) Determine the fraction of the initial kinetic energy of the balls that is lost during the collision.

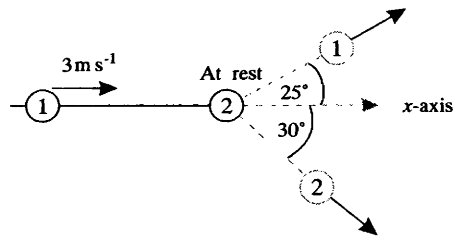


Figure 10.3

7

(d) A lift of mass 1000 kg starts from rest at the 4th floor of a building. When it passes the 10th floor, which is 32.0 m above the 4th floor, its speed is 2 m s^{-1}. The motor driving the lift operates with a constant power of 20 kW. How long does it take the lift to travel between the 4th and 10th floors?

4

20

Question 4 *Marks*

(a) (i) State the parallel axis theorem for calculating moments of inertia. If you quote a formula, clearly explain the meaning of any symbols in it.

(ii) Determine the moment of inertia of a circular hoop of mass 3.750 kg and radius 0.700 m, rotating about an axis, which is through a point on its circumference and at right angles to the plane of the hoop.

5

(b) A cylindrical bucket of radius 70 mm and height 150 mm contains 0.850 kg of water. The base of the bucket has a mass of 0.075 kg and the sides have a mass of 0.325 kg. The water is stirred with an electric stirrer until it is rotating with a uniform angular velocity of 3.700 rad s^{-1} while the bucket is held stationary. The electric stirrer is removed and the bucket is free to rotate about the axis of the cylinder. After a time the liquid and the bucket will rotate at the same angular speed. What will be the new angular velocity of the liquid?

7

(c) A uniform board of mass 25 kg is held up by three ropes as shown in the diagram. The man on the board has mass 100 kg. Find the tension in each of the three ropes.

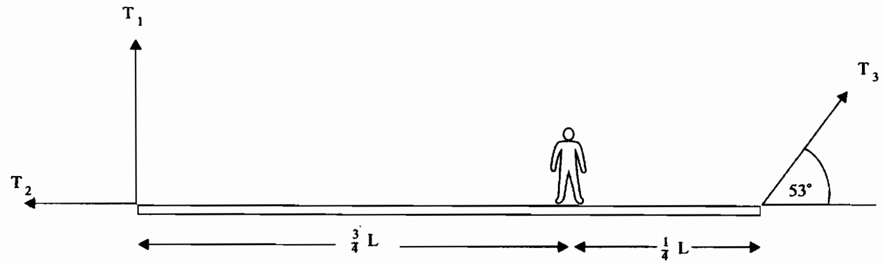


Figure 10.4

8

$\underline{\overline{20}}$

Section B

(Attempt any *three* questions in this section)

Question 5 *Marks*

(a) A block of aluminium of volume 50.00 cm^3 is placed in a glass beaker of volume 200.0 cm^3, which is then completely filled with water. Calculate the amount of water that flows out of the beaker if the temperature of the system is raised by 30°C. (The coefficients of volume expansion are: water $34.3\times 10^{-5}(°C)^{-1}$; aluminium $7.2\times 10^{-5}\ (°C)^{-1}$; glass $2.0\times 10^{-5}\ (°C)^{-1}$)

4

(b) 50 g of ice at 0°C is placed into 200 ml of water at 5°C. Determine the amount of ice that melts and the final temperature of the system. Assume the system of ice and water is thermally isolated. (Heat of fusion of ice: 334×10^{3} J kg^{-1}; specific heat capacity of water: 4190 J kg $(°C)^{-1}$; density of water: 1000 kg m^{-3})

4

(c) The filament of a 60 W light globe operates at 2200°C. The diameter of the filament is 0.10 mm and its emissivity is 0.35. Calculate the length of the filament.

4

(d) A cylindrical rod consists of two equal parts joined end to end. The thermal conductivity of one part is k_1, that of the other is k_2. Show that the total thermal conductivity of the whole rod is:

$k_{tot} = 2k_1 k_2 / (k_1 + k_2)$

8

20

Question 6 *Marks*

(a) Show on a pressure–volume diagram each of the following types of processes. Clearly label which is which: Adiabatic, isothermal, isochoric and isobaric.

4

(b) 1.0 litre of hydrogen at 227°C and 27.25 atmospheres pressure expands adiabatically and slowly against a piston until its volume is 4 litres. (γ = 1.41 for H_2)

(i) Find the final pressure and temperature of the gas.
(ii) What is the external work done by the gas.

8

(c) A certain steam engine takes 100 kJ of heat in each cycle from its boiler at 600°C and rejects 80 kJ at 400°C.

(i) How much work is done by the engine each cycle?
(ii) How efficient is this engine?
(iii) What would be the efficiency of a Carnot engine working between these temperatures?
(iv) Briefly explain why no heat engine can be 100% efficient.

8

20

Question 7 *Marks*

(a) A 0.50 kg mass is connected to a spring (k = 20.0 N m^{-1}) as shown in the diagram. The spring is stretched 0.10 m and then the mass is given a push to start it moving at 2.50 m s^{-1} towards the fixed end of the spring.

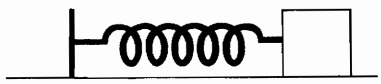


Figure 10.5

(i) Sketch a graph showing the position of the mass as a function of time.
(ii) Determine the initial acceleration of the mass.
(iii) Determine the amplitude of the resulting simple harmonic motion.
(iv) Determine the frequency of the resulting motion.
(v) Determine the time it takes for the mass to first reach the equilibrium position.

9

(b) Write down the expression for a progressive wave that has an amplitude of 6.0 cm and a wavelength of 0.50 m and that is propagating at 20.0 m s^{-1} along a rope stretched in the x direction. Assume that the direction of propagation is in the positive x direction and that the displacement is in the y direction. The displacement is zero at $t = 0, x = 0$, and increases with time at that point.

4

(c) A piano-tuner strikes a piano key and then hits a tuning fork whose frequency of vibration is 328 Hz. He hears 4 beats per second and with a slight increase in the tension he hears 3 beats per second. What was the frequency of the piano string when it was first struck?

3

(d) A standing wave is set up in an organ pipe that has one end closed and the other open.

