

Holevo Cramér-Rao bound: How close can we get without entangling measurements?

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In multi-parameter quantum metrology, the resource of entanglement can lead to an increase in efficiency of the estimation process. Entanglement can be used in the state preparation stage, or the measurement stage, or both, to harness this advantage—here we focus on the role of entangling measurements. Specifically, entangling or collective measurements over multiple identical copies of a probe state are known to be superior to measuring each probe individually, but the extent of this improvement is an open problem. It is also known that such entangling measurements, though resource-intensive, are required to attain the ultimate limits in multi-parameter quantum metrology and quantum information processing tasks. In this work we investigate the maximum precision improvement that collective quantum measurements can offer over individual measurements, calling this the ‘collective quantum enhancement’. We show that, whereas the maximum enhancement can, in principle, be a factor of n for estimating n parameters, this bound is not tight for large n . Instead, our results prove an enhancement linear in dimension of the qudit probe is possible using collective measurements and lead us to conjecture that this is the maximum collective quantum enhancement in any local estimation scenario.

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1 Introduction

Over half-a-century of advances in quantum metrology [1–3] has vastly improved our ability to measure, sense, image, and estimate with enhanced precision [4–9]. Of significant interest is the multi-parameter estimation scenario [3, 10–13], where two hall-mark quantum effects manifest themselves, playing opposing roles. On the one hand, incompatibility between the unknown parameters of a quantum system [14–17] hinders their simultaneous estimation from a single copy of an unknown state [3, 18–20]. On the other hand, given multiple identical copies of the state, an entangling measurement on all the copies, called a collective (or joint) measurement [21], can extract more information about the parameters than any measurement where the copies are measured individually [22–24]. As individual and separable measurements [25] can be recovered as special cases of collective measurements, it is clear that the latter can only lead to precision enhancements in estimation tasks [24, 26], but the extent of this improvement is a major open problem [23, 27]. In this work, we study *the maximum enhancement collective measurements stand to offer over individual measurements*, specifically in the context of parameter estimation and state tomography.

Despite their advantages, collective measurements are challenging to implement in any real estimation scenario and experimental demonstrations are few and far between [24, 26, 28–30]. Resultantly, the ratio between the optimal pre-

cisions attainable via collective versus individual measurement serves as a useful quantifier of both the quantum advantage offered by collective measurements, and the utility of performing complicated entangling measurements [31] and expending vast amounts of resources. If this ratio is small, then there is not much advantage to be gained from entangling measurements. But even if the ratio is large, our ability to perform the requisite measurements might be limited, meaning that the collective performance is just an overly optimistic goal that is far from being achievable.

In local estimation theory, where unknown parameters are assumed to be close to known true values, collective measurements on identical copies of a separable state do not offer any advantage for estimating a single parameter [1] or multiple parameters of a pure state [32]. Beyond this, except for some simple cases, not much is known about the optimal individual or collective measurement strategies or their performance relative to each other [19, 20]. One reason for this is that the analytic evaluation of the optimal performance of either class of measurements is notoriously difficult. In fact, instead of finding the optimal measurements, it is easier (and more common) to evaluate bounds on their precision. The most widely-used precision bounds for local estimation are quantum generalisations of the classical Cramér-Rao bound (CRB), called quantum CRBs (QCRBs) [14, 33–36]. These include the quantum Fisher information (QFI)-based CRBs [14, 33, 34], the Holevo CRB (HCRB) [35], the Nagaoka-Hayashi CRB (NHCRB) [36, 37], the Gill-Massar CRB (GM-CRB) [23], and the most informative CRB (MICRB) [36, 38, 39].

In general, QCRBs are not always attainable, especially in the multi-parameter setting [3], and the exploration of criteria for their attainability [39, 40] is an active albeit challenging area of research [41]. That said, in the collective measurement setting, the HCRB is known to be attainable in the asymptotic limit by performing collective measurements on a large number of identical copies of the unknown state [22, 42–44]. Additionally, the HCRB can be computed efficiently through a semi-definite program [45], making it amenable to both numerical and analytical techniques [40]. In contrast, the attainable bound for the individual measurement

setting, given by the MICRB [36], requires a conic program [38, 39] that is challenging to compute even numerically. Further, analytical solutions to the MICRB, as reformulated in Ref. [39], are only known for either two-level systems or single-parameter problems [39]. For exploring multi-parameter estimation from arbitrary finite-dimensional mixed quantum states, which is the problem we address in this work, these limitations render the MICRB a computationally-intractable choice for gauging individual precision.

As a result, other individual-measurement precision bounds such as the NHCRB [21, 36, 37] and the GMCRB [23, 40], which are more tractable computationally but less tight, are frequently used as substitutes for the MICRB [21, 40]. In particular, the NHCRB is a well-suited candidate to quantify individual measurement precision in multi-parameter qudit estimation. This is because of two reasons. First, the NHCRB is more analytically-tractable than the MICRB, because it can be efficiently computed through a semi-definite program that scales reasonably with system dimension and number of parameters [21]; this has, for instance, led to analytical lower and upper bounds to the NHCRB [46]. Second, the NHCRB is provably attainable in many cases of interest [21, 39, 47], including two-level systems [36, 37], and despite this attainability not extending to higher dimensions, the NHCRB is still very close to the tight MICRB (with a gap of less than 5% reported for random qudit models up to dimension 17, see Fig. 6 of Ref. [39]), and serves as a good approximation to it.

Armed with these bounds, we study how far off the collective-optimal precision can be from the individual-optimal one by looking at their ratio. Specifically, by investigating the maximum ratio between the NHCRB and the HCRB, we identify situations where collective quantum measurements are the most advantageous. Because the NHCRB is close to the MICRB, the ratio we study should be close to the attainable maximum collective enhancement. The majority of our results concern the NHCRB-to-HCRB ratio, which we represent using the symbol $\mathcal{R}^{\text{NH}}$ and refer to as *the ratio*. To distinguish from this the attainable collective enhancement, given by the ratio between the MICRB and the HCRB, we refer to the latter ratio as *the true ratio* and use the symbol $\mathcal{R}^{\text{MI}}$.

Our first result is preliminary and shows that the ratio of precisions $\mathcal{R}^{\text{NH}}$ is at most equal to the number of unknown parameters being estimated, n . This result can be intuitively expected and agrees with empirical data for estimating a few parameters (blue line for $n = 1, 2, 3$ in Fig. 1) but deviates with increasing n . To find a tighter bound for large numbers of parameters, we focus on state tomography, where the number of parameters is maximal, $n = n_{\text{max}}$. For the qubit tomography case, we extrapolate existing results to find a decreasing trend of the ratio $\mathcal{R}^{\text{MI}}$ with purity [48]. Motivated by this, we propose a model of estimating the coefficients of the generalised Gell-Mann matrices (GMMs) [49], which extend the Pauli matrices to higher dimensions, in mixed d -dimensional qudit states [50]. This “linear GMM model” is symmetric enough to admit analytical results in the full-parameter case ($n = n_{\text{max}}$) for both $\mathcal{R}^{\text{NH}}$ and $\mathcal{R}^{\text{MI}}$. Further, this model is equivalent to the problem of tomography in arbitrary orthonormal basis (ONB) and is therefore representative of a large class of full parameter models. Then, via semi-definite programming arguments, we extend our results to tomography in non-orthonormal bases and to the $n < n_{\text{max}}$ case.

A summary of our main analytical and numerical results comparing the collective- and individual-optimal precisions for local estimation from smooth models on d -dimensional qudits now follows:

- for any model comprising n parameters, the ratio $\mathcal{R}^{\text{NH}}$ is at most n (blue line in Fig. 1), (proved in Sec. 5.1),
- for ONB tomography of the maximally-mixed state, the ratio $\mathcal{R}^{\text{NH}}$ is exactly $d + 1$ (green line in Fig. 1), (proved in Sec. 5.3), and the true ratio $\mathcal{R}^{\text{MI}}$ is exactly $d + 1$ (proved in Sec. 5.6),
- for ONB tomography of any state, the maximum ratio $\mathcal{R}^{\text{NH}}$ is upper-bounded by $d + 2$ (proved in Sec. 5.4) and the maximum true ratio $\mathcal{R}^{\text{MI}}$ is upper-bounded by $d + 2$ (proved in Sec. 5.6),
- for tomography of the maximally-mixed state in any non-orthogonal basis, the ratio $\mathcal{R}^{\text{NH}}$ is upper-bounded by $d + 1$ (proved in Appendix H),

- for tomography of any state in any non-orthogonal basis, the ratio $\mathcal{R}^{\text{NH}}$ is upper-bounded by $d + 2$ (numerical result in Appendix H),
- for estimating any number of GMM-coefficients of the maximally-mixed state, the ratio $\mathcal{R}^{\text{NH}}$ is at most $d + 1$ (proved in Sec. 5.5),
- for ONB tomography of any state, the maximum ratio $\mathcal{R}^{\text{NH}}$ at fixed (known) purity decreases with purity and is at most $d + 1$ (numerical result in Sec. 3.3),
- for any model comprising n parameters, the ratio $\mathcal{R}^{\text{NH}}$ is upper-bounded by $\min(n, d + 1)$ (red line in Fig. 1), (conjecture).

Notably, the $d + 2$ bounds in the third and fifth points above are loose and based on numerical evidence, we expect the attainable bound therein to be $d + 1$. The rest of our paper is structured as follows. In Sec. 2 we introduce and define precision bounds for the individual and the collective measurement scenarios and review relevant background on them. In Sec. 3, we present our results, formalising the key quantities $\mathcal{R}^{\text{NH}}$ and $\mathcal{R}^{\text{MI}}$ in Sec. 3.1, and presenting analytical and numerical results in Secs. 3.2 and 3.3, respectively. We follow up with a discussion of our results in Sec. 4. Finally, our methodology is presented in Sec. 5, whilst deferring mathematical proofs to Appendices A–I.

2 Background: Precision Bounds for Parameter Estimation

In this section, we present a brief recap on quantum parameter estimation and introduce precision bounds for the collective and the individual measurement scenarios. The general recipe to estimate n parameters $\theta := \{\theta_j\}_{j \in [n]} \in \Theta \subseteq \mathbb{R}^n$ (where we define $[n] := \{1, \dots, n\}$ and denote by Θ the set of all possible parameter values) of a quantum state ρ_θ belonging to a d -dimensional Hilbert space $\mathcal{H}_d$ involves two steps. First, one performs quantum measurements, generally positive operator-valued measures (POVMs) $\{\Pi_l\}_{l \in [m]}$ with m outcomes, on ρ_θ . Second, a classical estimator operator $\hat{\theta}_{jl}$ is constructed that assigns an estimated value to θ_j

for each measurement outcome $l \in [m]$, which occurs with probability $p_l := \text{Tr}(\rho_\theta \Pi_l)$. Here Tr (in serif font) denotes tracing over the quantum system.

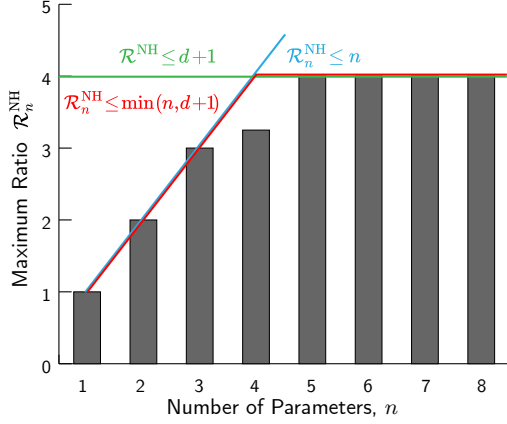


Figure 1: Summary of our main results on the maximum ratio quantities R_n^{NH} & $\mathcal{R}_n^{\text{NH}}$. Our $\mathcal{R}_n^{\text{NH}} \leq n$ (blue), $\mathcal{R}_n^{\text{NH}} \leq d+1$ (green) and $\mathcal{R}_n^{\text{NH}} \leq \min(n, d+1)$ (red) bounds are plotted against numerically and analytically found maximum collective enhancement values (bar chart) for n -parameter estimation from qutrits ($d=3$).

The performance of the estimator is quantified via its mean squared error (MSE) matrix

$$(V_\theta)_{jk} := \sum_{l \in [m]} (\hat{\theta}_{jl} - \theta_j)(\hat{\theta}_{kl} - \theta_k)p_l, \quad (1)$$

the trace of which gives the total average squared deviation $\text{Tr}(V_\theta) = \sum_{l,j} (\hat{\theta}_{jl} - \theta_j)^2 p_l$. Here Tr (in sans serif font) denotes tracing over the classical or parameter indices. In this work, we focus on the local setting, where the parameters of interest are close to their true values, $\theta^* := \{\theta_j^*\}_{j \in [n]}$, i.e., $\theta \approx \theta^*$. For locally-unbiased (LUB) estimators, which have zero bias at the true parameter values, V_θ is equivalent to the covariance matrix of parameter estimates and $\text{Tr}(V_\theta)$ is simply the sum of variances of each parameter.

Precision bounds lower-bound the uncertainties in estimating multiple (possibly) incompatible parameters. In this work, we focus on precision bounds on $\text{Tr}(V_\theta)$; the classical CRB yields a lower bound to this via

$$V_\theta \succcurlyeq J^{-1} \implies \text{Tr}(V_\theta) \geq \text{Tr}(J^{-1}), \quad (2)$$

where $A \succcurlyeq B$ denotes positive semi-definiteness of $A - B$, and $J \equiv J(\rho_\theta, \{\Pi_l\}_{l \in [m]})$ is the classical Fisher information (CFI) matrix. The CFI (defined later in Eq. (40)) is best understood as

a measure on the parameter space $\Theta \subseteq \mathbb{R}^n$ of the local sensitivity of measurements $\{\Pi_l\}_{l \in [m]}$ towards each θ_j when measuring state ρ_θ . Minimising $\text{Tr}(J^{-1})$ in Eq. (2) over all possible measurements $\{\Pi_l\}_{l \in [m]}$ yields the tight MICRB [36, 38, 39],

$$C_{\text{MI}} := \min_{\{\Pi_l\}_{l \in [m]}} \text{Tr}(J^{-1}), \quad (3)$$

stipulating the ultimate precision attainable via individual measurements. A recent reformulation of the MICRB [39] recasts Eq. (3) into a conic program (see Eq. (147) for definition) and presents SDP lower bounds to it. However, neither closed-form expressions nor efficient numerical techniques are known for evaluating the conic program (the SDP approximation's complexity scales as d^{12} for qudit tomography) and analytic solutions are known only for qubit models, single-parameter problems, and Gaussian models [39].

A different precision bound on $\text{Tr}(V_\theta)$ for the separable-measurement case, developed by Nagaoka [37] and Hayashi [16], is the NHCRB,

$$C_{\text{NHCRB}} := \min_{\mathbb{L}, \mathbb{X}} \left\{ \text{Tr}[\mathbb{S}_\theta \mathbb{L}] \mid \mathbb{L} \succcurlyeq \mathbb{X} \mathbb{X}^\top, \right. \\ \left. \mathbb{L}_{jk} = \mathbb{L}_{kj} \text{ Hermitian} \right\} - \text{Tr}(\theta \theta^\top). \quad (4)$$

Here $\mathbb{X} := \{X_1, \dots, X_n\}^\top$ are the Hermitian LUB operators that satisfy (abbreviating $\frac{\partial}{\partial \theta_j}$ as ∂_j)

$$\text{Tr}(\rho_\theta X_j) = \theta_j \quad \& \quad \text{Tr}(\partial_j \rho_\theta X_k) = \delta_{jk}, \quad (5)$$

and $\mathbb{S}_\theta = \mathbb{1}_n \otimes \rho_\theta$, Tr denotes trace over both classical and quantum subsystems, $(\cdot)^\top$ denotes transpose with respect to the classical (parameter) index, $\mathbb{1}_n$ denotes the n -dimensional identity matrix, and blackboard fonts represent classical-quantum matrices. The minimisation in Eq. (4) is a conic optimisation over the extended Hilbert space $\mathbb{H} := \mathbb{C}^n \otimes \mathcal{H}_d$, combining parameter space $\mathbb{C}^n$ and Hilbert space $\mathcal{H}_d$ of the qudit. The other optimisation variable $\mathbb{L}$ is a positive operator on $\mathbb{H}$ that is also symmetric under a partial transpose with respect to the first subspace $\mathbb{C}^n$, so as to reduce to valid covariance matrices on $\mathbb{C}^n$ under $\text{Tr}[\mathbb{S}_\theta \mathbb{L}]$ [39, 46].

Although the NHCRB is not always tight ($C_{\text{NHCRB}} \leq C_{\text{MI}}$), it is an analytically-tractable lower bound to C_{MI} that is often provably tight ($C_{\text{NHCRB}} = C_{\text{MI}}$) [21, 47], and always efficiently-computable (the SDP's complexity scales as $d^{4.5}$ for qudit tomography) [21]. Further, Ref. [39] reformulated the

tight bound and demonstrated the NHCRB to be a good approximation to the tight bound, with a gap of less than 5%, for large qudit dimensions. The GMCRB, another separable-measurement precision bound, is defined as $C_{\text{GMCRB}} := \left(\text{Tr}[J_{\text{SLD}}^{-1/2}] \right)^2 / (d-1)$ [23], where J_{SLD} is the symmetric logarithmic derivative (SLD) QFI (see Appendix D for definition). The GMCRB is generally not tight for d -dimensional states, and for $d > 2$ and $n \leq d-1$, the GMCRB is known to be weaker than the SLD CRB [23] (see Appendix D for definition). Nonetheless, the GMCRB is analytically-tractable and, for the problem of tomography in orthonormal bases, has been proven to be at most a factor of 2 away from the tight bound [40].

The HCRB is a collective-measurement precision bound on $\text{Tr}(V_\theta)$, defined as

$$C_{\text{HCRB}} := \min_{\mathbb{X}} \{ \text{Tr}(\mathbb{Z}_\theta[\mathbb{X}]) + \|\text{Im } \mathbb{Z}_\theta[\mathbb{X}]\|_1 \} - \text{Tr}(\theta\theta^\top),$$

$$\mathbb{Z}_\theta[\mathbb{X}]_{jk} := \text{Tr}(\rho_\theta X_j X_k), \quad (6)$$

where $\|X\|_1 := \text{Tr}(\sqrt{X^\dagger X})$ denotes the trace norm. An equivalent expression for C_{HCRB} , written in a similar form as Eq. (4), is

$$C_{\text{HCRB}} := \min_{\mathbb{L}, \mathbb{X}} \left\{ \text{Tr}[\mathbb{S}_\theta \mathbb{L}] \mid \text{Tr}[\mathbb{S}_\theta \mathbb{L}] \text{ real, symmetric,} \right. \\ \left. \text{Tr}[\mathbb{S}_\theta \mathbb{L}] \succcurlyeq \text{Tr}[\mathbb{S}_\theta \mathbb{X} \mathbb{X}^\top] \right\} - \text{Tr}(\theta\theta^\top). \quad (7)$$

Note that the minimisations in Eqs. (4), (6) and (7) have no explicit closed-form solution for general mixed states ρ_θ [51] and are typically evaluated numerically via SDPs [21, 45].

Besides $C_{\text{HCRB}} \leq C_{\text{NHCRB}}$, the following ordering between the various precision bounds is known

$$\max(C_{\text{SLD}}, C_{\text{RLD}}) \leq C_{\text{HCRB}} \leq C_{\text{NHCRB}} \leq C_{\text{MI}}. \quad (8)$$

Here C_{SLD} and C_{RLD} are, respectively, the SLD CRB and the right-logarithmic derivative (RLD) CRB (see Appendix D for definitions). By definition, C_{MI} is the tightest precision bound for separable measurements, and hence is greater than or equal to all four other CRBs in Eq. (8). We know that all three inequalities in Eq. (8) are saturated for single-parameter estimation [32]. Moreover, $C_{\text{NHCRB}} = C_{\text{HCRB}}$ for estimating any number of parameters from pure states [32]. On the

other hand, if the single-copy NHCRB and the HCRB are unequal, this gap persists between the finite-copy NHCRB and the HCRB, shrinking asymptotically with the number of copies [52]. As for the GMCRB, its position in Eq. (8) depends on model properties: for $d = 2$, we have $C_{\text{SLD}} \leq C_{\text{GMCRB}}$ but for $d > 2$ and $n \leq d-1$, we have $C_{\text{GMCRB}} \leq C_{\text{SLD}}$, whereas for $d > 2$ and $n > d-1$, no ordering can be specified.

Beyond ordering, some ratio relationships between the precision bounds are known. The HCRB is known to be at most twice the SLD CRB, i.e., $C_{\text{SLD}} \leq C_{\text{HCRB}} \leq 2C_{\text{SLD}}$ [53, 54] so that, up to a factor of 2, the SLD CRB also quantifies collective precision. For the problem of tomography in an orthonormal basis, the relation $C_{\text{GMCRB}} \leq C_{\text{MI}} \leq 2C_{\text{GMCRB}}$ was established in Ref. [40]. At the level of the Fisher information, the Gill-Massar trace $\text{Tr}[J_{\text{SLD}}^{-1}J] \leq d-1$ reduces to a ratio of the QFI J_{SLD} and the CFI J when they are proportional: $J \propto J_{\text{SLD}}$. This is the case for Fisher symmetric measurements [55] that attain a CFI such that $J_{\text{SLD}} = (d+1)J$, and thus $C_{\text{MI}}/C_{\text{SLD}} = d+1$, for tomography of the maximally-mixed state [56]. However, Fisher symmetric measurements exist only for maximally-mixed states and pure states [55, 56], so the ratio for general mixed states remains undecided. The recent reference [57] analysed the impact of dimensionality on precision and parameter incompatibility [58] by defining the normalised gap $\Delta = (C_{\text{HCRB}} - C_{\text{SLD}})/C_{\text{SLD}}$, which satisfies $0 \leq \Delta \leq 1$. By considering the estimation of $n = 2$ and 3 parameters encoded unitarily onto d -dimensional states, the authors showed that asymptotic incompatibility can vanish altogether when $d > n$, making the gap $\Delta = 0$. Surprisingly, a direct ratio relationship between C_{MI} and C_{HCRB} , or even between C_{NHCRB} and C_{HCRB} is not known [59], leading to a gap in our knowledge of the potential quantum advantage offered by collective measurements.

3 Results

3.1 Collective Quantum Enhancement

We first define a measure of collective enhancement for estimation tasks. Whereas the truly attainable ratio of collective and separable variances is given by $C_{\text{MI}}/C_{\text{HCRB}}$, the analytic intractability of C_{MI} renders this quantity be-

yond the reach of currently available tools. Instead, we consider the ratio between the NHCRB and the HCRB, which provides a lower bound to $C_{\text{MI}}/C_{\text{HCRB}}$. Specifically, in this work, we shall analyse the collective quantum enhancement

$$\mathcal{R}^{\text{NH}}[\{\rho_\theta | \theta \in \Theta\}] := \max_{\theta \in \Theta} \frac{C_{\text{NHCRB}}[\rho_\theta]}{C_{\text{HCRB}}[\rho_\theta]}, \quad (9)$$

for a given quantum statistical model $\{\rho_\theta | \theta \in \Theta\}$ with n parameters for d -dimensional qudit states, where the maximum is over all allowed parameter values for this model. This quantity can be interpreted as a measure of the maximum quantum enhancement obtainable from using collective measurements over separable measurements for this particular model.

A further maximisation over all quantum models with the same number of parameters, n , and for the same probe dimension, d , is possible,

$$\mathcal{R}_n^{\text{NH}} := \max_{n\text{-parameter models}} \mathcal{R}^{\text{NH}}[\{\rho_\theta | \theta \in \Theta\}], \quad (10)$$

where we have suppressed the dependence on d for conciseness. A final maximisation over all possible n for a given system dimension d ($1 \leq n \leq n_{\text{max}}$) then leads to

$$\mathcal{R}^{\text{NH}} := \max_{1 \leq n \leq n_{\text{max}}} \mathcal{R}_n^{\text{NH}}, \quad (11)$$

corresponding to the ultimate collective quantum enhancement in precision for this system dimension [3, 60]. Here n_{max} represents the maximum possible number of independent parameters and equals $2(d-1)$ for pure qudits and d^2-1 for mixed qudits. In this work we consider general mixed states and, thus, define $n_{\text{max}} := d^2-1$.

Physically, Eqs. (10) & (11) define model-free quantities that capture the maximum potential advantage of collective measurements across various quantum models, comparing their general utility beyond particular estimation scenarios. Practically, these two quantities, $\mathcal{R}_n^{\text{NH}}$ and $\mathcal{R}^{\text{NH}}$, offer insights into the behaviour of the maximum utility of collective measurements from two opposing extremes of the number of parameters. As depicted in Fig. 1, analysis using $\mathcal{R}_n^{\text{NH}}$ reveals a maximum utility linear in n for small n whereas analysis using $\mathcal{R}^{\text{NH}}$ reveals a plateauing utility, fixed by d instead of n , for large n . In the following we abbreviate the model representation $\{\rho_\theta | \theta \in \Theta\}$ to $\{\rho_\theta\}$ while specifying the model explicitly.

