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Algorithms in the Study of Multiperfect and Odd Perfect Numbers

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Certificate of Authorship/Originality

I certify that this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

Ronald Maurice Sorli
February 28, 2003

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Throughout this study my family has been very understanding of the demands of research. I am forever in their debt for their forbearance.

Abstract

A long standing unanswered question in number theory concerns the existence (or not) of odd perfect numbers. Over time many properties of an odd perfect number have been established and refined. The initial goal of this research was to improve the lower bound on the number of distinct prime factors of an odd perfect number, if one exists, to at least 9.

Previous approaches to this problem involved the analysis of a carefully chosen set of special cases with each then being eliminated by a contradiction. This thesis applies an algorithmic, factor chain approach to the problem. The implementation of such an approach as a computer program allows the speed, accuracy and flexibility of modern computer technology to not only assist but even direct the discovery process.

In addition to considering odd perfect numbers, several related problems were investigated, concerned with (i) harmonic, (ii) even multiperfect and (iii) odd triperfect numbers. The aim in these cases was to demonstrate the correctness and versatility of the computer code and to fine tune its efficiency while seeking improved properties of these types of numbers.

As a result of this work, significant improvements have been made to the understanding of harmonic numbers. The introduction of harmonic seeds, coupled with a straightforward procedure for generating most harmonic numbers below a chosen bound, expands the opportunities for further investigations of harmonic numbers and in particular allowed the determination of all harmonic numbers below 10^{12} and a proof that there are no odd harmonic numbers below 10^{15}.

When considering even multiperfect numbers, a search procedure was implemented to find the first 10-perfect number as well as several other new ones. As a fresh alternative to the factor chain search, a 0–1 linear programming model was constructed and used to show that all multiperfect numbers divisible by 2^a for $a \leq 65$, subject to a modest constraint, are known in the literature.

Odd triperfect numbers (if they exist) have properties which are similar to, but simpler than, those for odd perfect numbers. An extended test on the possible prime factors of such a number was developed that, with minor differences, applies to both odd triperfect and odd perfect numbers. When applicable, this test allows an earlier determination of a contradiction within a factor chain and so reduces the effort required. It was also shown that an odd triperfect number must be greater than 10^{128}.

While the goal of proving that an odd perfect number must have at least 9 distinct prime factors was not achieved, due to mainly practical limitations, the algorithmic approach was able to show that for an odd perfect number with 8 distinct prime factors, (i) if it is exactly divisible by 3^{2a} then $a = 1, 2, 3, 5, 6$ or $a \geq 31$ (ii) if the special component is π^α, $\pi < 10^6$ and $\pi^{\alpha+1} < 10^{40}$, then $\alpha = 1$.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Mathematical preliminaries

In general, roman letters will be used to denote positive integers, with p, q and r reserved for primes. The canonical decomposition of a positive integer n as a product of powers of distinct primes p_i, $i = 1, \ldots, t$, can be written as

$$n = \prod_{i=1}^{t} p_i^{a_i}.$$

We write $p^a \parallel n$ to mean that p^a is an exact divisor of n, that is, $p^a \mid n$ but $p^{a+1} \nmid n$, call p^a a *component* of n. More generally, we write $m \parallel n$ if $m \mid n$ and $(m, n/m) = 1$. Then m is called a unitary divisor of n and proper if $m > 1$.

The number of distinct prime factors of n is given by

$$\omega(n) = t$$

and the total number of prime factors is given by

$$\Omega(n) = \sum_{i=1}^{t} a_i.$$

The number of divisors function is defined as

$$\tau(n) = \sum_{d|n} 1.$$

For p prime and natural number a, we have $\tau(p^a) = a + 1$. Further, τ is a multiplicative function, that is, $\tau(mn) = \tau(m)\tau(n)$ if $(m, n) = 1$ and so we have

$$\tau(n) = \prod_{i=1}^{t} \tau(p_i^{a_i}) = \prod_{i=1}^{t} (a_i + 1).$$

The sum of divisors function is defined as

$$\sigma(n) = \sum_{d|n} d.$$

For prime p and natural number a, we have

$$\sigma(p^a) = 1 + p + p^2 + \cdots + p^a = \frac{p^{a+1} - 1}{p - 1}.$$

In particular, $\sigma(p) = p + 1$. It is easy to show that σ is also a multiplicative function. So, given the canonical decomposition of n we have

$$\sigma(n) = \prod_{i=1}^{t} \sigma({p_i}^{a_i}) = \prod_{i=1}^{t} \frac{{p_i}^{a_i+1} - 1}{p_i - 1}.$$

It will be convenient to define an index function $S(n) = \sigma(n)/n$. This is also is multiplicative and so we have

$$S(n) = \prod_{i=1}^{t} S(p_i^{a_i}) = \prod_{i=1}^{t} \frac{p_i^{a_i+1} - 1}{p_i^{a_i}(p_i - 1)},$$

for $n > 1$ and with $S(1) = 1$.

Notice that,

$$\begin{aligned} S(p^a) = \frac{\sigma(p^a)}{p^a} &= \frac{1 + p + p^2 + \cdots + p^a}{p^a} \\ &= \frac{(1 + p + p^2 + \cdots + p^{a-1}) + p^a}{p \cdot p^{a-1}} = 1 + \frac{S(p^{a-1})}{p}. \end{aligned} \tag{1.1}$$

Also,

$$S(p^a) = \frac{1 + p(1 + p + \cdots + p^{a-1})}{p \cdot p^{a-1}} = \frac{1}{p^a} + S(p^{a-1}).$$

We also introduce

$$S(n^\infty) = \prod_{i=1}^{t} \frac{p_i}{p_i - 1}.$$

We observe that $S(p^\infty) = \lim_{a\to\infty} S(p^a)$. Hence we can use S as a function in an extended sense and this function is also multiplicative, i.e. $S(m^a n^b) = S(m^a)S(n^b)$ for $(m, n) = 1$ where $a = 1$ or ∞, $b = 1$ or ∞.

The index function $S(p^a)$ is monotonic decreasing in p but monotonic increasing in a. A few simple relations assist in manipulating indices [98]:

if $0 \le a < b$ then $1 \le S(p^a) < S(p^b) < S(p^\infty) \le 2$,

if $p > q, 0 \le a, 1 \le b$ then $S(p^a) < S(q^b)$,

if $2 < p < q$ then $S(q^\infty) = q/(q - 1) < (p + 1)/p = S(p)$.

If $S(n) = k,\ k \in \mathbb{N}$, then n is said to be multiperfect, or to be specific a k-perfect number. However, if $S(n) < k$ (or $S(n) > k$), then n is said to be k-deficient (or k-abundant). In the case that $k = 2$, then we simply refer to n as being perfect, deficient or abundant.

The extension of the divisor function to

$$\sigma_k(n) = \sum_{d|n} d^k, \quad k \in \mathbb{Z},$$

allows the number of divisors and index functions to be represented as $\tau(n) = \sigma_0(n)$ and $S(n) = \sigma_{-1}(n)$ respectively. The use of $\sigma_{-1}(n)$ for the index function appears in the work of Iannucci [72, 73] and Iannucci and Sorli [74].

1.2 A brief history

The early development of perfect numbers is touched upon in several books on the history of mathematics. Dickson [38] gives a detailed chronology of the discoveries and developments in number theory (see also [3, 95, 96, 110, 118]). While the history of perfect numbers stretches back to the early Greek times, multiperfect numbers are of 17th century origin. In contrast, harmonic numbers (defined below) are a little over 50 years old. Only the highlights in the history of these types of numbers will be mentioned here.

1.2.1 Perfect numbers

A perfect number is one that is the sum of its proper divisors or aliquot parts (i.e. excluding the number itself). So, for a perfect number n, $\sigma(n) = 2n$ or $S(n) = 2$. The two smallest perfect numbers, $6 = 1+2+3$ and $28 = 1+2+4+7+14$, have been known in many early societies and often associated with magical or religious significance (creation and the lunar cycle respectively). Even perfect numbers were investigated firstly by Euclid [40]. The definitions of perfect ($\sigma(n) = 2n$), abundant ($\sigma(n) > 2n$) and deficient ($\sigma(n) < 2n$) numbers also appear in his work (although the terminology is modern). Euclid proved that a number of the form $2^{p-1}(2^p - 1)$ is a perfect number if $2^p - 1$ is a prime. The first four such numbers, $6, 28, 496$ and 8128 (corresponding to $p = 2, 3, 5$ and 7) were found at that time. It was conjectured that numbers of the form $2^p - 1$ were prime for every odd value of p. It was not until the 16th century that the two smallest exceptions, $2^9 - 1 = 511 = 7 \cdot 73$ and $2^{11} - 1 = 2047 = 23 \cdot 89$, were published. Other early conjectures were that perfect numbers end, alternately, in 6 or 8 (later improved to 28) and that there is one, and only one, perfect number between successive powers of ten and that every perfect number is triangular (e.g. $28 = 1 + 2 + 3 + 4 + 5 + 6 + 7$, the 7th triangular number). The first conjecture is partially true – perfect numbers do end in 6 or 8 but not in an alternating pattern (the smallest exception is the sixth perfect number 8589869056). Lucas in 1891 showed that perfect numbers end in $16, 28, 36, 56$ or 76. The second conjecture has been shown to be false (the gap between the fifth, 33550336 ($p = 13$) and sixth, 8589869056 ($p = 17$), perfect numbers is greater then expected). The third conjecture is true and was improved to state that every even perfect number is hexagonal (e.g. $28 = 1 + 5 + 9 + 13$, the 4th hexagonal number).

The 17th and 18th centuries saw a burst of study in the primality of numbers of the form $2^p - 1$. Mathematicians such as Descartes, Fermat and Mersenne applied their considerable talents to the problem. While Fermat's difference of squares method of factorisation was a great improvement over the then common, tedious and error prone method of factorisation by trial division up to the square root of the number, the practical limitation on their work was the lack of an efficient method for testing primality.

It was Euler in the 19th century who proved that all even perfect numbers are of the form studied by Euclid. This reduced the problem of finding even perfect numbers to that of finding primes of the form $2^p - 1$, now commonly referred to as Mersenne primes, M_p. In the 19th century a much simpler procedure for primality testing of Mersenne numbers (as distinct from factorisation) was introduced by Lucas. This allowed the existing list of known Mersenne primes (and hence even perfect numbers) to be corrected and extended. The search for Mersenne and Fermat primes (having the form $2^{2^n} + 1$) continues today and spurs the development of new techniques for primality testing and factorisation [13, 14, 83, 93, 111].

The Lucas-Lehmer test has been successfully used to find several subsequent Mersenne primes (and hence the largest known perfect numbers). Primality testing was significantly advanced by the work of Adleman et al. [1] and improved by Cohen and Lenstra [34]. This method is suitable for testing numbers in general with up to 1000 digits.

With the introduction of high-speed computing (and modern supercomputers), this limit has been steadily extended. In 1996 Slowinski and Gage found $M_{1257787}$, a number with 378632 digits, to be prime using a supercomputer. A very different approach was adopted by Woltman [119] who is coordinating a systematic, cooperative world-wide search (called the Global Internet Mersenne Prime Search, GIMPS) to complete and extend the range of tested exponents. An early success of this project was the discovery in 1996 by Armengaud, Woltman et al. of $M_{1398269}$, a number with 420921 digits. Subsequently, four more Mersenne primes have been found through this project, $M_{2976221}$ (Spence et al. 1997), $M_{3021377}$ (Clarkson et al. 1998),$M_{6972593}$ (Hajratwala et al. 1999) and $M_{13466917}$ (Cameron et al. 2001), the currently largest known Mersenne prime with over four million digits. As of December 2002, 39 Mersenne primes (and hence even perfect numbers) have been found, corresponding to $p =$ 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593 and 13466917. All Mersenne numbers with exponents up to 10000000 have been tested at least once and those with exponents up to 6000000 have been double checked.

The existence (or not) of odd perfect numbers was speculated upon but it was Euler who first produced a significant result. He proved that any odd perfect number must be of the form $\pi^{\alpha}m^2$, where π is the so-called special (or distinguished) prime satisfying the conditions $\pi \equiv \alpha \equiv 1 \pmod 4$ and $\pi \nmid m$ [41]. Research since has been to further uncover the form of odd perfect numbers, if any exist. General references are Guy [57] and Ribenboim [102]. Some of these characteristics are discussed below.

The number of distinct prime factors of an odd perfect number n, $\omega(n)$, was shown to be at least 5 by Sylvester [115] (and others), at least 6 by Webber [117] (and others), at least 7 by Pomerance [98] (and Robbins [103] which includes a technical summary of earlier methods) and at least 8 by Hagis [59] (and Chein [24]). All used the method of proof by contradiction of an increasing number of special cases. Cohen and Sorli [27, 32] have developed an algorithmic search approach and illustrated its use by providing alternative proofs that there are at least 6 and then 7 distinct prime factors.

With the assumption that $3 \nmid n$, Sylvester [115] showed that there must be at least 8 distinct prime factors. Kishore [79] showed that the lower bound was at least 10. This bound was improved by Hagis [61] (and Kishore [81]) to at least 11.

Related results concern the total number of prime factors, $\Omega(n)$. Using an algorithmic approach, Cohen [25] showed the total number of prime factors to be at least 23. This bound was improved by Sayers [106] to 29 and recently to 37 by Iannucci and Sorli [74].

An upper bound on the size of an odd perfect number with k distinct prime factors was shown to be $(4k)^{(4k)^{2^{k^2}}}$ by Pomerance [100]. This was much improved by Heath-Brown [66] to 4^{4^k} and more recently improved further to D^{4^k} where $D = (195)^{1/7} \approx 2.123$ by Cook [35]. A lower bound on the size of an odd perfect number has also been established and improved over time to 10^{300} [15, 58].

Numerous restrictions on the exponents of the prime factors, that is, the multiplicative structure, have been found [12, 33, 86, 87, 88, 89, 90, 91].

Another set of results concerns lower bounds on the sizes of the prime factors themselves. The largest prime factor of an odd perfect number was shown to be at least 100129 by Hagis and McDaniel [65]. This was improved to 10^6 by Hagis and Cohen [64]. Recently this has been increased to 10^7 by Jenkins [75]. The second largest prime factor was shown to be at least 139 by Pomerance [99] and at least 1009 by Hagis [60]. Iannucci [70, 72, 73] improved the bound on the second largest prime factor to 10^4 and established a lower bound of 100 on the third largest prime factor.

1.2.2 Multiperfect numbers

A k-perfect number is defined as one such that $\sigma(n) = kn$ or $S(n) = k, \quad k \in \mathbb{N}$. Such a number is also said to have multiplicity k. The case $k = 2$ corresponds to the previously discussed perfect numbers and the case $k > 2$ corresponds to proper multiperfect numbers. These numbers were first investigated in the 17th century by Mersenne and then by mathematicians such as Fermat, Descartes and Frenicle. The smallest proper multiperfect number is the 3-perfect number $120 = 2^3 \cdot 3 \cdot 5$. While 3-, 4-, 5- and 6-perfect numbers were found at that time, no general formulae were discovered. Descartes and Mersenne gave a few rules for generating new multiperfect numbers from certain known multiperfect numbers [38, 109]. The discovery of such substitutions continues today but (still) involves intuition, good luck and an analysis of known multiperfect numbers [22, 46].

Since this early work the search for multiperfect numbers has been intermittent. Carmichael [21] gave a list of all 15 multiperfect numbers less than 10^9. Later he extended this list to a total of 251 multiperfect numbers, together with the substitutions for generating some of these [22]. This list also included the first 7-perfect number discovered. Poulet [101] listed 334 multiperfect numbers including the first 8-perfect number. Franqui and Garcia [45, 46] and Brown [18] added over 60 new multiperfect numbers to this collection. Even in the last decade, the search was conducted mainly as an amateur pastime [17, 28] with the help of printed tables [16, 52]. Only relatively recently have previous results been collected and catalogued and extensive automated searching been carried out [44, 108, 112], resulting in a five-fold increase in the number of known multiperfect numbers to just over 4000 (as of December 2002), as well as the discovery of the first 9-, 10-perfect numbers and, so far, a single 11-perfect number.

While no odd multiperfect numbers have been found there have been some results concerning the form of such numbers paralleling some of the results for odd perfect numbers. Carmichael [20] showed that odd multiperfect numbers must have at least four distinct prime factors. Artuhov [4] showed that there are only finitely many odd multiperfect numbers with a given number of distinct prime factors.

Bounds have also been found for an odd multiperfect number. Cohen and Hagis [30] proved that for an odd multiperfect number n, $\omega(n) \geq 11$ and that $n > 10^{70}$. Building on the work for odd perfect numbers, they also proved that the largest prime factor must be at least 100109 and that the second largest prime factor must be at least 1000. Iannucci [71] has announced that the largest prime factor must be at least 10^7. Hagis [62] proved that the third largest prime factor must be greater than 100.

While there are few results for multiperfect numbers in general, there are several useful results for odd triperfect numbers in particular. Kanold [78] showed that, for an odd triperfect number n, $\omega(n) \geq 9$, n is a square and $n > 10^{20}$. The number of distinct prime factors of an odd triperfect number was shown to be at least 12 by Kishore [82] (and more simply by Hagis [63]). A lower bound of 10^{50} was established by Beck and Najar [10], improved to 10^{70} by the general multiperfect result as above [30].

1.2.3 Harmonic numbers

A natural number n is said to be harmonic if the harmonic mean of its positive divisors, i.e.

$$H(n) = \frac{\tau(n)}{\sum_{d|n} 1/d} = \frac{n\tau(n)}{\sigma(n)},$$

is an integer.

Compared to perfect and multiperfect numbers, harmonic numbers have only been studied for a relatively short time. Harmonic numbers were introduced by Ore [94], who found all those up to 10^5. Garcia [47] listed them all to 10^7. Cohen [29] has found all to 2×10^9 with a continuation up to 10^{10} reported in Cohen and Sorli [31]. All of these harmonic numbers are even. Finding new even harmonic numbers is not difficult, although finding all of them below some bound is challenging. On the other hand, relatively little is known about odd harmonic numbers. By introducing the concept of harmonic seeds (from which larger harmonic numbers can be derived), Cohen and Sorli were able to show that there are no odd harmonic numbers less than 10^{12}. What is known about odd harmonic numbers can be summarized as follows. If an odd harmonic number, n, exists then

(i) $n > 10^{12}$ [31],

(ii) if $p^a \parallel n$, p prime, then $p^a \equiv 1 \pmod 4$ [47, 92],

(iii) n has a component greater than 10^7 [92] and

(iv) $H(n) > 300$ [53].

The interest in harmonic numbers stems from the fact that all perfect numbers are harmonic, so if there are no odd harmonic numbers then there are no odd perfect numbers.

1.3 An overview of this thesis

The original goal of this thesis was to investigate the extension of the lower bound on the number of distinct prime factors of an odd perfect number from 8 to 9 using the computational approach developed by Cohen [27]. In addition to this task, several related problems were investigated, concerned with (i) harmonic numbers (ii) even multiperfect numbers and (iii) odd triperfect numbers.

The following four chapters provide the detailed working behind each of these investigations. The final chapter looks at the potential and, in some cases, the realisation of the potential for applying parallel computing techniques to improve the efficacy of the techniques and algorithms used in this research.

1.4 Achievements in this thesis

In each case some of the existing results have been improved upon. These can be summarized as follows:

harmonic numbers

- listed all harmonic numbers up to 10^{12} (previous bound 10^{10})
- found all harmonic seeds up to 10^{15} and consequently that if n is an odd harmonic number then $n > 10^{15}$ (previous bound 10^{12})

even multiperfect numbers

- found 27 large multiperfect numbers including the first 10-perfect number
- showed that, with some qualifications, all multiperfect numbers with a component 2^a, $0 \leq a \leq 65$ have been found

odd triperfect numbers

- showed that if n is an odd triperfect number then $n > 10^{128}$ (previous bound 10^{70})

odd perfect numbers

- showed that if n is an odd perfect number then $\Omega(n) \geq 37$ (previous bound 29)
- showed that if n is an odd perfect number with $\omega(n) = 8$ then
  - $3^a \parallel n$ for $a = 2, 4, 6, 10, 12$ or $a > 60$
  - if $\pi < 10^6$ and $\pi^{\alpha+1} < 10^{40}$ then $\pi \parallel N$

To achieve these improvements, the relevant theory was organized into computational algorithms. Hand in hand with this automation process, new theoretical results were obtained. These can be summarized as follows:

harmonic numbers

- introduced the concept of harmonic seed
- found simple rules for generating most, but not all, harmonic numbers from harmonic seeds

even multiperfect numbers

- introduced the concept of restricted sigma chaining
- developed and realised a 0–1 integer linear programming model for finding even multiperfect numbers as an alternate to factor chain searching

odd triperfect numbers

- generalized the formulae for calculating the bounds on the unknown prime factors of an odd triperfect number allowing earlier discrimination (and possible elimination) of cases

odd perfect numbers

- found an additional general restriction on the multiplicative structure of an odd perfect number (see Theorem 9)
- adapted the generalized bound calculations for odd triperfect numbers to odd perfect numbers

1.5 Contents of the accompanying Compact Disc

Due to the length of some of the computational proofs generated, they do not appear in printed form in the thesis. Rather, they are available on an accompanying Compact Disc with instructions for locating the relevant file given in a printed appendix of the thesis.

For convenience computer accessible versions of the full printed thesis are also included on the Compact Disc in both the original Postscript form as well as the more widely available Adobe PDF format.

The contents of the Compact Disc are as follows:

```
\Postscript\Thesis.ps
\Postscript\Evn12all.ps
\Postscript\Odd12all.ps
\Postscript\Otp128all.ps
\PDF\Thesis.pdf
\PDF\Evn12all.pdf
\PDF\Odd12all.pdf
\PDF\Otp128all.pdf
\Text\Opn7B21.txt
```

Chapter 2

Harmonic Numbers

2.1 Introduction

A natural number n is said to be harmonic if the harmonic mean of its positive divisors, that is

$$H(n) = \frac{\tau(n)}{\sum_{d|n} 1/d} = \frac{n\tau(n)}{\sigma(n)},$$

is an integer. For p prime, $H(p) = 2p/(p+1)$. From the multiplicative properties of τ and σ, H is also a multiplicative function.

Harmonic numbers are of interest because a) every perfect number, satisfying $\sigma(n) = 2n$, is harmonic, and (b) besides 1, no odd harmonic numbers have been found. A proof of the nonexistence of nontrivial odd harmonic numbers would therefore imply the nonexistence of odd perfect numbers.

In this chapter an algorithm is described for determining harmonic seeds and is applied to finding all harmonic seeds up to 10^{15} (with no odd ones except 1). This implies that there are no powerful harmonic numbers, odd harmonic numbers or deficient harmonic numbers less than 10^{15}. A process for generating most (but not all) harmonic numbers from harmonic seeds is given. Subsequently these results were supported by the results of a continuation, by the author, of an exhaustive search for harmonic numbers up to 10^{12}.

2.2 Basic theory

Definition 1 (Cohen and Sorli [31]). *The harmonic seed of* 1 *is* 1. *A harmonic seed of a number* $n > 1$, *is a proper unitary divisor of* n *which is harmonic and does not itself have a smaller proper unitary divisor which is harmonic.*

Definition 2. *We say* n *is powerful if* $p \mid n$ *implies* $p^2 \mid n$ *where* p *is prime.*

Once a harmonic seed has been found, more harmonic numbers can usually be generated in a straightforward manner. This approach is similar to the use of "breeders" in the generation of amicable numbers [11]. It is convenient to split the search for harmonic seeds into separate even and odd cases.

Known results on harmonic numbers include the following:

Lemma 1 (Ore [94]). *Besides* 1, *the only squarefree harmonic number is* 6.

Lemma 2 (Garcia [47], Mills [92]). *Let* n *be an odd harmonic number. If* $p^a \parallel n$, p *prime, then* $p^a \equiv 1 \pmod 4$.

Lemma 3 (Mills [92]). *If n is an odd harmonic number greater than 1, then n has a component greater than 10^7.*

Lemma 4 (Ore [94]). *There are no harmonic numbers of the form p^a, p prime.*

Lemma 5 (Callan [19], Pomerance [97]). *The only harmonic numbers of the form $p^a q^b$, p and q distinct primes, are even perfect numbers.*

Lemma 6 (Cohen [29]). *If $n = 2^{a-1}(2^a - 1)$ is perfect (so that $2^a - 1$ and a are primes), then $H(n) = a$.*

2.3 Even harmonic numbers

The method has similarities with the manual approach outlined by Garcia [47], applied here to first determine harmonic seeds.

2.3.1 An algorithm for finding even harmonic seeds

To find even harmonic numbers n less than some bound B we assume first that n has the prime factor decomposition

$$n = 2^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}.$$

From Lemma 5, it is known that, apart from the special case of even perfect numbers, all harmonic numbers have at least three distinct prime factors. With this, an upper bound can be placed on α_1. The worst case is $2^{\alpha_1} 3 \cdot 5$, and hence $\alpha_1 \leq b_1 = \lfloor \log_2(B/(3 \cdot 5)) \rfloor$. We need to investigate all cases arising from $\alpha_1 = 1, 2, \ldots, b_1$.

Starting with the assumption that n has a component 2^{α_1}, we calculate $H(2^{\alpha_1})$, which will be a rational number (reduced so that the numerator and denominator are relatively prime). Since $H(n)$ must be an integer, each prime divisor of the denominator of $H(2^{\alpha_1})$, say q, will either

(a) be a prime factor of n or

(b) be a divisor of $\tau(n)$ and hence appear as an exponent, $q - 1$ or higher, of a prime factor of n.

To see this suppose $H(2^{\alpha_1}) = x/y$, say, $(x, y) = 1$. Since $\sigma(n)H(n) = n\tau(n)$, then, after initially multiplying both sides by $H(2^{\alpha_1})$, we have $2^{\alpha_1}\tau(2^{\alpha_1})\sigma(n/2^{\alpha_1})H(n)y = n\tau(n)x$. Each of these possible decision branches is considered, leading, in general, to four possible conclusions

(i) n exceeds B,

(ii) an n with an integral $H(n)$ has been found (so n is a harmonic seed),

(iii) there is a contradiction as to the factors of n or

(iv) $H(n)$ is (still) not an integer.

The first case allows the search to be terminated for this set of assumed components and other alternative sets of assumed factors would then be tested. In a similar manner, the search is terminated for case (ii), since this corresponds to finding a harmonic seed. Case (iii) occurs when a factor of the denominator of $H(n)$ must be an additional factor of n but has previously been eliminated as a possible factor. Case (iv) leads to a continuation of the common factor-chain search method. The search continues until the factor-chains following from each of the assumed initial components 2^{α_1} have been resolved (terminated).

The various possibilities arising from q as a divisor of the denominator of $H(n)$, need to be clarified. For such a q, the continuation for $q^a,\ a = 1, 2, \ldots$ as components of n is straightforward. In addition, we must consider $q \mid \tau(p^{kq-1})$ for prime $p \neq q$, $k \geq 1$, which usually leads to a violation of the upper bound B. However if $q^a,\ a > 1$, is an exact divisor of the denominator, further possibilities need to be considered. For example, if 3^2 is such a divisor, then either $3^b,\ b > 0$ is a continuation or there are two additional components, say $p_1^{3k_1-1}$ and $p_2^{3k_2-1}$ for distinct p_1, p_2 and q, $k_1, k_2 \geq 1$ or there is a single additional component, p^{3^2k-1}. It is convenient to assume $p_1 < p_2$. If q^3 were the divisor then, in addition to considering q^b, there would be the possibility that (i) there are three additional prime factors, say $p_1^{qk_1-1}$, $p_2^{qk_2-1}$ and $p_3^{qk_3-1}$ or (ii) there are two additional prime factors, either $p_1^{qk_1-1}$, $p_2^{q^2k_2-1}$ or $p_1^{q^2k_1-1}$, $p_2^{qk_2-1}$ or (iii) there is a single additional prime factor, $p_1^{q^3k_1-1}$. In general, if $q^a,\ a > 0$, is an exact divisor then we need to consider every combination of primes and exponents of the form

$$\prod_{i=1}^{t} p_i^{q^{b_i}k_i-1}, \quad \sum_{i=1}^{t} b_i = a$$

with p_i prime, $p_i < p_{i+1}$, $p_i \neq q$, $b_i \geq 1$, $k_i \geq 1$. While these usually quickly lead to a violation of the bound B, the combinatorial nature of the number of such cases becomes a practical limitation on how far the bound can be raised.

It is convenient to summarize this search using a compact decision tree representation. For example, if we are searching for even harmonic numbers with a component of 2^2, then a partial decision tree is shown in Figure 2.1.

To further illustrate these ideas, some sample calculations for a bound $B = 10^{12}$ will now be discussed. The upper bound on the power of the initial component 2^{α_1} is

$$\alpha_1 \leq \lfloor \log_2(B/(3 \cdot 5)) \rfloor = \lfloor \log_2(10^{12}/(3 \cdot 5)) \rfloor = 35.$$

Looking at the case $\alpha_1 = 35$,

$$H(2^{35}) = \frac{2^{37}}{3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73 \cdot 109}.$$

It is easily seen that 109 must be a factor of n but then we have the contradiction $n \geq 2^{35}109 > B$ which is case (i) identified as TOO BIG in the computer generated proof given in Appendix C. (Alternatively, $109 \mid \tau(n)$, but then $2^{35}3^{108} > B$.) We have similar contradictions if $73, 37, \ldots, 3$ divide n. The conclusion is that 2^{35} cannot be a component of a harmonic number $n < 10^{12}$.

As a first example of the basic factor-chain method, consider the case $\alpha_1 = 20$. We have

$$H(2^{20}) = \frac{2^{20}3}{7 \cdot 127 \cdot 337}.$$

This is not an integer (of course) and $2^{20}337 < 10^{12}$, so both 337 and 127 must be prime factors of n, whereas 7 could be a prime factor or there could be a factor which is a prime to an exponent $7 - 1 = 6$ (or higher). A strategy of choosing the biggest possible new prime first is likely to lead to

$2^2 : \mathrm{H}=\frac{2^2 3}{7}$ [1]

$7 : \mathrm{H}=\frac{2^2 3}{7} \cdot \frac{7}{2^2} = 3$ (and so $2^2 7$ is HARMONIC)

$7^2 : \mathrm{H}=\frac{2^2 3}{7} \cdot \frac{7^2}{19} = \frac{2^2 3 \cdot 7}{19}$

$19 : \mathrm{H}=\frac{2^2 3 \cdot 7}{19} \cdot \frac{19}{2 \cdot 5} = \frac{2 \cdot 3 \cdot 7}{5}$

$5 : \mathrm{H}=\frac{2 \cdot 3 \cdot 7}{5} \cdot \frac{5}{3} = 2 \cdot 7 = 14$ (and so $2^2 5 \cdot 7^2 19$ is HARMONIC)

$5^2 : \mathrm{H}=\frac{2 \cdot 3 \cdot 7}{5} \cdot \frac{3 \cdot 5^2}{31} = \frac{2 \cdot 3^2 5 \cdot 7}{31}$

$31 : \mathrm{H}=\frac{2 \cdot 3^2 5 \cdot 7}{31} \cdot \frac{31}{2^4} = \frac{3^2 5}{2^3}$ [2]

... [3]

$5^3 : \mathrm{H}=\frac{2 \cdot 3 \cdot 7}{5} \cdot \frac{5^3}{3 \cdot 13} = \frac{2 \cdot 5^2 7}{13}$

$13 : \mathrm{H}=\frac{2 \cdot 5^2 7}{13} \cdot \frac{13}{7} = 2 \cdot 5^2 = 50$ (and so $2^2 5^3 7^2 13 \cdot 19$ is HARMONIC)

...

$5^4 : \mathrm{H}=\frac{2 \cdot 3 \cdot 7}{5} \cdot \frac{5^5}{11 \cdot 71} = \frac{2 \cdot 3 \cdot 5^4 7}{11 \cdot 71}$

... [4]

$3^4 : \mathrm{H}=\frac{2 \cdot 3 \cdot 7}{5} \cdot \frac{3^4 5}{11^2} = \frac{2 \cdot 3^5 7}{11^2}$

...

$11^4 : \mathrm{H}=\frac{2 \cdot 3 \cdot 7}{5} \cdot \frac{11^4}{3221} = \frac{2 \cdot 3 \cdot 7 \cdot 11^4}{5 \cdot 3221}$

$13^4 : \mathrm{H}=\frac{2 \cdot 3 \cdot 7}{5} \cdot \frac{5 \cdot 13^4}{30941} = \frac{2 \cdot 3 \cdot 7 \cdot 13^4}{30941}$

...

$3^9 : \mathrm{H}=\frac{2 \cdot 3 \cdot 7}{5} \cdot \frac{3^9 5}{2 \cdot 11^2 61} = \frac{3^{10} 7}{11^2 61}$

$11^9 : \mathrm{H}=\frac{2 \cdot 3 \cdot 7}{5} \cdot \frac{11^9}{2 \cdot 3 \cdot 3221 \cdot 13421} = \frac{7 \cdot 11^9}{5 \cdot 3221 \cdot 13421}$

$13^9 : \mathrm{H}=\frac{2 \cdot 3 \cdot 7}{5} \cdot \frac{5 \cdot 13^9}{7 \cdot 11 \cdot 2411 \cdot 30941} = \frac{2 \cdot 3 \cdot 13^9}{11 \cdot 2411 \cdot 30941}$

...

$3^{14} : \mathrm{H}=\frac{2 \cdot 3 \cdot 7}{5} \cdot \frac{3^{15} 5}{11^2 13 \cdot 4561} = \frac{2 \cdot 3^{16} 7}{11^2 13 \cdot 4561}$

$11^{14} : \mathrm{H}=\frac{2 \cdot 3 \cdot 7}{5} \cdot \frac{3 \cdot 11^{14}}{7 \cdot 19 \cdot 3221 \cdot 195019441} = \frac{2 \cdot 3^2 11^{14}}{5 \cdot 19 \cdot 3221 \cdot 195019441}$

$13^{14} : \mathrm{H}=\frac{2 \cdot 3 \cdot 7}{5} \cdot \frac{5 \cdot 13^{14}}{61 \cdot 4651 \cdot 30941 \cdot 161971} = \frac{2 \cdot 3 \cdot 7 \cdot 13^{14}}{61 \cdot 4651 \cdot 30941 \cdot 161971}$

...

$19^2 : \mathrm{H}=\frac{2^2 3 \cdot 7}{19} \cdot \frac{19^2}{127} = \frac{2^2 3 \cdot 7 \cdot 19}{127}$

$19^3 : \mathrm{H}=\frac{2^2 3 \cdot 7}{19} \cdot \frac{19^3}{2 \cdot 5 \cdot 181} = \frac{2 \cdot 3 \cdot 7 \cdot 19^2}{5 \cdot 181}$

$19^4 : \mathrm{H}=\frac{2^2 3 \cdot 7}{19} \cdot \frac{5 \cdot 19^4}{151 \cdot 911} = \frac{2^2 3 \cdot 5 \cdot 7 \cdot 19^3}{151 \cdot 911}$

...

$3^{18} : \mathrm{H}=\frac{2^2 3 \cdot 7}{19} \cdot \frac{3^{18} 19}{1597 \cdot 363889} = \frac{2^2 3^{19} 7}{1597 \cdot 363889}$

$5^{18} : \mathrm{H}=\frac{2^2 3 \cdot 7}{19} \cdot \frac{5^{18} 19}{191 \cdot 6271 \cdot 3981071} = \frac{2^2 3 \cdot 5^{18} 7}{191 \cdot 6271 \cdot 3981071}$

$11^{18} : \mathrm{H}=\frac{2^2 3 \cdot 7}{19} \cdot \frac{11^{18} 19}{6115909044841454629} = \frac{2^2 3 \cdot 7 \cdot 11^{18}}{6115909044841454629}$

...

$3^6 : \mathrm{H}=\frac{2^2 3}{7} \cdot \frac{3^6 7}{1093} = \frac{2^2 3^7}{1093}$

$5^6 : \mathrm{H}=\frac{2^2 3}{7} \cdot \frac{5^6 7}{19531} = \frac{2^2 3 \cdot 5^6}{19531}$

$11^6 : \mathrm{H}=\frac{2^2 3}{7} \cdot \frac{7 \cdot 11^6}{43 \cdot 45319} = \frac{2^2 3 \cdot 11^6}{43 \cdot 45319}$

...

$3^{13} : \mathrm{H}=\frac{2^2 3}{7} \cdot \frac{3^{13} 7}{2 \cdot 547 \cdot 1093} = \frac{2 \cdot 3^{14}}{547 \cdot 1093}$

$5^{13} : \mathrm{H}=\frac{2^2 3}{7} \cdot \frac{5^{13} 7}{3 \cdot 29 \cdot 449 \cdot 19531} = \frac{2^2 5^{13}}{29 \cdot 449 \cdot 19531}$

$11^{13} : \mathrm{H}=\frac{2^2 3}{7} \cdot \frac{7 \cdot 11^{13}}{2 \cdot 3 \cdot 43 \cdot 45319 \cdot 1623931} = \frac{2 \cdot 11^{13}}{43 \cdot 45319 \cdot 1623931}$

...

Figure 2.1: Partial decision tree for an even harmonic seed search with initial component 2^2

the bound B being exceeded sooner. Take $n = 2^{20}337^{\alpha_2}\cdots$ with $1 \le \alpha_2 \le 2 = \lfloor \log_{337}(10^{12}/(2^{20}3)) \rfloor$. For $\alpha_2 = 2$, we have

$$H(2^{20}337^2) = \frac{2^{20}3 \cdot 337}{7 \cdot 43 \cdot 127 \cdot 883}.$$

We find that 883 must be a prime factor of n but then $n \ge 2^{20}337^2 883 > 10^{12}$ (TOO BIG). We next consider $\alpha_2 = 1$, for which

$$H(2^{20}337) = \frac{2^{20}3}{7 \cdot 13^2 \cdot 127}.$$

Algorithmically we next consider 127 as a prime factor of n but then $n \ge 2^{20}337 \cdot 127 > 10^{12}$ (TOO BIG). From these two cases we conclude that 337^{α_2} cannot be a component of a harmonic number $n < 10^{12}$ with 2^{20} as a component and consequently that 2^{20} cannot be a component of such a harmonic number. This concludes the case for $\alpha_1 = 20$.

For $\alpha_1 = 18$ we have,

$$H(2^{18}) = \frac{2^{18}19}{524287}.$$

Obviously 524287 must be a prime factor of n rather than leading to an exponent of a prime factor. So we consider $2^{18}524287$, for which

$$H(2^{18}524287) = 19.$$

Since this is integral we have found a harmonic seed. This is case (ii) and is identified as HARMONIC!!! in the proof.

This case also illustrates the connection between perfect numbers and harmonic numbers (Lemmas 5 and 6) where we have $n = 2^{18}524287 = 2^{19-1}(2^{19}-1)$, with the Mersenne prime $M_{19} = 2^{19} - 1 = 524287$, and $\sigma(n) = 1048574 = 2n$ and hence that n is an even perfect number.

The first case where both decision branches are feasible occurs with $\alpha_1 = 13$, specifically for $2^{13}127 \cdot 43 \cdot 11$ (identified as TROUBLE!!! in the proof). Here,

$$H(2^{13}127 \cdot 43 \cdot 11) = \frac{2^6 7}{3^2}.$$

Either 3 is a factor or there are two additional prime factors each to the power $3 - 1 = 2$ (or higher) or there is one additional prime factor to the power $3^2 - 1 = 8$ (or higher). In the latter case, the worst case would be $n \ge 2^{13}127 \cdot 43 \cdot 11 \cdot 5^8 > 10^{12}$ which is TOO BIG. On the other hand, if there are two additional prime factors, then the only possibility (less than the bound B) is $2^{13}127 \cdot 43 \cdot 11 \cdot 5^2 7^2$ which gives

$$H(2^{13}127 \cdot 43 \cdot 11 \cdot 5^2 7^2) = \frac{2^6 5^2 7^3}{3 \cdot 19 \cdot 31}.$$

From which we find that 31 must be a prime factor but then $n \ge 2^{13}127 \cdot 43 \cdot 11 \cdot 5^2 7^2 31 > 10^{12}$ (TOO BIG).

Under the assumption that 3 is a prime factor of n we look at the cases $2^{13}127{\cdot}43{\cdot}11{\cdot}3^{\alpha_5}, 1 \le \alpha_5 \le 6$. Two more harmonic seeds are found from these cases as well several new branching cases. The new harmonic seeds are $66433720320 = 2^{13} \cdot 127 \cdot 43 \cdot 11 \cdot 3^3 \cdot 5$ and $57575890944 = 2^{13} \cdot 127 \cdot 43 \cdot 11 \cdot 3^2 \cdot 13$.

The longest (and the most involved) TROUBLEsome case involved $2^9 31 \cdot 11$ for which $H(2^9 31 \cdot 11) = 2^5 5/3^2$. The continuation $2^9 31 \cdot 11 \cdot 3$ leads to a search for one additional component of the form p^2, $p = 5, 7, 13, ..., 1381$. Another set of continuations involves a pair of squared primes $p^2 q^2$, $p = 5, 7, 13, ..., 43$ and $p < q$. Higher multiples of the exponents lead to the contradiction TOO BIG.

2.3.2 Generating harmonic numbers

Given a harmonic number n, further harmonic numbers may be generated from it by considering numbers of the form $n \cdot q_1^{\beta_1} q_2^{\beta_2} \cdots q_t^{\beta_t}$ where q_i are distinct odd primes, $q_i \nmid n$, $\beta_i > 0$ and $t \geq 1$. To assist in the derivation of additional harmonic numbers from known harmonic numbers the following is useful.

Theorem 1 (Cohen and Sorli [31]). *Suppose n and $n \cdot q_1 q_2 \dots q_t$ are harmonic numbers, where $q_1 < q_2 < \dots < q_t$ are primes not dividing n. Then $n \cdot q_1$ is harmonic, except when $t \geq 2$ and $q_1 q_2 = 6$, in which case $n \cdot q_1 q_2$ is harmonic. Further, if nq_1 is harmonic then $q_1 < 2H(n)$.*

Proof. We may assume $t \geq 2$. Suppose first that $q_1 \geq 3$. Since $n \cdot q_1 q_2 \dots q_t$ is harmonic and H is multiplicative,

$$H(n \cdot q_1 q_2 \dots q_t) = H(n) \prod_{i=1}^{t} H(q_i) = H(n) \prod_{i=1}^{t} \frac{2q_i}{q_i + 1} = h,$$

say, where h is an integer. Then

$$H(n) \prod_{i=1}^{t} q_i = h \prod_{i=1}^{t} \frac{q_i + 1}{2}.$$

Since

$$\frac{q_1 + 1}{2} < \frac{q_2 + 1}{2} < \dots < \frac{q_t + 1}{2} < q_t,$$

we have $q_t \mid h$, and then

$$H(n \cdot q_1 q_2 \dots q_{t-1}) = H(n) \prod_{i=1}^{t-1} \frac{2q_i}{q_i + 1} = \frac{h}{q_t} \cdot \frac{q_t + 1}{2} \in \mathbb{N}.$$

Applying the same argument to the harmonic number $n \cdot q_1 q_2 \dots q_{t-1}$, and repeating it as necessary, leads to our result in this case. If $q_1 = 2$, $H(2) = 4/3$ and so either $q_t \mid h$, which implies as before that $n \cdot q_1$ is harmonic, or $q_2 = 3$, in which case $n \cdot q_1 q_2 = 6n$ with $H(6) = 2$.

If nq_1 is harmonic then $H(nq_1) = H(n) \cdot 2q_1/(q_1 + 1)$. For this to be integral $(q_1 + 1)/2 \mid H(n)$ and so $q_1 < 2H(n)$ □

By assuming $\beta_i = 1$, for all i, Theorem 1 forms the basis of a one-prime-at-a-time procedure for generating harmonic squarefree multiples of a harmonic seed and the result $q_1 < 2H(n)$ implies that the search would be relatively short.

As an example of the repeated use of this idea, a list of all harmonic numbers that are squarefree multiples of the seed 2457000 is given in Table 2.1. It is not difficult to show that the list is complete, and in fact it seems clear that there are only finitely many harmonic squarefree multiples of any harmonic number. However a proof of this statement appears to be difficult.

As another example the harmonic seed $18620 = 2^2 5 \cdot 7^2 19$ gives rise to the harmonic numbers $55860 = 18620 \cdot 3$ and $242060 = 18620 \cdot 13$. These two harmonic numbers, in turn, give rise to the harmonic numbers $726180 = 55860 \cdot 13 = 242060 \cdot 3 = 18620 \cdot 13 \cdot 3$ and $2290260 = 55860 \cdot 41 = 18620 \cdot 3 \cdot 41$. Note the two distinct derivations for the harmonic number 726180. Importantly, we can demonstrate here a multiple other than a squarefree multiple which is also harmonic, namely $2178540 = 242060 \cdot 3^2 = 18620 \cdot 13 \cdot 3^2$. The harmonic number 2178540 leads to further harmonic numbers which are squarefree multiples. It is convenient to represent this sequence of derivations

Table 2.1: All harmonic numbers which are squarefree multiples of 2457000

n		$H(n)$	
27027000	$2^3 3^3 5^3 7 \cdot 11 \cdot 13$	110	$2 \cdot 5 \cdot 11$
513513000	$2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 19$	209	$11 \cdot 19$
18999981000	$2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37$	407	$11 \cdot 37$
1386998613000	$2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 73$	803	$11 \cdot 73$
1162161000	$2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 43$	215	$5 \cdot 43$
2945943000	$2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 109$	218	$2 \cdot 109$
2457000	$2^3 3^3 5^3 7 \cdot 13$	60	$2^2 3 \cdot 5$
46683000	$2^3 3^3 5^3 7 \cdot 13 \cdot 19$	114	$2 \cdot 3 \cdot 19$
1727271000	$2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 37$	222	$2 \cdot 3 \cdot 37$
126090783000	$2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73$	438	$2 \cdot 3 \cdot 73$
765181053000	$2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 37 \cdot 443$	443	443
5275179000	$2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 113$	226	$2 \cdot 113$
10597041000	$2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 227$	227	227
56511000	$2^3 3^3 5^3 7 \cdot 13 \cdot 23$	115	$5 \cdot 23$
12941019000	$2^3 3^3 5^3 7 \cdot 13 \cdot 23 \cdot 229$	229	229
5914045683000	$2^3 3^3 5^3 7 \cdot 13 \cdot 23 \cdot 229 \cdot 457$	457	457
71253000	$2^3 3^3 5^3 7 \cdot 13 \cdot 29$	116	$2^2 29$
144963000	$2^3 3^3 5^3 7 \cdot 13 \cdot 59$	118	$2 \cdot 59$

graphically as a directed acyclic graph as in Figure 2.2 where all nodes are harmonic numbers, the root node (18620) is a harmonic seed and edge labels are multiples.

Appendix A gives all values of a single feasible multiplier q_1 together with Q_p, the number of harmonic squarefree multiples (including the number itself), for each harmonic seed less than 10^{15}. The concept behind the listed Q_{p^a} values is introduced next.

Garcia [47] gave several harmonic numbers which are derived using one additional component, $q_1^{\beta_1}$, $\beta_1 > 1$ at a time. One example has been given above. Other examples include,

$$H(301953024) = H(2^{12} \cdot 8191) \cdot H(3^2) = 13 \cdot \frac{3^3}{13} = 27,$$

$$H(163390500) = H(2^2 5^3 7^2 \cdot 13 \cdot 19) \cdot H(3^3) = 50 \cdot \frac{3^3}{2 \cdot 5} = 135$$

and

$$H(80533908000) = H(2^5 3^2 7 \cdot 13^2 31 \cdot 61) \cdot H(5^3) = 117 \cdot \frac{5^3}{3 \cdot 13} = 375.$$

While there does not seem to be a general rule for deriving non-squarefree harmonic multiples of harmonic numbers, an upper bound on a single prime power multiple can be established, as follows.

Theorem 2. *Suppose n is harmonic, p is prime, $p \nmid n$ and $a \geq 2$ is an integer. Then $n \cdot p^a$ is harmonic if and only if $\sigma(p^a) \mid H(n)(a+1)$. Further, if $n \cdot p^a$ is harmonic then $p < \sqrt{3H(n)}$ and, where $p \geq 3$,*

$$a < \frac{\log(3H(n)/4)}{\log(p/2)}.$$

Proof. The first part is clear since

$$H(n \cdot p^a) = H(n)\,\frac{p^a(a+1)}{\sigma(p^a)}$$

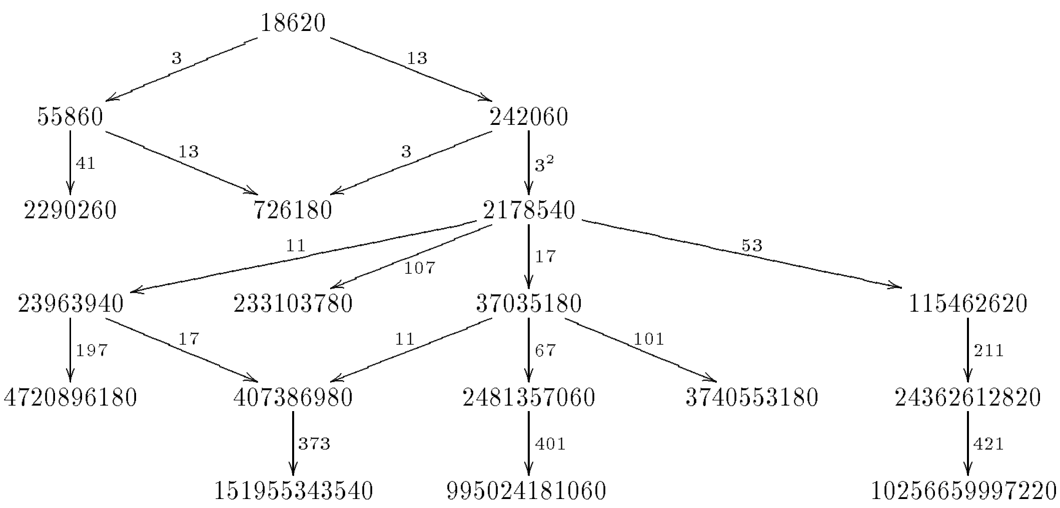

Figure 2.2: Derivation graph for the harmonic seed 18620

and $(p^a, \sigma(p^a)) = 1$. For the second part, observe that $p^{a-2} \geq 2^{a-2} \geq (a+1)/3$. Then

$$H(n) \geq \frac{\sigma(p^a)}{a+1} > \frac{p^a}{a+1} \geq \frac{p^a}{3 \cdot 2^{a-2}} = \frac{4}{3}\left(\frac{p}{2}\right)^a \geq \frac{4}{3}\left(\frac{p}{2}\right)^2 = \frac{p^2}{3}$$

so $p < \sqrt{3H(n)}$. From the above, we also have,

$$H(n) > \frac{4}{3}\left(\frac{p}{2}\right)^a$$

so $a < \log(3H(n)/4)/\log(p/2)$, provided $p \geq 3$. □

Note that if $p = 2$ then n is an odd harmonic number and we bound a from $H(n) > 2^a/(a+1)$.

This result indicates a relatively short search for possible p^a. For each such candidate we then check if $\sigma(p^a) \mid H(n)(a+1)$.

Appendix A includes Q_{p^a}, the number of harmonic numbers generated from each harmonic seed with one prime power at a time $(q^\beta, \beta > 0)$. In each case, it was demonstrable that all such multiples had been obtained and of these only a few were generated by a component q^β, $\beta > 1$ (as given by the number in parentheses following the Q_{p^a} value in the appendix).

While all harmonic numbers less than 10^7 can be generated from harmonic seeds using Theorems 1 and 2, mostly one prime at a time, there are harmonic numbers greater that 10^7 which are not so easily derived. One example (the smallest) can be found in the list of harmonic numbers less than 2×10^9 given by Cohen [29]. The harmonic seed of 13660770240 is $2^6 127$ since

$$H(13660770240) = H(2^6 127) \cdot H(3^2 5 \cdot 13^3 17) = 7 \cdot \frac{169}{7} = 169.$$

but no multiple of $2^6 127$ with a proper unitary factor of $3^2 5 \cdot 13^3 17$ is harmonic, as can be seen from

$$H(3^2 5 \cdot 13^3 17) = \frac{3^3}{13} \cdot \frac{5}{3} \cdot \frac{13^3}{5 \cdot 7 \cdot 17} \cdot \frac{17}{3^2} = \frac{13^2}{7}.$$

The simple (but time consuming) exhaustive search for harmonic numbers less than 10^{12} uncovered only one other example,

$$H(34482792960) = H(2^9 7 \cdot 11^2 19 \cdot 31) \cdot H(3^3 5) = 88 \cdot \frac{3^3}{2 \cdot 5} \cdot \frac{5}{3} = 396.$$

See Appendix C for the calculations for the search for even harmonic seeds less than 10^{12}.

2.4 Odd harmonic numbers

In order to find odd harmonic numbers (if any) below some bound, we shall seek odd harmonic seeds. The search for odd harmonic seeds follows the same basic factor chain approach as for the search for even harmonic seeds described previously. There are, though, some additional conditions that modify and improve the search. Firstly, the assumed initial (smallest) component takes the values $3^{\alpha_1}, 5^{\alpha_1}, \ldots, P^{\alpha_1}$ where P depends on the chosen bound (rather than the single case, 2^{α_1}, for the even harmonic number search). Then there is the restriction that $p^a \equiv 1 \pmod 4$ (Lemma 2) for all components p^a so that the set of allowable components, in increasing order is $\{5, 3^2, 13, 17, 5^2, 29, 37, 41, 7^2, 53, 61, 73, 3^4, 89, 97, \ldots\}$. Lastly, there is a lower bound of 10^7 on the largest component of an odd harmonic number (Lemma 3).

As for the even harmonic numbers previously discussed, the search bound for odd harmonic numbers was set at 10^{12}. Under these conditions the following results can be used to further restrict the search for odd harmonic numbers.

Lemma 7. *Suppose n is harmonic, $n > 1$. If $p^a \parallel n$ then $n\tau(n) > p^{2a}$.*

Proof. Since $H(n)\sigma(n) = n\tau(n)$ and $p^a \parallel n$ we have $H(n)\sigma(\frac{n}{p^a})\sigma(p^a) = \frac{n}{p^a}\tau(n)p^a$. Further, since $(\sigma(p^a), p^a) = 1$ we have $\sigma(p^a) \mid \frac{n}{p^a}\tau(n)$ and so $\frac{n}{p^a}\tau(n) \geq \sigma(p^a) > p^a$ which gives the result $n\tau(n) > p^{2a}$. □

Lemma 8. *Suppose n is an odd harmonic number. If $n < 10^{12}$ then $\omega(n) \leq 5$ and $\tau(n) > 100$.*

Proof. If $\omega(n) > 5$ and p^a is the largest component of n then by Lemma 3

$$n \geq 3^2 5 \cdot 13 \cdot 17 \cdot 29 \cdot p^a > 3^2 5 \cdot 13 \cdot 17 \cdot 29 \cdot 10^7 > 10^{12},$$

a contradiction. Also, using Lemmas 3 and 7, if $\tau(n) \leq 100$ then

$$n > \frac{p^{2a}}{\tau(n)} > \frac{10^{14}}{100} = 10^{12},$$

a contradiction. □

To illustrate these ideas, some sample calculations for a bound of $B = 10^{12}$ will now be discussed. We assume that, initially, the odd number n has the prime factor decomposition

$$n = 3^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}$$

with $3 < p_i$, $r \leq 5$. In calculating the bound on the exponent of the initial component 3^{α_1}, we need to ensure that α_1 is even so that $3^{\alpha_1} \equiv 1 \pmod 4$. Since there must be (at least) three distinct components, any additional components are selected, in increasing value, from the allowable set or 10^7 if all other components are less than 10^7.

If 3^{α_1} is the largest component then $3^{\alpha_1} > 10^7$ so $\alpha_1 > 7/\log(3)$ or $\alpha_1 \geq 16$. But then $n \geq 3^{\alpha_1}5 \cdot 13 > 10^{12}$ if $\alpha_1 > \log(10^{12}/(5 \cdot 13))/\log(3)$, so $\alpha_1 \leq 20$. If 3^{α_1} is not the largest component then $n \geq 3^{\alpha_1}5 \cdot 10^7 > 10^{12}$ if $\alpha_1 > \log(10^{12}/(5 \cdot 10^7))/\log(3)$, so $\alpha_1 \leq 8$. That is, for $B = 10^{12}$, we need only investigate 3^{α_1}, $2 \leq \alpha_1 \leq 8$ and $16 \leq \alpha_1 \leq 20$. For $B = 10^{15}$ this changes to $2 \leq \alpha_1 \leq 26$ (without any exclusions). As implemented the algorithm considers only a single interval.

Looking at the case $\alpha_1 = 20$,

$$H(3^{20}) = \frac{3^{21}7}{13 \cdot 1093 \cdot 368089}.$$

Obviously 368089 must be a prime factor of n (as must 1093 and 13) but then $n \geq 3^{20}368089 > 10^{12}$ (identified as TOO BIG in the proof).

For $\alpha_1 = 12$,

$$H(3^{12}) = \frac{3^{12}13}{797161}.$$

Since 797161 must be a prime factor of n and $3^{12}797161 \cdot 5 < 10^{12}$ (5 included to be the smallest third component) we continue the factor-chain with $3^{12}797161$, for which

$$H(3^{12}797161) = \frac{3^{12}13}{398581}.$$

Now 398581 must also be a prime factor of n, but then $n \geq 3^{12}797161 \cdot 398581 > 10^{12}$ (TOO BIG).

The first branching case occurs for $\alpha_1 = 4$,

$$H(3^4) = \frac{3^4 5}{11^2}.$$

Here 11 could be a prime factor or $11 - 1 = 10$ could be an exponent of a prime factor. Because the denominator contains 11^2, then if 11 is not a prime factor of n, n would have to contain either two prime factors to an exponent $11 - 1 = 10$ (or higher) or a single prime factor to an exponent $11^2 - 1 = 120$ (or higher). Since both of these cases lead to n becoming TOO BIG, 11 must be a factor and we investigate $3^4 11^{\alpha_2}, \alpha_2 \leq 9$ and even.

Having eliminated 3 as a possible factor, we next consider that the odd number n has the prime factor decomposition

$$n = 5^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}$$

with $5 < p_i$, $r \leq 5$. Since $5 \equiv 1 \pmod 4$, all powers of 5 must be considered.

If 5^{α_1} is the largest component then $\alpha_1 \geq 11$. But then $n \geq 5^{\alpha_1}13 \cdot 17 > 10^{12}$ if $\alpha_1 > 13$. If 5^{α_1} is not the largest component then $n \geq 5^{\alpha_1}7 \cdot 10^7 > 10^{12}$ if $\alpha_1 > 4$. That is, for $B = 10^{12}$, we need only investigate 5^{α_1}, $1 \leq \alpha_1 \leq 4$ and $11 \leq \alpha_1 \leq 13$. For $B = 10^{15}$ this changes to $1 \leq \alpha_1 \leq 9$ and $11 \leq \alpha_1 \leq 18$. For still larger values of B the gap disappears.

Using the same approach as was applied to 3 as the initial prime factor, it can be shown that 5 cannot be a prime factor of n. This is continued to cover $7, 11, 13, \ldots, P$, each in turn being assumed to be the initial (smallest) prime. By Lemma 3, for a bound of 10^{12}, $P = 313$ since $317 \cdot 331 \cdot 10^7 > 10^{12}$.

The case $5^7 313$ is the first time a contradiction (case (iii)) is used to terminate a branch of the search (identified as ALREADY in the proof). For $5^7 313$, we have $H(5^7) = 5^7/(3 \cdot 13 \cdot 313)$ and

$H(5^7 313) = 5^7/(3 \cdot 13 \cdot 157)$. Both 157 and 13 must be prime factors of n. Also 3^2 must be a factor of n since, if $3-1=2$ was an exponent, the worst case otherwise would be $n \geq 5^7 313 \cdot 157 \cdot 13 \cdot 7^2 > 10^{12}$ since $5^7 313 \cdot 157 \cdot 13^2 < 10^{12}$ but is not harmonic. However 3 has already been eliminated as a possible prime factor of n. This is the contradiction which terminates this branch of the search.

The longest case is for $p_1^{\alpha_1} = 5$ with $H(5) = 5/3$. Since 3 has already been eliminated as a possible prime factor of n, we consider the continuation $n = 5 \cdot p^{3k-1} \cdots$ for prime p and positive integer k. While the smallest value of p is 7, the largest value is given by the next smallest prime to the value $\lfloor\sqrt{B/(5 \cdot 13)}\rfloor$, which is 124021 for a bound $B = 10^{12}$ (and a substantial 124034723 for $B = 10^{18}$). We consider the square of every prime from 7 to 124021 (all 11650 of them) as the second component of n, then all $p^5, p \leq 107$, and $p^8, p \leq 17$ and lastly 7^{11} (since $5 \cdot 7^{14} 13 > 10^{12}$).

The most complicated case is for $p_1^{\alpha_1} = 17$, $H(17) = 17/3^2$. Since 3 has already been eliminated as a possible prime factor of n, then either there are two additional prime factors each to an exponent of $3k_i - 1$ or there is one additional prime factor to an exponent of $3^2 k - 1$. While the single additional prime to the power 8 ($k = 1$) leads to the conclusion TOO BIG, the case of pairs of additional prime factors leads to continuations of $19^2 23^2$, $19^2 29^2$, …, $19^2 12763^2$ and $23^2 29^2$, $23^2 31^2$, …, $23^2 10531^2$ up to $487^2 491^2$ and then the higher powers $19^5 23^2$, …, $31^5 37^2$, and $19^2 23^5$, …, $29^2 31^5$ (no higher powers are feasible). Where the termination is identical, these have been abbreviated in calculations shown in Appendix D where the details of the search for odd harmonic seeds less than 10^{12} can be found.