(i) Sketch a diagram showing the patterns of waves that are set up at the fundamental frequency.

(ii) If the fundamental frequency is 250 Hz and the pipe is 0.3 m long, determine the speed of sound.

4

20

Question 8 *Marks*

(a) A cylindrical slab, of silicone rubber, has a load of 157 kg placed so that the load is spread uniformly over the top of the slab. The slab has a radius of 230 mm. By how much will the rubber expand sideways given that it is 190 mm thick? (Young's modulus of elasticity = 5.97×10^5 N m^{-2}; Poisson's ratio for the material is 0.485.)

6

(b) A small source of sound radiates acoustic energy uniformly in all directions at a rate of 0.75 W. Find the sound intensity level in decibels at a point 20 m from the source. (Assume absorption is negligible.) ($I_0 = 1.0 \times 10^{-12}$ W m^{-2}at the threshold of hearing.)

5

(c) Explain what is meant by the term 'total internal reflection'.

2

(d) A layer of oil of thickness 0.897 cm lies on top of a block of flint glass of refractive index 1.670. Light is shone through the glass so that it strikes the glass–oil boundary at an angle of θ degrees to the normal to the surface.

What is the smallest angle θ at which no light emerges from the top surface of the oil? (The refractive index of oil is 1.235. The refractive index of air is 1.000.)

7

20

Section C

(Attempt any *three* questions in this section)

Question 1 *Marks*

(a) A point charge of $+5.0\ \mu C$ is placed at $x = 0$ mm on a line. A second point charge of $-1.0\ \mu C$ is placed at $x = 20$ mm. Find the x coordinates of the positions along the line (other than x = infinity) where:

(i) the field due to the two charges is zero;

(ii) the potential due to the two charges is zero.

7

(b) An α particle (mass $4 \times$ (proton mass), charge = +2e) with kinetic energy 0.2 MeV, approaches a gold nucleus (charge = −79e) directly along the line of centres of the two particles. If the gold nucleus does not move, what is the distance of closest approach of the α particle to the gold nucleus?

5

(c) Two parallel plates each of area 100 mm^2 are separated by an air space of 4.0 mm. The plates are charged so that one has charge $+2 \times 10^{-10}$ C and the other has charge -2×10^{-10} C.

Find the magnitude of the electric field between the plates.

4

(d) A 60 V power supply is used to charge an arrangement of three capacitors as shown in the diagram:

(i) Find the potential difference across the 50 μF capacitor.

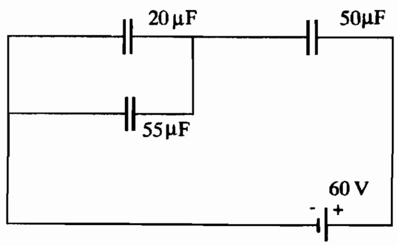


Figure 10.6

(ii) Find the charge on each of the capacitors.

4

20

Question 2 *Marks*

(a) In the circuit shown below, the battery has zero internal resistance.

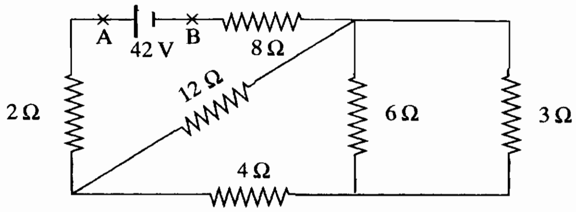


Figure 10.7

Find:

(i) the equivalent resistance from A to B through the network of resistors;
(ii) the current through the battery;
(iii) the current through the 12 Ω resistor;
(iv) the power loss in the 2 Ω resistor.

6

(b) (i) State Kirchhoff's current law (also otherwise known as the point or junction rule).

(ii) The circuit for a typical household lighting system is shown below. A light can be turned on by closing the switch in its line.

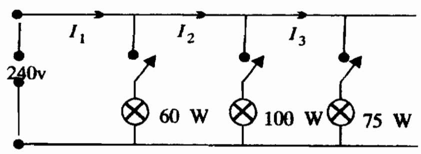


Figure 10.8

If the light globes are rated 60 W/240 V, 100 W/240 V and 75 W/240 V, as indicated in the diagram what will be the values of I_1, I_2 and I_3 when all the lights are turned on?

6

(c) For the following circuit use Kirchhoff's laws and assume that all the batteries have no internal resistance. Hence:

(i) write down sufficient equations to determine the current through the 8 V battery; and

(ii) solve these equations to find the current through the 8 V battery.

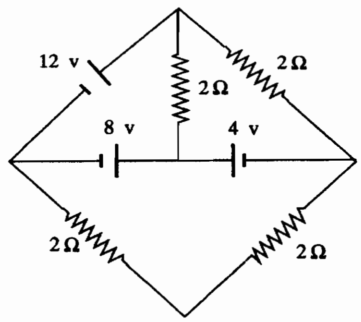


Figure 10.9

8

20

Questions 3 *Marks*

(a)

(i) State Faraday's Law of Electromagnetic Induction indicating the meaning of any symbols.

(ii) Calculate the average emf induced in a square wire loop 80 mm on a side placed perpendicular to a magnetic field of 5×10^{-3} T if the field drops to zero in 0.1 s.

5

(b) Magnetic and electric fields, of 1.2×10^{-2} T and 3.4×10^{4} V m^{-1} respectively, act on a charged particle in such a way that it passes through the crossed fields undeflected.

(i) Show on a diagram the electric field, magnetic field and velocity vectors.

(ii) What is the velocity of the particle?

5

(c) A long coil (solenoid) is constructed by winding 1250 turns of insulated wire evenly along a cardboard tube of radius 15 mm and length 230 mm.

(i) Calculate the self-inductance of the coil.

(ii) The coil is connected to a battery so that a current of 0.750 A flows through the coil. How much energy is stored in the magnetic field within the coil under steady state conditions?

(iii) If the battery is shorted using a wire of resistance 5 Ω across its terminals, what is the time rate of change of current through the coil as soon as the wire is connected?

(iv) Around the first coil is wound another 3300 turns of wire connected to a voltmeter. What voltage will be immediately measured by the voltmeter when the current is changing at a rate calculated in part (iii)?

10

20

Question 4 *Marks*

(a) (i) What does the term 'RMS' mean when applied to AC currents and voltages?

