Portfolio Credit Risk Modelling and CDO Pricing - Analytics and Implied Trees from CDO Tranches

A Thesis Submitted for the Degree of
Doctor of Philosophy

by

Tao Peng
B.Sc.(Southwest University of Finance and Economics, China)
M.Sc. (University of Queensland)
tao.peng-1@student.uts.edu.au

in

School of Finance and Economics
University of Technology, Sydney
PO Box 123 Broadway
NSW 2007, Australia

May 10, 2010

Certificate

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

Signature of Candidate . . . . . .

Date . . . . . .

Acknowledgements

I would like to thank my supervisor, Professor Erik Schlögl for his guidance, advice and patience through my PhD studies. His rigorous approach to academic research and the principles of research I learnt from him are critical for my academic career.

I have also benefited greatly from working with Peter Cotton at Julius Finance in New York and am grateful for his comments on the models and for sharing his insight on credit modeling. I would also like to give special thanks to Richard Stone at ANZ for his help on data collection and his knowledge on model testing.

To all of the staff at the School of Finance and Economics, I would like to take the opportunity to thank you for all your support and assistance. My research experience was enriched from participation and networking at the Quantitative Methods in Finance conference and the workshops organized by the Quantitative Finance Research Centre. Additionally, I learnt substantially from the master classes and seminars organized by the Financial Integrity Research Network where I had the opportunity to discuss with world renowned scholars in credit risk and relevant areas in quantitative finance. In particular the financial assistance for my research and traveling from both the school and the centre, the research scholarship I received from ARC and the generous resources provided with the help of Dzung Le at ANZ are also greatly appreciated.

Finally I would like thank my parents and my sister who have provided me for many years with endless support and continual encouragement.

Contents

Abstract

One of the most successful and most controversial innovative financial products in recent years has been collateralised debt obligations (CDOs). The dimensionality of dependency embedded in a typical CDO structure poses great challenges for researchers - in both generating realistic default dynamics and correlation, and in the mean time achieving fast and accurate model calibration.

The research presented in this thesis contributes to the class of *bottom-up* models, which, as opposed to *top-down* models, start by modelling the individual obligor default process and then moving them up through the dependency structures to build up the loss distributions at the portfolio level.

The Gaussian model (Li 2000) is a *static* copula model. It has only on correlation parameter, which can be calibrated to one CDO tranche at a time. Its simplicity achieves wide spread industry application even though it suffers from the problem of 'correlation smile'. In other words, it cannot fit the market in an arbitrage-free manner in the capital-structure dimension.

The first contribution of this thesis is the sensitivities analysis with regard to model parameters of expected losses of CDO tranches in the Gaussian and NIG copula models. The study provided substantial insight into the essence of the dependency structure. In addition, we apply the intensity approach to credit modelling in order to imply market distributions non-parametrically in the form of a binomial lattice.

Under the same framework, we developed a series of three models. The static binomial model can be calibrated to the CDS index tranches exactly, with one set of parameters. The model can be seen as a non-parametric copula model that is arbitrage free in the capital-structure dimension.

Static models are not suitable to price portfolio credit derivatives that are dynamic in nature. The static model can be naturally developed into a *dynamic* binomial model and satisfies no-arbitrage conditions in the time dimension. This setup, however, reduces model flexibility and calibration speed. The computational complexity comes from the non-Markovian character of the default process in the dynamic model.

Inspired by Mortensen (2006), in which the author defines the intensity integral as a conditioning variable, we modify the dynamic model into a *Markovian* model by modelling the intensity integral directly, which greatly reduces the computational time and increases model fit in calibration. We also show that, when stochastic recovery rates are involved, there is a third no-arbitrage condition for the expected loss process that needs to be built into the Markovian model. For all binomial models, we adopt a unique optimisation algorithm for model calibration - the Cross Entropy method. It is particularly advantageous in solving large-scale non-linear optimsation problems with multiple local extrema, as encountered in our model.