2.5 Implementation

During the development of the algorithm several simple computer programs were written in UBASIC (a dialect of BASIC which includes multiprecision and rational arithmetic as well as many useful number-theoretic functions). The differences between the even and odd searches were easily handled in a combined program with the even or odd case selected via a simple programmed switch. While the factor-chain approach (essentially a decision tree) is ideally suited to a recursive implementation, this had to be emulated in non-recursive UBASIC. The most significant consequence of this was that the branching cases had to be separately investigated and then the results combined. The algorithm for finding even harmonic seeds has since been re-implemented in *Mathematica* as a set of recursive functions. This version was used to extend the calculations to a bound of 10^{15}, Appendix A.

2.6 Results

To give some idea of the growth in the size of the search, Table 2.2 gives some bounds on the initial prime factors (value, power and number). Qualitatively, the branching factor and the depth of the search increase as the bound B is increased, leading to a large overall increase in the total number of cases to be considered. For the odd harmonic number case, even with the reduction in the density of the search trees (Lemma 2) and in the depth of each search tree (Lemma 3), there is still an overall increase in the work involved. For this reason complete details for the even and odd searches are given for a bound of 10^{12} but subsequently the work was carried out for $B = 10^{15}$. (In that case, Lemma 8 is replaced by: if $n \leq 10^{15}$ then $\omega(n) \leq 7$.) The result of the searches can be given as follows. Since an odd harmonic number must have an odd harmonic seed, we have.

Theorem 3. *There are no odd harmonic numbers less than* 10^{15}.

Guy [57] asked if a perfect square could be harmonic. Since a powerful hamonic number must have a powerful harmonic seed, observation of Appendix A gives

Theorem 4. *There are no powerful harmonic numbers less than* 10^{15}.

Goto and Shibata [53] noticed that no known harmonic number is deficient. Since a deficient harmonic number must have a deficient harmonic seed, observation of Appendix A gives

Theorem 5. *There are no deficient harmonic numbers less than* 10^{15}.

The growth in the number of harmonic seeds and resultant harmonic numbers can be seen in Table 2.3 together with an indication of the search effort involved. An analysis of the complete results up to 10^{12} and exploration of the even harmonic seeds up to 10^{15} suggest the following:

Conjecture 1. *Any harmonic number has a unique harmonic seed.*

Conjecture 2. *There are only finitely many harmonic multiples of any harmonic number.*

Table 2.2: Bounds concerned with the smallest prime factor in determining all harmonic seeds less than 10^n. (Here $\pi(m)$ denotes the number of primes not exceeding m)

	Even	Odd		
n	$\max(\alpha_1), p_1 = 2$	$\max(\alpha_1), p_1 = 3$	$\max(p_1)$	$\pi(\max(p_1)) - 1$
9	25	14	167	38
12	35	20	991	166
15	45	26	5591	737
18	55	32	31583	3398
21	65	40	177791	16140
24	75	46	999961	78495

Table 2.3: Number of harmonic seeds and harmonic numbers less than 10^n with time to find seeds relative to the $n = 6$ case

n	seeds	numbers	rel. time
4	9	12	0.2
5	12	18	0.5
6	13	31	1.0
7	19	46	2.3
8	26	72	6.3
9	33	115	16.7
10	42	181	56.6
11	56	295	250.1
12	74	425	948.9
13	101	>617	3763.0
14	138	>917	14111.9
15	174	>1397	56436.3

Chapter 3

Even Multiperfect Numbers

3.1 Some basic facts

A natural number N is said to be multiperfect if it satisfies the condition $\sigma(N) = kN,\ k \in \mathbb{N}$ (or $S(N) = k$). Such a number is said to be k-perfect (or have multiplicity k). Perfect numbers are 2-perfect while proper multiperfect numbers have multiplicity $k > 2$. Perfect numbers were discussed in Section 1.2.1 and odd perfect numbers will be further investigated in Chapter 5. We call 3-perfect numbers triperfect. Odd triperfect numbers will be further investigated in Chapter 4.

The smallest proper multiperfect number is the 3-perfect number 120:

$$\begin{aligned}\sigma(120) &= 1 + 2 + 3 + 4 + 5 + 6 + 8 + 10 + 12 + 15 + 20 + 24 \\ &\quad + 30 + 40 + 60 + 120 \\ &= 360 = 3 \cdot 120\end{aligned}$$

or

$$S(120) = S(2^3 3 \cdot 5) = S(2^3)S(3)S(5) = \frac{3 \cdot 5}{2^3} \cdot \frac{2^2}{3} \cdot \frac{2 \cdot 3}{5} = 3.$$

The smallest 4-perfect number is $30240 = 2^5 3^3 5 \cdot 7$. The largest multiperfect number known (as of December 2001) is a 11-perfect number that is greater than 2.519×10^{1906} and has 2^{468}, 3^{140}, 5^{66} and 7^{49} as components.

No odd multiperfect number other than 1 has been found. As for harmonic numbers, if a proof of the non-existence of odd multiperfect numbers can be found then this will also be a proof that no odd perfect numbers exist.

The theoretical results for even multiperfect numbers are sparse. Indeed, while the existence of even multiperfect numbers with multiplicities $3 \leq k \leq 11$ has been demonstrated, there is no proof that there are a finite number with a given multiplicity even though this is strongly believed to be the case, particularly for triperfect numbers.

3.2 A basic search procedure

If N is to be an even multiperfect number with 2^n as a component then we want to find the $p_i^{a_i}$ such that $N = 2^n p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$ with an integral $S(N)$. A systematic search approach involves the construction of a sequence $N_0, N_1, N_2, \ldots$ where $N_0 = 2^n$ and $N_{i+1} = N_i \cdot p_{i+1}^{a_{i+1}} = 2^n p_1^{a_1} p_2^{a_2} \ldots p_{i+1}^{a_{i+1}}$ for a sequence of distinct primes $P = \{p_1, p_2, \ldots\}$ and exponents $E = \{a_1, a_2, \ldots\}$. A common heuristic is to choose p_{i+1} so that it is the largest prime divisor of the numerator of $S(N_i)$ [101]. This is based on the principle that if $\sigma(N) = kN$ and $p^a \parallel N$ then $\sigma(p^a) \mid \sigma(N)$ so $q \mid N$ if $q \mid \sigma(p^a)$ (and $q \nmid k$). The determination of the corresponding a_i is more difficult and will be discussed later.

For example, let $N_0 = 2^8$. Then

$$S(N_0) = \frac{2^9 - 1}{2^8(2-1)} = \frac{511}{256} = \frac{7 \cdot 73}{2^8}.$$

Using the "largest new prime divisor next" heuristic and with all $a_i = 1$ we let $p_1 = 73$, so $N_1 = 2^8 73$ with

$$S(N_1) = S(2^8)S(73) = \frac{7 \cdot 73}{2^8} \cdot \frac{2 \cdot 37}{73} = \frac{7 \cdot 37}{2^7}.$$

This suggests $p_2 = 37$, $N_2 = 2^8 73 \cdot 37$ with

$$S(N_2) = S(2^8 73)S(37) = \frac{7 \cdot 37}{2^7} \cdot \frac{2 \cdot 19}{37} = \frac{7 \cdot 19}{2^6}$$

leading to $p_3 = 19$, $N_3 = 2^8 73 \cdot 37 \cdot 19$ with

$$S(N_3) = S(2^8 73 \cdot 37)S(19) = \frac{7 \cdot 19}{2^6} \cdot \frac{2^2 5}{19} = \frac{5 \cdot 7}{2^4}.$$

Continuing with the search chain we have $p_4 = 7$, $N_4 = 2^8 73 \cdot 37 \cdot 19 \cdot 7$ with

$$S(N_4) = S(2^8 73 \cdot 37 \cdot 19)S(7) = \frac{5 \cdot 7}{2^4} \cdot \frac{2^3}{7} = \frac{5}{2}.$$

Finally we have $p_5 = 5$, $N_5 = 2^8 73 \cdot 37 \cdot 19 \cdot 7 \cdot 5$ with

$$S(N_5) = S(2^8 73 \cdot 37 \cdot 19 \cdot 7)S(5) = \frac{5}{2} \cdot \frac{2 \cdot 3}{5} = 3.$$

We have found the 3-perfect number $2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$.

Unfortunately this simple search did not find (or even suggest) several other multiperfect numbers with 2^8 as a component. They can be found by searching higher powers of known prime factors (excluding 2) and by using simple substitutions. These substitutions will be described in detail in Section 3.4. In this example, one of the earliest discovered substitutions [38] can be used. For $N = 2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$ with $S(N) = 3$, $3 \nmid N$, we find that $3N = 2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$ is a 4-perfect number. Searching with a higher power of 19 leads to the investigation of $p_3 = 19$, $a_3 = 2$, that is $N_3 = 2^8 73 \cdot 37 \cdot 19^2$ with

$$S(N_3) = S(2^8 73 \cdot 37)S(19^2) = \frac{7 \cdot 19}{2^6} \cdot \frac{3 \cdot 127}{19^2} = \frac{3 \cdot 7 \cdot 127}{2^4 19}.$$

Continuing in this manner eventually leads to the 4-perfect number $2^8 3^2 7^2 13 \cdot 19^2 37 \cdot 73 \cdot 127$. A more recent substitution [101] which is also applicable to this example is the replacement of $19^2 127$ by $19^4 151 \cdot 911$ (since $S(19^2 127) = S(19^4 151 \cdot 911) = 2^7 3/19^2$) which gives $2^8 3^2 7^2 13 \cdot 19^4 37 \cdot 73 \cdot 151 \cdot 911$, another 4-perfect number. In this way we have found all four known multiperfect numbers with 2^8 as a component.

While the p_i's are usually determined from earlier σ calculations in the search chain, the corresponding powers, a_i, may change as a result of "backing up" in the search procedure [101]. This occurs when no continuation, p_{i+1}, can be found (and we have not found a multiperfect number). At this point alternate continuations are given by the primes (> 2) in the numerator of $S(N_i)$ (i.e. in

$\sigma(N_i)$). This situation is commonly referred to as having an excess of those particular primes. This means that if p_i is one such prime then the assumption that $p_i^{a_i}$ is a component of N is incorrect and, since the lower powers have been investigated, that a higher power of p_i may be needed. For example, if $p_i^{a_i} \parallel N_j$ and there is no direct continuation of this sequence, and we have $p_i \mid \sigma(N_j)$ then an alternative is to continue with $N_i = N_{i-1} \cdot p_i^{a_i+1}$.

Another example will illustrate this. A compact "tree" representation can be used to summarize the relevant calculations and will be used in any further examples. With $N_0 = 2^{10}$ as the starting component and writing S for $S(N_i)$, we have Figure 3.1.

In this case the search has found all three known multiperfect numbers with 2^{10} as a component. When there are several alternate continuations for "backing up", each will lead to distinct "subtrees". In a sequential investigation of these alternatives (as here) it is convenient to consider them in a most-recent-first (stack) order.

While a basic search procedure has been illustrated, several practical consideration concerned with "where", "when" and "howfar" have not been addressed. They will be taken up in Section 3.6.

3.3 Current results

Table 3.1 lists the first known multiperfect numbers N with given values of multiplicity k [38, 108]. Where several multiperfect numbers of a given index were reported at the same time, the first listed is credited with being the first. This is usually the one with the smallest exponent of 2.

Table 3.2 lists the smallest known multiperfect numbers N with given values of multiplicity k. The smallest were not always the first to be found and do not always correspond to the k-perfect number with the smallest power of 2 as a factor.

The largest known multiperfect numbers N with given values of multiplicity k are listed in Table 3.3. The largest values do not always correspond to those with the largest power of 2 as a factor.

In 1997 the author used the factor chain type of search to be described in Section 3.6 to find the then largest multiperfect number known, a 9-perfect number, $2^{238}3^{60}5^{30}7^{30}\cdots$, which exceeds 5.536×10^{935}. A more significant discovery by the author was that of the first 10-perfect number (an 11-perfect number has since been discovered by Woltman).

While new 9- and 10-perfect numbers continue to be discovered, it is very unlikely that there may be smaller ones with index less than 9. There is common agreement that all multiperfect numbers with indices less than 8 have probably been found, even though, in particular, the small set of known triperfect numbers has not been proven to be complete.

$$2^{10} : S = \frac{23 \cdot 89}{2^{10}}$$

$$89 : S = \frac{23 \cdot 89}{2^{10}} \cdot \frac{2 \cdot 3^2 5}{89} = \frac{3^2 5 \cdot 23}{2^9}$$

$$23 : S = \frac{3^2 5 \cdot 23}{2^9} \cdot \frac{2^3 3}{23} = \frac{3^3 5}{2^6}$$

$$5 : S = \frac{3^3 5}{2^6} \cdot \frac{2 \cdot 3}{5} = \frac{3^4}{2^5}$$

$$3 : S = \frac{3^4}{2^5} \cdot \frac{2^2}{3} = \frac{3^3}{2^3}$$

backup on 3

$$3^2 : S = \frac{3^4}{2^5} \cdot \frac{13}{3^2} = \frac{3^2 13}{2^5}$$

$$13 : S = \frac{3^2 13}{2^5} \cdot \frac{2 \cdot 7}{13} = \frac{3^2 7}{2^4}$$

$$7 : S = \frac{3^2 7}{2^4} \cdot \frac{2^3}{7} = \frac{3^2}{2}$$

backup on 3

$$3^3 : S = \frac{3^4}{2^5} \cdot \frac{2^3 5}{3^3} = \frac{3 \cdot 5}{2^2}$$

backup now on 3 (and later on 5)

$$3^4 : S = \frac{3^4}{2^5} \cdot \frac{11^2}{3^4} = \frac{11^2}{2^5}$$

$$11 : S = \frac{11^2}{2^5} \cdot \frac{2^2 3}{11} = \frac{3 \cdot 11}{2^3}$$

backup now on 11 (and later on 3)

$$11^2 : S = \frac{11^2}{2^5} \cdot \frac{7 \cdot 19}{11^2} = \frac{7 \cdot 19}{2^5}$$

$$19 : S = \frac{7 \cdot 19}{2^5} \cdot \frac{2^2 5}{19} = \frac{5 \cdot 7}{2^3}$$

$$7 : S = \frac{5 \cdot 7}{2^3} \cdot \frac{2^3}{7} = 5$$

found the 5-perfect number $2^{10} 89 \cdot 23 \cdot 5 \cdot 3^4 11^2 19 \cdot 7$

follow most recent unexplored alternate, 3

$$3^5 : S = \frac{3^4}{2^5} \cdot \frac{2^2 7 \cdot 13}{3^5} = \frac{7 \cdot 13}{2^3 3}$$

$$13 : S = \frac{7 \cdot 13}{2^3 3} \cdot \frac{2 \cdot 7}{13} = \frac{7^2}{2^2 3}$$

$$7 : S = \frac{7^2}{2^2 3} \cdot \frac{2^3}{7} = \frac{2 \cdot 7}{3}$$

backup on 7

$$7^2 : S = \frac{7^2}{2^2 3} \cdot \frac{3 \cdot 19}{7^2} = \frac{19}{2^2}$$

$$19 : S = \frac{19}{2^2} \cdot \frac{2^2 5}{19} = 5$$

found the 5-perfect number $2^{10} 89 \cdot 23 \cdot 5 \cdot 3^5 13 \cdot 7^2 19$

follow most recent unexplored alternate, 5

$$5^2 : S = \frac{3^3 5}{2^6} \cdot \frac{31}{5^2} = \frac{3^3 31}{2^6 5}$$

$$31 : S = \frac{3^3 31}{2^6 5} \cdot \frac{2^5}{31} = \frac{3^3}{2 \cdot 5}$$

$$3 : S = \frac{3^3}{2 \cdot 5} \cdot \frac{2^2}{3} = \frac{2 \cdot 3^2}{5}$$

backup on 3

$$3^2 : S = \frac{3^3}{2 \cdot 5} \cdot \frac{13}{3^2} = \frac{3 \cdot 13}{2 \cdot 5}$$

$$13 : S = \frac{3 \cdot 13}{2 \cdot 5} \cdot \frac{2 \cdot 7}{13} = \frac{3 \cdot 7}{5}$$

$$7 : S = \frac{3 \cdot 7}{5} \cdot \frac{2^3}{7} = \frac{2^3 3}{5}$$

backup on 3

$$3^3 : S = \frac{3^3}{2 \cdot 5} \cdot \frac{2^3 5}{3^3} = 2^2$$

found the 4-perfect number $2^{10} 89 \cdot 23 \cdot 5^2 31 \cdot 3^3$

All alternates have been examined

Figure 3.1: Example decision tree for an even multiperfect number search

Table 3.1: First known multiperfect numbers

k	Found	N	
1	-	1	
2	Pythagoras 500BC?	6	$2 \cdot 3$
3	Mersenne 1631	120	$2^3 3 \cdot 5$
4	Descartes 1638	30240	$2^5 3^3 5 \cdot 7$
5	Descartes 1638	1.418×10^{10}	$2^7 3^4 5 \cdot 7 \cdot 11^2 17 \cdot 19$
6	Fermat 1643	3.491×10^{40}	$2^{23} 3^7 5^3 7^4 11^3 13^3 17^2 31 \cdots$
7	Mason 1911	1.413×10^{55}	$2^{32} 3^{11} 5^4 7^5 11^2 13^2 17 \cdot 19^4 \cdots$
8	Poulet 1929	7.349×10^{172}	$2^{62} 3^{22} 5^{10} 7^{11} 11^3 13^7 17^2 19 \cdots$
9	Helenius 1992	7.984×10^{465}	$2^{130} 3^{49} 5^{15} 7^{20} 11^{10} 13^7 17^6 19^6 \ldots$
10	Sorli 1997	2.869×10^{923}	$2^{240} 3^{77} 5^{41} 7^{30} 11^{13} 13^{19} 17^{11} 19^{14} \ldots$
11	Woltman 2001	2.519×10^{1906}	$2^{468} 3^{140} 5^{66} 7^{49} 11^{40} 13^{31} 17^{11} 19^{12} \ldots$

Table 3.2: Smallest known multiperfect numbers

k	Found	N	
1	-	1	
2	Pythagoras 500BC?	6	$2 \cdot 3$
3	Mersenne 1631	120	$2^3 3 \cdot 5$
4	Descartes 1638	30240	$2^5 3^3 5 \cdot 7$
5	Descartes 1638	1.418×10^{10}	$2^7 3^4 5 \cdot 7 \cdot 11^2 17 \cdot 19$
6	Carmichael 1907	1.543×10^{20}	$2^{15} 3^5 5^2 7^2 11 \cdot 13 \cdot 17 \cdot 19 \cdots$
7	Mason 1911	1.413×10^{56}	$2^{32} 3^{11} 5^4 7^5 11^2 13^2 17 \cdot 19^3 \cdots$
8	Gretton 1990	8.268×10^{132}	$2^{62} 3^{15} 5^9 7^7 11^3 13^3 17^2 19 \cdot 23 \cdots$
9	Helenius 1995	5.613×10^{286}	$2^{104} 3^{43} 5^9 7^{12} 11^6 13^4 17 \cdot 19^4 \ldots$
10	Moxham 1998	8.103×10^{684}	$2^{209} 3^{77} 5^{23} 7^{26} 11^{14} 13^{11} 17^9 19^{12} \ldots$
11	Woltman 2001	2.519×10^{1906}	$2^{468} 3^{140} 5^{66} 7^{49} 11^{40} 13^{31} 17^{11} 19^{12} \ldots$

Table 3.3: Largest known multiperfect numbers

k	Found	N	
1	-	1	
2	Cameron et al. 2001	$4.278 \times 10^{8107891}$	$2^{13466916} \cdot M_{13466917}$
3	Fermat 1643	51001180160	$2^{14} 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151$
4	Carmichael 1911	1.598×10^{45}	$2^{37} 3^6 7 \cdot 11 \cdot 23 \cdot 83 \cdot 137 \cdot 331 \cdots$
5	Mason 1911	1.886×10^{100}	$2^{40} 3^{12} 7^8 11^2 17^2 19^5 31 \cdot 37^2 43 \cdots$
6	Gretton 1992	1.451×10^{192}	$2^{92} 3^{18} 5^{10} 7^4 13^3 17^4 19 \cdot 29 \cdots$
7	Helenius 1993	1.185×10^{312}	$2^{102} 3^{25} 5^{11} 7^{17} 11^8 13^5 19^9 \cdot 29 \cdots$
8	Helenius 1996	1.955×10^{613}	$2^{168} 3^{62} 5^{27} 7^{16} 11^9 13^9 17^{10} 19^5 \ldots$
9	Flammenkamp 2001	7.641×10^{1165}	$2^{274} 3^{103} 5^{41} 7^{34} 11^{17} 13^{14} 17^{10} 19^{16} \ldots$
10	Woltman 2001	4.380×10^{1877}	$2^{534} 3^{147} 5^{67} 7^{52} 11^{40} 13^{27} 17^{13} 19^{10} \ldots$
11	Woltman 2001	2.519×10^{1906}	$2^{468} 3^{140} 5^{66} 7^{49} 11^{40} 13^{31} 17^{11} 19^{12} \ldots$

3.4 Substitutions

Under certain conditions new multiperfect numbers can be generated from some existing multiperfect numbers. The simplest type of generator, the first used in the example in Section 3.2, was discovered by Descartes [38] and can be defined as follows.

Lemma 9. *If $S(N) = pk$, p prime, $k \in \mathbb{N}$ and $p \nmid N$ then $S(pN) = (p+1)k$.*

Proof.

$$S(pN) = S(p)S(N) = \left(\frac{p+1}{p}\right) \cdot pk = (p+1)k.$$

□

Far more common is the process of generating new multiperfect numbers by substitution. This involves the substitution of two or more components in a multiperfect number. The most common (and useful) substitutions transform one multiperfect number into another with the same multiplicity [22, 101]. The second substitution used before is of this type and can be defined as follows.

Lemma 10. *Let $P < N$, $P \parallel N$, $\left(\frac{N}{P}, Q\right) = 1$ and $S(P) = S(Q)$, then $S(N) = S\left(\frac{N}{P}Q\right)$*

Proof.

$$S(N) = S\left(\frac{N}{P}P\right) = S\left(\frac{N}{P}\right) S(P) = S\left(\frac{N}{P}\right) S(Q) = S\left(\frac{N}{P}Q\right).$$

□

Such P and Q are referred to as a pair of factor sets. Table 3.4 gives some of the known substitutions of this type, grouped according to author [22, 101, 107, 54, 116, 44] respectively.

Less common are substitutions resulting in a change in multiplicity. Such substitutions (other than the type described by Lemma 9) involve a pair of factor sets P and Q which obey the following.

Lemma 11. *Let $P < N$, $P \parallel N$, $\left(\frac{N}{P}, Q\right) = 1$ and $S(P) = rS(Q)$, $r \in \mathbb{Q}$, then $S(N) = rS\left(\frac{N}{P}Q\right)$*

Proof.

$$S(N) = S\left(\frac{N}{P}P\right) = S\left(\frac{N}{P}\right) S(P) = rS\left(\frac{N}{P}\right) S(Q) = rS\left(\frac{N}{P}Q\right).$$

□

Table 3.5 gives some of the known substitutions of this type (grouped according to author [38],[22]).

The literature is unclear as to the importance of substitutions in the discovery of multiperfect numbers, with the exception of the pair of factor sets $19^2 127$ and $19^4 151 \cdot 911$ [18, 101] which has been applied in Section 3.2.

Table 3.4: Same index substitutions for multiperfect numbers

P	$S(P)=S(Q)$	Q
$2^7 17$	$\frac{3^3 5}{2^6}$	$2^{10} 23 \cdot 89$
$2^7 17$	$\frac{3^3 5}{2^6}$	$2^{25} 19 \cdot 683 \cdot 2731 \cdot 8191$
$2^9 31$	$\frac{3 \cdot 11}{2^4}$	$2^{13} 43 \cdot 127$
$2^{11} 3^5$	$\frac{5 \cdot 7^2 13^2}{2^9 3^3}$	$2^{20} 3^3 127 \cdot 337$
$2^{28} 7 \cdot 23 \cdot 233 \cdot 1103 \cdot 2089$	$\frac{3^4 5 \cdot 11 \cdot 13 \cdot 19}{2^{16} 7}$	$2^{36} 7^5 43 \cdot 223 \cdot 7019 \cdot 112303$
		$\cdot 898423 \cdot 616318177$
$2^{33} 131071$	$\frac{3 \cdot 43691}{2^{16}}$	$2^{37} 174763 \cdot 524287$
$2^{38} 53 \cdot 229 \cdot 8191 \cdot 121369$	$\frac{3^3 5^2 7 \cdot 23 \cdot 79}{2^{22}}$	$2^{61} 59 \cdot 157 \cdot 43331 \cdot 3033169$
		$\cdot 715827883 \cdot 2147483647$
$3^4 11^3 13$	$\frac{2^4 7 \cdot 61}{3^3 11 \cdot 13}$	$3^5 11 \cdot 13^2$
$3^6 137 \cdot 547 \cdot 1093$	$\frac{2^4 23}{3^5}$	$3^{10} 107 \cdot 3851$
$3^7 23 \cdot 41$	$\frac{2^8 5 \cdot 7}{3^5 23}$	$3^{10} 23^2 79 \cdot 107 \cdot 3851$
$5^2 7^2 19 \cdot 31$	$\frac{2^7 3}{5 \cdot 7^2}$	$5^3 7^3 13$
$5^2 7^2 19^2 127$	$\frac{2^7 3^2 31}{5^2 7^2 19}$	$5^5 7^3 19$
$5^2 13^2 31^2 61 \cdot 83 \cdot 331$	$\frac{2^5 3^3 7}{5^2 13^2}$	$5^3 13^3 17$
$5^3 7^4 13^3 17^2 307 \cdot 467 \cdot 2801$	$\frac{2^9 3^4 11}{5^2 7^2 13 \cdot 17}$	$5^4 7^3 13 \cdot 17 \cdot 71$
$3 \cdot 7 \cdot 13$	$\frac{2^6}{3 \cdot 13}$	$3^2 13^2 31 \cdot 61$
$3^2 5^2 31$	$\frac{2^5 13}{3^2 5^2}$	$3^3 5^3$
$3^3 5^2 13$	$\frac{2^4 7 \cdot 31}{3^3 5 \cdot 13}$	$3^5 5 \cdot 13^2 61$
$3^4 7 \cdot 11^2$	$\frac{2^3 19}{3^4}$	$3^5 7^2 13$
$3^5 7 \cdot 13$	$\frac{2^6 7}{3^5}$	$3^7 5 \cdot 41$
$5^2 7^2 11 \cdot 31$	$\frac{2^7 3^2 19}{5^2 7^2 11}$	$5^4 7^3 11^2 71$
$5^3 7 \cdot 13^2 61$	$\frac{2^6 3^2 31}{5^3 7 \cdot 13}$	$5^5 7^3 13$
$5^3 11 \cdot 13$	$\frac{2^5 3^2 7}{5^3 11}$	$5^4 11^2 19 \cdot 71$
$19^2 127$	$\frac{2^7 3}{19^2}$	$19^4 151 \cdot 911$
$2^3 5$	$\frac{3^2}{2^2}$	$2^5 7$
$2^{19} 7^2 41$	$\frac{3^3 5^2 11 \cdot 19 \cdot 31}{2^{18} 7}$	$2^{29} 7^3 83 \cdot 151 \cdot 331$
$3^2 7 \cdot 13$	$\frac{2^4}{3^2}$	$3^3 5$
$3^5 5^2 31$	$\frac{2^7 7 \cdot 13}{3^5 5^2}$	$3^7 5^3 41$
$5^2 7^5 19^2 127$	$\frac{2^{10} 3^2 31 \cdot 43}{5^2 7^5 19}$	$5^5 7^7 19 \cdot 601 \cdot 1201$
$13 \cdot 31^2 61 \cdot 83 \cdot 331$	$\frac{2^6 3^2 7^2}{13 \cdot 31 \cdot 61}$	$13^2 31 \cdot 61^2 97$
$23 \cdot 37^3 73 \cdot 137^2$	$\frac{2^6 3 \cdot 5 \cdot 7 \cdot 19}{23 \cdot 37 \cdot 137}$	$23^2 37 \cdot 79 \cdot 137$
$3^3 5^2 31$	$\frac{2^8}{3^3 5}$	$3^6 5 \cdot 23 \cdot 137 \cdot 547 \cdot 1093$
$3^4 11^3 31 \cdot 61$	$\frac{2^9}{3^3 11}$	$3^6 11 \cdot 23 \cdot 137 \cdot 547 \cdot 1093$
$3^5 13^2 31 \cdot 61$	$\frac{2^8 7}{3^4 13}$	$3^6 13 \cdot 23 \cdot 137 \cdot 547 \cdot 1093$
$3^6 23^2 79 \cdot 137 \cdot 547 \cdot 1093$	$\frac{2^8 5 \cdot 7}{3^5 23}$	$3^7 23 \cdot 41$
$3^{12} 1093 \cdot 797161$	$\frac{2^2 547 \cdot 398581}{3^{12} 1093}$	$3^{13} 1093^2$
$3^{18} 17 \cdot 19^3 47 \cdot 181 \cdot 607 \cdot 1213$	$\frac{2^{18} 5^3 7 \cdot 13}{3^{14} 19^2}$	$3^{20} 19^2 23 \cdot 41 \cdot 127 \cdot 137 \cdot 409$

Table 3.4: Same index substitutions (contd.)

P	$S(P)=S(Q)$	Q
$\cdot 1597 \cdot 36389 \cdot 363889$		$\cdot 547 \cdot 1093 \cdot 36809 \cdot 368089$
$3^{18}17^{2}19^{3}181 \cdot 307 \cdot 607 \cdot 1213$	$\frac{2^{15}5^{3}7^{2}11\cdot 13}{3^{17}17\cdot 19^{2}47}$	$3^{22}17 \cdot 19^{2}127 \cdot 139 \cdot 3613$
$\cdot 1597 \cdot 36389 \cdot 363889$		$\cdot 2384579 \cdot 1001523179$
$5 \cdot 7^{2}19 \cdot 31$	$\frac{2^{8}3^{2}}{7^{2}31}$	$5^{2}7^{3}31^{2}83 \cdot 331$
$5^{2}7^{2}13 \cdot 19^{2}127$	$\frac{2^{8}3^{2}31}{5^{2}7\cdot 13\cdot 19}$	$5^{3}7 \cdot 13^{2}19 \cdot 61$
$2 \cdot 5 \cdot 19 \cdot 31 \cdot 61$	$\frac{2^{8}3^{2}}{19\cdot 61}$	$3^{2}7^{2}13^{2}19^{2}61^{2}97 \cdot 127$
$2 \cdot 7 \cdot 11^{2}19$	$\frac{2^{4}3\cdot 5}{11^{2}}$	$3^{4}5 \cdot 11^{4}179 \cdot 3221$
$2 \cdot 13 \cdot 31$	$\frac{2^{5}3\cdot 7}{13\cdot 31}$	$3^{2}13^{2}31^{2}61 \cdot 83 \cdot 331$
$2 \cdot 7^{2}13 \cdot 19^{3}181$	$\frac{2^{4}3^{2}5}{19^{2}}$	$2^{3}19^{2}127$
$2 \cdot 19^{2}127$	$\frac{2^{6}3^{2}}{19^{2}}$	$5 \cdot 7^{2}13 \cdot 19^{3}181$
$2^{2}13^{2}31 \cdot 61$	$\frac{2^{4}3\cdot 7}{13^{2}}$	$3^{2}5 \cdot 13^{3}17$
$2^{2}31 \cdot 61$	$\frac{2^{4}7}{61}$	$3^{2}7 \cdot 13^{2}61^{2}97$
$2^{3}31 \cdot 61$	$\frac{2^{3}3\cdot 5}{61}$	$3^{2}7^{2}13^{2}19 \cdot 61^{2}97$
$3 \cdot 5 \cdot 7 \cdot 19$	$\frac{2^{8}}{7\cdot 19}$	$3^{2}7^{2}13 \cdot 19^{2}127$
$3^{2}7^{2}13 \cdot 19 \cdot 31 \cdot 61$	$\frac{2^{9}5}{3\cdot 7\cdot 61}$	$3^{7}7^{4}13^{2}41 \cdot 61^{2}97 \cdot 467 \cdot 2801$
$3^{3}5^{2}19 \cdot 31$	$\frac{2^{10}}{3^{3}19}$	$3^{4}7 \cdot 11^{2}19^{2}127$
$3^{5}7^{2}13 \cdot 19^{2}127$	$\frac{2^{10}}{3^{3}19}$	$3^{6}5 \cdot 19 \cdot 23 \cdot 137 \cdot 547 \cdot 1093$
$3^{5}7^{3}13$	$\frac{2^{7}5^{2}}{3^{5}7}$	$3^{7}7^{2}19 \cdot 41$
$3^{6}5 \cdot 23 \cdot 31 \cdot 41 \cdot 137 \cdot 547 \cdot 1093$	$\frac{2^{14}7}{3^{2}5\cdot 31\cdot 41}$	$3^{7}5^{2}31^{2}41^{2}83 \cdot 331 \cdot 431 \cdot 1723$
$3^{6}7 \cdot 19 \cdot 23 \cdot 137 \cdot 547 \cdot 1093$	$\frac{2^{12}5}{3^{4}7\cdot 19}$	$3^{7}7^{2}19^{2}41 \cdot 127$
$3^{6}37^{3}73 \cdot 137^{2}547 \cdot 1093$	$\frac{2^{6}5\cdot 7\cdot 19}{3^{6}37}$	$3^{7}37 \cdot 41$
$5 \cdot 7^{2}13 \cdot 19 \cdot 61$	$\frac{2^{5}3^{2}31}{7\cdot 13\cdot 61}$	$5^{2}7^{3}13^{2}61^{2}97$
$5^{2}7^{2}11 \cdot 13 \cdot 17 \cdot 31$	$\frac{2^{9}3^{4}19}{5^{2}7\cdot 11\cdot 13\cdot 17}$	$5^{3}7^{4}11^{2}13^{3}17^{2}307 \cdot 467 \cdot 2801$
$5^{3}7^{2}13 \cdot 19^{2}127$	$\frac{2^{10}3^{3}}{5^{3}7\cdot 19}$	$5^{4}7 \cdot 11 \cdot 19 \cdot 71$
$7^{4}13^{4}31 \cdot 43 \cdot 61 \cdot 30941$	$\frac{2^{9}3^{4}11\cdot 191\cdot 2801}{7^{4}13^{4}43\cdot 61}$	$7^{9}13^{5}43^{2}61^{2}79^{2}97 \cdot 157 \cdot 631$
$11 \cdot 17^{2}29 \cdot 307$	$\frac{2^{5}3^{2}5\cdot 7}{17^{2}29}$	$13 \cdot 17^{3}29^{2}67$
$5^{4}7^{4}11^{3}17 \cdot 29 \cdot 61 \cdot 71 \cdot 467 \cdot 2801$	$\frac{2^{12}3^{9}13\cdot 31}{5^{3}7^{4}11^{2}17\cdot 29}$	$5^{5}7^{6}11^{4}17^{2}29^{2}67 \cdot 179 \cdot 263$
		$\cdot 307 \cdot 3221 \cdot 47334$
$7^{5}17 \cdot 37 \cdot 43$	$\frac{2^{7}3^{3}11\cdot 19^{2}}{7^{5}17\cdot 37}$	$7^{8}17^{2}37^{2}67 \cdot 307 \cdot 1063$
$13^{11}139 \cdot 157^{2}181^{2}191 \cdot 229^{2}$	$\frac{2^{15}3^{7}5^{3}7^{2}17\cdot 23\cdot 31\cdot 61\cdot 79\cdot 97}{13^{11}157\cdot 181\cdot 191\cdot 229}$	$13^{14}157 \cdot 181 \cdot 191^{2}199 \cdot 229$
$\cdot 827 \cdot 8269 \cdot 14197 \cdot 28393$		$\cdot 397 \cdot 1163 \cdot 4651 \cdot 30941$
		$\cdot 40493 \cdot 161971$

3.5 Restricted sigma chaining

It can be observed from known multiperfect numbers that small prime factors can (and usually do) appear with exponents greater than 1, whereas large prime factors occur as a consequence of having a small prime to a modest exponent or from having a larger prime factor (also with an exponent of 1).

Table 3.5: Different index substitutions for multiperfect numbers

$S(N)$	P	$S(P)$	$S\left(\frac{N}{P}Q\right)$	Q	$S(Q)$
3	3	$\frac{2^2}{3}$	4	$3^3 5$	$\frac{2^4}{3^2}$
3	3	$\frac{2^2}{3}$	4	$3^2 7 \cdot 13$	$\frac{2^4}{3^2}$
5	$3^3 5 \cdot 7^3 13$	$\frac{2^9 5^2}{3^2 7^2 13}$	4	$3^{10} 7 \cdot 23 \cdot 107 \cdot 3851$	$\frac{2^{10}}{3^4 7}$
5	$3^4 7 \cdot 11^2 19$	$\frac{2^5 5}{3^4}$	4	$3^6 23 \cdot 137 \cdot 547 \cdot 1093$	$\frac{2^7}{3^4}$
5	$5 \cdot 7$	$\frac{2^4 3}{5 \cdot 7}$	6	$5^3 7^2 13 \cdot 19$	$\frac{2^5 3^2}{5^2 7}$
5	$5^2 31$	$\frac{2^5}{5^2}$	6	$5^3 7 \cdot 13$	$\frac{2^6 3}{5^3}$

More directly, if N is a multiperfect number with $p_i^{a_i} \parallel N$, $q \mid \sigma(p_i^{a_i})$ and, for a suitably large bound B, if $q \geq B$ then $q \parallel N$. Alternatively, given a multiperfect number N and a prime q_2 we can find primes q_1 and q_3 and exponent $b > 1$ such that $q_1 < q_2 < q_3$ (q_1 much less than q_2) and $q_1^b q_2 q_3 \parallel N$ with either $q_2 \parallel \sigma(q_1^b)$ or $q_2 \parallel \sigma(q_3)$. Advantage can be taken of this by effectively combining the *effects* of such related primes into a single result.

Most known multiperfect numbers can be written as

$$N = 2^n p_1^{a_1} p_2^{a_2} \dots p_k^{a_k} p_{k+1} p_{k+2} \dots p_r$$

where $p_i < p_{i+1}$ and $a_k > 1$ (but $a_i \geq 1, i < k$). The few exceptions are the perfect and triperfect numbers and two 4-perfect numbers with $a_i = 1$ for all i. From this a bound, $B > p_k$, can be selected. Consequently whenever a new component, $p_i^{a_i}$, is introduced the necessary calculations for the σ of the large prime factors of $\sigma(p_i^{a_i}) > B$ can be "chained" to $p_i^{a_i}$ and a single result from several sequential steps found. This will be called restricted sigma chaining from now on and is defined as follows. For any component p^a and bound $B > 1$, set

$$\sigma_{eff}(p^a, B) = \sigma(p^a) \cdot \prod_i S(q_i)$$

for all $q_i \geq B$ where $q_i \parallel \sigma(p^a)$ or $q_i \parallel \sigma(q_j), i > j$ with the special case $\sigma_{eff}(p^a, \infty) = \sigma(p^a)$.

For example, if we asume $3^6 \parallel N$ and a target bound of $B = 100$ then since $\sigma(3^6) = 1093 > B$, we assume that $1093 \parallel N$ and so we can chain the calculations together and consider

$$\sigma_{eff}(3^6, 1093) = \sigma(3^6) \cdot S(1093) = 1093 \cdot \frac{2 \cdot 547}{1093} = 2 \cdot 547.$$

We continue the chain while any prime that arises is greater than the bound B, thus

$$\sigma_{eff}(3^6, 100) = \sigma(3^6) \cdot S(1093)\, S(547)\, S(137) = 1093 \cdot \frac{2 \cdot 547}{1093} \cdot \frac{2^2 137}{547} \cdot \frac{2 \cdot 3 \cdot 23}{137} = 2^4 3 \cdot 23$$

since $\sigma(3^6) = 1093$, $\sigma(1093) = 2 \cdot 547$, $\sigma(547) = 2^2 137$ and $1093 > 100$, $547 > 100$, $137 > 100$ while $\sigma(137) = 2 \cdot 3 \cdot 23$. Further examples are given later in Table 3.10.

3.6 An improved search procedure

The basic search procedure of Section 3.2 was less than comprehensive in specifying "where", "when" and "how far". In an attempt to find answers to these questions (or at least some recommendations),

	2^{10}	3^3	5^2	23	31	89
2^{10}				23		89
23	2^3	3				
89	2	3^2	5			
3^3	2^3		5			
5^2					31	
31	2^5					

Figure 3.2: Reconstruction of Poulet tableau for initial component 2^{10}

$$
\begin{aligned}
2^{10} &: S = \tfrac{23 \cdot 89}{2^{10}} \\
89 &: S = \tfrac{23 \cdot 89}{2^{10}} \cdot \tfrac{2 \cdot 3^2 5}{89} = \tfrac{3^2 5 \cdot 23}{2^9} \\
23 &: S = \tfrac{3^2 5 \cdot 23}{2^9} \cdot \tfrac{2^3 3}{23} = \tfrac{3^3 5}{2^6} \\
3^3 &: S = \tfrac{3^3 5}{2^6} \cdot \tfrac{2^3 5}{3^3} = \tfrac{5^2}{2^3} \\
5^2 &: S = \tfrac{5^2}{2^3} \cdot \tfrac{31}{5^2} = \tfrac{31}{2^3} \\
31 &: S = \tfrac{31}{2^3} \cdot \tfrac{2^5}{31} = 2^2
\end{aligned}
$$

found the 4-perfect number $2^{10}3^35^2 \cdot 23 \cdot 31 \cdot 89$

Figure 3.3: Decision tree corresponding to tableau for initial component 2^{10}

we will now look at strategies that have been used to find the great majority of known multiperfect numbers.

Very little has been published on this matter. With the exception of Poulet [101], most early reports consisted merely of lists of new discoveries (or compilations of known results). The most successful searches have been the modern computerised ones. Some information on the methods used has been disseminated via an electronic mailing list [108] or on personal web sites [44, 108]. The following description of the work of the most active researchers – Helenius, Schroeppel, Moxham, Flammenkamp, Woltman (and to a lesser extent, the author) is based on such email information. Any omission or misinterpretation of their work is unintentional and regretted.

Poulet gave the first published algorithmic approach to finding multiperfect numbers. He gave an example of systematically constructing a work-in-progress tableau which led to a 4-perfect number. This tableau is reconstructed in Figure 3.2 to show the sequence of steps (row-wise downwards) in the process. This should be compared to the corresponding parts of the decision tree in Figure 3.3.

On the problem of which new prime factor to continue with, Poulet gave no explicit heuristic although the previous example seems to follow a largest component first strategy. However he recognised that a large prime factor usually occurs "à la première puissance" and sorted possible prime factors into three "bandes" (i) those less than 100, (ii) those between 100 and 1000 and (iii) those greater than 1000.

The prime factors in this last band were dealt with immediately (with an exponent of one). This idea is implemented in all the recent computational searches. Although the number of and/or bounds for each band have been adjusted to accommodate larger multiperfect numbers (for example 398581^2 is a component of several multiperfect numbers and so 398581 should not be dealt with immediately).

Poulet should also be considered the originator of factor chaining – considering the net effect of

$$2^{57} : S = \frac{3 \cdot 59 \cdot 233 \cdot 1103 \cdot 2089 \cdot 3033169}{2^{57}}$$
$$3033169 : S = \frac{3 \cdot 59 \cdot 233 \cdot 1103 \cdot 2089 \cdot 3033169}{2^{57}} \cdot \frac{2 \cdot 5 \cdot 7 \cdot 43331}{3033169} = \frac{3 \cdot 59 \cdot 233 \cdot 1103 \cdot 2089 \cdot 43331}{2^{56}}$$
$$43331 : S = \frac{3 \cdot 59 \cdot 233 \cdot 1103 \cdot 2089 \cdot 43331}{2^{56}} \cdot \frac{2^2 3 \cdot 23 \cdot 157}{43331} = \frac{3^2 5 \cdot 7 \cdot 23 \cdot 59 \cdot 157 \cdot 233 \cdot 1103 \cdot 2089}{2^{54}}$$
$$2089 : S = \frac{3^2 5 \cdot 7 \cdot 23 \cdot 59 \cdot 157 \cdot 233 \cdot 1103 \cdot 2089}{2^{54}} \cdot \frac{2 \cdot 5 \cdot 11 \cdot 19}{2089} = \frac{3^2 5^2 7 \cdot 11 \cdot 19 \cdot 23 \cdot 59 \cdot 157 \cdot 233 \cdot 1103}{2^{53}}$$
$$1103 : S = \frac{3^2 5^2 7 \cdot 11 \cdot 19 \cdot 23 \cdot 59 \cdot 157 \cdot 233 \cdot 1103}{2^{53}} \cdot \frac{2^4 3 \cdot 23}{1103} = \frac{3^3 5^2 7 \cdot 11 \cdot 19 \cdot 23^2 59 \cdot 157 \cdot 233}{2^{49}}$$

Figure 3.4: Decision tree for initial component 2^{57}

all "large"prime factors first. He illustrated this idea with the example of 2^{57} with $S(2^{57}1103 \cdot 2089 \cdot 43331 \cdot 3033169) \cdot 2^{49} = 2^8 3^3 5^2 7 \cdot 11 \cdot 19 \cdot 23^2 59 \cdot 157 \cdot 233$ which can be represented as the decision tree in Figure 3.4. This idea of determining the net effect of factor chaining is formalised by the author as $\sigma_{eff}(p^a, b)$ in Section 3.5.

Besides publishing many new multiperfect numbers, Poulet also detailed several new substitutions, in particular the pair $19^2 127$, $19^4 151 \cdot 911$.

The next algorithmic advance came with the successful automation of searching by Helenius in 1992. An early success was his discovery of the first 9-perfect number. This work also marks the debut of the notion of effective exponent (although later researchers have used an extended definition of this notion). His definition discounts the actual 2-exponent by any Mersenne exponent (i.e. p is a Mersenne exponent if $M_p = 2^p - 1$ is a Mersenne prime). For example the component 2^{104} would have an effective exponent of $104 - (3 + 5 + 7) = 89$ since $104 = 3 \cdot 5 \cdot 7 - 1$ and $\sigma(2^{104}) = \sigma(M_3)\sigma(M_5)\sigma(M_7)\sigma(2^{89})$.

The basis of Helenius' search strategy is the systematic odometer algorithm whereby all combinations of restricted ranges of exponents of a set of prime bases (a preselected set of prime divisors) are investigated. Because of the search time involved, the 2-exponent is normally fixed and the exponents of other chosen prime bases are systematically varied over individual ranges. The combination of prime bases and corresponding exponent ranges was called a prime-profile by Schroeppel. When the selected initial components for a search involve the k smallest primes, the search is referred to as a k-prime search. The illustrative search in Section 3.2 is a 1-prime search with 2^{10} as the initial component. A 3-prime search for even multiperfect numbers would involve a factor of the form $2^a 3^b 5^c$ for values of a, b and c in certain ranges. Helenius incorporated formulae for the search bounds exponents b, c, $\ldots$ as multiples of the effective exponent corresponding to a chosen exponent a.

Much discussion among researchers has been concerned with the exponent ranges and centred on seeking to establish long-term values for the ratios of the 2-exponent to the 3-exponent (and similarly for larger primes). This ratio was considered both heuristically and statistically. Schroeppel gave two heuristic estimates for this ratio, "E2:E3". The first estimate of "roughly 8:3" comes from "the observation that most of an MPFN is 'random' primes to exponent 1" (where MPFN is used as an abbreviation for multiperfect number). The second estimate of "roughly log 3:log 2 = 1.58" comes from a "complicated attempt to estimate how many powers of 2 (or 3) are generated long term by each prime". (This coincides with the author's notion of effective exponent.)

A statistical approach to establishing exponent ranges uses a linear regression analysis of known multiperfect numbers. The results of such an analysis by the author can been seen in Section 3.7. As more (larger) multiperfect numbers are discovered the underlying linear assumption is questionable, but no replacement model has been championed.

Following the success of Helenius, Moxham produced a search heuristic whose prominent feature was the use of three categories of possible prime factors which he identified as sml, med and imm (for small, medium and immediate respectively). His definition of imm is as follows: "imms are defined as primes p large, such that sigma(p^n) with n>=2 has not been attempted yet, so that we know that they must go up and to the first power only". These are the largest primes which are invariably eliminated first (immediately) in most search heuristics.

To assist the practical implementation of his heuristic, Moxham used not only factorisation hints (from factorisation tables, as did Helenius) but also, in recognition of the increasing need for factorisation of larger $\sigma(p)$, linked his code to an efficient implementation of the multiple polynomial quadratic sieve factorisation algorithm (MPQS in UBASIC, while Helenius incorporated only Pollard's ρ method). One other distinguishing feature of his implementation was the use of an expanded table of substitutions. This table was generated by considering all pairs of known multiperfect numbers. A few of the multiperfect numbers discovered by Helenius were found by manually applying the $19^2 127$, $19^4 151 \cdot 911$ substitution.

As the search for larger multiperfect numbers expanded, the data structures for tables and lists, with factorisations leading to many more distinct prime factors, has become a performance concern.

The earlier multiperfect number search work by Woltman concentrated on an efficient implementation of a basic odometer search strategy. To this end, the time-consuming multiprecision arithmetic that was used by Helenius and Moxham was eliminated in favour of native 32-bit arithmetic only. This was achieved by restricting the maximum 2-exponent and using sigma chaining for the few cases where a prime greater than $2^{31} - 1$ resulted from a σ factorisation. (This is the same tack taken by the author in developing sigma chaining and applying it in the linear programming model for multiperfect numbers, to be described in Section 3.8.) While this approach is efficient, the modest bound on the 2-exponent proved too restrictive. Subsequently Woltman has implemented a search procedure which has most of the features of Moxham's program (including multiprecision arithmetic, a database of substitutions and three categories of possible prime factors) but continues to use a table of σ factorisation rather than simple hints (Helenius) or on-demand factorisation (Moxham). Rather than categorise the primes when needed, Woltman enhanced his table of σ factorisations by statically categorising the factors as small, medium and large.

The author has adopted a three tiered approach to facilitating factorisations:

(i) a table of complete σ factorisations for primes less than 10000 for all needed exponents

(ii) a list of prime factor hints for known "hard" cases and

(iii) a battery of simple factorisation methods such as Pollard ρ, Pollard $p - 1$, continued fraction and elliptic curve (with separate UBASIC ECM and MPQS programs used to generate those "hard" hints).

While Woltman's implementation represents the state-of-the-art as regards efficiency, the fundamental problem of where best/first to look has been tackled by several investigators (including Woltman) by modifying the odometer approach to allow specific components to be fixed (for example it was found to be productive to fix 19^{10} as a component).

Table 3.6: Lower bounds for even multiperfect numbers

$S(n)$	$\omega(n) \geq$	$n \geq$
3	2	6
4	4	210
5	6	30030
6	9	2.230×10^{8}
7	14	1.308×10^{16}
8	22	3.217×10^{30}
9	35	1.492×10^{57}
10	55	1.651×10^{103}
11	89	1.010×10^{190}
12	142	2.523×10^{338}

3.7 Where to search

Crude lower bounds for the value and the number of distinct prime factors were established by Sylvester [115]. If n is to be a multiperfect number of index $S(n)$ then $\omega(n) \geq k$ such that $\prod_{i=1}^{k} S(p_i) < S(n)$ where p_i are successive primes with $p_1 = 2$, and $n \geq \prod_{i=1}^{k} p_i$. Values of these lower bounds for various indices can be found in Table 3.6. While those bounds are good for small indices they quickly loose their relevance for larger indices. This can be easily seen by comparing these lower bounds for a given index with the currently known smallest multiperfect numbers given in Table 3.2.

Beyond a modest bound, an exhaustive search is not likely to be efficient. A better approach has been to use some variation of a systematic search as described in Section 3.6. An obvious question is "What are promising initial components?" Promising is usually taken to mean more likely to result in a discovery or to require fewer cases to be considered.

Since only even multiperfect numbers have been discovered, it is reasonable to ask what components of the form 2^n are promising initial components. As no multiperfect number with an index greater than 11 has been found, most (or possibly all) of the 2's of the initial component, 2^n, will need to be cancelled in the calculation of the index of the multiperfect number. Certain values of the exponent n can be shown, a priori, to be promising (as mentioned by Moxham and Woltman). If the assumed component is of the form 2^n where $n = kp - 1$, p a Mersenne exponent, then $\sigma(2^{kp-1}) = 2^{kp} - 1$, $(2^p - 1) \mid \sigma(2^{kp-1})$ and $\sigma(2^p - 1) = 2^p$. This leads to an immediate $1/k$ reduction in the outstanding 2-exponent. A bigger a priori reduction occurs when $n = pq - 1$ where both p and q are Mersenne exponents since then $(2^p - 1)(2^q - 1) \mid 2^n$. The corresponding reduction is $(p + q)/pq$. Table 3.7 enumerates n, together with the percentage reduction in parentheses, for various practical values of p and k (or q). As can be seen from the table, the size of the reduction decreases as n increases.

It was such information that helped direct the search for larger multiperfect numbers (the two largest known multiperfect numbers have a component of 2^{427} and $427 = 4 \cdot 107 - 1$). Once the factorisation of $2^{521} + 1$ became available through an update to the Cunningham tables [16], a time consuming investigation, starting with the component 2^{1041}, was begun by the author. So far it has not been fruitful.

Table 3.7: Promising 2-exponents of the form $n = kp - 1$ for multiperfect numbers with the percentage reduction in the 2-exponent given in parentheses

	k										
p	2	3	4	5	6	7	8	9	13	17	19
13	25 (52)	38 (42)	51 (25)	64 (28)	77 (16)	90 (22)	103 (12)	116 (11)	168 (15)		
17	33 (51)	50 (40)	67 (25)	84 (26)	101 (16)	118 (20)	135 (12)	152 (11)	220 (13)	288 (11)	
19	37 (51)	56 (39)	75 (25)	94 (25)	113 (16)	132 (19)	151 (12)	170 (11)	246 (13)	322 (11)	360 (10)
31	61 (50)	92 (36)	123 (25)	154 (23)	185 (16)	216 (17)	247 (12)	278 (11)	402 (10)	526 (9)	588 (8)
61	121 (50)	182 (35)	243 (25)	304 (21)	365 (16)	426 (15)	487 (12)	548 (11)	792 (9)	1036 (7)	
89	177 (50)	266 (34)	355 (25)	444 (21)	533 (16)	622 (15)	711 (12)	800 (11)	1156 (8)		
107	213 (50)	320 (34)	427 (25)	534 (20)	641 (16)	748 (15)	855 (12)	962 911)	1390 (8)		
127	253 (50)	380 (34)	507 (25)	634 (20)	761 (16)	888 (15)	1015 (12)				
521	1041 (50)										

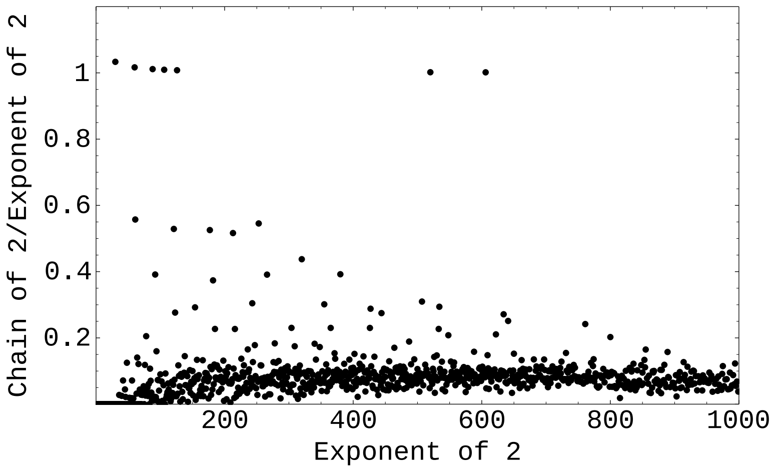

Figure 3.5: Effects of chaining on $\sigma(2^n)$

Multiperfect numbers are not exclusively found with such components. They have been found with a component 2^n for all $n < 331$. As of December 2001, the first four exponents of 2 for which a multiperfect number has not yet been found, are $n = 331$, 335, 336 and 345.

While the exponents in Table 3.7 are promising they are not necessarily the most fruitful. By far the most frequent 2-exponent among known multiperfect numbers, is 298 ($= 13 \cdot 23 - 1$) which does not appear in the table.

Figure 3.5 shows the ratio of the 2-exponent of $\sigma_{eff}(2^a, 10^6)$ to the initial exponent a. The seven points with a ratio of approximately 1 correspond to the first seven perfect numbers. The group of five points with a ratio about 0.5 correspond to those promising exponents in Table 3.7 with $k = 2$. Figure 3.5 shows that the promising cases have a distribution similar to that of the Mersenne primes with restricted sigma chaining providing a modest 10% reduction from the initial 2-exponent.

Since it is likely that there will also be a 3-component, i.e. a component 3^a, in all but the smaller multiperfect numbers, we might ask what boost in the reduction of the initial 2-exponent will come from having a 3-component. Figure 3.6 shows the ratio of the 2-exponent of $\sigma_{eff}(3^a, 10^6)$ to the initial exponent a. The few promising exponents of 3 correspond to small values on a. In general, the inclusion of a 3-component will result in only an immediate 5% reduction in the initial 2-exponent. Higher p-components provide an ever diminishing reduction in the initial 2-exponent.

The conclusion from these observations is that few "easy" cases remain unexplored within current resource constraints. Unless there are some theoretical advances, finding more multiperfect numbers will depend on the amount of computing resources devoted to systematic searching. The depth of such search trees and the magnitude of the numbers involved increase in a worse-than-linear (possibly a power) fashion as the initial exponent of 2 is increased.

An analysis of known proper multiperfect numbers provides some guidance in selecting combinations of components. The following analysis is based on the list of known multiperfect numbers as of December 2001 [44]. Figure 3.7 shows the exponent of 3 versus the exponent of 2 for known proper multiperfect numbers. A least-squares regression line is overlaid. Figures 3.8 and 3.9 show the results

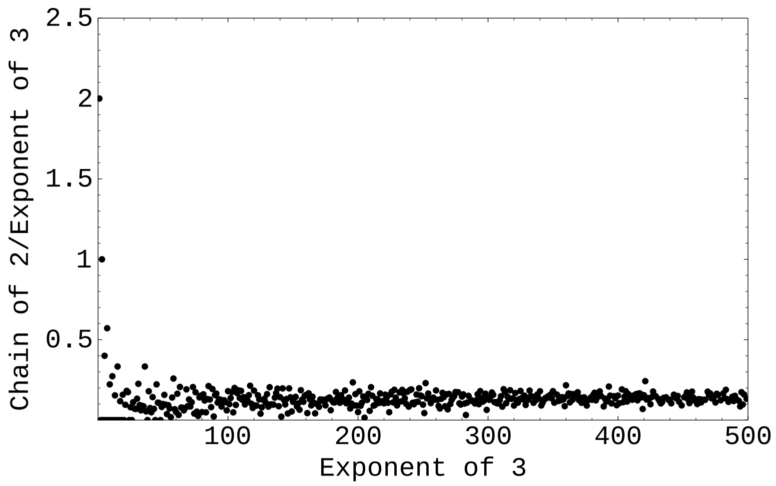

Figure 3.6: Effects of chaining on $\sigma(3^n)$

Table 3.8: Linear regression analysis of exponents of p^m given a component 2^n

	$m = \alpha n + \beta$			lower 95%		upper 95%	
p	α	β	R	α	β	α	β
3	0.298	3.007	0.933	0.295	2.427	0.301	3.587
5	0.150	-0.063	0.906	0.148	-0.417	0.151	0.291
7	0.102	1.953	0.856	0.100	1.643	0.104	2.264
11	0.060	0.092	0.846	0.059	-0.282	0.061	0.098
13	0.052	-0.033	0.836	0.051	-0.206	0.053	0.140
17	0.032	-0.112	0.764	0.031	-0.249	0.033	0.026
19	0.034	1.028	0.760	0.033	0.881	0.035	1.175

for the exponents of 5 and 7 respectively versus the exponent of 2.

A summary of a linear regression analysis for these three data sets and those for the exponents of 11, 13, 17 and 19, is given in Table 3.8. Besides the parameters, α and β, for the line of best fit, the table includes the regression coefficient, R, for each regression line. This value gives an indication of the strength of the correlation or the goodness of the fit. The closer this value is to 1, the stronger is the positive correlation. To provide assistance in bounding exponent searches, the parameters for the lower and upper 95% bounding lines of values are also given – 90% of known values fall between these lines.

For example, to carry out a 4-prime search with a component of 2^{380}, it is suggested that you search with the additional components $3^a5^b7^c$ with $114 < a < 118$, $55 < b < 58$ and $39 < c < 42$.