The NHCRB is generally close to the MICRB [39], but whether $\mathcal{R}_n^{\text{NH}}$ (or $\mathcal{R}^{\text{NH}}$) corresponds to the true maximum collective enhancement depends on the NHCRB's attainability for the model maximising $C_{\text{MI}}/C_{\text{HCRB}}$ in the setting of Eq. (10) (Eq. (11)). The non-attainability of the NHCRB in specific cases [39] means that the quantities $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}]$, $\mathcal{R}_n^{\text{NH}}$ and $\mathcal{R}^{\text{NH}}$ could be smaller than the corresponding true maximum collective enhancements, denoted $\mathcal{R}^{\text{MI}}[\{\rho_\theta\}]$, $\mathcal{R}_n^{\text{MI}}$ and $\mathcal{R}^{\text{MI}}$. These are defined similarly to Eqs. (9)–(11) through $\mathcal{R}^{\text{MI}}[\{\rho_\theta\}] := \max_{\theta \in \Theta} C_{\text{MI}}[\rho_\theta]/C_{\text{HCRB}}[\rho_\theta]$, $\mathcal{R}_n^{\text{MI}} := \max_{n\text{-param. models}} \mathcal{R}^{\text{MI}}[\{\rho_\theta\}]$ and

$$\mathcal{R}^{\text{MI}} := \max_{1 \leq n \leq n_{\text{max}}} \mathcal{R}_n^{\text{MI}}. \quad (12)$$

The same construction of maximum ratio quantities applies to other choices of separable measurement bounds, leading to $\mathcal{R}^{\text{GM}}[\{\rho_\theta\}]$ and $\mathcal{R}_n^{\text{GM}}$ for the GMCRB, studied later and depicted in Fig. 2.

The quantities $\mathcal{R}^{\text{NH}}$ and $\mathcal{R}^{\text{MI}}$ can only increase with d . This is because of two reasons: (i) higher dimensions allow more incompatible parameters to estimate, and (ii) a lower-dimensional model can always be embedded in a higher dimension keeping the ratio invariant. Note that we only consider full-rank models in defining Eqs. (9), (10), & (11), i.e., we assume ρ_θ to be non-singular or non-degenerate for all $\theta \in \Theta$.¹ Our definitions imply that for any n -parameter model over d -dimensional states, $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}] \leq \mathcal{R}_n^{\text{NH}} \leq \mathcal{R}^{\text{NH}}$ and $\mathcal{R}^{\text{MI}}[\{\rho_\theta\}] \leq \mathcal{R}_n^{\text{MI}} \leq \mathcal{R}^{\text{MI}}$.

Below, in Sec. 3.2, we present our analytical results. First, in Sec. 3.2.1, we present our results on the ratio quantities $\mathcal{R}_n^{\text{NH}}$ and $\mathcal{R}^{\text{NH}}$ and on $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}]$ for the linear GMM model. The GMM model results apply to tomography in arbitrary ONB, and we conjecture that this model at $\theta = 0$ maximises Eq. (11), thereby attaining $\mathcal{R}^{\text{NH}}$; we prove the attainability of the NHCRB at this point. Then, in Sec. 3.2.2, we present our analytical result using the MICRB to upper-bound $\mathcal{R}^{\text{MI}}[\{\rho_\theta\}]$ for the problem of ONB tomography and compare this to an upper bound derived from $\mathcal{R}_n^{\text{GM}}$. Finally, in Sec. 3.3, we present our numerical results including the extension of $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}]$ to tomography in non-orthogonal bases and to tomography under fixed

¹Singular or degenerate states can be approximated arbitrarily well by non-degenerate ones via regularisation [50].

probe purity, as well as random sampling experiments addressing $\mathcal{R}_n^{\text{NH}}$ and $\mathcal{R}^{\text{NH}}$ for generic qudit estimation models.

3.2 Analytical Results

3.2.1 Results for NHCRB ratio

In Sec. 5.1, we prove a model-independent and attainable upper bound on the collective precision enhancement. Specifically, by using an upper bound to C_{NHCRB} [39, 46] and using C_{SLD} to lower-bound C_{HCRB} , we prove that for estimating n independent parameters of a quantum state, the collective enhancement is at most a factor of n , i.e., $\mathcal{R}_n^{\text{NH}} \leq n$. We state this as Theorem 1 (see Sec. 5.1 for proof):

Theorem 1. *For estimating n parameters of any qudit state ρ_θ , the collective enhancement $C_{\text{NHCRB}}[\rho_\theta]/C_{\text{HCRB}}[\rho_\theta] \leq n$, i.e., $\mathcal{R}_n^{\text{NH}} \leq n$.*

The factor-of- n smaller attainable precision intuitively aligns with the approach of dividing the multi-parameter problem into n single-parameter problems, where the SLD-optimal separable measurement for each parameter θ_j individually is performed on a fraction $1/n$ of the number of available state copies.² Due to technical difficulties in guaranteeing the local-unbiasedness of parameter estimates within this approach [61], we instead use the analytic upper bound to C_{NHCRB} developed in Ref. [46] (summarised in Appendix G) to prove Theorem 1.

The upper bound in Theorem 1 can be a tight relation for models with small n . For example, a ratio $\mathcal{R}_{n=2}^{\text{NH}} = 2$ is attained for $d = 3$ by the model of estimating the coefficients of λ_1 & λ_2 (see Appendix A for definitions) in the maximally-mixed qutrit state, and a ratio $\mathcal{R}_{n=3}^{\text{NH}} = 3$ is attained for $d = 2$ by the model of estimating the three Pauli coefficients of the maximally-mixed qubit state [48]. We also expect the ratio $\mathcal{R}_n^{\text{NH}}$ to increase with n , because having more parameters to estimate can lead to higher incompatibility—this implies $\mathcal{R}_n^{\text{NH}} \leq \mathcal{R}_{n_{\text{max}}}^{\text{NH}}$. On the other hand, at the $n = n_{\text{max}} = d^2 - 1$ limit,

²However, this does not constitute a locally-unbiased multi-parameter estimation strategy, which must assign locally-unbiased estimates to all the parameters in each trial. When performing the SLD-optimal measurement for θ_j , the procedure for assigning locally-unbiased estimates for θ_k ($k \neq j$) is discussed in Ref. [61].

the upper bound from Theorem 1 would imply an enhancement $\mathcal{R}_{n_{\text{max}}}^{\text{NH}} \leq d^2 - 1$ —quadratic in d —is attainable; this is not the case, as we subsequently establish.

In Sec. 5.2, we introduce the ‘linear GMM model’: an n_{max} -parameter family of d -dimensional qudit states given by $\rho_\theta = \mathbb{1}_d/d + \sum_{j=1}^{n_{\text{max}}} \theta_j \lambda_j$ [50]. Here, the n_{max} parameters of interest, $\{\theta_j\}_{j \in [n_{\text{max}}]}$, are the coefficients of the GMMs $\Lambda_d := \{\lambda_j\}_{j \in [n_{\text{max}}]}$. The GMMs Λ_d are traceless, Hermitian generalisations of the qubit Pauli operators (see Appendix A) and the parameterisation is valid for any qudit state. Estimating $\{\theta_j\}_{j \in [n_{\text{max}}]}$ thus corresponds to tomography in the GMM basis, which is an ONB given that $\text{Tr}(\lambda_j \lambda_k) = \delta_{jk}$. In fact, the unweighted GMM model at any ρ_θ is equivalent to tomography of ρ_θ in any other ONB, as we show using Lemma 1 in Sec. 5.2. This means that the QCRBs as well as ratios between them are invariant to the particular choice of ONB and the following results derived for the GMM basis hold for any ONB.

For the GMM model, we prove that the maximum collective enhancement $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}]$ is a factor linear in dimension d , as opposed to quadratic. Specifically, we show that the collective enhancement is exactly $d + 1$ for estimating from the maximally-mixed state, $\rho_m = \mathbb{1}_d/d$, in Theorem 2 (see Sec. 5.3 for proof):

Theorem 2. *For ONB tomography of the maximally-mixed qudit state ρ_m , the collective enhancement $C_{\text{NHCRB}}[\rho_m]/C_{\text{HCRB}}[\rho_m] = d + 1$.*

The factor of $d + 1$ here previously appeared in the relation $J_{\text{SLD}} = (d + 1)J$ satisfied by Fisher SIC measurements for ρ_m [56]. This is because for the particular model considered in Theorem 2, the NHCRB is tight (see below) and the SLD CRB equals the HCRB. Therefore, the ratio $C_{\text{NHCRB}}/C_{\text{HCRB}}$ reduces to the ratio $C_{\text{MI}}/C_{\text{SLD}}$, which equals the proportionality constant between the QFI J_{SLD} and the CFI J . However, the true merit of Theorem 2 lies in our novel method for its proof, which directly extends to upper-bound the collective enhancement in several settings beyond Theorem 2. Specifically, this lets us prove that the maximum enhancement $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}]$ is less than $d + 2$ for estimating from arbitrary states in Theorem 3 (see Sec. 5.4 for proof):

Theorem 3. *For ONB tomography of arbitrary d -dimensional qudit state ρ_θ , the maximum collective enhancement $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}] \leq d + 2$.*

Although we prove an upper bound of $d + 2$, based on numerical evidence shown in Fig. 3, we expect the attainable bound to be $d + 1$ and expect $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}]$ to be maximised at $\theta = 0$ over Θ . Thus for the linear GMM model, we propose $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}] \leq d + 1$. Further, for the maximally-mixed case, we prove the SIC POVM to be an optimal individual measurement. This proves the NHCRB to be attainable or tight in this case, implying $C_{\text{MI}}/C_{\text{HCRB}} = d + 1$ as well at this maximum point.

We then treat two extensions of the linear GMM model. The first is the weighted version, where an arbitrary, full-rank, parameter-independent weight matrix W is included in the cost function (Appendix H). Importantly, weighted models are equivalent to reparameterisations of the unweighted model, i.e., estimating parameters that are not coefficients in any particular ONB [13, 21, 61]. Our results in Appendix H prove that for the maximally-mixed state, ρ_m , and for estimating any n_{max} independent parameters, the collective enhancement is at most $d + 1$. We also numerically show (Fig. 6) that the collective enhancement for estimating from any state $\rho_\theta \neq \rho_m$ under a weight W is smaller than the collective enhancement for estimating from ρ_m under the same weight W . This suggests the maximum collective enhancement $\mathcal{R}_{n_{\text{max}}}^{\text{NH}}$ for any full-parameter problem to be at most $d + 1$, i.e.,

$$\mathcal{R}_{n_{\text{max}}}^{\text{NH}} = \max_{n_{\text{max}}\text{-parameter models}} \mathcal{R}^{\text{NH}}[\{\rho_\theta\}] \leq d + 1, \quad (13)$$

and this upper bound is attained by the model studied in Sec. 5.3.

In Sec. 5.5, we treat the extension of the linear GMM model to the $n < n_{\text{max}}$ case, assuming the remaining $n_{\text{max}} - n$ GMM coefficients to be zero. In this case, we prove that for estimating from the maximally-mixed state, the maximum collective enhancement is $d + 1$ in Theorem 4 (Sec. 5.5):

Theorem 4. *For estimating fewer-than- n_{max} coefficients of GMMs of the maximally-mixed qudit state ρ_m , the collective enhancement $C_{\text{NHCRB}}[\rho_m]/C_{\text{HCRB}}[\rho_m] \leq d + 1$.*

We do not analytically solve this model for other states, or for states with the remaining $n_{\text{max}} - n$ GMM coefficients non-zero. However, the n bound in Sec. 5.1 and numerical results in Sec. 3.3 suggest that the maximum enhancement $\mathcal{R}_n^{\text{NH}}$ is non-decreasing with increasing n at fixed d . Having analysed the $n = n_{\text{max}}$ case in depth, we expect that for any $n < n_{\text{max}}$ model, the same bound of $d + 1$ should hold, i.e., $\mathcal{R}_n^{\text{NH}} \leq \mathcal{R}_{n_{\text{max}}}^{\text{NH}} \leq d + 1$.

In summary, we find that there are two different upper bounds on the maximum collective enhancement in the low n and the high n regimes. Taking the example of qutrits ($d = 3$), for $n = 1, 2$ & 3, models attaining $\mathcal{R}_n^{\text{NH}} = n$ are found, whereas for $n = 5, 6, 7$ & 8, models attaining $\mathcal{R}_n^{\text{NH}} = d + 1$ are found. Combining these two cases, we propose $\mathcal{R}_n^{\text{NH}} \leq \min(n, d + 1)$ and $\mathcal{R}^{\text{NH}} \leq d + 1$. Figure 1 summarises these results along with numerically-found maximum ratios and analytically-found ratios from known models.

3.2.2 Results for MICRB & GMCRB ratios

The results on $\mathcal{R}_n^{\text{NH}}$ and $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}]$ in Sec. 3.2.1 lower-bound the corresponding quantities $\mathcal{R}_n^{\text{MI}}$ and $\mathcal{R}^{\text{MI}}[\{\rho_\theta\}]$. In particular, for the linear GMM model that addresses ONB tomography, we have proved $\mathcal{R}^{\text{MI}}[\{\rho_\theta\}] \geq d + 1$. Based on analytical and numerical results, we conjecture that $\mathcal{R}^{\text{NH}} \leq d + 1$; accordingly, Theorem 2 implies $\mathcal{R}^{\text{MI}} \geq d + 1$. In Sec. 5.6, we use the NHCRB solution behind Theorem 2 to solve the MICRB for the linear GMM model at $\theta = 0$, obtaining a ratio of $d + 1$. We then extend this solution to upper-bound the true ratio for arbitrary θ by $d + 2$ in Theorem 5 (see Appendix I for proof):

Theorem 5. *For ONB tomography of arbitrary d -dimensional qudit state ρ_θ , the maximum true collective enhancement $\mathcal{R}^{\text{MI}}[\{\rho_\theta\}] \leq d + 2$.*

Theorem 5 therefore constrains the true maximum enhancement for ONB tomography as

$$d + 1 \leq \mathcal{R}^{\text{MI}}[\{\rho_\theta\}] \leq d + 2. \quad (14)$$

A different way to quantify collective precision enhancement is to consider the ratio $C_{\text{GMCRB}}/C_{\text{SLD}}$. As noted previously, the SLD CRB can capture collective performance up to a factor, and the GMCRB, though not

tight for qudit problems with small n , can be directly computed given the SLD QFI J_{SLD} . This significantly simplifies the evaluation of the ratio $C_{\text{GMCRB}}/C_{\text{SLD}}$, which then upper-bounds $C_{\text{GMCRB}}/C_{\text{HCRB}}$, compared to the ratios considered in Sec. 3.2.1.

Let us denote the maximum GMCRB-to-HCRB ratio for a fixed model as $\mathcal{R}^{\text{GM}}[\{\rho_\theta\}]$, for any n -parameter d -dimensional model as $\mathcal{R}_n^{\text{GM}}$, and for any d -dimensional model as $\mathcal{R}^{\text{GM}}$ (similar to Eqs. (9), (10) and (11) for the NHCRB). A straightforward application of the Cauchy-Schwartz inequality to the eigenvalues $\{\nu_j\}_{j \in [n]}$ of $J_{\text{SLD}}^{-1/2}$ (the positive square-root of the inverse of the SLDQFI matrix) leads to

$$\frac{(\text{Tr}[J_{\text{SLD}}^{-1/2}])^2}{\text{Tr}[J_{\text{SLD}}^{-1}]} = \frac{(\sum_j \nu_j)^2}{\sum_j \nu_j^2} \leq n, \quad (15)$$

with equality attained if J_{SLD} is a scalar matrix. The ratio in Eq. (15) proves the upper bound

$$\begin{aligned} \mathcal{R}_n^{\text{GM}} &:= \max_{\substack{\theta \in \Theta \\ n\text{-parameter} \\ \text{models } \{\rho_\theta\}}} \frac{C_{\text{GMCRB}}[\rho_\theta]}{C_{\text{HCRB}}[\rho_\theta]} \\ &\leq \max_{\substack{\theta \in \Theta \\ n\text{-parameter} \\ \text{models } \{\rho_\theta\}}} \frac{C_{\text{GMCRB}}[\rho_\theta]}{C_{\text{SLD}}[\rho_\theta]} \leq \frac{n}{d-1} \end{aligned} \quad (16)$$

for any n -parameter d -dimensional model, reducing to $\mathcal{R}_{n_{\text{max}}}^{\text{GM}} \leq n_{\text{max}}/(d-1) = d+1$ at the maximum number of parameters. Moreover, Eq. (16) is a tight bound: a ratio $\mathcal{R}_n^{\text{GM}} = n/(d-1)$ is attained by the model of estimating any n GMM coefficients of the maximally-mixed state. This is because $J_{\text{SLD}} = d\mathbf{1}_n$ here, so that $C_{\text{GMCRB}} = n^2/(d(d-1))$, dividing which by $C_{\text{SLD}} = C_{\text{HCRB}} = n/d$ (see Methods Sec. 5.5) produces the ratio $n/(d-1)$, thereby proving $\mathcal{R}_n^{\text{GM}} = n/(d-1)$.

In Fig. 2, we compare the maximum collective enhancement predicted by $\mathcal{R}_n^{\text{GM}}$ and by $\mathcal{R}_n^{\text{NH}}$ for generic n parameter models ($1 \leq n \leq n_{\text{max}}$) on qutrits. The comparison with the NHCRB, which is a tighter bound, reveals that whereas Eq. (16) upper-bounding $\mathcal{R}_n^{\text{GM}}$ is tight, $\mathcal{R}_n^{\text{GM}}$ itself generally underestimates collective enhancement. This is because the GMCRB can be far from the attainable individual precision for qudit problems, especially for small n , but also for full-parameter models (see Fig. 9 in Appendix J). Nonetheless,

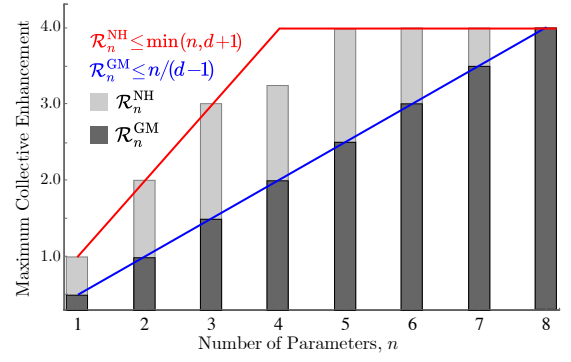


Figure 2: Comparison of the collective enhancements specified by the GMCRB ($\mathcal{R}_n^{\text{GM}}$) and by the NHCRB ($\mathcal{R}_n^{\text{NH}}$) for n -parameter models ($1 \leq n \leq n_{\text{max}}$). The maximum GMCRB ratios (dark gray bar chart) satisfy $\mathcal{R}_n^{\text{GM}} = n/(d-1)$ (blue line). The maximum NHCRB ratios (light gray bar chart) satisfy $\mathcal{R}_n^{\text{NH}} \leq \min(n, d+1)$ (red line). The ratios are plotted against the number of parameters, n , and include numerically found maximum ratios from random-sampling experiments (1300 samples for each n for $d = 3$) as well as analytically found ratios. The NHCRB ratio $\mathcal{R}_n^{\text{NH}}$ generally predicts a larger collective enhancement than the GMCRB ratio $\mathcal{R}_n^{\text{GM}}$, except at the maximum number of parameters ($n = n_{\text{max}}$).

if it holds that the GMCRB is separably attainable up to a scaling factor g (that is independent of d and n , e.g., $g = 2$ for ONB tomography [40]), implying $C_{\text{MI}} \leq g C_{\text{GMCRB}}$, then we can reason that

$$\frac{C_{\text{MI}}}{C_{\text{HCRB}}} \leq g \frac{C_{\text{GMCRB}}}{C_{\text{SLD}}} \leq g \frac{n}{d-1},$$

so that in general, $\mathcal{R}_n^{\text{NH}} \leq \mathcal{R}_n^{\text{MI}} \leq g \mathcal{R}_n^{\text{GM}} = g n/(d-1)$. For ONB tomography, which includes the GMM model as a special case, this proves that

$$\begin{aligned} d+1 &\leq \mathcal{R}^{\text{NH}}[\{\rho_\theta\}] \leq \mathcal{R}^{\text{MI}}[\{\rho_\theta\}] \\ &\leq 2 \mathcal{R}^{\text{GM}}[\{\rho_\theta\}] = 2(d+1). \end{aligned} \quad (17)$$

However, no such scaling factor g is currently known for general models beyond the setting of tomography.

3.3 Numerical Results

In this section, we report our numerical results on the effect of probe-purity on the collective enhancement and its maximum, and on the dependence of the maximum collective enhancement $\mathcal{R}_n^{\text{NH}}$ on d and n for randomly-sampled n -parameter, d -dimensional models. For both of these analyses, we rely on random sampling to

study the dependence of the maximum enhancement on state or model properties because directly maximising the ratio $C_{\text{NHCRB}}/C_{\text{HCRB}}$ over the state space $\mathcal{H}_d$ or the model space is computationally demanding for large d . For each randomly-generated problem instance, we solve the NHCRB and HCRB SDPs numerically and compute their ratio [21]. Further details of the random-sampling procedure used for subsequent results are presented in Appendix L.

In Fig. 3, we plot our random-sampling results for the ratio versus purity in the linear GMM model for $d = 2, 3$ and 4. For each d , the overall maximum ratio observed is $d + 1$. Interestingly, whereas for qubits the ratio is uniquely determined by purity, the higher dimensionality of the qudit state space allows for a range of ratios at any given purity. We find that the ratio at a given purity is maximised by full-rank depolarised pure states, $p|\phi\rangle\langle\phi| + (1-p)\mathbb{1}_d/d$ for any pure state $|\phi\rangle$ and $p \in [0, 1]$. To simplify computation, we choose the more specific family $\rho_{\text{max}}(p) = p|+\rangle\langle+|_d + (1-p)\mathbb{1}_d/d$, where $|+\rangle_d = (|0\rangle + \dots + |d-1\rangle)/\sqrt{d}$ and calculate the HCRB to be

$$C_{\text{HCRB}}[\rho_{\text{max}}(p)] = \frac{d^2 - 1}{d} + p(d-1) - \frac{d-1}{d}p^2. \quad (18)$$

For the NHCRB, based on numerical evidence for $d = 3$ to 8, the analytic solutions at the boundary cases ($(d^2 - 1)(d + 1)/d$ at $p = 0$ and $2(d-1)$ at $p = 1$), and the analytic solution for $d = 2$ (Eq. (24)), we find that

$$C_{\text{NHCRB}}[\rho_{\text{max}}(p)] = \frac{d^2 + 1}{2} - \frac{d^2 - 4d + 5}{2}p^2 + \frac{d^3 + 2d^2 - 3d - 2}{2d}\sqrt{1 - p^2}. \quad (19)$$

Accordingly, the maximum collective enhancement at a fixed purity P^* is $C_{\text{NHCRB}}[\rho_{\text{max}}(p^*)]/C_{\text{HCRB}}[\rho_{\text{max}}(p^*)]$ with $p^* = \sqrt{P^*d-1}$. Eqs. (18) and (19) reveal that the HCRB grows at most linearly with d , whereas the NHCRB grows at most quadratically, so that the maximum enhancement at fixed purity (dark red line in Fig. 3) grows at most linearly with dimension and is at most $d + 1$.

In contrast to the maximum ratio, the minimum-ratio states (blue dots in Fig. 3) are

rank-deficient states³ of the form $\rho_{\text{min}}^{(2)}(p) = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ for purity greater than $1/2$, $\rho_{\text{min}}^{(3)}(p) = p|0\rangle\langle 0| + p|1\rangle\langle 1| + (1-2p)|2\rangle\langle 2|$ for purity between $1/3$ and $1/2$, and so on,⁴ down to $\rho_{\text{min}}^{(d)}$ for purity between $1/d$ and $1/(d-1)$. This change in form of the minimum-ratio state reflects as the points of non-differentiability in the minimum-ratio curve in Fig. 3.

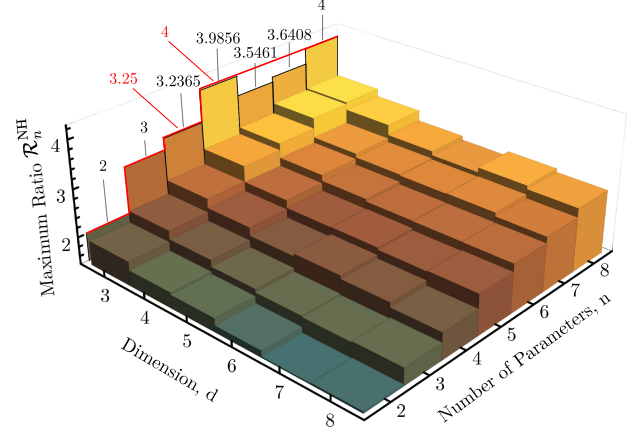


Figure 4: Maximum ratio $\mathcal{R}_n^{\text{NH}}$ between the NHCRB and the HCRB over 10,000 random models for each dimension, d , from three to eight and for each number of parameters, n , from two to eight. (See Fig. 11 in Appendix L for the distribution of ratios for each d and n .) The bar chart (with black callouts) on the back panel depicts the maximum ratio for estimating n GMM coefficients from the maximally-mixed qudit (Table 1 in Appendix F). The red line on the bar chart (with red callouts) represents the maximum ratio for each n from known analytic models, applicable for all $d \geq 3$.