(ii) If an AC signal is sinusoidally varying between –2.8 A and +2.8 A what is the RMS value of the current?

Do laboratory multimeters read RMS or maximum current/voltage values?

3

(b) (i) Show that the average power absorbed by an RLC series circuit can be expressed as $\text{Power}_{\text{average}} = \dfrac{(E_{\text{RMS}})^2 R}{R^2 + (\omega L - \frac{1}{\omega C})^2}$

where ω is the angular frequency of the external power supply.

(ii) Use this equation to determine the resonant frequency.

(iii) What is the power absorbed at the resonant frequency?
At what frequency is the power half that absorbed at resonance?

7

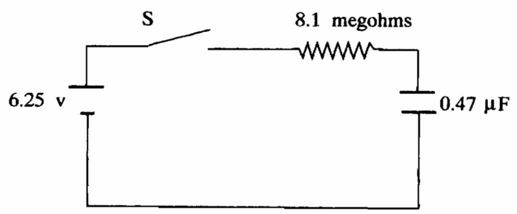


Figure 10.10

(c) The capacitor in the above circuit is uncharged when the switch S is closed. What voltage will be across the capacitor after 5 seconds?

5

(d) **Show clearly on a graph of voltage against time what output will appear across the terminals A–B of the circuit below if a symmetrical square wave is applied to the terminals C–D with amplitude 5.00 V and frequency 10 Hz, 100 Hz and 1000 Hz?**

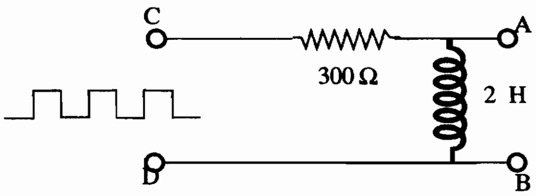


Figure 10.11

$\frac{5}{20}$

Section D

(Attempt any *three* questions in this section)

Question 5 *Marks*

(a) Explain the difference between a real and a virtual image.

2

(b) A biconvex lens is made of glass which has a refractive index 1.6. The faces of the lens are ground to produce spherical surfaces with radii of curvature of 0.25 m and 0.30 m. Determine the focal length of the lens.

3

(c) When an object is placed 370 mm in front of a mirror, a real, magnified image is produced which is 3.23 times the size of the object. What is the radius of curvature of the mirror?

5

(d) (i) An object is placed 5.12 mm in front of a biconvex lens of focal length 5.00 mm. Determine the nature and position of the image using the lens equation.

(ii) A second biconvex lens, of focal length 40.00 mm, is placed 248.00 mm behind the first lens. Determine the nature and position of the final image.

(iii) What is the total magnification?

(iv) Draw a rough ray diagram for the lens combination.

(v) Briefly explain how a microscope works. You may wish to use the diagram in part (iv).

10

20

Question 6 *Marks*

(a) (i) Define the coefficient of surface tension, γ. Clearly describe the force involved including its direction.

(ii) Derive the formula for γ in terms of W, the work done against surface tension forces.

(iii) Explain why small drops of liquid falling freely tend to be spherical.

6

(b) A helium-filled balloon has a volume of 1050 m^3. Let M kg be the mass of the balloon together with the equipment and passengers it is to lift (but not including the helium). Determine the maximum value M can have if the balloon is to rise. (Data: density of air = 1.29 kg m^{-3}; density of helium = 0.18 kg m^{-3})

5

(c) Water (density = 1000 kg m^{-3}) flows along a pipe as in the diagram:

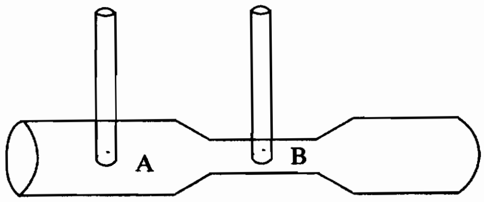


Figure 10.12

The pressure difference between A and B is 600 Pa. The velocity at A is 0.40 m s^{-1}. Treating water as an ideal fluid, calculate:

(i) the velocity at B; and
(ii) the cross-sectional area at A, given that the cross-sectional area at B is 1.00×10^{-3} m^2.

6

(d) Explain what is meant by Reynolds number.

3
20

Question 7 *Marks*

(a) (i) Determine the dimensions of η the coefficient of viscosity. You may use either Poiseuilles' Law, stated below, or the definition of the coefficient of viscosity.

$$\frac{dV}{dt} = \frac{\pi r^4(\Delta p/l)}{8\eta}$$

(ii) An artery is severed 150 mm from the heart. The patient is kept in a horizontal position, so that the wound is at the same vertical height as the heart. The artery is known to have a radius of 1.2 mm and blood a coefficient of viscosity at that temperature of 5.5×10^{-3} Pa s. If in 18 seconds 350 ml of blood flows from the wound determine the heart (gauge) pressure in mm Hg. (Density of mercury = 13600 kg m^{-3})

5

(b) In Young's double-slit arrangement the screen is placed 0.60 m from the slits, which are 0.80 mm apart. Light from a sodium spectral lamp is shone onto the slits and the fringes are observed to be 0.44 mm apart on the screen. What is the wavelength of the sodium light in nanometres?

3

(c) (i) Three polarizing filters are stacked with the polarizing axes of the second and third filters at angles θ and 90°, respectively, with that of the first. If unpolarized light of intensity I_o is incident on the stack show that the intensity I of light transmitted through the stack is given by: $I = \frac{1}{2} I_o \cos^2 \theta \sin^2 \theta$.

(ii) At what angle of incidence will the light reflected from water ($n = 1.33$) be completely polarized?

4

(d) Light is deviated in the first order through an angle of 20° by a transmission grating having 6000 lines cm^{-1}. What is the wavelength of the light? Assume normal incidence.

3

(e) A laser emits light of wavelength 5.0×10^{-7} m. This light is used to irradiate a metal with a work function of 2.1 eV.

(i) What is the energy of the photons emitted by the laser?
(ii) What is the maximum kinetic energy of electrons given off by the metal when the laser light shines on it?

5

20

Question 8 *Marks*

(a) Calculate the velocity of an electron having been accelerated through a voltage difference of 500 V.