An analysis of known multiperfect numbers can also be used to suggest what the index of a multiperfect number might be for a given magnitude or with a given component 2^n (Figures 3.10 and 3.12 respectively). These figures strongly suggest that there are bounds on the values (and exponent of 2) for proper multiperfect numbers of a given index. Conversely, if you are looking for multiperfect numbers with a particular index, these figures are helpful. For example, if you choose

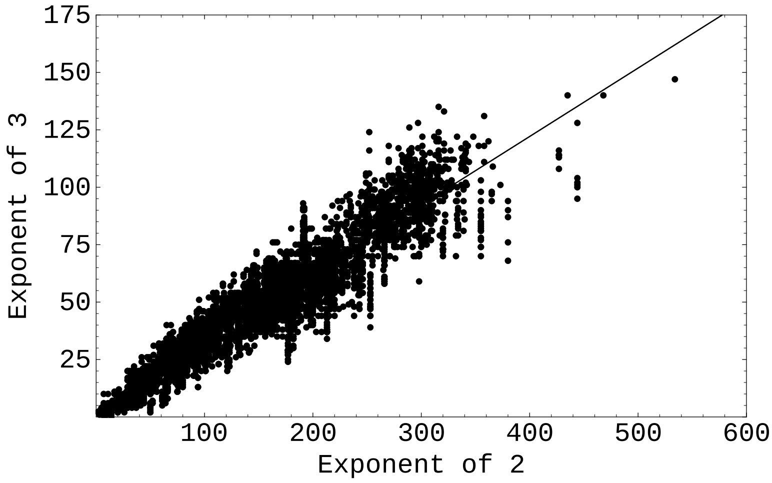

Figure 3.7: Correlation between exponent of 3 and exponent of 2

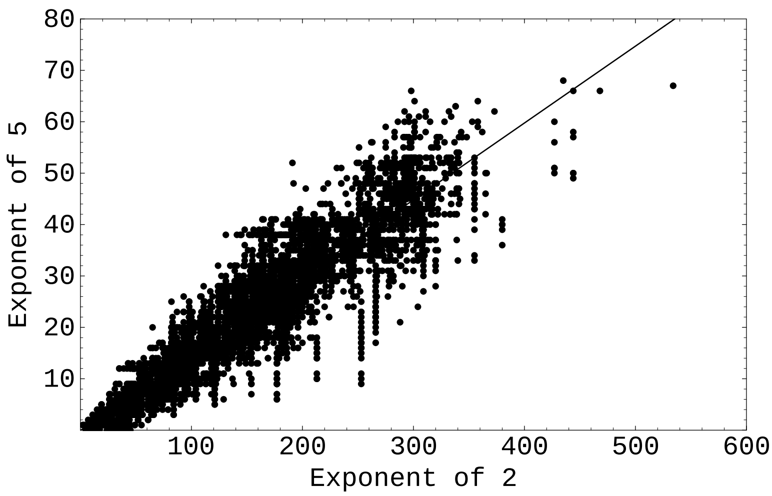

Figure 3.8: Correlation between exponent of 5 and exponent of 2

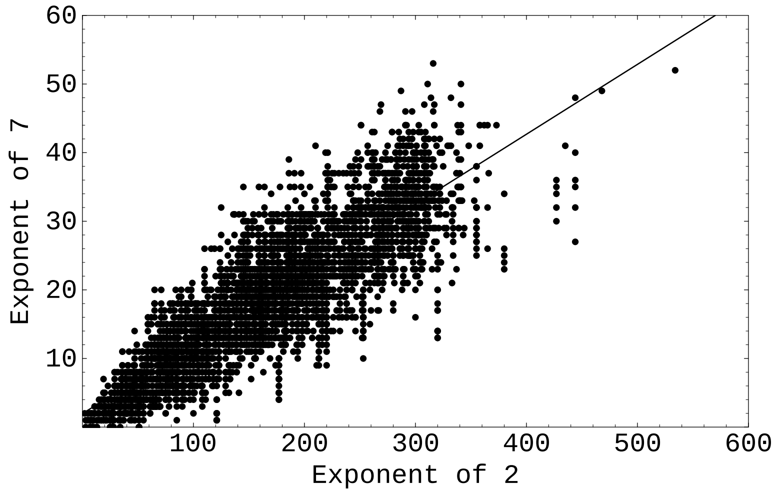

Figure 3.9: Correlation between exponent of 7 and exponent of 2

2^{380} as a component, any multiperfect numbers found are most likely to be 10-perfect, with a small possibility of finding an 11-perfect number. However, searching with a component of 2^{1041}, if or when successful, is likely to produce an 11-perfect (or even a 12-perfect) number.

Figure 3.11, the plot of the index of known multiperfect numbers versus the log log of their corresponding values, suggests that there may be an upper bound on the index of multiperfect numbers. While discoveries are still being made of 9-perfect and 10-perfect numbers (there is only one 11-perfect number known), it is believed that almost all the multiperfect numbers with an index less than 9 have been found.

3.8 0–1 linear programming model formulation

As an alternative to the algorithmic search method (as described in Sections 3.2 and 3.6) we now introduce a different approach to determining multiperfect numbers. It is based on the construction, and solution, of a linear programming decision model. Within chosen limits the corresponding linear programming model aims to be equivalent to an exhaustive search.

The variables used in defining the decision model are described as follows. Let xp_a be the decision variable (value 1 or 0) corresponding to whether p^a is, or is not, respectively, a component of the multiperfect number N. The particular p and a values will be specified later. This leads to the simple exclusivity condition that there can be at most one factor involving p, i.e. $xp\_1 + xp\_2 + xp\_3 + \cdots + xp\_k \leq 1$, where the choice of k will be discussed later. This naturally partitions the decision variables into sets, referred to as special ordered sets or generalized upper bounding sets. Problems where the variables can be so partitioned are classified as Multiple Choice Integer Programs [8].

Let tot_p, the total p-exponent, be the sum of the p-exponents for all possible components of N.

There is a dependency between components stemming from the condition that $S(N) \in \mathbb{N}$. If $p_i \mid N$ then there is a $p_j^{a_j} \parallel N$ where $p_i \mid \sigma(p_j^{a_j})$. This idea is the foundation of most multiperfect number search procedures. These dependencies lead to a set of "balance" equations (on the exponents), one

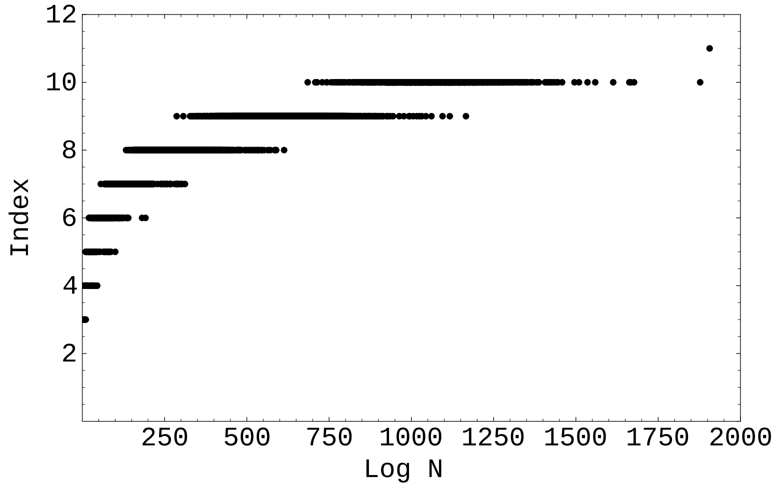

Figure 3.10: Index of known proper multiperfect numbers versus log of the number

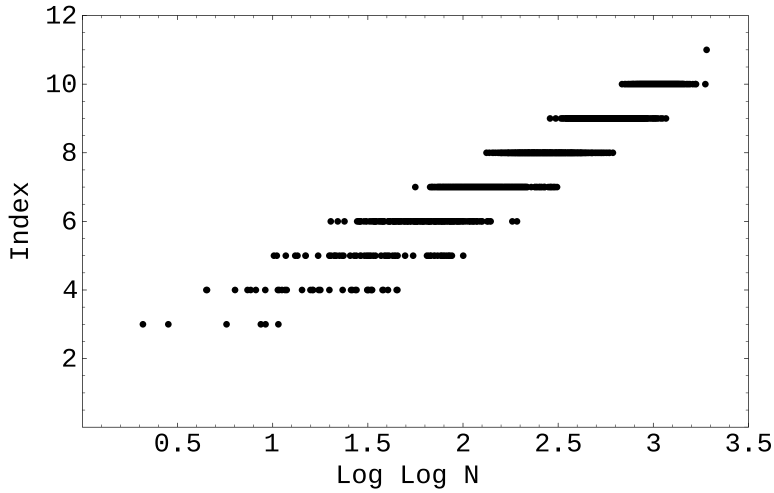

Figure 3.11: Index of known proper multiperfect numbers versus log log of the number

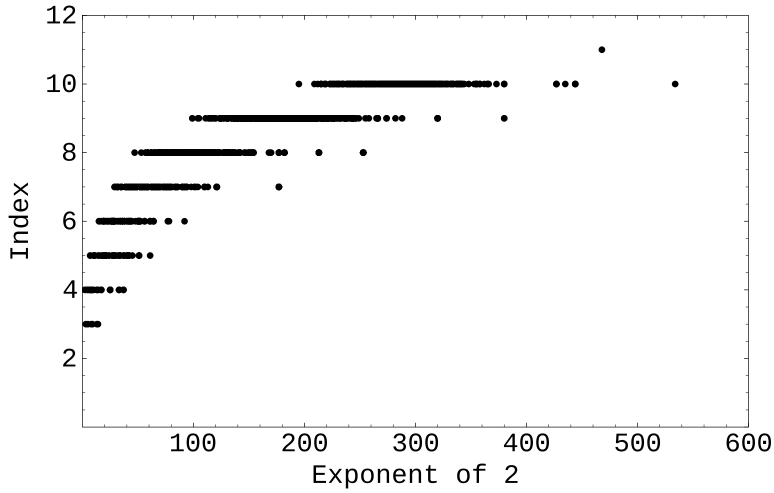

Figure 3.12: Index of known proper multiperfect numbers versus exponent of 2

for each value of p. For example, since $\sigma(3^5) = 2^2 7 \cdot 13$, $\sigma(5^5) = 2 \cdot 3^2 7 \cdot 31$, $\sigma(11^2) = 7 \cdot 19$, $\sigma(13) = 2 \cdot 7$ and $\sigma(13^3) = 2^2 5 \cdot 7 \cdot 17$, we have the total exponent of 2 is the sum of the exponent of 2 for all possible components, given (in part) as

$$tot\_2 = \cdots + 2x3\_5 + \cdots + x5\_5 + \cdots + x13\_1 + 2x13\_3 + \ldots$$

The total exponent of 7 is likewise given (in part) as

$$tot\_7 = x3\_5 + x5\_5 - x7\_1 - 2x7\_2 - 3x7\_3 - \cdots + x11\_2 + x13\_1 + x13\_3 + \ldots$$

Since the variables $xp\_a \in \{0, 1\}$ and the constraint coefficients are integral, the variables tot_p are also integral, without the need to manipulate them as such.

Having specified the constraints on the decision variables, we would like to find *all* feasible solutions (i.e. multiperfect numbers). However the linear programming model needs to be completed by specifying a linear objective function to be optimized (maximized or minimized). While *any* feasible non-zero linear combination of *any* of the variables would be acceptable, it is a simple matter to construct an objective function that will aim to find the multiperfect number with largest index, $S(n)$, subject to the exponent constraints. As yet no multiperfect number has been found with an index greater than 11. This allows us for practical purposes to limit the variables in the objective function to a linear combination of tot_2 through to tot_11, i.e. $w_2 tot\_2 + w_3 tot\_3 + w_5 tot\_5 + w_7 tot\_7 + w_{11} tot\_11$. The variables $tot\_13, tot\_17, \ldots$ appearing in the "balance" equations, are expected to have a final value of 0 for the "covered" cases and so can simply be replaced by the constant 0. Note that it would still be possible to obtain an optimal 12-perfect number (or with index $14, 15, 16, 18, \ldots$ but not any with index $13, 17, 19, 23, \ldots$). A 3-perfect number would have $tot\_3 = 1$ (and all other $tot\_p = 0$) whereas an 8-perfect number would have $tot\_2 = 3$ since $8 = 2^3$. A 10-perfect number would have $tot\_2 = tot\_5 = 1$ since $10 = 2 \cdot 5$.

An interesting problem is to determine the weights, w_i. In order to favour multiperfect numbers with larger indices, we have the weight ranking conditions

$$w_2 < w_3 < 2w_2 < w_5 < w_2 + w_3 < w_7 < 3w_2 < 2w_3 < w_2 + w_5 < w_{11}.$$

The solution involving the smallest positive integer weights can be found by solving the following integer linear programming problem:

minimize: w_{11}

subject to:

$$\begin{aligned} w_3 - w_2 &\geq 1 \\ 2w_2 - w_3 &\geq 1 \\ w_5 - 2w_2 &\geq 1 \\ w_2 + w_3 - w_5 &\geq 1 \\ w_7 - w_2 - w_3 &\geq 1 \\ 3w_2 - w_7 &\geq 1 \\ 2w_3 - 3w_2 &\geq 1 \\ w_2 + w_5 - 2w_3 &\geq 1 \\ w_{11} - w_2 - w_5 &\geq 1 \\ w_2 &\geq 1 \\ w_i &\in \mathbb{N} \end{aligned}$$

The unique solution is $w_2 = 5$, $w_3 = 8$, $w_5 = 12$, $w_7 = 14$ and $w_{11} = 18$.

Allowing for multiperfect numbers with any index less than 17 can be achieved by finding the weights w_i, for $i = 2, 3, 5, 7, 11$ and 13. Using the previous approach, the additional constraints are

$$\begin{aligned} 2w_2 + w_3 - w_{11} &\geq 1 \\ w_{13} - 2w_2 - w_3 &\geq 1 \\ w_2 + w_7 - w_{13} &\geq 1 \\ w_3 + w_5 - w_2 - w_7 &\geq 1 \\ 4w_2 - w_3 - w_5 &\geq 1 \end{aligned}$$

with the objective being to minimize w_{13}. Solving for the smallest such positive integer weights gives $w_2 = 9$, $w_3 = 14$, $w_5 = 21$, $w_7 = 25$, $w_{11} = 31$ and $w_{13} = 33$.

3.9 Reducing the number of decision variables

An obvious way to reduce the number of decision variables is to “fix” one or more components. In this study the component 2^k was specified and so all decision variables of the form $x2\_a$ were eliminated. This had the additional benefit of simplifying each “balance” constraint.

Restricted sigma chaining, introduced in Section 3.5, provides a tool for further significantly reducing the number of prime factors that need to be explicitly modelled. With a bound set at $B = 100$, if a decision is made to introduce the component 3^6, we would actually introduce $3^6 1093 \cdot 547 \cdot 137$ in *one* step (ideally by using a table to lookup 3^6 to give $\sigma_{eff}(3^6, 100) = 2^4 3 \cdot 23$ while keeping track of the eliminated factors (137, 547 and 1093) which are needed to reconstruct the complete result). While this reduces the number of primes (and hence decision variables) that must be considered, an equally important result is that the need to manipulate very large (multiprecision) integers can be completely eliminated. Almost all linear programming packages (in particular those used in this study) are unable to handle very large integer values. For the known multiperfect numbers, a conservative bound of $B = 10^6$ is sufficient to allow restricted sigma chaining to eliminate the need for multiprecision integer arithmetic.

To illustrate the complete structure, consider the model to "cover" the cases with 2^n for any specified $0 \le n < 20$ where the possible consequent components are restricted to any combination of $3^a, a \le 9$; $5^a, a \le 6$; $7^a, a \le 5$; $11^a, a \le 3$; $13^a, a \le 3$; $17^a, a \le 3$; $19^a, a \le 3$; $23^a, a \le 2$. This set of possible prime factors will be referred to by the number

$$C_{spec} = 2^{19} 3^9 5^6 7^5 11^3 13^3 17^3 19^3 23^2.$$

This set is not selfcontained in that other prime factors are necessary. The set of auxiliary prime factors will be referred to by the number C_{aux}. These additional prime factors appear in the σ's of the elements of C_{spec}. However, the number of such primes can be significantly reduced by applying restricted sigma chaining.

In order to cover the consequences of any exponent of a particular prime p, a simple, but conservative, bound can be determined from the values of $\sigma(p^i)$, $i \le a$. To allow for any combination of different prime factors, an overall cover can be determined as the product of the individual bounds. This can be defined as follows. Given $C = \prod_i p_i^{a_i}$ we define

$$B(p, a, b) = [\sigma_{eff}(p, b), \sigma_{eff}(p^2, b), \ldots, \sigma_{eff}(p^a, b)]$$

$$Cover(C) = \prod_i B(p_i, a_i, b_i)$$

for some specified b_i, where $[\ldots]$ denotes the lowest common multiple function of its arguments.

In order to seek a selfcontained (stable) cover (which may not exist) we can define a sequence of covers as follows:

$$Cover_1(C) = \prod_i B(p_i, a_i, \infty)$$

$$Cover_k(C) = B(p_j, a_j, b_{k,j}) \cdot \prod_{i \ne j} B(p_i, a_i, b_{k,i}), \quad k \ge 2$$

with $b_{k,j}$ being the largest prime satisfying $b_{k,j} \parallel Cover_{k-1}(C)$ and where j is determined by $b_{k,j} \parallel B(p_j, a_j, b_{k-1,j})$ with $b_{k,i} = b_{k-1,i}$ for $i \ne j$ and $b_{1,j} = \infty$ for all j.

For the proposed C_{spec} the bounds $B(p_i, a_i, \infty)$ (before any reduction by restricted sigma chaining) are given in Table 3.9. For example,

$$\begin{aligned} B(5, 6, \infty) &= [\sigma(5), \sigma(5^2), \ldots, \sigma(5^6)] \\ &= [2 \cdot 3, 31, 2^2 3 \cdot 13, 11 \cdot 71, 2 \cdot 3^2 7, 19531] \\ &= 2^2 3^2 7 \cdot 11 \cdot 13 \cdot 31 \cdot 71 \cdot 19531. \end{aligned}$$

Table 3.9: Bounds for the $2^n, n < 20$ model before restricted sigma chaining

p	a	$B(p,a,\infty)$
2	19	$3^3 5^2 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 41 \cdot 43 \cdot 73 \cdot 89 \cdot 127 \cdot 151 \cdot 257 \cdot 8191 \cdot 131071 \cdot 524287$
3	9	$2^4 5 \cdot 7 \cdot 11^2 13 \cdot 41 \cdot 61 \cdot 757 \cdot 1093$
5	6	$2^2 3^2 7 \cdot 11 \cdot 13 \cdot 31 \cdot 71 \cdot 19531$
7	5	$2^4 3 \cdot 5^2 19 \cdot 43 \cdot 2801$
11	3	$2^3 3 \cdot 7 \cdot 19 \cdot 61$
13	3	$2^2 3 \cdot 5 \cdot 7 \cdot 17 \cdot 61$
17	3	$2^2 3^2 5 \cdot 29 \cdot 307$
19	3	$2^3 3 \cdot 5 \cdot 127 \cdot 181$
23	2	$2^3 3 \cdot 7 \cdot 79$

Consequently

$$\begin{aligned} Cover_1(C_{spec}) &= B(2,19,\infty)\, B(3,9,\infty) \ldots B(23,2,\infty) \\ &= 2^{23} 3^{12} 5^8 7^6 11^4 13^3 17^2 19^3 23 \cdot 29 \cdot 31^2 41^2 43^2 61^3 71 \cdot 73 \cdot 79 \cdot \\ &\quad 89 \cdot 127^2 151 \cdot 181 \cdot 257 \cdot 307 \cdot 757 \cdot 1093 \cdot 2801 \cdot 8191 \cdot \\ &\quad 19531 \cdot 131071 \cdot 524287. \end{aligned}$$

From this it follows that 524287 (a component only of $\sigma(2^{18})$) could be a divisor of some multiperfect number covered by C_{spec}. If we assume that 524287 is a component of such a multiperfect number then it can be eliminated (by chaining to $\sigma(2^{18})$) as a necessary decision variable of the LP model. Thus the updated values are

$$\sigma_{eff}(2^{18}, 524287) = \sigma(2^{18}) \cdot S(524287) = 524287 \cdot \frac{2^{19}}{524287} = 2^{19}$$

$$\begin{aligned} B(2,19,524287) = 2^{19} 3^3 5^2 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 41 \cdot 43 \cdot 73 \cdot 89 \cdot \\ 127 \cdot 151 \cdot 257 \cdot 8191 \cdot 131071 \end{aligned}$$

and

$$\begin{aligned} Cover_2(C_{spec}) &= B(2,19,524287)\, B(3,9,\infty) \ldots B(23,2,\infty) \\ &= 2^{42} 3^{12} 5^8 7^6 11^4 13^3 17^2 19^3 23 \cdot 29 \cdot 31^2 41^2 43^2 61^3 71 \cdot 73 \cdot 79 \cdot \\ &\quad 89 \cdot 127^2 151 \cdot 181 \cdot 257 \cdot 307 \cdot 757 \cdot 1093 \cdot 2801 \cdot 8191 \cdot \\ &\quad 19531 \cdot 131071. \end{aligned}$$

While we have reduced the particular bound $B(2,19,\infty)$ to $B(2,19,524287)$ we can in fact set $b_{2,i} = 524287$ for all i i.e. there is a current global bound of 524287.

Likewise, 131071, being a component only of $\sigma(2^{16})$, can be eliminated to give the updated values,

$$\sigma_{eff}(2^{16}, 131071) = \sigma(2^{16}) \cdot S(131071) = 131071 \cdot \frac{2^{17}}{131071} = 2^{17}$$

$$B(2,19,131071) = 2^{19}3^3 5^2 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 41 \cdot 43 \cdot 73 \cdot 89 \cdot$$
$$127 \cdot 151 \cdot 257 \cdot 8191$$

and

$$\begin{aligned} Cover_3(C_{spec}) &= B(2,19,131071)\, B(3,9,\infty) \dots B(23,2,\infty) \\ &= 2^{42}3^{12}5^8 7^6 11^4 13^3 17^2 19^3 23 \cdot 29 \cdot 31^2 41^2 43^2 61^3 71 \cdot 73 \cdot 79 \cdot 89 \cdot \\ &\quad 127^2 151 \cdot 181 \cdot 257 \cdot 307 \cdot 757 \cdot 1093 \cdot 2801 \cdot 8191 \cdot 19531 \end{aligned}$$

with a global bound of $b_{3,i} = 131071$ for all i.

This process of eliminating the largest prime in the current *Cover* can be continued for $19531 \parallel \sigma(5^6)$, $8191 \parallel \sigma(2^{12})$, $\dots$, $379 \parallel \sigma(757)$ and $307 \parallel \sigma(17^2)$ with a final global bound of $b_{10,i} = 307$ for all i and

$$Cover_{10}(C_{spec}) = 2^{42}3^{14}5^8 7^7 11^5 13^4 17^2 19^5 23 \cdot 29 \cdot 31^2 41^2 43^2 61^3 71 \cdot 73 \cdot 79 \cdot 89 \cdot$$
$$127^2 137 \cdot 151 \cdot 181 \cdot 257^2.$$

From this *Cover* we have the largest factor 257^2. This implies that we cannot eliminate 257. In fact we need to allow for a factor of $257^a, a \le 2$. The case where $2^{15}5^6$ is an exact factor requires a decision variable for 257^2 since $\sigma_{eff}(2^{15}, 307) = 3 \cdot 5 \cdot 17 \cdot 257$ and $\sigma_{eff}(5^6, 307) = 2^2 19 \cdot 257$.

The covering set is expanded to include

$$C_{aux} = 257^2$$

with

$$B(257,2,\infty) = 2 \cdot 3 \cdot 43 \cdot 61 \cdot 1087.$$

While this would introduce the larger prime $1087 \parallel \sigma(257^2)$ into the *Cover*, it can be eliminated to give

$$\sigma_{eff}(257^2, 307) = \sigma(257^2) \cdot S(1087) = 61 \cdot 1087 \cdot \frac{2^6 17}{1087} = 2^6 17 \cdot 61$$

and

$$B(257,2,307) = 2^6 3 \cdot 17 \cdot 43 \cdot 61$$

and so give

$$\begin{aligned} Cover_{11}(C_{spec} \cdot C_{aux}) &= Cover_{10}(C_{spec}) \cdot Cover(C_{aux}) \\ &= 2^{48}3^{15}5^8 7^7 11^5 13^4 17^3 19^5 23 \cdot 29 \cdot 31^2 41^2 43^3 61^4 71 \cdot 73 \cdot 79 \cdot 89 \cdot \\ &\quad 127^2 137 \cdot 151 \cdot 181 \cdot 257^2. \end{aligned}$$

Note that we could have a global bound of $257 + 1$ instead of 307, but any future reductions will be to individual $b_{k,i}$ only.

Continuing the process of elimination we consider $181 \parallel \sigma(19^3)$, $151 \parallel \sigma(2^{14})$ and then $137 \parallel \sigma_{eff}(3^6, 547)$ without any complications. From this *Cover* we have 127^2 as the largest factor (in fact this factor also appeared in the original $Cover_1(C_{spec})$). The case where either $2^6 19^2$ or $2^{13}19^2$ is an exact factor requires a decision variable for 127^2.

The covering set is expanded again to give

$$C_{aux} = 127^2 257^2$$

with

$$B(127, 2, \infty) = 2^7 3 \cdot 5419.$$

While this introduces the larger prime $5419 \parallel \sigma(127^2)$ into the *Cover*, it (and the consequent $271 \parallel \sigma(5419)$) can be eliminated to give

$$\sigma_{eff}(127^2, 271) = \sigma(127^2) \cdot S(5419) \cdot S(271) = 3 \cdot 5419 \cdot \frac{2^2 5 \cdot 271}{5419} \cdot \frac{2^4 17}{271} = 2^6 3 \cdot 5 \cdot 17$$

$$B(127, 2, 271) = 2^7 3 \cdot 5 \cdot 17$$

and

$$Cover_{15}(C_{spec} \cdot C_{aux}) = 2^{56} 3^{17} 5^9 7^8 11^5 13^5 17^4 19^5 23^2 29 \cdot 31^2 41^2 43^3 61^4$$
$$71 \cdot 73 \cdot 79 \cdot 89 \cdot 127^2 257^2.$$

The elimination process continues with $89 \parallel \sigma(2^{10})$, $79 \parallel \sigma(23^2)$, $73 \parallel \sigma(2^8)$ (or $73 \parallel \sigma(2^{17})$) and $71 \parallel \sigma(5^4)$. We then find that the cover needs to be further expanded to include 61^4 for the case where $3^9 11^3 13^2 257^2$ is an exact factor of the multiperfect number. This leads to

$$C_{aux} = 61^4 127^2 257^2$$

$$B(61, 4, \infty) = 2^2 3 \cdot 5 \cdot 13 \cdot 31 \cdot 97 \cdot 131 \cdot 1861 \cdot 21491.$$

Next $21491 \parallel \sigma(61^4)$, $1861 \parallel \sigma(61^3)$, $199 \parallel \sigma_{eff}(61^4, 21491)$ and $97 \parallel \sigma(61^2)$ are eliminated to give

$$\sigma_{eff}(61^2, 71) = \sigma(61^2) \cdot S(97) = 3 \cdot 13 \cdot 97 \cdot \frac{2 \cdot 7^2}{97} = 2 \cdot 3 \cdot 7^2 13$$

$$\sigma_{eff}(61^3, 71) = \sigma(61^3) \cdot S(1861) = 2^2 31 \cdot 1861 \cdot \frac{2 \cdot 7^2 19}{1891} = 2^3 7^2 19 \cdot 31$$

$$\sigma_{eff}(61^4, 71) = \sigma(61^4) \cdot S(21491) \cdot S(199) \cdot S(131)$$
$$= 5 \cdot 131 \cdot 21491 \cdot \frac{2^2 3^3 199}{21491} \cdot \frac{2^3 5^2}{199} \cdot \frac{2^2 3 \cdot 11}{131} = 2^7 3^4 5^3 11$$
$$B(61, 4, 71) = 2^7 3^4 5^3 7^2 11 \cdot 13 \cdot 19 \cdot 31$$

and

$$Cover_{19}(C_{spec} \cdot C_{aux}) = 2^{65} 3^{21} 5^{13} 7^{10} 11^6 13^6 17^4 19^6 23^2 29 \cdot 31^3 37 \cdot 41^2 43^3 61^4$$
$$127^2 257^2.$$

It is then necessary to extend the cover to include 43^3 for the case where $2^{13} 7^5 257$ is an exact factor of the multiperfect number. This is followed by the elimination of $631 \parallel \sigma(43^2)$ to give

$$C_{aux} = 43^3 61^4 127^2 257^2$$

$$\sigma_{eff}(43^2, 631) = \sigma(43^2) \cdot S(631) = 3 \cdot 631 \cdot \frac{2^3 79}{631} = 2^3 3 \cdot 79$$

$$B(43,3,631) = 2^3 3 \cdot 5^2 11 \cdot 37 \cdot 79.$$

Earlier 79 $\|$ $\sigma(23^2)$ was eliminated but with the expansion of the cover to include 43^3 we must now "backup" and allow for 79^2 as a possible factor. Therefore we revert to $\sigma_{eff}(23^2, \infty) = 7 \cdot 79$ and $B(23, 2, \infty) = 2^3 3 \cdot 7 \cdot 79$. This results in

$$C_{aux} = 43^3 61^4 79^2 127^2 257^2$$

$$B(79,2,\infty) = 2^4 3 \cdot 5 \cdot 7^2 43.$$

Previously only $43^a, a \leq 3$ was catered for but with the introduction of a possible factor of 79^2 we must now increase the maximum exponent of 43 to $a \leq 4$ (to allow for $2^{13} 7^5 79^2 257$ as a factor). The updated values are

$$C_{aux} = 43^4 61^4 79^2 127^2 257^2$$

$$B(43,4,631) = 2^{10} 3^2 5^2 7 \cdot 11 \cdot 37 \cdot 79$$

and hence

$$Cover_{22}(C_{spec} \cdot C_{aux}) = 2^{78} 3^{24} 5^{15} 7^{13} 11^7 13^6 17^4 19^6 23^2 29 \cdot 31^3 37^2 41^2 43^4 61^4 \\ 79^2 127^2 257^2.$$

This process of elimination, updating and, more frequently, backing up leads eventually to a selfcontained (stable) cover of

$$C_{aux} = 29^3 31^4 37^4 41^4 43^4 53^2 61^4 67^3 73^2 79^2 89^2 127^2 131^2 137^2 257^2$$

with the corresponding values of the restricted sigma chains for these bounds given in Table 3.10 (where $\Rightarrow$ indicates one iteration of the chaining process). The B values are given in Table 3.11 with the final *Cover* being

$$Cover_{final}(C_{spec} \cdot C_{aux}) = 2^{119} 3^{39} 5^{24} 7^{21} 11^{12} 13^8 17^6 19^7 23^3 29^3 31^4 37^4 41^4 43^4 \\ 53^2 61^4 67^3 73^2 79^2 89^2 127^2 131^2 137^2 257^2.$$

$C_{spec} \cdot C_{aux}$ determines the decision variables of the LP model and the extent of the restricted sigma chaining.

Table 3.10: Restricted sigma chains for the $2^n, n < 20$ model

p^a	$\sigma_{eff}(p^a, b)$
2	3
2^2	7
2^3	$3 \cdot 5$
2^4	31
2^5	$3^2 7$
2^6	127
2^7	$3 \cdot 5 \cdot 17$
2^8	$7 \cdot 73$
2^9	$3 \cdot 11 \cdot 31$
2^{10}	$23 \cdot 89$
2^{11}	$3^2 5 \cdot 7 \cdot 13$
2^{12}	$8191 \Rightarrow 2^{13}$
2^{13}	$3 \cdot 43 \cdot 127$
2^{14}	$7 \cdot 31 \cdot 151 \Rightarrow 2^3 7 \cdot 19 \cdot 31$
2^{15}	$3 \cdot 5 \cdot 17 \cdot 257$
2^{16}	$131071 \Rightarrow 2^{17}$
2^{17}	$3^3 7 \cdot 19 \cdot 73$
2^{18}	$524287 \Rightarrow 2^{19}$
2^{19}	$3 \cdot 5^2 11 \cdot 31 \cdot 41$
3	2^2
3^2	13
3^3	$2^3 5$
3^4	11^2
3^5	$2^2 7 \cdot 13$
3^6	$1093 \Rightarrow 2 \cdot 547 \Rightarrow 2^3 137$
3^7	$2^4 5 \cdot 41$
3^8	$13 \cdot 757 \Rightarrow 2 \cdot 13 \cdot 379 \Rightarrow 2^3 5 \cdot 13 \cdot 19$
3^9	$2^2 11^2 61$
5	$2 \cdot 3$
5^2	31
5^3	$2^2 3 \cdot 13$
5^4	$11 \cdot 71 \Rightarrow 2^3 3^2 11$
5^5	$2 \cdot 3^2 7 \cdot 31$
5^6	$19531 \Rightarrow 2^2 19 \cdot 257$
7	2^3
7^2	$3 \cdot 19$
7^3	$2^4 5^2$
7^4	$2801 \Rightarrow 2 \cdot 3 \cdot 467 \Rightarrow 2^3 3^3 13$

Table 3.10: Restricted sigma chains for the 2^n (contd.)

p^a	$\sigma_{eff}(p^a, b)$
7^5	$2^3 3 \cdot 19 \cdot 43$
11	$2^2 3$
11^2	$7 \cdot 19$
11^3	$2^3 3 \cdot 61$
13	$2 \cdot 7$
13^2	$3 \cdot 61$
13^3	$2^2 5 \cdot 7 \cdot 17$
17	$2 \cdot 3^2$
17^2	$307 \Rightarrow 2^2 7 \cdot 11$
17^3	$2^2 3^2 5 \cdot 29$
19	$2^2 5$
19^2	$3 \cdot 127$
19^3	$2^3 5 \cdot 181 \Rightarrow 2^4 5 \cdot 7 \cdot 13$
23	$2^3 3$
23^2	$7 \cdot 79$
29	$2 \cdot 3 \cdot 5$
29^2	$13 \cdot 67$
29^3	$2^2 3 \cdot 5 \cdot 421 \Rightarrow 2^3 3 \cdot 5 \cdot 211 \Rightarrow 2^5 3 \cdot 5 \cdot 53$
31	2^5
31^2	$3 \cdot 331 \Rightarrow 2^2 3 \cdot 83 \Rightarrow 2^4 3^2 7$
31^3	$2^6 13 \cdot 37$
31^4	$5 \cdot 11 \cdot 17351 \Rightarrow 2^3 3^2 5 \cdot 11 \cdot 241 \Rightarrow 2^4 3^2 5 \cdot 11^3$
37	$2 \cdot 19$
37^2	$3 \cdot 7 \cdot 67$
37^3	$2^2 5 \cdot 19 \cdot 137$
37^4	$11 \cdot 41 \cdot 4271 \Rightarrow 2^4 3 \cdot 11 \cdot 41 \cdot 89$
41	$2 \cdot 3 \cdot 7$
41^2	$1723 \Rightarrow 2^2 431 \Rightarrow 2^6 3^3$
41^3	$2^2 3 \cdot 7 \cdot 29^2$
41^4	$5 \cdot 579281 \Rightarrow 2 \cdot 3 \cdot 5 \cdot 11 \cdot 67 \cdot 131$
43	$2^2 11$
43^2	$3 \cdot 631 \Rightarrow 2^3 3 \cdot 79$
43^3	$2^3 5^2 11 \cdot 37$
43^4	$3500201 \Rightarrow 2 \cdot 3 \cdot 583367 \Rightarrow 2^4 3^2 109 \cdot 223 \Rightarrow 2^{10} 3^2 5 \cdot 7 \cdot 11$
53	$2 \cdot 3^3$
53^2	$7 \cdot 409 \Rightarrow 2 \cdot 5 \cdot 7 \cdot 41$
61	$2 \cdot 31$
61^2	$3 \cdot 13 \cdot 97 \Rightarrow 2 \cdot 3 \cdot 7^2 13$
61^3	$2^2 31 \cdot 1861 \Rightarrow 2^3 7^2 19 \cdot 31$

Table 3.10: Restricted sigma chains for the 2^n (contd.)

p^a	$\sigma_{eff}(p^a, b)$
61^4	$5 \cdot 131 \cdot 21491 \Rightarrow 2^2 3^3 5 \cdot 131 \cdot 199 \Rightarrow 2^5 3^3 5^3 131$
67	$2^2 17$
67^2	$3 \cdot 7^2 31$
67^3	$2^3 5 \cdot 17 \cdot 449 \Rightarrow 2^4 3^2 5^3 17$
73	$2 \cdot 37$
73^2	$3 \cdot 1801 \Rightarrow 2 \cdot 3 \cdot 17 \cdot 53$
79	$2^4 5$
79^2	$3 \cdot 7^2 43$
89	$2 \cdot 3^2 5$
89^2	$8011 \Rightarrow 2^2 2003 \Rightarrow 2^4 3 \cdot 167 \Rightarrow 2^7 3^2 7$
127	2^7
127^2	$3 \cdot 5419 \Rightarrow 2^2 3 \cdot 5 \cdot 271 \Rightarrow 2^6 3 \cdot 5 \cdot 17$
131	$2^2 3 \cdot 11$
131^2	$17293 \Rightarrow 2 \cdot 8647 \Rightarrow 2^4 23 \cdot 47 \Rightarrow 2^8 3 \cdot 23$
137	$2 \cdot 3 \cdot 23$
137^2	$7 \cdot 37 \cdot 73$
257	$2 \cdot 3 \cdot 43$
257^2	$61 \cdot 1087 \Rightarrow 2^6 17 \cdot 61$

The resulting complete model with 2^{14} as the specified initial component (whose effects appear as constants in the appropriate balance equations), leads to an integer linear programming model with 48 rows (47 constraints and 1 objective) and 80 variables (of which 78 are 0–1) as illustrated in Appendix E. It is straightforward to modify the model into a pure 0–1 form by transforming each of the remaining integer variables, *tot_2* and *tot_3*, into a set of 0–1 variables using

$$tot\_p = \sum_{i \geq 0} \left(2^i \; tot\_p\_i\right), \quad tot\_p\_i \in \{0, 1\}.$$

3.10 Results

3.10.1 Base model

Initial results for this limited model, using the LP software package *LINGO*, were promising. For initial values corresponding to each $2^n, n = 1, 2, \ldots, 19$ in turn, the model produced the optimal solutions, or "raw" even multiperfect numbers, given in Table 3.12. (No odd multiperfect numbers were found when 2^0 was specified.) This table includes all feasible solutions and, within each 2^n group, are shown in the order in which they were uncovered using a one-(optimal)-solution-at-a-time approach. The blank entries in column N for $n = 12, 16$ and 18 result from restricted sigma chaining for these perfect numbers. The columns labeled Depth and Nodes give an indication of the size of the search tree examined. The computational effort expended is directly related to the number of pivots required (as seen in the Pivots column) and the dimensions of the problem.

Table 3.11: Bounds for the $2^n, n < 20$ model after restricted sigma chaining

p	a	$B(p,a,b)$
2	19	$2^{19}3^3 5^2 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 41 \cdot 43 \cdot 73 \cdot 89 \cdot 127 \cdot 257$
3	9	$2^4 5 \cdot 7 \cdot 11^2 13 \cdot 19 \cdot 41 \cdot 61 \cdot 137$
5	6	$2^3 3^2 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 257$
7	5	$2^4 3^3 5^2 13 \cdot 19 \cdot 43$
11	3	$2^3 3 \cdot 7 \cdot 19 \cdot 61$
13	3	$2^2 3 \cdot 5 \cdot 7 \cdot 17 \cdot 61$
17	3	$2^2 3^2 5 \cdot 7 \cdot 11 \cdot 29$
19	3	$2^4 3 \cdot 5 \cdot 7 \cdot 13 \cdot 127$
23	2	$2^3 3 \cdot 7 \cdot 79$
29	3	$2^5 3 \cdot 5 \cdot 13 \cdot 53 \cdot 67$
31	4	$2^6 3^2 5 \cdot 7 \cdot 11^3 13 \cdot 37$
37	4	$2^4 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 41 \cdot 67 \cdot 89 \cdot 137$
41	4	$2^6 3^3 5 \cdot 7 \cdot 11 \cdot 29^2 67 \cdot 131$
43	4	$2^{10} 3^2 5^2 7 \cdot 11 \cdot 37 \cdot 79$
53	2	$2 \cdot 3^3 5 \cdot 7 \cdot 41$
61	4	$2^5 3^3 5^3 7^2 13 \cdot 19 \cdot 31 \cdot 131$
67	3	$2^4 3^2 5^3 7^2 17 \cdot 31$
73	2	$2 \cdot 3 \cdot 17 \cdot 37 \cdot 53$
79	2	$2^4 3 \cdot 5 \cdot 7^2 43$
89	2	$2^7 3^2 5 \cdot 7$
127	2	$2^7 3 \cdot 5 \cdot 17$
131	2	$2^8 3 \cdot 11 \cdot 23$
137	2	$2 \cdot 3 \cdot 7 \cdot 23 \cdot 37 \cdot 73$
257	2	$2^6 3 \cdot 17 \cdot 43 \cdot 61$

Table 3.12: Raw multiperfect numbers for the $2^n, n < 20$ model

n	$S(N)$	N	Depth	Nodes	Pivots
1	2	3			
2	4	$3^2 5 \cdot 7^2 13 \cdot 19$	3	7	188
	2	7	27	85	1060
3	4	$3^2 5 \cdot 7 \cdot 13$	26	79	1240
	3	$3 \cdot 5$	27	121	1964
4	2	31			
5	4	$3^4 7^2 11^2 19^2 127$	27	171	2569
	4	$3^3 5 \cdot 7$	27	173	2874
	3	$3 \cdot 7$	30	203	3472
6	2	127			
7	5	$3^4 5 \cdot 7 \cdot 11^2 17 \cdot 19$	21	73	1905
	5	$3^5 5 \cdot 7^2 13 \cdot 17 \cdot 19$	29	151	3394
	4	$3^6 5 \cdot 17 \cdot 23 \cdot 137$	30	199	4243
	4	$3^3 5^2 17 \cdot 31$	30	203	4531
8	4	$3^2 7^2 13 \cdot 19^2 37 \cdot 73 \cdot 127$	27	113	2144
	4	$3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$	27	113	2439
	3	$5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$	29	137	3207
9	4	$3^3 5 \cdot 11 \cdot 31$	27	193	3502
	4	$3^4 7 \cdot 11^3 31^2 61$	28	217	4468
	4	$3^2 7 \cdot 11 \cdot 13 \cdot 31$	28	217	4767
	3	$3 \cdot 11 \cdot 31$	30	243	5540
10	5	$3^4 5 \cdot 7 \cdot 11^2 19 \cdot 23 \cdot 89$	21	63	1480
	5	$3^5 5 \cdot 7^2 13 \cdot 19 \cdot 23 \cdot 89$	27	141	2786
	4	$3^3 5^2 23 \cdot 31 \cdot 89$	30	215	3811
11	5	$3^6 5 \cdot 7^2 13 \cdot 19 \cdot 23 \cdot 137$	30	263	5100
	5	$3^3 5^2 7^2 13 \cdot 19 \cdot 31$	30	265	5884
	5	$3^5 5 \cdot 7^2 13^2 19 \cdot 31 \cdot 61$	30	267	5975
	5	$3^5 5^2 7^3 13^2 31^2 61$	30	267	5997
	5	$3^5 5^3 7^3 13^3 17$	30	267	6234
12	2				
13	4	$3^2 7 \cdot 11 \cdot 13 \cdot 43 \cdot 127$	27	201	3778
	4	$3^3 5 \cdot 11 \cdot 43 \cdot 127$	27	201	4270
	3	$3 \cdot 11 \cdot 43 \cdot 127$	27	217	4692
14	5	$3^2 5^2 7^3 13 \cdot 19 \cdot 31^2$	24	163	2912
	4	$3^2 7^2 13 \cdot 19^2 31 \cdot 127$	31	229	4340
	4	$3 \cdot 5 \cdot 7 \cdot 19 \cdot 31$	31	237	4689
	3	$5 \cdot 7 \cdot 19 \cdot 31$	31	255	5400
15	6	$3^5 5^3 7^4 11^2 13^3 17^2 19 \cdot 43 \cdot 257$	27	299	6491

Table 3.12: Raw multiperfect numbers for the 2^n model (contd.)

n	$S(N)$	N	Depth	Nodes	Pivots
	6	$3^5 5^4 7^3 11^2 13 \cdot 17 \cdot 19 \cdot 43 \cdot 257$	27	319	7541
	6	$3^7 5^3 7^2 11 \cdot 13 \cdot 17 \cdot 19 \cdot 41 \cdot 43 \cdot 257$	31	331	8121
	6	$3^5 5^2 7^2 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43 \cdot 257$	31	431	10898
	5	$3^7 5 \cdot 7 \cdot 11 \cdot 17 \cdot 41 \cdot 43 \cdot 257$	31	513	13389
16	2				
17	6	$3^4 5^3 7^3 11^2 13^2 19^3 31 \cdot 37 \cdot 61 \cdot 73$	33	215	4698
	5	$3^5 5 \cdot 7^3 13 \cdot 19^2 37 \cdot 73 \cdot 127$	32	277	6603
	4	$3^6 7 \cdot 19^2 23 \cdot 37 \cdot 73 \cdot 127 \cdot 137$	35	325	7916
18	2				
19	6	$3^4 5^5 7^3 11^3 13 \cdot 19 \cdot 31^3 37 \cdot 41 \cdot 61$	31	619	14923
	6	$3^9 5^3 7^4 11^3 13^3 17 \cdot 31 \cdot 41 \cdot 61^2$	31	623	16033
	6	$3^4 5^2 7^2 11^3 13 \cdot 19^2 31^3 37 \cdot 41 \cdot 61 \cdot 127$	31	619	17298
	6	$3^5 5^2 7^2 11 \cdot 13^2 19^2 31^3 37 \cdot 41 \cdot 61 \cdot 127$	31	627	18169
	6	$3^5 5^5 7^3 11 \cdot 13^2 19 \cdot 31^3 37 \cdot 41 \cdot 61$	31	619	18189
	6	$3^9 5^3 7^4 11^3 13^2 19 \cdot 31^2 41 \cdot 61^3$	31	621	18616
	6	$3^5 5^2 7^5 11^2 13 \cdot 19^2 31^2 41 \cdot 43 \cdot 127$	31	629	19297
	6	$3^8 5^3 7^2 11 \cdot 13^2 19^2 31^2 41 \cdot 61 \cdot 127$	31	631	20630
	6	$3^6 5^3 7^2 11 \cdot 13 \cdot 19 \cdot 23 \cdot 31 \cdot 41 \cdot 137$	31	629	20923
	5	$3^6 5 \cdot 7 \cdot 11 \cdot 23 \cdot 31 \cdot 41 \cdot 137$	31	723	24089
	5	$3^7 5^2 7 \cdot 11 \cdot 31^2 41^2$	33	745	25408

The factors eliminated from the model through the use of restricted sigma chaining, have to be reintroduced to give the full multiperfect numbers. For example, for 2^{14} the right-hand side constants are $3 - 14 = -11$ for *tot_2*, 1 for *tot_7*, 1 for *tot_19* and 1 for *tot_31*, since $\sigma(2^{14}) = 7 \cdot 31 \cdot 151$ and $\sigma_{eff}(2^{14}, b) = 2^3 7 \cdot 19 \cdot 31$ (as given in Table 3.10). As a consequence of this and the restricted sigma chain for $\sigma(31^2)$, the first, final, full multiperfect number produced for 2^{14} is $(2^{14} 83 \cdot 151 \cdot 331) \cdot (3^2 5^2 7^3 13 \cdot 19 \cdot 31^2)$. From the raw multiperfect numbers we obtain the final, full multiperfect numbers shown in Table 3.13. Of the 19 known multiperfect numbers with 2^{19} as a component, only 11 were covered (and found) by the model. The 8 known multiperfect numbers which were not found by the model included the component 19^4 (part of the substitution of $19^2 127$ with $19^4 151 \cdot 911$) or components such as 3^{10}, 7^7 or 11^6, which were not covered in this example.

Table 3.13: Final multiperfect numbers for the $2^n, n < 20$ model

n	$S(N)$	N
1	2	$(2) \cdot (3)$
2	4	$(2^2) \cdot (3^2 5 \cdot 7^2 13 \cdot 19)$
	2	$(2^2) \cdot (7)$
3	4	$(2^3) \cdot (3^2 5 \cdot 7 \cdot 13)$
	3	$(2^3) \cdot (3 \cdot 5)$
4	2	$(2^4) \cdot (31)$
5	4	$(2^5) \cdot (3^4 7^2 11^2 19^2 127)$
	4	$(2^5) \cdot (3^3 5 \cdot 7)$
	3	$(2^5) \cdot (3 \cdot 7)$
6	2	$(2^6) \cdot (127)$
7	5	$(2^7) \cdot (3^4 5 \cdot 7 \cdot 11^2 17 \cdot 19)$
	5	$(2^7) \cdot (3^5 5 \cdot 7^2 13 \cdot 17 \cdot 19)$
	4	$(2^7 547 \cdot 1093) \cdot (3^6 5 \cdot 17 \cdot 23 \cdot 137)$
	4	$(2^7) \cdot (3^3 5^2 17 \cdot 31)$
8	4	$(2^8) \cdot (3^2 7^2 13 \cdot 19^2 37 \cdot 73 \cdot 127)$
	4	$(2^8) \cdot (3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73)$
	3	$(2^8) \cdot (5 \cdot 7 \cdot 19 \cdot 37 \cdot 73)$
9	4	$(2^9) \cdot (3^3 5 \cdot 11 \cdot 31)$
	4	$(2^9 83 \cdot 331) \cdot (3^4 7 \cdot 11^3 31^2 61)$
	4	$(2^9) \cdot (3^2 7 \cdot 11 \cdot 13 \cdot 31)$
	3	$(2^9) \cdot (3 \cdot 11 \cdot 31)$
10	5	$(2^{10}) \cdot (3^4 5 \cdot 7 \cdot 11^2 19 \cdot 23 \cdot 89)$
	5	$(2^{10}) \cdot (3^5 5 \cdot 7^2 13 \cdot 19 \cdot 23 \cdot 89)$
	4	$(2^{10}) \cdot (3^3 5^2 23 \cdot 31 \cdot 89)$
11	5	$(2^{11} 547 \cdot 1093) \cdot (3^6 5 \cdot 7^2 13 \cdot 19 \cdot 23 \cdot 137)$
	5	$(2^{11}) \cdot (3^3 5^2 7^2 13 \cdot 19 \cdot 31)$
	5	$(2^{11}) \cdot (3^5 5 \cdot 7^2 13^2 19 \cdot 31 \cdot 61)$
	5	$(2^{11} 83 \cdot 331) \cdot (3^5 5^2 7^3 13^2 31^2 61)$
	5	$(2^{11}) \cdot (3^5 5^3 7^3 13^3 17)$
12	2	$(2^{12} 8191)$
13	4	$(2^{13}) \cdot (3^2 7 \cdot 11 \cdot 13 \cdot 43 \cdot 127)$
	4	$(2^{13}) \cdot (3^3 5 \cdot 11 \cdot 43 \cdot 127)$
	3	$(2^{13}) \cdot (3 \cdot 11 \cdot 43 \cdot 127)$
14	5	$(2^{14} 83 \cdot 151 \cdot 331) \cdot (3^2 5^2 7^3 13 \cdot 19 \cdot 31^2)$
	4	$(2^{14} 151) \cdot (3^2 7^2 13 \cdot 19^2 31 \cdot 127)$
	4	$(2^{14} 151) \cdot (3 \cdot 5 \cdot 7 \cdot 19 \cdot 31)$
	3	$(2^{14} 151) \cdot (5 \cdot 7 \cdot 19 \cdot 31)$
15	6	$(2^{15} 307 \cdot 467 \cdot 2801) \cdot (3^5 5^3 7^4 11^2 13^3 17^2 19 \cdot 43 \cdot 257)$

Table 3.13: Final multiperfect numbers for the 2^n model (contd.)

n	$S(N)$	N
	6	$(2^{15}71)\cdot(3^5 5^4 7^3 11^2 13\cdot 17\cdot 19\cdot 43\cdot 257)$
	6	$(2^{15})\cdot(3^7 5^3 7^2 11\cdot 13\cdot 17\cdot 19\cdot 41\cdot 43\cdot 257)$
	6	$(2^{15})\cdot(3^5 5^2 7^2 11\cdot 13\cdot 17\cdot 19\cdot 31\cdot 43\cdot 257)$
	5	$(2^{15})\cdot(3^7 5\cdot 7\cdot 11\cdot 17\cdot 41\cdot 43\cdot 257)$
16	2	$(2^{16}131071)$
17	6	$(2^{17}181)\cdot(3^4 5^3 7^3 11^2 13^2 19^3 31\cdot 37\cdot 61\cdot 73)$
	5	$(2^{17})\cdot(3^5 5\cdot 7^3 13\cdot 19^2 37\cdot 73\cdot 127)$
	4	$(2^{17}547\cdot 1093)\cdot(3^6 7\cdot 19^2 23\cdot 37\cdot 73\cdot 127\cdot 137)$
18	2	$(2^{18}524287)$
19	6	$(2^{19})\cdot(3^4 5^5 7^3 11^3 13\cdot 19\cdot 31^3 37\cdot 41\cdot 61)$
	6	$(2^{19}97\cdot 467\cdot 2801)\cdot(3^9 5^3 7^4 11^3 13^3 17\cdot 31\cdot 41\cdot 61^2)$
	6	$(2^{19})\cdot(3^4 5^2 7^2 11^3 13\cdot 19^2 31^3 37\cdot 41\cdot 61\cdot 127)$
	6	$(2^{19})\cdot(3^5 5^2 7^2 11\cdot 13^2 19^2 31^3 37\cdot 41\cdot 61\cdot 127)$
	6	$(2^{19})\cdot(3^5 5^5 7^3 11\cdot 13^2 19\cdot 31^3 37\cdot 41\cdot 61)$
	6	$(2^{19}83\cdot 331\cdot 467\cdot 1861\cdot 2801)\cdot(3^9 5^3 7^4 11^3 13^2 19\cdot 31^2 41\cdot 61^3)$
	6	$(2^{19}83\cdot 331)\cdot(3^5 5^2 7^5 11^2 13\cdot 19^2 31^2 41\cdot 43\cdot 127)$
	6	$(2^{19}83\cdot 331\cdot 379\cdot 757)\cdot(3^8 5^3 7^2 11\cdot 13^2 19^2 31^2 41\cdot 61\cdot 127)$
	6	$(2^{19}547\cdot 1093)\cdot(3^6 5^3 7^2 11\cdot 13\cdot 19\cdot 23\cdot 31\cdot 41\cdot 137)$
	5	$(2^{19}547\cdot 1093)\cdot(3^6 5\cdot 7\cdot 11\cdot 23\cdot 31\cdot 41\cdot 137)$
	5	$(2^{19}83\cdot 331\cdot 431\cdot 1723)\cdot(3^7 5^2 7\cdot 11\cdot 31^2 41^2)$

3.10.2 All solutions

The solution of a linear programming problem is an optimal solution, if one exists. An indication of the possible existence of alternate optimal solutions can also be given. Ideally, we would like *all* feasible solutions. This can be achieved by solving a sequence of problems where the optimal solutions of previous problems are explicitly disallowed by the addition of one constraint for each such solution. Such additional constraints are commonly called Dantzig cuts.

If the optimal solution of a 0–1 integer linear programming problem is partitioned into two disjoint sets given by $S_1 = \{i|x_i = 1\}$ and $S_0 = \{i|x_i = 0\} = \overline{S_1}$ then a suitable constraint to exclude this solution is given by

$$\sum_{i\in S_1}(1 - x_i) + \sum_{j\in S_0} x_j \geq 1$$

i.e. at least *one* variable must have a changed value.

To illustrate this simple approach, all solutions for the base model, with 2^{10} as a component, were found as follows.

The first optimal solution was given by $x3\_4 = x5\_1 = x7\_1 = x11\_2 = x19\_1 = x23\_1 = x89\_1 = 1$ and all other variables $xp\_a = 0$. The appropriate constraint to exclude this solution is thus

$$\begin{aligned}
&(1 - x3_4) + (1 - x5_1) + (1 - x7_1) + (1 - x11_2) + (1 - x19_1) \\
&+ (1 - x23_1) + (1 - x89_1) \\
&+ x3_1 + x3_2 + x3_3 + x3_5 + x3_6 + x3_7 + x3_8 + x3_9 \\
&+ x5_2 + x5_3 + x5_4 + x5_5 + x5_6 + x7_2 + x7_3 + x7_4 + x7_5 \\
&+ x11_1 + x11_3 + x13_1 + x13_2 + x13_3 + x17_1 + x17_2 + x17_3 \\
&+ x19_2 + x19_3 + x23_2 + x29_1 + x29_2 + x29_3 \\
&+ x31_1 + x31_2 + x31_3 + x31_4 + x37_1 + x37_2 + x37_3 + x37_4 \\
&+ x41_1 + x41_2 + x41_3 + x41_4 + x43_1 + x43_2 + x43_3 + x43_4 \\
&+ x53_1 + x53_2 + x61_1 + x61_2 + x61_3 + x61_4 + x67_1 + x67_2 + x67_3 \\
&+ x73_1 + x73_2 + x79_1 + x79_2 + x89_2 + x127_1 + x127_2 \\
&+ x131_1 + x131_2 + x137_1 + x137_2 + x257_1 + x257_2 \geq 1.
\end{aligned}$$

Solving the augmented model gives the alternate solution, the 5-perfect number $2^{10}3^5 5 \cdot 7^2 \cdot 13 \cdot 19 \cdot 23 \cdot 89$. Continuing this sequence of problems, we next find the 4-perfect number $2^{10}3^3 5^2 23 \cdot 31 \cdot 89$. Notice the expected monotonic decrease in the index of the multiperfect number found. When this solution is also excluded and the problem re-attempted, we find that no feasible integer solution exists – we have found *all* feasible solutions. In this case the coverage of the model was sufficient to allow *all* known such multiperfect numbers to be found.

Tables 3.12 and 3.13 include *all* feasible solutions found using this approach.

3.10.3 Scalability

To investigate the *scalability* of the model, the rate of growth in the size of the model (and hence the effort required to solve it) as the coverage is expanded, the coverage was increased initially to

$$C_{spec} = 2^{29}3^{25}5^{12}7^{10}11^8 13^8 17^4 19^{10}23^6.$$

This was sufficient to include the three smallest known 7-perfect numbers. When the process of finding a selfcontained C_{aux} was attempted no stable set was found. There was a positive feedback effect where increases in the exponents of some (usually small) primes caused a cycle (chain reaction) of increases. Three approaches can be used to overcome this problem. Firstly, by initially "fixing" more possible primes. This led to the choice

$$\begin{aligned}
C_{spec} = {} & 2^{29}3^{25}5^{12}7^{10}11^8 13^8 17^4 19^{10}23^6 29^6 31^6 37^4 41^4 43^4 47^6 \\
& 53^3 59^3 61^4 67^3 71^3 73^3 79^3 83^3 89^3 97^3.
\end{aligned}$$

Secondly, the LP model developed for 2^n, $n < 20$ allowed for the consequences of *any* 2^n to be taken into account. For example the model allowed for 73^2 even though this would only be necessary if $2^{17}137^2$ was an exact factor. For this small model this flexibility (extra decision variables) did not greatly impair efficiency. For larger models it is desirable to minimize the number of decision variables. This can be achieved by tailoring the cover to a particular 2^n. For the present model this can be achieved by

$$Cover(C_{spec}) = \sigma(2^{29}) \cdot Cover(C'_{spec})$$

where $2 \nmid C'_{spec}$. This was not done for this *Cover*.

Thirdly, the maximum exponent of any auxiliary factor can be limited to a specified maximum value (for example $p^a, a \leq K$, for any p prime and small integer K). For the expanded C_{spec} it was not necessary to use this third approach.

The tedious calculations for determining the additional necessary primes give

$$\begin{aligned} C_{aux} = {} & 103^3 109^4 127^2 131^3 137^2 139^2 151^2 157^3 163^2 167^2 181^4 \\ & 191^3 193^2 197^2 211^2 223^4 263^2 271^2 281^3 307^2 317^2 331^2 \\ & 367^2 379^2 409^2 433^2 523^2 547^2 601^2 613^2 757^2 829^2 \\ & 1093^2 398581^3. \end{aligned}$$

The last factor, 398581^3, originates from the restricted sigma chains $\sigma(3^{25}) = 2^2 398581 \cdot 797161 \Rightarrow 2^3 398581^2$ and $\sigma(1093^2) = 3 \cdot 398581$.

As a result of this expanded coverage the model increases to 118 rows with 228 variables (of which 226 are 0–1). The restricted sigma chain for 2^{29}, with respect to $C_{spec} \cdot C_{aux}$, is $3^2 7 \cdot 11 \cdot 31 \cdot 151 \cdot 331$. The PC version of *LINGO* used for the smaller model was unable to handle the size of the expanded model for $2^n, n \leq 29$. The widely available academic linear programming package *lp_solve* is able to handle larger problems on a wide range of more powerful hardware. Unfortunately numerical stability problems prevented *lp_solve* from finding all solutions for this coverage. Consequently the constraint logic programming package *opbdp* was used to find all solutions, corresponding to *all* 20 known multiperfect numbers with 2^{29} as a component, as shown in Table 3.14. The *opbdp* approach will be discussed later. While the expansion in the coverage of the model led to more possible multiperfect numbers, the increased size of the model resulted in a significant increase in the solution time.

This expanded cover was also used to find all the remaining eight (known) multiperfect numbers with 2^{19} as an exact factor which were not found with the original cover. This simply required using the restricted sigma chain for 2^{19} $(3 \cdot 5^2 11 \cdot 11 \cdot 41)$ instead of that for 2^{29}.