Whereas our analytical results in Secs. 5.3–5.6 apply to the linear GMM model and its extensions, we now consider arbitrary smooth models of full-rank qudit states, i.e., we address the quantities $\mathcal{R}_n^{\text{NH}}$ and $\mathcal{R}^{\text{NH}}$ from Eqs. (10) and (11). Specifically, we estimate n arbitrary independent parameters from arbitrary full-rank d -dimensional qudit states. In this case, the parameter derivatives $\partial_j \rho_\theta$ are arbitrary traceless Hermitian operators. Our results for this model are shown in Fig. 4 (and Fig. 11 in Appendix L). The 3D bar chart in Fig. 4 clearly depicts the increase in maximum ratio with number of pa-

³Although we have treated only full-rank states until now, rank-deficient states can be approximated arbitrarily well by full-rank ones [50].

⁴We only provide analytic expressions for $\rho_{\text{min}}^{(2)}$ and $\rho_{\text{min}}^{(3)}$.

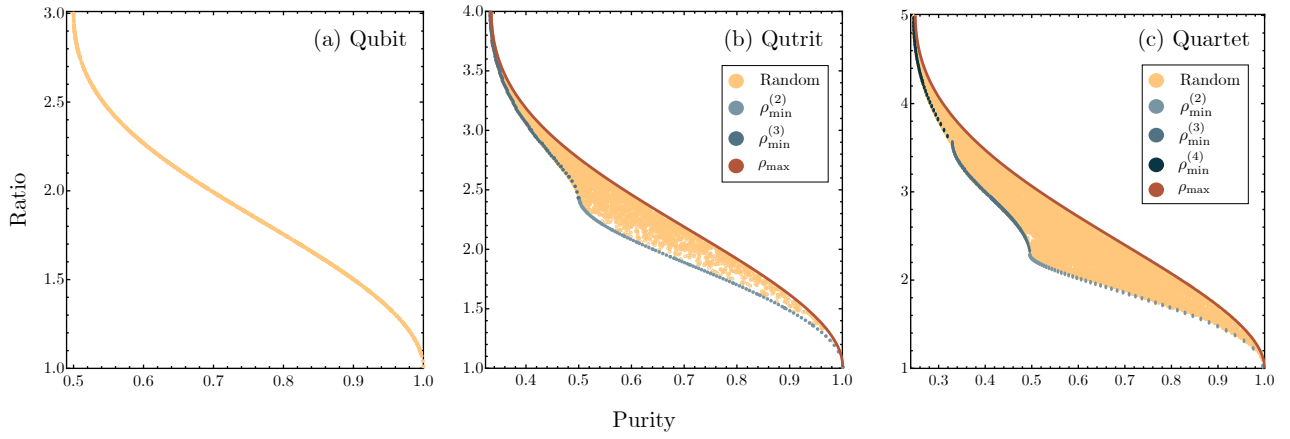


Figure 3: Ratio between the NHCRB and the HCRB versus purity for estimating all $d^2 - 1$ GMMs from arbitrary states. For qubits (a), we find a one-to-one dependence between ratio and purity (10,000 samples). However, for qutrits (b) and quartets (c), there is a region of allowed ratios at any given purity (15,000 and 25,000 samples, respectively). The ratio at any fixed purity is maximised by the state $\rho_{\max}$, which is a depolarised pure state, and minimised by the states $\rho_{\min}^{(2)}, \rho_{\min}^{(3)}$ (and $\rho_{\min}^{(4)}$ in (c)), which are rank-deficient classical states

rameters, in agreement with Secs. 5.1 and 5.5. This suggests $\mathcal{R}_n^{\text{NH}} \leq \mathcal{R}_{n+1}^{\text{NH}}$. On the other hand, in Fig. 4, the maximum ratio for a given number of parameters seems to decrease with increasing dimension—this is an anomaly stemming from having an increasingly larger sample space of models but a fixed number of samples (10^4) from them. In fact, the maximum ratio cannot decrease with increasing d ; any model in d dimensions can be extended to $d + 1$ dimensions by padding the state and its derivatives with zeros. This preserves both the individual and the collective precision bounds, and thus their ratio.

4 Discussion

In this work, we explored the ratio between separable- and collective-measurement precision bounds in order to identify the maximum collective quantum enhancement in a range of settings. Considering the ratio between the NHCRB and the HCRB, we established that for estimating any n independent parameters, the maximum precision-enhancement from collective measurements can, in principle, be n . However, at the maximum value of n , we proved this maximum enhancement to be only $O(d)$ or $O(\sqrt{n_{\max}})$. Specifically, for the model of qudit tomography in the Gell-Mann basis, we proved the maximum collective quantum enhancement to be $d + 2$, revealing the SIC POVM to be an individual-

optimal measurement for the maximally-mixed case. Based on the specific examples provided and our numerical results, we expect the attainable maximum enhancement to be $d + 1$ instead. We also established a maximum enhancement of $d + 1$ for tomography in any other basis, i.e., for estimating any other $n_{\max}$ parameters, as well as for estimating fewer than $n_{\max}$ parameters of the maximally-mixed state. Finally, we numerically demonstrated a maximum enhancement in $O(d)$ for states of a fixed known purity. Our work thus suggests supplementing the known ratio result $C_{\text{SLD}} \leq C_{\text{HCRB}} \leq 2 C_{\text{SLD}}$ with $C_{\text{HCRB}} \leq C_{\text{NHCRB}} \leq (d + 1) C_{\text{HCRB}}$. Throughout, we have demonstrated our analytical findings via numerics and figures for specific d .

It is, however, important to note that, though our methodology leveraging the NHCRB as a separable-measurement precision bound leads to several analytical results, the NHCRB is a substitute here for the most informative bound C_{MI} . The NHCRB can overestimate the best attainable separable precision whereas C_{MI} is tight by definition, so the attainable maximum advantage from collective measurements could be larger than $d + 1$, as in Eq. (12). However, numerical evidence for large qudit dimensions [39] suggests the gap $(C_{\text{MI}} - C_{\text{NHCRB}})/C_{\text{NHCRB}}$ to be small, so that the NHCRB-maximum ratio should be close to the MICRB-maximum ratio [47]. Indeed, for qudit tomography in orthonormal bases, we prove the attainable maximum collective enhancement

to be bounded between $d+1$ and $d+2$ —the same bound that applies for the NHCRB-maximum ratio. We then compare this result to a bound based on the GMCRB, demonstrating that our approach imposes a stronger constraint on attainable enhancement. Notably, both approaches predict a maximum enhancement linear in dimension, and not quadratic, at the maximum number of parameters. However, beyond full-parameter models, the NHCRB reflects more accurately a non-linear dependence of maximum collective enhancement on number of parameters n , whereas the GMCRB predicts a linear relationship (Fig. 2). These factors justify our choice of the NHCRB as the individual measurement precision bound.

Our choice of the Gell-Mann basis for tomography was motivated by its symmetry and simplicity, apart from being a generalisation of Pauli matrices. The orthonormality of the basis leads to (local) parameter orthogonality of the basis coefficients for estimating from the maximally-mixed state [62], making the classical and quantum (SLD) Fisher information matrices diagonal. As further captured by the theory of Fisher SIC measurements [56], the two Fisher information matrices are proportional. In general, any full-parameter model is D-invariant [13, 35, 63], leading to $C_{\text{HCRB}} = C_{\text{RLD}}$. Additionally, for GMM tomography, the SLD and RLD CRBs equal the HCRB, which is a signature of a locally-classical model [13, 63]. In fact, the theory of quantum local asymptotic normality [42, 44, 64, 65] predicts that in the asymptotic limit of number of copies, this model becomes a completely classical Gaussian model [51]. This theory therefore offers a physical explanation for why collective enhancement might be maximised in the orthonormal tomography setting: the single-copy measurements are subject to maximal parameter incompatibility whereas all the parameters can be estimated simultaneously in the asymptotic limit.

However, the significance of considering the Gell-Mann basis cannot be overstated. Generally, any parameter estimation problem may be linearised about the true parameter values as $\rho_\theta \approx \rho_{\theta^*} + \sum (\theta_j - \theta_j^*) \partial_j \rho_{\theta^*}$, with the partial derivatives $\partial_j \rho_\theta$ necessarily traceless and Hermitian, meaning they are combinations of GMMs (Sec. VC in [23]). This estimation problem can then be linearly transformed to the equiv-

alent problem of estimating some number of GMM coefficients [13], precisely the model solved in Sec. 5.5. For example, the simple result of $C_{\text{HCRB}}[\rho_m] = n/d$ from Sec. 5.3 can be directly linearly transformed to obtain a closed-form expression for the HCRB for estimating any $d^2 - 1$ independent parameters from any full-rank qudit state. Lastly, although our main analysis was specific to GMM tomography, we proved that the bounds and their ratios are invariant for tomography in any other orthonormal basis for the same space. These results lay a foundation for the future exploration of the enhancement potential of entangling measurements in multi-parameter quantum estimation.

Our approach in this work was to study quantities $\mathcal{R}^{\text{NH}}$ and $\mathcal{R}^{\text{MI}}$ to assess collective quantum enhancement. The collective- and separable-optimal precisions are identical for single-parameter problems, but can grow increasingly farther apart with increasing number of parameters, disagreeing the most at the maximum number of incompatible parameters. Therefore, the maximum ratio of precisions effectively compares how parameter incompatibility affects separable versus collective measurement strategies. Our results motivate defining a finite incompatibility measure through the gap $\delta = (C_{\text{NHCRB}} - C_{\text{HCRB}})/C_{\text{HCRB}}$, which complements the normalised gap $\Delta = (C_{\text{HCRB}} - C_{\text{SLD}})/C_{\text{SLD}}$ that was connected to the asymptotic incompatibility between parameters in Refs. [53, 57]. In contrast to Ref. [57], our results show that even if the asymptotic incompatibility vanishes [53], the individual and collective attainable precisions can still disagree maximally and the finite incompatibility δ can be as large as the dimension d , a significant difference from the asymptotic case.

The results in Figs. 1, 2 and 4 indicate that the maximum collective enhancement $\mathcal{R}_n^{\text{NH}}$ increases with number of parameters, and hence is largest for state tomography. Moreover, for tomography, the maximum enhancement decreases with purity (Fig. 3) and is maximised by minimum-purity states. Based on this, we conjecture that maximum enhancement $\mathcal{R}^{\text{NH}}$ is attained for orthonormal tomography of the maximally-mixed state. This case was studied analytically to find a ratio of $d+1$. Hence, we conjecture that $C_{\text{NHCRB}} \leq (d+1)C_{\text{HCRB}}$ for all smooth full-rank models in the local estimation setting, i.e., $\mathcal{R}^{\text{NH}} = d+1$.

Notably, both the n bound (Sec. 5.1) and the $d+1$ bound (Secs. 5.3 & 5.4) can be tight, and for qubit tomography (Eq. (24)), they are tight and identical. Resultantly, if our conjecture holds true, we would also expect $\mathcal{R}_n^{\text{NH}} \leq \min(n, d+1)$ (red line in Figs. 1 & 2) to hold for qudits. For qutrits, Fig. 1 shows this conjectured upper bound to be attainable for all n except $n = 4$. The value of $\mathcal{R}_4^{\text{NH}}$ for qutrits remains an open problem (we report a ratio of 3.25 in Fig. 4), motivating further inquiry into the utility of collective measurements in multi-parameter quantum estimation.

In conclusion, we find that for local estimation problems involving n parameters, the optimal collective measurements are at most n times more precise than the optimal individual measurements. Although this suggests that a collective quantum enhancement of n is possible, and that the utility of collective measurements scales with the number of estimated parameters, our further analysis indicates otherwise. By taking the probe dimension d into account, we upper-bound the collective enhancement by $d+1$, which is a tighter bound for large n ($n > d$). Our investigation into the utility of collective measurements thus reveals a diminishing payoff in the asymptotic limit. Whereas collective measurements on two copies, three copies, and so on, are practically feasible and outperform the optimal individual measurements, the optimal collective measurements that saturate the HCRB require entangling measurements on asymptotically-large number of copies but only enhance precision by a factor at most linear in dimension, underscoring their non-utility.

Our results apply to multi-parameter quantum metrology and quantum sensing, where a judicious choice between measurement strategies would be resource-wise beneficial. Conversely, our work questions the choice of the HCRB when benchmarking the performance of real-world quantum measurements, and suggests the (finite-copy) NHCRB as a more suitable alternative. Investigating the advantage offered by finite-copy collective measurements, or extending to Bayesian settings could offer valuable insights into the potential of entangling measurements.

5 Methods

In this section, we present our methodology for analytically deriving the $\mathcal{R}_n^{\text{NH}} \leq n$ and the $\mathcal{R}^{\text{NH}} \leq d+1$ bounds on the maximum collective enhancement. First, in Sec. 5.1, we establish a problem-independent upper bound of n on the collective enhancement $\mathcal{R}_n^{\text{NH}}$. Then, over Secs. 5.2–5.5, we establish the upper bound of $d+1$ on $\mathcal{R}^{\text{NH}}$. We start by introducing the ‘linear GMM model’ and identifying some of its simplifying features in Sec. 5.2. Then, in Sec. 5.3, we solve the full linear GMM model, corresponding to GMM-basis tomography, for maximally-mixed qudit states, showing the enhancement here to be exactly $d+1$. Next, in Sec. 5.4, we extend our arguments for the full linear GMM model to arbitrary qudit states, establishing a maximum collective enhancement of $d+2$. Finally, in Sec. 5.5, we explore the related model of estimating fewer than n_{max} parameters of the maximally-mixed state, proving that the maximum enhancement remains $d+1$. We also establish the maximum enhancement to be $d+1$ for the weighted linear GMM model in Appendix H and for a different choice of the individual-measurement precision bound in Appendix J.

5.1 Ratio of n

We now establish a model-agnostic (or problem-independent) upper bound of n on the collective enhancement $\mathcal{R}_n^{\text{NH}}$.

Theorem 1. *For estimating n parameters of any qudit state ρ_θ , the collective enhancement $C_{\text{NHCRB}}[\rho_\theta]/C_{\text{HCRB}}[\rho_\theta] \leq n$, i.e.,*

$$\mathcal{R}_n^{\text{NH}} \leq n. \quad (20)$$

Proof. Using Ref. [46]’s upper bound (based on Ref. [39]) to the NHCRB (see summary in Appendix G),

$$C_{\text{NHCRB}}[\rho_\theta] \leq \min_{\mathbb{X}} \left\{ \text{Tr}(\mathbb{Z}_\theta[\mathbb{X}]) + \sum_{j,k \in [n]} \|\rho_\theta[X_j, X_k]\|_1 \right\}, \quad (21)$$

and $\|\rho_\theta[X_j, X_k]\|_1 \leq 1/2 \text{Tr}[\rho_\theta(X_j^2 + X_k^2)]$, we

get

$$C_{\text{NHCRB}}[\rho_\theta] \leq n \min_{\mathbb{X}} \text{Tr}(\mathbb{Z}_\theta[\mathbb{X}]) = n C_{\text{SLD}}[\rho_\theta]. \quad (22)$$

On the other hand, from Eq. (6), we have $C_{\text{HCRB}}[\rho_\theta] \geq \min_{\mathbb{X}} \text{Tr}(\mathbb{Z}_\theta[\mathbb{X}])$, so that $C_{\text{NHCRB}}[\rho_\theta]/C_{\text{HCRB}}[\rho_\theta] \leq n$ for all smooth models and, thus, $\mathcal{R}_n^{\text{NH}} \leq n$. $\square$

Note that under the assumption of independent parameters θ , we have $n \leq n_{\text{max}}$ [23], so that Eq. (20) implies $\mathcal{R}^{\text{NH}} \leq n_{\text{max}} = d^2 - 1$.

5.2 Model: Estimating GMMs from Qudits

We now introduce our quantum statistical model, which is an n_{max} -parameter family of d -dimensional qudit states. This model, which we call the ‘linear GMM model’, involves estimating the n_{max} coefficients $\{\theta_j\}_{j \in [n_{\text{max}}]}$ of the GMMs $\Lambda_d := \{\lambda_j\}_{j \in [n_{\text{max}}]}$ from the Bloch representation of a qudit state [50],

$$\rho_\theta = \mathbb{1}_d/d + \sum_{j=1}^{n_{\text{max}}} \theta_j \lambda_j. \quad (23)$$

The GMMs Λ_d are traceless, Hermitian generalisations of the qubit Pauli operators (see Appendix A), and the decomposition in Eq. (23) is a one-to-one map between the Hilbert space $\mathcal{H}_d$ of ρ_θ and the parameter space $\Theta \subset \mathbb{R}^{n_{\text{max}}}$. Estimating θ is thus equivalent to qudit state tomography. Note that we adopt the convention of normalising the GMMs such that $\text{Tr}(\lambda_j \lambda_k) = \delta_{jk}$.⁵

It is useful to summarise the $d = 2$ case results here [48]; the HCRB and NHCRB are

$$\begin{aligned} C_{\text{HCRB}} &= C_{\text{RLD}} = 3 - r^2 + 2r, \\ C_{\text{NHCRB}} &= C_{\text{GMCRB}} = 5 - r^2 + 4\sqrt{1 - r^2} \end{aligned} \quad (24)$$

with $r^2 = \sum_j \theta_j^2 = \text{Tr}(\rho^2) - 1/2$ the squared length of the Bloch vector. In this case, the NHCRB is attained by measuring informationally-complete (IC) POVMs, simplifying to symmetric informationally-complete (SIC) POVMs (see Eq. (39) for definition) at $r = 0$ [48]. It is straightforward⁶ to see from Eq. (24)

⁵Some authors [66] instead normalise as $\text{Tr}(\lambda_j \lambda_k) = 2\delta_{jk}$ to be consistent with the $d = 2$ case for Pauli matrices. Our convention rescales the parameter values and bounds, but leaves their ratios invariant.

⁶ C_{HCRB} increases with r whereas C_{NHCRB} decreases.

that the ratio $C_{\text{NHCRB}}/C_{\text{HCRB}}$ is maximised at $r = 0$, corresponding to estimating parameters of the maximally-mixed state. Thus, for the qubit tomography model, the maximum enhancement $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}]$ is three, and this ratio is attained when estimating the three Pauli matrix coefficients of the maximally-mixed qubit state [48].

An important simplifying feature of the linear GMM model is that the LUB operators $\mathbb{X} = \{X_1, \dots, X_n\}^\top$ are uniquely fixed to be the GMMs themselves, i.e.,

$$X_j = \lambda_j. \quad (25)$$

That there is exactly one feasible solution for the LUB operators significantly simplifies the evaluation of the bounds.⁷ To see this unique feature of our model, consider that the true (unknown) state is

$$\rho_\theta^* = \mathbb{1}_d/d + \sum_{j=1}^{n_{\text{max}}} \theta_j^* \lambda_j.$$

The LUB constraints (Eq. (5)) at θ^* are then

$$\begin{aligned} \text{Tr}(\rho_\theta X_k)|_{\theta=\theta^*} &= \theta_k^*, \\ \text{Tr}(\partial_j \rho_\theta X_k)|_{\theta=\theta^*} &= \delta_{jk}. \end{aligned} \quad (26)$$

Writing $X_j = \sum_k c_{jk} \lambda_k$, where c_{jk} are unknown real numbers (to preserve Hermiticity of X_j), reduces Eq. (26) to

$$\sum_j c_{kj} \theta_j^* = \theta_k^* \quad \& \quad c_{kj} = \delta_{jk},$$

which immediately implies $X_j = \lambda_j$, as claimed.

The simplification from having a unique solution for feasible LUB operators, $\mathbb{X}$, extends more generally to tomography in any ONB ($n = n_{\text{max}}$). The model here,

$$\rho_{\theta'} = \mathbb{1}_d + \sum_{j=1}^{n_{\text{max}}} \theta'_j \mathcal{B}_j, \quad (27)$$

is called the ONB model, and corresponds to tomography in the basis $\{\mathcal{B}_j\}_{j \in [n_{\text{max}}]}$, which is orthonormal, $\text{Tr}(\mathcal{B}_j \mathcal{B}_k) = \delta_{jk}$. Further, $\mathcal{B}_j$ are traceless and Hermitian and, therefore, can be

⁷This is not generally true; in most cases there are multiple feasible LUB operators $\mathbb{X}$, which need to be optimised over to calculate the bounds, see for example Refs. [21, 26] and Appendix F.

written as a real linear combinations of the GMMs λ_k through $\mathcal{B}_j = \sum_k \eta_{jk} \lambda_k$, so that

$$\begin{bmatrix} \mathcal{B}_1 \\ \vdots \\ \mathcal{B}_n \end{bmatrix} = \begin{bmatrix} \eta_{11} & \dots & \eta_{1n} \\ \vdots & \ddots & \vdots \\ \eta_{n1} & \dots & \eta_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}, \quad (28)$$

or in short, with vectorised notation $\mathcal{B}_d$ for the left hand side,

$$\mathcal{B}_d = (\eta \otimes \mathbb{1}_d) \Lambda_d. \quad (29)$$

Now imposing the condition that $\mathcal{B}_d$ is orthonormal, i.e.,

$$\text{Tr}(\mathcal{B}_j \mathcal{B}_k) = \delta_{jk}, \quad (30)$$

implies that the transformation matrix η is orthogonal, so that the inner-products between its different rows (or different columns) is zero. This is stated and proved as Lemma 1 below.

Lemma 1. *For transforming from the GMM basis to any other ONB for the space of $d \times d$ traceless Hermitian matrices over reals, the transformation matrix η is real and orthogonal, meaning*

$$\eta \eta^\top = \eta^\top \eta = \mathbb{1}_n. \quad (31)$$

Proof. The orthonormality condition from Eq. (30) in the GMM basis simplifies to

$$\begin{aligned} \text{Tr}(\mathcal{B}_j \mathcal{B}_k) &= \sum_{a,b=1}^{n_{\max}} \eta_{ja} \eta_{kb} \text{Tr}(\lambda_a \lambda_b) \\ &= \sum_{a=1}^{n_{\max}} \eta_{ja} \eta_{ka} = (\eta \eta^\top)_{jk} = \delta_{jk}, \end{aligned}$$

so that $\eta \eta^\top = \mathbb{1}_n$. As both $\mathcal{B}_d$ and Λ_d are bases for the space, the matrix η is full-rank and therefore invertible. This makes $\eta^\top$ the inverse of η , so that $\eta \eta^\top = \eta^\top \eta = \mathbb{1}_n$, as claimed. $\square$

Clearly, the GMM model is an ONB model, and Lemma 1 says any two ONB models are related by a real unitary (orthogonal) transformation of the parameter derivatives. Conversely, this connects the ONB parameters θ' (in Eq. (27)) to the GMM parameters θ (in Eq. (23)) through

$$\theta_k = \sum_{j=1}^{n_{\max}} \eta_{jk} \theta'_j \implies \theta = \eta^\top \theta' \implies \theta' = \eta \theta.$$

This relation enables a much stronger connection between different ONB models. In particular, any

ONB model can now be considered as a reparameterised GMM model with $\theta' = \eta \theta$ [13, 21, 61]. From Sec. V of Ref. [61], we know that such a reparameterisation of any model is equivalent to the weighted version of the original model with weight matrix $W = \eta^\top \eta$. For the reparameterised GMM model, the orthogonality of η (Lemma 1) makes this weight $W = \eta^\top \eta = \mathbb{1}_n$, thereby proving that the GMM model and the ONB model are completely equivalent. This means that both individual (NHCRB, MICRB, GMCRB) and collective (SLD CRB, HCRB) precision bounds are invariant to the particular choice of ONB for tomography. Therefore, the ratios $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}]$ and $\mathcal{R}^{\text{MI}}[\{\rho_\theta\}]$ obtained below for the GMM model also hold for the model of tomography in any other ONB.

5.3 Ratio of $d + 1$: Maximally-mixed State

We now investigate the parameter estimation of θ ($n = n_{\max}$) for the maximally-mixed qudit state $\rho_\theta^* = \mathbb{1}_d/d =: \rho_m$ in d dimensions (corresponding to $\theta^* = 0$). For ρ_m , we calculate the SLD and RLD CRBs, the HCRB, the NHCRB, and the GMCRB. We also find the SIC-POVM in d dimensions to be an optimal individual measurement that attains the NHCRB, thus establishing $C_{\text{MI}} = C_{\text{NHCRB}}$ for this case [37]. Choosing $\rho_\theta^* = \rho_m$ simplifies the evaluation of various CRBs as this choice of ρ_θ^* commutes with every linear operator.