3

(b) The emission spectrum of hydrogen gas consists of a series of lines, which are given by the Rydberg equation. A hydrogen atom absorbs a photon causing the electron to jump from its normal (innermost) orbit to the 6th orbit. Find the wavelength of the photon absorbed.

4

(c) Calculate the velocity of an electron which has a de Broglie wavelength of 0.36 nm.

3

(d) **A carbon specimen found in a cave believed to have been inhabited by cave people contains 1/7 as much ^{14}C as an equal amount of carbon in living matter. Find the approximate age of the specimen (the half-life of ^{14}C is 5568 years).**

5

(e) **Calculate the energy released in joules when two deuterium atoms (^{2}H) fuse to form helium (^{4}He), given that the mass of the helium nucleus is 6.645×10^{-27} kg and the mass of the deuterium nucleus is 3.344×10^{-27} kg.**

5

20

Answers Sections A and B

Question 1

(a) p[M L^{-1} T^{-2}]; $\frac{1}{2}\rho v^2$ [M L^{-3} (LT^{-1})^2 = M L^{-1} T^{-2}]; ρgh [M L^{-3} LT^{-2} L = M L^{-1} T^{-2}]

(b) (i) 30° S of E; (ii) 0.433 m s^{-1}; (iii) 231 s.

(c) (i) $t = 25 + 50.3 = 75.3$ s; (ii) s = 625 + 2515 = 3140 m

(d) 21.8 m s^{-1}

Question 2

(a) If not banked

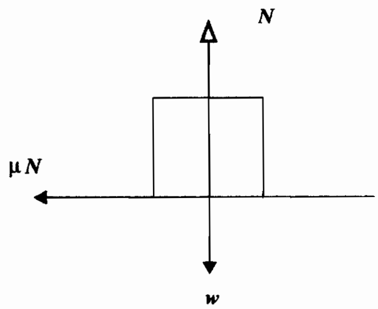


Figure 10.13

μN must be large to provide centripetal acceleration.

If banked

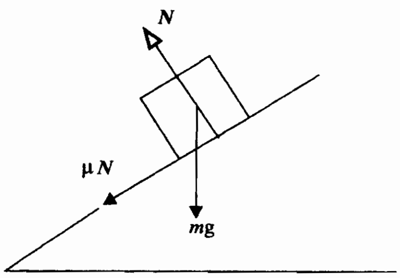


Figure 10.14

The normal reaction N has a component towards the centre, which adds to the component of μN towards the centre.

(b) $F = \dfrac{mg(\sin 25 + 0.25 \cos 25)}{\cos 25 - 0.25 \sin 25} = 74.9 \text{ N}$

(c) (i) $a = 0.47 \text{ m s}^{-2}$; (ii) $T = 9.2$ kN; (iii) $T_1 = 990 \text{ g} = 9.7$ kN, $T_2 = 900 \text{ g} = 8.8$ kN; (iv) $a = 0.20 \text{ m s}^{-2}$, $N = 500$ N

Question 3

(a) Momentum of water per second = 418 kg m s^{-1}
with 1 officer (70 kg) $v_{\text{recoil}} = 5.97 \text{ m s}^{-1}$
with 2 officers $v_{\text{recoil}} = 2.99 \text{ m s}^{-1}$

(b) (i) $\Delta E = 175$ kJ; (ii) Crumpling of the car, heat energy, sound energy.

(c) (i) $v_1 = 1.83$, $v_2 = 1.55$; (ii) fraction lost = 0.36

(d) $t = 15.8$ s

Question 4

(a) (i) $I_A = I_{\text{cm}} + Md^2$; (ii) $I = 3.675 \text{ kg m}^2$

(b) $I_{\text{water}} = 2.082 \times 10^{-3} \text{ kg m}^2$; $I_{\text{bucket}} = 1.777 \times 10^{-3} \text{ kg m}^2$; $\omega_f = 2.0 \text{ rad s}^{-1}$

(c) $T_1 = 368$ N, $T_2 = 646$ N, $T_3 = 1074$ N

Question 5

(a)
$\Delta V_{\text{al}} = 0.108 \text{ cm}^3$
$\Delta V_{\text{water}} = 1.544 \text{ cm}^3$
$\Delta V_{\text{glass}} = 0.12 \text{ cm}^3$
$\Delta V = 1.532 \text{ cm}^3$

(b) $m = 0.013$ kg, $T_f = 0.0°\text{C}$

(c) $l = 0.26$ m

(d) $$\frac{dQ}{dt} = k_1 A \frac{T_1 - T_m}{\frac{L}{2}} = k_2 A \frac{T_m - T_2}{\frac{L}{2}} = k_{\text{tot}} A \frac{T_1 - T_2}{L}$$

$$T_m = \frac{k_1 T_1 + k_2 T_2}{k_1 + k_2}$$

Question 6

(a)

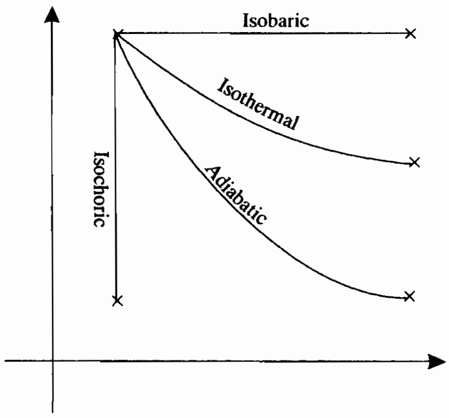


Figure 10.15

(b) (i) 3.86 atmospheres, 283 K; (ii) 2920 J

(c) (i) 20 kJ; (ii) 20%; (iii) 0.77; (iv) some energy is lost in any real thermodynamic process.