To further test the scalability of the model, the coverage was increased to include "small" cases announced by G. Woltman (with 2^{65} as a component) and by J. Moxham (with 2^{62} as a component), with the aim of either finding new multiperfect numbers or of showing that all have been found (within the bounds of the cover). This coverage is sufficient to include the smaller known 8-perfect numbers. The largest individual components of known multiperfect numbers with a component $2^n, n \leq 65$, are 3^{40}, 5^{20}, 7^{20}, 11^{10}, 13^9, 17^5, 19^9, 23^4, 29^4, 31^4, 37^5, 41^2, 43^4, $47^2, \ldots$. Consequently the initial coverage was set beyond this to

$$\begin{aligned} C_{spec} = {} & 2^{65} 3^{45} 5^{30} 7^{25} 11^{15} 13^{10} 17^6 19^{10} 23^6 29^6 31^6 37^6 41^4 43^4 47^4 \\ & \cdot 53^4 59^4 61^4 67^4 71^4 73^4 79^4 83^4 89^4 97^4. \end{aligned} \tag{3.1}$$

Table 3.14: Final multiperfect numbers for the 2^{29} model

$S(N)$	N
7	$(2^{29}617 \cdot 911 \cdot 2617 \cdot 4733 \cdot 5233 \cdot 36809 \cdot 368089) \cdot (3^{20}5^{4}7^{6}11^{5}13^{2}17^{2}19^{4}$ $\cdot 29 \cdot 31^{2}37 \cdot 41 \cdot 61 \cdot 71 \cdot 103 \cdot 151^{2}263 \cdot 307 \cdot 331^{2}409 \cdot 1093^{2}398581)$
7	$(2^{29}2617 \cdot 4733 \cdot 5233 \cdot 36809 \cdot 368089) \cdot (3^{20}5^{5}7^{6}11^{3}13^{2}17 \cdot 19 \cdot 23 \cdot 29$ $\cdot 31^{2}41 \cdot 61^{2}97 \cdot 137 \cdot 151 \cdot 263 \cdot 331^{2}409 \cdot 547 \cdot 1093)$
7	$(2^{29}) \cdot (3^{17}5^{2}7^{6}11^{3}13^{2}17 \cdot 19^{4} \cdot 23 \cdot 29 \cdot 31^{2}37 \cdot 61^{2}97 \cdot 137 \cdot 151^{2}263$ $\cdot 331^{2}379 \cdot 547 \cdot 757 \cdot 1093)$
6	$(2^{29}) \cdot (3^{6}5^{3}7^{3}11 \cdot 13 \cdot 19 \cdot 23 \cdot 31 \cdot 83 \cdot 137 \cdot 151 \cdot 331 \cdot 547 \cdot 1093)$
6	$(2^{29}) \cdot (3^{10}5^{3}7^{3}11 \cdot 13 \cdot 19 \cdot 23 \cdot 31 \cdot 83 \cdot 151 \cdot 331)$
6	$(2^{29}467 \cdot 719 \cdot 1871 \cdot 2617 \cdot 2801 \cdot 5233 \cdot 34511) \cdot (3^{16}5 \cdot 7^{4}11^{2}13^{2}17 \cdot 19^{2}31^{2}61$ $\cdot 127 \cdot 151 \cdot 331^{2})$
6	$(2^{29}) \cdot (3^{5}5^{2}7^{3}11 \cdot 13^{2}19^{2}31^{3}37 \cdot 61 \cdot 83 \cdot 127 \cdot 151 \cdot 331)$
6	$(2^{29}) \cdot (3^{4}5^{2}7^{3}11^{3}13 \cdot 19^{2}31^{3}37 \cdot 61 \cdot 83 \cdot 127 \cdot 151 \cdot 331)$
6	$(2^{29}) \cdot (3^{6}5 \cdot 7^{5}11^{2}13 \cdot 19^{3}23 \cdot 31 \cdot 43 \cdot 83 \cdot 137 \cdot 151 \cdot 181 \cdot 331 \cdot 547 \cdot 1093)$
6	$(2^{29}) \cdot (3^{10}5 \cdot 7^{5}11^{2}13 \cdot 19^{3}23 \cdot 31 \cdot 43 \cdot 83 \cdot 151 \cdot 181 \cdot 331)$
6	$(2^{29}) \cdot (3^{8}5 \cdot 7^{5}11^{2}13 \cdot 19^{4}23 \cdot 31 \cdot 43 \cdot 83 \cdot 137 \cdot 151^{2}331 \cdot 379 \cdot 547 \cdot 757 \cdot 1093)$
6	$(2^{29}45319) \cdot (3^{4}5^{4}7^{3}11^{6}13 \cdot 19 \cdot 31 \cdot 43 \cdot 71 \cdot 83 \cdot 103 \cdot 151 \cdot 331)$
6	$(2^{29}) \cdot (3^{9}5 \cdot 7^{8}11^{3}13 \cdot 19^{4}31 \cdot 37^{3}61^{2}73 \cdot 83 \cdot 97 \cdot 137^{2}151^{2}331 \cdot 547 \cdot 1093)$
6	$(2^{29}179 \cdot 2617 \cdot 3221 \cdot 5233) \cdot (3^{13}5^{2}7^{5}11^{4}17 \cdot 19^{2}31^{2}41 \cdot 43 \cdot 127 \cdot 151 \cdot 163$ $\cdot 307 \cdot 331^{2}547^{2}613 \cdot 1093)$
6	$(2^{29}179 \cdot 613 \cdot 1201 \cdot 2617 \cdot 3221 \cdot 5233) \cdot (3^{13}5^{5}7^{7}11^{4}17 \cdot 19 \cdot 31^{2}41 \cdot 43$ $\cdot 151 \cdot 163 \cdot 307 \cdot 331^{2}547^{2}601 \cdot 1093)$
5	$(2^{29}467 \cdot 719 \cdot 1871 \cdot 2617 \cdot 2801 \cdot 5233 \cdot 34511) \cdot (3^{16}7^{4}11^{2}13^{2}17 \cdot 19^{2}31^{2}61$ $\cdot 127 \cdot 151 \cdot 331^{2})$
5	$(2^{29}) \cdot (3^{6}7^{5}11^{2}13 \cdot 19^{3}23 \cdot 31 \cdot 43 \cdot 83 \cdot 137 \cdot 151 \cdot 181 \cdot 331 \cdot 547 \cdot 1093)$
5	$(2^{29}) \cdot (3^{10}7^{5}11^{2}13 \cdot 19^{3}23 \cdot 31 \cdot 43 \cdot 83 \cdot 151 \cdot 181 \cdot 331)$
5	$(2^{29}) \cdot (3^{8}7^{5}11^{2}13 \cdot 19^{4}23 \cdot 31 \cdot 43 \cdot 83 \cdot 137 \cdot 151^{2}331 \cdot 379 \cdot 547 \cdot 757 \cdot 1093)$
5	$(2^{29}) \cdot (3^{9}7^{8}11^{3}13 \cdot 19^{4}31 \cdot 37^{3}61^{2}73 \cdot 83 \cdot 97 \cdot 137^{2}151^{2}331 \cdot 547 \cdot 1093)$

The tedious calculations for determining the additional necessary primes (with the exponents now constrained to be ≤ 4) gave

$$\begin{aligned} C_{aux} = {} & 101^4 103^4 107^4 109^4 113^4 127^4 131^4 137^4 139^4 149^4 151^4 \\ & \cdot 157^4 163^3 167^4 173^3 179^3 181^4 191^4 193^4 197^4 199^4 211^4 223^4 \\ & \cdot 227^4 229^4 233^4 239^4 241^2 251^4 257^4 263^4 271^4 277^3 281^4 283^4 \\ & \cdot 293^3 307^4 311^3 313^4 317^4 331^4 337^2 347^2 349^4 353^2 359^3 367^4 \\ & \cdot 373^2 379^4 397^2 409^4 419^3 431^3 433^4 439^3 443^2 449^4 457^2 487^4 499^3 \\ & \cdot 521^3 523^3 547^2 557^4 569^2 571^3 577^4 601^3 613^2 619^2 631^3 641^2 \\ & \cdot 653^2 661^3 673^3 683^3 691^3 701^2 709^2 743^3 751^2 757^3 797^2 811^2 \qquad (3.2) \\ & \cdot 823^2 829^3 859^3 877^4 911^2 941^2 967^3 991^4 1009^2 1013^2 1051^2 1069^2 \\ & \cdot 1087^2 1093^4 1103^2 1109^2 1117^2 1123^2 1277^2 1279^2 1321^2 1361^2 \\ & \cdot 1471^2 1609^2 1613^2 1619^2 1621^3 1699^3 1741^2 1801^2 1861^2 1871^2 \\ & \cdot 1973^2 1993^2 2003^2 2083^2 2089^2 2113^2 2141^2 2143^2 2203^2 2243^2 \\ & \cdot 2383^3 2393^2 2521^2 2551^2 3217^2 3833^2 4051^2 5419^2 6271^2 7193^2 \\ & \cdot 7621^2 8191^2 8971^2 15901^2 398581^3. \end{aligned}$$

As a result of this expanded coverage the model increased to 497 rows with 624 variables (of which 622 are 0–1). The restricted sigma chain for 2^{65}, with respect to $C_{spec} \cdot C_{aux}$, is $2^5 3^2 5^2 7^3 \cdot 23 \cdot 67 \cdot 89 \cdot 149 \cdot 683 \cdot 2141$. By way of comparison, a model was constructed with the C_{spec} as in (3.1) but where 2^{65} was a given component, This model had only 214 rows and 470 variables.

Exploration of each $2^n, n \leq 65$, for this cover did not produce any new multiperfect numbers. Therefore, given the generous bounds on the exponents in the cover, it is very likely that all multiperfect numbers with 2^n, $n \leq 65$, as a component have indeed been found. Our construction ensures the following:

Theorem 6. *Let C_{spec} and C_{aux} be as in (3.1) and (3.2), and $C_{spec} \cdot C_{aux} = \prod_{i=1}^{t} q_i^{b_i}$, q_i distinct primes. All multiperfect numbers of the form $\prod_{i=1}^{t} q_i^{c_i} \cdot l$, $0 \leq c_i \leq b_i$ and $q_i \nmid l$, l squarefree, are known in the literature.*

Since the largest prime factor of an odd perfect number exceeds 10^7 [75] we have:

Corollary 1. *There are no odd perfect numbers of the form πm^2 for prime $\pi \equiv 1 \pmod 4$ and $m^2 \mid C_{spec} \cdot C_{aux}/2^{65}$ where C_{spec} and C_{aux} are as in (3.1) and (3.2).*

One final attempt at expansion was to construct C_{spec} and C_{aux} for 2^n, $65 \leq n \leq 110$. The aim was to look for the smallest 9-perfect number and any remaining small 7- or 8-perfect numbers. An analysis of known multiperfect numbers 2^n, $n < 110$, suggested a conservative bound

$$\begin{aligned} C_{spec} = {} & 2^{110} 3^{60} 5^{45} 7^{35} 11^{25} 13^{20} 17^{15} 19^{15} 23^{10} 29^7 31^8 37^7 41^7 43^7 47^6 \\ & \cdot 53^5 59^4 61^5 67^4 71^4 73^4 79^4 83^4 89^4 97^4. \end{aligned}$$

The tedious calculations for determining the additional necessary primes (with the exponents constrained to be ≤ 4) gave

$$\begin{aligned} C_{aux} = {} & 101^4 103^4 107^4 109^4 113^4 127^4 131^4 137^4 139^4 149^4 151^4 \\ & \cdot 157^4 163^4 167^4 173^4 179^4 181^4 191^4 193^4 197^4 199^4 211^4 223^4 \\ & \cdot 227^4 229^4 233^4 239^4 241^4 251^4 257^4 263^4 269^2 271^4 277^4 281^4 283^4 \\ & \cdot 293^3 307^4 311^4 313^4 317^4 331^4 337^4 347^3 349^4 353^4 359^4 367^4 \\ & \cdot 373^4 379^4 383^2 397^4 401^2 409^4 419^4 421^4 431^4 433^4 439^4 443^3 449^4 \\ & \cdot 457^3 461^4 463^2 467^3 487^4 491^4 499^4 503^2 509^2 521^4 523^4 547^4 557^3 \\ & \cdot 563^3 569^4 571^4 577^4 587^3 599^4 601^4 607^3 613^3 617^3 619^4 631^4 641^2 \\ & \cdot 647^2 653^4 661^4 673^4 677^2 683^4 691^3 701^4 709^4 719^3 733^2 743^4 751^4 \\ & \cdot 757^3 761^3 769^3 773^3 797^3 811^4 823^3 829^4 853^3 859^4 863^2 877^4 881^2 \\ & \cdot 883^2 911^4 919^2 937^3 941^2 953^3 967^2 991^4 997^2 1009^3 1013^3 1021^2 1051^4 \\ & \cdot 1061^2 1063^2 1069^3 1087^4 1093^4 1103^2 1109^2 1117^2 1123^2 1151^2 1153^2 1163^2 1171^4 \\ & \cdot 1201^2 1213^2 1249^2 1277^2 1279^3 1301^3 1307^2 1321^2 1399^2 1471^3 1481^2 1489^2 \\ & \cdot 1499^2 1511^2 1523^2 1597^2 1607^3 1609^3 1613^3 1621^3 1657^2 1693^2 1699^3 1723^2 \\ & \cdot 1741^2 1759^2 1801^2 1861^2 1871^2 1873^2 1889^2 1951^2 1973^2 1993^2 2003^2 2017^2 \\ & \cdot 2081^2 2083^2 2089^2 2113^2 2141^2 2143^3 2203^2 2213^2 2221^3 2243^2 2267^2 2311^2 \\ & \cdot 2383^3 2393^2 2437^2 2521^2 2551^3 2593^2 2617^2 2693^2 2767^2 2789^2 2851^2 3019^3 3121^2 \\ & \cdot 3187^2 3203^2 3217^2 3391^2 3607^2 3631^2 3659^2 3673^2 3769^2 3833^2 3877^2 3917^2 \\ & \cdot 4021^2 4051^3 4099^2 4177^3 4201^3 4447^2 4591^2 4909^2 4987^2 5167^2 5281^3 5419^2 \\ & \cdot 5749^2 5879^2 6271^3 6529^2 7129^2 7369^2 7621^2 8101^2 8191^2 8269^2 8971^2 9181^2 \\ & \cdot 9511^2 9967^2 10039^3 15607^2 15901^2 19441^2 19501^2 22129^2 23971^2 30781^2 38923^2 \\ & \cdot 46601^2 100801^2 398581^3. \end{aligned}$$

As a result of this expanded coverage the model increased to 845 rows with 1033 variables (of which 1031 are 0–1). The size of this model is such that the resources available to the author are unable to solve it within a realistic time limit.

3.11 Specialized 0–1 algorithms

One approach described in the literature for solving large scale 0–1 integer programming problems is branch-and-bound with problem pre-conditioning, prioritized variable search order and modified branch bounding conditions [2, 36, 56, 76, 104, 105]. Branch-and-cut has been shown to be a better approach in scheduling and related 0–1 optimisation problems [68, 69]. The related knapsack problem has also yielded new efficient algorithms [84, 85]. High performance computing techniques such as vector and parallel processing have been employed in solving large-scale integer programming problems [39, 49] and combinatorial optimisation problems [42]. Advances have also been made in the application of artificial intelligence (AI) techniques developed for automatic theorem proving, to solving pseudo-boolean (i.e. 0–1) optimisation problems. This area of research is referred to as Constraint Logic Programming [6, 7, 9, 23].

The linear programming packages (*LINGO* and *lp_solve*) employed here used the general branch-and-bound algorithm without any pre-conditioning to solve for the 0–1 variables. A more specialized solution algorithm, such as one of the implicit enumeration algorithms [5, 50, 51, 120, 121], could prove to be an improvement.

The literature describes several approaches to accelerating the solution of 0–1 integer programming problems:

(i) fix the values of some of the variables during a pre-processing phase,

(ii) pre-condition the problem by tightening some of the constraints through coefficient reduction,

(iii) use surrogate constraints to more decisively "prune" the search and

(iv) aggregate constraints, thus reducing the size of the problem without affecting the optimal solution. The ultimate reduced problem would have a single constraint – this is called a knapsack problem.

It was not deemed appropriate to investigate such specialised acceleration techniques at the time.

While the branch-and-bound implementation in *lp_solve* failed to find all solutions for the 2^{29} model due to numerical stability problems, the logic-based branch-and-cut implementation in ***opbdp*** was able to find all solutions (and in fact *all* the known multiperfect numbers with 2^{29} as an exact factor) in significantly less time. Indeed ***opbdp*** would typically find an optimal solution quickly and then spend the rest of the time confirming that the solution was optimal. From this it appears that the type of model developed here is more amenable to a solution method which takes advantage of the model's logical structure rather than its polyhedral structure. Advantage was taken of any option to display intermediate feasible solutions to accelerate the finding of all feasible solutions.

Conveniently using ***opbdp*** required only minor reformatting of the model data as used by *lp_solve* (or *LINGO*). In particular the integer variables *tot_2* and *tot_3* need to be replaced by a weighted sum of a set of 0–1 variables as previously described.

Chapter 4

Odd Triperfect Numbers

4.1 Introduction

A positive integer N is said to be triperfect (or 3-perfect) if $\sigma(N) = 3N$ (or $S(N) = 3$). In this chapter N will always denote an odd triperfect number assuming such exist.

The smallest triperfect number is 120 with

$$\begin{aligned}\sigma(120) &= 1 + 2 + 3 + 4 + 5 + 6 + 8 + 10 + 12 + 15 + 20 + 24 \\ &\quad + 30 + 40 + 60 + 120 \\ &= 360 = 3 \cdot 120\end{aligned}$$

or

$$S(120) = S(2^3 3 \cdot 5) = S(2^3)S(3)S(5) = \frac{3 \cdot 5}{2^3} \cdot \frac{2^2}{3} \cdot \frac{2 \cdot 3}{5} = 3.$$

There are six known 3-perfect numbers, all of which are even ($2^3 3 \cdot 5$, $2^5 3 \cdot 7$, $2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$, $2^9 3 \cdot 11 \cdot 31$, $2^{13} 3 \cdot 11 \cdot 43 \cdot 127$ and $2^{14} 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151$). It is not known whether or not any odd triperfect numbers exist. If an odd triperfect number N exists it must satisfy the following conditions:

(i) $\omega(N) \geq 12$ [63, 82],

(ii) N is a square [78],

(iii) $N > 10^{70}$ [30].

Further it is known that:

(iv) the largest prime factor exceeds 100109 [30],

(v) the second largest prime factor exceeds 1000 [30],

(vi) the third largest prime factor exceeds 100 [62].

4.2 A lower bound on the size of an odd triperfect number

In this section we will show that the lower bound for odd triperfects can be improved from $N > 10^{70}$ to $N > 10^{128}$ by using a systematic computerized search modelled after the approach outlined by Brent et al [15]. The proof by Beck and Najar [10] while essentially algorithmic, includes the frequent use of special cases to shorten their search.

4.2.1 Theoretical Background

Beck and Najar showed that, if N is triperfect and $(N, 6) = 1$, then N exceeds 10^{108} and has at least 32 distinct prime factors. Using the lower bounds on the larger prime factors found by Cohen and Hagis, this can be easily improved as follows.

Lemma 12. *If N is an odd triperfect number and $3 \nmid N$ then $\omega(N) \geq 35$ and $N > 10^{128}$.*

Proof. Given that the lower bounds on the two largest prime factors of an odd triperfect number are 1000 and 100109 (the lower bound of 100 on the third largest prime factor is not decisive here), then if $\omega(N) \leq 34$ it follows that

$$3 = S(N) < S(5^\infty 7^\infty 11^\infty \cdots 137^\infty 139^\infty 1009^\infty 100109^\infty) < 3.$$

This contradiction completes the first part of the proof. (There are 32 primes from 5 to 139, inclusive.) With $\omega(N) \geq 35$ we have

$$N \geq 5^2 7^2 11^2 \cdots 139^2 149^2 1009^2 100109^2 > 10^{128}.$$

□

If the bound of 10000019 on the largest prime factor announced by Iannucci [71] is used instead then we have that $N > 10^{132}$ in these circumstances.

4.2.2 An algorithm for finding a lower bound on the size of N

Let $N < 10^{128}$. We can assume that $3 \mid N$, since if $3 \nmid N$ then by Lemma 12, $N > 10^{128}$. Further since N is a square, we can write

$$N = 3^{2n} p_1^{2a_1} p_2^{2a_2} \dots p_t^{2a_t} \tag{4.1}$$

with p_i distinct primes and $n, a_i \in \mathbb{N}$.

A systematic search involves the construction of a sequence $N_0, N_1, N_2, \dots$ where $N_0 = 3^{2n}$ is an assumed initial component and

$$N_{i+1} = N_i \cdot p_{i+1}^{2a_{i+1}} = 3^{2n} p_1^{2a_1} p_2^{2a_2} \dots p_{i+1}^{2a_{i+1}}, \quad i \geq 0$$

for a sequence of distinct primes $P = \{p_1, p_2, \dots\}$ and exponents $E = \{a_1, a_2, \dots\}$. The next p_{i+1} is chosen so that

$$p_{i+1} = \max\{p : p \mid \sigma(p_i^{2a_i}),\ p \neq p_j \text{ for } j \leq i\} \tag{4.2}$$

while such primes exist. The corresponding a_{i+1} take the values $1, 2, 3, \dots$. The sequences are extended until either:

(i) $S(N_i) > 3$ (or $S(N_i) = 3$, an odd triperfect has been found)

(ii) $N_i > 10^{128}$ (the proposed lower bound has been exceeded) or

(iii) the power of 3 that divides $\sigma(N_i)$ for $j \leq i$ exceeds $2n + 1$ (a contradiction of the assumed component 3^{2n}).

While Beck and Najar used special cases to terminate some of their sequences, those circumstances arise so rarely that they were not implemented in the computerized search.

The algorithm can also be described as a decision tree process. Since $3 \mid N$, the even powers of 3 are the roots of the distinct decision trees. If $3^2 \parallel N$, then since $\sigma(N) = 3N$, $\sigma(3^2) = 13 \mid N$, and so the children of the root 3^2 are labeled with different even powers of 13 (i.e. 13^2, 13^4, $\ldots$). Each of these possibilities leads to further σ factorisations and further subtrees. Having terminated all of these subtrees, by methods to be described, we then assume $3^4 \parallel N$, beginning a second tree (with 3^4 as its root). We continue in this manner until all appropriate powers of 3 have been investigated. Each path in the decision trees is terminated by one of the three conditions mentioned in the preceding paragraph.

Since $\omega(N) \geq 12$ and given the lower bounds on the three largest prime factors of N, we have the worst case

$$3^{2n} \cdot 5^2 7^2 11^2 13^2 17^2 19^2 23^2 29^2 101^2 1009^2 100109^2 > 10^{128}$$

for $n > 94$. It is sufficient to examine sequences for which $N_0 = 3^{2n}, n = 1, 2, \ldots, 94$. However, given the following lemma, a better bound on the number of sequences can be calculated.

Lemma 13. *If N is an odd triperfect number, B a given bound, p prime and $p^{2a} \parallel N$ then $N > B$ if (i) $p \equiv 1 \pmod 3$ and $a > \frac{\log 3B}{4 \log p}$ or (ii) $p \equiv 2 \pmod 3$ or $p = 3$ and $a > \frac{\log B}{4 \log p}$.*

Proof. For case (i), we have, $\sigma(p^{2a}) \mid 3N$, both p^{2a} and $\sigma(p^{2a})$ divide $3N$ and so $3N \geq p^{2a}\sigma(p^{2a}) > p^{4a} > 3B$ if $a > \frac{\log 3B}{4 \log p}$.

For case (ii), since $p \equiv 2 \pmod 3$ or $p = 3$, $3 \nmid \sigma(p^{2a})$, both p^{2a} and $\sigma(p^{2a})$ divide N and so $N \geq p^{2a}\sigma(p^{2a}) > p^{4a} > B$ if $a > \frac{\log B}{4 \log p}$. □

Using this lemma, we have that, if $3^{2n} \parallel N$, $N > B = 10^{128}$ for $n \geq 68$.

For N_{i+1} we have p_{i+1} given by (4.2) and the corresponding a_{i+1} is restricted so that $N_i \cdot p_{i+1}^{a_{i+1}} < 10^{128}$.

By considering more than one known factor, the bound can be tested for earlier. For example (Beck and Najar, case 2-15), given $3^2 13^2 61^4 \parallel N$ and that $\sigma(61^4) = 5 \cdot 131 \cdot 21491$, $\sigma(21491^2) = 421 \cdot 1097113$, $\sigma(1097113^2) = 3 \cdot 37 \cdot 5743 \cdot 1888171$ and noting that each of these is squarefree and they are relatively prime, we can state that $\left(\sigma(61^4)\sigma(21491^2)\sigma(1097113^2)\right)^2 \mid N$ and hence that known factors include $\left(\sigma(1097113^2)\right)^2 > 10^{24}$; $\left(\sigma(21491^2)\right)^2 > 10^{17}$; $\left(\sigma(61^4)\right)^2 > 10^{14}$; whose product exceeds 10^{55} (beyond the limit set by Beck and Najar) thus eliminating this case. Despite this being an improvement, the added complexity in testing and the infrequency with which it can be applied led to the exclusion of this type of extended testing in this study.

The complete search for odd triperfect numbers $< 10^{128}$ can be found in Appendix F. No odd triperfect numbers were found. The details of the search constitute a proof of the following theorem.

Theorem 7. *If N is an odd triperfect number then $N > 10^{128}$.*

Assuming that the largest prime factor of an odd triperfect number exceeds 10^7 (as claimed by Iannucci), the search was also carried out for a bound of 10^{132}. No odd triperfect numbers were found and so Theorem 7 would be amended to be $N > 10^{132}$.

4.3 The number of distinct prime factors of an odd triperfect number

In this section we discuss an approach to improving the known result $\omega(N) \geq 12$. The development of the algorithm and its supporting theory follows that of Cohen [27] and as updated in Cohen and Sorli [32] for the odd perfect number case but here specialised for the simpler triperfect case and extended to provide additional powerful decision criteria.

4.3.1 Theoretical background

In the following, p, q, r will denote primes and m will denote any positive integer. Let $\overline{m}$ denote a proper divisor of m (except that $\overline{1} = 1$).

It will be convenient to consider N to have the prime decomposition

$$N = \prod_{i=1}^{u} p_i^{a_i} \cdot \prod_{i=1}^{v} q_i^{b_i} \cdot \prod_{i=1}^{w} r_i^{c_i} = \lambda \cdot \mu \cdot \nu \tag{4.3}$$

which is interpreted as follows: each $p_i^{a_i}$ is a known component of N, each q_i is a known prime factor of N but the exponent b_i is unknown, and each prime factor r_i of N and exponent c_i are unknown. By "known", we mean explicitly postulated or the consequence of such an assumption. Any of u, v, w may be zero, in which case we set λ, μ, ν, respectively, equal to 1.

Lemma 14. *For any odd triperfect number $N = \lambda\mu\nu$, as given by (4.3), we have*

$$S(\lambda\overline{\mu}) \leq 3 \leq S(\lambda\nu\mu^{\infty}). \tag{4.4}$$

Both inequalities are strict if $v > 0$; the left-hand inequality is strict if $w > 0$.

Proof. Since

$$S(N) = S(\lambda\mu\nu) = S(\lambda)S(\mu)S(\nu) = 3$$

and noting that $S(\overline{\mu}) \leq S(\mu)$ and $S(\nu) \geq 1$, with strict inequality if $v > 0$ or $w > 0$, respectively, $S(\mu) < S(\mu^{\infty})$ if $v > 0$, we have

$$\begin{aligned} S(\lambda\overline{\mu}) &= S(\lambda)S(\overline{\mu}) \\ &\leq S(\lambda)S(\mu) \\ &\leq S(\lambda)S(\mu)S(\nu) = 3 \\ &\leq S(\lambda)S(\nu)S(\mu^{\infty}) = S(\lambda\nu\mu^{\infty}). \end{aligned}$$

□

Lemma 15. *Suppose $w \geq 1$, and assume $r_1 < r_2 < \cdots < r_w$. Then*

$$\frac{3}{3 - S(\lambda\overline{\mu})} - \frac{3 - S(\lambda\overline{\mu})}{3} < r \tag{4.5}$$

for $r = r_1$. Further, if $S(\lambda\mu^{\infty}) < 3$, then

$$r < \frac{3 + S(\lambda\mu^{\infty})(w-1)}{3 - S(\lambda\mu^{\infty})} \tag{4.6}$$

for $r = r_1$.

Proof. : Since, using (1.1),

$$S(\nu) \geq S(r_1^{c_1}) = 1 + \frac{S(r_1^{c_1-1})}{r_1}, \tag{4.7}$$

we have

$$\begin{aligned} 3 = S(N) &= S(\lambda)S(\mu)S(\nu) \\ &\geq S(\lambda)S(\overline{\mu})\left(1 + \frac{S(r_1^{c_1-1})}{r_1}\right) = S(\lambda\overline{\mu}) + \frac{S(\lambda\overline{\mu}r_1^{c_1-1})}{r_1}. \end{aligned}$$

That is,

$$3 \geq S(\lambda\overline{\mu}) + \frac{S(\lambda\overline{\mu}r_1^{c_1-1})}{r_1}$$

which may be rearranged to give

$$\frac{S(\lambda\overline{\mu}r_1^{c_1-1})}{3 - S(\lambda\overline{\mu})} \leq r_1 \tag{4.8}$$

with strict inequality if $v \geq 1$ or $w \geq 2$. Since N is a square, $c_1 \geq 2$ and

$$r_1 \geq \frac{S(\lambda\overline{\mu}r_1^{c_1-1})}{3 - S(\lambda\overline{\mu})} \geq \frac{S(\lambda\overline{\mu}r_1)}{3 - S(\lambda\overline{\mu})} = \frac{S(\lambda\overline{\mu})}{3 - S(\lambda\overline{\mu})}\left(1 + \frac{1}{r_1}\right)$$

or simply

$$\frac{S(\lambda\overline{\mu})}{3 - S(\lambda\overline{\mu})}\left(1 + \frac{1}{r_1}\right) \leq r_1.$$

This inequality can be reorganized to give

$$\frac{S(\lambda\overline{\mu})}{3 - S(\lambda\overline{\mu})} \leq \frac{r_1^2}{r_1 + 1} = r_1 - 1 + \frac{1}{r_1 + 1} \tag{4.9}$$

so that

$$\frac{3}{3 - S(\lambda\overline{\mu})} - \frac{1}{r_1 + 1} \leq r_1.$$

Rather than solve this quadratic inequality directly, giving rise to concerns for computational precision, we can estimate r_1 on the left using the strict inequality

$$\frac{S(\lambda\overline{\mu})}{3 - S(\lambda\overline{\mu})} < r_1$$

to give, after simplification, (4.5). This will only involve rational arithmetic. While (4.5) is less accurate than (4.9), the difference is insignificant in practice.

We now derive (4.6). For $i = 2, 3, \ldots, w$, we have

$$r_i > r_{i-1} + 1 > r_{i-2} + 2 > \cdots > r_2 + i - 2 > r_1 + i - 1$$

and

$$\frac{r_i}{r_i - 1} < \frac{r_1 + i - 1}{r_1 + i - 2}.$$

Then

$$\begin{aligned} S(\nu) < S(\nu^\infty) &= \prod_{i=1}^{w} \frac{r_i}{r_i - 1} \\ &\leq \prod_{i=1}^{w} \frac{r_1 + i - 1}{r_1 + i - 2} \\ &= \frac{r_1}{r_1 - 1} \cdot \frac{r_1 + 1}{r_1} \cdot \frac{r_1 + 2}{r_1 + 1} \cdots \frac{r_1 + w - 1}{r_1 + w - 2} \\ &= \frac{r_1 + w - 1}{r_1 - 1} = 1 + \frac{w}{r_1 - 1} \end{aligned}$$

or simply

$$S(\nu) < 1 + \frac{w}{r_1 - 1}. \tag{4.10}$$

Therefore,

$$3 = S(N) = S(\lambda)S(\mu)S(\nu) < S(\lambda\mu^\infty)\left(1 + \frac{w}{r_1 - 1}\right).$$

That is

$$3 < S(\lambda\mu^\infty)\left(1 + \frac{w}{r_1 - 1}\right)$$

and (4.6) follows after rearrangement. □

When $w \geq 3$, (4.6) can be improved. For $i = 1, 2, \ldots, w$, we have

$$r_i \geq r_{i-1} + 2 \geq r_{i-2} + 4 \geq \cdots \geq r_2 + 2i - 4 \geq r_1 + 2i - 2$$

and

$$\frac{r_i}{r_i - 2} \leq \frac{r_1 + 2i - 2}{r_1 + 2i - 4}.$$

Then

$$\begin{aligned}
(S(\nu))^2 < (S(\nu^\infty))^2 &= \prod_{i=1}^{w}\left(\frac{r_i}{r_i - 1}\right)^2 \\
&< \prod_{i=1}^{w}\left(\frac{r_i}{r_i - 1}\cdot\frac{r_i - 1}{r_i - 2}\right) = \prod_{i=1}^{w}\frac{r_i}{r_i - 2} \\
&< \prod_{i=1}^{w}\frac{r_1 + 2i - 2}{r_1 + 2i - 4} \\
&= \frac{r_1}{r_1 - 2}\cdot\frac{r_1 + 2}{r_1}\cdot\frac{r_1 + 4}{r_1 + 2}\cdots\frac{r_1 + 2w - 2}{r_1 + 2w - 4} \\
&= \frac{r_1 + 2w - 2}{r_1 - 2} = 1 + \frac{2w}{r_1 - 2}
\end{aligned}$$

or simply

$$(S(\nu))^2 < 1 + \frac{2w}{r_1 - 2}. \tag{4.11}$$

Therefore,

$$9 = (S(N))^2 = (S(\lambda)S(\mu)S(\nu))^2 < (S(\lambda\mu^\infty))^2\left(1 + \frac{2w}{r_1 - 2}\right).$$

That is

$$9 < (S(\lambda\mu^\infty))^2\left(1 + \frac{2w}{r_1 - 2}\right)$$

and after rearrangement we have

$$r_1 < \frac{2\left(9 + (S(\lambda\mu^\infty))^2(w - 1)\right)}{9 - (S(\lambda\mu^\infty))^2}. \tag{4.12}$$

Corollary 2. *Suppose $w \geq 2$, and assume $r_1 < r_2 < \cdots < r_w$. If l_1 is the largest prime r not dividing λ or μ, which satisfies (4.6), then*

$$\frac{3}{3 - S(\lambda\overline{\mu}l_1^2)} - \frac{3 - S(\lambda\overline{\mu}l_1^2)}{3} < r \tag{4.13}$$

for $r = r_2$. Further, if $S(\lambda\mu^\infty s_1^\infty) < 3$, where s_1 is the smallest prime r not dividing λ or μ, which satisfies (4.5), then

$$r < \frac{3 + S(\lambda\mu^\infty s_1^\infty)(w-2)}{3 - S(\lambda\mu^\infty s_1^\infty)} \tag{4.14}$$

for $r = r_2$.

Proof. By using a slight variation on the approach used in the proof of the first part of Lemma 15, we have for (4.13)

$$S(\nu) \geq S(r_1^{c_1} r_2^{c_2}) = S(r_1^{c_1})\left(1 + \frac{S(r_2^{c_2-1})}{r_2}\right) \tag{4.15}$$

Therefore,

$$\begin{aligned} 3 = S(N) &= S(\lambda)S(\mu)S(\nu) \\ &\geq S(\lambda)S(\overline{\mu})S(r_1^{c_1})\left(1 + \frac{S(r_2^{c_2-1})}{r_2}\right) = S(\lambda\overline{\mu}r_1^{c_1}) + \frac{S(\lambda\overline{\mu}r_1^{c_1}r_2^{c_2-1})}{r_2}. \end{aligned}$$

That is,

$$3 \geq S(\lambda\overline{\mu}r_1^{c_1}) + \frac{S(\lambda\overline{\mu}r_1^{c_1}r_2^{c_2-1})}{r_2}$$

which may be rearranged to give

$$\frac{S(\lambda\overline{\mu}r_1^{c_1}r_2^{c_2-1})}{3 - S(\lambda\overline{\mu}r_1^{c_1})} \leq r_2.$$

But $l_1 \geq r_1$, $c_1 \geq 2$, $c_2 \geq 2$ and $S(r_1^{c_1}) \geq S(r_1^2) \geq S(l_1^2)$ which gives

$$\frac{S(\lambda\overline{\mu}l_1^2 r_2)}{3 - S(\lambda\overline{\mu}l_1^2)} = \frac{S(\lambda\overline{\mu}l_1^2)}{3 - S(\lambda\overline{\mu}l_1^2)}\left(1 + \frac{1}{r_2}\right) \leq r_2.$$

This last inequality can be reorganized to give

$$\frac{3}{3 - S(\lambda\overline{\mu}l_1^2)} - \frac{1}{r_2 + 1} \leq r_2.$$

We can estimate r_2 on the left-hand side using

$$\frac{S(\lambda\overline{\mu}l_1^2)}{3 - S(\lambda\overline{\mu}l_1^2)} < r_2$$

and then, after some rearrangement, obtain the result. As before, using the estimate (4.13) for r_2 will only involve rational arithmetic.

The proof for (4.14) follows the approach used in the proof of the second part of Lemma 15. Thus

$$\begin{aligned} S(\nu) < S(\nu^\infty) &= \prod_{i=1}^{w} S(r_i^\infty) \\ &= S(r_1^\infty)\prod_{i=2}^{w} S(r_i^\infty) \\ &= S(r_1^\infty)\prod_{i=2}^{w} \frac{r_i}{r_i - 1} \\ &\leq S(r_1^\infty)\prod_{i=2}^{w} \frac{r_2 + i - 2}{r_2 + i - 3} \\ &= S(r_1^\infty)\left(\frac{r_2}{r_2 - 1}\cdot\frac{r_2 + 1}{r_2}\cdot\frac{r_2 + 2}{r_2 + 1}\cdots\frac{r_2 + w - 2}{r_2 + w - 1}\right) \\ &= S(r_1^\infty)\left(\frac{r_2 + w - 2}{r_2 - 1}\right) = S(r_1^\infty)\left(1 + \frac{w - 1}{r_2 - 1}\right). \end{aligned}$$

That is,

$$S(\nu) < S(r_1^\infty)\left(1 + \frac{w-1}{r_2-1}\right). \tag{4.16}$$

Therefore,

$$3 = S(N) = S(\lambda)S(\mu)S(\nu) < S(\lambda)S(\mu^\infty)S(r_1^\infty)\left(1 + \frac{w-1}{r_2-1}\right)$$

or simply

$$3 < S(\lambda\mu^\infty r_1^\infty)\left(1 + \frac{w-1}{r_2-1}\right)$$

which can be rearranged to give

$$r_2 < \frac{3 + S(\lambda\mu^\infty r_1^\infty)(w-2)}{3 - S(\lambda\mu^\infty r_1^\infty)}.$$

In order to complete the proof we use the fact that $r_1 \geq s_1$ so $S(r_1^\infty) \leq S(s_1^\infty)$. □

It was observed that while both bounds on r_1 (as given by Lemma 15 and possibly (4.12)) were monotonic decreasing, with respect to an increasing value of the most recently selected prime divisor of N, as was the lower bound on r_2 (as given by (4.13)), the upper bound on r_2 (as given by (4.14)) has an erratic pattern with sometimes large changes in value.

Corollary 2 can be extended in an iterative way so that, given bounds on $r_1, r_2, \ldots, r_{k-1}$, we may be able to give bounds on r_k. When applicable, this provides a more powerful decision criterion (that is, a contradiction may be detected earlier in a search path).

Corollary 3. *Assume $r_1 < r_2 < \cdots < r_w$ and let l_1, s_1 be as in Corollary 2. Take $k = 2, 3, \ldots$ in turn and suppose $w \geq k$. Then*

$$\frac{3}{3 - S(\lambda\overline{\mu}l_1^2 l_2^2 \cdots l_{k-1}^2)} - \frac{3 - S(\lambda\overline{\mu}l_1^2 l_2^2 \cdots l_{k-1}^2)}{3} < r, \tag{4.17}$$

for $r = r_k$, where l_{k-1} is the largest prime r not dividing λ or μ, which satisfies (4.18) for $k = 1, 2, \ldots$, respectively. Further, if $S(\lambda\mu^\infty s_1^\infty s_2^\infty \ldots s_{k-1}^\infty) < 3$, then

$$r < \frac{3 + S(\lambda\mu^\infty s_1^\infty s_2^\infty \ldots s_{k-1}^\infty)(w-k)}{3 - S(\lambda\mu^\infty s_1^\infty s_2^\infty \ldots s_{k-1}^\infty)}, \tag{4.18}$$

for $r = r_k$, where s_{k-1} is the smallest prime r not dividing λ or μ, which satisfies (4.17) for $k = 1, 2, \ldots$, respectively. Note: $l_i < l_{i+1}$ and $s_i < s_{i+1}$ for all i.

Proof. We use induction on k. The case $k = 2$ coincides with Corollary 2. Now fix any $k \geq 3$. By the approach used in the proof of Corollary 2, we have for (4.17)

$$S(\nu) \geq S(r_1^{c_1} \ldots r_{k-1}^{c_{k-1}} r_k^{c_k}) = S(r_1^{c_1} \ldots r_{k-1}^{c_{k-1}})\left(1 + \frac{S(r_k^{c_k - 1})}{r_k}\right)$$

which leads to

$$\frac{S(\lambda\overline{\mu}r_1^{c_1} \ldots r_{k-1}^{c_{k-1}} r_k^{c_k-1})}{3 - S(\lambda\overline{\mu}r_1^{c_1} \ldots r_{k-1}^{c_{k-1}})} \leq r_k.$$

But $l_i \geq r_i$, $c_i \geq 2$ and $S(r_i^{c_i}) \geq S(r_i^2) \geq S(l_i^2)$ for $i < k$ which gives

$$\frac{S(\lambda\overline{\mu}l_1^2 \ldots l_{k-1}^2 r_k^{c_k-1})}{3 - S(\lambda\overline{\mu}l_1^2 \ldots l_{k-1}^2)} \leq r_k.$$

Since $c_k \geq 2$ and by estimating r_k using

$$\frac{S(\lambda\overline{\mu}l_1^2 \ldots l_{k-1}^2)}{3 - S(\lambda\overline{\mu}l_1^2 \ldots l_{k-1}^2)} \leq r_k$$

the result is obtained.

Likewise for (4.18) we have

$$S(\nu) < \left(\prod_{i=1}^{k-1} S(r_i^{\infty})\right)\left(\prod_{i=k}^{w} \frac{r_k + i - k}{r_k + i - k - 1}\right) = \left(\prod_{i=1}^{k-1} S(r_i^{\infty})\right)\left(1 + \frac{w - k + 1}{r_k - 1}\right). \quad (4.19)$$

Therefore,

$$3 < S(\lambda\mu^{\infty}r_1^{\infty}r_2^{\infty} \ldots r_{k-1}^{\infty})\left(1 + \frac{w - k + 1}{r_k - 1}\right)$$

which can be rearranged to give

$$r_k < \frac{3 + S(\lambda\mu^{\infty}r_1^{\infty}r_2^{\infty} \ldots r_{k-1}^{\infty})(w - k)}{3 - S(\lambda\mu^{\infty}r_1^{\infty}r_2^{\infty} \ldots r_{k-1}^{\infty})}.$$

But $r_i \geq s_i$ and $S(r_i^{\infty}) \leq S(s_i^{\infty})$ for $i < k$ and we have the result. □

Lemma 15 and Corollary 3, together with the lower bounds on the three largest prime factors of an odd triperfect number, provide a new powerful test, not withstanding the erratic changes in the upper bound on r_k, $k \geq 2$.

As a demonstration of the effectiveness of this extended testing approach, a proof that $\omega(N) \geq 10$ is as follows. Assuming $\lambda = 1$, $\overline{\mu} = 3^2$ and $w = 8$ (that is, 3 is the only known prime factor but its even exponent is unknown and $\omega(N) = 9$), we can calculate

$$1.4 < r_1 < 7.4, \quad 1.8 < r_2 < 12.7, \quad 2.2 < r_3 < 17.2, \quad 2.5 < r_4 < 21.3,$$
$$2.9 < r_5 < 27.6, \quad 3.3 < r_6 < 37.1, \quad r_7 < 77.5.$$

The last inequality, $r_7 < 77.5$, is a contradiction of the result that the second largest prime factor of an odd triperfect number exceeds 1000. Since $r_6 < 37.1$ we also have a contradiction that the third largest prime factor of an odd triperfect number exceeds 100. Hence $\omega(N) \geq 10$.

4.3.2 Results for $\omega(N) = 12$

It has been shown previously that $\omega(N) \geq 12$ [82, 63]. We now look at applying the theory developed in the previous section to examine the case where an odd triperfect number has exactly 12 distinct prime factors. By Lemma 12, $3 \mid N$.

The approach with least information assumes only that $3^2 \mid N$. Then $\lambda = 1$, $\overline{\mu} = 3^2$ and $w = 11$. We calculate $1.4 < r_1 < 9.4$ using (4.5) and (4.12) and, using Corollary 2 with $s_1 = 5$ and $l_1 = 7$, that $1.8 < r_2 < 17.7$. Subsequently, we have, using Corollary 3, $2.0 < r_3 < 25.3$ (with $s_2 = 7$ and $l_2 = 17$), $2.2 < r_4 < 33.5$ (with $s_3 = 11$ and $l_3 = 23$), $2.4 < r_5 < 47.5$ (with $s_4 = 13$ and $l_4 = 31$), $2.5 < r_6 < 73.2$ (with $s_5 = 17$ and $l_5 = 47$) and $2.6 < r_7 < 192.3$ (with $s_6 = 19$ and $l_6 = 73$). Corollary 3 cannot be applied to find bounds on r_8 since

$$S(\lambda\mu^{\infty}s_1^{\infty} \ldots s_7^{\infty}) = S(3^{\infty}5^{\infty}7^{\infty}11^{\infty}13^{\infty}17^{\infty}19^{\infty}23^{\infty}) > 3,$$

Table 4.1: Bounds $[s_i, l_i]$ on r_i for an odd triperfect number N with $\omega(N) = 12$

λ	$\overline{\mu}$	r_1	r_2	r_3	r_4	r_5	r_6	r_7
1	3^2	[5,7]	[7,17]	[11,23]	[13,31]	[17,47]	[19,73]	[23,191]
3^2	13^2	[5,7]	[7,17]	[11,23]	[17,31]	[19,47]	[23,73]	[29,199]
3^4	11^2	[5,7]	[7,19]	[13,31]	[17,43]	[19,67]	[23,163]	
3^6	1093^2	[5,7]	[7,13]	[11,19]	[17,29]	[19,37]	[23,61]	[29,151]
3^8	$13^2 757^2$	[5,7]	[7,17]	[11,23]	[17,41]	[19,61]	[23,157]	
3^{10}	$23^2 3851^2$	[5,7]	[7,13]	[11,23]	[13,31]	[17,47]	[19,109]	
3^{12}	797161^2	[5,7]	[7,13]	[11,19]	[13,29]	[17,37]	[19,61]	[23,151]
3^{14}	$11^2 13^2 4561^2$	[5,7]	[7,19]	[17,37]	[19,61]	[23,151]		
3^{16}	$1871^2 34511^2$	[5,7]	[7,13]	[11,19]	[13,23]	[17,31]	[19,47]	[23,113]
3^{18}	$1597^2 363889^2$	[5,7]	[7,13]	[11,19]	[13,23]	[17,31]	[19,47]	[23,113]
3^{20}	$13^2 1093^2 368089^2$	[5,7]	[7,11]	[11,23]	[17,31]	[19,47]	[23,113]	
1	3^{22}	[5,7]	[7,17]	[11,23]	[13,31]	[17,47]	[19,73]	[23,191]

although we could have found a lower bound for r_8. Unlike the situation for $\omega(N) = 9$, there is no contradiction with regards any of the lower bounds on the three largest prime factors. However the calculations do show that either 5 or 7 is the second smallest prime factor. See the first row of Table 4.1. This result can also be shown directly by considering $\omega(N) = 12$, $3 \mid N$, $5, 7 \nmid N$ for which we have the contradiction:

$$3 = S(N) < S(3^\infty 11^\infty 13^\infty 17^\infty 19^\infty 23^\infty 29^\infty 31^\infty 37^\infty 101^\infty 1009^\infty 100109^\infty) < 2.4.$$

By assuming an exact component $3^{2a} \parallel N$, one or more consequent prime factors of N are revealed and so the bounds on r_i can be tightened, but not necessarily the corresponding interval of primes $[s_i, l_i]$. For example (see the second row of Table 4.1), if $3^2 \parallel N$, so that $\sigma(3^2) = 13 \mid N$, then we calculate $1.6 < r_1 < 9.5$, or $5 \le r_1 \le 7$. Note in this case r_6 would be the eighth smallest prime factor and $r_6 \le 73$, corresponding with $r_7 \le 191$ assuming only $3^2 \mid N$. The results for a range of such assumed components are given in Table 4.1. None of these particular cases leads to a contradiction nor any further restriction on the second smallest prime factor – still more exact components need to be assumed for further accuracy.

Rather than treating such exact components as further special cases, an algorithmic approach, similar to that to be described in Section 5.3.2 for the odd perfect number problem, can be applied. Since an odd triperfect number, if one exists, must be a square, the generation of exponents is simplified and no additional logic is required to manage a special prime.

Preliminary investigations with such a modified algorithm when $\omega(N) = 12$ have not uncovered any results significantly beyond those of Table 4.1 which would restrict further either the prime factors of N or the exponent of the component 3^{2a} of an odd triperfect number. This was not pursued as priority was given to the corresponding algorithm for odd perfect numbers.

Chapter 5

Odd Perfect Numbers

5.1 Introduction

A positive integer N is said to be perfect if $\sigma(N) = 2N$ (or $S(N) = 2$) and in this chapter N will always denote an odd perfect number assuming such exist. While 39 even perfect numbers have been found, each corresponding to a Mersenne prime, it is not known whether or not an odd perfect number exists. If an odd perfect number, N, does exist then it must satisfy the following conditions:

(i) N is of the form $\pi^\alpha m^2$, where π is the special prime satisfying the conditions $\pi \equiv \alpha \equiv 1 \pmod 4$ and $\pi \nmid m$ [41]. We will call π^α the special component of N.

(ii) $\omega(N) \geq 8$ [24, 59],

(iii) $\Omega(N) \geq 37$ [74]

(iv) $N > 10^{300}$ [15].

Further we have lower bounds on the three largest prime factors as given by the following:

Lemma 16 (Jenkins [75]). *The largest prime factor of an odd perfect number exceeds* 10^7.

Lemma 17 (Iannuci [72]). *The second largest prime factor of an odd perfect number exceeds* 10^4.

Lemma 18 (Iannuci [73]). *The third largest prime factor of an odd perfect number exceeds* 100.

5.2 The total number of prime factors of an odd perfect number

The first significant result concerning a lower bound on the total number of prime factors of an odd perfect number was obtained by Cohen [25] in 1982 when he proved that $\Omega(n) \geq 23$. In 1986, Sayers [106] improved this result to obtain $\Omega(n) \geq 29$. As a result of joint work with Iannucci, this lower bound was improved to 37, and is stated here:

Theorem 8 (Iannucci and Sorli [74]). *If N is an odd perfect number then $\Omega(N) \geq 37$.*

This section details this joint effort.

5.2.1 Preliminaries

It is due to Euler, and well known, that N has the shape given by

$$N = \pi^{\alpha} q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_k^{2\beta_k}, \tag{5.1}$$

where π, q_1, ..., q_k are distinct primes, α, β_1, ..., β_k are positive integers, and $\pi \equiv \alpha \equiv 1 \pmod 4$. From (5.1), it follows that $\omega(N) = k + 1$ and $\Omega(N) = \alpha + 2\sum_{j=1}^{k} \beta_j$. We will assume that $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_k$.

McDaniel [90] proved that if $\beta_1 = 2$ for all j then N has no prime factor less than 101. This result was extended by Cohen [26] who showed that N has no prime factor less than 739 under the same condition. Cohen's result implies that then $\omega(N) \geq 47326$. This follows from the properties of the index function since if $\omega(N) < 47326$ then we have the contradiction

$$2 = S(N) < S(739^{\infty}743^{\infty} \ldots 578309^{\infty}) = \prod_{739 \leq p \leq 578309} \frac{p}{p-1} < 2,$$

where the product is taken over the 47325 consecutive primes indicated. It follows (from McDaniel's result, in fact) that

$$\text{If } \beta_j = 1 \text{ or } 2 \text{ for all } j, \text{ then } \Omega(N) > 35. \tag{5.2}$$

Many similar results regarding the exponents β_j have appeared in the literature. We will apply some of these results to prove Theorem 8.

We first introduce some notation for the sake of brevity. For nonnegative integers m, j, $a_i \geq 1$ ($1 \leq i \leq m$ if $m > 0$) and $b_i \geq 1$ ($1 \leq i \leq m + j$ if $m + j > 0$), we let the expression

$$b_1(a_1), b_2(a_2), \ldots, b_m(a_m), b_{m+1}(*), \ldots, b_{m+j}(*)$$

represent the following: Of the set $\{\beta_1, \beta_2, \ldots, \beta_k\}$ of decreasing numbers,

(i) at most a_i can equal b_i, $1 \leq i \leq m$,

(ii) any element not equal to b_i, $1 \leq i \leq m$, must belong to $\{b_{m+1}, \ldots, b_{m+j}\}$ for which each member can occur an unrestricted number of times.

Then for $x = 1$, 3, or 5, we let the expression

$$[\, x : b_1(a_1), b_2(a_2), \ldots, b_m(a_m), b_{m+1}(*), \ldots, b_{m+j}(*)\,] \tag{5.3}$$

represent the following:

If $x = 1$, then items (i) and (ii) above are impossible.

If $x = 3$ and (i) and (ii) are true, then $3 \nmid N$.

If $x = 5$ and (i) and (ii) are true, then $3 \nmid N$ and $5 \nmid N$.

Using this notation, some of the results we shall apply are given as follows.

McDaniel [89] showed it is impossible to have $\beta_j \equiv 1 \pmod 3$ for all j, $1 \leq j \leq k$. This implies, sufficient for our purposes,

$$[\, 1 : 10(*), 7(*), 4(*), 1(*)\,], \tag{5.4}$$

that is, the exponents $2\beta_1, \ldots, 2\beta_k$ cannot all belong to $\{2, 8, 14, 20\}$. (Steuerwald [114] had previously obtained $[\,1 : 1(*)\,]$.) Cohen and Willams [33] showed it is impossible to have $\beta_1 = 5$ or 6 and $\beta_j = 1$ for all j, $2 \le j \le k$. These results give us, respectively,

$$[\,1 : 5(1), 1(*)\,], \tag{5.5}$$

$$[\,1 : 6(1), 1(*)\,]. \tag{5.6}$$

Brauer [12] showed that $\beta_1 = 2$, $\beta_j = 1$ for all j, $2 \le j \le k$, is impossible, and Kanold [77] showed that $\beta_1 = 3$, $\beta_j = 1$ for all j, $2 \le j \le k$, is impossible. Cohen [25] showed that $\beta_1 = 3$, $\beta_2 = 2$, $\beta_j = 1$ for all j, $3 \le j \le k$, is impossible. These three results, combined with that of Steuerwald, give us

$$[\,1 : 3(1), 2(1), 1(*)\,], \tag{5.7}$$

Steuerwald's result, along with Theorem 2 in Sayers [106], gives us

$$[\,1 : 3(3), 1(*)\,]. \tag{5.8}$$

5.2.2 Problem specification

We first assume that $\Omega(N) = 35$. The possibilities $29 \le \Omega(N) \le 33$ will be automatically catered for. Then (5.2) becomes equivalent to

$$[\,1 : 2(*), 1(*)\,], \tag{5.9}$$

and this, in conjunction with Theorem 3 in Sayers [106], then gives us

$$[\,1 : 4(1), 2(4), 1(*)\,]. \tag{5.10}$$

There are exactly 686 possible cases for the exponents in (5.1) when $\Omega(N) = 35$; these range from $\alpha = 33$, $k = 1$, $\beta_1 = 1$, to $\alpha = 1$, $k = 17$, $\beta_1 = \cdots = \beta_{17} = 1$. The condition $\omega(N) \ge 8$ eliminates exactly 439 of these cases. Of the remaining 247 cases, exactly 81 are further eliminated by conditions (5.4) through (5.10). This leaves 166 cases to consider.

Table 5.1 shows the increase in the number of cases (and the diminishing effectiveness of conditions (5.4) through (5.10)) as $\Omega(N)$ increases. Table 5.2 illustrates the application of these contradictions and some to be mentioned in the case $\alpha = 13$.

Of the 166 cases remaining when $\omega(N) = 35$, exactly 136 satisfy $\omega(N) \le 10$. Recalling that

Table 5.1: Number of possible cases for $\Omega(N)$

$\Omega(N)$	Total	$\omega(N) \geq 8$	Cond
21	82	9	0
23	112	17	1
25	159	28	5
27	213	47	12
29	294	73	29
31	389	113	57
33	525	168	99
35	686	247	166
37	910	354	259
39	1176	502	390

Table 5.2: Enumeration of possible cases for $\Omega(N) = 35$, $\omega(N) \geq 8$ and $\alpha = 13$

k	$\beta_1, \ldots, \beta_k$		Excluded by lemma
7	$\beta_1 = 5, \beta_2 = \cdots = \beta_7 = 1$	5(1),1(6)	(5.8)
	$\beta_1 = 4, \beta_2 = 2, \beta_3 = \cdots = \beta_7 = 1$	4(1),2(1),1(5)	(5.13)
	$\beta_1 = \beta_2 = 3, \beta_3 = \cdots = \beta_7 = 1$	3(2),1(5)	(5.11)
	$\beta_1 = 3, \beta_2 = \beta_3 = 2, \beta_4 = \cdots = \beta_7 = 1$	3(1),2(2),1(4)	
	$\beta_1 = \cdots = \beta_4 = 2, \beta_5 = \cdots = \beta_7 = 1$	2(4),1(3)	(5.12)
8	$\beta_1 = 4, \beta_2 = \cdots = \beta_8 = 1$	4(1),1(7)	
	$\beta_1 = 3, \beta_2 = 2, \beta_3 = \cdots = \beta_8 = 1$	3(1),2(1),1(6)	(5.10)
	$\beta_1 = \cdots = \beta_3 = 2, \beta_4 = \cdots = \beta_8 = 1$	2(3),1(5)	(5.12)
9	$\beta_1 = 3, \beta_2 = \cdots = \beta_9 = 1$	3(1),1(8)	(5.11)
	$\beta_1 = \beta_2 = 2, \beta_3 = \cdots = \beta_9 = 1$	2(2),1(7)	(5.12)
10	$\beta_1 = 2, \beta_2 = \cdots = \beta_{10} = 1$	2(1),1(9)	(5.12)
11	$\beta_1 = \cdots = \beta_{11} = 1$	1(11)	(5.12)

$\omega(N) \geq 11$ if $3 \nmid N$, 120 of these 136 cases are eliminated once we prove the following fourteen results:

$$[\,3 : 3(5), 1(*)\,], \tag{5.11}$$
$$[\,3 : 4(1), 3(3), 2(1), 1(*)\,], \tag{5.12}$$
$$[\,3 : 5(2), 3(1), 1(*)\,], \tag{5.13}$$
$$[\,3 : 5(2), 2(2), 1(*)\,], \tag{5.14}$$
$$[\,3 : 5(1), 4(2), 1(*)\,], \tag{5.15}$$
$$[\,3 : 5(1), 3(3), 2(2), 1(*)\,], \tag{5.16}$$
$$[\,3 : 6(2), 1(*)\,], \tag{5.17}$$
$$[\,3 : 6(1), 5(1), 4(1), 3(1), 2(*), 1(*)\,], \tag{5.18}$$
$$[\,3 : 6(1), 3(2), 2(1), 1(*)\,], \tag{5.19}$$
$$[\,3 : 7(1), 5(1), 1(*)\,], \tag{5.20}$$
$$[\,3 : 7(1), 3(2), 1(*)\,], \tag{5.21}$$
$$[\,3 : 8(1), 3(1), 2(*), 1(*)\,], \tag{5.22}$$
$$[\,3 : 9(1), 3(1), 2(*), 1(*)\,], \tag{5.23}$$
$$[\,3 : 11(1), 1(*)\,]. \tag{5.24}$$

(For example, (5.11) states: If, in (5.1), $2\beta_1 = \cdots = 2\beta_l = 6$ for $1 \leq l \leq 5$ and $2\beta_{l+1} = \cdots = 2\beta_k = 2$, then $3 \nmid N$.)