From their definitions, (see Eqs. (87) and (88) in Appendix D), we find both the SLD and the RLD operators to be simply

$$L_j^{\text{SLD}} = L_j^{\text{RLD}} = d \lambda_j. \quad (32)$$

The two resulting QFI matrices are equal and diagonal (see Appendix D),

$$J^{(\text{SLD})} = J^{(\text{RLD})} = \begin{bmatrix} d & 0 & \dots & 0 \\ 0 & d & \dots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \dots & 0 & d \end{bmatrix}_{n \times n}, \quad (33)$$

which is a sign that our model is “locally classical” [13, 63]. The two QFIs then yield their respective CRBs,

$$C_{\text{SLD}} = C_{\text{RLD}} = \frac{n_{\max}}{d} = \frac{d^2 - 1}{d}. \quad (34)$$

As expected of a locally classical model, the HCRB coincides with the SLD CRB and RLD CRB [13, 63]. In fact, any full parameter model ($n = n_{\max}$) with linearly-independent parameter derivatives constitutes a “D-invariant” model, for which $C_{\text{HCRB}} = C_{\text{RLD}}$ is known to hold [35, 63]. Nonetheless, and more directly, note that the minimisation over $\mathbb{X}$ in the definition in Eq. (6) is trivial due to the uniqueness discussed in Sec. 5.2. Thus, $(\mathbb{Z}_\theta[\mathbb{X}])_{jk} = \frac{1}{d} \text{Tr}(\lambda_j \lambda_k) = \delta_{jk}/d$, which is exactly $J^{(\text{SLD})^{-1}}$. Correspondingly,

$$\begin{aligned} C_{\text{HCRB}} &= \text{Tr}(\mathbb{Z}_\theta[\mathbb{X}]) = \text{Tr}(J^{(\text{SLD})^{-1}}) \\ &= \frac{d^2 - 1}{d} = C_{\text{SLD}}. \end{aligned}$$

We write this result as Lemma 2, and defer the detailed proof to Appendix C. Note that, more generally, $C_{\text{SLD}} = \min_{\mathbb{X}} \{\text{Tr}(\mathbb{Z}_\theta[\mathbb{X}])\}$. Additionally, when $\mathbb{X}$ is uniquely fixed,

$$C_{\text{HCRB}} \geq \text{Tr}(\mathbb{Z}_\theta[\mathbb{X}]) = \text{Tr}(\mathbb{S}_\theta \mathbb{X} \mathbb{X}^\top) = C_{\text{SLD}}. \quad (35)$$

Lemma 2. *The HCRB for estimating $\theta \approx 0$ from ρ_m is*

$$C_{\text{HCRB}}[\rho_m] = \frac{d^2 - 1}{d}. \quad (36)$$

The NHCRB is not as trivial to compute because despite $\mathbb{X}$ being uniquely fixed, there is still a minimisation over $nd \times nd$ matrix $\mathbb{L}$ in Eq. (4) [21]. Moreover, directly proving the optimality of a candidate $\mathbb{L}$ is difficult—for this purpose we turn to the SDP formulation of the NHCRB [21] (see Appendix E for definition). The SDP approach offers a simple way to prove optimality via duality: if we can find a primal-feasible solution and a dual-feasible solution such that the primal objective value equals the dual objective value, then the solutions are optimal. In Appendix E, we present a pair of such solutions and prove their optimality using this approach. The optimal argument $\mathbb{L}^*$ we find to the SDP is

$$\mathbb{L}_{jk}^* = \left(\frac{d+1}{d+2} \right) (\{\lambda_j, \lambda_k\} + \delta_{jk} \mathbb{1}_d) \quad (37)$$

where $j, k \in [n]$ and $\{\cdot, \cdot\}$ denotes the anti-commutator. Directly computing $\text{Tr}[\mathbb{S}_\theta \mathbb{L}^*]$ then leads to the following lemma.

Lemma 3. *The NHCRB for estimating $\theta_j \approx 0$ from ρ_m is*

$$C_{\text{NHCRB}}[\rho_m] = \frac{(d^2 - 1)(d + 1)}{d}. \quad (38)$$

Our first main result now follows straightforwardly from Lemmas 2 and 3.

Theorem 2. *For ONB tomography of the maximally-mixed qudit state ρ_m , the collective enhancement $C_{\text{NHCRB}}[\rho_m]/C_{\text{HCRB}}[\rho_m] = d + 1$.*

The HCRB is already known to be asymptotically attainable, so we now prove the attainability or tightness of the NHCRB for our model. Specifically, we show that the NHCRB in Lemma 3 can be attained by measuring any rank-one symmetric informationally-complete (SIC) POVM in d dimensions (assuming one exists). The SIC POVM is a set of d^2 measurement operators $\{\Pi_l\}_{l \in [d^2]}$ that form a POVM and are completely symmetric between themselves under the trace inner product,

$$\text{Tr}(\Pi_{l_1} \Pi_{l_2}) = \frac{1}{d^2(d+1)} \quad \forall l_1 \neq l_2, l_1, l_2 \in [d^2]. \quad (39)$$

To prove that measuring SIC POVMs attains the NHCRB, we show that the measured probabilities $\text{Tr}(\rho_m \Pi_l)$ directly yield a variance equal to C_{NHCRB} from Lemma 3, establishing $C_{\text{MI}} = C_{\text{NHCRB}}$ in this case. The CFI matrix J_{jk} ($j, k \in [n]$), which in the multi-parameter case is given by

$$J_{jk} \left[\{\Pi_l\}_{l \in [m]} \right] = \sum_{l=1}^m \frac{\text{Tr}[\partial_j \rho_\theta \Pi_l] \text{Tr}[\partial_k \rho_\theta \Pi_l]}{\text{Tr}[\rho_\theta \Pi_l]}, \quad (40)$$

simplifies to (see Lemma 9 and proof in Appendix E.5)

$$\begin{aligned} J_{jk} &= d^2 \sum_{l=1}^{d^2} \text{Tr}[\lambda_j \Pi_l] \text{Tr}[\lambda_k \Pi_l] \\ &= \delta_{jk} \frac{d}{d+1} \end{aligned} \quad (41)$$

in this case, so that Eq. (2) then leads to

$$\text{Tr}(J^{-1}) = \frac{(d^2 - 1)(d + 1)}{d} = C_{\text{NHCRB}}. \quad (42)$$

From Eq. (3), we then have $C_{\text{MI}} \leq \text{Tr}(J^{-1}) = C_{\text{NHCRB}} \leq C_{\text{MI}}$ with the last inequality from Eq. (8). This proves $C_{\text{MI}} = C_{\text{NHCRB}}$, meaning

that the ultimate individual precision is attained for this model by measuring SIC POVMs. Notably, any rank-one SIC POVM in d dimensions, irrespective of its orientation, constitutes an optimal individual measurement in this scenario.

An alternative proof of this attainability can be furnished using the GM inequality for individual measurements [23],

$$\text{Tr}[J^{(\text{SLD})^{-1}}J] \leq d-1. \quad (43)$$

For the SLD QFI $J^{(\text{SLD})}$ in Eq. (33), the inequality in Eq. (43) implies that

$$\text{Tr}(J^{-1}) \geq (d^2-1)(d+1)/d, \quad (44)$$

which is saturated by the SIC POVM CFI J from Eq. (41), as seen in Eq. (42).

5.4 Ratio of $d+2$: Extension to Arbitrary States

In this section, we extend Lemmas 2 & 3 and Theorem 2 for ρ_m to arbitrary qudit states $\rho_\theta \neq \rho_m$. Such a qudit state can still be written as in Eq. (23), but now the true parameter values θ^* are non-zero and $\theta \approx \theta^*$. In this case, we show that $C_{\text{HCRB}}[\rho_\theta] \geq C_{\text{HCRB}}[\rho_m] - \sum_{j \in [n]} \theta_j^{*2}$ and that $C_{\text{NHCRB}}[\rho_\theta] \leq C_{\text{NHCRB}}[\rho_m] - \sum_{j \in [n]} \theta_j^{*2}$, which, we then show, imply

$$\frac{C_{\text{NHCRB}}[\rho_\theta]}{C_{\text{HCRB}}[\rho_\theta]} \leq d+2.$$

This establishes the maximum collective quantum enhancement $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}]$ for the linear GMM model to be $d+2$. We also argue that the optimal individual measurements are now IC POVMs, supported by numerical results in Appendix K.

The HCRB and the NHCRB involve an additional $-\text{Tr}(\theta^* \theta^{*\top}) = -\sum_{j \in [n]} \theta_j^{*2}$ term for non-zero θ^* (Eqs. (4), (6) & (7)). For the HCRB, it is simple to see from Eq. 83 in Appendix C that $\text{Tr}(\mathbb{S}_\theta \mathbb{X} \mathbb{X}^\top)$ still lower-bounds $\min_{\mathbb{L}, \mathbb{X}} \text{Tr}(\mathbb{S}_\theta \mathbb{L})$ (see also Remark 1 in Appendix C) so that

$$C_{\text{HCRB}}[\rho_\theta] \geq \text{Tr}(\mathbb{S}_\theta \mathbb{X} \mathbb{X}^\top) - \sum_{j \in [n]} \theta_j^{*2} \quad (45)$$

despite $\mathbb{L} = \mathbb{X} \mathbb{X}^\top$ not being the optimal solution anymore. Note also that the purity of the true state ρ_θ^* is

$$P(\rho_\theta^*) = \text{Tr}((\rho_\theta^*)^2) = 1/d + \sum_{j \in [n]} \theta_j^{*2}.$$

By explicit calculation, we find $\text{Tr}(\mathbb{S}_\theta \mathbb{X} \mathbb{X}^\top) = d^2-1/d = C_{\text{HCRB}}[\rho_m]$, and hence,

$$C_{\text{HCRB}}[\rho_\theta] \geq \frac{d^2-1}{d} - \sum_{j \in [n]} \theta_j^{*2} = d - P(\rho_\theta^*). \quad (46)$$

From Lemma 2, we know that this inequality is saturated by the maximally-mixed state ρ_m , which has purity $1/d$. Figure 5 (a) depicts how $d - P(\rho_\theta^*)$ compares with the actual HCRB for qutrit states.

For the NHCRB, our key insight is that the optimal argument $\mathbb{L}_{jk}^* = d+1/d+2 (\{\lambda_j, \lambda_k\} + \delta_{jk} \mathbb{1}_d)$ from Lemma 4 in Sec. 5.3 is still feasible: $\mathbb{L}^*$ satisfies the constraints $\mathbb{L}_{jk} = \mathbb{L}_{kj}$ Hermitian and $\mathbb{L} \succcurlyeq \mathbb{X} \mathbb{X}^\top$, which are all independent of ρ_θ . However, $\mathbb{L}^*$ is not optimal so $\text{Tr}[\mathbb{S}_\theta \mathbb{L}^*]$ only upper-bounds $\min_{\mathbb{L}} \text{Tr}[\mathbb{S}_\theta \mathbb{L}]$ in Eq. (4). Again, we explicitly calculate $\text{Tr}[\mathbb{S}_\theta \mathbb{L}^*]$ to find

$$\text{Tr}[\mathbb{S}_\theta \mathbb{L}^*] = \frac{(d^2-1)(d+1)}{d} = C_{\text{NHCRB}}[\rho_m] \quad (47)$$

so that we can upper-bound the NHCRB as

$$\begin{aligned} C_{\text{NHCRB}}[\rho_\theta] &\leq \frac{(d^2-1)(d+1)}{d} - \sum_j \theta_j^{*2} \\ &= d^2 + d - 1 - P(\rho_\theta^*). \end{aligned} \quad (48)$$

From Lemma 3, we see that the inequality in Eq. (48) is saturated by the maximally-mixed state ρ_m . Figure 5 (b) depicts how $d^2 + d - 1 - P(\rho_\theta^*)$ compares with the actual NHCRB for qutrit states.

Theorem 3. *For ONB tomography of arbitrary d -dimensional qudit state ρ_θ , the maximum collective enhancement $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}] \leq d+2$.*

Proof. Combining the lower bound for the HCRB in Eq. (46) with the upper bound for the NHCRB in Eq. (48), we get

$$\frac{C_{\text{NHCRB}}[\rho_\theta]}{C_{\text{HCRB}}[\rho_\theta]} \leq \frac{d^2 + d - 1 - P(\rho_\theta)}{d - P(\rho_\theta)}. \quad (49)$$

Then, using $1/d \leq P(\rho_\theta) \leq 1$, we find the maximum of the right hand side of Eq. (49) to be $d+2$, attained when $P(\rho_\theta) = 1$, i.e., when ρ_θ is pure. $\square$

Theorem 3 establishes a loose upper bound that we expect to never be attained. The ratio between the two bounds is actually maximised by

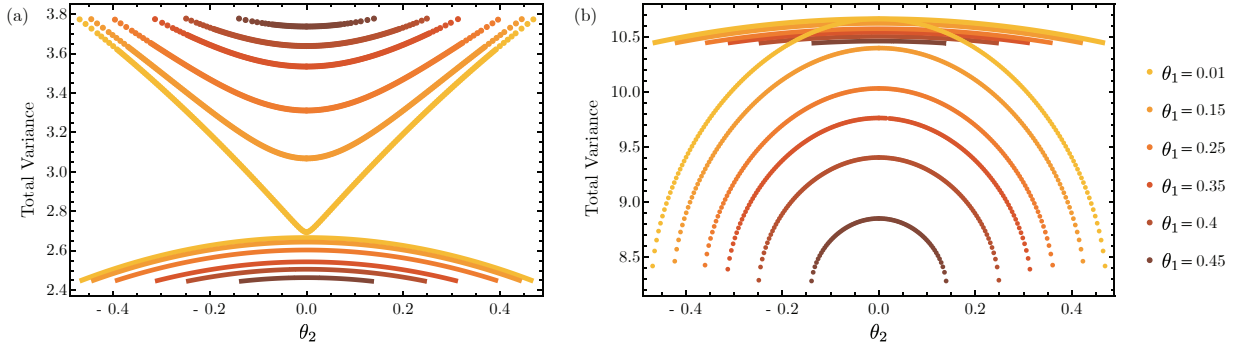


Figure 5: Comparison of the HCRB and the NHCRB to their lower and upper bounds, respectively. (a) HCRB and its lower bound $d - P(\rho_\theta)$ (from Eq. (46)). The lower solid parabolic curves show the lower bound and the upper triangular curves (beginning and ending with dots) show the numerically-computed HCRB. (b) NHCRB and its upper bound $d^2 + d - 1 - P(\rho_\theta)$ (from Eq. (48)). The lower dotted curves show the numerically-computed NHCRB and the upper solid curves show the upper bound. The state chosen in both (a) and (b) is a mixed qutrit $\rho_\theta = \mathbb{1}_d/d + \theta_1\lambda_2 + \theta_2\lambda_4$.

maximally-mixed states and minimised by pure states, as seen in Fig. 3. This discrepancy is due to the HCRB lower bound in Eq. (45) being a decreasing function of purity, whereas numerical results (Fig. 3 and weighted version in Appendix H) and analytical results for the qubit case (Eq. (24)) show the HCRB to be an increasing function of purity for this model. Figure 5 (a) and (b) numerically compare the lower and upper bounds from Eqs. (46) and (48) with the true HCRB and NHCRB, respectively, for $d = 3$ and ρ_θ close to ρ_m . It is evident that the lower and upper bounds are valid for all ρ_θ but saturated only for ρ_m . It is also clear from Fig. 5 (a) that the HCRB lower bound in Eq. (46) is a decreasing function of purity whereas the true HCRB is an increasing function of purity. And, in fact, using $C_{\text{HCRB}}[\rho_\theta] \geq C_{\text{HCRB}}[\rho_m]$ instead of Eq. (45) in the proof of Theorem 3 leads to an upper bound of $d + 1$ on the ratio. This is a tight bound and is saturated by the model considered in Sec. 5.3. Nonetheless, Theorems 2 and 3 together establish that for the linear GMM model, and any other ONB model, $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}] \leq d + 2$.

We extend the upper bounds on the ratio proved in Secs. 5.3 and 5.4 to arbitrary, full-rank, parameter-independent weight matrices W in Appendix H. Arbitrary weight matrices correspond to reparameterisations of the model [13, 21, 61], i.e., estimating parameters that are not coefficients of any particular ONB. Our results in Appendix H prove that for the maximally-mixed state ρ_m , and for estimating any n_{max} independent parameters locally, the maximum collective

enhancement is at most $d + 1$. We also numerically show that when estimating from any other state ρ_θ , the ratio is smaller than when estimating from ρ_m with the same weight W . This suggests the maximum enhancement from collective measurements over individual measurements in any local tomography problem is $d + 1$, i.e.,

$$\mathcal{R}_{n_{\text{max}}}^{\text{NH}} = \max_{n_{\text{max}}\text{-parameter models}} \mathcal{R}^{\text{NH}}[\{\rho_\theta\}] \leq d + 1, \quad (50)$$

and this upper bound is saturated by the model studied in Sec. 5.3.

Finally, the optimal separable measurements, assuming all the θ_j to be independent, are IC-POVMs. This is because to estimate $d^2 - 1$ independent parameters, one needs $d^2 - 1$ independent probabilities which can only arise from measuring a POVM with at least d^2 linearly independent elements. Having any more than d^2 POVM elements is also redundant, as the extra elements cannot be linearly independent from the first d^2 elements. In Appendix K, we depict the transition from SIC POVMs to IC POVMs as the purity of ρ_θ increases from $1/d$ for the maximally-mixed state to 1 for pure states. This result is in line with previous findings that IC POVMs are optimal for state estimation and tomography [67, 68].

5.5 Related Model: Estimating $n < n_{\text{max}}$ GMMs

In Secs. 5.3 and 5.4, we studied the full-parameter linear GMM model for the cases $\theta^* = 0$ and $\theta^* \neq 0$. In this section, we study the GMM model

with $n < n_{\max}$ parameters, with the remaining $n_{\max} - n$ parameters set to zero, i.e., estimating $\{\theta_j\}_{j \in [n]} \in \Theta$ from

$$\rho_\theta = \mathbb{1}_d/d + \sum_{j \in [n]} \theta_j \lambda_j. \quad (51)$$

The case with $\{\theta_j\}_{j \in [n_{\max}] \setminus [n]} \neq 0$ is also interesting but we do not study that here. Moreover, we only provide analytic results for the true state $\rho_\theta^* = \rho_m$. This is because, numerically, we see that when the parameters not estimated are set to zero, the ratio is maximised by ρ_m . Although we have not specified which n GMMs we choose to estimate, and despite the NHCRB (but not the HCRB) depending on this choice,⁸ the bounds we provide on the ratio are independent of this choice.

The differentiating factor for this model is that $X_j = \lambda_j$ are not the sole possible LUB operators. Nonetheless, for the HCRB, this choice is still optimal, and the HCRB is n/d , the same form as Lemma 2. For the NHCRB, we find that linearly modifying Λ_d and bilinearly modifying $\mathbb{L}_{jk}^*$ from Lemma 4 gives us an upper bound of $(d+1)n/d$, also the same form as Lemma 3.

For estimating parameters of ρ_m , we find the SLD-optimal $\mathbb{X}$, i.e., $\arg \min_{\mathbb{X}} \text{Tr}[\mathbb{S}_\theta \mathbb{X} \mathbb{X}^\top]$, to be $X_j = \lambda_j$ for $j \in [n]$ (see Appendix F). This establishes that $C_{\text{SLD}} = n/d$. Moreover, for this choice of $\mathbb{X}$, $\mathbb{L} = \mathbb{X} \mathbb{X}^\top$ satisfies all the HCRB constraints (Eq. (7)) and yields

$$\text{Tr}[\mathbb{S}_\theta \mathbb{L}] = \text{Tr}[\mathbb{S}_\theta \mathbb{X} \mathbb{X}^\top] = \frac{n}{d},$$

so that $C_{\text{HCRB}}[\rho_m] = C_{\text{SLD}}[\rho_m] = n/d$, as claimed.

For the NHCRB, the X_j are linear combinations of the $d^2 - 1$ GMMs, and can be written as

$$\mathbb{X} = \mathbb{C}^{(2)} \Lambda_d, \quad (52)$$

where $\mathbb{C}^{(2)}$ is a real matrix. Then, $\mathbb{X} \mathbb{X}^\top = \mathbb{C}^{(2)} \Lambda \Lambda^\top \mathbb{C}^{(2)\top}$. We similarly modify $\mathbb{L}^*$ from Eq. (37) to define $\mathbb{L}^{**} := \mathbb{C}^{(2)} \mathbb{L}^* \mathbb{C}^{(2)\top}$, which ensures $\mathbb{L}^{**} \succcurlyeq \mathbb{X} \mathbb{X}^\top$ because of $\mathbb{L}^* \succcurlyeq \Lambda \Lambda^\top$ from Lemma 4 in Appendix E. The NHCRB in Eq. (4) then becomes a minimisation over $\mathbb{L}$ and $\mathbb{C}^{(2)}$. However, if we choose our ansatz $\mathbb{L}^{**}$ for $\mathbb{L}$ and

minimise only over $\mathbb{C}^{(2)}$, we should get a larger value, i.e.,

$$\begin{aligned} C_{\text{NHCRB}}[\rho_m] &= \min_{\mathbb{L}, \mathbb{C}^{(2)}} \left\{ \text{Tr}[\mathbb{S}_\theta \mathbb{L}] \mid \mathbb{L}_{jk} = \mathbb{L}_{kj}, \right. \\ &\quad \left. \mathbb{L}_{jk} \text{ Hermitian}, \right. \\ &\quad \left. \mathbb{L} \succcurlyeq \mathbb{C}^{(2)} \Lambda \Lambda^\top \mathbb{C}^{(2)\top} \right\} \\ &\leq \min_{\mathbb{C}^{(2)}} \left\{ \text{Tr}[\mathbb{S}_\theta \mathbb{L}^{**}] \mid \mathbb{L}^{**} = \mathbb{C}^{(2)} \Lambda \Lambda^\top \mathbb{C}^{(2)\top} \right\} \\ &= \frac{(d+1)n}{d}. \end{aligned} \quad (53)$$

The inequality in Eq. (53) holds because the second minimisation is performed over a subset of the set over which the first minimisation is performed and the last equality follows after some algebra (see Appendix F). Combining this upper bound on the NHCRB with $C_{\text{HCRB}} = n/d$, we get the following theorem.

Theorem 4. *For estimating fewer-than- $n_{\max}$ coefficients of GMMs of the maximally-mixed qudit state ρ_m , the collective enhancement $C_{\text{NHCRB}}[\rho_m]/C_{\text{HCRB}}[\rho_m] \leq d+1$.*

Numerically, we see this ratio actually depends on n : as n increases from 2 to $d^2 - 1$, the ratio increases from 2 to $d+1$. Table 1 in Appendix F depicts this increase, though not monotonic, by listing the two bounds and their ratios for $d = 3$. As proven here, the HCRB only depends on n and d . Interestingly, when the true values of the parameters not being estimated are non-zero, the maximally-mixed state is no longer the ratio-maximising state. However, the n bound in Sec. 5.1 and numerical results in Sec. 3.3 suggest that the maximum enhancement $\mathcal{R}_n^{\text{NH}}$ cannot decrease with increasing n at fixed d , meaning $\mathcal{R}_n^{\text{NH}} \leq \mathcal{R}_{n+1}^{\text{NH}}$. And we have analysed the $n = n_{\max}$ case in depth, so we expect that for any $n < n_{\max}$ model, the same bound of $d+1$ should hold, i.e., $\mathcal{R}_n^{\text{NH}} \leq \mathcal{R}_{n_{\max}}^{\text{NH}} \leq d+1$.

5.6 True Ratio for GMM Model using MICRB

To establish an upper bound to the true collective enhancement $\mathcal{R}^{\text{MI}}[\{\rho_\theta\}]$, we use the MICRB, which quantifies the optimal precision attainable with separable measurements [36, 38, 39]. The MICRB is formulated in Ref. [39] as a conic optimisation over an $(n+1)d \times (n+1)d$ -sized operator X that lies in a separable cone, $\mathcal{S}_{\text{SEP}}$, within

⁸See Table 1 in Appendix F

the space of positive operators of this size (see definition in Eq. (147) in Appendix I). Additionally, the operator X satisfies linear trace constraints arising from the POVM condition and from the local unbiasedness condition (Eqs. (148), (149) in Appendix I). Subject to these constraints, the MICRB minimises the objective $\text{Tr}[(W \otimes \rho_\theta)X]$, where the typically $n \times n$ weight matrix W is modified to be $(n+1) \times (n+1)$ -sized by padding its first row and column with zeros ($W_{1j} = W_{j1} = 0$ for $j \in [n+1]$).

It is important to note that one of the main contributions of Ref. [39] was showing that various precision bounds, including the NHCRB and the HCRB, can all be recast as the minimisation of this same objective but subject to different inclusion constraints on operator X . In particular, the NHCRB was shown to require the positive semi-definiteness of X , whereas the MICRB was shown to require X to also be separable, over an $(n+1)$ -dimensional real space and a d -dimensional complex space. Our key observation is to find that for the linear GMM model, at the maximally-mixed state, a solution to the MICRB is given by

$$X_{\text{sol}} := \begin{pmatrix} \mathbb{1}_d & \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda_1 & \mathbb{L}_{11}^* & \mathbb{L}_{12}^* & \dots & \mathbb{L}_{1n}^* \\ \lambda_2 & \mathbb{L}_{21}^* & \mathbb{L}_{22}^* & \dots & \mathbb{L}_{2n}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_n & \mathbb{L}_{n1}^* & \mathbb{L}_{n2}^* & \dots & \mathbb{L}_{nn}^* \end{pmatrix}, \quad (54)$$

where $\mathbb{L}_{jk}^*$ is the NHCRB-optimal argument presented in Eq. (37) of Sec. 5.3. This connection between the NHCRB solution and the MICRB solution is not surprising, given that two bounds minimise the same objective subject to different constraints, and are equal for the linear GMM model at the maximally-mixed state.