Question 7

(a) (i)

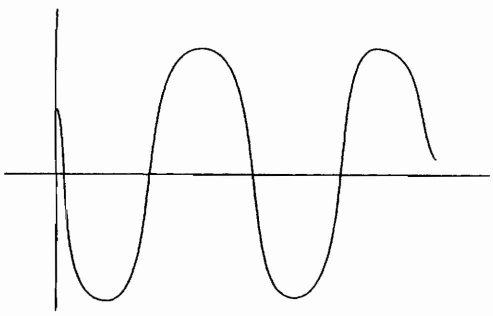


Figure 10.16

(ii) $a = 4\ \mathrm{m\ s^{-2}}$; (iii) $A = 0.41$ m; (iv) $f = 1.01$ Hz; (v) $t = 0.039$ s

(b) $y = 0.06 \sin 2\pi(2x - 40t)$

(c) $f = 324$ Hz

(d) (i)

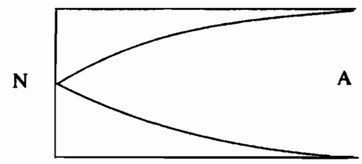


Figure 10.17

(ii) $c = 300\ \mathrm{m\ s^{-1}}$

Question 8

(a) $\Delta l = 2.95 \times 10^{-3}$ m, $\Delta w = 3.46 \times 10^{-3}$ m

(b) $I = 1.49 \times 10^{-4}\ \mathrm{W\ m^{-2}}$; $n = 81.7$ dB

(c) At angles greater than the critical angle there are no refracted rays.

(d) $i_{\mathrm{crit}}(\mathrm{oil/air}) = 54.1°$

$\theta = 36.8°$

Answers Sections C and D

Question 1

(a) (i) $x = 36.2$ mm; (ii) $x = 16.7$ mm

(b) 1.14×10^{-12} m

(c) $C = 2.21 \times 10^{-13}$ F
$V = 904$ V
$E = 2.26 \times 10^{5}$ V m^{-1}

(d) (i) 36.0 V;
(ii) $q_{20\,\mu F} = 4.80 \times 10^{-4}$ C; $q_{55\,\mu F} = 1.32 \times 10^{-3}$ C; $q_{50\,\mu F} = 1.80 \times 10^{-3}$ C

Question 2

(a) (i) $R_{eq} = 14\ \Omega$; (ii) $i = 3.0$ A; (iii) $i = 1.0$ A; (iv) $P = 18$ W

(b) (i) The algebraic sum of currents flowing into a junction is equal to the algebraic sum of the current flowing out of the junction: $\sum i = 0$

(ii)

$I_{60\,w} = 0.25$ A	$I_3 = 0.313$ A
$I_{100\,w} = 0.417$ A	$I_2 = 0.729$ A
$I_{75\,w} = 0.313$ A	$I_1 = 0.979$ A

(c)

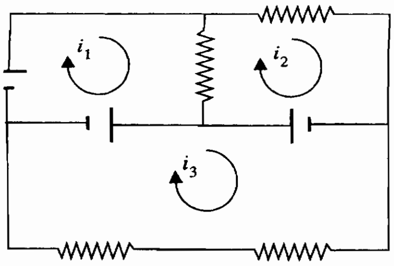


Figure 10.18

$i_1 - i_2 = 2$	$i_1 = 6$ A	
$2i_2 - i_1 = 2$	$i_2 = 4$ A	
$i_3 = 1$	$i_3 = 1$ A	$i_{8v}=5A$

Question 3

(a) (i) The emf induced in a circuit is proportional to the rate of change of magnetic flux through the circuit: $\varepsilon = -N\dfrac{d\varphi}{dt}$

(ii) $\varphi = 32 \times 10^{-6}$ Wb

$\varepsilon = 3.20 \times 10^{-4}$ V

(b) (i)

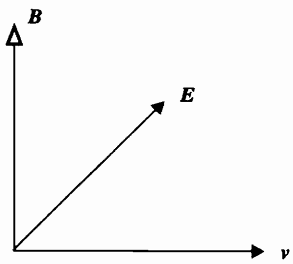


Figure 10.19

B, E, and v are all perpendicular to each other.

(ii) $v = 2.83 \times 10^{6}$ m s^{-1}

(c) (i) $L = 6.03 \times 10^{-3}$ H; (ii) Energy $= 1.7 \times 10^{-3}$ J; (iii) $\dfrac{dI}{dt} = -621$ A s^{-1};

(iv) $\dfrac{d\varphi}{dt} = -3.00 \times 10^{-3}$ Wb s^{-1}; $\varepsilon = 9.9$ V

Question 4

(a) (i) RMS — means root mean square, $V_{\mathrm{rms}} = \dfrac{V_{\max}}{\sqrt{2}}$ for AC signal;

(ii) $i_{\mathrm{rms}} = 2.0$ A; (iii) RMS

(b) (i) $P = V_{\mathrm{rms}}\, i_{\mathrm{rms}} \cos\varphi = \dfrac{E^2_{\mathrm{rms}}}{Z}\cos\varphi$

$$\tan\varphi = \frac{X_L - X_C}{R}$$

$$\therefore \quad \cos\varphi = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{R}{Z}$$

(ii) Power is a maximum if $X_L = X_C$

$$\text{i.e. } \omega L = \frac{1}{\omega C} \quad \text{or} \quad \omega^2 = \frac{1}{LC}$$

(iii) Maximum power $= \dfrac{E^2_{\mathrm{RMS}}}{R}$

(iv) Half power: $R = X_L - X_C$

$$\therefore \quad \omega = \frac{RC \pm \sqrt{R^2C^2 + 4LC}}{2LC}$$

(c) $\tau = 3.81$ s, $V_c = 4.6$ V

(d) $\tau = 6.7$ m s

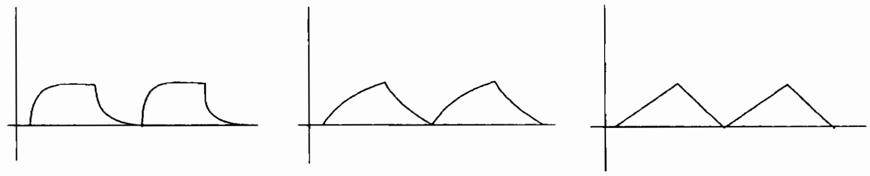


Figure 10.20

Question 5

(a) Real image — rays of light actually pass through image.
Virtual image — rays of light only appear to pass through image.