The remaining sixteen cases (of the 136 mentioned above) are eliminated once we prove three further results, namely

$$[\,3 : 4(3), 2(1), 1(*)\,], \tag{5.25}$$
$$[\,3 : 7(1), 4(1), 2(1), 1(*)\,], \tag{5.26}$$
$$[\,3 : 10(1), 2(1), 1(*)\,], \tag{5.27}$$

and these are all special cases of the following:

Theorem 9. *If $N = \pi^\alpha \prod_{j=1}^k q_j^{2\beta_j}$ is an odd perfect number and $\beta_j \equiv 1 \pmod 3$ or $\beta_j \equiv 2 \pmod 5$ for all $j = 1, 2, \ldots, k$, then $3 \nmid N$.*

This leaves exactly 30 cases to consider. In each of these remaining cases, we have $\omega(N) \leq 14$. It follows from the properties of the index function that $\omega(N) \geq 15$ if $3 \nmid N$ and $5 \nmid N$ since we have the contradiction

$$2 = S(N) < S(7^\infty 11^\infty \ldots 59^\infty) = \prod_{7 \leq p \leq 59} \frac{p}{p-1} < 2,$$

the product being over the 14 consecutive primes indicated. The remaining 30 cases are then eliminated once we prove the following five results:

$$[\,5:3(4),2(*),1(*)\,], \tag{5.28}$$

$$[\,5:4(2),3(2),2(*),1(*)\,], \tag{5.29}$$

$$[\,5:6(1),5(1),4(1),3(1),2(*),1(*)\,], \tag{5.30}$$

$$[\,5:7(1),3(1),2(*),1(*)\,], \tag{5.31}$$

$$[\,5:8(1),4(1),1(*)\,]. \tag{5.32}$$

Results (5.4) through (5.32) will also eliminate every possible case for the exponents in (5.1) if we assume either of $\Omega(N) = 29$, 31, or 33. Therefore, recalling that it is known that $\Omega(N) \geq 29$ (Sayers [12]), it suffices, for the proof of Theorem 8, to prove the results stated in (5.11) through (5.32); we outline these proofs in the following section.

The results (5.11) through (5.32) are all independent and quite specific for our purposes, although they all contain some generality in allowing an unrestricted number of exponents equal to 2, and in some cases an unrestricted number of exponents equal to 4. Theorem 9, by which we prove (5.25) through (5.27), would have greater applicability.

5.2.3 An algorithm for testing the lower bound on $\Omega(N)$

Theorem 9 and (5.11) through (5.24) are all proved by contradiction. For results (5.11) through (5.24), we assume separately in each case that $3 \mid N$ and obtain a contradiction at the end of a chain of factorisations, in manners to be described shortly.

The factorisation chains are constructed systematically, one component at a time, beginning with 3^2, or 3^4, or ..., with, in practice, each new component implying at least one additional candidate prime divisor (since an odd perfect number was not found!). For example, if N is an odd perfect number and, by assumption, $3^2 \parallel N$ then $13 \mid N$ since $\sigma(3^2) = 13$ and $S(3^2 \cdot 13^a) < 2$ for any natural number a. Then $13 \parallel N$ (since 13 may be the special prime) so $7 \mid N$ since $\sigma(13) = 2 \cdot 7$ and $S(3^2 \cdot 13 \cdot 7^a) < 2$; or $13^2 \parallel N$ so $61 \mid N$ since $\sigma(13^2) = 3 \cdot 61$ and $S(3^2 \cdot 13^2 \cdot 61^a) < 2$; or $13^4 \parallel N$ so ... If there are more than one candidate prime divisors available, then choosing the smallest as the basis for the next component of N results in the greatest increase in $S(N')$ (and hence usually the shortest path to a contradiction), where N' is the product of the components so far assumed or as yet unexplored (and in the latter case, for the purpose of calculating $S(N')$, they are given their smallest possible exponent). For each prime chosen to continue a chain, exponents are investigated as allowed by the exponent pattern for the lemma under consideration. (If the candidate prime might be the special prime, then only the exponent 1 is considered for it, since $\pi + 1 = \sigma(\pi) \mid \sigma(\pi^\alpha)$, when $\pi \equiv \alpha \equiv 1 \pmod 4$.)

If $q^{2\beta}$ (or π^α) is the new additional assumed component of N then the set of assumed prime divisors of N needs to be updated with the prime divisors of $\sigma(q^{2\beta})$ (or of $\sigma(\pi)/2$). A chain is continued while $S(N') < 2$. However, it can be observed that the larger assumed primes make little contribution to the value of $S(N')$. We can take advantage of this by only finding the "small" prime divisors of $\sigma(q^{2\beta})$ (or of $\sigma(\pi)/2$), perhaps leaving a single "hard" composite. Any such composites are easily identified and then excluded from the calculation of $S(N')$. This underestimates the value of $S(N')$ and may lead to slightly longer chains in some cases but this is more than offset by the substantial reduction in factorisation time. If there is no unexplored prime available from earlier factorisations with which to continue the chain, then it is necessary to factor one of the carried forward composites (and in practice the most recently added composite was used).

For the proofs of the results (5.11) through (5.24), an additional constraint, that each (non-special) prime factor q of N occurred to a given exponent 2β, so that $q^b \parallel \sigma(N')$ for $b \le 2\beta$, was employed to allow another contradiction that could terminate a chain. A violation of this constraint (when so many primes q arose from factorisations as to imply $b > 2\beta$), was described as saying there was an excess of the prime q.

For the proofs of each of the results (5.28) through (5.32), we first showed that $3 \nmid N$, as above, and then assumed that $5 \mid N$; in a similar manner, this was also shown to lead to a contradiction.

The proof of Theorem 9 was accomplished by assuming $3 \mid N$, ignoring the second possible contradiction (of an excess of primes), and employing the facts that $\sigma(q^2) \mid \sigma(q^{2\beta})$ when $\beta \equiv 1 \pmod 3$ and $\sigma(q^4) \mid \sigma(q^{2\beta})$ when $\beta \equiv 2 \pmod 5$. Only exponents with $\beta = 1$ or 2 were assumed (on non-special primes), and the only contradiction used to terminate a chain was $S(N') > 2$.

5.2.4 Implementation

The most novel feature of the algorithm is the effective use of incomplete factorisations. This was implemented as follows. If the composite was less than a chosen bound, usually 10^{15}, then the complete factorisation was carried out (with minimal effort). For composites greater than the bound, a stored list of complete factorisations was searched. If the desired factorisation was not found then an incomplete factorisation was carried out using the `FactorComplete->False` option of the `FactorInteger[]` function of *Mathematica*. (*Maple* has a similar `` `easy` `` option for its `ifactor()` function.) In *Mathematica*, for incomplete factorisation, only the trial division, Pollard $p-1$, Pollard ρ and continued fraction methods of factorisation are applied to find "small" factors, in some combination not detailed in the accompanying documentation.

To help clarify the algorithm, Appendix G shows the computational proof of $[3 : 5(1), 2(1), 1(*)]$ (which is in fact subsumed by (5.14)). In this example, full factorisations are shown although the opportunity for partial factorisation is noted.

If it became necessary for the continuation of a chain to have the full factorisation of a composite, then this was established separately either by looking up known tables or by calculation. A list of needed, complete factorisations would then be updated within the program.

The most difficult factorisation required was that of $\sigma(\sigma(61^6)^{16})$, the product of a 73-digit prime and a 100-digit prime which occurred in the proof of (5.25). The factorisation of this "hard" composite was realized through the assistance of Herman te Riele and Peter Montgomery at the Centrum voor Wiskunde en Informatica (CWI) in Amsterdam, to whom we are most grateful.

5.2.5 Economisation

The patterns of exponents represented by (5.11) through (5.24) are the result of amalgamation through generalisation of the patterns of the original 166 cases. Initially, we experimented with generalisations of the form $[3 : \ldots, 1(*)]$, that is, an unrestricted number of components with an exponent of 2. Then patterns like $[3 : \ldots, 2(*), 1(*)]$ were selectively tried. This was followed by generalisations such

Table 5.3: Enumeration of possible cases for (5.14) $[\,3 : 5(2), 2(2), 1(*)\,]$ for $\Omega(N) = 35$

α	$\beta_1, \ldots, \beta_k$	
1	$\beta_1 = \beta_2 = 5, \beta_3 = \beta_4 = 2, \beta_5 = \cdots = \beta_7 = 1$	5(2),2(2),1(3)
	$\beta_1 = \beta_2 = 5, \beta_3 = 2, \beta_4 = \cdots = \beta_8 = 1$	5(2),2(1),1(5)
	$\beta_1 = 5, \beta_2 = \beta_3 = 2, \beta_4 = \cdots = \beta_{11} = 1$	5(1),2(2),1(8)
	$\beta_1 = 5, \beta_2 = 2, \beta_3 \cdots = \beta_{12} = 1$	5(1),2(1),1(10)

as $[3 : 3(1), 2(*), 1(*)]$, ..., $[3 : 3(4), 2(*), 1(*)]$. Another series of generalisations investigated was $[3 : 3(1), 2(*), 1(*)]$, ..., $[3 : 6(1), 5(1), 4(1), 3(1), 2(*), 1(*)]$. This last pattern (5.18) involved the generation of a computational proof of almost nine million lines. In each case, practical considerations determined how comprehensive the generalisation could be made.

Table 5.3 illustrates the possible cases for $\Omega(N) = 35$ covered by (5.14).

5.3 The number of distinct prime factors of an odd perfect number

In this section we will focus on the problem of extending the lower bound on the number of distinct prime factors of an odd perfect number to $\omega(N) \geq 9$. The theory developed here closely parallels that for odd triperfect numbers (Section 4.3) which in turn was based on Cohen [27] and updated in Cohen and Sorli [32]. Indeed several of the intermediate results of Section 4.3 are used here. The similarity between the odd triperfect case and the present odd perfect case inevitably leads to some repetition in the exposition of the theory.

5.3.1 Theoretical background

In the following, N denotes an odd perfect number, assuming one exists, with the prime decomposition (4.3), but repeated here,

$$N = \prod_{i=1}^{u} p_i^{a_i} \cdot \prod_{i=1}^{v} q_i^{b_i} \cdot \prod_{i=1}^{w} r_i^{c_i} = \lambda \cdot \mu \cdot \nu$$

which is interpreted as follows: each $p_i^{a_i}$ is a known component of N, each q_i is a known prime factor of N but the exponent b_i is unknown, and each prime factor r_i of N and exponent c_i are unknown. By "known", we mean explicitly postulated or the consequence of such an assumption. Any of u, v, w may be zero, in which case we set λ, μ, ν, respectively, equal to 1. We have previously let $\overline{m}$ denote a proper divisor of m (except $\overline{1} = 1$).

Lemma 19. *For any odd perfect number $N = \lambda\mu\nu$, as given as above, we have*

$$S(\lambda\overline{\mu}) \leq 2 \leq S(\lambda\nu\mu^{\infty}). \tag{5.33}$$

Both inequalities are strict if $v > 0$; the left-hand inequality is strict if $w > 0$.

Proof. We need only note that $S(\overline{\mu}) \leq S(\mu)$ and $S(\nu) \geq 1$, with strict inequality if $v > 0$ or $w > 0$, respectively, $S(\mu) < S(\mu^{\infty})$ if $v > 0$, and $2 = S(\lambda\mu\nu) = S(\lambda)S(\mu)S(\nu)$. □

Lemma 20. *Suppose $w \geq 1$, and assume $r_1 < r_2 < \cdots < r_w$. Then*

$$\frac{S(\lambda\overline{\mu}r_1^{c_1-1})}{2 - S(\lambda\overline{\mu})} \leq r \tag{5.34}$$

for $r = r_1$, with strict inequality if $v \geq 1$ or $w \geq 2$. Further, if $S(\lambda\mu^\infty) < 2$, then

$$r < \frac{2 + S(\lambda\mu^\infty)(w-1)}{2 - S(\lambda\mu^\infty)} \tag{5.35}$$

for $r = r_1$.

Proof. Using (4.7), we have

$$\begin{aligned} 2 = S(N) &= S(\lambda)S(\mu)S(\nu) \\ &\geq S(\lambda)S(\overline{\mu})\left(1 + \frac{S(r_1^{c_1-1})}{r_1}\right) = S(\lambda\overline{\mu}) + \frac{S(\lambda\overline{\mu}r_1^{c_1-1})}{r_1}. \end{aligned}$$

That is,

$$2 \geq S(\lambda\overline{\mu}) + \frac{S(\lambda\overline{\mu}r_1^{c_1-1})}{r_1}$$

which may be rearranged to give (5.34). The remark concerning strict inequality is clear.

We now derive (5.35). Using (4.10), we have

$$2 = S(N) = S(\lambda)S(\mu)S(\nu) < S(\lambda\mu^\infty)\left(1 + \frac{w}{r_1 - 1}\right)$$

and (5.35) follows after rearrangement.

In practice, we mostly must assume $c_1 \geq 1$ and replace (5.34) by

$$\frac{S(\lambda\overline{\mu})}{2 - S(\lambda\overline{\mu})} \leq r_1. \tag{5.36}$$

When the special prime is known to divide λ, we can assume $c_1 \geq 2$ and so from (5.34), as in the derivation of (4.5), we have

$$\frac{2}{2 - S(\lambda\overline{\mu})} - \frac{2 - S(\lambda\overline{\mu})}{2} < r_1. \tag{5.37}$$

which only involves rational arithmetic. □

When $w \geq 3$, (5.35) can be improved to

$$r_1 < \frac{2\left(4 + (S(\lambda\mu^\infty))^2 (w-1)\right)}{4 - (S(\lambda\mu^\infty))^2}. \tag{5.38}$$

which mirrors the improvement (4.12). In [27, 32] it was incorrectly stated that there is an improvement also when $w = 2$.

Corollary 4. *Suppose $w \geq 2$, and assume $r_1 < r_2 < \cdots < r_w$. If $S(\lambda\overline{\mu}l_1) < 2$, where l_1 is the largest prime r not dividing λ or μ, which satisfies (5.35), then*

$$\frac{S(\lambda\overline{\mu}l_1)}{2 - S(\lambda\overline{\mu}l_1)} < r \tag{5.39}$$

for $r = r_2$. Further, if $S(\lambda\mu^\infty s_1^\infty) < 2$, where s_1 is the smallest prime r not dividing λ or μ, which satisfies (5.36), or if possible, (5.37), then

$$r < \frac{2 + S(\lambda\mu^\infty s_1^\infty)(w-2)}{2 - S(\lambda\mu^\infty s_1^\infty)} \tag{5.40}$$

for $r = r_2$.

Proof. By using a slight variation on the approach used in the proof of Lemma 15, we have for (5.39), using (4.15)

$$\begin{aligned} 2 = S(N) &= S(\lambda)S(\mu)S(\nu) \\ &\geq S(\lambda)S(\overline{\mu})S(r_1^{c_1})\left(1 + \frac{S(r_2^{c_2-1})}{r_2}\right) = S(\lambda\overline{\mu}r_1^{c_1}) + \frac{S(\lambda\overline{\mu}r_1^{c_1}r_2^{c_2-1})}{r_2}. \end{aligned}$$

That is,

$$2 \geq S(\lambda\overline{\mu}r_1^{c_1}) + \frac{S(\lambda\overline{\mu}r_1^{c_1}r_2^{c_2-1})}{r_2}$$

which may be rearranged to give

$$\frac{S(\lambda\overline{\mu}r_1^{c_1}r_2^{c_2-1})}{2 - S(\lambda\overline{\mu}r_1^{c_1})} \leq r_2.$$

But $l_1 \geq r_1$, $c_1 \geq 1$, $c_2 \geq 1$ and $S(r_1^{c_1}) \geq S(r_1) \geq S(l_1)$ which gives the result.

The proof for (5.40) also follows the approach used in Lemma 20. Using (4.10) we have

$$2 = S(N) = S(\lambda)S(\mu)S(\nu) < S(\lambda)S(\mu^\infty)S(r_1^\infty)\left(1 + \frac{w-1}{r_2 - 1}\right)$$

or simply

$$2 < S(\lambda)S(\mu^\infty r_1^\infty)\left(1 + \frac{w-1}{r_2-1}\right)$$

which can be rearranged to give

$$r_2 < \frac{2 + S(\lambda)S(\mu^\infty r_1^\infty)(w-2)}{2 - S(\lambda)S(\mu^\infty r_1^\infty)}.$$

But $r_1 \geq s_1$ so $S(r_1^\infty) \leq S(s_1^\infty)$ and we have the result. □

When the special prime is known to divide λ, we can assume $c_1 \geq 2$, $c_2 \geq 2$ and obtain

$$\frac{2}{2 - S(\lambda\overline{\mu}l_1^2)} - \frac{2 - S(\lambda\overline{\mu}l_1^2)}{2} < r_2. \tag{5.41}$$

which parallels (4.13) for the odd triperfect case.

Corollary 4 can be generalized in an iterative way so that, given bounds on $r_1, r_2, \ldots, r_{k-1}$, we may be able to give bounds on r_k which, when applicable, provides a more powerful decision criterion.

Corollary 5. *Assume $r_1 < r_2 < \cdots < r_w$ and let l_1, s_1 be as in Corollary 4. Take $k = 2, 3, \ldots$ in turn and suppose $w \geq k$. If $S(\lambda\overline{\mu}l_1l_2\cdots l_{k-1}) < 2$, then*

$$\frac{S(\lambda\overline{\mu}l_1l_2\cdots l_{k-1})}{2 - S(\lambda\overline{\mu}l_1l_2\cdots l_{k-1})} < r \tag{5.42}$$

for $r = r_k$, where l_{k-1} is the largest prime r, not dividing λ or μ, which satisfies (5.43) for $k = 1, 2, \ldots$, respectively. Further, if $S(\lambda\mu^\infty s_1^\infty s_2^\infty \ldots s_{k-1}^\infty) < 2$, then

$$r < \frac{2 + S(\lambda\mu^\infty s_1^\infty s_2^\infty \ldots s_{k-1}^\infty)(w-k)}{2 - S(\lambda\mu^\infty s_1^\infty s_2^\infty \ldots s_{k-1}^\infty)}, \tag{5.43}$$

for $r = r_k$, where s_{k-1} is the smallest prime r not dividing λ or μ, which satisfies (5.42) for $k = 1, 2, \ldots$, respectively. Note: $l_i < l_{i+1}$ and $s_i < s_{i+1}$ for all i.

Proof. : By using a slight variation on the approach used in the proof of Corollary 4, we have for (5.42)

$$S(\nu) \geq S(r_1^{c_1} \ldots r_{k-1}^{c_{k-1}} r_k^{c_k}) = S(r_1^{c_1} \ldots r_{k-1}^{c_{k-1}}) \left(1 + \frac{S(r_k^{c_k - 1})}{r_k}\right)$$

which leads to

$$\frac{S(\lambda\overline{\mu} r_1^{c_1} \ldots r_{k-1}^{c_{k-1}} r_k^{c_k - 1})}{2 - S(\lambda\overline{\mu} r_1^{c_1} \ldots r_{k-1}^{c_{k-1}})} \leq r_k.$$

But $l_i \geq r_i$, $c_i \geq 1$ and $S(r_i^{c_i}) \geq S(r_i) \geq S(l_i)$ for $i < k$ which gives

$$\frac{S(\lambda\overline{\mu} l_1 \ldots l_{k-1} r_k^{c_k - 1})}{2 - S(\lambda\overline{\mu} l_1 \ldots l_{k-1})} \leq r_k.$$

Since $c_k \geq 1$, estimating r_k using

$$\frac{S(\lambda\overline{\mu} l_1 \ldots l_{k-1})}{2 - S(\lambda\overline{\mu} l_1 \ldots l_{k-1})} \leq r_k$$

as before the result is obtained.

Likewise for (5.43), by using (4.19), we have

$$2 = S(N) = S(\lambda)S(\mu)S(\nu) < S(\lambda)S(\mu^\infty) \left(\prod_{i=1}^{k-1} S(r_i^\infty)\right) \left(1 + \frac{w - k + 1}{r_k - 1}\right)$$

or simply

$$2 < S(\lambda)S(\mu^\infty r_1^\infty r_2^\infty \ldots r_{k-1}^\infty) \left(1 + \frac{w - k + 1}{r_k - 1}\right)$$

which can be rearranged to give

$$r_k < \frac{2 + S(\lambda)S(\mu^\infty r_1^\infty r_2^\infty \ldots r_{k-1}^\infty)(w-k)}{2 - S(\lambda)S(\mu^\infty r_1^\infty r_2^\infty \ldots r_{k-1}^\infty)}.$$

But $r_i \geq s_i$ and $S(r_i^\infty) \leq S(s_i^\infty)$ for $i < k$ so we have the result. □

Lemma 20 and Corollary 5, together with Lemmas 16, 17 and 18 provide a powerful test.

5.3.2 An algorithm for testing the lower bound on $\omega(N)$

We assume that N is an odd perfect number with $\omega(N) = t$ distinct prime factors. In brief, the algorithm may be described as a progressive sieve, or "coin-sorter", in which the sieve gets finer and finer, so that eventually nothing is allowed through. The essence of the algorithm is the factor chain common to many problems concerned with odd perfect numbers. We shall use the terminology of graph theory to describe the branching process. Since $3 \mid N$ if $t \leq 10$, for our present purposes the even powers of 3 are the roots of the trees. If 3^2 is an exact divisor of N, then, since $\sigma(N) = 2N$, $\sigma(3^2) = 13$ is a divisor of N, and so the children of the root 3^2 are labelled with different powers of 13. The first of these is 13^1, meaning that we assume 13 is an exact divisor of N (and hence that 13 is the special prime), the second 13^2, then 13^4, 13^5, ... Each of these possibilities leads to further factorisations and further subtrees. Having terminated all these, by methods to be described, we then assume that 3^4 is an exact divisor, beginning the second tree, and we continue in this manner. Only prime powers as allowed in Section (5.1) are considered, and notice is taken of whether the special prime has been specified earlier in any path. These powers are called Eulerian.

We distinguish between *initial components*, which label the nodes and comprise *initial primes* and *initial exponents*, and *consequent primes*, which arise within a tree through factorisation. It is necessary to maintain a count of the total number of distinct initial and consequent primes as they arise within a path, and we let k be this number.

Often, more than one new prime will arise from a single factorisation. All are included in the count, within k, and, whenever further branching is required, the smallest available consequent prime is used as the new initial prime. This preferred strategy will give the greatest increase in $S(\lambda\overline{\mu})$. On the other hand a strategy of selecting the largest available consequent prime will usually give a significant increase in k.

Appendices H and I show the application of the algorithm to a proof that $t \geq 6$ and $t \geq 7$ respectively. In these only, the prime factors of N are numbered as $p_1, p_2, \ldots, p_t$, distinct from the use elsewhere of this notation and successive indentations indicate the nodes. It is worthwhile considering these appendices in conjunction with the description of the algorithm.

To show that $t \geq \omega$, say, we build on earlier results which have presumably shown that $t \geq \omega - 1$, and we suppose that $t = \omega - 1$. If, within any path, we have $k > \omega - 1$, then there is clearly a contradiction, and that path is terminated. This is one of a number of possible contradictions that may arise and which terminate a path. The result will be proved when every path in every tree has been terminated with a contradiction (unless an odd perfect number has been found). The different possible contradictions are indicated with upper case letters.

In the contradiction just mentioned, we have too Many distinct prime factors of N: this is Contradiction *M1*. If there are too Many occurrences of a single prime this is Contradiction *M2*; that is, within a path an initial prime has occurred as a consequent prime more times than the initial exponent. (So counts must also be maintained within each path of the occurrences of each initial prime as a consequent prime.)

If $k = \omega - 3$ but none of these k primes exceeds 100, then Lemma 18 must be (about to be) violated: this is Contradiction *P3*. If $k = \omega - 2$ and none of these primes exceeds 10^4, then Lemma 17 is violated: Contradiction *P2*. Or if, in this case, one exceeds 10^4 but no other exceeds 100, then this is another version of Contradiction *P3*. If $k = \omega - 1$ and none of these primes exceeds 10^7, then Lemma 16 is violated: Contradiction *P1*. In this case, there are the following further possibilities: one prime exceeds 10^7 but no other exceeds 10^4, or one exceeds 10^7, another exceeds 10^4, but no other exceeds 100. These are other versions of contradictions *P2* and *P3*, respectively. These, and some of the other forms of contradiction below, require only counts or comparisons, and no calculations.

At the outset, a number B is chosen, and then the number of subtrees with a given initial prime p is bounded by taking as initial components Eulerian powers p^a with $p^{a+1} \leq B$. If possible, these trees are continued by factorising $\sigma(p^a)$. When a becomes so large that $p^{a+1} > B$, which may occur with $a = 0$, then we write q^b for p^a and we have one more subtree with this initial prime; it is distinguished by writing its initial component as q^∞. This tree must be continued differently. In the first place, the smallest available consequent prime, which is not already an initial prime, is used to begin a new subtree. If no such primes are available, then Lemma 20 is used, as described below.

The product of the u initial components p^a within a path is the number λ. Those initial primes q with exponents ∞, and all consequent primes which are not initial primes, are the v prime factors of μ. If $k < \omega - 1$ then there are $w = \omega - k - 1$ remaining prime factors of n, still to be found or postulated. These are the prime factors r of ν. The numbers u, v, w are not fixed: they vary as the path develops, for example, by taking a consequent prime as another initial prime.

If factorisation can no longer be used to provide further prime factors of N, so, in particular, there are no consequent primes which are not initial primes, then the inequalities of Lemma 20 are used. In that lemma, $\overline{\mu}$ is taken to be the product of powers q^β, where $q \mid \mu$ and β is given as follows. Let $b_0 = \min\{b : q^{b+1} > B\}$. If $b_0 = 0$, then we proceed in a manner to be described later. Otherwise, let

$$\beta = \begin{cases} b_0, & \text{if } b_0 \text{ is even } (b_0 > 0), \\ b_0 + 1, & \text{if } b_0 \text{ is odd,} \end{cases}$$

with one possible exception. If $\pi \nmid \lambda$, and the set $Q_1 = \{q : q \equiv b_0 \equiv 1 \pmod 4\}$ is nonempty, then take $\beta = b_0$ for $q = \min Q_1$. Values of $S(p^a)$ and $S(q^\beta)$ must be maintained, along with their product. This is the value of $S(\lambda\overline{\mu})$ to be used in Lemma 20.

Lemma 20 is used to provide an interval, the primes within which are considered in turn as possible divisors of ν. If there are No primes within the interval that have not been otherwise considered, then this is Contradiction *N*. New primes within the interval are taken in increasing order, giving still further factors of N either through factorisation or through further applications of Lemma 20. There will be occasions when no new primes arise through factorisation, all being used earlier in the same path. Then again Lemma 20 is used to provide further possible prime factors of N (or, if $k = \omega - 1$, we may have found an odd perfect number). This lemma specifically supplies the smallest possible candidate for the remaining primes; a still Smaller prime subsequently arising through factorisation gives us Contradiction *S*.

We also denote by q any consequent prime which is not an initial prime, and, for such primes, we let $Q_2 = \{q : q \equiv 1 \pmod 4\}$. Then, for such primes, we let $\beta = 2$ with the possible exception that, considering all primes q, we let $\beta = b_0$ or 1, as relevant, for $q = \min(Q_1 \cup Q_2)$, if this set is nonempty. Again, the value of $S(\lambda\overline{\mu})$, defined as before, must be maintained. If this value exceeds 2, we have an Abundant divisor of N, and the path is terminated: Contradiction A. This may well occur with $k < \omega - 1$. Values of $S(q^\infty)$ must also be maintained. These, multiplied with the values of $S(p^a)$, give values of $S(\lambda\mu^\infty)$. If this is less than 2 and $k = \omega - 1$ then, for all possible values of the exponents b, the postulated number N is Deficient: Contradiction D.

Contradictions A and D are in fact contradictions of Lemma 19. If, on the other hand, we have a postulated set of prime powers p^a and q^b, for which $S(\lambda\overline{\mu}) \leq 2 \leq S(\lambda\mu^\infty)$, then (5.33) is satisfied and we have candidates for an odd perfect number. If $v = w = 0$, so that we are talking only of known powers p^a, then their product *is* an odd perfect number. Our sieving principle arises when $v > 0$.

In every such case where we have a set of prime powers satisfying (5.33), with $v > 0$, we increase the value of B and investigate that set more closely. With the larger value of B, some prime powers shift from μ to λ, and allow further factorisation, often resulting quickly in Contradiction *M1* or *S*. The value of $S(\overline{\mu})$ increases, so the interval given by Lemma 20 shortens, and hopefully the case which led to our increasing B is no longer exceptional, or Contradiction A or D may be enforced. In that case, we revert to the earlier value of B and continue from where we were. Alternatively, it may be necessary to increase B still further, and later perhaps further again. When $w = 0$, since $S(\overline{\mu}) \to S(\mu^\infty)$ as $B \to \infty$, such cases must eventually be dispensed with, one way or the other.

Cohen [27] points out that if $b_0 = 0$ then

> ... B may immediately be increased, because, most often, the situation of the preceding paragraph will prevail. Suppose we have a node labelled q^∞, where $q > B$. If $q \mid \sigma(p^a)$, then $p^{a+1} > \sigma(p^a) \geq q > B$, so this node could not arise from its parent node by factorisation. Therefore, we must have $q = r_1$ following an application of Lemma 20. Assume that, in that application of Lemma 20, we had $w = 1$. Rearranging (5.36) and (5.35), we have, respectively,
>
> $$S(\lambda\overline{\mu})\left(1+\frac{1}{r_1}\right) \leq 2 \qquad \text{and} \qquad 2 < S(\lambda\mu^\infty)\left(\frac{r_1}{r_1-1}\right). \tag{5.44}$$
>
> These show that we now have a number, namely $\lambda\mu'$ with $\mu' = \mu r_1^{c_1}$, satisfying (5.33), with $v > 0$, and these are the conditions for increasing B. (If we are entitled to assume an exponent 2 for r_1, so that (5.37) may be used instead of (5.36), then we obtain a correspondingly adjusted form for the left-hand inequality in (5.44).) Suppose now that, following the earlier application of Lemma 20, we still have $w \geq 1$. Possibly, $S(\lambda\mu^\infty r_1^\infty) = S(\lambda\mu'^\infty) > 2$, so that we may argue as above. Otherwise, we may use Lemma 20 to find bounds for r_2 (as done in Corollary 4, in part). Since $r_2 > B$, we may then use the preceding argument, and this idea may be repeated as necessary. Thus, B must eventually be increased, and there is no harm in doing so immediately.

We summarise the various contradictions:

- *A* There is an Abundant divisor.
- *D* The number is Deficient.
- *M1* There are too Many prime factors.
- *M2* A single prime has occurred too Many times (an excess of that prime).
- *N* There is No new prime within the given interval.
- *P1* There is no Prime factor exceeding 10^7.
- *P2* There is at most one Prime factor exceeding 10^4.
- *P3* There are at most two Prime factors exceeding 100.
- *S* There is a prime Smaller than the purportedly smallest remaining prime.
- Π None of the primes can be the special prime.

One of these, Contradiction Π, was not discussed previously. It has not been used in either Appendices H or I. Within any path with $k = \omega - 1$, if π is not implicit in an initial component and if there is no prime $q \equiv 1 \pmod 4$, then Contradiction Π may be invoked.

While it is possible to use the constraint $\Omega(N) \geq 37$ as a test when $w = 0$, other conditions such as D, *M1* or one of the P are more easily applied. Further it is not applicable when a q^∞ is involved.

In some of the more recently generated proofs the numerical basis for the contradiction is also given (for example see Appendix G and its footnotes).

Cohen [27] has the following suggestion regarding the implementation of Contradiction A.

> It is possible to use the tables of primitive abundant numbers given by Dickson [37], and corrected by Ferrier [43] and Herzog [67], to create a look-up file. (The abundant number n is primitive if all proper divisors of n are deficient.) Within any path, if the product $\lambda\overline{\mu}$ is a multiple of one of the numbers in the file, then N is abundant: Contradiction A. There are about 500 numbers in Dickson's tables, but these may be adjusted "upwards" to have Eulerian components, in which case many would coincide (though they may no longer be primitive). All of these contain three or four components only. The microfiche supplement to Kishore [80] contains those primitive abundant numbers N with five components and satisfying $S(N) < 2 + 2/10^{10}$, and could perhaps be used similarly. Other such numbers could be appended to the list as they are found, so that subsequent runs of the program would be speeded up.

Considering the number of times Contradiction A was invoked in the various applications of the algorithm, this improvement was not deemed cost effective and hence was not implemented.

The algorithm has been almost fully automated to show that $\omega(N) \geq 7$, with completion not required if the salient features only are desired. It was apparent that the bound B needed to be extended to 10^{21} (incrementing the exponent on 10 from 6 in steps of 3). While the conditions for increasing B by some (fixed) increment are straightforward (even though the implementation of the actual process of handling the newly uncovered cases is not) the value of B to revert to when this case is resolved has several possibilities.

The strategy used in Appendix 1 of Cohen [27] was to revert to the ***initial*** value of B. A consequence of this, having returned from more than one level of incrementation, was the likelihood of again immediately incrementing B to the level used in the previous case. This produces a much longer proof (depending on the initial B and its increment) due to the repeated intermediate cases. A more conservative strategy is to revert to the ***previous*** value of B (backup one increment). The proof that $\omega(N) \geq 7$ in Appendix I has been manually optimised so that the value of B was only decremented when it was known that there would not be an immediate re-incrementation. However automating this "optimising" strategy would not be straightforward.

There are annotations in Appendix I, designed to further explain the algorithm and to give other points of interest.

An unabridged computational proof that $\omega(N) \geq 7$, generated by the latest version of the *Mathematica* code with a static bound of $B = 10^{21}$ can be found in Appendix J.

Table 5.4: Bounds $[s_i, l_i]$ on r_i for an odd perfect number N with $\omega(N) = 8$

λ	$\overline{\mu}$	r_1	r_2
1	3^2	[5,19]	[7,89]
$3^2 13$	7^2	[11,47]	[17,2153]
3^2	13^2	[5,19]	[7,211]
3^4	11^2	[5,23]	
$3^6 1093$	547^2	[5,13]	[7,61]
3^6	1093^2	[5,17]	[7,73]
$3^8 13$	$7^2 757^2$	[17,61]	
$3^8 757$	$7^2 13^2$	[11,17]	[11,59]
3^8	$13^2 757^2$	[5,19]	
3^{10}	$23^2 3851^2$	[5,17]	[7,199]
$3^{12} 797161$	398581^2	[5,13]	[7,61]
3^{12}	797161^2	[5,17]	[7,73]
$3^{14} 13$	$7^2 11^2 4561^2$		
$3^{14} 4561$	$11^2 13^2 2281^2$	[17,23]	[19,37]
3^{14}	$11^2 13^2 4561^2$	[17,31]	[19,53]
3^{16}	$1871^2 34511^2$	[5,13]	[7,61]
$3^{18} 1597$	$17^2 47^2 363889^2$	[7,13]	[11,37]
$3^{18} 363889$	$5^2 1597^2 36389^2$	[17,43]	[23,613]
3^{18}	$1597^2 363889^2$	[5,13]	[7,61]
$3^{20} 13$	$7^2 1093^2 368089^2$	[17,47]	
$3^{20} 1093$	$13^2 547^2 368089^2$	[7,11]	[11,37]
$3^{20} 368089$	$5^2 13^2 1093^2 36809^2$		
3^{20}	$13^2 1093^2 368089^2$	[5,17]	
1	3^{22}	[5,19]	[7,89]

5.3.3 Results for $\omega(N) = 8$

Let us consider now the application of the algorithm to showing that $\omega(N) \geq 9$. This requires showing that there is no odd perfect number with exactly eight distinct prime factors since earlier results have shown that $\omega(N) \geq 8$.

Since $3 \mid N$ if $\omega(N) = 8$, we can investigate initial components of N of the form 3^a, a even, using Lemma 20 and Corollaries 4 and 5. Table 5.4 gives bounds on r_1 and r_2 where possible. When $q \mid \sigma(3^a)$ the table includes separate cases corresponding to whether q is, or is not, the special prime π (to the power 1 only), and when $q = \pi$ then the divisors of $\sigma(q)/2$ have been taken into account.

From this information alone it is not possible to significantly restrict the second smallest prime factor. (Iannucci has claimed privately that the second smallest prime factor is either 5 or 7. If this is so, then Table 5.4 shows that if $3^{14} \parallel N$ then $\pi^\alpha = 13$.) The algorithm described in the previous section can be used to enhance these results.

As the implementation of the algorithm does not currently include adaptively changing B, limited experiments with static values of B were carried out with the aim of whittling down the possible components of N.

By assuming $3^a \parallel N$ for $a \leq 60$ it was shown that, in many cases, all paths were terminated in a contradiction thus eliminating this 3^a as a possible component of N. Figure 5.1 gives a proof that $3^{48} \nparallel N$. A longer proof that $3^{26} \nparallel N$ is given in Appendix K. The bound B was set initially to 10^{18}

10^{24} $3^{48} \Rightarrow 491 \cdot 1093 \cdot 4019 \cdot 8233 \cdot 51157 \cdot 131713$

$491^2 \Rightarrow 37 \cdot 6529$ M1=9

$491^4 \Rightarrow 5 \cdot 101 \cdot 191 \cdot 603791$ M1=11

$491^6 \Rightarrow 7 \cdot 617 \cdot 1051 \cdot 3093060713$ M1=11

491^∞

$1093^1 \Rightarrow 547$ D=1.5077909388967033211

$1093^2 \Rightarrow 3 \cdot 398581$ D=1.5050395001722643082

$1093^4 \Rightarrow 11 \cdot 31 \cdot 4189129561$ M1=10

$1093^5 \Rightarrow 3 \cdot 547 \cdot 398581 \cdot 1193557$ M1=10

$1093^6 \Rightarrow 7 \cdot 29 \cdot 14939 \cdot 562731116179$ M1=11

1093^∞

$4019^2 \Rightarrow 829 \cdot 19489$ M1=9

$4019^4 \Rightarrow 11 \cdot 131 \cdot 4051 \cdot 4391 \cdot 10181$ M1=12

4019^∞

$8233^1 \Rightarrow 23 \cdot 179$ M1=9

$8233^2 \Rightarrow 3 \cdot 409 \cdot 55249$ M1=9

$8233^4 \Rightarrow 11 \cdot 6691 \cdot 62431173781$ M1=10

$8233^5 \Rightarrow 3 \cdot 13 \cdot 23 \cdot 179 \cdot 409 \cdot 55249 \cdot 5213389$ M1=13

8233^∞

$51157^1 \Rightarrow 25579$ D=1.5050945657802078700

$51157^2 \Rightarrow 3 \cdot 9091 \cdot 95959$ M1=9

$51157^4 \Rightarrow 18993881 \cdot 360591138421$ M1=9

51157^∞

$131713^1 \Rightarrow 11 \cdot 5987$ M1=9

$131713^2 \Rightarrow 3 \cdot 31 \cdot 186542431$ M1=9

$131713^\infty : 3.04 < p_8 < 4.05$ N

Done - 26 lines

Figure 5.1: A proof that $3^{48} \nparallel N$ when $\omega(N) = 8$ taking $B = 10^{24}$

but increased as required to the smallest k such that $3^{a+1} < 10^{3k}$. As a result, $B = 10^{30}$ was used for the $3^{60} \nparallel N$ proof. The results of these computer runs can be summarised as follows.

Theorem 10. *If N is an odd perfect number with $\omega(N) = 8$ then $3^a \parallel N$ for $a = 2, 4, 6, 10, 12$ or $a > 60$.*

The effects on the length of the proof of increasing t for $\omega(N) = t$ can be seen in Table 5.5. The blank entries in the column for $t = 8$ reflect the difficulty in generating a proof within an acceptable time limit for those particular cases. The column with $t = 7$ in Table 5.5 duplicates the results of Chein [24] and Hagis [59] to the extent that if $\omega(N) = 7$ then $3^{38} \mid N$.

The choice of B for a static bound can have a dramatic effect on the length of the proof. When considering $3^{16} \parallel N$, $\omega(N) = 8$, a choice of $B = 10^{18}$ resulted in a proof of 26129 lines. This dropped to 23132 and 16340 lines as B was reduced to 10^{15} and 10^{12} respectively. However, when B was reduced to 10^9 the proof required 103599 lines. This behaviour, together with the variability displayed in Table 5.5, indicates not only the desirability of adaptively changing B but also the importance of choosing an appropriate initial B.

As a step in his proof that if an odd perfect number N exists then $\omega(N) \geq 8$, Hagis [59] showed that (Proposition 7.2) if $\pi = 5$ then $5 \parallel N$. With this as inspiration, another set of experiments was

Table 5.5: Number of lines of output for proof that $3^a \nmid\nmid N$ when $\omega(N) = t$ taking $B = 10^{18}$

	t		
a	6	7	8
2	273	479074	
4	251	1295641	
6	247	16306	
8	230	986	712680
10	82	6961	
12	466	42467	
14	65	207	1286
16	59	676	26129
18	100	770	52818
20	96	384	2194
22	33	914	151406
24	34	116	1727
26	1	32	135
28	18	176	2039
30	8	86	667
32	24	101	424
34	20	58	281
36	96	1592	99196

initiated based on assuming a special component π^α as the initial component. The results of these experiments are expressed as follows.

Theorem 11. *If N is an odd perfect number with $\omega(N) = 8$ and the special component is π^α, $\pi < 10^6$ and $\pi^{\alpha+1} < 10^{40}$, then $\pi \parallel N$.*

In most such cases a contradiction was quickly obtained (often a single line sufficed). Appendix L gives a proof that $61^5 \nmid\nmid N$. The largest search tree (approximately 81400 lines) occurred for the case $\pi^\alpha = 17^{13} \parallel N$. Theorem 11 suggests:

Conjecture 3. *If N is an odd perfect number with special prime π then $\pi \parallel N$.*

Cohen [27] argued as follows:

> It seems likely that the worst case, the hardest to dispense with, will be the path involving 3^∞, 5^∞, 17^∞, 257^∞, ..., for the reason that $\prod_{i=0}^{\infty} F_i / (F_i - 1) = 2$, where $F_i = 2^{2^i} + 1$ is the ith Fermat number. Since $F_7 < 3.5 \times 10^{38}$, numbers larger than this should not be encountered, and this is in our favour. Also working for us is the fact that F_5, F_6 and F_7 are composite. As suggested by Appendix 1 in ..., however, and easy enough to see in general, there will arise an inequality of the approximate form $F_6 < p_7 < 2F_6$. This interval includes about $\frac{1}{2}F_6 \approx 10^{19}$ odd numbers. Only the primes in this interval are required, so that the number of possibilities may be reduced somewhat by an incremental wheel. Here and elsewhere, it would not be necessary to check each number for primality. Certainly, probabilistic tests would be sufficient, with subsequent testing if necessary.

10^{36} $3^2 \Rightarrow 13$ *
......
$3^{74} \Rightarrow 11^2 13 \cdot 601 \cdot 4561 \cdot 8951 \cdot 9601 \cdot 391151 \cdot 2098303812601$ M1=9
$3^{\infty} : 3.75 < p_2 < 20.00$
$5^1 \Rightarrow 3 : 9.90 < p_3 < 53.16$ *
......
$5^{50} \Rightarrow 31 \cdot 409 \cdot 90271 \cdot 317731 \cdot 466344409 \cdot 654652168021$ D=1.9422111824916665943
$5^{\infty} : 15.93 < p_3 < 89.10$
$17^1 \Rightarrow 3^2 : 135.99 < p_4 < 674.51$ *
......
$17^{28} \Rightarrow 59 \cdot 7193 \cdot 6088087 \cdot 11658852700685942029849$ *
$17^{\infty} : 255.99 < p_4 < 1274.51$
$257^1 \Rightarrow 3 \cdot 43$ S=43
......
$257^{13} \Rightarrow 3 \cdot 29 \cdot 43 \cdot 883 \cdot 62273 \cdot 160174771 \cdot 325050960571$ M1=10
$257^{\infty} : 65535.99 < p_5 < 262140.01$
$65537^1 \Rightarrow 3^2 11 \cdot 331$ S=11
......
$65537^6 \Rightarrow 79236625382393117320177582087$ *
$65537^{\infty} : 4294967295 < p_6 < 12884901886$
$4294967311^2 \Rightarrow 3 \cdot 2083 \cdot 10837 \cdot 88093 \cdot 3092137$ M1=10
$4294967311^{\infty} : 1229782942255939580 < p_7 < 2459565884511879167$
$1229782942255939601^1 \Rightarrow 3 \cdot 841147 \cdot 243671823961$ M1=9
......
$18446744073709551629^{\infty} : 1317624581295290072 < p_8 < 1317624581295290076$ N
......

Figure 5.2: Extract from a decision tree for $S(3^{\infty}5^{\infty}17^{\infty}\dots)$ with $B = 10^{36}$ (* indicates continuation not shown)

The effects on the interval for r_1 of changing B and the target $\omega(N) = t$ can be seen in Table 5.6. (F_5' and F_6' are the next largest primes after the composite F_5 and F_6 respectively.) Changing B impacts on the lower bound of r_1 but has no effect on its upper bound. The converse is true when changing the target $\omega(N)$.

The path involving 3^{∞}, 5^{∞}, 17^{∞}, ... is the hardest to resolve not only because it will require a large value of B but also because of the total number of paths with 3^{∞} as their origin which will need to be considered. Figure 5.2 illustrates this by showing an extract from the decision tree with $B = 10^{36}$ (which appears to be sufficient to resolve the path involving 3^{∞}, 5^{∞}, 17^{∞}, ...).

Cohen has also argued that

> Since we must have $N > 10^{300}$, if $\omega(N) = 8$ then there is a component of N exceeding $10^{300/8} = 10^{37.5}$. This suggests that B may have to be taken to about this size, but this is a very rough argument.

Table 5.6 more directly supports this conclusion.

Table 5.6: Bounds on r_1 for $S(3^\infty 5^\infty 17^\infty \ldots)$ where F_i' corresponds to the next prime greater than F_i

$\lambda = 1, \quad \mu = F_0^\infty F_1^\infty F_2^\infty = 3^\infty 5^\infty 17^\infty$

B	$r_1 >$
10^6	254.87
10^9	254.99

$\omega(N)$	$r_1 <$
4	256
5	511
6	766
7	1021
8	1276

$\lambda = 1, \quad \mu = F_0^\infty F_1^\infty F_2^\infty F_3^\infty = 3^\infty 5^\infty 17^\infty 257^\infty$

B	$r_1 >$
10^6	58,270.01
10^9	65,526.18
10^{12}	65,534.99

$\omega(N)$	$r_1 <$
5	65,536
6	131,071
7	196,606
8	262,141

$\lambda = 1, \quad \mu = F_0^\infty F_1^\infty F_2^\infty F_3^\infty F_4^\infty$

B	$r_1 >$
10^6	525,513.31
10^9	397,167,826.74
10^{12}	4,257,534,466.95
10^{15}	4,294,947,490.45
10^{18}	4,294,967,260.55
10^{21}	4,294,967,294.98
10^{24}	4,294,967,294.99

$\omega(N)$	$r_1 <$
6	4,294,967,296
7	8,589,934,591
8	12,884,901,886

$\lambda = 1, \quad \mu = F_0^\infty F_1^\infty F_2^\infty F_3^\infty F_4^\infty F_5'^\infty$

B	$r_1 >$
10^6	525,577.61
10^9	437,637,400.33
10^{12}	$4.885007356 \times 10^{11}$
10^{15}	$9.307304977 \times 10^{14}$
10^{18}	$3.730479462 \times 10^{17}$
10^{21}	$1.228529518 \times 10^{18}$
10^{24}	$1.229778894 \times 10^{18}$
10^{27}	$1.229782939 \times 10^{18}$
10^{30}	$1.229782942 \times 10^{18}$

$\omega(N)$	$r_1 <$
7	$1.229782942 \times 10^{18}$
8	$2.459565884 \times 10^{18}$

$\lambda = 1, \quad \mu = F_0^\infty F_1^\infty F_2^\infty F_3^\infty F_4^\infty F_5'^\infty F_6'^\infty$

B	$r_1 >$
10^6	525,577.61
10^9	437,637,400.33
10^{12}	$4.885007485 \times 10^{11}$
10^{15}	$9.307774601 \times 10^{14}$
10^{18}	$3.807477974 \times 10^{17}$
10^{21}	$1.316185806 \times 10^{18}$
10^{24}	$1.317619934 \times 10^{18}$
10^{27}	$1.317624577 \times 10^{18}$
10^{30}	$1.317624581 \times 10^{18}$
10^{33}	$1.317624581 \times 10^{18}$

$\omega(N)$	$r_1 <$
8	$1.317624581 \times 10^{18}$

Chapter 6

Parallel algorithms

6.1 Introduction

The research described in previous chapters has looked at algorithms for problems concerned with harmonic, multiperfect, triperfect and odd perfect numbers and their application to improving some known results. In several cases the improvements obtained were not limited by the algorithm but by practical computing resources. In this chapter we look at the resource requirements of those algorithms and discuss alternatives for their implementation using high performance computing techniques.

6.2 High performance computing

There is a never ending quest in scientific computing to be able to work with larger, more accurate mathematical models. At the applied level, there is a desire to use existing models in a more timely fashion. A possible solution to both needs can be found in high performance computing. This covers the specification, implementation and execution of the relevant algorithms – both software and hardware.

There are several possibilities for completing calculations in a more timely fashion. At the software level we could (i) optimize the way the calculations are executed on a particular computer system (ii) use a better algorithm. At the hardware level we could (i) use a faster machine (ii) use multiple machines on the same problem. In this research, all of these possibilities have been used in varying circumstances.

Before discussing the use of such techniques for each problem area, there are common improvements across all applications. Computing hardware technology continues to advance at an astonishing rate, in particular for the ubiquitous personal computer. This coupled with standardized programming languages, such as C, has allowed the application software developed for this research to evolve over time while being able to take advantage of the latest (fastest) available hardware and software.

The greatest increase in available computing resource has occurred through parallel computing – applying multiple computers to the problem at hand. Such systems range from highly integrated Massively Parallel Processors (MPP) to loosely coupled Networks of Workstations (NOW). Techniques for harnessing large numbers of commodity off-the-shelf computers (COTS) have, in recent years, allowed the construction of coherent scalable networked supercomputers, so called Beowulf cluster systems [113]. Standard software such as the Linux operating system, the GNU compilers and programming tools and the PVM [48] and MPI [55] message passing libraries are integral to the convenient exploitation of such systems. Applications built on these foundations are also becoming more common. In this research, parallel computing was used in a very simple but still very effective way using the NOW architecture.

We will now consider each problem area in turn, looking at the opportunities for parallel computing as well as other ways of improving the efficiency, and hence reducing the computational time of the work.

6.3 Algorithms

6.3.1 Exhaustive search for harmonic numbers

The simplest algorithm applied in this study was for an exhaustive search for harmonic numbers less than a given bound B. As the search has not previously been discussed, and as an introductory example for parallel processing, several variations of the search will be considered. At its most basic the algorithm can be described thus

```
for n = 2, 3, 4, ... while n < B do
    if H(n) ∈ ℕ then
        n is harmonic
    endif
endfor
```

If the unit of work is the calculation of $H(n)$ then the total amount of work is $\mathcal{O}(B)$. The calculation of $H(n)$ implies the factorization of n. For a small bound B, such as 10^{12}, trial division by primes $\leq \sqrt{n}$ is an acceptable choice for a factorization algorithm. It is expedient to initially construct a table of primes less than $\sqrt{B}$ by sieving. As long as available software optimizations are used, such as 32-bit native arithmetic, there appear to be few opportunities for any significant improvements.

Since the calculation/testing of each $H(n)$ is essentially independent, there is a natural parallelism in the approach. If there are *nproc* identical (homogeneous) processors available then a simple *nproc*-way uniform partition of the B candidates for $n < B$ will lead to a *nproc*-fold reduction in the overall time to complete the calculations. This parallel version can be described as follows.

```
quota = B/nproc
for p = 1, 2, ..., nproc do in parallel
    for k = 1, 2, 3, ... while k ≤ quota do
        n = k + (p − 1) · quota
        if H(n) ∈ ℕ then
            n is harmonic
        endif
    endfor
endfor parallel
```

This ignores the slight complication that arises when $nproc \nmid B$ and assumes that the time taken to calculate $H(n)$ is independent of n.

When the computing resources are heterogeneous then there is a straightforward modification to the above algorithm. Let s_i be the relative speed of processor i, established using comparative benchmarking. If these relative speeds are normalized so that $s_i = 1$ for the slowest processor then the quota of work for each processor depends on its relative speed. The faster the processor the more candidates it can process in a given time with the overall time being determined by the time it takes the slowest processor to complete its single quota. The modified (heterogeneous) algorithm is as follows

$totalspeed = \sum_{i=1}^{nproc} s_i$
for $p = 1, 2, \ldots, nproc$ do
    $quota_p = B \cdot s_p / totalspeed$
endfor
$start_1 = 0$
for $p = 2, 3, \ldots, nproc$ do
    $start_p = \sum_{i=1}^{p-1} quota_i$
endfor
for $p = 1, 2, \ldots, nproc$ do in parallel
    for $k = 1, 2, 3, \ldots$ while $k \leq quota_p$ do
        $n = k + start_p$
        if $H(n) \in \mathbb{N}$ then
            n is harmonic
        endif
    endfor
endfor parallel

As a general observation, the amount of work required to factor n, as required in order to calculate $H(n)$, increases as the size of n. To equalize the distribution of small (easy) and large (harder) factorizations, a round-robin approach to the distribution of the candidates is preferred. This will still achieve an *nproc*-fold reduction in the overall time to complete the calculations. This (homogeneous) round-robin algorithm is described thus

for $p = 1, 2, \ldots, nproc$ do in parallel
    for $k = 0, 1, 2, \ldots$ while $k \cdot nproc \leq B$ do
        $n = p + k \cdot nproc$
        if $H(n) \in \mathbb{N}$ then
            n is harmonic
        endif
    endfor
endfor parallel

An alternative, highly non-uniform partitioning scheme is based on the form $n = 2^a \cdot m$, where $a \geq 0, m \geq 3$ and $(2, m) = 1$ (i.e. m is odd positive). The corresponding (homogeneous) algorithm is given as

```
for a = 0, 1, ... while 2^a < B do in parallel
    for m = 3, 5, 7, ... while 2^a · m < B do
        n = 2^a · m
        if H(n) ∈ ℕ then
            n is harmonic
        endif
    endfor
endfor parallel
```

An obvious optimization in this algorithm is to calculate $H(2^a)$ once only and hence test $H(2^a) \cdot H(m)$ rather than $H(n)$ directly. This also means that only the smaller (easier) m needs to be factored.

A finer partitioning would be based on the form $n = 2^a 3^b \cdot m$, where $a, b \geq 0, m \geq 5$ and $(6, m) = 1$ (i.e. m is odd positive and $3 \nmid m$). Here $m = 5, 7, 11, 13, 17, 19, 23, \ldots$. The corresponding (homogeneous) algorithm is given as

```
for a = 0, 1, ... while 2^a < B do in parallel
    for b = 0, 1, ... while 2^a 3^b < B do in parallel
        h = H(2^a 3^b)
        for m = 5, 7, 11, ... while 2^a 3^b · m < B do
            if h · H(m) ∈ ℕ then
                n is harmonic
            endif
        endfor
    endfor parallel
endfor parallel
```

Continuing in this fashion, the next finer partitioning would be based on the form $n = 2^a 3^b 5^c \cdot m$, where $a, b, c \geq 0, m \geq 7$ and $(30, m) = 1$ and then the form $n = 2^a 3^b 5^c 7^d \cdot m$, where $a, b, c, d \geq 0, m \geq 11$ and $(210, m) = 1$. The values of m are exactly those generated in the classical prime sieving process with a 2-, 6-, 30- and 210-wheel respectively.

It was this last type of implementation that was used to find all harmonic numbers less than 10^{12}. Initially it was assumed that $2^a \parallel N$. Decreasing values of a (rather than $a = 0, 1, \ldots$ as suggested above) were used to allow a shortest-first behaviour. These cases were manually distributed over a set of PC class workstations. As the individual cases were completed the next smaller exponent case was begun on that machine. When an individual case became too time-consuming (took more than a day) then the next finer partitioning was used for all subsequent cases. The finest partitioning used was $2^1 3^0 5^0 7^a$.

Once a case was begun it was run till completion. With a large set of workstations, a gradual increase in the number of workstations initially allocated cases would allow an approximate prediction as to how long the next case might take and so whether the next finer partitioning should be used thus avoiding having to abort very long running cases (resulting in a waste of effort).

Overall a set of 25 Pentium III 333MHz workstations was used continuously for almost 3 months on the exhaustive search for all harmonic numbers less than 10^{12}. This equates to over 6 years on a single workstation of that type and would not be practical for reliability reasons if not timeliness. Where possible, algorithms which are expected to be long running (and don't afford any opportunity for parallelism) should have checkpoints and program restart capabilities designed into their implementation.

6.3.2 Factor chain based algorithms

Most of the search algorithms used in this research have the factor chain method as their basis. Such algorithms were used for (a) finding harmonic seeds, (b) showing that there are no odd harmonic seeds less than 10^{15}, (c) finding even multiperfect numbers, (d) showing that, for an odd perfect number N, $\Omega(N) \geq 37$ and (e) showing that, for an odd perfect number there are restrictions on the special component π^α and the 3^a component.

Differences occur when specifying termination condition (an appropriate contradiction) and which prime or exponent to consider next based on the appropriate "index" (such as $H(n)$, $S(n)$ or $\sigma(n)$).

The following pseudocode illustrates a simplified structure of a recursive algorithm for the odd perfect number problem. Only minor modifications are needed when considering the corresponding problem for odd triperfect numbers. To initiate the algorithm for a complete problem, we assume $3 \mid N$ and so would use $\mathrm{DoP}(p, 1, 1)$, where the 1's represent λ and μ, the known components and the known consequent primes (other than 3) respectively.

The capability of conveniently specifying a general initial set of conditions (p, λ, μ) proved essential when investigating particular known components as in the proofs of Theorems 10 and 11.

With the use of tables of factorisations and individual hints, any difficult factorisations were once-only problems and so the overall computational time for a proof was proportional to the length (the number of lines of output) of the proof. Because of the great variability in the final lengths of the proofs, formulating an effective workload sharing strategy for a set of parallel resources appears difficult (unlike formulating a strategy for the harmonic number exhaustive search algorithm in the previous section). In practice the actual computational time for the various proofs varied from "instantaneous" to a few days in some extreme cases. It is hoped that with an adaptive bound B these few extreme cases would be dealt with expeditiously.

```
DoP(p, λ, μ)
begin
    for each a: allowable Eulerian exponent and p^(a+1) ≤ B do in parallel
        DoPA(p, a, λ, μ)
    endfor parallel
    DoPInf(p, a, λ, μ)
end

DoPA(p, a, λ, μ)
begin
    s = σ(p^a)
    λ' = λ p^a
    μ' = μ ∏_{t prime, t>2, t|s, t∤(λ'μ)} t      { update μ with new consequent primes }
    if no contradiction with (λ'μ') then
        if μ > 1 then
            p' = min(t : t | μ')
            μ' = μ'/p'
            DoP(p', λ', μ')
        else
            calculate {lb, ub} on r_1
            if no contradiction with (λμ', {lb, ub}) then
                for each r : lb < r < ub and r ∤ (λμ') do in parallel
                    DoP(r, λ, μ')
                endfor parallel
            else
                this case is terminated
            endif
        endif
    else
        this case is terminated
    endif
end
```

[continued ...]

```
DoPInf(p, a, λ, μ)
begin
    if μ > 1 then
        p' = min(t : t | μ')
        μ' = μ/p'
        DoP(p', λ, μ')
    else
        calculate {lb, ub} on r_1
        if no contradiction with (λμ', {lb, ub}) then
            for each r : lb < r < ub and r ∤ (λμ') do in parallel
                DoP(r, λ, μ')
            endfor parallel
        else
            this case is terminated
        endif
    endif
end
```

Appendix A

Harmonic seeds less than 10^{15}

Harmonic seeds n less than 10^{15} with initial squarefree multipliers, q_1 such that nq_1 is harmonic, the number of squarefree harmonic multiples, Q_p, and the number of one-component-at-a-time harmonic multiples, Q_{p^a}.

Note that limitations within *Mathematica* prevented the generation of the complete set of multiples for the harmonic seed 812193794048000.

