



University of Technology, Sydney

Algorithms in the Study of Multiperfect and Odd Perfect Numbers

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by

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Certificate of Authorship/Originality

I certify that this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

Ronald Maurice Sorli
February 28, 2003

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Abstract

A long standing unanswered question in number theory concerns the existence (or not) of odd perfect numbers. Over time many properties of an odd perfect number have been established and refined. The initial goal of this research was to improve the lower bound on the number of distinct prime factors of an odd perfect number, if one exists, to at least 9.

Previous approaches to this problem involved the analysis of a carefully chosen set of special cases with each then being eliminated by a contradiction. This thesis applies an algorithmic, factor chain approach to the problem. The implementation of such an approach as a computer program allows the speed, accuracy and flexibility of modern computer technology to not only assist but even direct the discovery process.

In addition to considering odd perfect numbers, several related problems were investigated, concerned with (i) harmonic, (ii) even multiperfect and (iii) odd triperfect numbers. The aim in these cases was to demonstrate the correctness and versatility of the computer code and to fine tune its efficiency while seeking improved properties of these types of numbers.

As a result of this work, significant improvements have been made to the understanding of harmonic numbers. The introduction of harmonic seeds, coupled with a straightforward procedure for generating most harmonic numbers below a chosen bound, expands the opportunities for further investigations of harmonic numbers and in particular allowed the determination of all harmonic numbers below 10^{12} and a proof that there are no odd harmonic numbers below 10^{15} .

When considering even multiperfect numbers, a search procedure was implemented to find the first 10-perfect number as well as several other new ones. As a fresh alternative to the factor chain search, a 0–1 linear programming model was constructed and used to show that all multiperfect numbers divisible by 2^a for $a \leq 65$, subject to a modest constraint, are known in the literature.

Odd triperfect numbers (if they exist) have properties which are similar to, but simpler than, those for odd perfect numbers. An extended test on the possible prime factors of such a number was developed that, with minor differences, applies to both odd triperfect and odd perfect numbers. When applicable, this test allows an earlier determination of a contradiction within a factor chain and so reduces the effort required. It was also shown that an odd triperfect number must be greater than 10^{128} .

While the goal of proving that an odd perfect number must have at least 9 distinct prime factors was not achieved, due to mainly practical limitations, the algorithmic approach was able to show that for an odd perfect number with 8 distinct prime factors, (i) if it is exactly divisible by 3^{2a} then $a = 1, 2, 3, 5, 6$ or $a \geq 31$ (ii) if the special component is π^α , $\pi < 10^6$ and $\pi^{\alpha+1} < 10^{40}$, then $\alpha = 1$.

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