(b) $f = 0.227$ m

(c) Real image, inverted, $m = -3.23$, $s' = 1195$ mm, $f = 283$ mm, $R = 566$ mm

(d) (i) real inverted, magnified, $m_1 = -41.7$, $s'_1 = 213.3$ mm
(ii) virtual, magnified, $m_2 = +7.55$, $s'_2 = -262$ mm
(iii) $M = -314.8$, final image inverted
(iv) Figure 10.21

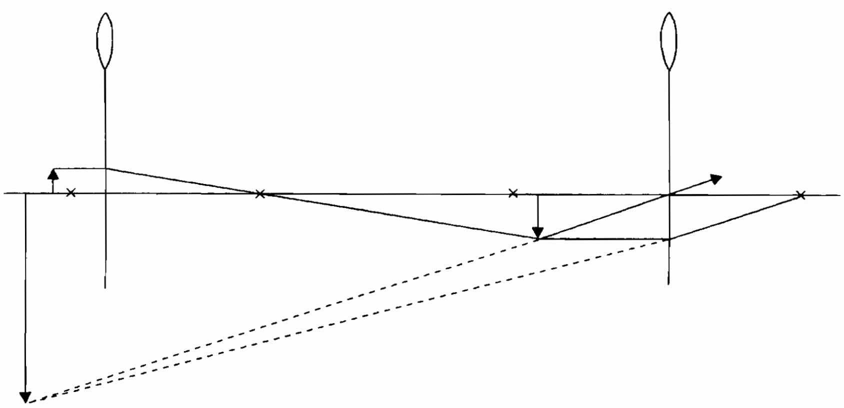


Figure 10.21

(v) Ray diagram shows how a microscope works.
1st lens – objective 2nd lens – eyepiece
Final image can be at $s'_2 = -250$ mm (least distance of distinct vision) or $s'_2 = -\infty$ (relaxed vision).

Question 6

(a) (i) $F = \gamma l$ (see 2.8.6)

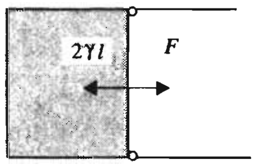


Figure 10.22

(ii) $W = \gamma \Delta A$;

(iii) Surface tension forces tend to make surface area a minimum, for a given volume this is achieved in a sphere.

(b) $M = 1170$ [kg]

(c) (i) $v_2 = 1.17$ m s^{-1}; (ii) $\frac{A_1}{A_2} = 2.93$; $A_1 = 2.93 \times 10^{-3}$ m^2

Question 7

(a) (i) M L^{-1} T^{-1}; (ii) $\Delta P = 1.97 \times 10^4$ Pa = 148 mm Hg

(b) $\lambda = 5.87 \times 10^{-7}$ m = 587 nm

(c) (i) $I = \frac{1}{2} I_0 \times \cos^2\theta \times \cos^2(90 - \theta) = \frac{1}{2} I_0 \cos^2\theta \times \sin^2\theta$

(ii) Brewster's Law $\tan\varphi_p = \frac{n'}{n}$

$\varphi_p = 53.1°$

(d) $\lambda = 5.70 \times 10^{-7}$ m

(e) (i) $E = 3.97 \times 10^{-19}$ J; (ii) $K_{max} = 6.09 \times 10^{-20}$ J

Question 8

(a) $v = 1.3 \times 10^{7}$ m s^{-1}

(b) $\lambda = 9.4 \times 10^{-8}$ m

(c) $v = 2.0 \times 10^{6}$ m s^{-1}

(d) $t = 15\,600$ years

(e) $E = 3.9 \times 10^{-12}$ J

Glossary

absolute zero The temperature at which each molecule has no kinetic energy. This temperature corresponds to 0 K, or to –273.15°C.

acceleration The time rate of change of velocity.

activity The number of radioactive disintegrations per second.

adiabatic A thermodynamic process in which no heat enters or leaves the system.

alternating current (AC) Electric current that continually reverses in direction.

amplitude The distance from the midpoint to the crest of a wave.

angle of incidence The angle between an incident ray and the normal to a surface.

angle of reflection The angle between a reflected ray and the normal to a surface.

angular momentum The product of the moment of inertia and angular velocity.

angular velocity The time rate of change of angle.

Archimedes' Principle An immersed object is buoyed up by a force equal to the weight of the fluid it displaces.

atom The smallest particle of an element that can be identified with that element. It consists of protons and neutrons in a nucleus surrounded by electrons.

atomic mass number The total number of nucleons (neutrons and protons) in the nucleus of an atom.

atomic number The number of protons in the nucleus of an atom.

average speed The total distance covered divided by the total time taken.

barometer An instrument for measuring atmospheric pressure.

beats A variation in the loudness of sound caused by interference when two tones of slightly different frequencies are sounded together.

Bernoulli's equation The equation for the conservation of energy for fluid flow.

binding energy The energy difference between a nucleus and its separate constituent nucleons.

Bohr theory The theory of the atom that says the electrons go around the nucleus in certain specified orbits.

boiling The change of state from liquid to gas that occurs beneath the surface of the liquid. The gas that forms beneath the surface occurs as bubbles, which rise to the surface and escape.

bulk modulus The ratio of the increase in pressure on an object to the fractional decrease in volume.

buoyant force The upward force exerted by a fluid on a submerged object.

capacitance The ratio of the charge on each plate of a capacitor to the potential difference between the plates.

Celsius scale The temperature scale in which the number 0 is assigned to the temperature at which water freezes (at standard pressure), and the number 100 is assigned to the temperature at which water boils (at standard pressure).

centre of gravity The point at the centre of an object's weight distribution, where the force of gravity can be considered to act.

centre of mass The point at the centre of an object's mass distribution, where all its mass can be considered to be concentrated.

chain reaction A self-sustaining nuclear reaction.

charge The property to which is attributed the mutual repulsion of two electrons or two protons and the mutual attraction of an electron and a proton.

circuit A completed path along which charge can flow.

coherent Type of light beam in which the waves all have the same frequency, phase and direction.

component One of a series of vectors whose vector sum is equal to a given vector.

conservation of angular momentum A system of objects will maintain a constant total angular momentum unless acted upon by an unbalanced external torque.

conservation of energy Energy cannot be created or destroyed; it may be transformed from one form into another, but the total amount of energy never changes.

conservation of momentum In the absence of a net external force, the total momentum of a system of objects remains unchanged.

constructive interference Addition of two waves when the crest of one wave overlaps the crest of another, so that their individual effects add together. The result is a wave of increased amplitude.

convection A means of heat transfer by movement of the heated substance itself, such as by currents in a fluid.

converging lens A lens that causes parallel rays of light to converge to a focus.