See Table 2.1 for details of all harmonic numbers which are squarefree multiples of 2457000 and see Figure 2.2 for a complete elaboration of the harmonic multiples for the harmonic seed 18620.

n		$H(n)$		q_1	Q_p	Q_{p^a}	
1		1			1	1	
6	$2 \cdot 3$	2	2		1	1	
28	$2^2 7$	3	3	5	2	2	
270	$2 \cdot 3^3 5$	6	$2 \cdot 3$	11	2	2	
496	$2^4 31$	5	5		1	1	
672	$2^5 3 \cdot 7$	8	2^3		1	1	
1638	$2 \cdot 3^2 7 \cdot 13$	9	3^2	5,17	4	4	
6200	$2^3 5^2 31$	10	$2 \cdot 5$	3,19	6	21	(3)
8128	$2^6 127$	7	7	13	2	11	(1)
18620	$2^2 5 \cdot 7^2 19$	14	$2 \cdot 7$	3,13	5	18	(1)
30240	$2^5 3^3 5 \cdot 7$	24	$2^3 3$	11,23,47	5	5	
32760	$2^3 3^2 5 \cdot 7 \cdot 13$	24	$2^3 3$	11,23,47	5	5	
173600	$2^5 5^2 7 \cdot 31$	25	5^2		1	1	
1089270	$2 \cdot 3^2 5 \cdot 7^2 13 \cdot 19$	42	$2 \cdot 3 \cdot 7$	11,41,83	5	5	
2229500	$2^2 5^3 7^3 13$	35	$5 \cdot 7$		1	1	
2457000	$2^3 3^3 5^3 7 \cdot 13$	60	$2^2 3 \cdot 5$	11,19,23,29,59	18	18	
4713984	$2^9 3^3 11 \cdot 31$	48	$2^4 3$	5,7,23,47	37	61	(2)
6051500	$2^2 5^3 7^2 13 \cdot 19$	50	$2 \cdot 5^2$	3	4	17	(1)
8506400	$2^5 5^2 7^3 31$	49	7^2	13,97	5	11	(1)
17428320	$2^5 3^2 5 \cdot 7^2 13 \cdot 19$	96	$2^5 3$	11,23,31,47,191	12	14	(1)
23088800	$2^5 5^2 7^2 19 \cdot 31$	70	$2 \cdot 5 \cdot 7$	3,13,139	11	68	(3)
29410290	$2 \cdot 3^5 5 \cdot 7^2 13 \cdot 19$	81	3^4	17,53	5	5	
33550336	$2^{12} 8191$	13	13		1	10	(1)
45532800	$2^7 3^3 5^2 17 \cdot 31$	96	$2^5 3$	7,11,23,47,191	39	39	
52141320	$2^3 3^4 5 \cdot 7 \cdot 11^2 19$	108	$2^2 3^3$	17,23,53,71,107	14	14	
81695250	$2 \cdot 3^3 5^3 7^2 13 \cdot 19$	105	$3 \cdot 5 \cdot 7$	29,41	4	4	
115048440	$2^3 3^2 5 \cdot 13^2 31 \cdot 61$	78	$2 \cdot 3 \cdot 13$	11	2	2	

n		$H(n)$		q_1	Q_p	Q_{p^a}	
142990848	$2^9 3^2 7 \cdot 11 \cdot 13 \cdot 31$	120	$2^3 3 \cdot 5$	5,19,23,29,47,59,79,239	41	41	
255428096	$2^9 7 \cdot 11^2 19 \cdot 31$	88	$2^3 11$	3,43	16	16	
326781000	$2^3 3^3 5^3 7^2 13 \cdot 19$	168	$2^3 3 \cdot 7$	11,23,41,47,83,167	18	34	(2)
459818240	$2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$	96	$2^5 3$	3,11,23,31,47,191	45	218	(11)
481572000	$2^5 3^3 5^3 7^3 13$	168	$2^3 3 \cdot 7$	11,23,41,47,83,167	18	34	(2)
644271264	$2^5 3^2 7 \cdot 13^2 31 \cdot 61$	117	$3^2 13$	5,17,233	6	10	(1)
1307124000	$2^5 3^3 5^3 7^2 13 \cdot 19$	240	$2^4 3 \cdot 5$	11,23,29,31,47,59,79,239,479	35	35	
1381161600	$2^7 3^2 5^2 7 \cdot 13 \cdot 17 \cdot 31$	240	$2^4 3 \cdot 5$	11,19,23,29,47,59,79,239,479	63	63	
1630964808	$2^3 3^4 11^3 31 \cdot 61$	99	$3^2 11$	5,17,197	8	8	
1867650048	$2^{10} 3^4 11 \cdot 23 \cdot 89$	128	2^7	7,31,127	16	408	(15)
2876211000	$2^3 3^2 5^3 13^2 31 \cdot 61$	150	$2 \cdot 3 \cdot 5^2$	11,19,29,59,149	16	29	(3)
8410907232	$2^5 3^2 7^2 13 \cdot 19^2 127$	171	$3^2 19$	5,17,37,113	18	18	
8589869056	$2^{16} 131071$	17	17		1	1	
8628633000	$2^3 3^3 5^3 13^2 31 \cdot 61$	195	$3 \cdot 5 \cdot 13$	29,389	3	3	
8698459616	$2^5 7^2 11^2 19^2 127$	121	11^2	241	2	59	(3)
10200236032	$2^{14} 7 \cdot 19 \cdot 31 \cdot 151$	96	$2^5 3$	3,5,11,23,47,191	71	158	(3)
14182439040	$2^7 3^4 5 \cdot 7 \cdot 11^2 17 \cdot 19$	384	$2^7 3$	23,31,47,127,191,383	31	57	(3)
19017782784	$2^9 3^2 7^2 11 \cdot 13 \cdot 19 \cdot 31$	336	$2^4 3 \cdot 7$	5,23,41,47,83,167,223	23	29	(1)
19209881600	$2^{11} 5^2 7^2 13 \cdot 19 \cdot 31$	256	2^8	3,127	22	22	
35032757760	$2^9 3^2 5 \cdot 7^3 11 \cdot 13 \cdot 31$	392	$2^3 7^2$	97	4	4	
43861478400	$2^{10} 3^3 5^2 23 \cdot 31 \cdot 89$	264	$2^3 3 \cdot 11$	7,11,43,47,131,263	43	53	(1)
57575890944	$2^{13} 3^2 11 \cdot 13 \cdot 43 \cdot 127$	192	$2^6 3$	5,7,23,31,47,191,383	83	772	(39)
57648181500	$2^2 3^2 5^3 7^3 13^3 17$	273	$3 \cdot 7 \cdot 13$	41,181	3	3	
66433720320	$2^{13} 3^3 5 \cdot 11 \cdot 43 \cdot 127$	224	$2^5 7$	7,13,31,223	21	26	(1)
71271827200	$2^8 5^2 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73$	270	$2 \cdot 3^3 5$	3,11,17,29,53,59,89,107,179,269	73	90	(5)
73924348400	$2^4 5^2 7 \cdot 31^2 83 \cdot 331$	125	5^3		1	1	
77924700000	$2^5 3^3 5^5 7^2 19 \cdot 31$	375	$3 \cdot 5^3$	29,149	4	4	

n		$H(n)$		q_1	Q_p	Q_{p^a}	
81417705600	$2^7 3 \cdot 5^2 7 \cdot 11^2 17 \cdot 19 \cdot 31$	484	$2^2 11^2$	43,241,967	5	5	
84418425000	$2^3 3^2 5^5 7^2 13 \cdot 19 \cdot 31$	375	$3 \cdot 5^3$	29,149	4	4	
109585986048	$2^9 3^7 7 \cdot 11 \cdot 31 \cdot 41$	324	$2^2 3^4$	5,17,23,53,71,107,647	157	157	
110886522600	$2^3 3 \cdot 5^2 7 \cdot 31^2 83 \cdot 331$	155	$5 \cdot 31$	61	2	5	(1)
123014892000	$2^5 3 \cdot 5^3 7^3 11^2 13 \cdot 19$	484	$2^2 11^2$	43,241,967	5	5	
124406100000	$2^5 3^2 5^5 7^3 13 \cdot 31$	375	$3 \cdot 5^3$	29,149	4	4	
137438691328	$2^{18} 524287$	19	19	37	3	25	(2)
156473635500	$2^2 3^2 5^3 7^2 13^3 17 \cdot 19$	390	$2 \cdot 3 \cdot 5 \cdot 13$	11,29,59,389	9	9	
183694492800	$2^7 3^2 5^2 7^2 13 \cdot 17 \cdot 19 \cdot 31$	672	$2^5 3 \cdot 7$	11,23,41,47,83,167,191,223	29	29	
206166804480	$2^{11} 3^2 5 \cdot 7 \cdot 13^2 31 \cdot 61$	384	$2^7 3$	11,23,47,127,191,383	20	34	(2)
221908282624	$2^8 7 \cdot 19^2 37 \cdot 73 \cdot 127$	171	$3^2 19$	5,17,113	9	9	
271309925250	$2 \cdot 3^7 5^3 7^2 13 \cdot 19 \cdot 41$	405	$3^4 5$	17,29,53,89,269,809	28	31	(1)
428440390560	$2^5 3^2 5 \cdot 7^2 13^2 19 \cdot 31 \cdot 61$	546	$2 \cdot 3 \cdot 7 \cdot 13$	11,41,83,181,1091	12	12	
443622427776	$2^7 3^4 11^3 17 \cdot 31 \cdot 61$	352	$2^5 11$	7,43	19	19	
469420906500	$2^2 3^3 5^3 7^2 13^3 17 \cdot 19$	507	$3 \cdot 13^2$	337,1013	4	4	
513480135168	$2^9 3^5 7^2 11 \cdot 13 \cdot 19 \cdot 31$	648	$2^3 3^4$	5,17,23,47,53,71,107,431,647	227	230	(1)
677701763200	$2^7 5^2 7 \cdot 11 \cdot 17^2 31 \cdot 307$	340	$2^2 5 \cdot 17$	3,19,67	29	44	(3)
830350521000	$2^3 3^4 5^3 7^3 11^2 13 \cdot 19$	756	$2^2 3^3 7$	17,23,41,53,71,83,107,167,251,503,1511	59	72	(3)
945884459520	$2^9 3^5 5 \cdot 7^3 11 \cdot 13 \cdot 31$	756	$2^2 3^3 7$	17,23,41,53,71,83,107,167,251,503,1511	59	65	(1)
997978703400	$2^3 3^3 5^2 7 \cdot 31^2 83 \cdot 331$	279	$3^2 31$	17,61,557	6	16	(2)
1058501001600	$2^7 3^7 5^2 7 \cdot 17 \cdot 31 \cdot 41$	648	$2^3 3^4$	11,23,47,53,71,107,431,647	77	77	
1085239701000	$2^3 3^7 5^3 7^2 13 \cdot 19 \cdot 41$	648	$2^3 3^4$	11,17,23,47,53,71,107,431,647	140	140	
1144136294400	$2^{13} 3 \cdot 5^2 11 \cdot 31 \cdot 43 \cdot 127$	350	$2 \cdot 5^2 7$	13,19,139,349	12	21	(3)
1179832600464	$2^4 3 \cdot 7^2 19 \cdot 31^2 83 \cdot 331$	217	$7 \cdot 31$	13,61,433	6	29	(3)
1330464844800	$2^{10} 3^2 5^2 7 \cdot 13 \cdot 23 \cdot 31 \cdot 89$	660	$2^2 3 \cdot 5 \cdot 11$	11,19,29,43,59,109,131,263,439,659,1319	74	74	
1480003190400	$2^7 3 \cdot 5^2 7 \cdot 17 \cdot 23^2 31 \cdot 79$	529	23^2		1	1	
1517389419000	$2^3 3 \cdot 5^3 7^2 13 \cdot 19 \cdot 23^2 79$	529	23^2		1	1	

n		$H(n)$		q_1	Q_p	Q_{p^a}	
1542738616320	$2^{13}3 \cdot 5 \cdot 11^2 19 \cdot 43 \cdot 127$	352	$2^5 11$	7,31	15	46	(3)
1553357978368	$2^8 7^2 19^2 37 \cdot 73 \cdot 127$	252	$2^2 3^2 7$	3,5,11,13,17,23,41,71,83,167,251,503	282	2419	(125)
1599300612000	$2^5 3^7 5^3 7^3 13 \cdot 41$	648	$2^3 3^4$	11,17,23,47,53,71,107,431,647	140	140	
2112394079250	$2 \cdot 3^5 5^3 7^2 13^3 17 \cdot 19$	585	$3^2 5 \cdot 13$	29,89,233,389	7	7	
2198278051200	$2^7 3^4 5^2 7 \cdot 11^2 17 \cdot 19 \cdot 31$	1080	$2^3 3^3 5$	23,29,47,53,59,71,79,89,107,179,239,269,359, 431,719	117	120	(1)
2236152828000	$2^5 3 \cdot 5^3 7^3 13 \cdot 23^2 79$	529	23^2		1	1	
2827553208480	$2^5 3^2 5 \cdot 7^5 11 \cdot 13 \cdot 19 \cdot 43$	686	$2 \cdot 7^3$	97	3	3	
3321402084000	$2^5 3^4 5^3 7^3 11^2 13 \cdot 19$	1080	$2^3 3^3 5$	17,23,29,47,53,59,71,79,89,107,179,239,269,359, 431,719	159	162	(1)
3622293071600	$2^4 5^2 7^3 31^2 83 \cdot 331$	245	$5 \cdot 7^2$	13,97	5	34	(1)
3946161492000	$2^5 3^2 5^3 7^3 13^2 31 \cdot 61$	735	$3 \cdot 5 \cdot 7^2$	29,41,97,293	10	13	(1)
4314435969536	$2^9 7^2 11^2 19^2 31 \cdot 127$	385	$5 \cdot 7 \cdot 11$	13,109,769	8	64	(2)
4409499089268	$2^2 3^3 7^4 13 \cdot 467 \cdot 2801$	147	$3 \cdot 7^2$	5,41,97,293	8	8	
4959751305600	$2^7 3^5 5^2 7^2 13 \cdot 17 \cdot 19 \cdot 31$	1296	$2^4 3^4$	11,23,47,53,71,107,431,647,863,2591	140	143	(1)
5111051997870	$2 \cdot 3^2 5 \cdot 7^2 13^2 19 \cdot 61^2 97$	366	$2 \cdot 3 \cdot 61$	11	2	4	(1)
5914410203520	$2^7 3^5 5 \cdot 7 \cdot 13^2 17 \cdot 31 \cdot 61$	936	$2^3 3^2 13$	11,23,47,71,103,233,311,467,1871	44	44	
6073712944992	$2^5 3^4 7^2 11^3 19 \cdot 31 \cdot 61$	693	$3^2 7 \cdot 11$	5,13,17,41,197,461	41	56	(2)
6844445080704	$2^7 3^4 7 \cdot 11^2 17 \cdot 19^2 127$	684	$2^2 3^2 19$	5,23,37,71,113,151,227,683,1367	80	80	
7322605472000	$2^8 5^3 7^3 13 \cdot 19 \cdot 37 \cdot 73$	672	$2^5 3 \cdot 7$	3,11,23,31,41,47,83,167,191,223	188	654	(15)
8449576317000	$2^3 3^5 5^3 7^2 13^3 17 \cdot 19$	936	$2^3 3^2 13$	11,23,47,71,103,233,311,467,1871	44	44	
9831938337200	$2^4 5^2 7^2 19 \cdot 31^2 83 \cdot 331$	350	$2 \cdot 5^2 7$	3,13,139,349	23	181	(5)
10297226649600	$2^{13} 3^3 5^2 11 \cdot 31 \cdot 43 \cdot 127$	630	$2 \cdot 3^2 5 \cdot 7$	13,17,19,29,41,59,83,89,139,179,251,419,1259	102	158	(11)
10461217539500	$2^2 5^3 7^3 13^2 61^2 97$	305	$5 \cdot 61$		1	1	
10711009764000	$2^5 3^2 5^3 7^2 13^2 19 \cdot 31 \cdot 61$	1050	$2 \cdot 3 \cdot 5^2 7$	11,29,41,59,83,139,149,349,419,1049,2099	30	30	
10881843388416	$2^{13} 3^5 7 \cdot 11 \cdot 13 \cdot 43 \cdot 127$	648	$2^3 3^4$	5,17,23,47,53,71,107,431,647	336	339	(1)
11484718245000	$2^3 3^5 5^4 7^2 11 \cdot 13 \cdot 19 \cdot 71$	1125	$3^2 5^3$	17,29,89,149,449	12	12	

n		$H(n)$		q_1	Q_p	Q_{p^a}	
11567890545120	$2^5 3^5 5 \cdot 7^2 13^2 19 \cdot 31 \cdot 61$	1053	$3^4 13$	17,53,233,701	10	10	
12452007204000	$2^5 3^5 5^3 7^3 13^3 17$	936	$2^3 3^2 13$	11,23,47,71,103,233,311,467,1871	44	44	
13661860101120	$2^{10} 3^4 5 \cdot 7 \cdot 11^2 19 \cdot 23 \cdot 89$	1056	$2^5 3 \cdot 11$	31,43,47,131,191,263,2111	31	62	(5)
14747907505800	$2^3 3 \cdot 5^2 7^2 19 \cdot 31^2 83 \cdot 331$	434	$2 \cdot 7 \cdot 31$	13,61,433	6	11	(1)
15462510336000	$2^{11} 3^3 5^3 7 \cdot 13^2 31 \cdot 61$	960	$2^6 3 \cdot 5$	11,19,23,29,47,59,79,127,191,239,383,479	129	176	(8)
16924847940000	$2^5 3^5 5^4 7^3 11 \cdot 13 \cdot 71$	1125	$3^2 5^3$	17,29,89,149,449	12	12	
20662005324800	$2^{10} 5^2 7 \cdot 23^2 31 \cdot 79 \cdot 89$	506	$2 \cdot 11 \cdot 23$	3,43	4	4	
21590959104000	$2^{13} 3^3 5^3 11 \cdot 13 \cdot 43 \cdot 127$	800	$2^5 5^2$	7,19,31,79,199	97	180	(12)
21733758429600	$2^5 3 \cdot 5^2 7^3 31^2 83 \cdot 331$	434	$2 \cdot 7 \cdot 31$	13,61,433	6	11	(1)
22385029489560	$2^3 3^{10} 5 \cdot 23 \cdot 107 \cdot 3851$	198	$2 \cdot 3^2 11$	11,17,43,131,197	13	13	
23885971200000	$2^{11} 3^3 5^5 7^3 13 \cdot 31$	960	$2^6 3 \cdot 5$	11,19,23,29,47,59,79,127,191,239,383,479	129	176	(8)
24613169545216	$2^{14} 7 \cdot 19^2 31 \cdot 127 \cdot 151$	285	$3 \cdot 5 \cdot 19$	5,29,37,113,569	13	13	
28103080287744	$2^9 3^3 7 \cdot 11 \cdot 31^2 83 \cdot 331$	496	$2^4 31$	61,991	5	171	(2)
30233275380000	$2^5 3^3 5^4 7^3 11^2 19 \cdot 71$	1155	$3 \cdot 5 \cdot 7 \cdot 11$	13,29,41,109,461,769,2309	23	23	
32133029292000	$2^5 3^3 5^3 7^2 13^2 19 \cdot 31 \cdot 61$	1365	$3 \cdot 5 \cdot 7 \cdot 13$	29,41,181,389,2729	8	8	
32752714995000	$2^3 3^2 5^4 7^3 11^2 13 \cdot 19 \cdot 71$	1155	$3 \cdot 5 \cdot 7 \cdot 11$	29,41,109,461,769,2309	10	10	
43180427911400	$2^3 5^2 13^2 19 \cdot 31^3 37 \cdot 61$	403	$13 \cdot 31$		1	6	(1)
43947421401888	$2^5 3^6 23 \cdot 137 \cdot 547 \cdot 1093$	216	$2^3 3^3$	5,7,11,17,47,53,71,107,431	359	862	(38)
46013471418096	$2^4 3^2 7^2 13 \cdot 19 \cdot 31^2 83 \cdot 331$	558	$2 \cdot 3^2 31$	5,11,17,61,557	30	138	(7)
46353444300800	$2^{11} 5^2 7^2 13 \cdot 19^2 31 \cdot 127$	760	$2^3 5 \cdot 19$	3,37,79,151,379	100	290	(4)
47911115564928	$2^7 3^4 7^2 11^2 17 \cdot 19^2 127$	1008	$2^4 3^2 7$	5,13,23,31,41,47,71,83,167,223,251,503	294	1622	(105)
53092467020880	$2^4 3^3 5 \cdot 7^2 19 \cdot 31^2 83 \cdot 331$	651	$3 \cdot 7 \cdot 31$	13,41,61,433,1301	10	20	(2)
54934276752360	$2^3 3^6 5 \cdot 23 \cdot 137 \cdot 547 \cdot 1093$	252	$2^2 3^2 7$	7,11,13,17,41,71,83,167,251,503	74	102	(5)
58991630023200	$2^5 3 \cdot 5^2 7^2 19 \cdot 31^2 83 \cdot 331$	620	$2^2 5 \cdot 31$	61,619	6	11	(1)
61015386432000	$2^9 3 \cdot 5^3 7^3 11^2 13 \cdot 19 \cdot 31$	1540	$2^2 5 \cdot 7 \cdot 11$	43,109,139,307,439,769,3079	22	61	(1)
72874680721920	$2^9 3^7 5 \cdot 7^2 11 \cdot 19 \cdot 31 \cdot 41$	1512	$2^3 3^3 7$	13,17,23,47,53,71,83,107,167,251,431,503, 1511,3023	189	228	(5)

n		$H(n)$		q_1	Q_p	Q_{p^a}	
76343936628960	$2^5 3^5 5 \cdot 7^5 11 \cdot 13 \cdot 19 \cdot 43$	1323	$3^3 7^2$	17,41,53,97,293,881	20	20	
77212128389760	$2^7 3^5 5 \cdot 7^2 13 \cdot 17 \cdot 19^2 127$	1368	$2^3 3^2 19$	11,23,37,47,71,113,151,227,683,911,1367	117	117	
78429196876800	$2^{10} 3 \cdot 5^2 7 \cdot 11^2 19 \cdot 23 \cdot 31 \cdot 89$	1331	11^3	241	2	2	
79708161843200	$2^{13} 5^2 11^2 19 \cdot 31 \cdot 43 \cdot 127$	660	$2^2 3 \cdot 5 \cdot 11$	3,7,23,29,59,109,131,263,439,659,1319	92	223	(7)
84761657875440	$2^4 3^2 5 \cdot 7^3 13 \cdot 31^2 83 \cdot 331$	651	$3 \cdot 7 \cdot 31$	41,61,433,1301	7	7	
90134334505600	$2^7 5^2 7^2 11 \cdot 17^2 19 \cdot 31 \cdot 307$	952	$2^3 7 \cdot 17$	3,13,67,271	56	176	(9)
118773204802560	$2^{10} 3^4 5 \cdot 7 \cdot 11 \cdot 23^2 79 \cdot 89$	1104	$2^4 3 \cdot 23$	31,47,137,367,1103,2207	15	23	(2)
125356165141536	$2^5 3^{10} 7 \cdot 23 \cdot 107 \cdot 3851$	297	$3^3 11$	5,17,53,197,593	23	23	
131010859980000	$2^5 3^2 5^4 7^3 11^2 13 \cdot 19 \cdot 71$	1650	$2 \cdot 3 \cdot 5^2 11$	29,43,59,109,131,149,659,3299	28	28	
134669267040000	$2^8 3 \cdot 5^4 7 \cdot 11 \cdot 19 \cdot 37 \cdot 71 \cdot 73$	1250	$2 \cdot 5^4$	499,1249	5	5	
147860255088000	$2^7 3^5 5^3 7 \cdot 13^2 17 \cdot 31 \cdot 61$	1800	$2^3 3^2 5^2$	11,19,23,29,47,59,71,79,89,149,179,199,239,359, 449,599,719	471	477	(2)
172292186816512	$2^{14} 7^2 19^2 31 \cdot 127 \cdot 151$	420	$2^2 3 \cdot 5 \cdot 7$	3,5,11,13,23,29,41,59,83,139,167,419,839	227	2442	(81)
176951824358400	$2^{10} 3^2 5^2 7^2 13 \cdot 19 \cdot 23 \cdot 31 \cdot 89$	1848	$2^3 3 \cdot 7 \cdot 11$	11,41,43,47,83,131,167,263,307,461,1231,1847	64	165	(3)
212066490636000	$2^5 3^3 5^3 7^5 11 \cdot 13 \cdot 19 \cdot 43$	1715	$5 \cdot 7^3$	97	3	3	
220626385182720	$2^{13} 3^7 5 \cdot 11 \cdot 41 \cdot 43 \cdot 127$	864	$2^5 3^3$	7,17,23,31,47,53,71,107,191,431,863	295	499	(28)
225855341712000	$2^7 3^4 5^3 7^3 11^2 13 \cdot 17 \cdot 19$	2688	$2^7 3 \cdot 7$	23,31,41,47,83,127,167,191,223,383,2687	110	120	(2)
228409599600000	$2^7 3^5 5^5 7^3 13 \cdot 17 \cdot 31$	1800	$2^3 3^2 5^2$	11,19,23,29,47,59,71,79,89,149,179,199,239,359, 449,599,719	471	477	(2)
271450207324800	$2^7 3^3 5^2 7 \cdot 17 \cdot 31^2 83 \cdot 331$	992	$2^5 31$	61,991	5	15	(1)
274350998756016	$2^4 3^2 7 \cdot 13^2 31^2 61 \cdot 83 \cdot 331$	585	$3^2 5 \cdot 13$	5,17,29,89,233,389	17	22	(2)
289197263628000	$2^5 3^5 5^3 7^2 13^2 19 \cdot 31 \cdot 61$	2025	$3^4 5^2$	17,29,53,89,149,269,449,809,4049	64	81	(2)
295185198672000	$2^7 3^7 5^3 7^2 13 \cdot 17 \cdot 19 \cdot 41$	2304	$2^8 3^2$	11,23,31,47,71,127,191,383,1151	235	322	(5)
325989310185000	$2^3 3^7 5^4 7^2 11 \cdot 19 \cdot 41 \cdot 71$	1575	$3^2 5^2 7$	13,17,29,89,149,349,449,1049	39	44	(1)
340851122110464	$2^{17} 3 \cdot 7 \cdot 19^2 37 \cdot 73 \cdot 127$	512	2^9	31	6	158	(4)
346024723524480	$2^7 3^2 5 \cdot 7^2 11^2 13 \cdot 17 \cdot 19^2 127$	1936	$2^4 11^2$	31,43,241,967	10	10	
383328899840250	$2 \cdot 3^3 5^3 7^2 13^2 19 \cdot 61^2 97$	915	$3 \cdot 5 \cdot 61$	29	2	2	

n		$H(n)$		q_1	Q_p	Q_{p^a}	
406940521854720	$2^8 3 \cdot 5 \cdot 7^2 17 \cdot 19 \cdot 37^2 67 \cdot 73$	1184	$2^5 37$	31	5	15	(2)
421167642255360	$2^{13} 3^2 5 \cdot 7 \cdot 11^2 13 \cdot 19 \cdot 43 \cdot 127$	1584	$2^4 3^2 11$	17,23,31,47,71,131,197,263,1583,3167	56	56	
427338875132928	$2^{10} 3^4 11^3 23 \cdot 31 \cdot 61 \cdot 89$	968	$2^3 11^2$	7,43,241,967	19	42	(1)
436539941600000	$2^8 5^5 7^3 19 \cdot 31 \cdot 37 \cdot 73$	1050	$2 \cdot 3 \cdot 5^2 7$	3,11,13,29,41,59,83,139,149,349,419,1049,2099	94	801	(28)
443254811126400	$2^7 3^2 5^2 7^2 13 \cdot 17 \cdot 19^2 31 \cdot 127$	1995	$3 \cdot 5 \cdot 7 \cdot 19$	29,37,41,113,569,797,3989	25	43	(2)
480405299220000	$2^5 3^7 5^4 7^3 11 \cdot 41 \cdot 71$	1575	$3^2 5^2 7$	13,17,29,89,149,349,449,1049	39	44	(1)
530924670208800	$2^5 3^3 5^2 7^2 19 \cdot 31^2 83 \cdot 331$	1116	$2^2 3^2 31$	11,17,23,61,71,557,743	32	135	(13)
549138477888000	$2^9 3^3 5^3 7^3 11^2 13 \cdot 19 \cdot 31$	2772	$2^2 3^2 7 \cdot 11$	17,23,41,43,71,83,131,167,197,251,263,307,461, 503,1847	100	183	(6)
575168392726200	$2^3 3^2 5^2 7^2 13 \cdot 19 \cdot 31^2 83 \cdot 331$	1116	$2^2 3^2 31$	11,17,23,61,71,557,743	32	45	(1)
666684905040000	$2^7 3^7 5^4 7 \cdot 11 \cdot 17 \cdot 41 \cdot 71$	2000	$2^4 5^3$	19,31,79,199,499,1999	58	61	(1)
685877496890040	$2^3 3^2 5 \cdot 7 \cdot 13^2 31^2 61 \cdot 83 \cdot 331$	806	$2 \cdot 13 \cdot 31$		1	1	
758961158661216	$2^5 3^3 7^2 19^4 151 \cdot 911$	399	$3 \cdot 7 \cdot 19$	5,13,37,41,113,797	23	41	(5)
779473415243250	$2 \cdot 3^7 5^3 7^2 13^3 17 \cdot 19 \cdot 41$	1521	$3^2 13^2$	233,337,1013,3041	7	7	
805018217203200	$2^9 3 \cdot 5^2 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31^3 37$	1550	$2 \cdot 5^2 31$	61,619,1549	7	7	
812193794048000	$2^{14} 5^3 7^3 13 \cdot 19 \cdot 31 \cdot 151$	1120	$2^5 5 \cdot 7$	3,79,139,223,2239	129	> 1900	
822207921882984	$2^3 3^2 7^2 13 \cdot 19^4 151 \cdot 911$	399	$3 \cdot 7 \cdot 19$	5,37,41,113,797	14	29	(2)
847616578754400	$2^5 3^2 5^2 7^3 13 \cdot 31^2 83 \cdot 331$	1116	$2^2 3^2 31$	11,17,23,61,71,557,743	32	45	(1)

Appendix B

Harmonic numbers less than 10^{12}

n		$H(n)$	n		$H(n)$
1		1	4713984	$2^9 3^3 11 \cdot 31$	48
6	$2 \cdot 3$	2	4754880	$2^6 3^2 5 \cdot 13 \cdot 127$	45
28	$2^2 7$	3	5772200	$2^3 5^2 7^2 19 \cdot 31$	49
140	$2^2 5 \cdot 7$	5	6051500	$2^2 5^3 7^2 13 \cdot 19$	50
270	$2 \cdot 3^3 5$	6	8506400	$2^5 5^2 7^3 31$	49
496	$2^4 31$	5	8872200	$2^3 3^3 5^2 31 \cdot 53$	53
672	$2^5 3 \cdot 7$	8	11981970	$2 \cdot 3^2 5 \cdot 7^2 11 \cdot 13 \cdot 19$	77
1638	$2 \cdot 3^2 7 \cdot 13$	9	14303520	$2^5 3^3 5 \cdot 7 \cdot 11 \cdot 43$	86
2970	$2 \cdot 3^3 5 \cdot 11$	11	15495480	$2^3 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 43$	86
6200	$2^3 5^2 31$	10	16166592	$2^6 3^2 13 \cdot 17 \cdot 127$	51
8128	$2^6 127$	7	17428320	$2^5 3^2 5 \cdot 7^2 13 \cdot 19$	96
8190	$2 \cdot 3^2 5 \cdot 7 \cdot 13$	15	18154500	$2^2 3 \cdot 5^3 7^2 13 \cdot 19$	75
18600	$2^3 3 \cdot 5^2 31$	15	23088800	$2^5 5^2 7^2 19 \cdot 31$	70
18620	$2^2 5 \cdot 7^2 19$	14	23569920	$2^9 3^3 5 \cdot 11 \cdot 31$	80
27846	$2 \cdot 3^2 7 \cdot 13 \cdot 17$	17	23963940	$2^2 3^2 5 \cdot 7^2 11 \cdot 13 \cdot 19$	99
30240	$2^5 3^3 5 \cdot 7$	24	27027000	$2^3 3^3 5^3 7 \cdot 11 \cdot 13$	110
32760	$2^3 3^2 5 \cdot 7 \cdot 13$	24	29410290	$2 \cdot 3^5 5 \cdot 7^2 13 \cdot 19$	81
55860	$2^2 3 \cdot 5 \cdot 7^2 19$	21	32997888	$2^9 3^3 7 \cdot 11 \cdot 31$	84
105664	$2^6 13 \cdot 127$	13	33550336	$2^{12} 8191$	13
117800	$2^3 5^2 19 \cdot 31$	19	37035180	$2^2 3^2 5 \cdot 7^2 13 \cdot 17 \cdot 19$	102
167400	$2^3 3^3 5^2 31$	27	44660070	$2 \cdot 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 41$	82
173600	$2^5 5^2 7 \cdot 31$	25	45532800	$2^7 3^3 5^2 17 \cdot 31$	96
237510	$2 \cdot 3^2 5 \cdot 7 \cdot 13 \cdot 29$	29	46683000	$2^3 3^3 5^3 7 \cdot 13 \cdot 19$	114
242060	$2^2 5 \cdot 7^2 13 \cdot 19$	26	50401728	$2^6 3^2 13 \cdot 53 \cdot 127$	53
332640	$2^5 3^3 5 \cdot 7 \cdot 11$	44	52141320	$2^3 3^4 5 \cdot 7 \cdot 11^2 19$	108
360360	$2^3 3^2 5 \cdot 7 \cdot 11 \cdot 13$	44	56511000	$2^3 3^3 5^3 7 \cdot 13 \cdot 23$	115
539400	$2^3 3 \cdot 5^2 29 \cdot 31$	29	69266400	$2^5 3 \cdot 5^2 7^2 19 \cdot 31$	105
695520	$2^5 3^3 5 \cdot 7 \cdot 23$	46	71253000	$2^3 3^3 5^3 7 \cdot 13 \cdot 29$	116
726180	$2^2 3 \cdot 5 \cdot 7^2 13 \cdot 19$	39	75038600	$2^3 5^2 7^2 13 \cdot 19 \cdot 31$	91
753480	$2^3 3^2 5 \cdot 7 \cdot 13 \cdot 23$	46	80832960	$2^6 3^2 5 \cdot 13 \cdot 17 \cdot 127$	85
950976	$2^6 3^2 13 \cdot 127$	27	81695250	$2 \cdot 3^3 5^3 7^2 13 \cdot 19$	105
1089270	$2 \cdot 3^2 5 \cdot 7^2 13 \cdot 19$	42	90409410	$2 \cdot 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 83$	83
1421280	$2^5 3^3 5 \cdot 7 \cdot 47$	47	108421632	$2^9 3^3 11 \cdot 23 \cdot 31$	92
1539720	$2^3 3^2 5 \cdot 7 \cdot 13 \cdot 47$	47	110583200	$2^5 5^2 7^3 13 \cdot 31$	91
2178540	$2^2 3^2 5 \cdot 7^2 13 \cdot 19$	54	115048440	$2^3 3^2 5 \cdot 13^2 31 \cdot 61$	78
2229500	$2^2 5^3 7^3 13$	35	115462620	$2^2 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 53$	106
2290260	$2^2 3 \cdot 5 \cdot 7^2 19 \cdot 41$	41	137891520	$2^6 3^2 5 \cdot 13 \cdot 29 \cdot 127$	87
2457000	$2^3 3^3 5^3 7 \cdot 13$	60	142990848	$2^9 3^2 7 \cdot 11 \cdot 13 \cdot 31$	120
2845800	$2^3 3^3 5^2 17 \cdot 31$	51	144963000	$2^3 3^3 5^3 7 \cdot 13 \cdot 59$	118
4358600	$2^3 5^2 19 \cdot 31 \cdot 37$	37	163390500	$2^2 3^3 5^3 7^2 13 \cdot 19$	135

n		$H(n)$	n		$H(n)$
164989440	$2^9 3^3 5 \cdot 7 \cdot 11 \cdot 31$	140	1352913408	$2^9 3^3 7 \cdot 11 \cdot 31 \cdot 41$	164
191711520	$2^5 3^2 5 \cdot 7^2 11 \cdot 13 \cdot 19$	176	1379454720	$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$	144
221557248	$2^9 3^3 11 \cdot 31 \cdot 47$	94	1381161600	$2^7 3^2 5^2 7 \cdot 13 \cdot 17 \cdot 31$	240
233103780	$2^2 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 107$	107	1509765120	$2^{12} 3^2 5 \cdot 8191$	45
255428096	$2^9 7 \cdot 11^2 19 \cdot 31$	88	1558745370	$2 \cdot 3^5 5 \cdot 7^2 13 \cdot 19 \cdot 53$	159
287425800	$2^3 3^3 5^2 17 \cdot 31 \cdot 101$	101	1630964808	$2^3 3^4 11^3 31 \cdot 61$	99
300154400	$2^5 5^2 7^2 13 \cdot 19 \cdot 31$	130	1632825792	$2^6 3^2 13 \cdot 17 \cdot 101 \cdot 127$	101
301953024	$2^{12} 3^2 8191$	27	1727271000	$2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 37$	222
318177800	$2^3 5^2 19 \cdot 31 \cdot 37 \cdot 73$	73	1862023680	$2^9 3^3 5 \cdot 11 \cdot 31 \cdot 79$	158
318729600	$2^7 3^3 5^2 7 \cdot 17 \cdot 31$	168	1867650048	$2^{10} 3^4 11 \cdot 23 \cdot 89$	128
326781000	$2^3 3^3 5^3 7^2 13 \cdot 19$	168	2008725600	$2^5 3 \cdot 5^2 7^2 19 \cdot 29 \cdot 31$	203
400851360	$2^5 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 23$	184	2140041600	$2^7 3^3 5^2 17 \cdot 31 \cdot 47$	188
407386980	$2^2 3^2 5 \cdot 7^2 11 \cdot 13 \cdot 17 \cdot 19$	187	2144862720	$2^9 3^3 5 \cdot 7 \cdot 11 \cdot 13 \cdot 31$	260
423184320	$2^6 3^2 5 \cdot 13 \cdot 89 \cdot 127$	89	2369162250	$2 \cdot 3^3 5^3 7^2 13 \cdot 19 \cdot 29$	203
428972544	$2^9 3^3 7 \cdot 11 \cdot 13 \cdot 31$	156	2481357060	$2^2 3^2 5 \cdot 7^2 13 \cdot 17 \cdot 19 \cdot 67$	201
447828480	$2^9 3^3 5 \cdot 11 \cdot 19 \cdot 31$	152	2701389600	$2^5 3^2 5^2 7^2 13 \cdot 19 \cdot 31$	270
459818240	$2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$	96	2705020500	$2^2 3 \cdot 5^3 7^2 13 \cdot 19 \cdot 149$	149
481572000	$2^5 3^3 5^3 7^3 13$	168	2716826112	$2^9 3^2 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31$	228
499974930	$2 \cdot 3^5 5 \cdot 7^2 13 \cdot 17 \cdot 19$	153	2738824704	$2^9 3^3 7 \cdot 11 \cdot 31 \cdot 83$	166
500860800	$2^7 3^3 5^2 11 \cdot 17 \cdot 31$	176	2763489960	$2^3 3^4 5 \cdot 7 \cdot 11^2 19 \cdot 53$	212
513513000	$2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 19$	209	2777638500	$2^2 3^3 5^3 7^2 13 \cdot 17 \cdot 19$	255
526480500	$2^2 3 \cdot 5^3 7^2 13 \cdot 19 \cdot 29$	145	2839922400	$2^5 3 \cdot 5^2 7^2 19 \cdot 31 \cdot 41$	205
540277920	$2^5 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 31$	186	2876211000	$2^3 3^2 5^3 13^2 31 \cdot 61$	150
559903400	$2^3 5^2 7^2 19 \cdot 31 \cdot 97$	97	2945943000	$2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 109$	218
623397600	$2^5 3^3 5^2 7^2 19 \cdot 31$	189	3134799360	$2^9 3^3 5 \cdot 7 \cdot 11 \cdot 19 \cdot 31$	266
644271264	$2^5 3^2 7 \cdot 13^2 31 \cdot 61$	117	3209343200	$2^5 5^2 7^2 19 \cdot 31 \cdot 139$	139
675347400	$2^3 3^2 5^2 7^2 13 \cdot 19 \cdot 31$	189	3221356320	$2^5 3^2 5 \cdot 7 \cdot 13^2 31 \cdot 61$	195
714954240	$2^9 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 31$	200	3288789504	$2^9 3^2 7 \cdot 11 \cdot 13 \cdot 23 \cdot 31$	230
758951424	$2^9 3^3 7 \cdot 11 \cdot 23 \cdot 31$	161	3328809120	$2^5 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 191$	191
766284288	$2^9 3 \cdot 7 \cdot 11^2 19 \cdot 31$	132	3349505250	$2 \cdot 3^3 5^3 7^2 13 \cdot 19 \cdot 41$	205
819131040	$2^5 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 47$	188	3506025600	$2^7 3^3 5^2 7 \cdot 11 \cdot 17 \cdot 31$	308
825120800	$2^5 5^2 7^3 31 \cdot 97$	97	3594591000	$2^3 3^3 5^3 7^2 11 \cdot 13 \cdot 19$	308
886402440	$2^3 3^4 5 \cdot 7 \cdot 11^2 17 \cdot 19$	204	3702033720	$2^3 3^4 5 \cdot 7 \cdot 11^2 19 \cdot 71$	213
900463200	$2^5 3 \cdot 5^2 7^2 13 \cdot 19 \cdot 31$	195	3740553180	$2^2 3^2 5 \cdot 7^2 13 \cdot 17 \cdot 19 \cdot 101$	202
995248800	$2^5 3^2 5^2 7^3 13 \cdot 31$	189	3831421440	$2^9 3 \cdot 5 \cdot 7 \cdot 11^2 19 \cdot 31$	220
1047254400	$2^7 3^3 5^2 17 \cdot 23 \cdot 31$	184	4143484800	$2^7 3^3 5^2 7 \cdot 13 \cdot 17 \cdot 31$	312
1162161000	$2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 43$	215	4146734592	$2^9 3^2 7 \cdot 11 \cdot 13 \cdot 29 \cdot 31$	232
1199250360	$2^3 3^4 5 \cdot 7 \cdot 11^2 19 \cdot 23$	207	4720896180	$2^2 3^2 5 \cdot 7^2 11 \cdot 13 \cdot 19 \cdot 197$	197
1265532840	$2^3 3^2 5 \cdot 11 \cdot 13^2 31 \cdot 61$	143	4738324500	$2^2 3^3 5^3 7^2 13 \cdot 19 \cdot 29$	261
1307124000	$2^5 3^3 5^3 7^2 13 \cdot 19$	240	5058000640	$2^8 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73$	176

n		$H(n)$	n		$H(n)$
5133201408	$2^{12}3^{2}17 \cdot 8191$	51	8154824040	$2^{3}3^{4}5 \cdot 11^{3}31 \cdot 61$	165
5275179000	$2^{3}3^{3}5^{3}7 \cdot 13 \cdot 19 \cdot 113$	226	8243595360	$2^{5}3^{2}5 \cdot 7^{2}11 \cdot 13 \cdot 19 \cdot 43$	344
5297292000	$2^{5}3^{3}5^{3}7^{3}11 \cdot 13$	308	8410907232	$2^{5}3^{2}7^{2}13 \cdot 19^{2}127$	171
5510647296	$2^{9}3^{3}7 \cdot 11 \cdot 31 \cdot 167$	167	8436460032	$2^{9}3^{2}7 \cdot 11 \cdot 13 \cdot 31 \cdot 59$	236
5579121240	$2^{3}3^{4}5 \cdot 7 \cdot 11^{2}19 \cdot 107$	214	8589869056	$2^{16}131071$	17
5943057120	$2^{5}3^{2}5 \cdot 7^{2}11 \cdot 13 \cdot 19 \cdot 31$	341	8628633000	$2^{3}3^{3}5^{3}13^{2}31 \cdot 61$	195
6720569856	$2^{9}3^{2}7 \cdot 11 \cdot 13 \cdot 31 \cdot 47$	235	8659696500	$2^{2}3^{3}5^{3}7^{2}13 \cdot 19 \cdot 53$	265
7279591410	$2 \cdot 3^{2}5 \cdot 7^{2}13 \cdot 19 \cdot 41 \cdot 163$	163	8696764800	$2^{7}3^{3}5^{2}17 \cdot 31 \cdot 191$	191
7330780800	$2^{7}3^{3}5^{2}7 \cdot 17 \cdot 23 \cdot 31$	322	8698459616	$2^{5}7^{2}11^{2}19^{2}127$	121
7515963000	$2^{3}3^{3}5^{3}7^{2}13 \cdot 19 \cdot 23$	322	9866368512	$2^{9}3^{3}7 \cdot 11 \cdot 13 \cdot 23 \cdot 31$	299
8104168800	$2^{5}3^{3}5^{2}7^{2}13 \cdot 19 \cdot 31$	351			

n		$H(n)$
10200236032	$2^{14}7 \cdot 19 \cdot 31 \cdot 151$	96
10575819520	$2^{8}5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73$	184
10597041000	$2^{3}3^{3}5^{3}7 \cdot 13 \cdot 19 \cdot 227$	227
10597759200	$2^{5}3^{3}5^{2}7^{2}17 \cdot 19 \cdot 31$	357
10952611488	$2^{5}3^{2}7 \cdot 13^{2}17 \cdot 31 \cdot 61$	221
10983408128	$2^{9}7 \cdot 11^{2}19 \cdot 31 \cdot 43$	172
11076156000	$2^{5}3^{3}5^{3}7^{3}13 \cdot 23$	322
11296276992	$2^{9}3^{2}7 \cdot 11 \cdot 13 \cdot 31 \cdot 79$	237
11480905800	$2^{3}3^{2}5^{2}7^{2}13 \cdot 17 \cdot 19 \cdot 31$	357
12941019000	$2^{3}3^{3}5^{3}7 \cdot 13 \cdot 23 \cdot 229$	229
13067913600	$2^{7}3^{3}5^{2}7 \cdot 17 \cdot 31 \cdot 41$	328
13073550336	$2^{10}3^{4}7 \cdot 11 \cdot 23 \cdot 89$	224
13398021000	$2^{3}3^{3}5^{3}7^{2}13 \cdot 19 \cdot 41$	328
13581986600	$2^{3}5^{2}7^{2}13 \cdot 19 \cdot 31 \cdot 181$	181
13584130560	$2^{9}3^{2}5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31$	380
13660770240	$2^{6}3^{2}5 \cdot 13^{3}17 \cdot 127$	169
14182439040	$2^{7}3^{4}5 \cdot 7 \cdot 11^{2}17 \cdot 19$	384
14254365440	$2^{8}5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73$	186
14378364000	$2^{5}3^{3}5^{3}7^{2}11 \cdot 13 \cdot 19$	440
14541754500	$2^{2}3^{3}5^{3}7^{2}13 \cdot 19 \cdot 89$	267
14980291200	$2^{7}3^{3}5^{2}7 \cdot 17 \cdot 31 \cdot 47$	329
15174001920	$2^{8}3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73$	264
15192777600	$2^{7}3^{2}5^{2}7 \cdot 11 \cdot 13 \cdot 17 \cdot 31$	440
15358707000	$2^{3}3^{3}5^{3}7^{2}13 \cdot 19 \cdot 47$	329
16003510272	$2^{12}3^{2}53 \cdot 8191$	53
16569653760	$2^{9}3^{3}5 \cdot 11 \cdot 19 \cdot 31 \cdot 37$	296
16919229600	$2^{5}3^{2}5^{2}7^{3}13 \cdot 17 \cdot 31$	357
17624538624	$2^{9}3 \cdot 7 \cdot 11^{2}19 \cdot 23 \cdot 31$	253
18999981000	$2^{3}3^{3}5^{3}7 \cdot 11 \cdot 13 \cdot 19 \cdot 37$	407
19017782784	$2^{9}3^{2}7^{2}11 \cdot 13 \cdot 19 \cdot 31$	336
19209881600	$2^{11}5^{2}7^{2}13 \cdot 19 \cdot 31$	256
19744452000	$2^{5}3^{3}5^{3}7^{3}13 \cdot 41$	328
20015559200	$2^{5}5^{2}7^{3}13 \cdot 31 \cdot 181$	181
20387256120	$2^{3}3^{4}5 \cdot 7 \cdot 11^{2}17 \cdot 19 \cdot 23$	391
21537014400	$2^{7}3^{3}5^{2}11 \cdot 17 \cdot 31 \cdot 43$	344
21611457280	$2^{8}5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73$	188
21943595520	$2^{9}3^{3}5 \cdot 7^{2}11 \cdot 19 \cdot 31$	392
22633884000	$2^{5}3^{3}5^{3}7^{3}13 \cdot 47$	329
22933532160	$2^{9}3^{3}5 \cdot 7 \cdot 11 \cdot 31 \cdot 139$	278
23450730240	$2^{8}3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 37 \cdot 73$	272

n		$H(n)$
23855232960	$2^6 3^2 5 \cdot 13 \cdot 29 \cdot 127 \cdot 173$	173
24362612820	$2^2 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 53 \cdot 211$	211
25559301600	$2^5 3^3 5^2 7^2 19 \cdot 31 \cdot 41$	369
25666007040	$2^{12} 3^2 5 \cdot 17 \cdot 8191$	85
26113432800	$2^5 3 \cdot 5^2 7^2 13 \cdot 19 \cdot 29 \cdot 31$	377
26242070400	$2^7 3^2 5^2 7 \cdot 13 \cdot 17 \cdot 19 \cdot 31$	456
26454556800	$2^7 3^3 5^2 7 \cdot 17 \cdot 31 \cdot 83$	332
27122823000	$2^3 3^3 5^3 7^2 13 \cdot 19 \cdot 83$	332
27689243400	$2^3 3^2 5^2 7^2 13 \cdot 19 \cdot 31 \cdot 41$	369
27726401736	$2^3 3^4 11^3 17 \cdot 31 \cdot 61$	187
29715285600	$2^5 3^2 5^2 7^2 11 \cdot 13 \cdot 19 \cdot 31$	495
30063852000	$2^5 3^3 5^3 7^2 13 \cdot 19 \cdot 23$	460
30600708096	$2^{14} 3 \cdot 7 \cdot 19 \cdot 31 \cdot 151$	144
31638321000	$2^3 3^2 5^3 11 \cdot 13^2 31 \cdot 61$	275
31727458560	$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73$	276
31766716800	$2^7 3^2 5^2 7 \cdot 13 \cdot 17 \cdot 23 \cdot 31$	460
32950224384	$2^9 3 \cdot 7 \cdot 11^2 19 \cdot 31 \cdot 43$	258
32956953120	$2^5 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 31 \cdot 61$	366
33040072800	$2^5 3^3 5^2 7^2 19 \cdot 31 \cdot 53$	371
34174812672	$2^9 3^2 7 \cdot 11 \cdot 13 \cdot 31 \cdot 239$	239
34482792960	$2^9 3^3 5 \cdot 7 \cdot 11^2 19 \cdot 31$	396
35032757760	$2^9 3^2 5 \cdot 7^3 11 \cdot 13 \cdot 31$	392
35793412200	$2^3 3^2 5^2 7^2 13 \cdot 19 \cdot 31 \cdot 53$	371
37906596000	$2^5 3^3 5^3 7^2 13 \cdot 19 \cdot 29$	464
39970476000	$2^5 3^3 5^3 7^3 13 \cdot 83$	332
40053686400	$2^7 3^2 5^2 7 \cdot 13 \cdot 17 \cdot 29 \cdot 31$	464
40520844000	$2^5 3^3 5^3 7^2 13 \cdot 19 \cdot 31$	465
40752391680	$2^9 3^3 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31$	494
40805200800	$2^5 3^2 5^2 7^3 13 \cdot 31 \cdot 41$	369
42054536160	$2^5 3^2 5 \cdot 7^2 13 \cdot 19^2 127$	285
42763096320	$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73$	279
43783188480	$2^{12} 3^2 5 \cdot 29 \cdot 8191$	87
43861478400	$2^{10} 3^3 5^2 23 \cdot 31 \cdot 89$	264
43952044500	$2^2 3^3 5^3 7^2 13 \cdot 19 \cdot 269$	269
44184172032	$2^9 3^3 7 \cdot 11 \cdot 13 \cdot 31 \cdot 103$	309
45578332800	$2^7 3^3 5^2 7 \cdot 11 \cdot 13 \cdot 17 \cdot 31$	572
45923623200	$2^5 3^2 5^2 7^2 13 \cdot 17 \cdot 19 \cdot 31$	510
50497467930	$2 \cdot 3^5 5 \cdot 7^2 13 \cdot 17 \cdot 19 \cdot 101$	303
51001180160	$2^{14} 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151$	160
52748186400	$2^5 3^2 5^2 7^3 13 \cdot 31 \cdot 53$	371

n		$H(n)$
53227843200	$2^7 3^3 5^2 7 \cdot 17 \cdot 31 \cdot 167$	334
53621568000	$2^9 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 31$	500
54572427000	$2^3 3^3 5^3 7^2 13 \cdot 19 \cdot 167$	334
54648009000	$2^3 3^2 5^3 13^2 19 \cdot 31 \cdot 61$	285
56481384960	$2^9 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 79$	395
57575890944	$2^{13} 3^2 11 \cdot 13 \cdot 43 \cdot 127$	192
57629644800	$2^{11} 3 \cdot 5^2 7^2 13 \cdot 19 \cdot 31$	384
57648181500	$2^2 3^2 5^3 7^3 13^3 17$	273
57897151488	$2^{10} 3^4 11 \cdot 23 \cdot 31 \cdot 89$	248
59388963480	$2^3 3^4 5 \cdot 7 \cdot 11^2 17 \cdot 19 \cdot 67$	402
61434828000	$2^5 3^3 5^3 7^2 13 \cdot 19 \cdot 47$	470
62487000576	$2^9 3^2 7 \cdot 11 \cdot 13 \cdot 19 \cdot 23 \cdot 31$	437
64834371840	$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73$	282
64914595200	$2^7 3^2 5^2 7 \cdot 13 \cdot 17 \cdot 31 \cdot 47$	470
66433720320	$2^{13} 3^3 5 \cdot 11 \cdot 43 \cdot 127$	224
67622100480	$2^9 3^3 5 \cdot 11 \cdot 19 \cdot 31 \cdot 151$	302
71271827200	$2^8 5^2 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73$	270
73924348400	$2^4 5^2 7 \cdot 31^2 83 \cdot 331$	125
77120316000	$2^5 3^3 5^3 7^2 13 \cdot 19 \cdot 59$	472
77924700000	$2^5 3^3 5^5 7^2 19 \cdot 31$	375
78340298400	$2^5 3^2 5^2 7^2 13 \cdot 19 \cdot 29 \cdot 31$	522
80422524000	$2^5 3^3 5^3 7^3 13 \cdot 167$	334
80533908000	$2^5 3^2 5^3 7 \cdot 13^2 31 \cdot 61$	375
80551516500	$2^2 3^3 5^3 7^2 13 \cdot 17 \cdot 19 \cdot 29$	493
81417705600	$2^7 3 \cdot 5^2 7 \cdot 11^2 17 \cdot 19 \cdot 31$	484
81488534400	$2^7 3^2 5^2 7 \cdot 13 \cdot 17 \cdot 31 \cdot 59$	472
83410119000	$2^3 3^2 5^3 13^2 29 \cdot 31 \cdot 61$	290
84418425000	$2^3 3^2 5^5 7^2 13 \cdot 19 \cdot 31$	375
87825283840	$2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73 \cdot 191$	191
89526646440	$2^3 3^4 5 \cdot 7 \cdot 11^2 17 \cdot 19 \cdot 101$	404
93419333280	$2^5 3^2 5 \cdot 7 \cdot 13^2 29 \cdot 31 \cdot 61$	377
95088913920	$2^9 3^2 5 \cdot 7^2 11 \cdot 13 \cdot 19 \cdot 31$	560
95300150400	$2^7 3^3 5^2 7 \cdot 13 \cdot 17 \cdot 23 \cdot 31$	598
97941285120	$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 71 \cdot 73$	284

n		$H(n)$
100383241728	$2^9 3\cdot 7\cdot 11^2 19\cdot 31\cdot 131$	262
100522566144	$2^9 3^2 7\cdot 11\cdot 13\cdot 19\cdot 31\cdot 37$	444
103262796000	$2^5 3^3 5^3 7^2 13\cdot 19\cdot 79$	474
108061356200	$2^3 5^2 7^2 19\cdot 31\cdot 97\cdot 193$	193
109111766400	$2^7 3^2 5^2 7\cdot 13\cdot 17\cdot 31\cdot 79$	474
109585986048	$2^9 3^7 7\cdot 11\cdot 31\cdot 41$	324
110886522600	$2^3 3\cdot 5^2 7\cdot 31^2 83\cdot 331$	155
112202596352	$2^{14} 7\cdot 11\cdot 19\cdot 31\cdot 151$	176
115987576320	$2^9 3^3 5\cdot 7\cdot 11\cdot 19\cdot 31\cdot 37$	518
123014892000	$2^5 3\cdot 5^3 7^3 11^2 13\cdot 19$	484
124406100000	$2^5 3^2 5^5 7^3 13\cdot 31$	375
126090783000	$2^3 3^3 5^3 7\cdot 13\cdot 19\cdot 37\cdot 73$	438
133410461184	$2^9 3^3 7\cdot 11\cdot 13\cdot 31\cdot 311$	311
134369095680	$2^{12} 3^2 5\cdot 89\cdot 8191$	89
137438691328	$2^{18} 524287$	19
137770869600	$2^5 3^3 5^2 7^2 13\cdot 17\cdot 19\cdot 31$	663
142275893760	$2^9 3^2 5\cdot 7\cdot 11\cdot 13\cdot 31\cdot 199$	398
142985422944	$2^5 3^2 7^2 13\cdot 17\cdot 19^2 127$	323
143173648800	$2^5 3^2 5^2 7^2 13\cdot 19\cdot 31\cdot 53$	530
147112449120	$2^5 3^2 5\cdot 7^2 13\cdot 19\cdot 23\cdot 367$	367
150115204512	$2^5 3^2 7\cdot 13^2 31\cdot 61\cdot 233$	233
150759100800	$2^7 3^3 5^2 7\cdot 11\cdot 17\cdot 31\cdot 43$	602
151955343540	$2^2 3^2 5\cdot 7^2 11\cdot 13\cdot 17\cdot 19\cdot 373$	373
153003540480	$2^{14} 3\cdot 5\cdot 7\cdot 19\cdot 31\cdot 151$	240
154567413000	$2^3 3^3 5^3 7^2 11\cdot 13\cdot 19\cdot 43$	602
156473635500	$2^2 3^2 5^3 7^2 13^3 17\cdot 19$	390
156798019840	$2^8 5\cdot 7\cdot 11\cdot 19\cdot 31\cdot 37\cdot 73$	341
159248314400	$2^5 5^2 7^3 31\cdot 97\cdot 193$	193
159381986400	$2^5 3^2 5^2 7^2 13\cdot 19\cdot 31\cdot 59$	531
164297299320	$2^3 3^4 5\cdot 7\cdot 11^2 19\cdot 23\cdot 137$	411
164751121920	$2^9 3\cdot 5\cdot 7\cdot 11^2 19\cdot 31\cdot 43$	430
169696449000	$2^3 3^2 5^3 13^2 31\cdot 59\cdot 61$	295
169956154368	$2^{10} 3^4 7\cdot 11\cdot 13\cdot 23\cdot 89$	416
183694492800	$2^7 3^2 5^2 7^2 13\cdot 17\cdot 19\cdot 31$	672
194743785600	$2^7 3^3 5^2 7\cdot 13\cdot 17\cdot 31\cdot 47$	611
201532767744	$2^9 3\cdot 7\cdot 11^2 19\cdot 31\cdot 263$	263
206166804480	$2^{11} 3^2 5\cdot 7\cdot 13^2 31\cdot 61$	384
213815481600	$2^8 3\cdot 5^2 7\cdot 19\cdot 31\cdot 37\cdot 73$	405
217494027520	$2^8 5\cdot 7\cdot 11\cdot 19\cdot 37\cdot 43\cdot 73$	344
220524885504	$2^9 3^3 7\cdot 11\cdot 31\cdot 41\cdot 163$	326

n		$H(n)$
220920860160	$2^9 3^3 5 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 103$	515
221908282624	$2^8 7 \cdot 19^2 37 \cdot 73 \cdot 127$	171
227783556000	$2^5 3^3 5^3 7^3 11 \cdot 13 \cdot 43$	602
234605428736	$2^{14} 7 \cdot 19 \cdot 23 \cdot 31 \cdot 151$	184
236489897160	$2^3 3^4 5 \cdot 11^3 29 \cdot 31 \cdot 61$	319
237191556096	$2^{10} 3^4 11 \cdot 23 \cdot 89 \cdot 127$	254
240423674400	$2^5 3^2 5^2 7^2 13 \cdot 19 \cdot 31 \cdot 89$	534
250230357000	$2^3 3^3 5^3 13^2 29 \cdot 31 \cdot 61$	377
271309925250	$2 \cdot 3^7 5^3 7^2 13 \cdot 19 \cdot 41$	405
280541488500	$2^2 3^3 5^3 7^2 13 \cdot 17 \cdot 19 \cdot 101$	505
285266741760	$2^9 3^3 5 \cdot 7^2 11 \cdot 13 \cdot 19 \cdot 31$	728
287879454720	$2^{13} 3^2 5 \cdot 11 \cdot 13 \cdot 43 \cdot 127$	320
288662774400	$2^7 3^2 5^2 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31$	836
289048687200	$2^5 3^2 5^2 7^2 13 \cdot 19 \cdot 31 \cdot 107$	535
292337717760	$2^9 3^3 5 \cdot 11 \cdot 31 \cdot 79 \cdot 157$	314
307001350656	$2^9 3^2 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 113$	452
307030348800	$2^{10} 3^3 5^2 7 \cdot 23 \cdot 31 \cdot 89$	462
311203567584	$2^5 3^2 7^2 13 \cdot 19^2 37 \cdot 127$	333
312402636000	$2^5 3^3 5^3 7^2 13 \cdot 19 \cdot 239$	478
321300067176	$2^3 3^4 11^3 31 \cdot 61 \cdot 197$	197
326196097920	$2^7 3^4 5 \cdot 7 \cdot 11^2 17 \cdot 19 \cdot 23$	736
330097622400	$2^7 3^2 5^2 7 \cdot 13 \cdot 17 \cdot 31 \cdot 239$	478
336607789056	$2^{14} 3 \cdot 7 \cdot 11 \cdot 19 \cdot 31 \cdot 151$	264
341519256000	$2^6 3^2 5^3 13^3 17 \cdot 127$	325
349002044160	$2^8 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 37 \cdot 73$	506
350280184800	$2^5 3 \cdot 5^2 7^2 13 \cdot 19 \cdot 31 \cdot 389$	389
362526484320	$2^5 3^2 5 \cdot 7^2 11 \cdot 13 \cdot 19 \cdot 31 \cdot 61$	671
384342364800	$2^7 3^3 5^2 17 \cdot 23 \cdot 31 \cdot 367$	367
403031236608	$2^{13} 3^2 7 \cdot 11 \cdot 13 \cdot 43 \cdot 127$	336
405280060416	$2^{10} 3^4 7 \cdot 11 \cdot 23 \cdot 31 \cdot 89$	434
410240742912	$2^9 3^2 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 151$	453
417624936960	$2^9 3 \cdot 5 \cdot 7 \cdot 11^2 19 \cdot 31 \cdot 109$	436
426778934400	$2^7 3^3 5^2 7 \cdot 13 \cdot 17 \cdot 31 \cdot 103$	618
428440390560	$2^5 3^2 5 \cdot 7^2 13^2 19 \cdot 31 \cdot 61$	546
428555439000	$2^3 3^2 5^3 13^2 31 \cdot 61 \cdot 149$	298
429520946400	$2^5 3^3 5^2 7^2 13 \cdot 19 \cdot 31 \cdot 53$	689
434508127200	$2^5 3^3 5^2 7^2 17 \cdot 19 \cdot 31 \cdot 41$	697
437409004032	$2^9 3^2 7^2 11 \cdot 13 \cdot 19 \cdot 23 \cdot 31$	644
439655610240	$2^7 3^4 5 \cdot 7 \cdot 11^2 17 \cdot 19 \cdot 31$	744
443622427776	$2^7 3^4 11^3 17 \cdot 31 \cdot 61$	352

n		$H(n)$
465036042240	$2^{13}3^{3}5\cdot7\cdot11\cdot43\cdot127$	392
469420906500	$2^{2}3^{3}5^{3}7^{2}13^{3}17\cdot19$	507
470717137800	$2^{3}3^{2}5^{2}7^{2}13\cdot17\cdot19\cdot31\cdot41$	697
479411093504	$2^{14}7\cdot19\cdot31\cdot47\cdot151$	188
482476262400	$2^{10}3^{3}5^{2}11\cdot23\cdot31\cdot89$	484
483548738400	$2^{5}3^{2}5^{2}7^{2}13\cdot19\cdot31\cdot179$	537
494122282290	$2\cdot3^{5}5\cdot7^{2}13\cdot19\cdot53\cdot317$	317
502612830720	$2^{9}3^{2}5\cdot7\cdot11\cdot13\cdot19\cdot31\cdot37$	740
505159855200	$2^{5}3^{2}5^{2}7^{2}11\cdot13\cdot17\cdot19\cdot31$	935
513480135168	$2^{9}3^{5}7^{2}11\cdot13\cdot19\cdot31$	648
518453342208	$2^{12}3^{2}17\cdot101\cdot8191$	101
520212037632	$2^{14}3\cdot7\cdot17\cdot19\cdot31\cdot151$	272
547929930240	$2^{9}3^{7}5\cdot7\cdot11\cdot31\cdot41$	540
583096381560	$2^{3}3^{4}5\cdot7\cdot11^{2}19\cdot53\cdot211$	422
586207480320	$2^{9}3^{3}5\cdot7\cdot11^{2}17\cdot19\cdot31$	748
603567619200	$2^{7}3^{2}5^{2}7\cdot13\cdot17\cdot19\cdot23\cdot31$	874
616719527424	$2^{9}3^{2}7\cdot11\cdot13\cdot19\cdot31\cdot227$	454
618269652000	$2^{5}3^{3}5^{3}7^{2}11\cdot13\cdot19\cdot43$	860
626112396000	$2^{5}3^{3}5^{3}7^{2}13\cdot19\cdot479$	479
633926092800	$2^{11}3\cdot5^{2}7^{2}11\cdot13\cdot19\cdot31$	704
652482082560	$2^{8}3\cdot5\cdot7\cdot11\cdot19\cdot37\cdot43\cdot73$	516
653289436800	$2^{7}3^{2}5^{2}7\cdot11\cdot13\cdot17\cdot31\cdot43$	860
661576406400	$2^{7}3^{2}5^{2}7\cdot13\cdot17\cdot31\cdot479$	479
666574634880	$2^{7}3^{4}5\cdot7\cdot11^{2}17\cdot19\cdot47$	752
677701763200	$2^{7}5^{2}7\cdot11\cdot17^{2}31\cdot307$	340
693688413600	$2^{5}3^{2}5^{2}7^{3}13\cdot17\cdot31\cdot41$	697
703816286208	$2^{14}3\cdot7\cdot19\cdot23\cdot31\cdot151$	276
704575228896	$2^{5}3^{4}7^{2}11^{2}19^{2}127$	405
713178090240	$2^{8}3\cdot5\cdot7\cdot11\cdot19\cdot37\cdot47\cdot73$	517
726673802400	$2^{5}3^{2}5^{2}7^{2}13\cdot19\cdot31\cdot269$	538
726972637440	$2^{8}3\cdot5\cdot7\cdot17\cdot19\cdot31\cdot37\cdot73$	527
753132796416	$2^{9}3^{2}7\cdot11\cdot13\cdot23\cdot31\cdot229$	458
765181053000	$2^{3}3^{3}5^{3}7\cdot13\cdot19\cdot37\cdot443$	443
779729094144	$2^{9}3^{2}7^{2}11\cdot13\cdot19\cdot31\cdot41$	656
783990099200	$2^{8}5^{2}7\cdot11\cdot19\cdot31\cdot37\cdot73$	495
793104238080	$2^{9}3^{3}5\cdot7\cdot11^{2}19\cdot23\cdot31$	759
819730138500	$2^{2}3^{3}5^{3}7^{2}13\cdot19\cdot29\cdot173$	519
830350521000	$2^{3}3^{4}5^{3}7^{3}11^{2}13\cdot19$	756
861743282400	$2^{5}3^{2}5^{2}7^{2}11\cdot13\cdot19\cdot29\cdot31$	957
863638364160	$2^{13}3^{3}5\cdot11\cdot13\cdot43\cdot127$	416

n		$H(n)$
869516291840	$2^8 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73$	366
888875820360	$2^3 3^4 5 \cdot 11^3 31 \cdot 61 \cdot 109$	327
888988066400	$2^5 5^2 7^2 19 \cdot 31 \cdot 139 \cdot 277$	277
893835790848	$2^9 3^2 7^2 11 \cdot 13 \cdot 19 \cdot 31 \cdot 47$	658
906550977024	$2^9 3^3 7 \cdot 11 \cdot 31 \cdot 83 \cdot 331$	331
945884459520	$2^9 3^5 5 \cdot 7^3 11 \cdot 13 \cdot 31$	756
950432517216	$2^5 3^2 7^2 13 \cdot 19^2 113 \cdot 127$	339
970956604800	$2^7 3^2 5^2 7 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 37$	888
995024181060	$2^2 3^2 5 \cdot 7^2 13 \cdot 17 \cdot 19 \cdot 67 \cdot 401$	401
997978703400	$2^3 3^3 5^2 7 \cdot 31^2 83 \cdot 331$	279

Appendix C

Computations for finding all even harmonic seeds less than 10^{12}

Most TROUBLEsome cases (except those for $2^a \parallel N,\ a < 4$) are explicitly eliminated as the result of merging the outputs from separate cases with that of the main output.