CHAPTER 1

Introduction

1.1. Literature Review

1.1.1. Collateralised Debt Obligations.

1.1.1.1. *Definition, economics and rating.* Collateralised debt obligations (CDOs) are asset-backed securities whose payments are subject to the incidence of default on the underlying portfolio of credit risky assets. Normally, a Special-Purpose Vehicle (SPV) is set up as a separate entity to hold the collateral and also issue liabilities of different priorities, which will comprise both debt and equity components.(Figure 1.1.1) These prioritised classes are typically known as senior, mezzanine and equity *tranches* depending on their seniority in the capital structure. Therefore what distinguishes CDO from other pooled investments, such as mutual funds is the way of partitioning of risk among investors not in proportion to the size of their investments but based on the priority level of the tranche they invest in. Investors with higher risk tolerance can choose to invest in riskier tranches with higher return, while risk-averse investors can choose more senior, but lower yielding, tranches. Therefore, the key feature of the innovation in assets like CDO is that it allows investors to invest in securities suitable to their risk-reward profile by transforming the risk characteristics of the assets that back up these securities.

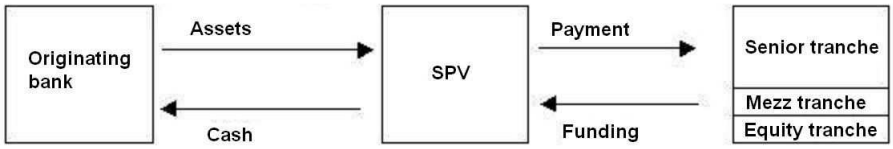


FIGURE 1.1.1

Traditionally a CDO is fully funded. The collateral manager is responsible for the selection and purchase of the collateral portfolio for a SPV. Using a mechanism called structural subordination, the scheduled coupon and principal payments on the different securities are paid to investors according to a set of rules known as *the waterfall*. The

interest and principal payments on the senior securities are paid first. Then the mezzanine security holders receive their coupon and principal. Lastly, the equity security holders receive their coupon and principal, and therefore are the first to absorb any loss.

There are different types of CDOs. In a *cash flow* CDO the manager is not actively trading the underlying portfolio and the interest and principal payments are subject to the number and timing of defaults of the collateral. In a *market value* CDO the underlying credits are actively traded and the payments received are subject to the trading performance of the manager.

Credit derivatives, including CDOs, are created to transfer the credit risk among market participants, which, presumably, has the effect of addressing market imperfections. They increase the liquidity of credit markets, lower credit-risk premia and offer investors an improved range of asset supply and hedging opportunities. (Duffie & Garleanu 2001)

Commercial banks have regulatory capital requirements that may restrict their lending activities and profit opportunities. A CDO, in the form of a collateralised loan obligation (CLO), can help them to securitise and sell some of their loan assets so that they are removed from their balance sheets, thus achieving capital relief and increasing assets valuation.

Investment banks are driven to exploit the differences between the cost of acquiring the collateral assets and the values generated from management fees and the sale of CDO securities. This results in increased market liquidity achieved by transforming the risk-return profile of a set of illiquid risky assets into a set of securities whose risk-return characteristics can be tailored to the requirements of different classes of investor.

Apart from sound regulatory policies, the development of the CDOs market relies on fair valuation and proper risk measurement of the CDO tranches. A sound pricing model is normally a prerequisite to the market development evidenced by the fact that the Black Scholes formula fueled the growth of the options market in the 1970s. However, all models should be used with caution. Proper understanding of the models' assumptions and limitations is important when applying them. The market values and the risks of these prioritised CDO tranches are sensitive to the default correlation[1] among the underlying

[1]Default correlation is the phenomenon that the likelihood of one obligor defaulting on its debt is affected by whether or not another obligor has defaulted on its debts.

names in the credit portfolio. Institutional investors tend to rely on the ratings of structured credit products, including CDOs, when making investment decisions. The sub-prime mortgage crisis that developed in 2007-2008 substantiated this point. In the sub-prime crisis even tranches rated AAA suffered substantial loss, indicating that even the ratings agencies are currently ill equipped to correctly value default correlation embedded in the underlying collateral.