Coulomb's Law The electrical force between two charges varies directly as the product of the charges and inversely as the square of the distance between them.

crest One of the places in a wave where the wave is highest or the disturbance is greatest.

critical angle The minimum angle of incidence at which a light ray is totally reflected within the medium.

density A property of a substance, equal to the mass divided by the volume.

destructive interference Addition of two waves when the crest of one wave overlaps the trough of another, so that their individual effects cancel each other. The result is a wave of decreased amplitude.

diffraction The spreading of a wave after passing through an aperture.

diffraction grating A series of closely-spaced parallel slits, which are used to separate wavelengths of light by interference.

dimensional analysis The method by which formulae are checked using the dimensions of each variable.

direct current (DC) Electric current whose flow of charge is always in one direction only.

displacement The change of position of a particle. It is a vector quantity.

diverging lens A lens that causes parallel rays of light to diverge.

Doppler effect The observed change in frequency of a wave due to the motion of the source or the observer.

efficiency The ratio of useful work output to total energy input.

elastic The term that applies to a material that returns to its original shape and size after it has been stretched or compressed.

elastic collision A collision in which there is no loss of the total kinetic energy.

elastic limit The extension or compression beyond which an elastic material will not return to its original state.

elasticity The property of a material by which it experiences a change in shape when a deforming force acts on it.

electrical resistance The resistance of a material to the flow of an electric current through it.

electric current The flow of electric charge.

electric field A force field that fills the space around every electric charge or group of charges. Another electric charge introduced into this region will experience an electric force.

electric potential The electric potential energy per unit charge at a location in an electric field.

electric potential energy The energy a charge possesses by virtue of its location in an electric field.

electromagnet A magnet whose field is produced by an electric current.

electromagnetic induction The phenomenon of inducing a voltage in a conductor by changing the magnetic field around the conductor.

electromagnetic spectrum The range of electromagnetic waves extending from radio waves to gamma rays.

electromagnetic wave A wave that is partly electric and partly magnetic and that carries energy emitted by vibrating electric charges in atoms.

electrostatics The study of electric charges at rest.

energy That property of an object or a system which enables it to do work.

entropy The measure of the disorder of a system.

equation of state The equation relating the pressure, the volume and the temperature of an ideal gas. It is also called the ideal gas equation.

equilibrium The state of a system that has no net force acting on it.

evaporation The change of state from liquid to gas that takes place at the surface of a liquid.

eyepiece The lens of an optical instrument that is closer to the eye.

Faraday's Law The induced voltage in a coil is proportional to the product of the number of loops and the rate at which the magnetic flux changes within those loops.

First Law of Thermodynamics The change in the internal energy of a system equals the heat entering the system minus the work done by the system.

flotation A floating object displaces a quantity of fluid of weight equal to its own weight.

fluid Anything that flows; in particular, any liquid or gas.

focal length The distance between the centre of a lens and either focal point.

focal point For a converging lens, the point at which a beam of light parallel to the principal axis converges. For a diverging lens, the point from which such a beam appears to diverge.

force The cause of the change in motion of a particle.

force field What fills the space around a mass, electric charge, or magnet, so that another mass, electric charge, or magnet introduced to this region will experience a force.

free fall Motion under the influence of gravitational force only.

frequency The number of vibrations per second.

friction The force that acts to resist the relative motion (or attempted motion) of objects or materials that are in contact.

gas This consists of widely separated rapidly moving molecules.

generator A machine that produces electric current by rotating a coil within a magnetic field.

gravitational field A force field that fills the space around every mass. Another mass in this region will experience a gravitational force.

half-life The time required for half the atoms of a radioactive isotope of an element to decay.

heat The process of energy transfer from one material to another because of a temperature difference between the materials.

ideal gas equation This is also called the equation of state.

impedance The AC equivalent of resistance, the ratio of voltage to current in an AC circuit.

impulse The product of force multiplied by the time interval during which the force acts.

inelastic collision A collision in which the total kinetic energy is not conserved.

inertia The resistance of any material object to a change in its state of motion.

in phase Term applied to two or more waves whose crests (and troughs) arrive at a place at the same time, so that their effects reinforce each other.

instantaneous speed The speed at any instant of time.

insulator A material that is a poor conductor of heat or electricity.

interference pattern A pattern formed by the overlapping of two or more coherent waves that arrive in a region at the same time.

internal energy The total energy inside a substance.

isothermal A thermodynamic process in which there is no change of temperature.

isotope Elements with the same atomic number but different atomic mass numbers.

Kelvin scale The temperature scale in which absolute zero (–273.15°C) is taken as 0 K.

kinetic energy The energy of motion.

Kirchhoff's First Law The algebraic sum of currents entering and leaving a junction in a circuit is zero.

Kirchhoff's Second Law The algebraic sum of the increase or decrease of potential around a closed loop in a circuit is zero.

laser An optical instrument that produces a beam of coherent light.

law of reflection When a wave strikes a surface, the angle of incidence is equal to the angle of reflection.

law of universal gravitation For any pair of objects, each attracts the other object with a force that is directly proportional to the mass of each object, and inversely proportional to the square of the distance between their centres of mass.

lens A piece of transparent material that can bend parallel rays of light so that they converge to or diverge from a single point.

Lenz's Law The induced emf and any resulting induced current are in such a direction as to oppose the change in flux that produced them.

liquid The molecules in a liquid can move rapidly and are not rigidly bound together. The motion is restricted by the surface of the liquid.

longitudinal wave A wave in which the vibration is in the same direction as that in which the wave is travelling.

magnetic domain A microscopic cluster of atoms with their magnetic fields aligned.

magnetic field A force field that fills the space around every magnet, moving charge or current-carrying wire. Another magnet, moving charge or current-carrying wire introduced into this region will experience a magnetic force.

magnetic flux The product of the area of a surface and the magnetic field perpendicular to that surface.

mass A measure of the inertia of an object, which may also be considered as the quantity of matter a body contains.

mechanical energy The name given to potential and kinetic energy.

microscope An optical instrument that produces a greatly magnified image of a small close object.

molar heat capacity The quantity of heat required to raise one mole of a substance through a temperature rise of one degree.

moment of inertia The rotational analogue of mass, which takes into account the distribution of the mass about the axis of rotation.

momentum The product of the mass and the velocity of an object.

natural frequency A frequency at which an elastic object naturally tends to vibrate.

net force The combination of all the forces that act on an object.

neutron An electrically neutral particle found in the nucleus of an atom.