In Appendix I, we rigorously prove that X_{sol} satisfies all the MICRB constraints, including the POVM constraint, the LUB constraints and the separability constraint (Lemma 10). The MICRB objective value for X_{sol} equals the NHCRB, $C_{\text{NHCRB}}[\rho_m] = n(d+1)/d$, as expected. Further, we show that X_{sol} is a feasible candidate (but not necessarily optimal) for the MICRB for arbitrary qudit states, thereby leading to the upper-bound,

$$C_{\text{MI}}[\rho_\theta] \leq n \left(\frac{d+1}{d} \right) - \sum_j \theta_j^2, \quad (55)$$

which was derived for the NHCRB in Eq. (48) of Sec. 5.4. Combining this upper bound for the MICRB with the lower bound $C_{\text{HCRB}}[\rho_\theta] \geq n/d - \sum_j \theta_j^2$, we prove in Theorem 5 of Appendix I that the true maximum ratio $\mathcal{R}^{\text{MI}}[\{\rho_\theta\}]$ in the linear GMM model is upper-bounded by $d+2$.

Theorem 5. *For ONB tomography of arbitrary d -dimensional qudit state ρ_θ , the maximum true collective enhancement $\mathcal{R}^{\text{MI}}[\{\rho_\theta\}] \leq d+2$.*

This theorem reproduces the upper bound derived on the maximum ratio $\mathcal{R}^{\text{NH}}[\{\rho_\theta\}]$ in Theorem 3 and similarly extends to tomography in any ONB. However, as with Theorem 3, we do not expect the upper-bound of $d+2$ to be attainable.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Code availability

The codes that support the findings of this study are available from the corresponding author upon reasonable request.

References

- [1] V. Giovannetti, S. Lloyd, and L. Maccone, *Phys. Rev. Lett.* **96**, 010401 (2006).
- [2] V. Giovannetti, S. Lloyd, and L. Maccone, *Nat. Photonics* **5**, 222–229 (2011).
- [3] M. Szczykulska, T. Baumgratz, and A. Datta, *Adv. Phys.: X* **1**, 621–639 (2016).
- [4] C. M. Caves, *Phys. Rev. D* **23**, 1693 (1981).
- [5] V. Giovannetti, S. Lloyd, and L. Maccone, *Nature* **412**, 417–419 (2001).
- [6] U. Dorner, R. Demkowicz-Dobrzanski, B. J. Smith, J. S. Lundeen, W. Wasilewski, K. Banaszek, and I. A. Walmsley, *Phys. Rev. Lett.* **102**, 040403 (2009).
- [7] M. Kacprowicz, R. Demkowicz-Dobrzański, W. Wasilewski, K. Banaszek, and I. A. Walmsley, *Nat. Photonics* **4**, 357–360 (2010).
- [8] H. Yonezawa, D. Nakane, T. A. Wheatley, K. Iwasawa, S. Takeda, H. Arao, K. Ohki, K. Tsumura, D. W. Berry, T. C. Ralph,

- H. M. Wiseman, E. H. Huntington, and A. Furusawa, *Science* **337**, 1514–1517 (2012).
- [9] M. Tsang, R. Nair, and X.-M. Lu, *Phys. Rev. X* **6**, 031033 (2016).
- [10] M. Paris and J. Rehacek, *Quantum State Estimation*, Lect. Notes Phys. (Springer Berlin Heidelberg, 2004).
- [11] M. Hayashi, *Asymptotic Theory of Quantum Statistical Inference: Selected Papers* (World Scientific, 2005).
- [12] M. G. A. Paris, *Int. J. Quantum Inf.* **07**, 125–137 (2009).
- [13] F. Albarelli, M. Barbieri, M. Genoni, and I. Gianani, *Phys. Lett. A* **384**, 126311 (2020).
- [14] H. Yuen and M. Lax, *IEEE Trans. Inf. Theory* **19**, 740 (1973).
- [15] C. Helstrom and R. Kennedy, *IEEE Trans. Inf. Theory* **20**, 16–24 (1974).
- [16] M. Hayashi, On simultaneous measurement of noncommutative physical values, in *Development of infinite-dimensional noncommutative analysis*, 1099 (RIMS Kokyuroku, Kyoto Univ., 1999) pp. 96–188.
- [17] J. S. Sidhu, Y. Ouyang, E. T. Campbell, and P. Kok, *Phys. Rev. X* **11**, 011028 (2021).
- [18] C. W. Helstrom, *J. Stat. Phys.* **1**, 231–252 (1969).
- [19] J. Yang, S. Pang, Y. Zhou, and A. N. Jordan, *Phys. Rev. A* **100**, 032104 (2019).
- [20] L. Pezzè, M. A. Ciampini, N. Spagnolo, P. C. Humphreys, A. Datta, I. A. Walmsley, M. Barbieri, F. Sciarrino, and A. Smerzi, *Phys. Rev. Lett.* **119**, 130504 (2017).
- [21] L. O. Conlon, J. Suzuki, P. K. Lam, and S. M. Assad, *Npj Quantum Inf.* **7**, 110 (2021).
- [22] S. Massar and S. Popescu, *Phys. Rev. Lett.* **74**, 1259 (1995).
- [23] R. D. Gill and S. Massar, *Phys. Rev. A* **61**, 042312 (2000).
- [24] A. Mansouri, R. A. Abrahao, and J. S. Lundeen, in *Frontiers in Optics + Laser Science 2022 (FIO, LS)* (Optica Publishing Group, 2022) p. FM3B.5.
- [25] S. Zhou, C.-L. Zou, and L. Jiang, *Quantum Sci. Technol.* **5**, 025005 (2020).
- [26] L. O. Conlon, T. Vogl, C. D. Marciniak, I. Pogorelov, S. K. Yung, F. Eilenberger, D. W. Berry, F. S. Santana, R. Blatt, T. Monz, P. K. Lam, and S. M. Assad, *Nat. Phys.* **19**, 351–357 (2023).
- [27] M. A. Ballester, *Phys. Rev. A* **69**, 022303 (2004).
- [28] Z. Hou, J.-F. Tang, J. Shang, H. Zhu, J. Li, Y. Yuan, K.-D. Wu, G.-Y. Xiang, C.-F. Li, and G.-C. Guo, *Nat. Commun.* **9**, 10.1038/s41467-018-03849-x (2018).
- [29] Y. Yuan, Z. Hou, J.-F. Tang, A. Streltsov, G.-Y. Xiang, C.-F. Li, and G.-C. Guo, *Npj Quantum Inf.* **6**, 10.1038/s41534-020-0280-6 (2020).
- [30] L. O. Conlon, F. Eilenberger, P. K. Lam, and S. M. Assad, *Commun. Phys.* **6**, 10.1038/s42005-023-01454-z (2023).
- [31] J. Pauwels, A. Pozas-Kerstjens, F. Del Santo, and N. Gisin, *Phys. Rev. X* **15**, 021013 (2025).
- [32] K. Matsumoto, *J. Phys. A* **35**, 3111–3123 (2002).
- [33] C. Helstrom, *Phys. Lett. A* **25**, 101–102 (1967).
- [34] C. Helstrom, *IEEE Trans. Inf. Theory* **14**, 234 (1968).
- [35] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (Springer, 2011).
- [36] H. Nagaoka, A new approach to Cramér-Rao bounds for quantum state estimation, in *Asymptotic Theory of Quantum Statistical Inference* (World Scientific, 2005) p. 100–112.
- [37] H. Nagaoka, A generalization of the simultaneous diagonalization of Hermitian matrices and its relation to quantum estimation theory, in *Asymptotic Theory of Quantum Statistical Inference* (World Scientific, 2005) p. 133–149.
- [38] M. Hayashi, A linear programming approach to attainable Cramér-Rao type bounds, in *Quantum Communication, Computing, and Measurement*, edited by O. Hirota, A. S. Holevo, and C. M. Caves (Springer US, Boston, MA, 1997) pp. 99–108.
- [39] M. Hayashi and Y. Ouyang, *Quantum* **7**, 1094 (2023).
- [40] H. Zhu, *Quantum State Estimation and Symmetric Informationally Complete POMs*, *Phd thesis*, National University of Singapore (2012).
- [41] L. O. Conlon, J. Suzuki, P. K. Lam, and S. M. Assad, *Phys. Lett. A* **542**, 130445 (2025).

- [42] J. Kahn and M. Guță, *Commun. Math. Phys.* **289**, 597–652 (2009).
- [43] K. Yamagata, A. Fujiwara, and R. D. Gill, *Ann. Stat.* **41**, 2197 (2013).
- [44] Y. Yang, G. Chiribella, and M. Hayashi, *Commun. Math. Phys.* **368**, 223–293 (2019).
- [45] F. Albarelli, J. F. Friel, and A. Datta, *Phys. Rev. Lett.* **123**, 200503 (2019).
- [46] L. O. Conlon, J. Suzuki, P. K. Lam, and S. Assad, *The Gap Persistence Theorem Between Nagaoka-Hayashi Bound and Holevo Bound for Quantum Multiparameter Estimation*, Tech. Rep. 123, 14 (IEICE, 2023).
- [47] J. Zhang and J. Suzuki [10.48550/arxiv.2403.20131](https://arxiv.org/abs/2403.20131) (2024).
- [48] B. Li, L. O. Conlon, P. K. Lam, and S. M. Assad, *Phys. Rev. A* **108**, 032605 (2023).
- [49] M. Gell-Mann, *Phys. Rev.* **125**, 1067 (1962).
- [50] Y. Watanabe, T. Sagawa, and M. Ueda, *Phys. Rev. A* **84**, 042121 (2011).
- [51] J. Suzuki, *J. Math. Phys.* **57**, 042201 (2016).
- [52] L. O. Conlon, J. Suzuki, P. K. Lam, and S. M. Assad [10.48550/arxiv.2208.07386](https://arxiv.org/abs/2208.07386) (2022).
- [53] A. Carollo, B. Spagnolo, A. A. Dubkov, and D. Valenti, *J. Stat. Mech.: Theory Exp.* **2019** (9), 094010.
- [54] M. Tsang, F. Albarelli, and A. Datta, *Phys. Rev. X* **10**, 031023 (2020).
- [55] N. Li, C. Ferrie, J. A. Gross, A. Kalev, and C. M. Caves, *Phys. Rev. Lett.* **116**, 180402 (2016).
- [56] H. Zhu and M. Hayashi, *Phys. Rev. Lett.* **120**, 030404 (2018).
- [57] A. Candeloro, Z. Pazhotan, and M. G. A. Paris, *Quantum Sci. Technol.* **9**, 045045 (2024).
- [58] S. Ragy, M. Jarzyna, and R. Demkowicz-Dobrzański, *Phys. Rev. A* **94**, 052108 (2016).
- [59] F. Belliard and V. Giovannetti, *New J. Phys.* **23**, 063055 (2021).
- [60] L. O. Conlon, P. K. Lam, and S. M. Assad, *Entropy* **25**, 1122 (2023).
- [61] A. Fujiwara and H. Nagaoka, *J. Math. Phys.* **40**, 4227–4239 (1999).
- [62] D. R. Cox and N. Reid, *J. R. Stat. Soc., B* **49**, 1 (1987).
- [63] J. Suzuki, *Entropy* **21**, [10.3390/e21070703](https://arxiv.org/abs/10.3390/e21070703) (2019).
- [64] R. Demkowicz-Dobrzański, W. Górecki, and M. Guță, *J. Phys. A* **53**, 363001 (2020).
- [65] A. Fujiwara and K. Yamagata, *Ann. Stat.* **51**, [10.1214/23-aos2285](https://arxiv.org/abs/10.1214/23-aos2285) (2023).
- [66] R. A. Bertlmann and P. Krammer, *J. Phys. A* **41**, 235303 (2008).
- [67] G. M. D’Ariano, L. Maccone, and M. G. A. Paris, *J. Phys. A* **34**, 93–103 (2000).
- [68] G. M. D’Ariano, P. Perinotti, and M. F. Sacchi, *J. Opt. B: Quantum Semiclass. Opt.* **6**, S487–S491 (2004).
- [69] H. E. Haber, *SciPost Phys. Lect. Notes*, 21 (2021).
- [70] V. I. Borodulin, R. N. Rogalyov, and S. R. Slabospitskii, *Core 3.2 (compendium of relations, version 3.2)* (2022), [arXiv:1702.08246 \[hep-ph\]](https://arxiv.org/abs/1702.08246).
- [71] A. E. Rastegin, *Eur. Phys. J. D* **67**, 269 (2013).
- [72] G. Tóth and I. Apellaniz, *J. Phys. A* **47**, 424006 (2014).

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Author Contributions

L.O.C. conceived the project. A.D., L.O.C., J.S., S.K.Y. and S.M.A. developed the theory and designed the numerical experiments. A.D. and L.O.C. wrote the proofs and A.D. performed the numerical simulations. A.D., L.O.C. and S.M.A. wrote the manuscript. All authors contributed to discussions regarding the results in this paper. S.M.A. and P.K.L. supervised the project.

Competing Interests

All authors declare no financial or non-financial competing interests.

A Gell-Mann Matrices and Tomography via Parameter Estimation

The main advantage of the Bloch representation for qubits,

$$\rho = \frac{1}{2} \left(\mathbb{1}_2 + \sum_{j \in \{x,y,z\}} \theta_j \sigma_j \right), \quad (56)$$

where $\mathcal{P} := \{\sigma_x, \sigma_y, \sigma_z\}$ is the Pauli basis, is the convenience of working with the real-valued Bloch vector $\theta := \{\theta_x, \theta_y, \theta_z\} \in \mathbb{R}^3$ instead of the equivalent complex operator $\rho \in \mathbb{C}^{2 \times 2}$. The same convenience is availed in three dimensions by replacing $\mathcal{P}$ with the GMMs, $\Lambda_3 := \{\lambda_j\}_{j=1}^8$. These constitute an orthonormal basis over the reals for the space of 3×3 traceless Hermitian matrices and generalise the Pauli matrices to three dimensions. So for a qutrit state ρ , we can write

$$\rho = \mathbb{1}_3/3 + \sum_{j=1}^8 \theta_j \lambda_j \quad (57)$$

with

$$\begin{aligned} \lambda_1 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \lambda_2 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \lambda_3 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \lambda_4 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \\ \lambda_5 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, & \lambda_6 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \\ \lambda_7 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, & \lambda_8 &= \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \end{aligned} \quad (58)$$

Note that we choose a different convention in Eq. (57) from that in Eq. (56), and we set $\text{Tr}(\lambda_j \lambda_k) = \delta_{jk}$ instead of the standard $2\delta_{jk}$ in Eq. (58) for convenience.

The eight GMMs in Eq. (58) for $d = 3$ can be extended to $d > 3$ leading to the generalised GMMs Λ_d (that we shall also refer to as GMMs). In fact, Λ_d consists of $\binom{d}{2}$ real, symmetric matrices that generalise σ_x , $\binom{d}{2}$ imaginary, skew-symmetric matrices that generalise σ_y , and $d-1$ real, diagonal matrices that generalise σ_z . In total, we have $d^2 - 1$ matrices, $\{\lambda_j\}_{j=1}^{d^2-1}$, in Λ_d , and, for arbitrary qudit density matrix ρ in d dimensions, we

can write

$$\rho = \mathbb{1}_d/d + \sum_{j=1}^{d^2-1} \theta_j \lambda_j \quad (59)$$

to get a one-to-one map between $\rho \leftrightarrow \theta$. Resultantly, a qudit state estimation or tomography problem can be treated as a parameter estimation problem with θ as the unknown parameter. Note that our convention in Eq. (59) is different from that used in some existing literature [66] but is equivalent up to a re-scaling of the parameters, which leaves the ratio unchanged.

B Proof of Generalised Gell-Mann Matrix Identities

In this appendix, we prove the following identities for $\Lambda_d = \{\lambda_j\}_{j=1}^{d^2-1}$.

1. $\sum_{j \in [n]} \lambda_j^2 = \frac{d^2-1}{d} \mathbb{1}_d$
2. $\sum_{m \in [n]} \lambda_m \lambda_j \lambda_m = -\frac{1}{d} \lambda_j$
3. $\sum_{j,k \in [n]} \lambda_j \lambda_k \lambda_j \lambda_k = -\frac{d^2-1}{d^2} \mathbb{1}_d$

Proof of Identity 1. It is known that $\sum_{j \in [n]} \lambda_j^2$ is a group invariant called the Casimir operator [69]. Thus, $\sum_{j \in [n]} \lambda_j^2 = C \mathbb{1}_d$ for some constant C . We use the trace condition $\text{Tr}(\lambda_j \lambda_k) = \delta_{jk}$

$$\text{Tr} \left(\sum_{j \in [n]} \lambda_j^2 \right) = \sum_{j \in [n]} \text{Tr}(\lambda_j^2) = n = Cd, \quad (60)$$

which implies $C = n/d$, proving

$$\sum_{j \in [n]} \lambda_j^2 = \frac{d^2-1}{d} \mathbb{1}_d.$$

□

Proof of Identity 2. For this proof, we use some properties of GMMs from Ref. [70] (see page 17, Sec. 4.6 *Gell-Mann Matrices in n-dimensions*). Writing the product $\lambda_m \lambda_j$ in terms of the commutator and the anti-commutator, we get

$$\begin{aligned} 2\lambda_m \lambda_j &= \{\lambda_m, \lambda_j\} + [\lambda_m, \lambda_j] \\ &= \frac{2}{d} \delta_{mj} \mathbb{1}_d + \sum_c d_{mjc} \lambda_c + \sum_c i f_{mjc} \lambda_c, \end{aligned} \quad (61)$$

where $d_{jkl} = \text{Tr}(\{\lambda_j, \lambda_k\} \lambda_l)$ and $f_{jkl} = -i \text{Tr}([\lambda_j, \lambda_k] \lambda_l)$ are the fully-symmetric

and fully-antisymmetric structure constants⁹ of $\mathfrak{su}(d)$ [69, 70]. Repeating the process after right-multiplying Eq. (61) by λ_m ,

$$\begin{aligned} 2\lambda_m\lambda_j\lambda_m &= \frac{2\delta_{mj}}{d}\lambda_m + \sum_c (d_{mjc} + if_{mjc})\lambda_c\lambda_m \\ &= \frac{2\delta_{mj}}{d}\lambda_m + \frac{1}{2}\sum_c (d_{mjc} + if_{mjc})\left(\frac{2\delta_{cm}}{d}\mathbb{1}_d \right. \\ &\quad \left. + \sum_p (d_{cmp} + if_{cmp})\lambda_p\right) \\ &= \frac{2}{d}\delta_{mj}\lambda_m + \frac{1}{d}(d_{mjm} + if_{mjm})\mathbb{1}_d \\ &\quad + \frac{1}{2}\sum_{c,p}(d_{mjc} + if_{mjc})(d_{cmp} + if_{cmp})\lambda_p \end{aligned} \quad (62)$$

Due to anti-symmetry, $f_{mjm} = 0$, and

$$\begin{aligned} (d_{mjc} + if_{mjc})(d_{cmp} + if_{cmp}) \\ = [(d_{mjc}d_{cmp} - f_{mjc}f_{cmp}) \\ + i(d_{mjc}f_{cmp} + f_{mjc}d_{cmp})]. \end{aligned} \quad (63)$$

Thus,

$$\sum_m \lambda_m\lambda_j\lambda_m = \frac{1}{d}\lambda_j + \frac{1}{2d}\underbrace{\sum_m d_{mjm}}_{\textcircled{1}}\mathbb{1}_d \quad (64)$$

$$+ \frac{1}{4}\sum_p \left[\underbrace{\sum_{m,c} d_{mjc}d_{cmp}}_{\textcircled{2}} - \underbrace{\sum_{m,c} f_{mjc}f_{cmp}}_{\textcircled{3}} \right] \quad (65)$$

$$+ i\left(\underbrace{\sum_{m,c} d_{mjc}f_{cmp}}_{\textcircled{4}} + \underbrace{\sum_{m,c} f_{mjc}d_{cmp}}_{\textcircled{5}} \right)\lambda_p. \quad (66)$$

Below we evaluate terms $\textcircled{1}, \textcircled{2}, \textcircled{3}, \textcircled{4}$ and $\textcircled{5}$ one by one, using properties of the GMMs listed in Ref. [70].

$$\begin{aligned} \textcircled{1}: \sum_m d_{mjm} &= \sum_m d_{jmm} \\ &= \frac{1}{4}\text{Tr}\left[\lambda_j \sum_m \{\lambda_m, \lambda_m\}\right] = \frac{1}{2}\text{Tr}\left[\lambda_j \frac{d^2 - 1}{d}\mathbb{1}_d\right] \\ &= \frac{d^2 - 1}{2d}\text{Tr}(\lambda_j) = 0 \end{aligned} \quad (67)$$

⁹Owing to different normalisation conventions, our d_{abc} and f_{abc} are scaled up by a factor of $\sqrt{2}$ compared to Ref. [70].

$$\textcircled{2}: \sum_{m,c} d_{mjc}d_{cmp} = \sum_{m,c} d_{jmc}d_{pmc} = 2\frac{d^2 - 4}{d}\delta_{jp} \quad (68)$$

$$\textcircled{3}: \sum_{m,c} f_{mjc}f_{cmp} = \sum_{m,c} f_{jmc}f_{pmc} = 2d\delta_{jp} \quad (69)$$

The Jacobi identity [70] reads

$$\sum_k d_{abk}f_{kcl} + d_{bck}f_{kal} + d_{cak}f_{kbl} = 0.$$

If we set $a = c$ and then sum over a , we get

$$2\sum_{a,k} d_{bak}f_{lak} = \sum_k \left(\sum_a d_{aak} \right) f_{blk}.$$

Using this to simplify $\textcircled{4}$, we get

$$\begin{aligned} \textcircled{4}: \sum_{m,c} d_{mjc}f_{cmp} &= -\sum_{m,c} d_{jmc}f_{pmc} \\ &= -\frac{1}{2}\sum_{m,c} d_{mmc}f_{jpc} \end{aligned} \quad (70)$$

and, similarly, for $\textcircled{5}$ we get

$$\begin{aligned} \textcircled{5}: \sum_{m,c} f_{mjc}d_{cmp} &= -\sum_{m,c} d_{pmc}f_{jmc} \\ &= -\frac{1}{2}\sum_{m,c} d_{mmc}f_{pjc} = \frac{1}{2}\sum_{m,c} d_{mmc}f_{jpc}, \end{aligned} \quad (71)$$

so that

$$\textcircled{4} + \textcircled{5}: \sum_{m,c} (d_{mjc}f_{cmp} + f_{mjc}d_{cmp}) = 0. \quad (72)$$

Combining the expressions for $\textcircled{1}, \textcircled{2}$ and $\textcircled{3}$, we get

$$\begin{aligned} \sum_m \lambda_m\lambda_j\lambda_m &= \frac{1}{d}\lambda_j + \frac{1}{2}\sum_p \left(\frac{d^2 - 4}{d} - d \right) \delta_{jp}\lambda_p \\ &= \frac{1}{d}\lambda_j - \frac{2}{d}\lambda_j = -\frac{1}{d}\lambda_j, \end{aligned} \quad (73)$$

thus proving Identity 2. $\square$

Corollary 1. By linearity, any $d \times d$ traceless Hermitian matrix A satisfies

$$\sum_m \lambda_m A \lambda_m = -\frac{1}{d}A. \quad (74)$$

Corollary 2. For any $j, k \in [n]$,

$$\sum_m \lambda_m \lambda_j \lambda_k \lambda_m = \delta_{jk} \mathbb{1}_d - \frac{1}{d} \lambda_j \lambda_k. \quad (75)$$

Proof. To see this, start with assuming $j \neq k$. From Eq. (61), this implies $\lambda_j \lambda_k$ is traceless Hermitian, and thus from Corollary 1,

$$\sum_m \lambda_m \lambda_j \lambda_k \lambda_m = -\frac{1}{d} \lambda_j \lambda_k.$$

Similarly, for $j = k$, $\lambda_j \lambda_k - \frac{1}{d} \mathbb{1}_d$ is a traceless, Hermitian matrix (see Eq. (61)). Thus, from Corollary 1,

$$\sum_m \lambda_m \left(\lambda_j^2 - \frac{1}{d} \mathbb{1}_d \right) \lambda_m = -\frac{1}{d} \left(\lambda_j^2 - \frac{1}{d} \mathbb{1}_d \right),$$

so that

$$\sum_m \lambda_m \lambda_j^2 \lambda_m = \mathbb{1}_d - \frac{1}{d} \lambda_j^2. \quad (76)$$

This concludes the proof of Corollary 2. $\square$

Proof of Identity 3. Using Identity 2, we have

$$\sum_j \lambda_j \lambda_k \lambda_j \lambda_k = \left(\sum_j \lambda_j \lambda_k \lambda_j \right) \lambda_k = -\frac{1}{d} \lambda_k^2. \quad (77)$$

Summing over k and using Identity 1, we find

$$\sum_{j,k} \lambda_j \lambda_k \lambda_j \lambda_k = -\frac{1}{d} \sum_k \lambda_k^2 = -\frac{d^2 - 1}{d^2} \mathbb{1}_d, \quad (78)$$

which proves Identity 3. $\square$

C Deferred Proofs: Proof of Lemma 2

Proof of Lemma 2. The proof is segmented into three parts. (i) First we establish that the X_j are completely and uniquely determined by the local unbiasedness conditions to be $X_j = \lambda_j$. This can be traced back to the trace orthonormality $\text{Tr}[\lambda_j \lambda_k] = \delta_{jk}$ of GMMs. (ii) We establish a lower bound on C_{HCRB} . (iii) We show this lower bound is achieved by valid choices of arguments $\mathbb{S}_\theta$ and $\mathbb{L}$, implying C_{HCRB} is equal to the lower bound.