The credit-rating agencies assess the risk of the tranches on the basis of the credit quality of the collateral and the mechanics of the waterfall. They use proprietary rating models which take into account the probability of default of individual obligors and their default correlation. Methodologies for rating CDOs, however, are still at a relatively crude stage of development. Correlation parameters used in ratings models tend to be based on rudimentary assumptions, for example, treating the names within a given industrial sector as homogenous and as though they have the same default correlation between them, and treating the correlations between the names from different industrial sectors the same as correlations between different sectors.(Duffie 2007)

1.1.1.2. *The Synthetic CDO.* Synthetic CDOs developed as an outgrowth of cash CDOs. Cash CDOs have a reference portfolio made up of cash assets such as corporate bonds or loans. The reference portfolio in a synthetic CDO is made up of credit default swaps. A loan originator, using credit derivatives technology, can free up economic capital and transfer credit risk without the need to remove assets from its balance sheet, which help it maintain client relationships. Therefore, the synthetic CDOs market not only relies on the growth of a broad and liquid credit default swap (CDS) market but also drives the development of the CDS market.

The explosive growth in the synthetic CDOs market prior to the financial crisis not only reflects the expediency in constructing these products but also a leap in faith, mostly blind faith, in the pricing models. Since it is difficult to estimate the underlying corporate default probabilities from historical default rates due to scarcity of data, one can infer the default probabilities from CDS prices. Since the CDS is a much more liquid market than corporate bonds, these default probabilities are assumed to be correct and plugged into

the pricing formula and spit out as tranche spreads. This practice spurred the invention of even more exotic and customised single-tranche synthetic CDOs (STCDO).[2]

A full capital-structure synthetic CDO still takes a long time to arrange, as it is hard to find investors who are interested in all the tranches in the capital structure. However, a STCDO has no SPV (Special Purpose Vehicle) and is not funded; it only has one CDO tranche issued; it can typically be issued within a few days of initial enquiry, which is assisted by the use of standard ISDA (International Swaps and Derivatives Association) documentation; the tranche is tailored to the exact requirements of the investor which include size of investment, the number and names of the credits in the reference portfolio, the level of structural subordination of the tranche, the desired rating and choice of ratings agencies, the maturity date and the currency. Therefore, a STCDO tranche is sometimes called a bespoke tranche.

In a STCDO, only a specific portion of the portfolios risk, rather than the entire capital structure, is transferred to the investor. The dealer retains a portion of the credit risk of the STCDO. Therefore, the STCDO is vulnerable to changes in credit spreads on the underlying portfolio of the CDO (market risk) and to defaults on this portfolio (default correlation risk). These risks must be hedged dynamically by the underlying CDSs or other STCDOs. However, correct hedging requirements rely on accurate pricing of the tranches. The popular copula model (Li 2000) does not give adequate hedging quantities, as evidenced by the ineffectiveness of tranche delta hedging which occurred in May, 2005 after the rating of General Motors (GM) debt was downgraded.

1.1.1.3. *The CDS indices and calibration issues.* The need for sourcing the risk and hedging the risk sensitivities, as well as providing a vehicle for expressing people's view on default correlation, led to the development of standard CDS indices. A CDS index is a standardised CDS portfolio which can be used to hedge credit risk or to take a position on a basket of credit entities. Like a single-name CDS, a CDS index is also an over-the-counter credit derivative. However, it is also a completely standardised credit security and may therefore be more liquid and traded at a smaller bid-offer spread. This means that it can be cheaper to hedge a portfolio of credit default swaps or bonds with a CDS

[2]Synthetic CDOs are derivatives of the underlying CDSs and since the major usage of the models discussed in this thesis is for pricing synthetic CDOs, the default probabilities mentioned, unless stated otherwise, are risk neutral probabilities.

index than it would be to buy many CDSs to achieve a similar effect. CDS indices are benchmarks for protecting investors who own bonds against default, and traders use them to speculate on changes in credit quality.

There are currently two main families of CDS indices: CDX and iTraxx. CDX contains North American and Emerging Market companies and is administered by CDS Index Company and marketed by Markit Group Limited, and iTraxx contains companies from the rest of the world and is managed by the International Index Company.

There are tranches that are based on the CDS indices. When valuing these CDO tranches, the industry copula models are generally calibrated to CDS index tranches. When calibrated to these index tranches, the industry-standard Gaussian copula model is internally inconsistent in that the correlation that makes the model spread match the price of one tranche of a CDO structure is typically much different from that which matches another tranche of the same structure.