Newton's First Law Every body continues in its state of rest, or constant velocity unless there is a net force acting on it.

Newton's Second Law The net force on an object is proportional to the time rate of change of momentum.

Newton's Third Law Whenever one body exerts a force on a second body, the second body exerts an equal and opposite force on the first.

node Any part of a standing wave that remains stationary.

normal A line that is perpendicular to a surface.

nuclear fission The splitting of heavy atomic nuclei into lighter nuclei accompanied by the release of energy.

nuclear fusion The combining of lighter nuclei into heavier nuclei accompanied by the release of energy.

nucleon The principal building block of the nucleus; a neutron or a proton.

nucleus The positively charged centre of an atom, which contains protons and neutrons and has almost all the mass of the entire atom but only a tiny fraction of the volume.

objective lens In an optical device using compound lenses, the lens closest to the object observed.

Ohm's Law The voltage across an ohmic resistor is proportional to the current through it.

parallel circuit An electric circuit in which devices are connected to the same two points of the circuit.

Pascal's principle Changes in pressure at any point in an enclosed fluid at rest are transmitted undiminished to all points in the fluid.

period The time required for a complete oscillation or orbit.

phasor diagram A vector diagram indicating the voltages across various components in an AC circuit.

photoelectric effect The emission of electrons from certain metals when exposed to light.

photon In the particle model of electromagnetic radiation, a particle that travels at the speed of light and whose energy is related to the frequency of the radiation.

Poisson's ratio The ratio of the fractional changes in diameter and in length when a body is in a state of tension.

polarization The filtering out of all vibrations in a transverse wave that are not in a given direction.

potential difference The difference in electric potential, or voltage, between two points.

potential energy Energy that an object has by virtue of its position.

power The rate at which work is done.

pressure The force per unit of surface area, where the force is perpendicular to the surface.

principal axis The line joining the centres of curvature of the surfaces of a lens.

proton A positively charged particle that is found in the nucleus of an atom.

quantum The smallest amount of anything.

radiant energy Any energy, including heat, light, and X-rays, that is transmitted by radiation. It occurs in the form of electromagnetic waves.

radiation (a) The transmission of energy by electromagnetic waves. (b) The particles given off by radioactive atoms.

radioactive The term applied to an atom with a nucleus that is unstable and that can spontaneously emit a particle and become the nucleus of another element.

rarefaction A disturbance in air (or other matter) in which the pressure is lowered.

ray A thin beam of light.

ray diagram A diagram showing the rays that can be drawn to determine the size and location of an image formed by a mirror or lens.

real image An image that is formed by converging light rays and that can be displayed on a screen.

reflection The bouncing back of a wave that strikes the boundary between two media.

refraction the change in direction of a wave as it crosses the boundary between two media in which it travels at different speeds.

resolution The process of breaking up a vector into components.

resonance A phenomenon that occurs when the frequency of forced vibrations on an object matches the object's natural frequency, and an increase in amplitude results.

rotation The spinning motion that takes place when an object moves about an axis.

scalar quantity A quantity in physics, such as mass, that can be completely specified by a statement of its magnitude, without reference to direction.

Second Law of Thermodynamics One among many statements of the law says that it is not possible for a heat engine working in a cycle to produce no other effect than converting heat energy to work.

series circuit An electric circuit in which devices are arranged so that the current flows through each in turn.

shear modulus The ratio of shear force applied to the angular deformation.

simple harmonic motion The periodic motion that occurs when the acceleration is proportional to the displacement from the equilibrium position and in the opposite direction.

Snell's Law For light passing from one medium to another, the ratio of sine of the angle of incidence to the sine of the angle of refraction is a constant.

solid In a solid the forces of molecular attraction are capable of locking molecules together in a rigid ordered structure.

specific heat capacity The quantity of heat required to raise the temperature of a unit mass of a substance by one degree.

speed The distance moved per unit of time.

standing wave Wave in which parts of the wave remain stationary and the wave appears not to be travelling. The result of interference between an incident wave and a reflected wave.

surface tension The ratio of the work done in extending a surface, to the area extended.

telescope Optical instrument that forms enlarged images of very distant objects.

temperature The property of a material that tells how warm or cold it is with respect to some standard.

terminal speed An object reaches a constant speed when all the forces acting on it are in equilibrium.

thermal conduction A means by which heat is transferred from atom to neighbouring atom.

thermal equilibrium The state of two or more objects in thermal contact when they have reached a common temperature.

torque The cause of a change of rotational motion about an axis; the product of a force and the perpendicular distance from the line of action of the force to the axis.

total internal reflection The reflection (with no transmission) of light that strikes the boundary between two media at an angle greater than the critical angle.

transformer A device for increasing or decreasing voltage by means of electromagnetic induction.

transverse wave A wave in which the vibration is at right angles to the direction in which the wave is travelling.

trough One of the places in a wave where the wave is lowest or the disturbance is greatest in the opposite direction from a crest.

vector An arrow whose length represents the magnitude of a quantity and whose direction represents the direction of the quantity.

vector quantity A quantity in physics, such as force or velocity, that has both magnitude and direction.

velocity The time rate of change of displacement.

vibration A repeating, to-and-fro motion of something (such as the particles of an elastic body) when displaced from its position of equilibrium.

virtual image An image formed that can be seen by an observer looking into a mirror or through a lens, but that cannot be projected on a screen because light from the object does not actually pass through it.

viscosity A measure of the frictional force exerted by fluids.

wave A disturbance that repeats regularly in space and time and that is transmitted progressively from one particle or region in a medium to the next with no actual transport of matter.

wave speed The speed at which the crest of a wave appears to move.

wavelength The distance from the top of a crest of a wave to the top of the following crest.

weight The force on a body of matter due to the gravitational attraction of the earth.

work The product of the distance moved by an object and the component of the force acting in the direction moved.

Young's modulus The ratio of the force applied per unit area to the fractional change in length.

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