Part (i): The local unbiased conditions

$$\text{Tr}[\rho_\theta X_j] = \theta_j \quad \text{and} \quad \text{Tr}[\partial_j \rho_\theta X_k] = \delta_{jk} \quad (79)$$

for $j, k \in [d^2 - 1]$ at $\theta = 0$ become

$$\text{Tr}[X_j] = 0 \quad \text{and} \quad \text{Tr}[\lambda_j X_k] = \delta_{jk}. \quad (80)$$

It is simple to verify that the only solution to this is $X_j = \lambda_j$. By virtue of being traceless and Hermitian, each X_j must be a linear combination of λ_k s, i.e.,

$$X_j = \sum_k c_{jk} \lambda_k \quad c_{jk} \in \mathbb{R}, \quad (81)$$

which then means c_{jk} must satisfy $\text{Tr}[\lambda_j X_k] = c_{kj} = \delta_{jk}$, implying $X_j = \lambda_j$.

Part (ii): Tracing over the parameter indices (Tr) in

$$\text{Tr}[\mathbb{S}_\theta \mathbb{L}] \succcurlyeq \text{Tr}[\mathbb{S}_\theta \mathbb{X} \mathbb{X}^\top] \quad (82)$$

yields

$$\text{Tr}[\mathbb{S}_\theta \mathbb{L}] \geq \text{Tr}[\mathbb{S}_\theta \mathbb{X} \mathbb{X}^\top]. \quad (83)$$

Part (iii): The choice $\mathbb{L} = \mathbb{X} \mathbb{X}^\top$ leads to

$$(\text{Tr}[\mathbb{S}_\theta \mathbb{L}])_{j,k} = \text{Tr}[\mathbb{1}/d \lambda_j \lambda_k] = \delta_{jk}/d, \quad (84)$$

which is real, symmetric and has trace (over parameter indices)

$$\text{Tr}[\mathbb{S}_\theta \mathbb{L}] = \text{Tr}[\mathbb{S}_\theta \mathbb{X} \mathbb{X}^\top] = \frac{d^2 - 1}{d}. \quad (85)$$

Finally, from *part (ii)* we know that a lower $\text{Tr}[\mathbb{S}_\theta \mathbb{L}]$ is not possible, thus proving Eq. (36). $\square$

Remark 1. When there exists a unique set of LUB operators $\mathbb{X}$, the inequality in Eq. (83) holds regardless of whether $\mathbb{L} = \mathbb{X} \mathbb{X}^\top$ is a valid choice according to the HCRB constraints in Eq. (7). We utilise this fact to solve the full-parameter linear GMM model for arbitrary states in Sec. 5.4 and for arbitrary weight matrices in Appendix H.

D SLD & RLD CRBs

The two simplest quantum Cramér-Rao bounds, the SLD and the RLD, generalise the logarithmic derivative of a parameterised probability distribution,

$$\partial_\theta p_\theta(x) = p_\theta(x) \partial_\theta [\log(p_\theta(x))] , \quad (86)$$

to linear operators acting on the density matrix ρ_θ . The SLD version produces Hermitian operators $\{L_j^{(\text{SLD})}\}$ and the RLD version produces operators $\{L_j^{(\text{RLD})}\}$ defined implicitly via

$$2\partial_j \rho_\theta =: L_j^{(\text{SLD})} \rho_\theta + \rho_\theta L_j^{(\text{SLD})}, \quad (87)$$

$$\partial_j \rho_\theta =: \rho_\theta L_j^{(\text{RLD})}. \quad (88)$$

Once Eqs. (87) and (88) are solved for $\{L_j^{(\text{SLD})}\}$ and $\{L_j^{(\text{RLD})}\}$, the corresponding QFIs can be computed using

$$[J^{(\text{SLD})}]_{jk} := \text{Re} \left[\text{Tr} \left[\rho_\theta L_j^{(\text{SLD})} L_k^{(\text{SLD})} \right] \right], \quad (89)$$

$$[J^{(\text{RLD})}]_{jk} := \text{Tr} \left[\rho_\theta L_k^{(\text{RLD})} L_j^{(\text{RLD})\dagger} \right]. \quad (90)$$

Notably, $J^{(\text{SLD})}$ is real and symmetric and $J^{(\text{RLD})}$ is complex and Hermitian. Finally, the traced versions of the SLD and RLD QFI matrix inequalities $V_\theta \succcurlyeq J^{(\text{SLD})^{-1}}$ and $V_\theta \succcurlyeq J^{(\text{RLD})^{-1}}$ yield the scalar SLD and RLD CRBs

$$\text{Tr}(V_\theta) \geq C_{\text{SLD}} \quad \& \quad \text{Tr}(V_\theta) \geq C_{\text{RLD}}, \quad (91)$$

with

$$C_{\text{SLD}} = \text{Tr} \left[J^{(\text{SLD})^{-1}} \right], \quad (92)$$

$$C_{\text{RLD}} = \text{Tr} \left[\text{Re} \left[J^{(\text{RLD})} \right]^{-1} \right] + \left\| \text{Im} \left[J^{(\text{RLD})} \right]^{-1} \right\|_1, \quad (93)$$

where $\|X\|_1 := \text{Tr}(\sqrt{X^\dagger X})$ denotes the trace norm. The SLD and RLD CRBs are not attainable in general, especially in multi-parameter contexts. For more details on the SLD and RLD CRB, see Ref. [13].

For the model in Sec. 5.3, $\rho_\theta^* = \mathbb{1}_d/d$ and $\partial_j \rho_\theta = \lambda_j$, so Eqs. (87) & (88) become

$$\begin{aligned} 2\lambda_j &= 2/d L_j^{\text{SLD}}, \\ \lambda_j &= 1/d L_j^{\text{RLD}} \end{aligned} \quad (94)$$

implying $L_j^{\text{SLD}} = L_j^{\text{RLD}} = d\lambda_j$. A direct computation of Eqs. (89), (90), (92) & (93) then yields the QFIs

$$J^{(\text{SLD})} = J^{(\text{RLD})} = \begin{bmatrix} d & 0 & \dots & 0 \\ 0 & d & \dots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \dots & 0 & d \end{bmatrix}_{n \times n}, \quad (95)$$

and the scalar CRBs

$$C_{\text{SLD}} = C_{\text{RLD}} = \frac{d^2 - 1}{d}, \quad (96)$$

as in main text Eqs. (33) and (34).

E Deferred Proofs: Proof of Lemma 3

E.1 SDP Formulation and Dual Problem

The SDP formulation of the NHCRB [21] is

$$\begin{aligned} C_{\text{NHCRB}} &:= \min_{\mathbb{Y}} \left[\text{Tr}[\mathbb{F}_0 \mathbb{Y}] | \text{Tr}[\mathbb{F}_k \mathbb{Y}] = c_k, \right. \\ &\quad \left. \mathbb{F}_0 = \begin{pmatrix} \mathbb{S}_\theta & 0 \\ 0 & 0 \end{pmatrix}, \mathbb{Y} \succcurlyeq 0 \right], \end{aligned} \quad (97)$$

where $\mathbb{F}_k$ are constant matrices and c_k are constants, as defined in Ref. [21] (supplementary note 4). The dual problem to the SDP in Eq. (97) reads

$$\tilde{C}_{\text{NHCRB}} := \max_y \left[\sum_k y_k c_k \mid \sum_k y_k \mathbb{F}_k \preceq \mathbb{F}_0 \right]. \quad (98)$$

In the following two lemmas, we present solutions to the primal and dual problems.

Lemma 4. *The optimal $\mathbb{L}$ for the primal problem in Eq. (4) is*

$$\mathbb{L}_{jk}^* = d^{+1/d+2} (\{\lambda_j, \lambda_k\} + \delta_{jk} \mathbb{1}_d) \quad (99)$$

where $j, k \in [n]$ and $\{\lambda_j, \lambda_k\}$ is the anti-commutator.

Lemma 5. *The optimal y for the dual problem in Eq. (98), y^* , is such that*

$$\sum_k y_k^* \mathbb{F}_k = \begin{bmatrix} 0 & \mathbb{G}_{12}^{(1)} & \dots & \mathbb{G}_{1n}^{(1)} & \mathbb{G}_1^{(2)} \\ \mathbb{G}_{21}^{(1)} & 0 & \dots & \mathbb{G}_{2n}^{(1)} & \vdots \\ \vdots & \vdots & \ddots & \mathbb{G}_{(n-1)n}^{(1)} & \\ \mathbb{G}_{n1}^{(1)} & \dots & \mathbb{G}_{n(n-1)}^{(1)} & 0 & \mathbb{G}_n^{(2)} \\ \mathbb{G}_1^{(2)} & \dots & & \mathbb{G}_n^{(2)} & \mathbb{G}^{(3)} \end{bmatrix} \quad (100)$$

with

$$\begin{aligned} \mathbb{G}_{jk}^{(1)} &= -1/d [\lambda_j, \lambda_k] \\ \mathbb{G}_j^{(2)} &= d^{+1/d} \lambda_j \\ \mathbb{G}^{(3)} &= -(d^2-1)(d+1)/d^2 \mathbb{1}_d, \end{aligned} \quad (101)$$

where $j, k \in [n]$ and $[\lambda_j, \lambda_k]$ is the commutator.

The proof of Lemmas 4 and 5 is broken up into the following three subsections. In Subsec. E.2, we prove the feasibility of $\mathbb{L}^*$ from Lemma 4. In Subsec. E.3 we prove the feasibility of y^* from Lemma 5. Finally in Subsec. E.4 we prove that the primal-objective value from $\mathbb{L}^*$ equals the dual objective value from y^* , thus establishing their optimality and proving Lemma 3 from the main text.

E.2 Feasibility of $\mathbb{L}^*$ for the Primal Problem

The $\mathbb{L}_{jk}^*$ from Lemma 4 is easily seen to be symmetric in j and k , meaning $\mathbb{L}_{jk}^* = \mathbb{L}_{kj}^*$. $\mathbb{L}_{jk}^*$ is also seen to be Hermitian. To show $\mathbb{L}^*$ is feasible, it only remains to show $\mathbb{L}^* - \mathbb{X}\mathbb{X}^\top \succcurlyeq 0$. Writing $\mathbb{L}^* - \mathbb{X}\mathbb{X}^\top$ as a block matrix,

$$\begin{aligned} (\mathbb{L}^* - \mathbb{X}\mathbb{X}^\top)_{jk} &= \frac{d+1}{d+2} (\{\lambda_j, \lambda_k\} + \delta_{jk} \mathbb{1}_d) - \lambda_j \lambda_k \\ &= \frac{d+1}{d+2} (\delta_{jk} \mathbb{1}_d + \mathbb{N}_{1jk} - \mathbb{N}_{2jk}) \end{aligned} \quad (102)$$

$$\begin{bmatrix} 0 & \frac{1}{d}[\lambda_2, \lambda_1] & \dots & \frac{1}{d}[\lambda_n, \lambda_1] & \frac{d+1}{d}\lambda_1 \\ \frac{1}{d}[\lambda_1, \lambda_2] & 0 & \dots & \frac{1}{d}[\lambda_n, \lambda_2] & \vdots \\ \vdots & \vdots & \ddots & \frac{1}{d}[\lambda_n, \lambda_{n-1}] & \frac{d+1}{d}\lambda_n \\ \frac{1}{d}[\lambda_1, \lambda_n] & \dots & \frac{1}{d}[\lambda_{n-1}, \lambda_n] & 0 & -\frac{n(d+1)}{d^2} \mathbb{1}_d \\ \frac{d+1}{d}\lambda_1 & \dots & \frac{d+1}{d}\lambda_n & -\frac{n(d+1)}{d^2} \mathbb{1}_d & \end{bmatrix}, \quad (104)$$

satisfies $\mathbb{F}_0 - \sum_k y_k^* \mathbb{F}_k \succcurlyeq 0$. Note that this $\sum_k y_k^* \mathbb{F}_k$ corresponds to y_j^* values

$$y_j^{(1)} = 0, \quad y_{jk}^{(2)} = \frac{d+1}{d} \delta_{jk}, \quad y_{jk}^{(3)} = 0 \quad (105)$$

$$y_{jkl}^{(4)} = -f_{jkl}/d, \quad y_j^{(5)} = -\frac{n(d+1)}{d\sqrt{d}} \delta_{j1} \quad (106)$$

we see that we need to prove $\mathbb{N} := \mathbb{1}_{nd} + \mathbb{N}_1 - \mathbb{N}_2 \succcurlyeq 0$, where we have defined block matrices

$$(\mathbb{N}_1)_{jk} := \lambda_k \lambda_j \quad \& \quad (\mathbb{N}_2)_{jk} := \frac{\lambda_j \lambda_k}{d+1}.$$

We first evaluate $(\mathbb{N}_1 - \mathbb{N}_2)^2 = \mathbb{N}_1^2 + \mathbb{N}_2^2 - \mathbb{N}_1 \mathbb{N}_2 - \mathbb{N}_2 \mathbb{N}_1$ to find

$$\begin{aligned} (\mathbb{N}_1^2)_{jk} &= \delta_{jk} \mathbb{1}_d - 1/d \lambda_j \lambda_k \\ (\mathbb{N}_2^2)_{jk} &= \frac{d-1}{d(d+1)} \lambda_j \lambda_k \\ (\mathbb{N}_1 \mathbb{N}_2)_{jk} &= -\frac{1}{d(d+1)} \lambda_j \lambda_k \\ (\mathbb{N}_2 \mathbb{N}_1)_{jk} &= -\frac{1}{d(d+1)} \lambda_j \lambda_k, \end{aligned} \quad (103)$$

where we have used the identities proven in Appendix B. Combining these results we arrive at $((\mathbb{N}_1 - \mathbb{N}_2)^2)_{jk} = \delta_{jk} \mathbb{1}_d$ which means $(\mathbb{N}_1 - \mathbb{N}_2)^2 = \mathbb{1}_{nd}$. From this, and using that $\mathbb{N}_1 - \mathbb{N}_2$ is Hermitian, we can conclude that the eigenvalues of $\mathbb{N}_1 - \mathbb{N}_2$ are ± 1 . Hence the eigenvalues of $\mathbb{N} = \mathbb{1}_{nd} + \mathbb{N}_1 - \mathbb{N}_2$ are 0 and 2. This proves that $\mathbb{N}$ is a positive semi-definite operator, and that $\mathbb{L}^* - \mathbb{X}\mathbb{X}^\top \geq 0$. $\square$

E.3 Feasibility of y^* for the Dual Problem

We need to show that the matrix $\sum_k y_k^* \mathbb{F}_k$, which explicitly is

whereas the corresponding c_j values are

$$c_j^{(1)} = 0, \quad c_{jk}^{(2)} = 2\delta_{jk}, \quad c_{jk}^{(3)} = 0 \quad (107)$$

$$c_{jkl}^{(4)} = 0, \quad c_j^{(5)} = \sqrt{d} \delta_{j1} \quad (108)$$

so that the dual objective value is

$$\begin{aligned} \sum_j 2y_{jj}^{(2)} + \sqrt{d} y_j^{(5)} &= \frac{2n(d+1)}{d} - \frac{n(d+1)}{d} \\ &= \frac{n(d+1)}{d}. \end{aligned} \quad (109)$$

To show this y^* is feasible, note that proving $\mathbb{F}_0 - \sum_k y_k^* \mathbb{F}_k \succcurlyeq 0$ is equivalent to showing

$$\begin{bmatrix} \mathbb{1}_d & [\lambda_1, \lambda_2] & \dots & [\lambda_1, \lambda_n] & -(d+1)\lambda_1 \\ [\lambda_2, \lambda_1] & \mathbb{1}_d & \dots & [\lambda_2, \lambda_n] & \vdots \\ \vdots & \vdots & \ddots & [\lambda_{n-1}, \lambda_n] & \\ [\lambda_n, \lambda_1] & \dots & [\lambda_n, \lambda_{n-1}] & \mathbb{1}_d & -(d+1)\lambda_n \\ -(d+1)\lambda_1 & \dots & & -(d+1)\lambda_n & \frac{n(d+1)}{d}\mathbb{1}_d \end{bmatrix}$$

is positive semi-definite. Using Schur's comple-

ment lemma, this can be simplified to showing

$$\begin{bmatrix} \mathbb{1}_d & [\lambda_1, \lambda_2] & \dots & [\lambda_1, \lambda_n] \\ [\lambda_2, \lambda_1] & \mathbb{1}_d & \dots & [\lambda_2, \lambda_n] \\ \vdots & \vdots & \ddots & [\lambda_{n-1}, \lambda_n] \\ [\lambda_n, \lambda_1] & \dots & [\lambda_n, \lambda_{n-1}] & \mathbb{1}_d \end{bmatrix} - \frac{d}{d-1} \begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \dots & \lambda_1\lambda_n \\ \lambda_2\lambda_1 & \ddots & \dots & \vdots \\ \vdots & \vdots & & \vdots \\ \lambda_n\lambda_1 & \lambda_n\lambda_2 & \dots & \lambda_n^2 \end{bmatrix} \succcurlyeq 0 \quad (110)$$

We rewrite the left hand side of Eq. (110) in the block-matrix representation as

whereas

$$\begin{aligned} & \delta_{jk}\mathbb{1}_d + [\lambda_j, \lambda_k] - \frac{d}{d-1}\lambda_j\lambda_k \\ &= \delta_{jk}\mathbb{1}_d - \frac{1}{d-1}\lambda_j\lambda_k - \lambda_k\lambda_j \\ &= (\mathbb{1}_{nd})_{jk} - \left((\mathbb{M}_1)_{jk} + (\mathbb{M}_2)_{jk} \right), \end{aligned}$$

where we have defined

$$(\mathbb{M}_1)_{jk} := \frac{1}{d-1}\lambda_j\lambda_k \quad \& \quad (\mathbb{M}_2)_{jk} := \lambda_k\lambda_j. \quad (111)$$

Thus, we finally need to prove the following theorem to establish the feasibility of y^* .

Lemma 6. *The operator $\mathbb{M} := \mathbb{1}_{nd} - (\mathbb{M}_1 + \mathbb{M}_2)$ is positive semi-definite.*

Before we can prove Lemma 6, we first need to prove the following two lemmas.

Lemma 7. $\mathbb{M}_1$ and $\mathbb{M}_2$ commute, i.e., $\mathbb{M}_1\mathbb{M}_2 = \mathbb{M}_2\mathbb{M}_1$.

Proof. Using Identity 2 from Appendix B, we have

$$\begin{aligned} (\mathbb{M}_1\mathbb{M}_2)_{jk} &= \sum_l (\mathbb{M}_1)_{jl}(\mathbb{M}_2)_{lk} \\ &= \frac{1}{d-1} \sum_l \lambda_j\lambda_l\lambda_k\lambda_l \\ &= \frac{1}{d-1} \lambda_j \left(\frac{-1}{d} \lambda_k \right) \\ &= -\frac{1}{d(d-1)} \lambda_j\lambda_k, \end{aligned} \quad (112)$$

$$\begin{aligned} (\mathbb{M}_2\mathbb{M}_1)_{jk} &= \sum_l (\mathbb{M}_2)_{jl}(\mathbb{M}_1)_{lk} \\ &= \frac{1}{d-1} \sum_l \lambda_l\lambda_j\lambda_l\lambda_k \\ &= \frac{1}{d-1} \left(\frac{-1}{d} \lambda_j \right) \lambda_k \\ &= -\frac{1}{d(d-1)} \lambda_j\lambda_k. \end{aligned} \quad (113)$$

Hence $\mathbb{M}_1\mathbb{M}_2 = \mathbb{M}_2\mathbb{M}_1$, which also implies that $\mathbb{M}_1$ and $\mathbb{M}_2$ share some eigenvectors. $\square$

Lemma 8. $\mathbb{M}_1 + \mathbb{M}_2$ satisfies $(\mathbb{M}_1 + \mathbb{M}_2)^2 = \mathbb{1}_{nd}$ or, equivalently,

$$\left[(\mathbb{M}_1 + \mathbb{M}_2)^2 \right]_{jk} = \delta_{jk}\mathbb{1}_d$$

where $j, k \in [n]$.

Proof.

$$\begin{aligned}
& \left[(\mathbb{M}_1 + \mathbb{M}_2)^2 \right]_{jk} \\
&= \sum_l (\mathbb{M}_1 + \mathbb{M}_2)_{jl} (\mathbb{M}_1 + \mathbb{M}_2)_{lk} \\
&= \sum_l \left(\frac{1}{d-1} \lambda_j \lambda_l + \lambda_l \lambda_j \right) \left(\frac{1}{d-1} \lambda_l \lambda_k + \lambda_k \lambda_l \right) \\
&= \frac{1}{(d-1)^2} \lambda_j \left(\sum_l \lambda_l^2 \right) \lambda_k + \sum_l \lambda_l \lambda_j \lambda_k \lambda_l \\
&+ \frac{1}{d-1} \left[\lambda_j \left(\sum_l \lambda_l \lambda_k \lambda_l \right) + \left(\sum_l \lambda_l \lambda_j \lambda_l \right) \lambda_k \right]
\end{aligned}$$

so using Corollary 2 and Identities 1 and 2,

$$\begin{aligned}
&= \delta_{jk} \mathbb{1}_d \\
&+ \left(\frac{d^2 - 1}{d(d-1)^2} - \frac{1}{d} - \frac{1}{d(d-1)} - \frac{1}{d(d-1)} \right) \lambda_j \lambda_k \\
&= \delta_{jk} \mathbb{1}_d.
\end{aligned}$$

□

Now we can prove Lemma 6 as follows.

Proof of Lemma 6. From Lemma 8, the eigenvalues of $(\mathbb{M}_1 + \mathbb{M}_2)^2$ must all be 1. As $(\mathbb{M}_1 + \mathbb{M}_2)$ is Hermitian, its eigenvalues must be ± 1 . It follows that the eigenvalues of $\mathbb{M} = \mathbb{1}_{nd} - (\mathbb{M}_1 + \mathbb{M}_2)$ must be either 2 or 0. Hence $\mathbb{M}$, being a Hermitian matrix with non-negative eigenvalues, must be positive semi-definite. □

E.4 Optimality of Solutions & Proof of Lemma 3

Proof of Lemma 3. Note that, by direct calculation,

$$\text{Tr}(\mathbb{S}_\theta \mathbb{L}^*) = \sum_k y_k^* c_k = \frac{(d^2 - 1)(d + 1)}{d}. \quad (114)$$

In other words, $\mathbb{L}^*$ is primal-feasible and y^* is dual-feasible and the primal value equals the dual value. This lets us conclude that $(d^2 - 1)(d + 1)/d$ is the true optimal value of the primal and dual problems, and that $\mathbb{L}^*$ and y^* are optimal solutions to the primal and dual problems, respectively. As a result, we have

$$C_{\text{NHCRB}} = \frac{(d^2 - 1)(d + 1)}{d}. \quad (115)$$

□

E.5 Attainability of NHCRB via SIC POVMs

Lemma 9. *The CFI matrix for estimating all GMMs from the maximally-mixed state ρ_m by measuring the SIC POVM in d dimensions is*

$$J = \begin{bmatrix} \frac{d}{d+1} & 0 & \cdots & 0 \\ 0 & \frac{d}{d+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{d}{d+1} \end{bmatrix}_{n \times n}. \quad (116)$$

Proof of Lemma 9. In the multi-parameter case, the CFI matrix J_{jk} ($j, k \in [n]$) is given by

$$J_{jk}[\{\Pi_l\}] = \sum_{l=1}^{d^2} \frac{\text{Tr}[\partial_j \rho_\theta \Pi_l] \text{Tr}[\partial_k \rho_\theta \Pi_l]}{\text{Tr}[\rho_\theta \Pi_l]}, \quad (117)$$

From [71] we have that for any (rank-one) SIC POVM $\{\Pi_l\}_{l=1}^{d^2}$,

$$\sum_{l=1}^{d^2} \text{Tr}[\rho \Pi_l]^2 = \frac{\text{Tr}[\rho^2] + 1}{d(d+1)} \quad (118)$$

for arbitrary density matrix ρ . For the diagonal elements in Eq. (117), substituting $\rho = \mathbb{1}/d + \theta_j \lambda_j$ into Eq. (118) and using $\text{Tr}(\rho^2) = 1/d + \theta_j^2$ gives

$$J_{jj} = d^2 \sum_{l=1}^{d^2} \text{Tr}[\lambda_j \Pi_l]^2 = \frac{d}{d+1}, \quad (119)$$

whereas for the off-diagonal elements, substituting $\rho = \mathbb{1}/d + \theta_j \lambda_j + \theta_k \lambda_k$ into Eq. (118) and using $\text{Tr}(\rho^2) = 1/d + \theta_j^2 + \theta_k^2$ gives

$$J_{jk} = d^2 \sum_{l=1}^{d^2} \text{Tr}[\lambda_j \Pi_l] \text{Tr}[\lambda_k \Pi_l] = 0 \quad (j \neq k), \quad (120)$$

thus proving Eq. (116). □

F Estimating a Subset of GMMs

Consider estimating a subset $\{\lambda_j\}_{j \in K}$ of GMMs from the maximally-mixed state ρ_m . Here K denotes a subset of n indices from 1 to $n_{\max}$ ($K \subseteq [n_{\max}]$, $|K| = n$). We denote the maximum NHCRB-to-HCRB ratio here as $\mathcal{R}_n^{\text{NH}}[\{\rho_m\}]$, referring to the model $\{\rho_m\}$ of estimating n GMM coefficients from ρ_m . Now, the corresponding unbiased operators can be written as

$$X_j = \lambda_j + \sum_{m \in [n_{\max}] \setminus K} c_{jm} \lambda_m, \quad j \in K, \quad c_{jm} \in \mathbb{R}, \quad (121)$$

which follows from the unbiasedness conditions in Eq. (26). Specifically, $\text{Tr}(\partial_j \rho_\theta X_k) = \text{Tr}(\lambda_j X_k) = \delta_{jk}$ forces each X_j to contain a unit contribution from λ_j due to the orthonormality of the GMMs and $\text{Tr}(\rho_\theta X_j) = \theta_j$ implies the only other GMMs contributing to X_j must be the ones not being estimated.