The text of this proof is available on the companion Compact Disc as the equivalent files `\Postscript\Evn12all.ps` and `\PDF\Evn12all.pdf`.

Appendix D

Computations for proof of non-existence of odd harmonic numbers less than 10^{12}

The text of this proof is available on the companion Compact Disc as the equivalent files `\Postscript\Odd12all.ps` and `\PDF\Odd12all.pdf`.

Appendix E

0–1 linear programming model for multiperfect numbers

The particular syntax is that used by *LINGO* (a PC-based linear programming package). In particular the inequality condition $\leq$ is represented as $<$. The condition $@BIN(x3\_2)$ declares the variable $x3\_2$ (corresponding to the possible component 3^2) to be a binary, or 0–1, variable.

This syntax is very similar to that accepted by *lp_solve*, a freely available, academic linear programming package. Indeed *lp_solve* was the preferred package when the upward scalability of the model was first tested, particularly since it has an option (*-i*) to display intermediate feasible solutions.

The changes required to use the constraint logic programming package *opbdp* involved transforming the few remaining integer variables, *tot_p*, into sets of binary variables and ensuring that the constraints had constants on the right hand side.

```
! Simplified 0-1 LP model for Multiperfect Numbers (MPFNs);
! MUST specify factors (as constants) for a chosen 2^n, 0<=n<=19;
! n=0 corresponds to odd perfect;
! e.g. 2^19 = 3.5^2.11.31.41;
!       2^17 = 3^3.7.19.73;
! and shown here for;
!       2^14 = 2^3.7.19.31 (after chaining);
! Last modified by R.M. Sorli on 23 July 2001;

model

max =
   5*tot_2+8*tot_3+12*tot_5+14*tot_7+18*tot_11;

x3_1+x3_2+x3_3+x3_4+x3_5+x3_6+x3_7+x3_8+x3_9<1;
x5_1+x5_2+x5_3+x5_4+x5_5+x5_6<1;    x7_1+x7_2+x7_3+x7_4+x7_5<1;
x11_1+x11_2+x11_3<1;  x13_1+x13_2+x13_3<1;    x17_1+x17_2+x17_3<1;
x19_1+x19_2+x19_3<1;  x23_1+x23_2<1;  x29_1+x29_2+x29_3<1;
x31_1+x31_2+x31_3+x31_4<1;  x37_1+x37_2+x37_3+x37_4<1;
x41_1+x41_2+x41_3+x41_4<1;  x43_1+x43_2+x43_3+x43_4<1;
x53_1+x53_2<1;  x61_1+x61_2+x61_3+x61_4<1;    x67_1+x67_2+x67_3<1;
x73_1+x73_2<1;  x79_1+x79_2<1;  x89_1+x89_2<1;    x127_1+x127_2<1;
x131_1+x131_2<1;  x137_1+x137_2<1;  x257_1+x257_2<1;

tot_2 = 3-14
   +2*x3_1+3*x3_3+2*x3_5+3*x3_6+4*x3_7+3*x3_8+2*x3_9
   +x5_1+2*x5_3+3*x5_4+x5_5+2*x5_6  +3*x7_1+4*x7_3+3*x7_4+3*x7_5
   +2*x11_1+3*x11_3  +x13_1+2*x13_3  +x17_1+2*x17_2+2*x17_3
   +2*x19_1+4*x19_3  +3*x23_1  +x29_1+5*x29_3
   +5*x31_1+4*x31_2+6*x31_3+4*x31_4  +x37_1+2*x37_3+4*x37_4
   +x41_1+6*x41_2+2*x41_3+x41_4  +2*x43_1+3*x43_2+3*x43_3+10*x43_4
   +x53_1+x53_2  +x61_1+x61_2+3*x61_3+5*x61_4
   +2*x67_1+4*x67_3  +x73_1+x73_2  +4*x79_1  +x89_1+7*x89_2
   +7*x127_1+6*x127_2  +2*x131_1+8*x131_2  +x137_1  +x257_1+6*x257_2;
tot_3 =
   -x3_1-2*x3_2-3*x3_3-4*x3_4-5*x3_5-6*x3_6-7*x3_7-8*x3_8-9*x3_9
   +x5_1+x5_3+2*x5_4+2*x5_5  +x7_2+3*x7_4+x7_5  +x11_1+x11_3
   +x13_2  +2*x17_1+2*x17_3  +x19_2  +x23_1  +x29_1+x29_3
   +2*x31_2+2*x31_4  +x37_2+x37_4  +x41_1+3*x41_2+x41_3+x41_4
   +x43_2+2*x43_4  +3*x53_1  +x61_2+3*x61_4  +x67_2+2*x67_3
   +x73_2  +x79_2  +2*x89_1+2*x89_2  +x127_2  +x131_1+x131_2
   +x137_1  +x257_1;
```

```
tot_5 =
   -x5_1-2*x5_2-3*x5_3-4*x5_4-5*x5_5-6*x5_6
   +x3_3+x3_7+x3_8  +2*x7_3  +x13_3  +x17_3  +x19_1+x19_3
   +x29_1+x29_3  +x31_4  +x37_3  +x41_4  +2*x43_3+x43_4  +x53_2
   +3*x61_4  +3*x67_3  +x79_1  +x89_1  +x127_2;
tot_7 = 1
   -x7_1-2*x7_2-3*x7_3-4*x7_4-5*x7_5
   +x3_5  +x5_5  +x11_2  +x13_1+x13_3  +x17_2  +x19_3  +x23_2
   +x31_2  +x37_2  +x41_1+x41_3  +x43_4  +x53_2  +2*x61_2+2*x61_3
   +2*x67_2  +2*x79_2  +x89_2  +x137_2;
tot_11 =
   -x11_1-2*x11_2-3*x11_3
   +2*x3_4+2*x3_9  +x5_4  +x17_2  +3*x31_4  +x37_4  +x41_4
   +x43_1+x43_3+x43_4  +x131_1;
0 =
   -x13_1-2*x13_2-3*x13_3
   +x3_2+x3_5+x3_8  +x5_3  +x7_4  +x19_3  +x29_2  +x31_3  +x61_2;
0 =
   -x17_1-2*x17_2-3*x17_3
   +x13_3  +x67_1+x67_3  +x73_2  +x127_2  +x257_2;
0 = 1
   -x19_1-2*x19_2-3*x19_3
   +x3_8  +x5_6  +x7_2+x7_5  +x11_2  +x37_1+x37_3  +x61_3;
0 =
   -x23_1-2*x23_2    +x131_2  +x137_1;
0 =
   -x29_1-2*x29_2-3*x29_3    +x17_3  +2*x41_3;
0 = 1
   -x31_1-2*x31_2-3*x31_3-4*x31_4    +x5_2+x5_5  +x61_1+x61_3  +x67_2;
0 =
   -x37_1-2*x37_2-3*x37_3-4*x37_4    +x31_3  +x43_3  +x73_1  +x137_2;
0 =
   -x41_1-2*x41_2-3*x41_3-4*x41_4    +x3_7  +x37_4  +x53_2;
0 =
   -x43_1-2*x43_2-3*x43_3-4*x43_4    +x7_5  +x79_2  +x257_1;
0 =
   -x53_1-2*x53_2    +x29_3  +x73_2;
0 =
   -x61_1-2*x61_2-3*x61_3-4*x61_4    +x3_9  +x11_3  +x13_2  +x257_2;
0 =
   -x67_1-2*x67_2-3*x67_3    +x29_2  +x37_2  +x41_4;
0 =
```

```
   -x73_1-2*x73_2    +x137_2;
0 =
   -x79_1-2*x79_2    +x23_2  +x43_2;
0 =
   -x89_1-2*x89_2    +x37_4;
0 =
   -x127_1-2*x127_2  +x19_2;
0 =
   -x131_1-2*x131_2  +x41_4  +x61_4;
0 =
   -x137_1-2*x137_2  +x3_6  +x37_3;
0 =
   -x257_1-2*x257_2  +x5_6;

@BIN(x3_1); @BIN(x3_2); @BIN(x3_3); @BIN(x3_4); @BIN(x3_5); @BIN(x3_6);
   @BIN(x3_7);@BIN(x3_8);@BIN(x3_9);
@BIN(x5_1); @BIN(x5_2); @BIN(x5_3); @BIN(x5_4); @BIN(x5_5); @BIN(x5_6);
@BIN(x7_1); @BIN(x7_2); @BIN(x7_3); @BIN(x7_4); @BIN(x7_5);
@BIN(x11_1); @BIN(x11_2); @BIN(x11_3);
@BIN(x13_1); @BIN(x13_2); @BIN(x13_3);
@BIN(x17_1); @BIN(x17_2); @BIN(x17_3);
@BIN(x19_1); @BIN(x19_2); @BIN(x19_3);
@BIN(x23_1); @BIN(x23_2);
@BIN(x29_1); @BIN(x29_2); @BIN(x29_3);
@BIN(x31_1); @BIN(x31_2); @BIN(x31_3); @BIN(x31_4);
@BIN(x37_1); @BIN(x37_2); @BIN(x37_3); @BIN(x37_4);
@BIN(x41_1); @BIN(x41_2); @BIN(x41_3); @BIN(x41_4);
@BIN(x43_1); @BIN(x43_2); @BIN(x43_3); @BIN(x43_4);
@BIN(x53_1); @BIN(x53_2);
@BIN(x61_1); @BIN(x61_2); @BIN(x61_3); @BIN(x61_4);
@BIN(x67_1); @BIN(x67_2); @BIN(x67_3);
@BIN(x73_1); @BIN(x73_2);
@BIN(x79_1); @BIN(x79_2);
@BIN(x89_1); @BIN(x89_2);
@BIN(x127_1); @BIN(x127_2);
@BIN(x131_1); @BIN(x131_2);
@BIN(x137_1); @BIN(x137_2);
@BIN(x257_1); @BIN(x257_2);

end
```

Appendix F

Computations for proof of non-existence of odd triperfect numbers less than 10^{128}

The text of this proof is available on the companion Compact Disc as the equivalent files `\Postscript\Otp128all.ps` and `\PDF\Otp128all.pdf`.

Appendix G

Proof of $[\,3 : 5(1), 2(1), 1(*)\,]$ for odd perfect numbers

- $3^2 \Rightarrow 13$ [1]
  - $13^1 \Rightarrow 7$ [2]
    - $7^2 \Rightarrow 3 \cdot 19$
      - $19^2 \Rightarrow 3 \cdot 127$
        - $127^2 \Rightarrow 3 \cdot 5419$ xs=3 [3]
        - $127^4 \Rightarrow 262209281$
          - $262209281^2 \Rightarrow 13 \cdot 1231 \cdot 4296301150081$ [4]
            - $1231^2 \Rightarrow 3 \cdot 13 \cdot 37 \cdot 1051$ xs=3
            - $1231^{10} \Rightarrow 23 \cdot 67 \cdot 3323 \cdot 38237 \cdot 40842910222965466771$ S=2.04507 [5]
          - $262209281^{10} \Rightarrow 23 \cdot 67 \cdot 947 \cdot 15334282166504555262919920768081 \cdot$ $68653792009551353915243849650830737287676 74853$ S=2.04490 [6]
        - $127^{10} \Rightarrow 23 \cdot 47834644354838156839$ S=2.01226
      - $19^4 \Rightarrow 151 \cdot 911$
        - $151^2 \Rightarrow 3 \cdot 7 \cdot 1093$
          - $911^2 \Rightarrow 830833$ [7]
            - $1093^2 \Rightarrow 3 \cdot 39858$ xs=3
            - $1093^{10} \Rightarrow 23 \cdot 6491 \cdot 2608387 \cdot 6254429058851062673$ S=2.01438
          - $911^{10} \Rightarrow 67 \cdot 472319 \cdot 50390258557 \cdot 247174661801$
            - $67^2 \Rightarrow 3 \cdot 7^2 \cdot 31$ xs=3
        - $151^{10} \Rightarrow 23 \cdot 14864609 \cdot 18145704541823$ S=2.01223
      - $19^{10} \Rightarrow 104281 \cdot 62060021$
        - $104281^2 \Rightarrow 3 \cdot 7 \cdot 43 \cdot 67 \cdot 179743$
          - $43^2 \Rightarrow 3 \cdot 631$ xs=3
          - $43^4 \Rightarrow 3500201$
            - $67^2 \Rightarrow 3 \cdot 7^2 \cdot 31$ xs=3
        - $104281^4 \Rightarrow 5 \cdot 41 \cdot 3181 \cdot 181345750520141$ S=2.42843
    - $7^4 \Rightarrow 2801$
      - $2801^2 \Rightarrow 37 \cdot 43 \cdot 4933$
        - $37^2 \Rightarrow 3 \cdot 7 \cdot 67$
          - $43^2 \Rightarrow 3 \cdot 631$
            - $67^2 \Rightarrow 3 \cdot 7^2 \cdot 31$ xs=3
            - $67^{10} \Rightarrow 11 \cdot 89 \cdot 1890149702927663$ S=2.15947
          - $43^{10} \Rightarrow 60381099 \cdot 3664405207$
            - $67^2 \Rightarrow 3 \cdot 7^2 \cdot 31$ S=2.00408

[1] Convenient notation for odd prime factors of $\sigma(3^2) = 13$;
further factorisations in the chain are indicated by indentations.
[2] 13 could be the special prime.
[3] Contradiction: an excess of 3's.
[4] Although $262209281 \equiv 1 \pmod 4$, the smallest possible exponent is 2,
since 13 is currently the special prime.
[5] Contradiction: $S(3^2 13^1 7^2 19^2 127^4 262209281^2 1231^{10} \cdot 23^2 67^2 \ldots) > 2$.
[6] The program would not need to fully factorise $\sigma(262209281^{10})$:
it is sufficient to calculate $S(3^2 13^1 7^2 19^2 127^4 262209281^{10} \cdot 23^2 67^2 947^2) > 2$.
[7] The smallest unexplored prime, 911 here rather than 1093, is used to continue the chain.

$37^{10} \Rightarrow 2663 \cdot 1855860368209$

$43^2 \Rightarrow 3 \cdot 631$

$631^2 \Rightarrow 3 \cdot 307 \cdot 433$

$307^2 \Rightarrow 3 \cdot 43 \cdot 733$ xs=3

$2801^{10} \Rightarrow 23 \cdot 1372537 \cdot 10196539 \cdot 45437789 \cdot 2033125998091$

$23^2 \Rightarrow 7 \cdot 79$

$79^2 \Rightarrow 3 \cdot 7^2 \cdot 43$

$43^2 \Rightarrow 3 \cdot 631$

$631^2 \Rightarrow 3 \cdot 307 \cdot 433$ xs=3

$7^{10} \Rightarrow 1123 \cdot 293459$

$1123^2 \Rightarrow 3 \cdot 127 \cdot 3313$

$127^2 \Rightarrow 3 \cdot 5419$

$3313^2 \Rightarrow 3 \cdot 7^2 \cdot 19 \cdot 3931$ xs=3

$3313^4 \Rightarrow 257611 \cdot 467791631$

$5419^2 \Rightarrow 3 \cdot 31 \cdot 313 \cdot 1009$ xs=3

$127^4 \Rightarrow 262209281$

$3313^2 \Rightarrow 3 \cdot 7^2 \cdot 19 \cdot 3931$

$19^2 \Rightarrow 3 \cdot 127$ xs=3

$1123^4 \Rightarrow 41 \cdot 19571 \cdot 1983851$

$41^2 \Rightarrow 1723$

$1723^2 \Rightarrow 3 \cdot 990151$

$19571^2 \Rightarrow 43 \cdot 283 \cdot 31477$

$43^2 \Rightarrow 3 \cdot 631$

$283^2 \Rightarrow 3 \cdot 73 \cdot 367$ xs=3

$13^2 \Rightarrow 3 \cdot 61$

$61^1 \Rightarrow 31$

$31^2 \Rightarrow 3 \cdot 331$

$331^2 \Rightarrow 3 \cdot 7 \cdot 5233$ xs=3

$331^4 \Rightarrow 5 \cdot 37861 \cdot 63601$ S=2.04318

$331^{10} \Rightarrow 11 \cdot 23 \cdot 89 \cdot 703207451121161180833$

$11^2 \Rightarrow 7 \cdot 19$ S=2.35075

$11^4 \Rightarrow 5 \cdot 3221$ S=2.37678

$31^4 \Rightarrow 5 \cdot 11 \cdot 17351$ S=2.23914

$31^{10} \Rightarrow 23 \cdot 397 \cdot 617 \cdot 150332843$

$23^2 \Rightarrow 7 \cdot 79$ S=2.03164

$23^4 \Rightarrow 292561$

$397^2 \Rightarrow 3 \cdot 31 \cdot 1699$

$617^2 \Rightarrow 97 \cdot 3931$

$97^2 \Rightarrow 3 \cdot 3169$ xs=3

$61^2 \Rightarrow 3 \cdot 13 \cdot 97$

$97^1 \Rightarrow 7^2$

$7^2 \Rightarrow 3 \cdot 19$ xs=3

$7^4 \Rightarrow 2801$

$2801^2 \Rightarrow 37 \cdot 43 \cdot 4933$

$37^2 \Rightarrow 3 \cdot 7 \cdot 67$ xs=3

$37^{10} \Rightarrow 2663 \cdot 1855860368209$

$43^2 \Rightarrow 3 \cdot 631$ xs=3

$2801^{10} \Rightarrow 23 \cdot 1372537 \cdot 10196539 \cdot 45437789 \cdot 2033125998091$

$23^2 \Rightarrow 7 \cdot 79$

$79^2 \Rightarrow 3 \cdot 7^2 \cdot 43$ xs=3

$7^{10} \Rightarrow 1123 \cdot 293459$

$1123^2 \Rightarrow 3 \cdot 127 \cdot 3313$ xs=3

$1123^4 \Rightarrow 41 \cdot 19571 \cdot 1983851$

$41^2 \Rightarrow 1723$

$1723^2 \Rightarrow 3 \cdot 990151$ xs=3

$97^2 \Rightarrow 3 \cdot 3169$ xs=3

$97^4 \Rightarrow 11 \cdot 31 \cdot 262321$

$11^2 \Rightarrow 7 \cdot 19$ S=2.24044

$11^{10} \Rightarrow 15797 \cdot 1806113$

$31^2 \Rightarrow 3 \cdot 331$ xs=3

$97^{10} \Rightarrow 89 \cdot 837197335075000483$

$89^1 \Rightarrow 3^2 \cdot 5$ xs=9

$89^2 \Rightarrow 8011$

$8011^2 \Rightarrow 3 \cdot 13 \cdot 1645747$ xs=39

$8011^4 \Rightarrow 5 \cdot 11 \cdot 41 \cdot 61 \cdot 151 \cdot 3121 \cdot 63541$ S=2.21259

$89^4 \Rightarrow 131 \cdot 691 \cdot 701$

$131^2 \Rightarrow 17293$

$691^2 \Rightarrow 3 \cdot 19 \cdot 8389$ xs=3

$61^4 \Rightarrow 5 \cdot 131 \cdot 21491$

$5^1 \Rightarrow 3$

$131^2 \Rightarrow 17293$

$17293^2 \Rightarrow 3 \cdot 13 \cdot 7668337$ xs=3

$17293^{10} \Rightarrow 11 \cdot 261581 \cdot 2279749583 \cdot 364621693279346988621236431$ S=2.11381

$131^{10} \Rightarrow 23 \cdot 67 \cdot 353 \cdot 1453 \cdot 15401 \cdot 123210869$ S=2.04801

$5^2 \Rightarrow 31$ S=2.05324

$5^{10} \Rightarrow 12207031$ S=2.00310

$61^{10} \Rightarrow 199 \cdot 859 \cdot 4242586390571$

$199^2 \Rightarrow 3 \cdot 13267$

$859^2 \Rightarrow 3 \cdot 246247$ xs=3

$859^4 \Rightarrow 181 \cdot 631 \cdot 4772771$

$181^1 \Rightarrow 7 \cdot 13$

$7^2 \Rightarrow 3 \cdot 19$ xs=3

$181^2 \Rightarrow 3 \cdot 79 \cdot 139$ xs=3

$199^4 \Rightarrow 71 \cdot 22199431$

$71^2 \Rightarrow 5113$

$859^2 \Rightarrow 3 \cdot 246247$

$5113^1 \Rightarrow 2557$

$2557^2 \Rightarrow 3 \cdot 7 \cdot 13^2 \cdot 19 \cdot 97$ xs=39

$5113^2 \Rightarrow 3 \cdot 8715961$ xs=3

$13^4 \Rightarrow 30941$

$30941^1 \Rightarrow 3^4 \cdot 191$ xs=9

$30941^2 \Rightarrow 157 \cdot 433 \cdot 14083$

$157^1 \Rightarrow 79$

$79^2 \Rightarrow 3 \cdot 7^2 \cdot 43$

$7^2 \Rightarrow 3 \cdot 19$ S=2.00960

$7^{10} \Rightarrow 1123 \cdot 293459$

$43^2 \Rightarrow 3 \cdot 631$

$433^2 \Rightarrow 3 \cdot 37 \cdot 1693$ xs=3

$79^{10} \Rightarrow 5479 \cdot 1750258119644519$

$433^2 \Rightarrow 3 \cdot 37 \cdot 1693$

$37^2 \Rightarrow 3 \cdot 7 \cdot 67$

$7^2 \Rightarrow 3 \cdot 19$ xs=3

$157^2 \Rightarrow 3 \cdot 8269$

$433^1 \Rightarrow 7 \cdot 31$

$7^2 \Rightarrow 3 \cdot 19$ S=2.00290

$7^{10} \Rightarrow 1123 \cdot 293459$

$31^2 \Rightarrow 3 \cdot 331$

$331^2 \Rightarrow 3 \cdot 7 \cdot 5233$ xs=3

$433^2 \Rightarrow 3 \cdot 37 \cdot 1693$

$37^1 \Rightarrow 19$

$19^2 \Rightarrow 3 \cdot 127$ xs=3

$19^{10} \Rightarrow 104281 \cdot 62060021$

$1693^2 \Rightarrow 3 \cdot 13 \cdot 151 \cdot 487$ xs=3

$37^2 \Rightarrow 3 \cdot 7 \cdot 67$ xs=3

$37^{10} \Rightarrow 2663 \cdot 1855860368209$

$1693^1 \Rightarrow 7 \cdot 11^2$ S=2.07684

$1693^2 \Rightarrow 3 \cdot 13 \cdot 151 \cdot 487$ xs=3

$433^{10} \Rightarrow 947 \cdot 245207152504994694307129$

$947^2 \Rightarrow 7 \cdot 277 \cdot 463$

$7^2 \Rightarrow 3 \cdot 19$

$19^2 \Rightarrow 3 \cdot 127$ xs=3

$157^{10} \Rightarrow 28447 \cdot 321910472390668481$

$433^1 \Rightarrow 7 \cdot 31$

$7^2 \Rightarrow 3 \cdot 19$ S=2.00272

$433^2 \Rightarrow 3 \cdot 37 \cdot 1693$

$37^1 \Rightarrow 19$

$19^2 \Rightarrow 3 \cdot 127$
$127^2 \Rightarrow 3 \cdot 5419$ xs=3
$37^2 \Rightarrow 3 \cdot 7 \cdot 67$
$7^2 \Rightarrow 3 \cdot 19$ xs=3
$30941^{10} \Rightarrow 23 \cdot 4027 \cdot 868256471236283976942975961084961132329$1
$23^2 \Rightarrow 7 \cdot 79$
$7^2 \Rightarrow 3 \cdot 19$ S=2.03461
$13^{10} \Rightarrow 23 \cdot 419 \cdot 859 \cdot 18041$
$23^2 \Rightarrow 7 \cdot 79$
$7^2 \Rightarrow 3 \cdot 19$ S=2.04140
$7^4 \Rightarrow 2801$
$79^2 \Rightarrow 3 \cdot 7^2 \cdot 43$
$43^2 \Rightarrow 3 \cdot 631$
$419^2 \Rightarrow 13 \cdot 13537$
$631^2 \Rightarrow 3 \cdot 307 \cdot 433$ xs=3
$23^4 \Rightarrow 292561$
$419^2 \Rightarrow 13 \cdot 13537$
$859^2 \Rightarrow 3 \cdot 246247$
$13537^1 \Rightarrow 7 \cdot 967$
$7^2 \Rightarrow 3 \cdot 19$ S=2.01798
$13537^2 \Rightarrow 3 \cdot 523 \cdot 116803$
$523^2 \Rightarrow 3 \cdot 13 \cdot 7027$ xs=3
$3^4 \Rightarrow 11^2$
$11^2 \Rightarrow 7 \cdot 19$ S=2.01587
$11^{10} \Rightarrow 15797 \cdot 1806113$
$15797^1 \Rightarrow 3 \cdot 2633$
$2633^2 \Rightarrow 19 \cdot 365017$
$19^2 \Rightarrow 3 \cdot 127$
$127^2 \Rightarrow 3 \cdot 5419$
$5419^2 \Rightarrow 3 \cdot 31 \cdot 313 \cdot 1009$
$31^2 \Rightarrow 3 \cdot 331$ xs=3
$15797^2 \Rightarrow 249561007$
$1806113^1 \Rightarrow 3 \cdot 17 \cdot 17707$
$17^2 \Rightarrow 307$
$307^2 \Rightarrow 3 \cdot 43 \cdot 733$
$43^2 \Rightarrow 3 \cdot 631$
$631^2 \Rightarrow 3 \cdot 307 \cdot 433$
$433^2 \Rightarrow 3 \cdot 37 \cdot 1693$ xs=3
$1806113^2 \Rightarrow 19 \cdot 171686630257$
$19^2 \Rightarrow 3 \cdot 127$
$127^2 \Rightarrow 3 \cdot 5419$
$5419^2 \Rightarrow 3 \cdot 31 \cdot 313 \cdot 1009$

$31^2 \Rightarrow 3 \cdot 331$
$313^1 \Rightarrow 157$
$157^2 \Rightarrow 3 \cdot 8269$ xs=3
$313^2 \Rightarrow 3 \cdot 181^2$ xs=3
$3^{10} \Rightarrow 23 \cdot 3851$
$23^2 \Rightarrow 7 \cdot 79$
$7^2 \Rightarrow 3 \cdot 19$
$19^2 \Rightarrow 3 \cdot 127$
$79^2 \Rightarrow 3 \cdot 7^2 \cdot 43$ S=2.01255
$79^4 \Rightarrow 39449441$
$127^2 \Rightarrow 3 \cdot 5419$
$3851^2 \Rightarrow 13 \cdot 1141081$ S=2.11738
$19^4 \Rightarrow 151 \cdot 911$
$79^2 \Rightarrow 3 \cdot 7^2 \cdot 43$ S=2.01251
$7^4 \Rightarrow 2801$
$79^2 \Rightarrow 3 \cdot 7^2 \cdot 43$
$43^2 \Rightarrow 3 \cdot 631$
$631^2 \Rightarrow 3 \cdot 307 \cdot 433$
$307^2 \Rightarrow 3 \cdot 43 \cdot 733$
$433^1 \Rightarrow 7 \cdot 31$
$31^2 \Rightarrow 3 \cdot 331$
$331^2 \Rightarrow 3 \cdot 7 \cdot 5233$
$733^2 \Rightarrow 3 \cdot 19 \cdot 9439$ S=2.09450
$433^2 \Rightarrow 3 \cdot 37 \cdot 1693$
$37^1 \Rightarrow 19$ S=2.07612
$37^2 \Rightarrow 3 \cdot 7 \cdot 67$
$67^2 \Rightarrow 3 \cdot 7^2 \cdot 31$ S=2.06489
$23^4 \Rightarrow 292561$
$3851^2 \Rightarrow 13 \cdot 1141081$
$13^1 \Rightarrow 7$
$7^2 \Rightarrow 3 \cdot 19$ S=2.07391
$13^2 \Rightarrow 3 \cdot 61$
$61^1 \Rightarrow 31$
$31^2 \Rightarrow 3 \cdot 331$
$331^2 \Rightarrow 3 \cdot 7 \cdot 5233$ S=2.08179
$61^2 \Rightarrow 3 \cdot 13 \cdot 97$
$97^1 \Rightarrow 7^2$ S=2.02947
$97^2 \Rightarrow 3 \cdot 3169$
$3169^1 \Rightarrow 5 \cdot 317$ S=2.17110
$3169^2 \Rightarrow 3 \cdot 3348577$
$292561^1 \Rightarrow 19 \cdot 7699$
$19^2 \Rightarrow 3 \cdot 127$

$$127^2 \Rightarrow 3 \cdot 5419$$
$$5419^2 \Rightarrow 3 \cdot 31 \cdot 313 \cdot 1009$$
$$31^2 \Rightarrow 3 \cdot 331$$
$$313^2 \Rightarrow 3 \cdot 181^2$$
$$181^2 \Rightarrow 3 \cdot 79 \cdot 139$$
$$79^2 \Rightarrow 3 \cdot 7^2 \cdot 43 \text{ xs=3}$$
$$292561^2 \Rightarrow 3 \cdot 13^2 \cdot 168820969$$
$$1141081^1 \Rightarrow 337 \cdot 1693$$
$$337^2 \Rightarrow 3 \cdot 43 \cdot 883$$
$$43^2 \Rightarrow 3 \cdot 631$$
$$631^2 \Rightarrow 3 \cdot 307 \cdot 433$$
$$307^2 \Rightarrow 3 \cdot 43 \cdot 733$$
$$433^2 \Rightarrow 3 \cdot 37 \cdot 1693$$
$$37^2 \Rightarrow 3 \cdot 7 \cdot 67 \text{ xs=3}$$
$$1141081^2 \Rightarrow 3 \cdot 7^2 \cdot 53233 \cdot 166393 \text{ S=2.03038}$$

End of proof.

Appendix H

A proof that an odd perfect number has at least six distinct prime factors

The *only* contradictions used here are *P1*, *P2* and *P3*, and $B = 10^6$ is sufficient throughout.

While 10^6 was used as the lower bound on the largest prime factor (that is for the *P1* contradiction), the recent improvement to 10^7 has no effect on this proof.

10^6 $3^2 \Rightarrow 13$ [1]

$13^1 \Rightarrow 7$ P3 [2]

$13^2 \Rightarrow 3, 61$ P3

$13^4 \Rightarrow 30941$

$30941^\infty : p_4 < 8.2$ P3 [3]

$13^\infty : p_3 < 11.8$ P3

$3^4 \Rightarrow 11^2$

$11^2 \Rightarrow 7, 19$ P2 [4]

$11^4 \Rightarrow 5, 3221$ P2

$11^\infty : p_3 < 14.9$ P3

$3^6 \Rightarrow 1093$

$1093^\infty : 3.005 < p_3 < 10.1$, $p_4 < 31.3$ P2 [5]

$3^8 \Rightarrow 13, 757$

$13^1 \Rightarrow 7$ P2

$13^2 \Rightarrow 3, 61$ P2

$13^4 \Rightarrow 30941$

$757^1 \Rightarrow 379$ P1 [6]

757^∞

$30941^\infty : p_5 < 5.4$ P1

13^∞

$757^1 \Rightarrow 379$ P2

$757^\infty : p_4 < 9.8$ P2

$3^{10} \Rightarrow 23, 3851$

$23^2 \Rightarrow 7, 79$ P1

23^∞

$3851^\infty : p_4 < 8.3$ P2

$3^\infty : 2.9 < p_2 < 13$, $p_3 < 46$ P3

End of proof.

[1] Convenient notation for odd prime factors of $\sigma(3^2) = 13$;
further factorisations in the chain are indicated by indentations.
[2] 13 could be the special prime; $\lambda = 3^2 13$, $\mu = 7$
[3] The upper bound in Lemma 20 should be calculated first;
there may be no need to calculate the lower bound; $\lambda = 3^2 13^4$, $\mu = 30941$
[4] $\lambda = 3^4 11^2$, $\mu = 7.19$
[5] An example of the use of Corollary 4; $\lambda = 3^6$, $\mu = 1093$, $s_1 = 5, l_1 = 7, l_2 = 31$
[6] $\lambda = 3^8 13^4 757$, $\mu = 379.30941$

Appendix I

A proof that an odd perfect number has at least seven distinct prime factors

VERSION = R.M.Sorli, August 1999

MAX-PRIMES = 7

INITIAL-B-EXP = 6

Uses known bounds on three largest primes: P1=1000000, P2=10000, P3=100

Contradictions *M2* and Π not needed. Extended r_i intervals (Corollary 5) not significant here. The improved lower bound of 10^7 for *P1* not applied here but few instances found where it would be significant.

Bound B manually adjusted and results spliced.

Rechecked with *Mathematica* and with long sequences abbreviated, September 2002

10^6 $3^2 \Rightarrow 13$ [1]

    $13^1 \Rightarrow 7$ [2]

        $7^2 \Rightarrow 3 \cdot 19$ P3 [3]

        $7^4 \Rightarrow 2801$

            $2801^\infty : p_5 < 20.67$ P2 [4]

        $7^6 \Rightarrow 29 \cdot 4733$ P2

        $7^\infty : p_4 < 30.40$ P3

    $13^2 \Rightarrow 3 \cdot 61$

        $61^1 \Rightarrow 31$ P3

        $61^2 \Rightarrow 3 \cdot 13 \cdot 97$ P3

        $61^\infty : p_4 < 12.65$ P3

    $13^4 \Rightarrow 30941$

        $30941^\infty : 3.59 < p_4 < 11.79, p_5 < 90.07$ P3 [5]

    $13^\infty : 3.59 < p_3 < 15.39$

        $5^1 \Rightarrow 3 : p_4 < 47.10$ P3

        $5^2 \Rightarrow 31$ P3

        $5^4 \Rightarrow 11 \cdot 71$ P2

        $5^5 \Rightarrow 3^2 7 \cdot 31$ P2

        $5^6 \Rightarrow 19531$

            $19531^\infty : p_5 < 90.11$ P3

        $5^\infty : 44.47 < p_4 < 134.43, p_5 < 2739.97$ P2

        $7^2 \Rightarrow 3 \cdot 19$ P3

        $7^4 \Rightarrow 2801$

            $2801^\infty : p_5 < 22.01$ P2

        $7^6 \Rightarrow 29 \cdot 4733$ P2

        $7^\infty : p_4 < 32.41$ P3

        $11^2 \Rightarrow 7 \cdot 19$ S [6]

        $11^4 \Rightarrow 5 \cdot 3221$ S

        $11^\infty : p_4 < 19.53$ P3

$3^4 \Rightarrow 11^2$

    $11^2 \Rightarrow 7 \cdot 19$ P3

    $11^4 \Rightarrow 5 \cdot 3221$

        $5^1 \Rightarrow 3$ [7]

            $3221^\infty : 71.60 < p_5 < 144.21, p_6 < 13084.48$ P1 [8]

        5^2 A [9]

---

[1] Convenient notation for odd prime factors of $\sigma(3^2) = 13$;
further factorisations in the chain are indicated by indentations.
[2] 13 could be the special prime.
[3] $\lambda = 3^2 13.7^2,\ \mu = 19$
[4] The upper bound in Lemma 20 should be calculated first;
there may be no need to calculate the lower bound; $\lambda = 3^2 13.7^4,\ \mu = 2801$
[5] Lemma 20 and Corollary 4
[6] The factor 7 is smaller than 11.
[7] no new primes, use Lemma 20
[8] $\lambda = 3^4 11^4 5,\ \mu = 3221$
[9] The number $3^4 11^4 5^2$ is Abundant.

$11^\infty : 4.60 < p_3 < 19.42, p_4 < 70.36$ P3

$3^6 \Rightarrow 1093$

  $1093^\infty : 3.00 < p_3 < 13.03$

    $5^1 \Rightarrow 3 : 9.94 < p_4 < 28.13$

      $11^2 \Rightarrow 7 \cdot 19$ S

      $11^4 \Rightarrow 5 \cdot 3221$ P2

      $11^\infty : p_5 < 208.51$ P2

      $13^2 \Rightarrow 3 \cdot 61$ P2

      $13^4 \Rightarrow 30941$

        $30941^\infty : p_6 < 40.78$ P1

      $13^\infty : p_5 < 80.46$ P2

      $17^2 \Rightarrow 307$ P2

      $17^\infty : p_5 < 45.18$ P2

      $19^2 \Rightarrow 3 \cdot 127$ P2

      $19^\infty : p_5 < 39.36$ P2

      $23^2 \Rightarrow 7 \cdot 79$ S

      $23^\infty : p_5 < 33.10$ P2

    $5^2 \Rightarrow 31$

      $31^2 \Rightarrow 3 \cdot 331$ P2

      $31^\infty : p_5 < 50.87$ P2

    $5^4 \Rightarrow 11 \cdot 71$ P2

    $5^5 \Rightarrow 3^2 7 \cdot 31$ P2

    $5^6 \Rightarrow 19531$

      $19531^\infty : 15.11 < p_5 < 31.25, p_6 < 293.12$ P1

    $5^\infty : 15.11 < p_4 < 46.34, p_5 < 578.72$ P2

    $7^2 \Rightarrow 3 \cdot 19$

      $19^2 \Rightarrow 3 \cdot 127$ P2

      $19^\infty : p_5 < 24.43$ P2

    $7^4 \Rightarrow 2801$

      $2801^\infty : p_5 < 15.09$ P2

    $7^6 \Rightarrow 29 \cdot 4733$ P2

    $7^\infty : 7.02 < p_4 < 22.08, p_5 < 52.97$ P2

    $11^2 \Rightarrow 7 \cdot 19$ S

    $11^4 \Rightarrow 5 \cdot 3221$ S

    $11^\infty : 4.72 < p_4 < 15.18, p_5 < 17.90$ P2

    $13^1 \Rightarrow 7$ S

    $13^2 \Rightarrow 3 \cdot 61$

      $61^1 \Rightarrow 31$ P2

      $61^2 \Rightarrow 3 \cdot 13 \cdot 97$ P2

      $61^\infty : p_5 < 10.50$ P2

    $13^4 \Rightarrow 30941$

      $30941^\infty : 4.34 < p_5 < 9.69, p_6 < 19.38$ P1

13^{∞} : $4.34 < p_4 < 14.04$, $p_5 < 37.72$ P2

$3^8 \Rightarrow 13 \cdot 757$

$13^1 \Rightarrow 7$

$7^2 \Rightarrow 3 \cdot 19$ P2

$7^4 \Rightarrow 2801$ P2

$7^6 \Rightarrow 29 \cdot 4733$ P1

7^{∞}

757^{∞} : $p_5 < 34.41$ P2

$13^2 \Rightarrow 3 \cdot 61$

$61^1 \Rightarrow 31$ P2

$61^2 \Rightarrow 3 \cdot 13 \cdot 97$ P2

61^{∞}

$757^1 \Rightarrow 379$ P2

757^{∞} : $p_5 < 10.55$ P2

$13^4 \Rightarrow 30941$

$757^1 \Rightarrow 379$

379^{∞}

30941^{∞} : $p_6 < 5.43$ P1

757^{∞}

30941^{∞} : $4.36 < p_5 < 9.73$, $p_6 < 19.67$ P1

13^{∞}

$757^1 \Rightarrow 379$

379^{∞} : $p_5 < 9.86$ P2

757^{∞} : $4.36 < p_4 < 14.09$, $p_5 < 38.31$ P2

$3^{10} \Rightarrow 23 \cdot 3851$

$23^2 \Rightarrow 7 \cdot 79$ P2

23^{∞}

3851^{∞} : $3.63 < p_4 < 11.91$, $p_5 < 100.85$ P2

3^{∞} : $2.99 < p_2 < 16.00$

$5^1 \Rightarrow 3$: $9.90 < p_3 < 37.00$

$11^2 \Rightarrow 7 \cdot 19$ S

$11^4 \Rightarrow 5 \cdot 3221$

3221^{∞} : $p_5 < 205.22$ P2

11^{∞} : $99.98 < p_4 < 298.00$

101^{∞} : $9837.77 < p_5 < 19999.00$

9839^{∞} P2

..... [10]

9973^{∞} P2

10007^{∞} : $581703.22 < p_6 < 14294285.72$

581729^{∞} P1

.....

[10] for primes p, $9839 \leq p \leq 9973$ contradiction *P2* applies

999983^{∞} P1

$1000003^{\infty} \longrightarrow$ [11]

....

10^9 3^{∞} 5^1 11^{∞} 101^{∞} 10007^{∞} $1000003 \leq p_6 < 14294285.72$

$3^{12} \Rightarrow 797161$ M1

$3^{14} \Rightarrow 11^2 13 \cdot 4561$ M1

$3^{16} \Rightarrow 1871 \cdot 34511$ M1

$3^{\infty} : 2.99 < p_2 < 16.00$

$5^1 \Rightarrow 3 : 9.89 < p_3 < 37.00$

$11^6 \Rightarrow 43 \cdot 45319$ M1

$11^{\infty} : 99.98 < p_4 < 298.00$

$101^2 \Rightarrow 10303$ M1

$101^{\infty} : 9999.86 < p_5 < 19999.00$

$10007^{\infty} : 14017645.09 < p_6 < 14294285.72$

$14017667^{\infty} \longrightarrow$

....

10^{12} 3^{∞} 5^1 11^{∞} 101^{∞} 10007^{∞} $14017667 \leq p_6 < 14294285.72$

$3^{18} \Rightarrow 1597 \cdot 363889$ M1

$3^{20} \Rightarrow 13 \cdot 1093 \cdot 368089$ M1

$3^{22} \Rightarrow 47 \cdot 1001523179$ M1

$3^{24} \Rightarrow 11^2 8951 \cdot 391151$ M1

$3^{\infty} : 2.99 < p_2 < 16.00$

$5^1 \Rightarrow 3 : 9.89 < p_3 < 37.00$

$11^8 \Rightarrow 7 \cdot 19 \cdot 1772893$ S

$11^{10} \Rightarrow 15797 \cdot 1806113$ M1

$11^{\infty} : 99.98 < p_4 < 298.00$

$101^4 \Rightarrow 5 \cdot 31 \cdot 491 \cdot 1381$ S

$101^{\infty} : 9999.99 < p_5 < 19999.00$

$10007^{\infty} : 14294047.20 < p_6 < 14294285.72$

$14294051^{\infty} \longrightarrow$

....

10^{15} 3^{∞} 5^1 11^{∞} 101^{∞} 10007^{∞} $14294051 \leq p_6 < 14294285.72$

$3^{26} \Rightarrow 13 \cdot 109 \cdot 433 \cdot 757 \cdot 8209$ M1

$3^{28} \Rightarrow 59 \cdot 28537 \cdot 20381027$ M1

$3^{30} \Rightarrow 683 \cdot 102673 \cdot 4404047$ M1

$3^{\infty} : 2.99 < p_2 < 16.00$

$5^1 \Rightarrow 3 : 9.89 < p_3 < 37.00$

$11^{12} \Rightarrow 1093 \cdot 3158528101$ M1

$11^{\infty} : 99.98 < p_4 < 298.00$

$101^6 \Rightarrow 71 \cdot 15100497917$ S

[11] The inequalities (5.33) are satisfied with $B = 10^6$, so a "finer" sieve, with $B = 10^9$, is tried to decide this case. Movement to a finer sieve or back to a previous one are highlighted with arrows and dots.

101^{∞} : $9999.99 < p_5 < 19999.00$

$10007^2 \Rightarrow 7 \cdot 31 \cdot 461521$ S

10007^{∞} : $14294285.62 < p_6 < 14294285.72$ N $\longleftarrow$ [12]

$10009^2 \Rightarrow 3 \cdot 73 \cdot 79 \cdot 5791$ S

10009^{∞} : $1111999.99 < p_6 < 11120000.00$ N $\longleftarrow$

$10037^2 \Rightarrow 37 \cdot 103 \cdot 26437$ S

10037^{∞} : $2712432.42 < p_6 < 2712432.44$ N $\longleftarrow$

.... [13]

10^{12} 10039^{∞} : $2573838.48 < p_6 < 2573846.16$ N $\longleftarrow$

10061^{∞} : $1649177.20 < p_6 < 1649180.33$ N $\longleftarrow$

10067^{∞} : $1502385.46 < p_6 < 1502388.06$ N $\longleftarrow$

10069^{∞} : $1459127.99 < p_6 < 1459130.44$ N $\longleftarrow$

10079^{∞} : $1275694.33 < p_6 < 1275696.21$ N $\longleftarrow$

10091^{∞} : $1108789.80 < p_6 < 1108791.21$ N $\longleftarrow$

10093^{∞} : $1085159.94 < p_6 < 1085161.30$ N $\longleftarrow$

10099^{∞} : $1019998.81 < p_6 < 1020000.00$ N $\longleftarrow$

....

10^6 10103^{∞} : $p_6 < 980776.70$ P1

.....

19997^{∞} : $p_6 < 20002.01$ P1

103^{∞} : $p_5 < 6799.00$ P2

.....

293^{∞} : $p_5 < 301.59$ P2

$13^2 \Rightarrow 3 \cdot 61$ P3

$13^4 \Rightarrow 30941$

30941^{∞} : $p_5 < 79.10$ P3

13^{∞} : $39.97 < p_4 < 118.00$, $p_5 < 3199.00$ P2

$17^2 \Rightarrow 307$

307^{∞} : $p_5 < 48.00$ P2

17^{∞} : $p_4 < 66.58$ P3

$19^2 \Rightarrow 3 \cdot 127$

127^{∞} : $p_5 < 45.95$ P2

19^{∞} : $p_4 < 58.00$ P3

$23^2 \Rightarrow 7 \cdot 79$ S

23^{∞} : $p_4 < 48.77$ P3

$29^2 \Rightarrow 13 \cdot 67$ S

29^{∞} : $p_4 < 42.22$ P3

$31^2 \Rightarrow 3 \cdot 331$

331^{∞} : $p_5 < 28.76$ P2

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[12] With the finer sieve, this case is now void. We delay reverting to an earlier bound since in this case it was found the the bound was immediately increased again.

[13] Now reverting to an earlier bound will continue to be appropriate.

$31^\infty : p_4 < 40.86$ P3

$5^2 \Rightarrow 31$

$31^2 \Rightarrow 3 \cdot 331$

$331^\infty : p_5 < 54.38$ P2

$31^\infty : p_4 < 74.93$ P3

$5^4 \Rightarrow 11 \cdot 71$ P3

$5^5 \Rightarrow 3^2 7 \cdot 31$ P3

$5^6 \Rightarrow 19531$

$19531^\infty : 15.00 < p_4 < 46.03$

$17^1 \Rightarrow 3^2 : 136.69 < p_5 < 272.42, p_6 < 63835.77$ P1

$17^2 \Rightarrow 307$

$307^\infty : p_6 < 1225.37$ P1

$17^\infty : 258.44 < p_5 < 516.07, p_6 < 15159.14$ P1

$19^2 \Rightarrow 3 \cdot 127$

$127^\infty : p_6 < 374.54$ P1

$19^\infty : 95.34 < p_5 < 191.71, p_6 < 14263.61$ P1

$23^2 \Rightarrow 7 \cdot 79$ S

$23^\infty : p_5 < 99.77$ P3

$29^1 \Rightarrow 3 \cdot 5 : 33.15 < p_5 < 65.37, p_6 < 313.04$ P1

$29^2 \Rightarrow 13 \cdot 67$ S

$29^\infty : p_5 < 68.02$ P3

$31^2 \Rightarrow 3 \cdot 331$

$331^\infty : p_6 < 35.33$ P1

$31^\infty : p_5 < 63.08$ P3

$37^1 \Rightarrow 19$ S

$37^2 \Rightarrow 3 \cdot 7 \cdot 67$ S

$37^\infty : p_5 < 53.92$ P3

$41^1 \Rightarrow 3 \cdot 7$ S

$41^2 \Rightarrow 1723$

$1723^\infty : p_6 < 25.99$ P1

$41^\infty : p_5 < 50.25$ P3

$43^2 \Rightarrow 3 \cdot 631$

$631^\infty : p_6 < 25.89$ P1

$43^\infty : p_5 < 48.83$ P3

$5^\infty : 14.99 < p_3 < 61.00$

$17^1 \Rightarrow 3^2 : 135.97 < p_4 < 406.00$

$137^\infty : 17987.62 < p_5 < 36991.00$

$17989^\infty \longrightarrow$

....

10^9 $3^\infty\ 5^\infty\ 17^1\ 137^\infty\ 17989 \le p_5 < 36991.00$

$3^{12} \Rightarrow 797161$ P1

$3^{14} \Rightarrow 11^2 13 \cdot 4561$ M1

$3^{16} \Rightarrow 1871 \cdot 34511$ M1
3^{∞} : $2.99 < p_2 < 16.00$
$5^{8} \Rightarrow 19 \cdot 31 \cdot 829$ M1
$5^{9} \Rightarrow 3 \cdot 11 \cdot 71 \cdot 521$ M1
$5^{10} \Rightarrow 12207031$
$17^{1} \Rightarrow 3^{2}$: $135.99 < p_4 < 406.00$
$137^{2} \Rightarrow 7 \cdot 37 \cdot 73$ M1
137^{∞}
12207031^{∞} : $18516.73 < p_6 < 18517.05$
18517^{∞} $\longrightarrow$

....

10^{12} 3^{∞} 5^{10} 17^{1} 137^{∞} 12207031^{∞} $18517 \leq p_6 < 18517.05$
$3^{18} \Rightarrow 1597 \cdot 363889$ M1
$3^{20} \Rightarrow 13 \cdot 1093 \cdot 368089$ M1
$3^{22} \Rightarrow 47 \cdot 1001523179$ M1
$3^{24} \Rightarrow 11^{2} 8951 \cdot 391151$ M1
3^{∞} : $2.99 < p_2 < 16.00$
$5^{10} \Rightarrow 12207031$
$17^{1} \Rightarrow 3^{2}$: $135.99 < p_4 < 406.00$
$137^{4} \Rightarrow 11 \cdot 101 \cdot 319411$ M1
137^{∞}
12207031^{∞} : $18517.04 < p_6 < 18517.05$ N $\longleftarrow$

....

10^{9} 5^{∞} : $14.99 < p_3 < 61.00$
$17^{1} \Rightarrow 3^{2}$: $135.99 < p_4 < 406.00$
$137^{2} \Rightarrow 7 \cdot 37 \cdot 73$ M1
137^{∞} : $18495.41 < p_5 < 36991.00$
18503^{∞} : $45135383.03 < p_6 < 48887570.29$
45135403^{∞} $\longrightarrow$

....

10^{12} 3^{∞} 5^{∞} 17^{1} 137^{∞} 18503^{∞} $45135403 \leq p_6 < 48887570.29$
$3^{18} \Rightarrow 1597 \cdot 363889$ M1
$3^{20} \Rightarrow 13 \cdot 1093 \cdot 368089$ M1
$3^{22} \Rightarrow 47 \cdot 1001523179$ M1
$3^{24} \Rightarrow 11^{2} 8951 \cdot 391151$ M1
3^{∞} : $2.99 < p_2 < 16.00$
$5^{12} \Rightarrow 305175781$ M1
$5^{13} \Rightarrow 3 \cdot 29 \cdot 449 \cdot 19531$ M1
$5^{14} \Rightarrow 11 \cdot 31 \cdot 71 \cdot 181 \cdot 1741$ M1
$5^{16} \Rightarrow 409 \cdot 466344409$ M1
5^{∞} : $14.99 < p_3 < 61.00$
$17^{1} \Rightarrow 3^{2}$: $135.99 < p_4 < 406.00$

$137^4 \Rightarrow 11 \cdot 101 \cdot 319411$ M1

137^∞ : $18495.41 < p_5 < 36991.00$

18503^∞ : $48886751.65 < p_6 < 48887570.29$

$48886759^\infty \longrightarrow$

....

10^{15} 3^∞ 5^∞ 17^1 137^∞ 18503^∞ $48886759 \le p_6 < 48887570.29$

$3^{18} \Rightarrow 1597 \cdot 363889$ M1

$3^{20} \Rightarrow 13 \cdot 1093 \cdot 368089$ M1

$3^{22} \Rightarrow 47 \cdot 1001523179$ M1

$3^{24} \Rightarrow 11^2 8951 \cdot 391151$ M1

3^∞ : $2.99 < p_2 < 16.00$

$5^{17} \Rightarrow 3^3 7 \cdot 19 \cdot 31 \cdot 829 \cdot 5167$ M1

$5^{18} \Rightarrow 191 \cdot 6271 \cdot 3981071$ M1

$5^{20} \Rightarrow 31 \cdot 379 \cdot 19531 \cdot 519499$ M1

5^∞ : $14.99 < p_3 < 61.00$

$17^1 \Rightarrow 3^2$: $135.99 < p_4 < 406.00$

$137^6 \Rightarrow 8933 \cdot 745603139$ M1

137^∞ : $18495.41 < p_5 < 36991.00$

$18503^2 \Rightarrow 7^3 421 \cdot 2371$ M1

18503^∞ : $48887569.65 < p_6 < 48887570.29$ N $\longleftarrow$

$18517^2 \Rightarrow 3 \cdot 7 \cdot 19 \cdot 859393$ M1

18517^∞ : $16308187.35 < p_6 < 16308187.43$ N $\longleftarrow$

$18521^2 \Rightarrow 13 \cdot 67 \cdot 393853$ M1

18521^∞ : $13701836.75 < p_6 < 13701836.81$ N $\longleftarrow$

$18523^2 \Rightarrow 3 \cdot 163 \cdot 541 \cdot 1297$ M1

18523^∞ : $12688255.95 < p_6 < 12688256.01$ N $\longleftarrow$

$18539^2 \Rightarrow 43 \cdot 7993327$ M1

18539^∞ : $7973926.68 < p_6 < 7973926.70$ N $\longleftarrow$

$18541^2 \Rightarrow 3 \cdot 13 \cdot 79 \cdot 241 \cdot 463$ M1

18541^∞ : $7620351.98 < p_6 < 7620352.01$ N $\longleftarrow$

....

10^{12} 18553^∞ : $6019948.89 < p_6 < 6019961.27$ N $\longleftarrow$

18583^∞ : $3950485.16 < p_6 < 3950490.49$ N $\longleftarrow$

18587^∞ : $3777650.70 < p_6 < 3777655.57$ N $\longleftarrow$

18593^∞ : $3545125.95 < p_6 < 3545130.23$ N $\longleftarrow$

18617^∞ : $2845629.77 < p_6 < 2845632.53$

$2845631^\infty \longrightarrow$

....

10^{15} 3^∞ 5^∞ 17^1 137^∞ 18617^∞ $2845631 \le p_6 < 2845632.53$

$3^{26} \Rightarrow 13 \cdot 109 \cdot 433 \cdot 757 \cdot 8209$ M1

$3^{28} \Rightarrow 59 \cdot 28537 \cdot 20381027$ M1

$3^{30} \Rightarrow 683 \cdot 102673 \cdot 4404047$ M1

3^{∞} : $2.99 < p_2 < 16.00$

$5^{17} \Rightarrow 3^3 7 \cdot 19 \cdot 31 \cdot 829 \cdot 5167$ M1

$5^{18} \Rightarrow 191 \cdot 6271 \cdot 3981071$ M1

$5^{20} \Rightarrow 31 \cdot 379 \cdot 19531 \cdot 519499$ M1

5^{∞} : $14.99 < p_3 < 61.00$

$17^1 \Rightarrow 3^2$: $135.99 < p_4 < 406.00$

$137^6 \Rightarrow 8933 \cdot 745603139$ M1

137^{∞} : $18495.41 < p_5 < 36991.00$

$18617^2 \Rightarrow 7 \cdot 49515901$ M1

18617^{∞} : $2845632.52 < p_6 < 2845632.53$ N ⟵

....