Notice that at the block-matrix level, this can be rewritten as

$$\mathbb{X} = \left[\mathbf{1}_n \mid \mathbb{C}_{n \times (n_{\max} - n)}^{(1)} \right]_{n \times n_{\max}} \Lambda_d, \quad (122)$$

where $\mathbb{X} := [X_1, \dots, X_n]^\top$, $\Lambda_d := [\lambda_1, \dots, \lambda_{n_{\max}}]^\top$ and $\mathbb{C}_{ab}^{(1)} = c_{ab}$. For convenience, we also define

$$\mathbb{C}^{(2)} := \left[\mathbf{1}_n \mid \mathbb{C}_{n \times (n_{\max} - n)}^{(1)} \right]_{n \times n_{\max}} \quad (123)$$

so that $\mathbb{X}\mathbb{X}^\top = \mathbb{C}^{(2)} \Lambda_d \Lambda_d^\top \mathbb{C}^{(2)\top}$.

We can now use $C_{\text{HCRB}}[\rho_m] \geq C_{\text{SLD}}[\rho_m] = \min_{\mathbb{X}} \text{Tr}[\mathbb{S}_\theta \mathbb{X}\mathbb{X}^\top]$ to get

$$\begin{aligned} C_{\text{HCRB}}[\rho_m] &\geq \frac{1}{d} \min_{\mathbb{C}^{(1)}} \left(n + \sum_{a,b} (\mathbb{C}_{ab}^{(1)})^2 \right) \\ &= \frac{n}{d} = C_{\text{SLD}}[\rho_m]. \end{aligned} \quad (124)$$

Moreover, as in Appendix C, $\mathbb{L} = \mathbb{X}\mathbb{X}^\top$ is a valid choice leading to

$$(\text{Tr}[\mathbb{S}_\theta \mathbb{L}])_{j,k} = \frac{1}{d} \left(\delta_{jk} + \sum_{l \in [n_{\max}] \setminus K} \mathbb{C}_{jl}^{(1)} \mathbb{C}_{kl}^{(1)} \right), \quad (125)$$

which is real, symmetric and gives $\text{Tr}[\mathbb{S}_\theta \mathbb{L}] = \text{Tr}[\mathbb{S}_\theta \mathbb{X}\mathbb{X}^\top]$. This proves

$$C_{\text{HCRB}}[\rho_m] = \frac{n}{d} \quad (126)$$

following the same arguments as in Appendix C. For estimating all $n_{\max}$ parameters this reduces to Lemma 2. Numerical checks also verify this result, as shown, e.g., in Table 1.

Notice that $\Lambda_d \Lambda_d^\top$ is the same as $\mathbb{X}\mathbb{X}^\top$ from Lemma 4, so that, using the fact that $\mathbb{C}^{(2)\top} \mathbb{C}^{(2)} \succcurlyeq 0$, we can modify $\mathbb{L}^*$ from Lemma 4 as

$$\mathbb{L}^{**}(\mathbb{C}^{(1)}) := \mathbb{C}^{(2)} \mathbb{L}^* \mathbb{C}^{(2)\top}. \quad (127)$$

It then follows from Lemma 4 that

$$\begin{aligned} \mathbb{L}^* - \Lambda \Lambda^\top &\succcurlyeq 0 \\ \Rightarrow \mathbb{C}^{(2)} (\mathbb{L}^* - \Lambda \Lambda^\top) \mathbb{C}^{(2)\top} &\succcurlyeq 0 \\ \Rightarrow \mathbb{L}^{**} - \mathbb{X}\mathbb{X}^\top &\succcurlyeq 0. \end{aligned} \quad (128)$$

That this $\mathbb{L}^{**}$ satisfies the other NHCRB constraints ($\mathbb{L}_{jk}^{**} = \mathbb{L}_{kj}^{**}$ Hermitian from Eq. (4)) for all $\mathbb{C}^{(1)}$ is also easy to check.

Note that $C_{\text{NHCRB}}[\rho_m]$ is now defined by the following minimisation:

$$\begin{aligned} C_{\text{NHCRB}} &:= \min_{\mathbb{L}, \mathbb{C}^{(1)}} \left\{ \text{Tr}[\mathbb{S}_\theta \mathbb{L} \mid \mathbb{L}_{jk} = \mathbb{L}_{kj} \text{ Hermitian}, \right. \\ &\quad \left. \mathbb{L} \succcurlyeq [\mathbf{1}_n \mid \mathbb{C}^{(1)}] \Lambda \Lambda^\top [\mathbf{1}_n \mid \mathbb{C}^{(1)}]^\top \right\}, \end{aligned} \quad (129)$$

whereas if we restrict the minimisation over $\mathbb{L}$ to a minimisation over our ansatz $\mathbb{L}^{**}(\mathbb{C}^{(1)})$, we should get a larger value than C_{NHCRB} , i.e.,

$$\begin{aligned} &\min_{\mathbb{L}, \mathbb{C}^{(1)}} \left\{ \text{Tr}[\mathbb{S}_\theta \mathbb{L} \mid \mathbb{L}_{jk} = \mathbb{L}_{kj} \text{ Hermitian}, \right. \\ &\quad \left. \mathbb{L} \succcurlyeq [\mathbf{1}_n \mid \mathbb{C}^{(1)}] \Lambda \Lambda^\top [\mathbf{1}_n \mid \mathbb{C}^{(1)}]^\top \right\} \\ &\leq \min_{\mathbb{C}^{(1)}} \left\{ \text{Tr}[\mathbb{S}_\theta \mathbb{L}^{**} \mid \mathbb{L}_{jk}^{**} = \mathbb{L}_{kj}^{**} \text{ Hermitian}, \right. \\ &\quad \left. \mathbb{L}^{**} \succcurlyeq [\mathbf{1}_n \mid \mathbb{C}^{(1)}] \Lambda \Lambda^\top [\mathbf{1}_n \mid \mathbb{C}^{(1)}]^\top \right\}. \end{aligned} \quad (130)$$

This is because the minimisation on the RHS of Eq. (130) is over a subset of the set over which the minimisation on the LHS is performed. The quantity on the RHS of Eq. (130) can then be simplified to

$$\min_{\mathbb{C}^{(1)}} \left\{ \frac{d+1}{d} \left(n + \sum_{a,b} (\mathbb{C}_{ab}^{(1)})^2 \right) \right\} = \frac{(d+1)n}{d}.$$

This lets us upper-bound $C_{\text{NHCRB}}[\rho_m]$ as

$$C_{\text{NHCRB}}[\rho_m] \leq \frac{(d+1)n}{d}, \quad (131)$$

which for estimating all $n_{\max}$ parameters reduces to Eq. (48) from Subsec. 5.4. Combining with $C_{\text{HCRB}}[\rho_m] = n/d$, we find

$$\mathcal{R}_n^{\text{NH}}[\{\rho_m\}] = \max_{K \subseteq [n_{\max}], |K|=n} \frac{C_{\text{NHCRB}}[\rho_m]}{C_{\text{HCRB}}[\rho_m]} \leq d+1, \quad (132)$$

as claimed in Theorem 4. Numerically, we see the ratio $\mathcal{R}_n^{\text{NH}}[\{\rho_m\}]$ actually depends on n : as n increases up to $n_{\max}$, the ratio increases up to $d+1$.

Table 1 lists out C_{HCRB} , the minimum and maximum values of C_{NHCRB} and the maximum ratio $\mathcal{R}_n^{\text{NH}}[\{\rho_m\}]$ for estimating a given number, n , of GMMs from qutrits. The HCRB only depends on n but not on which GMMs are chosen and is equal to n/d .

n	C _{HCRB}	Range:C _{NHCRB}	Max Ratio $\mathcal{R}_n^{\text{NH}}[\{\rho_m\}]$
2	2/3	(2/3, 4/3)	2
3	1	(3/2, 3)	3
4	4/3	(2.8270, 4.3154)	3.2365
5	5/3	(25/6, 6.6427)	3.9856
6	2	(6, 7.0921)	3.5461
7	7/3	(8.4369, 8.4951)	3.6408
8	8/3	32/3	4

Table 1: HCRB and NHCRB for estimating a subset $\{\lambda_j\}_{j \in K} \subseteq \Lambda_3$ of GMMs from the maximally-mixed qudit state ρ_m . The HCRB depends only on the number of parameters, $|K| = n$, but the NHCRB depends on the subset K chosen, so we tabulate its range in the third column, as (Min NHCRB, Max NHCRB). The fourth column lists the maximum ratio, $\mathcal{R}_n^{\text{NH}}[\{\rho_m\}]$, between the NHCRB and the HCRB, taking into account all possible subsets $\{\lambda_j\}_{j \in K}$.

G Summary of Relevant Results from Ref. [46]

In this appendix, we summarize the approach and results from Ref. [46] that are relevant for proving the upper bound to the NHCRB in main-text Eq. (21),

$$\begin{aligned} \text{C}_{\text{NHCRB}}[\rho_\theta] \leq \min_{\mathbb{X}} \left\{ \text{Tr}(\mathbb{Z}_\theta[\mathbb{X}]) \right. \\ \left. + \sum_{j,k \in [n]} \|\rho_\theta[X_j, X_k]\|_1 \right\}. \end{aligned} \quad (133)$$

Ref. [46] analyses the concept of gap persistence between the NHCRB and the HCRB with increasing number of copies in multi-parameter quantum estimation. In doing so, the authors upper-bound and lower-bound the NHCRB (Secs. 3.2 & 3.3, pg. 58 in [46]).

Say the d -dimensional Hilbert space of the qudit, $\mathcal{H}_d$ is combined with the n -dimensional (complex) parameter space to define an extended Hilbert space $\mathbb{H} = \mathbb{C}^n \otimes \mathcal{H}_d$. Ref. [46] then defines the NHCRB as

$$\text{C}_{\text{NHCRB}} := \min_{\mathbb{X}} F_{\text{NH}}(\mathbb{X}) \quad (134)$$

subject to $\mathbb{X}$ being LUB operators, and with the

Nagaoka-Hayashi (NH) function $F_{\text{NH}}(\mathbb{X})$ given by

$$F_{\text{NH}}(\mathbb{X}) := \min_{\mathbb{L}} \left\{ \text{Tr}[\mathbb{S}\mathbb{L}] \mid \mathbb{L} \in \mathcal{L}_{+, \text{sym}}(\mathbb{H}), \right. \\ \left. \mathbb{L} \geq \mathbb{X}\mathbb{X}^\top \right\}, \quad (135)$$

where $\mathcal{L}_{+, \text{sym}}(\mathbb{H})$ denotes the set of all positive semidefinite operators $\mathbb{L} \succcurlyeq 0$ on $\mathbb{H}$ that are also symmetric under the partial transpose with respect to the first Hilbert space, i.e., $\mathbb{L}_{jk} = \mathbb{L}_{kj}$ for all $j, k \in [n]$.

Ref. [46] then proves that the NH function can be rewritten as

$$\begin{aligned} F_{\text{NH}}(\mathbb{X}) &= \text{Tr}\{\text{Re } \mathbb{Z}_\theta[\mathbb{X}]\} + F_{\text{NH},2}(\mathbb{X}), \\ F_{\text{NH},2}(\mathbb{X}) &:= \min_{\mathbb{V}} \left\{ \text{Tr}[\mathbb{V}] \mid \mathbb{V} \in \mathcal{L}_{+, \text{sym}}(\mathbb{H}), \right. \\ &\quad \left. \mathbb{V} \geq \text{sym}_-(\sqrt{\mathbb{S}_\theta} \mathbb{X} \mathbb{X}^\top \sqrt{\mathbb{S}_\theta}) \right\}, \end{aligned} \quad (136)$$

where $\mathbb{Z}_\theta[\mathbb{X}]_{jk} = \text{Tr}(\rho_\theta X_j X_k)$ as in Eq. (6), $\mathbb{S}_\theta = \mathbb{1}_n \otimes \rho_\theta$, and $\text{sym}_-(\mathbb{A}) = \frac{1}{2}(\mathbb{A} - \mathbb{A}^\top)$, with $^\top$ denoting partial transpose with respect to parameter space. This rearrangement makes

$$\text{C}_{\text{NHCRB}} = \min_{\mathbb{X}} \left\{ \text{Tr}\{\text{Re } \mathbb{Z}_\theta[\mathbb{X}]\} + F_{\text{NH},2}(\mathbb{X}) \right\}, \quad (137)$$

in which the first term is equal to the Holevo objective function in Eq. (6).

Finally, the authors prove as one of their results (Theorem 1, Sec. 3.2 in Ref. [46]) that the second term of the NH function is bounded from above as

$$F_{\text{NH},2}(\mathbb{X}) \leq \sum_{j,k} \|\sqrt{\rho_\theta}[X_j, X_k]\sqrt{\rho_\theta}\|_1, \quad (138)$$

so that the NHCRB can be upper-bounded as

$$\begin{aligned} \text{C}_{\text{NHCRB}} \leq \min_{\mathbb{X}} \left\{ \text{Tr}\{\text{Re } \mathbb{Z}_\theta[\mathbb{X}]\} \right. \\ \left. + \sum_{j,k} \|\sqrt{\rho_\theta}[X_j, X_k]\sqrt{\rho_\theta}\|_1 \right\}, \end{aligned} \quad (139)$$

from which main-text Eq. (21) follows. Physically, this argument reveals the difference between the HCRB and the NHCRB to originate from the second term, $F_{\text{NH},2}(\mathbb{X})$; this term captures the non-commutativity inherent in quantum measurements for multi-parameter estimation. This result from Ref. [46] forms a basis of our proof of Theorem 1 showing that the ratio $\text{C}_{\text{NHCRB}}[\rho_\theta]/\text{C}_{\text{HCRB}}[\rho_\theta]$ is upper-bounded by n .

For further details, readers are encouraged to consult Ref. [46], which presents an in-depth exploration of the relationship between these two bounds.

H Extension to Arbitrary Weight Matrices

In this section, we extend the ratio bound of $d + 1$ for the linear GMM model to arbitrary, parameter-independent, positive weight matrices W . For fair comparison with the unweighted case, corresponding to $W = \mathbb{1}_n$, we trace-normalise $\text{Tr}(W) = n$. Additionally, W must be real, symmetric and positive ($W \succ 0$). Below, we shall refer to estimating from ρ_θ under weight matrix W as estimating from (ρ_θ, W) and denote the corresponding precision bounds by $C_{\text{HCRB}}^W[\rho_\theta]$ and $C_{\text{NHCRB}}^W[\rho_\theta]$.

This weighted model corresponds to reparameterisations of the linear GMM model [13, 21, 61], i.e., estimating any n_{max} parameters that are not necessarily coefficients of the GMMs. Similar to the other cases where all n_{max} parameters are estimated, the unbiased operators are uniquely fixed to be $X_j = \lambda_j$. We first bound the weighted HCRB and the weighted NHCRB to prove that the collective enhancement is at most $d+1$ for estimating from (ρ_m, W) for any W . Then, to extend to arbitrary states $\rho_\theta \neq \rho_m$, we numerically demonstrate that the collective enhancement for estimating from (ρ_θ, W) is always smaller than the collective enhancement for estimating from (ρ_m, W) . However, we do not prove this.

The weighted HCRB is defined via [45]

$$C_{\text{HCRB}}^W[\rho_\theta] := \min_{\substack{V \in \mathbb{R}^{n \times n}, \\ V = V^\top}} \{ \text{Tr}[WV] \mid V \succcurlyeq \mathbb{Z}_\theta[\mathbb{X}] \}, \quad (140)$$

where, by explicit computation for the maximally-mixed case, $\mathbb{Z}_\theta[\mathbb{X}]_{jk} = \text{Tr}[\rho_m X_j X_k] = \delta_{jk}/d$ or $\mathbb{Z}_\theta[\mathbb{X}] = 1/d \mathbb{1}_n$. Then, it follows from the positivity of W that $V \succcurlyeq 1/d \mathbb{1}_n$ implies

$$WV \succcurlyeq \frac{1}{d}W \implies \text{Tr}[WV] \geq \text{Tr}[W]/d = \frac{n}{d}. \quad (141)$$

This proves $C_{\text{HCRB}}^W[\rho_m] \geq n/d$.

The weighted NHCRB is defined via [21]

$$C_{\text{NHCRB}}^W[\rho_\theta] := \min_{\mathbb{L}} \left\{ \text{Tr}[WV] \mid V = \text{Tr}[\mathbb{S}_\theta \mathbb{L}] \right. \\ \left. \mathbb{S}_\theta = \mathbb{1}_n \otimes \rho_\theta, \mathbb{L}_{jk} = \mathbb{L}_{kj}^{\text{Hermitian}}, \right. \\ \left. \mathbb{L} \succcurlyeq \mathbb{X}\mathbb{X}^\top \right\}. \quad (142)$$

Notably, the feasibility constraints on $\mathbb{L}$ are unchanged from the unweighted case, i.e., the optimal $\mathbb{L}^*$ from Lemma 4 still satisfies $\mathbb{L}_{jk}^* = \mathbb{L}_{kj}^*$ Hermitian and $\mathbb{L}^* \succcurlyeq \mathbb{X}\mathbb{X}^\top$, despite not being optimal for the minimisation in Eq. (142). This sub-optimal $\mathbb{L}^*$ thus yields an upper bound to the minimum in Eq. (142),

$$C_{\text{NHCRB}}^W[\rho_m] \leq \text{Tr} \left[W \text{Tr} \left[\frac{1}{d} \mathbb{1}_{nd} \mathbb{L}^* \right] \right] \\ = \frac{d+1}{d} \text{Tr}[W] = \frac{n(d+1)}{d}, \quad (143)$$

which proves $C_{\text{NHCRB}}^W[\rho_m] \leq n(d+1)/d$. Combining with $C_{\text{HCRB}}^W[\rho_m] \geq n/d$ then proves the claim,

$$\frac{C_{\text{NHCRB}}^W[\rho_m]}{C_{\text{HCRB}}^W[\rho_m]} \leq d+1. \quad (144)$$

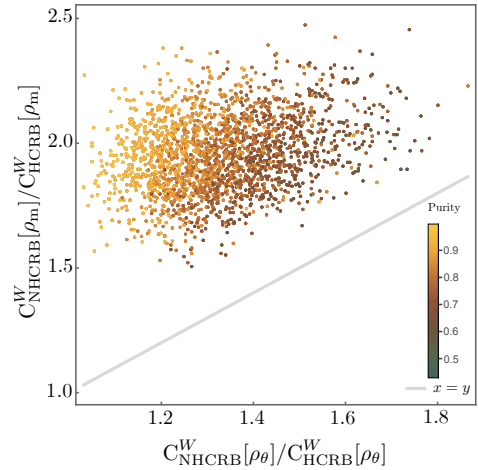


Figure 6: Comparison of the weighted ratio $C_{\text{NHCRB}}^W/C_{\text{HCRB}}^W$ for estimating from (ρ_θ, W) to that from (ρ_m, W) , for the full-parameter linear GMM model (5000 samples). The points are colour-coded by the purity of ρ_θ and the gray line corresponds to $y = x$.

So far, in this weighted tomography setting, which is equivalent to full tomography in arbitrary basis, we have established the ratio to be at

most $d + 1$ only for the maximally-mixed state. We do not prove the bound for arbitrary states but numerically demonstrate its validity in Fig. 6. By generating random full-rank, real, symmetric and positive W , such that $\text{Tr}(W) = n$, and random full-rank states ρ_θ , we compare the ratio for (ρ_θ, W) , $C_{\text{NHCRB}}^W[\rho_\theta]/C_{\text{HCRB}}^W[\rho_\theta]$, to the ratio for (ρ_m, W) , $C_{\text{NHCRB}}^W[\rho_m]/C_{\text{HCRB}}^W[\rho_m]$. Repeating this over 5000 random samples of W and ρ_θ , we find the ratio for (ρ_θ, W) to always be smaller than the ratio for (ρ_m, W) . This means

$$\frac{C_{\text{NHCRB}}^W[\rho_\theta]}{C_{\text{HCRB}}^W[\rho_\theta]} \leq \frac{C_{\text{NHCRB}}^W[\rho_m]}{C_{\text{HCRB}}^W[\rho_m]} \leq d + 1, \quad (145)$$

thus establishing the upper bound of $d + 1$ for arbitrary full-parameter estimation (or tomography in arbitrary basis) from any state,

$$\mathcal{R}_{n_{\max}}^{\text{NH}} \leq d + 1. \quad (146)$$

I True Maximum Collective Enhancement in GMM Model

The tight bound for individual precision called the MICRB C_{MI} was reformulated in Ref. [39] using the following conic optimisation problem (Eq. (39) in Ref. [39]),

$$S(P_1) := \min_{X \in \mathcal{S}_{SEP}} \{ \text{Tr}[(W \otimes \rho) X] \mid C_1, C_2 \text{ hold.} \}, \quad (147)$$

where W is the $(n + 1) \times (n + 1)$ weight matrix defined with $W_{11} = W_{1j} = W_{j1} = 0$ for all $j \in \{1, \dots, n + 1\}$, and C_1, C_2 refer to two equality constraints on the $(n + 1)d \times (n + 1)d$ -sized operator X ,

$$C_1 : \text{Tr}_{\mathcal{X}}[(|0\rangle\langle 0| \otimes \mathbb{1}_d) X] = \mathbb{1}_d, \quad (148)$$

and

$$C_2 : \text{Tr} \left[\left(\frac{|0\rangle\langle j| + |j\rangle\langle 0|}{2} \otimes \partial_k \rho \right) X \right] = \delta_{jk} \\ (\text{for all } 1 < j, k \leq n + 1), \quad (149)$$

which ensure the POVM constraint and the local unbiasedness constraint, respectively. The constraint C_1 involves partial tracing over space $\mathcal{X}$, which denotes an $(n + 1)$ -dimensional real vector space spanned by $\{|0\rangle, \dots, |n\rangle\}$. Lastly, the optimisation domain in Eq. (147) is the separable

cone $\mathcal{S}_{SEP}$, denoting the convex hull of $(n + 1)d$ -dimensional operators that are tensor products of real, symmetric, positive-semidefinite operators on the space $\mathcal{X}$ (denoted $\mathcal{M}_{rs,+}(\mathcal{X})$ in Ref. [39]), and complex, Hermitian, positive-semi-definite operators on the d -dimensional Hilbert space $\mathcal{H}_d$ (denoted $\mathcal{B}_{sa,+}(\mathcal{H}_d)$ in Ref. [39]). In the notation of Ref. [39], this cone is denoted $\mathcal{S}_{SEP} := \text{conv}(\mathcal{M}_{rs,+}(\mathcal{X}) \otimes \mathcal{B}_{sa,+}(\mathcal{H}_d))$.

For the problem of GMM tomography of arbitrary qudit states, the derivatives appearing in constraint C_2 , $\partial_k \rho = \lambda_k$, are parameter independent. Thus, feasibility for the MICRB minimisation (comprising inclusion in the separable cone $X \in \mathcal{S}_{SEP}$, and constraints C_1 & C_2) does not depend on the state ρ being estimated, i.e., the true values of the parameters θ . Let us define a candidate solution X_{sol} to the MICRB in a block-wise manner,

$$(X_{\text{sol}})_{jk} := \begin{cases} \mathbb{1}_d & j = k = 1 \\ \lambda_j & j > k = 1 \\ \lambda_k & k > j = 1 \\ \frac{d+1}{d+2} (\{\lambda_j, \lambda_k\} + \delta_{jk} \mathbb{1}_d) & j, k > 1 \end{cases}, \quad (150)$$

where block indices j, k run from 1 to $n + 1$. In fact, this candidate solution can be rewritten as

$$X_{\text{sol}} = \begin{bmatrix} \mathbb{1}_d & \mathbb{X}^\top \\ \mathbb{X} & \mathbb{L}^* \end{bmatrix} \quad (151)$$

where $\mathbb{X} = \{\lambda_1, \dots, \lambda_n\}^\top$ (from Eq. (25)) and $\mathbb{L}^*$ (from Eq. (37)) are the NHCRB-optimal solutions (see Lemma 4). The NHCRB optimisation constraint was the positivity of X_{sol} above [39], which was proven through $\mathbb{L}^* - \mathbb{X}\mathbb{X}^\top \succcurlyeq 0$ in Lemma 4 (Appendix E). Thus, Lemma 4 solved the NHCRB for the maximally-mixed state.

It can be easily checked that X_{sol} satisfies both constraints C_1 , through the $j = k = 1$ term, and C_2 , through the $j > k = 1$ and the $k > j = 1$ terms. The proof that X_{sol} belongs to the separable cone $\mathcal{S}_{SEP}$ (deferred to Lemma 10 below) follows by expressing X_{sol} as the sum over d^2 operators, $\sum_{l \in [d^2]} \Xi_l \otimes \Pi_l$, where each Ξ_l is an estimator matrix in $\mathcal{M}_{rs,+}(\mathcal{X})$ and Π_l is the l^{th} element of a SIC POVM, therefore belonging to $\mathcal{B}_{sa,+}(\mathcal{H}_d)$. Thus, X_{sol} is feasible for the MICRB minimisation for GMM tomography of arbitrary qudits.

In fact, while X_{sol} is feasible (not necessarily optimal) for all true states, it is the MICRB-optimal solution for the maximally-mixed state.

Tomography in the GMM basis corresponds to the identity-weighted full GMM model, so the appropriate matrix W is

$$W_{\text{id}} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix}_{(n+1) \times (n+1)}. \quad (152)$$

In this case, the objective in Eq. (147) equals $\text{Tr}[(\mathbb{1}_n \otimes \rho)X_{22}]$ where X_{22} denotes the block of X starting from row and column indices $d+1$ up to indices $(n+1)d$. For the candidate solution, X_{22} is $\mathbb{L}^*$, so the objective equals $\text{Tr}(\mathbb{S}_\theta \mathbb{L}^*)$ where $\mathbb{S}_\theta = \mathbb{1}_n \otimes \rho$ as previously defined. This objective value is identical to the NHCRB objective (see Eq. (4)) that was computed in Lemma 3 (and proved in Appendix E.4). For the maximally-mixed state $\rho_m = \mathbb{1}_d/d$, the objective value attained by X_{sol} therefore equals

$$\begin{aligned} \text{Tr}[(W_{\text{id}} \otimes \rho_m)X_{\text{sol}}] &= \frac{\text{Tr}[\mathbb{L}^*]}{d} \\ &= \frac{n(d+1)}{d} = C_{\text{NHCRB}}[\rho_m]. \end{aligned} \quad (153)$$

As X_{sol} is feasible, this upper-bounds the minimum in Eq. (147) as $C_{\text{MI}}[\rho_m] \leq C_{\text{NHCRB}}[\rho_m]$, whereas, by definition, $C_{\text{NHCRB}}[\rho_m] \leq C_{\text{MI}}[\rho_m]$, thus proving $C_{\text{MI}}[\rho_m] = C_{\text{NHCRB}}[\rho_m]$ and the optimality of X_{sol} . Further, since X_{sol} is feasible for any ρ , the objective $\text{Tr}[(W_{\text{id}} \otimes \rho)X_{\text{sol}}]$ also upper-bounds the minimisation in Eq. (147) for any other qudit state ρ . By direct computation (see Eq. (47)) we have

$$\text{Tr}[(W_{\text{id}} \otimes \rho)X_{\text{sol}}] = \frac{n(d+1)}{d},$$

so that, taking into account the correction for non-zero true parameter values, this proves $C_{\text{MI}}[\rho_\theta] \leq \frac{n(d+1)}{d} - \sum_j \theta_j^2$, similar to Eq. (48) for the NHCRB.

Then, the same argument used to prove Theorem 3 for the NHCRB can be used to prove the analogue theorem below for the MICRB.

Theorem 5. *For tomography on arbitrary d -dimensional qudit states ρ_θ , the maximum true collective enhancement $\mathcal{R}^{\text{MI}}[\{\rho_\theta\}] = \max_{\theta \in \Theta} C_{\text{MI}}[\rho_\theta]/C_{\text{HCRB}}[\rho_\theta] \leq d+2$.*

Proof. Combining the lower bound for the HCRB in Eq. (46) with the upper bound for the MICRB

given by $C_{\text{MI}}[\rho_\theta] \leq \frac{n(d+1)}{d} - \sum_j \theta_j^2$, we get

$$\frac{C_{\text{MI}}[\rho_\theta]}{C_{\text{HCRB}}[\rho_\theta]} \leq \frac{d^2 + d - 1 - P(\rho_\theta)}{d - P(\rho_\theta)}. \quad (154)$$

Then, using $1/d \leq P(\rho_\theta) \leq 1$, we find the maximum of the right hand side of Eq. (154) to be $d+2$, attained when $P(\rho_\theta) = 1$, i.e., when ρ_θ is pure. $\square$

We now prove that X_{sol} belongs to the separable cone $\mathcal{S}_{\text{SEP}}$. The proof proceeds utilising the fact that $\mathbb{X}$ and $\mathbb{L}^*$ appearing in X_{sol} originate from the optimal solution to the NHCRB for the maximally-mixed state, where SIC POVMs (if they exist) are the optimal measurements (proved in Lemma 9, Appendix E.5). This fact connects the SIC POVM elements, denoted Π_l^* for $1 \leq l \leq d^2$, to the optimal unbiased operators, $X_j = \lambda_j$, and the optimal operator $\mathbb{L}^*$ through a real, linear transform. This transformation can then be used to construct classical matrices $\Xi_l \in \mathcal{M}_{rs,+}(\mathcal{X})$ that prove $X_{\text{sol}} = \sum_{l \in [d^2]} \Xi_l \otimes \Pi_l^*$.

Lemma 10. *The candidate solution X_{sol} defined in Eq. (150) belongs to the separable cone $\mathcal{S}_{\text{SEP}}$.*

Proof. Let $\{\Pi_1^*, \dots, \Pi_{d^2}^*\}$ denote an optimal SIC POVM (if one exists) attaining the NHCRB for the identity-weighted full GMM model on maximally-mixed state. Then, from the NHCRB definition [21], there exists an $n \times d^2$ real matrix ξ (given by $\hat{\theta}_{jl}^* - \theta_j$, where $\hat{\theta}_{jl}$ is the classical estimator function that assigns a value to θ_j based on outcome l) that simultaneously satisfies the relations (Eqs. (12) & (13) in Ref. [21])

$$\mathbb{L}^* = \sum_{l \in [d^2]} \begin{pmatrix} \xi_{1l} \\ \xi_{2l} \\ \vdots \\ \xi_{nl} \end{pmatrix} (\xi_{1l} \quad \xi_{2l} \quad \dots \quad \xi_{nl}) \otimes \Pi_l^* \quad (155)$$

and (Eq. (17) in Ref. [21])

$$X_j = \lambda_j = \sum_{l \in [d^2]} \xi_{jl} \Pi_l^* \quad (\text{for all } j \in [n]), \quad (156)$$

thus connecting the optimal solution $\mathbb{L}^*$, the optimal unbiased operators $\mathbb{X}$ and the optimal SIC POVM $\{\Pi_l^*\}$. Finally, we can decompose $\mathbb{1}_d = \sum_{l \in [d^2]} \Pi_l^*$ using the POVM. This lets us rewrite

the candidate solution as

$$\begin{aligned}
X_{\text{sol}} &= \begin{bmatrix} \mathbb{1}_d & \mathbb{X} \\ \mathbb{X}^\top & \mathbb{L}^* \end{bmatrix} \\
&= \sum_{l \in [d^2]} \begin{bmatrix} \Pi_l^* & \xi_{1l} \Pi_l^* & \cdots & \xi_{nl} \Pi_l^* \\ \xi_{1l} \Pi_l^* & \xi_{1l}^2 \Pi_l^* & \cdots & \xi_{1l} \xi_{nl} \Pi_l^* \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{nl} \Pi_l^* & \xi_{1l} \xi_{nl} \Pi_l^* & \cdots & \xi_{nl}^2 \Pi_l^* \end{bmatrix} \\
&= \sum_{l \in [d^2]} \underbrace{\begin{bmatrix} 1 & \xi_{1l} & \cdots & \xi_{nl} \\ \xi_{1l} & \xi_{1l}^2 & \cdots & \xi_{1l} \xi_{nl} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{nl} & \xi_{1l} \xi_{nl} & \cdots & \xi_{nl}^2 \end{bmatrix}}_{\Xi_l \in \mathcal{M}_{rs,+}(\mathcal{X})} \otimes \underbrace{\Pi_l^*}_{\mathcal{B}_{sa,+}(\mathcal{H}_d)}
\end{aligned} \tag{157}$$

thus proving $X_{\text{sol}} \in \mathcal{S}_{SEP}$. $\square$

J Gill-Massar Cramér-Rao Bound

In fact, the Gill-Massar CRB (GMCRB) [23],

$$C_{\text{GMCRB}}[\rho_\theta] := \frac{(\text{Tr}[J_{\text{SLD}}^{-1/2}])^2}{d-1}, \tag{158}$$

which is obtained by inserting the classical CRB, $\text{Tr}(V_\theta) \geq \text{Tr}(J^{-1})$, into Eq. (43), is identical to the NHCRB for the full-parameter linear GMM model. This follows from the inequality in Eq. (43) being saturated in this case (Sec. VC below Eq. (54) in [23]). We further numerically verify this equivalence in Fig. 7, where we plot the two bounds for estimating all 8 GMM coefficients from 2000 random qutrit states. Both bounds agree for this model, as evidenced by the points all lying on the $y = x$ line. However, this equivalence raises the question of why we choose the NHCRB over the GMCRB as our main tool to quantify finite-copy precision, which we now answer.

In short, the NHCRB is generally tighter than the GMCRB across a range of qudit estimation models (e.g., models comprising a few parameters, $n < n_{\text{max}}$, see Figs. 2, 9) and thus is better-suited for the exploration of model-free quantities like $\mathcal{R}_n^{\text{MI}}$ and $\mathcal{R}^{\text{MI}}$. In Fig. 2, we compare the GMCRB-to-HCRB ratio $\mathcal{R}_n^{\text{GM}}$ (dark gray bars, blue line) and the NHCRB-to-HCRB ratio $\mathcal{R}_n^{\text{NH}}$ (light gray bars, red line) by combining ratios from known analytic models and from 1300 randomly-sampled numerical models for each n

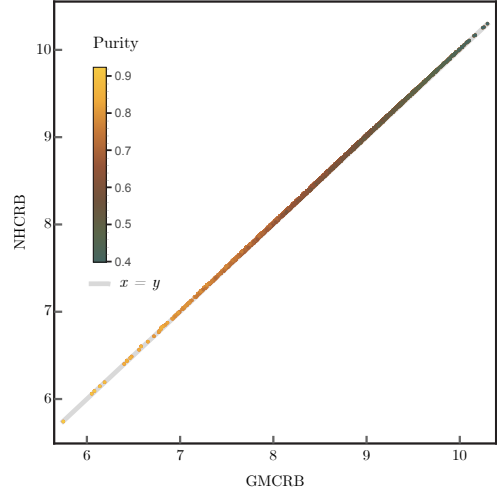


Figure 7: Comparison of the NHCRB and the GMCRB for tomography in the GMM basis ($n = 8$, $d = 3$). The two bounds are equal for the 2000 randomly-generated states and are color-coded by purity of the state.

from 1 to $n_{\text{max}} = 8$ and $d = 3$. The NHCRB ratio satisfies $\mathcal{R}_n^{\text{NH}} \leq n$, whereas the GMCRB ratio satisfies $\mathcal{R}_n^{\text{GM}} \leq n/(d-1)$, proved in Sec. 3.2.2.

Similarly, the NHCRB shows a sub-additive scaling with number of copies similar to C_{MI} , whereas the standard GMCRB is additive with number of copies. This means that NHCRB ratios can be directly used to compare multi-copy collective precision to separably-attainable precision, whereas the GMCRB requires a modification for the two-copy setting [56]. The multi-copy GMCRB [23] is defined via

$$\begin{aligned}
C_{\text{GMCRB}}[\rho_\theta^{\otimes k}] &:= \min_{V_\theta^{(k)} \succ 0} \left\{ \text{Tr}(V_\theta^{(k)}) \right. \\
&\quad \left. \text{Tr}(J_{\text{SLD}}^{-1}(kV_\theta^{(k)})^{-1}) \leq d-1 \right\},
\end{aligned} \tag{159}$$

where (k) represents k -copy quantities. Rephrasing the minimisation in Eq. (159) in terms of $kV_\theta^{(k)}$ directly leads to

$$C_{\text{GMCRB}}[\rho_\theta^{\otimes k}] = \frac{1}{k} C_{\text{GMCRB}}[\rho_\theta], \tag{160}$$

meaning the GMCRB is additive for measuring k copies of ρ_θ simultaneously. This complements the well-known additivity of the SLD QFI (Eqs. (72) & (73) in [72]), on which the GMCRB is based. We note that the minimisation in Eq. (159) has the closed-form solution:

$$C_{\text{GMCRB}}[\rho_\theta^{\otimes k}] = \frac{(\text{Tr}[J_{\text{SLD}}^{-1/2}])^2}{k(d-1)}. \tag{161}$$

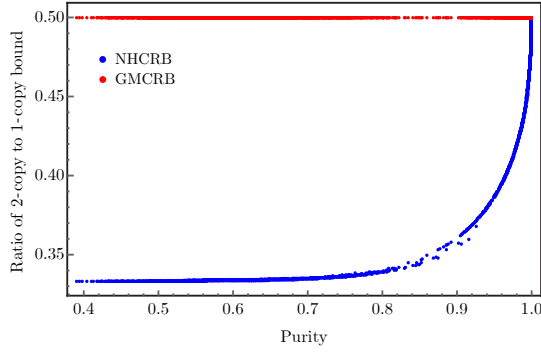


Figure 8: Comparison of the two-copy to one-copy ratio for the NHCRB and the GMCRB. Bounds correspond to tomography in the GMM basis ($n = 8$, $d = 3$) for 5000 random states. The GMCRB is additive and underestimates the two-copy enhancement except for pure states, where the two bounds agree and there is no two-copy enhancement.

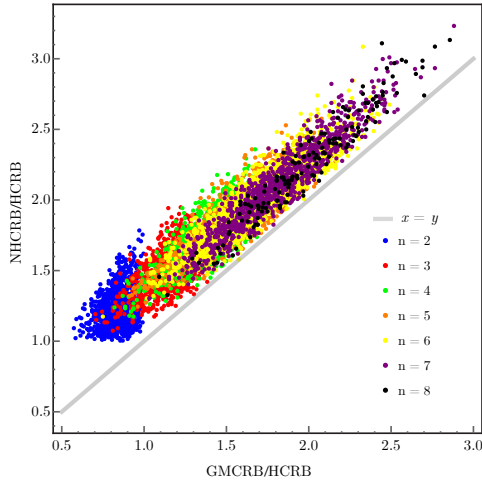


Figure 9: Comparison of the NHCRB and the GMCRB, normalised by the HCRB, for estimating fewer than $n_{\max}$ independent (but non-orthogonal) parameters via individual measurements. The bounds are calculated for 2000 random qutrit states with number of parameters n ranging from two to eight. The NHCRB is tighter than the GMCRB in this case, even for full-parameter models (black points).

For $k = 2$, Eq. (160) implies that the ratio of the two-copy bound to the one-copy bound is exactly half for the GMCRB, as can be seen in Fig. 8. In Fig. 8, we compare the ratio of two-copy to one-copy bounds for the NHCRB and the GMCRB over 5000 randomly generated qutrit states. It is clear that the NHCRB is not additive with respect to number of copies; instead, the two-copy NHCRB is always smaller than the two-copy GMCRB, except for pure states where the two bounds agree. This subadditivity of the NHCRB and additivity of the GMCRB can be attributed to the fact that the k -copy GMCRB considers individually measuring each of the k copies, whereas the k -copy NHCRB considers measuring the k -copies simultaneously or collectively. As a result, the gap between two-copy NHCRB and GMCRB represents the increase in precision from two-copy measurements compared to one-copy measurements. Notably the optimal Fisher information is also not additive under tensoring.

Moreover, for estimating fewer than $n_{\max}$ parameters, the NHCRB is strictly higher than the GMCRB even in the one-copy case, i.e., the former is a tighter bound. Figure 9 depicts this by considering the estimation of 2 to 8 arbitrary parameters from 2000 randomly generated qutrit states (following the same methodology as used in Fig. 4 to generate the states and parameters). The GMCRB and NHCRB are computed for this model and are both normalised by the HCRB. It is clear that all the plotted points lie above the $y = x$ line, numerically demonstrating that the NHCRB is tighter than the GMCRB in this case. Nonetheless, Fig. 9 also reveals an increasing trend of the ratio between the individual-optimal and collective-optimal precisions with number of parameters, irrespective of the particular choice of the individual-precision bound.

K Optimal IC POVMs for Arbitrary States

In this section we numerically investigate the optimal POVMs saturating the NHCRB for the full-parameter linear GMM model and for arbitrary states ρ_θ^* . As the purity of ρ_θ^* increases from $1/d$ to 1, the optimal individual measurements evolve from SIC POVMs to distorted IC POVMs. This transition is depicted in Fig. 10, where the inner

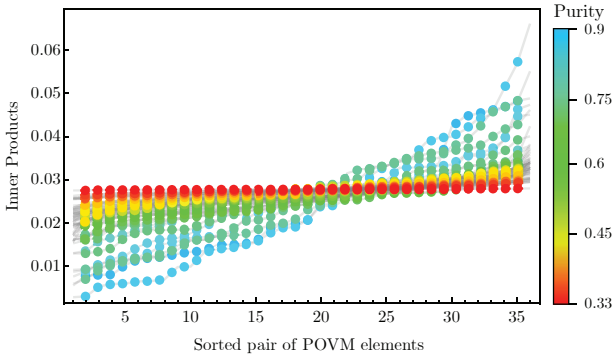


Figure 10: Evolution of the qutrit individual-optimal measurements from SIC POVM (red) to IC POVMs (all other colours) as purity of ρ_θ increases from $1/3$ for the maximally-mixed state to 1 for pure states.

products between the POVM vectors are equal at minimum purity but spread out with increasing purity. For Fig. 10, we first generate 500 random mixed qutrit states by uniformly-randomly choosing the parameters $\{\theta_j\}$ and rejection-sampling to ensure the positivity of ρ_θ . For each state, we numerically solve for the optimal one-copy, d^2 -element, rank-one POVM and ensure that it saturates the NHCRB. Then we compute the inner-product between every pair of elements of this optimal POVM. We then bin the states into 57 purity intervals and average the sorted list of inner products over each interval. Finally, we plot these sorted inner-products for each purity interval, colour-coded by the average purity of that interval.

L Random-Sampling of States and Parameters

For the random-sampling experiments in Fig. 3, we generate random mixed qudit states by first generating an entry-wise random $d \times d$ complex matrix S , and then assigning $\rho_\theta = SS^\dagger / \text{Tr}(SS^\dagger)$. This procedure ensures $\rho_\theta = \rho_\theta^\dagger$, $\rho_\theta \succcurlyeq 0$ and $\text{Tr}(\rho_\theta) = 1$. The true GMM coefficients (θ^* for the GMM model) can be found via $\text{Tr}(\rho_\theta \lambda_j)$. Unfortunately, this procedure generates low-purity states with a much higher probability than high-purity states, which becomes a problem for $d = 3$ and 4. We circumvent this issue by generating additional samples of the form $(1-p)\rho_\theta + p\mathbb{1}_d/d$ and $(1-p)\rho_\theta + p|+\rangle\langle+|_d$, where $p \in [0, 1]$. This sampling method is non-uniform but our aim here is not to sample uni-

formly according to some measure, but rather to find models with extremal properties. We compute the ratio for the full-parameter linear GMM model for all these states, the random samples and their convex combinations, to produce the yellow points in Fig. 3. The ratio-maximising (red) and ratio-minimising (blue) states at fixed purity are found by numerically maximising and minimising the ratio over the state space.

For the random-sampling experiments in Figs. 4 and 11, we generate random mixed qudit states by the following technique. For each d and n , we uniformly-randomly choose $n_{\max}$ coefficients $\{\phi_j\}_{j \in [n_{\max}]}$ from the interval $[-\sqrt{(d-1)/d}, \sqrt{(d-1)/d}]$. These define a random state $\rho_\theta = \mathbb{1}_d + \sum_{j \in [n_{\max}]} \phi_j \lambda_j$ which is guaranteed to be trace-one and Hermitian, but not positive. We ensure the positive semi-definiteness of ρ_θ by rejection sampling (discarding if it is not positive). This process generates a valid random qudit state. Next we generate the n arbitrary parameters $\{\theta_j\}_{j \in [n]}$ by generating at random the parameter derivatives $\partial_j \rho_\theta$, which must be Hermitian and traceless. We do this by writing each $\partial_j \rho_\theta$ in the GMM basis and randomly generating the coefficients in this basis. Then we rejection-sample to ensure the n parameter derivatives are linearly-independent, and lead to a valid model.

Figure 11 indicates that the 10^4 number of samples is relatively small for higher d and n —the minimum ratio observed, which should be close to one, is much larger for large d and n . This is because our sampling method generates states with low purity with higher probability and states with high purity with lower probability. As a result, the increasing or decreasing trends of the maximum observed ratio with n or d are not perfect for large d and n in Fig. 4.

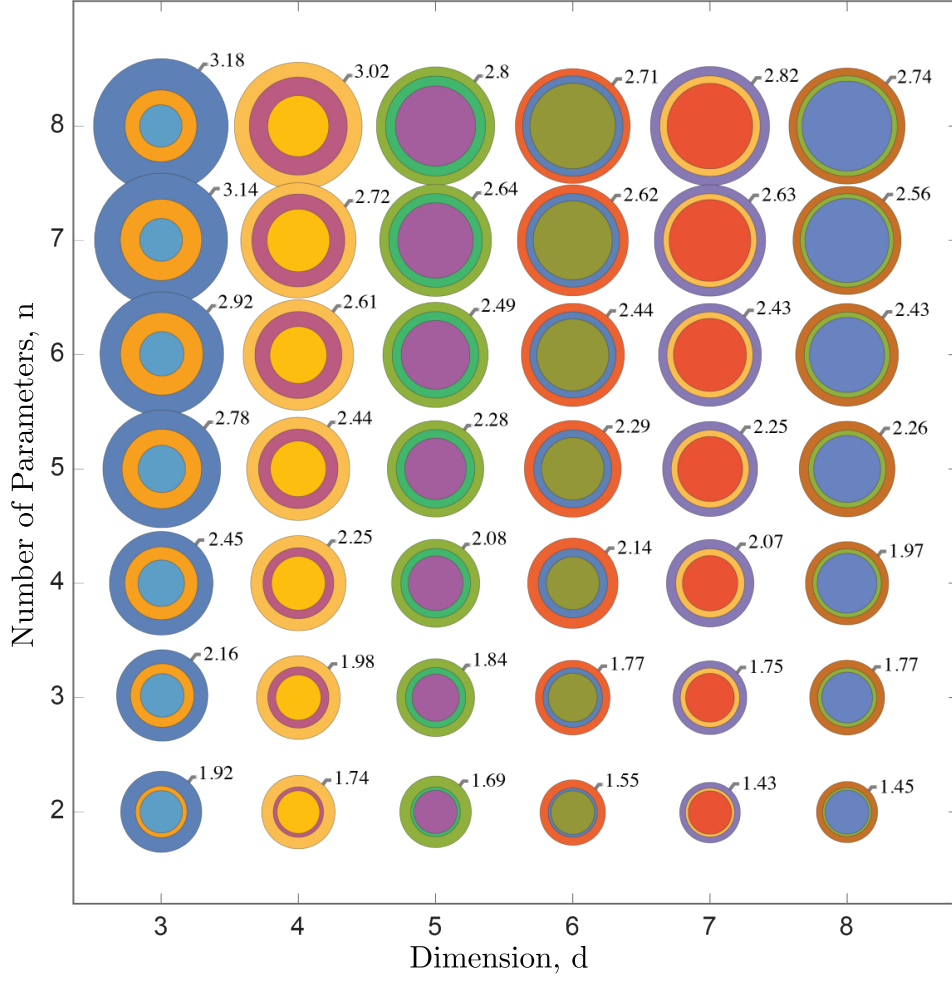


Figure 11: Bubble plot of random sampling data for the ratio between the NHCRB and the HCRB for estimating arbitrary parameters from arbitrary qudit states. Bubbles are plotted on a grid over qudit dimension d and number of parameters n . The size (diameter) of the bubbles indicate the minimum, the average and the maximum ratios sampled for each d and n , and the numerical labels are the maximum ratio up to three significant figures.