10^{12} 18637^{∞} : $2444618.22 < p_6 < 2444620.26$ N ⟵

18661^{∞} : $2091727.97 < p_6 < 2091729.46$ N ⟵

18671^{∞} : $1973257.65 < p_6 < 1973258.98$ N ⟵

18679^{∞} : $1887803.64 < p_6 < 1887804.86$ N ⟵

18691^{∞} : $1772769.39 < p_6 < 1772770.47$ N ⟵

18701^{∞} : $1687195.13 < p_6 < 1687196.10$ N ⟵

18713^{∞} : $1594916.89 < p_6 < 1594917.76$ N ⟵

18719^{∞} : $1552502.00 < p_6 < 1552502.82$ N ⟵

18731^{∞} : $1474169.82 < p_6 < 1474170.56$ N ⟵

18743^{∞} : $1403448.85 < p_6 < 1403449.53$ N ⟵

18749^{∞} : $1370604.14 < p_6 < 1370604.78$ N ⟵

18757^{∞} : $1329160.23 < p_6 < 1329160.83$ N ⟵

18773^{∞} : $1253454.02 < p_6 < 1253454.56$ N ⟵

18787^{∞} : $1194040.26 < p_6 < 1194040.75$ N ⟵

18793^{∞} : $1170291.90 < p_6 < 1170292.37$ N ⟵

18797^{∞} : $1154985.65 < p_6 < 1154986.10$ N ⟵

18803^{∞} : $1132774.13 < p_6 < 1132774.57$ N ⟵

18839^{∞} : $1015823.70 < p_6 < 1015824.05$ N ⟵

....

10^{6} 18859^{∞} : $p_6 < 960874.85$ P1

.....

36979^{∞} : $p_6 < 37004.01$ P1

139^{∞} : $6197.40 < p_5 < 12511.00$

6199^{∞} P2

.....

9997^{∞} P2

10007^{∞} : $p_6 < 16688.23$ P1

.....

12503^{∞} : $p_6 < 12520.02$ P1

149^{∞} : $p_5 < 3095.62$ P2

.....

$401^\infty : p_5 < 409.57$ P2
$17^2 \Rightarrow 307$
$307^\infty : p_5 < 2339.69$ P2
$17^\infty : 254.87 < p_4 < 766.00$
$257^1 \Rightarrow 3 \cdot 43$ S
$257^\infty : 58270.01 < p_5 < 131071.00$
$58271^\infty \longrightarrow$

. . . .

10^9 $3^\infty\ 5^\infty\ 17^\infty\ 257^\infty\ 58271 \leq p_5 < 131071.00$
$3^{12} \Rightarrow 797161$ P1
$3^{14} \Rightarrow 11^2 13 \cdot 4561$ M1
$3^{16} \Rightarrow 1871 \cdot 34511$ M1
$3^\infty : 2.99 < p_2 < 16.00$
$5^8 \Rightarrow 19 \cdot 31 \cdot 829$ M1
$5^9 \Rightarrow 3 \cdot 11 \cdot 71 \cdot 521$ M1
$5^{10} \Rightarrow 12207031$
$17^4 \Rightarrow 88741$ D
$17^5 \Rightarrow 3^3 7 \cdot 13 \cdot 307$ M1
$17^6 \Rightarrow 25646167$ D
$17^\infty : 254.99 < p_4 < 766.00$
$257^2 \Rightarrow 61 \cdot 1087$ M1
257^∞
$12207031^\infty : 65796.18 < p_6 < 65800.95$ N
$5^\infty : 14.99 < p_3 < 61.00$
$17^4 \Rightarrow 88741$
$257^2 \Rightarrow 61 \cdot 1087$ M1
257^∞
$88741^\infty : 212943.48 < p_6 < 213020.72$ P1
$17^5 \Rightarrow 3^3 7 \cdot 13 \cdot 307$ M1
$17^6 \Rightarrow 25646167$
$257^2 \Rightarrow 61 \cdot 1087$ M1
257^∞
$25646167^\infty : 65692.37 < p_6 < 65693.38$ N
$17^\infty : 254.99 < p_4 < 766.00$
$257^2 \Rightarrow 61 \cdot 1087$ M1
$257^\infty : 65527.74 < p_5 < 131071.00$
$65537^1 \Rightarrow 3^2 11 \cdot 331$ S
$65537^\infty : 520358966.88 < p_6 < 4294967296.00$
$520358983^\infty \longrightarrow$

. . . .

10^{12} $3^\infty\ 5^\infty\ 17^\infty\ 257^\infty\ 65537^\infty\ 520358983 \leq p_6 < 4294967296.00$
$3^{18} \Rightarrow 1597 \cdot 363889$ M1

$3^{20} \Rightarrow 13 \cdot 1093 \cdot 368089$ M1

$3^{22} \Rightarrow 47 \cdot 1001523179$ M1

$3^{24} \Rightarrow 11^2 8951 \cdot 391151$ M1

3^{∞} : $2.99 < p_2 < 16.00$

$5^{12} \Rightarrow 305175781$ M1

$5^{13} \Rightarrow 3 \cdot 29 \cdot 449 \cdot 19531$ M1

$5^{14} \Rightarrow 11 \cdot 31 \cdot 71 \cdot 181 \cdot 1741$ M1

$5^{16} \Rightarrow 409 \cdot 466344409$ M1

5^{∞} : $14.99 < p_3 < 61.00$

$17^{8} \Rightarrow 19 \cdot 307 \cdot 1270657$ M1

17^{∞} : $254.99 < p_4 < 766.00$

257^{∞} : $65534.99 < p_5 < 131071.00$

$65537^{2} \Rightarrow 37 \cdot 116085511$ S

65537^{∞} : $4274620823.30 < p_6 < 4294967296.00$

4274620829^{∞} $\longrightarrow$

....

10^{15} 3^{∞} 5^{∞} 17^{∞} 257^{∞} 65537^{∞} $4274620829 \leq p_6 < 4294967296$

$3^{26} \Rightarrow 13 \cdot 109 \cdot 433 \cdot 757 \cdot 8209$ M1

$3^{28} \Rightarrow 59 \cdot 28537 \cdot 20381027$ M1

$3^{30} \Rightarrow 683 \cdot 102673 \cdot 4404047$ M1

3^{∞} : $2.99 < p_2 < 16.00$

$5^{17} \Rightarrow 3^3 7 \cdot 19 \cdot 31 \cdot 829 \cdot 5167$ M1

$5^{18} \Rightarrow 191 \cdot 6271 \cdot 3981071$ M1

$5^{20} \Rightarrow 31 \cdot 379 \cdot 19531 \cdot 519499$ M1

5^{∞} : $14.99 < p_3 < 61.00$

$17^{9} \Rightarrow 3^2 11 \cdot 71 \cdot 101 \cdot 88741$

$17^{10} \Rightarrow 2141993519227$ M1

17^{∞} : $254.99 < p_4 < 766.00$

$257^{4} \Rightarrow 11 \cdot 398137391$ M1

$257^{5} \Rightarrow 3^2 7 \cdot 13 \cdot 43 \cdot 61 \cdot 241 \cdot 1087$ M1

257^{∞} : $65534.99 < p_5 < 131071.00$

65537^{∞} : $4294960317.71 < p_6 < 4294967296.00$

4294960321^{∞} $\longrightarrow$

....

10^{18} 3^{∞} 5^{∞} 17^{∞} 257^{∞} 65537^{∞} $4294960321 \leq p_6 < 4294967296.00$

$3^{32} \Rightarrow 13 \cdot 23 \cdot 3851 \cdot 2413941289$ M1

$3^{34} \Rightarrow 11^2 71 \cdot 1093 \cdot 2664097031$ M1

$3^{36} \Rightarrow 13097927 \cdot 17189128703$ M1

3^{∞} : $2.99 < p_2 < 16.00$

$5^{21} \Rightarrow 3 \cdot 23 \cdot 67 \cdot 5281 \cdot 12207031$ M1

$5^{22} \Rightarrow 8971 \cdot 332207361361$ M1

$5^{24} \Rightarrow 11 \cdot 71 \cdot 101 \cdot 251 \cdot 401 \cdot 9384251$ M1

$5^{\infty}:\ 14.99 < p_3 < 61.00$

$17^{12} \Rightarrow 212057 \cdot 2919196853$ M1

$17^{13} \Rightarrow 3^2 22796593 \cdot 25646167$ M1

$17^{\infty}:\ 254.99 < p_4 < 766.00$

$257^{6} \Rightarrow 29 \cdot 62273 \cdot 160174771$ M1

$257^{\infty}:\ 65534.99 < p_5 < 131071.00$

$65537^{\infty}:\ 4294967281.52 < p_6 < 4294967296.00$

$4294967291^{\infty}\ \longrightarrow$

. . . .

10^{21} $3^{\infty}\ 5^{\infty}\ 17^{\infty}\ 257^{\infty}\ 65537^{\infty}\ 4294967291 \le p_6 < 4294967296.00$

$3^{38} \Rightarrow 13^2 313 \cdot 6553 \cdot 7333 \cdot 797161$ M1

$3^{40} \Rightarrow 83 \cdot 2526913 \cdot 86950696619$ M1

$3^{42} \Rightarrow 431 \cdot 380808546861411923$ M1

$3^{\infty}:\ 2.99 < p_2 < 16.00$

$5^{25} \Rightarrow 3 \cdot 5227 \cdot 38923 \cdot 305175781$ M1

$5^{26} \Rightarrow 19 \cdot 31 \cdot 109 \cdot 271 \cdot 829 \cdot 4159 \cdot 31051$ M1

$5^{28} \Rightarrow 59 \cdot 35671 \cdot 22125996444329$ M1

$5^{29} \Rightarrow 3^2 7 \cdot 11 \cdot 31 \cdot 61 \cdot 71 \cdot 181 \cdot 521 \cdot 1741 \cdot 7621$ M1

$5^{\infty}:\ 14.99 < p_3 < 61.00$

$17^{14} \Rightarrow 307 \cdot 88741 \cdot 6566760001$ M1

$17^{16} \Rightarrow 10949 \cdot 1749233 \cdot 2699538733$ M1

$17^{\infty}:\ 254.99 < p_4 < 766.00$

$257^{\infty}:\ 65534.99 < p_5 < 131071.00$

$65537^{4} \Rightarrow 3571 \cdot 37693451 \cdot 137055701$ S

$65537^{\infty}:\ 4294967294.98 < p_6 < 4294967296.00$ N $\longleftarrow$

. . . .

10^{18} $65539^{2} \Rightarrow 3 \cdot 31 \cdot 397 \cdot 116341$ S

$65539^{\infty}:\ 1431699453.50 < p_6 < 1431699456.00$ N $\longleftarrow$

$65543^{2} \Rightarrow 7^2 6199 \cdot 14143$ S

$65543^{\infty}:\ 613622929.13 < p_6 < 613622930.29$ N $\longleftarrow$

$65551^{2} \Rightarrow 3 \cdot 127 \cdot 11278213$ S

$65551^{\infty}:\ 286392318.94 < p_6 < 286392320.00$ N $\longleftarrow$

$65557^{1} \Rightarrow 32779$ S

$65557^{2} \Rightarrow 3 \cdot 7 \cdot 19 \cdot 10771393$ S

$65557^{\infty}:\ 204584666.39 < p_6 < 204584667.42$ N $\longleftarrow$

$65563^{2} \Rightarrow 3 \cdot 67 \cdot 199 \cdot 107467$ S

$65563^{\infty}:\ 159135960.97 < p_6 < 159135971.56$

159135961^{∞} A $\longleftarrow$

. . . .

10^{15} $65579^{2} \Rightarrow 8329 \cdot 516349$ S

$65579^{\infty}:\ 99946964.15 < p_6 < 99946972.28$ N

$65581^{1} \Rightarrow 11^2 271$ S

$65581^2 \Rightarrow 3 \cdot 13 \cdot 43 \cdot 433 \cdot 5923$ S

65581^∞ : $95507789.82 < p_6 < 95507797.33$ N

$65587^2 \Rightarrow 3 \cdot 7 \cdot 204843817$ S

65587^∞ : $84279289.93 < p_6 < 84279296.00$ N

$65599^2 \Rightarrow 3 \cdot 7 \cdot 19 \cdot 547 \cdot 19717$ S

65599^∞ : $68238575.49 < p_6 < 68238579.81$ N

$65609^1 \Rightarrow 3^8 5$ A

$65609^2 \Rightarrow 367 \cdot 11729173$ S

65609^∞ : $58899803.21 < p_6 < 58899806.68$ N

$65617^1 \Rightarrow 7 \cdot 43 \cdot 109$ S

$65617^2 \Rightarrow 3 \cdot 433 \cdot 3314593$ S

65617^∞ : $53089011.51 < p_6 < 53089014.52$ N

$65629^1 \Rightarrow 5 \cdot 6563$ S

$65629^2 \Rightarrow 3 \cdot 7^2 29300893$ S

65629^∞ : $46247272.83 < p_6 < 46247275.35$ N

$65633^1 \Rightarrow 3 \cdot 10939$ S

$65633^2 \Rightarrow 13 \cdot 19^2 97 \cdot 9463$ S

65633^∞ : $44342871.33 < p_6 < 44342873.73$ N

$65647^2 \Rightarrow 3 \cdot 5641 \cdot 254659$ S

65647^∞ : $38758342.58 < p_6 < 38758344.65$ N

$65651^2 \Rightarrow 4310119453$ D [14]

65651^∞ : $37412505.83 < p_6 < 37412507.83$ N

$65657^1 \Rightarrow 3 \cdot 31 \cdot 353$ S

$65657^2 \Rightarrow 7 \cdot 103 \cdot 5979067$ S

65657^∞ : $35560589.97 < p_6 < 35560591.87$ N

$65677^1 \Rightarrow 32839$ S

$65677^2 \Rightarrow 3 \cdot 1437844669$ D

65677^∞ : $30525830.51 < p_6 < 30525832.17$ N

$65687^2 \Rightarrow 15901 \cdot 271357$ S

65687^∞ : $28508592.43 < p_6 < 28508594.01$

  28508593^1 D

  28508593^∞ A

$65699^2 \Rightarrow 7 \cdot 1597 \cdot 386119$ S

65699^∞ : $26414625.05 < p_6 < 26414626.55$ N

$65701^1 \Rightarrow 7 \cdot 13 \cdot 19^2$ S

$65701^2 \Rightarrow 3 \cdot 37 \cdot 38889073$ S

65701^∞ : $26095242.15 < p_6 < 26095243.64$

  26095243^∞ A

$65707^2 \Rightarrow 3 \cdot 2539 \cdot 566821$ S

65707^∞ : $25181919.11 < p_6 < 25181920.56$ N

$65713^1 \Rightarrow 11 \cdot 29 \cdot 103$ S

[14] The number $3^a 5^b 17^c 257^d 65651^2 4310119453^e$ is Deficient, for all $a \geq 32, b \geq 21, c \geq 12, d \geq 6, e \geq 2$.

$65713^2 \Rightarrow 3 \cdot 7 \cdot 19 \cdot 2437 \cdot 4441$ S
65713^∞ : $24330516.27 < p_6 < 24330517.69$
24330517^1 A
$65717^1 \Rightarrow 3^3 1217$ S
$65717^2 \Rightarrow 151 \cdot 28601257$ S
65717^∞ : $23794273.60 < p_6 < 23794275.01$ N
$65719^2 \Rightarrow 3 \cdot 1213 \cdot 1186879$ S
65719^∞ : $23534943.13 < p_6 < 23534944.52$ N
$65729^1 \Rightarrow 3 \cdot 5 \cdot 7 \cdot 313$ S
$65729^2 \Rightarrow 4177 \cdot 1034323$ S
65729^∞ : $22318911.64 < p_6 < 22318912.99$ N
$65731^2 \Rightarrow 3 \cdot 13 \cdot 1093 \cdot 101359$ S
65731^∞ : $22090671.88 < p_6 < 22090673.23$ N
$65761^1 \Rightarrow 131 \cdot 251$ S
$65761^2 \Rightarrow 3 \cdot 1441524961$ D
65761^∞ : $19153987.00 < p_6 < 19153988.27$ N
$65777^1 \Rightarrow 3 \cdot 19 \cdot 577$ S
$65777^2 \Rightarrow 241 \cdot 1783 \cdot 10069$ S
65777^∞ : $17886703.90 < p_6 < 17886705.13$ N
$65789^1 \Rightarrow 3^2 5 \cdot 17 \cdot 43$ S
$65789^2 \Rightarrow 13 \cdot 19 \cdot 139 \cdot 126067$ S
65789^∞ : $17041431.08 < p_6 < 17041432.28$ N
$65809^1 \Rightarrow 5 \cdot 6581$ S
$65809^2 \Rightarrow 3 \cdot 7 \cdot 13 \cdot 277 \cdot 57271$ S
65809^∞ : $15797775.70 < p_6 < 15797776.88$ N
$65827^2 \Rightarrow 3 \cdot 19 \cdot 97 \cdot 783733$ S
65827^∞ : $14824647.42 < p_6 < 14824648.58$ N
$65831^2 \Rightarrow 67 \cdot 757 \cdot 85447$ S
65831^∞ : $14624523.86 < p_6 < 14624525.02$ N
$65837^1 \Rightarrow 3 \cdot 10973$ S
$65837^2 \Rightarrow 7^3 73 \cdot 331 \cdot 523$ S
65837^∞ : $14334311.46 < p_6 < 14334312.61$ N
$65839^2 \Rightarrow 3 \cdot 7 \cdot 43 \cdot 4800487$ S
65839^∞ : $14240128.12 < p_6 < 14240129.27$ N
$65843^2 \Rightarrow 139 \cdot 337 \cdot 92551$ S
65843^∞ : $14055442.87 < p_6 < 14055444.01$ N
$65851^2 \Rightarrow 3 \cdot 7 \cdot 206496193$ S
65851^∞ : $13700143.63 < p_6 < 13700144.76$ N
$65867^2 \Rightarrow 7 \cdot 13 \cdot 397 \cdot 120091$ S
65867^∞ : $13041068.90 < p_6 < 13041070.02$ N
$65881^1 \Rightarrow 32941$ S
$65881^2 \Rightarrow 3 \cdot 7 \cdot 109 \cdot 271 \cdot 6997$ S

65881^{∞} : $12514525.50 < p_6 < 12514526.61$ N

$65899^2 \Rightarrow 3 \cdot 19 \cdot 109 \cdot 698977$ S

65899^{∞} : $11897220.19 < p_6 < 11897221.29$ N

$65921^1 \Rightarrow 3 \cdot 10987$ S

$65921^2 \Rightarrow 7 \cdot 193 \cdot 3216613$ S

65921^{∞} : $11221123.90 < p_6 < 11221124.99$ N

$65927^2 \Rightarrow 61 \cdot 73 \cdot 157 \cdot 6217$ S

65927^{∞} : $11049938.39 < p_6 < 11049939.48$ N

$65929^1 \Rightarrow 5 \cdot 19 \cdot 347$ S

$65929^2 \Rightarrow 3 \cdot 1448899657$ D

65929^{∞} : $10994038.12 < p_6 < 10994039.21$ N

$65951^2 \Rightarrow 7 \cdot 15859 \cdot 39181$ S

65951^{∞} : $10414695.79 < p_6 < 10414696.87$ N

$65957^1 \Rightarrow 3 \cdot 10993$ S

$65957^2 \Rightarrow 67 \cdot 64931221$ S

65957^{∞} : $10267201.81 < p_6 < 10267202.89$ N

$65963^2 \Rightarrow 7 \cdot 3691 \cdot 168409$ S

65963^{∞} : $10123852.87 < p_6 < 10123853.94$ N

$65981^1 \Rightarrow 3 \cdot 7 \cdot 1571$ S

$65981^2 \Rightarrow 37 \cdot 2029 \cdot 57991$ S

65981^{∞} : $9716999.56 < p_6 < 9717000.63$ N

$65983^2 \Rightarrow 3 \cdot 73 \cdot 283 \cdot 70249$ S

65983^{∞} : $9673816.28 < p_6 < 9673817.34$ N

$65993^1 \Rightarrow 3 \cdot 17 \cdot 647$ S

$65993^2 \Rightarrow 7 \cdot 31 \cdot 20069779$ S

65993^{∞} : $9463569.42 < p_6 < 9463570.49$ N

..... [15]

$70111^2 \Rightarrow 3 \cdot 67 \cdot 1867 \cdot 13099$ S

70111^{∞} : $1004311.34 < p_6 < 1004312.34$ N

$70117^1 \Rightarrow 35059$ S

$70117^2 \Rightarrow 3 \cdot 19 \cdot 6961 \cdot 12391$ S

70117^{∞} : $1003081.77 < p_6 < 1003082.77$ N

$70121^1 \Rightarrow 3 \cdot 13 \cdot 29 \cdot 31$ S

$70121^2 \Rightarrow 7 \cdot 19 \cdot 36970111$ S

70121^{∞} : $1002263.85 < p_6 < 1002264.85$ N

$70123^2 \Rightarrow 3 \cdot 7 \cdot 127 \cdot 571 \cdot 3229$ S

70123^{∞} : $1001855.42 < p_6 < 1001856.42$ N

$70139^2 \Rightarrow 43 \cdot 283 \cdot 404269$ S

70139^{∞} : $p_6 < 998601.77$ P1

$70141^1 \Rightarrow 17 \cdot 2063$ S

$70141^2 \Rightarrow 3 \cdot 37 \cdot 44322793$ S

[15] Abbreviated for convenience. Contradiction N applies for all $p, 65993 \leq p \leq 70111$

70141^{∞} : $p_6 < 998196.53$ P1
$70157^1 \Rightarrow 3 \cdot 11 \cdot 1063$ S
$70157^2 \Rightarrow 13^2 29124703$ S
70157^{∞} : $p_6 < 994967.24$ P1
$70163^2 \Rightarrow 7 \cdot 61 \cdot 11529079$ S
70163^{∞} : $p_6 < 993762.01$ P1
$70177^1 \Rightarrow 35089$ S
$70177^2 \Rightarrow 3 \cdot 7 \cdot 13 \cdot 859 \cdot 21001$ S
70177^{∞} : $p_6 < 990961.93$ P1
$70181^1 \Rightarrow 3^2 7 \cdot 557$ S
$70181^2 \Rightarrow 4925442943$ D
70181^{∞} : $p_6 < 990165.01$ P1
$70183^2 \Rightarrow 3 \cdot 13 \cdot 126300607$ S
70183^{∞} : $p_6 < 989767.07$ P1
$70199^2 \Rightarrow 37 \cdot 103 \cdot 619 \cdot 2089$ S
70199^{∞} : $p_6 < 986595.78$ P1
.....
$131009^1 \Rightarrow 3 \cdot 5 \cdot 11 \cdot 397$ S
$131009^2 \Rightarrow 7 \cdot 2451927013$ S
131009^{∞} : $p_6 < 131134.06$ P1
$131011^2 \Rightarrow 3 \cdot 31 \cdot 97 \cdot 373 \cdot 5101$ S
131011^{∞} : $p_6 < 131132.06$ P1
$131023^2 \Rightarrow 3 \cdot 7 \cdot 13 \cdot 62883361$ S
131023^{∞} : $p_6 < 131120.04$ P1
$131041^1 \Rightarrow 65521$ S
$131041^2 \Rightarrow 3 \cdot 61 \cdot 1879 \cdot 49939$ S
131041^{∞} : $p_6 < 131102.01$ P1
$131059^2 \Rightarrow 3 \cdot 4933 \cdot 1160659$ S
131059^{∞} : $p_6 < 131084.00$ P1
$131063^2 \Rightarrow 7 \cdot 109 \cdot 22513291$ S
131063^{∞} : $p_6 < 131080.00$ P1

....

10^6 263^{∞} : $9409.54 < p_5 < 19162.43$
9413^{∞} P2
.....
9997^{∞} P2
10007^{∞} : $p_6 < 225435.82$ P1
.....
19157^{∞} : $p_6 < 19168.87$ P1
$269^1 \Rightarrow 3^3 5$: $p_5 < 9836.72$ P2
269^{∞} : $5224.29 < p_5 < 10554.08$
5227^{∞} P2

.....

9997^∞ P2

$10007^\infty : p_6 < 11165.56$ P1

.....

$10531^\infty : p_6 < 10578.26$ P1

$271^\infty : p_5 < 9215.00$ P2

$277^1 \Rightarrow 139$ S

$277^\infty : p_5 < 6728.15$ P2

$281^1 \Rightarrow 3 \cdot 47$ S

$281^\infty : p_5 < 5733.40$ P2

$283^\infty : p_5 < 5346.56$ P2

$293^1 \Rightarrow 3 \cdot 7^2$ S

$293^\infty : p_5 < 4039.65$ P2

$307^\infty : p_5 < 3071.00$ P2

$311^\infty : p_5 < 2884.82$ P2

$313^1 \Rightarrow 157$ S

$313^\infty : p_5 < 2801.53$ P2

$317^1 \Rightarrow 3 \cdot 53$ S

$317^\infty : p_5 < 2651.33$ P2

$331^\infty : p_5 < 2251.80$ P2

$337^1 \Rightarrow 13^2$ S

$337^\infty : p_5 < 2122.86$ P2

$347^\infty : p_5 < 1945.73$ P2

$349^1 \Rightarrow 5^2 7$ S

$349^\infty : p_5 < 1914.88$ P2

$353^1 \Rightarrow 3 \cdot 59$ S

$353^\infty : p_5 < 1856.98$ P2

$359^\infty : p_5 < 1778.58$ P2

$367^\infty : p_5 < 1687.22$ P2

$373^1 \Rightarrow 11 \cdot 17$ S

$373^\infty : p_5 < 1626.90$ P2

$379^\infty : p_5 < 1572.47$ P2

$383^\infty : p_5 < 1539.04$ P2

$389^1 \Rightarrow 3 \cdot 5 \cdot 13$ S

$389^\infty : p_5 < 1492.66$ P2

$397^1 \Rightarrow 199$ S

$397^\infty : p_5 < 1436.96$ P2

$401^1 \Rightarrow 3 \cdot 67$ S

$401^\infty : p_5 < 1411.42$ P2

$409^1 \Rightarrow 5 \cdot 41$ S

$409^\infty : p_5 < 1364.34$ P2

$419^\infty : p_5 < 1311.99$ P2

$421^1 \Rightarrow 211$ S

$421^\infty : p_5 < 1302.28$ P2

$431^\infty : p_5 < 1257.06$ P2

$433^1 \Rightarrow 7 \cdot 31$ S

$433^\infty : p_5 < 1248.63$ P2

$439^\infty : p_5 < 1224.45$ P2

$443^\infty : p_5 < 1209.19$ P2

$449^1 \Rightarrow 3^2 5^2 : p_5 < 1183.99$ P2

$449^\infty : p_5 < 1187.48$ P2

$457^1 \Rightarrow 229$ S

$457^\infty : p_5 < 1160.56$ P2

$461^1 \Rightarrow 3 \cdot 7 \cdot 11$ S

$461^\infty : p_5 < 1147.88$ P2

$463^\infty : p_5 < 1141.73$ P2

$467^\infty : p_5 < 1129.77$ P2

$479^\infty : p_5 < 1096.48$ P2

$487^\infty : p_5 < 1076.20$ P2

$491^\infty : p_5 < 1066.58$ P2

$499^\infty : p_5 < 1048.29$ P2

$503^\infty : p_5 < 1039.59$ P2

$509^1 \Rightarrow 3 \cdot 5 \cdot 17 : p_5 < 1025.02$ P2

$509^\infty : p_5 < 1027.05$ P2

$521^1 \Rightarrow 3^2 29$ S

$521^\infty : p_5 < 1003.68$ P2

$523^\infty : p_5 < 999.99$ P2

$541^1 \Rightarrow 271$ S

$541^\infty : p_5 < 969.11$ P2

$547^\infty : p_5 < 959.66$ P2

$557^1 \Rightarrow 3^2 31$ S

$557^\infty : p_5 < 944.76$ P2

$563^\infty : p_5 < 936.28$ P2

$569^1 \Rightarrow 3 \cdot 5 \cdot 19$ S

$569^\infty : p_5 < 928.13$ P2

$571^\infty : p_5 < 925.48$ P2

$577^1 \Rightarrow 17^2 : p_5 < 916.47$ P2

$577^\infty : p_5 < 917.73$ P2

$587^\infty : p_5 < 905.45$ P2

$593^1 \Rightarrow 3^3 11$ S

$593^\infty : p_5 < 898.42$ P2

$599^\infty : p_5 < 891.65$ P2

$601^1 \Rightarrow 7 \cdot 43$ S

$601^\infty : p_5 < 889.44$ P2

607^{∞} : $p_5 < 882.97$ P2
$613^1 \Rightarrow 307$ S
613^{∞} : $p_5 < 876.72$ P2
$617^1 \Rightarrow 3 \cdot 103$ S
617^{∞} : $p_5 < 872.67$ P2
619^{∞} : $p_5 < 870.67$ P2
631^{∞} : $p_5 < 859.16$ P2
$641^1 \Rightarrow 3 \cdot 107$ S
641^{∞} : $p_5 < 850.12$ P2
643^{∞} : $p_5 < 848.37$ P2
647^{∞} : $p_5 < 844.92$ P2
$653^1 \Rightarrow 3 \cdot 109$ S
653^{∞} : $p_5 < 839.87$ P2
659^{∞} : $p_5 < 834.98$ P2
$661^1 \Rightarrow 331$ S
661^{∞} : $p_5 < 833.38$ P2
$673^1 \Rightarrow 337$ S
673^{∞} : $p_5 < 824.10$ P2
$677^1 \Rightarrow 3 \cdot 113$ S
677^{∞} : $p_5 < 821.12$ P2
683^{∞} : $p_5 < 816.77$ P2
691^{∞} : $p_5 < 811.14$ P2
$701^1 \Rightarrow 3^3 13$ S
701^{∞} : $p_5 < 804.40$ P2
$709^1 \Rightarrow 5 \cdot 71$ S
709^{∞} : $p_5 < 799.22$ P2
719^{∞} : $p_5 < 792.99$ P2
727^{∞} : $p_5 < 788.20$ P2
$733^1 \Rightarrow 367$ S
733^{∞} : $p_5 < 784.72$ P2
739^{∞} : $p_5 < 781.32$ P2
743^{∞} : $p_5 < 779.10$ P2
751^{∞} : $p_5 < 774.76$ P2
$757^1 \Rightarrow 379$ S
757^{∞} : $p_5 < 771.60$ P2
$761^1 \Rightarrow 3 \cdot 127$ S
761^{∞} : $p_5 < 769.54$ P2
$19^2 \Rightarrow 3 \cdot 127$
127^{∞} : $p_5 < 737.49$ P2
19^{∞} : $95.98 < p_4 < 286.00$
$97^1 \Rightarrow 7^2$ S
$97^2 \Rightarrow 3 \cdot 3169$ P2

$97^\infty : p_5 < 18431.00$ P3

$101^1 \Rightarrow 3 \cdot 17$ S

$101^\infty : p_5 < 3839.00$ P2

$103^\infty : p_5 < 2796.72$ P2

$107^\infty : p_5 < 1849.19$ P2

$109^1 \Rightarrow 5 \cdot 11$ S

$109^\infty : p_5 < 1594.08$ P2

$113^1 \Rightarrow 3 \cdot 19 : p_5 < 1204.34$ P2

$113^\infty : p_5 < 1263.95$ P2

$127^\infty : p_5 < 779.39$ P2

$131^\infty : p_5 < 712.15$ P2

$137^1 \Rightarrow 3 \cdot 23$ S

$137^\infty : p_5 < 635.88$ P2

$139^\infty : p_5 < 615.19$ P2

$149^1 \Rightarrow 3 \cdot 5^2 : p_5 < 528.78$ P2

$149^\infty : p_5 < 535.16$ P2

$151^\infty : p_5 < 522.64$ P2

$157^1 \Rightarrow 79$ S

$157^\infty : p_5 < 490.02$ P2

$163^\infty : p_5 < 463.24$ P2

$167^\infty : p_5 < 447.91$ P2

$173^1 \Rightarrow 3 \cdot 29$ S

$173^\infty : p_5 < 427.89$ P2

$179^\infty : p_5 < 410.76$ P2

$181^1 \Rightarrow 7 \cdot 13$ S

$181^\infty : p_5 < 405.59$ P2

$191^\infty : p_5 < 383.00$ P2

$193^1 \Rightarrow 97$ S

$193^\infty : p_5 < 379.05$ P2

$197^1 \Rightarrow 3^2 11$ S

$197^\infty : p_5 < 371.60$ P2

$199^\infty : p_5 < 368.09$ P2

$211^\infty : p_5 < 349.61$ P2

$223^\infty : p_5 < 334.63$ P2

$227^\infty : p_5 < 330.24$ P2

$229^1 \Rightarrow 5 \cdot 23$ S

$229^\infty : p_5 < 328.15$ P2

$233^1 \Rightarrow 3^2 13$ S

$233^\infty : p_5 < 324.14$ P2

$239^\infty : p_5 < 318.56$ P2

$241^1 \Rightarrow 11^2$ S

$241^\infty : p_5 < 316.80$ P2

$251^{\infty} : p_5 < 308.68$ P2

$257^1 \Rightarrow 3 \cdot 43$ S

$257^{\infty} : p_5 < 304.30$ P2

$263^{\infty} : p_5 < 300.23$ P2

$269^1 \Rightarrow 3^3 5 : p_5 < 295.83$ P2

$269^{\infty} : p_5 < 296.44$ P2

$271^{\infty} : p_5 < 295.23$ P2

$277^1 \Rightarrow 139$ S

$277^{\infty} : p_5 < 291.78$ P2

$281^1 \Rightarrow 3 \cdot 47$ S

$281^{\infty} : p_5 < 289.60$ P2

$283^{\infty} : p_5 < 288.55$ P2

$23^2 \Rightarrow 7 \cdot 79$ S

$23^{\infty} : 49.28 < p_4 < 148.86, p_5 < 1925.74$ P2

$29^1 \Rightarrow 3 \cdot 5 : 33.11 < p_4 < 97.43, p_5 < 617.67$ P2

$29^2 \Rightarrow 13 \cdot 67$ S

$29^{\infty} : 33.46 < p_4 < 101.39, p_5 < 976.46$ P2

$31^2 \Rightarrow 3 \cdot 331$

$331^{\infty} : p_5 < 69.56$ P2

$31^{\infty} : p_4 < 94.00$ P3

$37^1 \Rightarrow 19$ S

$37^2 \Rightarrow 3 \cdot 7 \cdot 67$ S

$37^{\infty} : p_4 < 80.29$ P3

$41^1 \Rightarrow 3 \cdot 7$ S

$41^2 \Rightarrow 1723$

$1723^{\infty} : p_5 < 50.93$ P2

$41^{\infty} : p_4 < 74.80$ P3

$43^2 \Rightarrow 3 \cdot 631$

$631^{\infty} : p_5 < 50.73$ P2

$43^{\infty} : p_4 < 72.67$ P3

$47^2 \Rightarrow 37 \cdot 61$ S

$47^{\infty} : p_4 < 69.23$ P3

$53^1 \Rightarrow 3^3 : 22.27 < p_4 < 64.95, p_5 < 69.57$ P2

$53^2 \Rightarrow 7 \cdot 409$ S

$53^{\infty} : p_4 < 65.46$ P3

$59^2 \Rightarrow 3541$

$3541^{\infty} : p_5 < 42.42$ P2

$59^{\infty} : p_4 < 62.75$ P3

$7^2 \Rightarrow 3 \cdot 19$

$19^2 \Rightarrow 3 \cdot 127$

$127^{\infty} : p_5 < 26.82$ P2

$19^{\infty} : p_4 < 35.94$ P3

$7^4 \Rightarrow 2801$

2801^∞ : $7.01 < p_4 < 22.06$, $p_5 < 52.75$ P2

$7^6 \Rightarrow 29 \cdot 4733$

$29^1 \Rightarrow 3 \cdot 5$ S

$29^2 \Rightarrow 13 \cdot 67$ P1

29^∞

4733^∞ : $p_5 < 20.38$ P2

7^∞ : $6.99 < p_3 < 29.00$, $p_4 < 78.00$ P3

$11^2 \Rightarrow 7 \cdot 19$ S

$11^4 \Rightarrow 5 \cdot 3221$ S

11^∞ : $4.71 < p_3 < 19.86$, $p_4 < 26.24$ P3

$13^1 \Rightarrow 7$ S

$13^2 \Rightarrow 3 \cdot 61$

$61^1 \Rightarrow 31$ P3

$61^2 \Rightarrow 3 \cdot 13 \cdot 97$ P3

61^∞ : $p_4 < 15.21$ P3

$13^4 \Rightarrow 30941$

30941^∞ : $4.33 < p_4 < 14.01$, $p_5 < 13.64$ P3

13^∞ : $4.33 < p_3 < 18.34$, $p_4 < 19.95$ P3

End of proof.

Appendix J

An alternative proof that an odd perfect number has at least seven distinct prime factors

This proof consists of the output produced by the latest version of the *Mathematica* code and saved as (plain) text. It uses a static bound of $B = 10^{21}$ and 10^7 for the lower bound on the largest prime factor (for the *P1* contradiction).

This output has been edited only to the extent that “InvisibleSpace” has been removed and so its appearance matches that of *Mathematica* itself. Some of the numeric values are expressed in scientific format. In this text output no attempt has been made to show the exponent as a superscript (but it is easily inferred).

NO CASES HAVE BEEN ABBREVIATED.

The text of this proof is available on the companion Compact Disc as the plain ASCII text file `\Text\Opn7B21.txt`.

Appendix K

A proof that $3^{26} \nparallel N$ when $\omega(N) = 8$ taking $B = 10^{18}$

10^{18} $3^{26} \Rightarrow 13 \cdot 109 \cdot 433 \cdot 757 \cdot 8209$ [1]

$13^1 \Rightarrow 7$

$7^2 \Rightarrow 3 \cdot 19$ A=2.0091207268419272780 [2]

$7^4 \Rightarrow 2801$ D=1.9097910448205824644 [3]

$7^6 \Rightarrow 29 \cdot 4733$ M1=9 [4]

$7^8 \Rightarrow 3^2 \cdot 19 \cdot 37 \cdot 1063$ M1=10

$7^{10} \Rightarrow 1123 \cdot 293459$ M1=9

$7^{12} \Rightarrow 16148168401$ D=1.9092228170809066546

$7^{14} \Rightarrow 3 \cdot 19 \cdot 31 \cdot 2801 \cdot 159871$ M1=11

$7^{16} \Rightarrow 14009 \cdot 2767631689$ M1=9

$7^{18} \Rightarrow 419 \cdot 4534166740403$ M1=9

$7^{20} \Rightarrow 3 \cdot 19 \cdot 29 \cdot 4733 \cdot 11898664849$ M1=11

7^{∞}

$109^2 \Rightarrow 3 \cdot 7 \cdot 571$ D=1.9125708538396263617

$109^4 \Rightarrow 31 \cdot 191 \cdot 24061$ M1=10

$109^6 \Rightarrow 113 \cdot 281 \cdot 53306107$ M1=10

109^{∞}

$433^2 \Rightarrow 3 \cdot 37 \cdot 1693$ M1=9

$433^4 \Rightarrow 11 \cdot 1811 \cdot 1768661$ M1=10

433^{∞}

$757^2 \Rightarrow 3 \cdot 13 \cdot 14713$ M2=13

$757^4 \Rightarrow 11 \cdot 191 \cdot 2521 \cdot 62081$ M1=11

757^{∞}

$8209^2 \Rightarrow 3 \cdot 31^2 \cdot 97 \cdot 241$ M1=10

8209^{∞} : $p_8 < 22.04$ P2 [5]

$13^2 \Rightarrow 3 \cdot 61$

$61^1 \Rightarrow 31$ D=1.7281913285275361414

$61^2 \Rightarrow 3 \cdot 13 \cdot 97$ D=1.6903113223996837594

$61^4 \Rightarrow 5 \cdot 131 \cdot 21491$ M1=10

$61^5 \Rightarrow 3 \cdot 7 \cdot 13 \cdot 31 \cdot 97 \cdot 523$ M1=11

$61^6 \Rightarrow 52379047267$ D=1.6728928026981493943

$61^8 \Rightarrow 3^2 \cdot 13 \cdot 19 \cdot 97 \cdot 903870199$ M1=10

$61^9 \Rightarrow 5 \cdot 11 \cdot 31 \cdot 131 \cdot 21491 \cdot 1238411$ M1=13

61^{∞}

$109^1 \Rightarrow 5 \cdot 11$ M1=9

$109^2 \Rightarrow 3 \cdot 7 \cdot 571$ M1=9

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[1] Convenient notation for odd prime factors of $\sigma(3^{26}) = 13 \cdot 109 \cdot 433 \cdot 757 \cdot 8209$; further factorisations in the chain are indicated by indentations.

[2] The number $3^{26}13 \cdot 7^2 \cdot 19^2 109^2 433^2 757^2 8209^2$ is Abundant.

[3] The number $3^{26}13 \cdot 7^4 \cdot 109^{\infty} 433^{\infty} 757^{\infty} 2801^{\infty} 8209^{\infty}$ is Deficient.

[4] There are too Many (9) prime factors.

[5] $\lambda = 3^{26}13,\ 7 \cdot 109 \cdot 433 \cdot 757 \cdot 8209 \mid \mu$

$109^4 \Rightarrow 31 \cdot 191 \cdot 24061$ M1=10

$109^5 \Rightarrow 3 \cdot 5 \cdot 7 \cdot 11 \cdot 61 \cdot 193 \cdot 571$ M1=12

$109^6 \Rightarrow 113 \cdot 281 \cdot 53306107$ M1=10

109^∞

$433^1 \Rightarrow 7 \cdot 31$ M1=9

$433^2 \Rightarrow 3 \cdot 37 \cdot 1693$ M1=9

$433^4 \Rightarrow 11 \cdot 1811 \cdot 1768661$ M1=10

$433^5 \Rightarrow 3 \cdot 7 \cdot 13 \cdot 31 \cdot 37 \cdot 1693 \cdot 14389$ M1=12

433^∞

$757^1 \Rightarrow 379$ D=1.6773155179948220951

$757^2 \Rightarrow 3 \cdot 13 \cdot 14713$ D=1.6730065082175634046

$757^4 \Rightarrow 11 \cdot 191 \cdot 2521 \cdot 62081$ M1=11

$757^5 \Rightarrow 3 \cdot 13 \cdot 163 \cdot 379 \cdot 3511 \cdot 14713$ M1=11

757^∞

$8209^1 \Rightarrow 5 \cdot 821$ M1=9

$8209^2 \Rightarrow 3 \cdot 31^2 \cdot 97 \cdot 241$ M1=10

$8209^\infty : p_8 < 6.12$ P2

$13^4 \Rightarrow 30941$

$109^1 \Rightarrow 5 \cdot 11$ M1=9

$109^2 \Rightarrow 3 \cdot 7 \cdot 571$ M1=9

$109^4 \Rightarrow 31 \cdot 191 \cdot 24061$ M1=10

$109^5 \Rightarrow 3 \cdot 5 \cdot 7 \cdot 11 \cdot 61 \cdot 193 \cdot 571$ M1=13

$109^6 \Rightarrow 113 \cdot 281 \cdot 53306107$ M1=10

109^∞

$433^1 \Rightarrow 7 \cdot 31$ M1=9

$433^2 \Rightarrow 3 \cdot 37 \cdot 1693$ M1=9

$433^4 \Rightarrow 11 \cdot 1811 \cdot 1768661$ M1=10

$433^5 \Rightarrow 3 \cdot 7 \cdot 13 \cdot 31 \cdot 37 \cdot 1693 \cdot 14389$ M1=12

433^∞

$757^1 \Rightarrow 379$ D=1.6506187276121187655

$757^2 \Rightarrow 3 \cdot 13 \cdot 14713$ D=1.6463783016698907180

$757^4 \Rightarrow 11 \cdot 191 \cdot 2521 \cdot 62081$ M1=11

$757^5 \Rightarrow 3 \cdot 13 \cdot 163 \cdot 379 \cdot 3511 \cdot 14713$ M1=11

757^∞

$8209^1 \Rightarrow 5 \cdot 821$ M1=9

$8209^2 \Rightarrow 3 \cdot 31^2 \cdot 97 \cdot 241$ M1=10

8209^∞

$30941^1 \Rightarrow 3^4 \cdot 191$ D=1.6549309642021578139

$30941^2 \Rightarrow 157 \cdot 433 \cdot 14083$ M1=9

$30941^\infty : p_8 < 5.66$ P1 [6]

$13^5 \Rightarrow 3 \cdot 7 \cdot 61 \cdot 157$ M1=9

[6] $\lambda = 3^{26}13^4,\ 109 \cdot 433 \cdot 757 \cdot 8209 \cdot 30941 \mid \mu$

$13^6 \Rightarrow 5229043$

$109^1 \Rightarrow 5 \cdot 11$ M1=9

$109^2 \Rightarrow 3 \cdot 7 \cdot 571$ M1=9

$109^4 \Rightarrow 31 \cdot 191 \cdot 24061$ M1=10

$109^5 \Rightarrow 3 \cdot 5 \cdot 7 \cdot 11 \cdot 61 \cdot 193 \cdot 571$ M1=13

$109^6 \Rightarrow 113 \cdot 281 \cdot 53306107$ M1=10

109^∞

$433^1 \Rightarrow 7 \cdot 31$ M1=9

$433^2 \Rightarrow 3 \cdot 37 \cdot 1693$ M1=9

$433^4 \Rightarrow 11 \cdot 1811 \cdot 1768661$ M1=10

$433^5 \Rightarrow 3 \cdot 7 \cdot 13 \cdot 31 \cdot 37 \cdot 1693 \cdot 14389$ M1=12

433^∞

$757^1 \Rightarrow 379$ D=1.6505701151290084073

$757^2 \Rightarrow 3 \cdot 13 \cdot 14713$ D=1.6463298140718498485

$757^4 \Rightarrow 11 \cdot 191 \cdot 2521 \cdot 62081$ M1=11

$757^5 \Rightarrow 3 \cdot 13 \cdot 163 \cdot 379 \cdot 3511 \cdot 14713$ M1=11

757^∞

$8209^1 \Rightarrow 5 \cdot 821$ M1=9

$8209^2 \Rightarrow 3 \cdot 31^2 \cdot 97 \cdot 241$ M1=10

8209^∞

$5229043^\infty : p_8 < 5.66$ P1 [7]

$13^8 \Rightarrow 3^2 \cdot 61 \cdot 1609669$ D=1.6736556331540876847

$13^9 \Rightarrow 7 \cdot 11 \cdot 2411 \cdot 30941$ M1=10

$13^{10} \Rightarrow 23 \cdot 419 \cdot 859 \cdot 18041$ M1=10

$13^{12} \Rightarrow 53 \cdot 264031 \cdot 1803647$ M1=9

$13^{13} \Rightarrow 7^2 \cdot 29 \cdot 22079 \cdot 5229043$ M1=10

$13^{14} \Rightarrow 3 \cdot 61 \cdot 4651 \cdot 30941 \cdot 161971$ M1=10

13^∞

$109^1 \Rightarrow 5 \cdot 11$ A=2.2435649333980129222

$109^2 \Rightarrow 3 \cdot 7 \cdot 571$ D=1.9239552041601003280

$109^4 \Rightarrow 31 \cdot 191 \cdot 24061$ M1=9

$109^5 \Rightarrow 3 \cdot 5 \cdot 7 \cdot 11 \cdot 61 \cdot 193 \cdot 571$ M1=12

$109^6 \Rightarrow 113 \cdot 281 \cdot 53306107$ M1=9

109^∞

$433^1 \Rightarrow 7 \cdot 31$ D=1.9845962279384374350

$433^2 \Rightarrow 3 \cdot 37 \cdot 1693$ D=1.6929458477667729817

$433^4 \Rightarrow 11 \cdot 1811 \cdot 1768661$ M1=9

$433^5 \Rightarrow 3 \cdot 7 \cdot 13 \cdot 31 \cdot 37 \cdot 1693 \cdot 14389$ M1=11

433^∞

$757^1 \Rightarrow 379$

$379^2 \Rightarrow 3 \cdot 61 \cdot 787$ M1=9

[7] uses 10^7 for *P1* bound

$379^4 \Rightarrow 11 \cdot 41 \cdot 45869891$ M1=10

379^∞

$8209^2 \Rightarrow 3 \cdot 31^2 \cdot 97 \cdot 241$ M1=10

$8209^\infty : p_8 < 5.73$ P2

$757^2 \Rightarrow 3 \cdot 13 \cdot 14713$

$8209^1 \Rightarrow 5 \cdot 821$ M1=9

$8209^2 \Rightarrow 3 \cdot 31^2 \cdot 97 \cdot 241$ M1=10

8209^∞

$14713^1 \Rightarrow 7 \cdot 1051$ M1=9

$14713^2 \Rightarrow 3 \cdot 19 \cdot 3798019$ M1=9

$14713^\infty : p_8 < 5.66$ P1

$757^4 \Rightarrow 11 \cdot 191 \cdot 2521 \cdot 62081$ M1=10

$757^5 \Rightarrow 3 \cdot 13 \cdot 163 \cdot 379 \cdot 3511 \cdot 14713$ M1=10

757^∞

$8209^1 \Rightarrow 5 \cdot 821$ A=2.0437992331848772244

$8209^2 \Rightarrow 3 \cdot 31^2 \cdot 97 \cdot 241$ M1=9

$8209^\infty : p_7 < 10.31$ P2

Done - 135 lines

Appendix L

A proof that $61^5 \nparallel N$ when $\omega(N) = 8$ taking $B = 10^{40}$

10^{40} $61^5 \Rightarrow 3 \cdot 7 \cdot 13 \cdot 31 \cdot 97 \cdot 523$ [1]

$3^2 \Rightarrow 13$

$7^2 \Rightarrow 3 \cdot 19$ A=2.0421960462495938029 [2]

$7^4 \Rightarrow 2801$ D=1.9421809329899115352 [3]

$7^6 \Rightarrow 29 \cdot 4733$ M1=9 [4]

$7^8 \Rightarrow 3^2 \cdot 19 \cdot 37 \cdot 1063$ M1=10

$7^{10} \Rightarrow 1123 \cdot 293459$ M1=9

$7^{12} \Rightarrow 16148168401$ D=1.9416030681577419028

$7^{14} \Rightarrow 3 \cdot 19 \cdot 31 \cdot 2801 \cdot 159871$ M1=10

$7^{16} \Rightarrow 14009 \cdot 2767631689$ M1=9

$7^{18} \Rightarrow 419 \cdot 4534166740403$ M1=9

$7^{20} \Rightarrow 3 \cdot 19 \cdot 29 \cdot 4733 \cdot 11898664849$ M1=11

$7^{22} \Rightarrow 47 \cdot 3083 \cdot 31479823396757$ M1=10

$7^{24} \Rightarrow 2551 \cdot 2801 \cdot 31280679788951$ M1=10

$7^{26} \Rightarrow 3^3 \cdot 19 \cdot 37 \cdot 109 \cdot 811 \cdot 1063 \cdot 2377 \cdot 2583253$ M1=14

$7^{28} \Rightarrow 59 \cdot 127540261 \cdot 71316922984999$ M1=10

$7^{30} \Rightarrow 311 \cdot 21143 \cdot 3999088279399464409$ M1=10

$7^{32} \Rightarrow 3 \cdot 19 \cdot 1123 \cdot 3631 \cdot 293459 \cdot 1532917 \cdot 12323587$ M1=13

$7^{34} \Rightarrow 29 \cdot 2801 \cdot 4733 \cdot 2127431041 \cdot 77192844961$ M1=12

$7^{36} \Rightarrow 223 \cdot 2887 \cdot 4805345109492315767981401$ M1=10

$7^{38} \Rightarrow 3 \cdot 19 \cdot 486643 \cdot 7524739 \cdot 44975113 \cdot 16148168401$ M1=12

$7^{40} \Rightarrow 83 \cdot 20515909 \cdot 4362139336229068656094783$ M1=10

$7^{42} \Rightarrow 166003607842448777 \cdot 2192537062271178641$ M1=9

$7^{44} \Rightarrow 3^2 \cdot 19 \cdot 31 \cdot 37 \cdot 1063 \cdot 2801 \cdot 159871 \cdot 1527007411 \cdot 125096112091$ M1=14

$7^{46} \Rightarrow 13722816749522711 \cdot 6368151199641855045948 7$ M1=9

7^{∞}

$13^2 \Rightarrow 3 \cdot 61$

$31^2 \Rightarrow 3 \cdot 331$ D=1.9465349418253326609

$31^4 \Rightarrow 5 \cdot 11 \cdot 17351$ M1=10

$31^6 \Rightarrow 917087137$ D=1.9407193181377494680

$31^8 \Rightarrow 3^2 \cdot 331 \cdot 3637 \cdot 81343$ M1=10

$31^{10} \Rightarrow 23 \cdot 397 \cdot 617 \cdot 150332843$ M1=11

$31^{12} \Rightarrow 42407 \cdot 2426789 \cdot 7908811$ M1=10

$31^{14} \Rightarrow 3 \cdot 5 \cdot 11 \cdot 331 \cdot 2521 \cdot 17351 \cdot 327412201$ M1=13

$31^{16} \Rightarrow 751670559138758105956097$ D=1.9407193160921110563

$31^{18} \Rightarrow 571 \cdot 14251 \cdot 8877066633261076216 9$ M1=10

$31^{20} \Rightarrow 3 \cdot 43 \cdot 331 \cdot 6301 \cdot 917087137 \cdot 2813432694367$ M1=12

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[1] Convenient notation for odd prime factors of $\sigma(61^5) = 3 \cdot 7 \cdot 13 \cdot 31 \cdot 97 \cdot 523$; further factorisations in the chain are indicated by indentations.

[2] The number $61^5 3^2 7^2 \cdot 13^2 19^2 31^2 97^2 523^2$ is Abundant.

[3] The number $61^5 3^2 7^4 \cdot 13^{\infty} 31^{\infty} 97^{\infty} 523^{\infty} 2801^{\infty}$ is Deficient.

[4] There are too Many (9) prime factors.

$31^{22} \Rightarrow 1509997 \cdot 61562537 \cdot 7176374761323733117$ M1=10

$31^{24} \Rightarrow 5^2 \cdot 11 \cdot 101 \cdot 4951 \cdot 17351 \cdot 13277801 \cdot 20235942281002951$ M1=14

31^{∞}

$97^2 \Rightarrow 3 \cdot 3169$ D=1.9413297898025716172

$97^4 \Rightarrow 11 \cdot 31 \cdot 262321$ M1=9

$97^6 \Rightarrow 43 \cdot 967 \cdot 20241187$ M1=10

$97^8 \Rightarrow 3^2 \cdot 1153 \cdot 3169 \cdot 240813217$ M1=10

$97^{10} \Rightarrow 89 \cdot 837197335075000483$ M1=9

$97^{12} \Rightarrow 53 \cdot 79 \cdot 20359 \cdot 8224356155341457$ M1=11

$97^{14} \Rightarrow 3 \cdot 11 \cdot 31 \cdot 3169 \cdot 244471 \cdot 262321 \cdot 31728277831$ M1=12

$97^{16} \Rightarrow 620652129019588680550123276746411$ D=1.9407193160921110563

$97^{18} \Rightarrow 21433 \cdot 280314943 \cdot 9719915834548161102949 3$ M1=10

97^{∞}

$523^2 \Rightarrow 3 \cdot 13 \cdot 7027$ D=1.9409955221818529029

$523^4 \Rightarrow 3491 \cdot 21472771$ M1=9

$523^6 \Rightarrow 20504128695797293$ D=1.9407193160921111508

$523^8 \Rightarrow 3^2 \cdot 13 \cdot 7027 \cdot 6821641334623519$ M1=9

$523^{10} \Rightarrow 89 \cdot 199 \cdot 397 \cdot 2003 \cdot 108926496542584733$ M1=12

$523^{12} \Rightarrow 53 \cdot 7917272313053609286926516514677$ M1=9

$523^{\infty} : p_8 < 33.74$ P2 [5]

$13^4 \Rightarrow 30941$ D=1.9416605924029926254

$13^6 \Rightarrow 5229043$ D=1.9416034084263569143

$13^8 \Rightarrow 3^2 \cdot 61 \cdot 1609669$ D=1.9416042740878253532

$13^{10} \Rightarrow 23 \cdot 419 \cdot 859 \cdot 18041$ M1=11

$13^{12} \Rightarrow 53 \cdot 264031 \cdot 1803647$ M1=10

$13^{14} \Rightarrow 3 \cdot 61 \cdot 4651 \cdot 30941 \cdot 161971$ M1=10

$13^{16} \Rightarrow 103 \cdot 443 \cdot 15798461357509$ M1=10

$13^{18} \Rightarrow 12865927 \cdot 9468940004449$ M1=9

$13^{20} \Rightarrow 3 \cdot 43 \cdot 61 \cdot 337 \cdot 547 \cdot 2714377 \cdot 5229043$ M1=12

$13^{22} \Rightarrow 1381 \cdot 2519545342349331183143$ M1=9

$13^{24} \Rightarrow 701 \cdot 9851 \cdot 30941 \cdot 2752135920929651$ M1=11

$13^{26} \Rightarrow 3^3 \cdot 61 \cdot 650971 \cdot 1609669 \cdot 57583418699431$ M1=10

$13^{28} \Rightarrow 1973 \cdot 2843 \cdot 3539 \cdot 846041103974872866961$ M1=11

$13^{30} \Rightarrow 311 \cdot 1117 \cdot 817050901143136340856815036 9$ M1=10

$13^{32} \Rightarrow 3 \cdot 23 \cdot 61 \cdot 419 \cdot 859 \cdot 18041 \cdot 1755103211998167904672 9$ M1=12

$13^{34} \Rightarrow 211 \cdot 30941 \cdot 5229043 \cdot 3357897971 \cdot 707179356161321$ M1=12

13^{∞}

$31^2 \Rightarrow 3 \cdot 331$ D=1.9474213420720655082

$31^4 \Rightarrow 5 \cdot 11 \cdot 17351$ M1=10

$31^6 \Rightarrow 917087137$ D=1.9416030701041145633

$31^8 \Rightarrow 3^2 \cdot 331 \cdot 3637 \cdot 81343$ M1=10

[5] $\lambda = 61^5 3^2 13^2,\ 7 \cdot 31 \cdot 97 \cdot 523 \mid \mu$

$31^{10} \Rightarrow 23 \cdot 397 \cdot 617 \cdot 150332843$ M1=11

$31^{12} \Rightarrow 42407 \cdot 2426789 \cdot 7908811$ M1=10

$31^{14} \Rightarrow 3 \cdot 5 \cdot 11 \cdot 331 \cdot 2521 \cdot 17351 \cdot 327412201$ M1=13

$31^{16} \Rightarrow 751670559138758105956097$ D=1.9416030680575446224

$31^{18} \Rightarrow 571 \cdot 14251 \cdot 88770666332610762169$ M1=10

$31^{20} \Rightarrow 3 \cdot 43 \cdot 331 \cdot 6301 \cdot 917087137 \cdot 2813432694367$ M1=12

$31^{22} \Rightarrow 1509997 \cdot 61562537 \cdot 7176374761323733117$ M1=10

$31^{24} \Rightarrow 5^2 \cdot 11 \cdot 101 \cdot 4951 \cdot 17351 \cdot 13277801 \cdot 20235942281002951$ M1=14

31^{∞}

$97^2 \Rightarrow 3 \cdot 3169$ D=1.9422138197614981071

$97^4 \Rightarrow 11 \cdot 31 \cdot 262321$ M1=9

$97^6 \Rightarrow 43 \cdot 967 \cdot 20241187$ M1=10

$97^8 \Rightarrow 3^2 \cdot 1153 \cdot 3169 \cdot 240813217$ M1=10

$97^{10} \Rightarrow 89 \cdot 837197335075000483$ M1=9

$97^{12} \Rightarrow 53 \cdot 79 \cdot 20359 \cdot 8224356155341457$ M1=11

$97^{14} \Rightarrow 3 \cdot 11 \cdot 31 \cdot 3169 \cdot 244471 \cdot 262321 \cdot 31728277831$ M1=12

$97^{16} \Rightarrow 6206521290195886805501232767464 1$ D=1.9416030680575446224

$97^{18} \Rightarrow 21433 \cdot 280314943 \cdot 9719915834548161 1029493$ M1=10

97^{∞}

$523^2 \Rightarrow 3 \cdot 13 \cdot 7027$ D=1.9418793999241943660

$523^4 \Rightarrow 3491 \cdot 21472771$ M1=9

$523^6 \Rightarrow 20504128695797293$ D=1.9416030680575447169

$523^8 \Rightarrow 3^2 \cdot 13 \cdot 7027 \cdot 6821641334623519$ M1=9

$523^{10} \Rightarrow 89 \cdot 199 \cdot 397 \cdot 2003 \cdot 108926496542584733$ M1=12

$523^{12} \Rightarrow 53 \cdot 7917272313053609286926516514677$ M1=9

$523^{\infty} : p_8 < 34.25$ P2

$3^4 A = 2.0011478003505957535$ [6]

Done - 105 lines

[6]The number $61^5 3^4 \cdot 7^2 13^2 31^2 97^2 523^2$ is Abundant.

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