1.1.1.4. *Default correlation.* To understand the correlation parameter matching problem, we will first examine how correlation affects default. The default correlation is fully reflected in the portfolio loss process. The portfolio loss can be defined as

$$L(t) = \sum_{i=1}^{N} (1 - R_i) I_i \tag{1.1.1}$$

where $I_i = 1_{\tau_i < t}$ is the counting process for a defaultable obligor with notional of 1, τ_i is the default time and R_i is the recovery rate which is assumed to be constant under market standard assumption.[3]

Then the expected portfolio loss is given by

$$\mathbb{E}[L(t)] = \sum_{i=1}^{N} (1 - R_i) P_i(0, t)$$

where $P_i(0, t)$ is the default probability of issuer i from time zero to time t.

Therefore, the expected loss of the portfolio is independent of the default correlation. However, for an investor who holds a particular tranche on the capital structure, the default correlation can make dramatic difference to the expected loss for different subordinations.

[3]Here the recovery rate is not dependent on economic scenarios and later on this assumption is relaxed in the dynamic models.

One common observation is that where an equity tranche investor would benefit from a higher correlation, the expected loss on the equity tranche being lower if correlation increases, a senior tranche investor would benefit from a lower correlation since a higher correlation results in extreme co-movements.

For example, there is a credit portfolio of 100 names, each with a notional of 1 and a default probability of 0.05 and also assuming zero recovery rates for all names. The two tranches for which we calculate expected loss are the $[0-1]\%$ equity tranche and the $[99-100]\%$ supersenior tranche. We consider two extreme cases for default correlation. The first is the zero correlation and the second is the 100% correlation case. We now calculate the expected loss for two cases.

In the zero correlation case, if one credit defaults, the $[0-1]\%$ tranche will be wiped out. The expected loss for the tranche is $1 \times (1-(1-0.05)^{100}) \approx 0.9941$. For the $[99-100]\%$ tranche, it can only be wiped out if all credits default together. Therefore, its expected loss is $1 \times 0.05^{100} \approx 0$.

In the 100% correlation case, if one credit defaults, all credits default together, therefore the probability of all defaulting is 0.05. The expected loss for both tranches is the same, which is $1 \times 0.05 = 0.05$. Compared to the first case, the equity tranche investor would prefer a more correlated portfolio and the supersenior tranche investor would prefer a less correlated portfolio.

1.1.1.5. *Pricing of STCDOs.* The payment mechanism of STCDOs is not a waterfall in the traditional sense. It is solely dependent on the loss of the underlying credits. There are no interest and principal flows from the reference portfolio paid to the tranches.

We first define the subordinations of a tranche by,

1. $\underline{L}$: as the attachment point or subordination or lower strike, below which, $L(t) < \underline{L}$, tranche loss is zero.

2. $\overline{H}$: is the detachment point or upper strike, above which, $L(t) > \overline{H}$, tranche loss is 100%

where $L(t)$ represents the cumulative loss by time t.

Therefore the loss for tranche $[\underline{L}, \overline{H}]$ by time t is given by

$$L(t, \underline{L}, \overline{H}) = \text{max}(\text{min}(L(t), \overline{H}) - \underline{L}, 0) = \text{min}(L(t), \overline{H}) - \text{min}(L(t), \underline{L})$$

The expected loss of tranche $[\underline{L}, \overline{H}]$ is given by

$$EL_t(\underline{L}, \overline{H}) = \mathbb{E}[L(t, \underline{L}, \overline{H})] = \sum_{l=0}^{N} p(\ell, t)\text{max}(\text{min}(L(t), \overline{H}) - \underline{L}, 0)$$

where N is the maximum number of losses and $p(\ell, t)$ is the loss distribution $P\{L(t) = \ell\}$.

The *premium leg* is a series of cash flows made by the tranche protection buyer to the tranche protection seller. They are calculated by the remaining notional of the tranche times the contractual tranche spread and paid on a quarterly basis using an Actual 360 day count convention.

The *contingent leg* contains the loss payments made by the protection seller to the protection buyer to cover the default loss on the tranche. The size of the loss is simply the change in the loss $L(t, \underline{L}, \overline{H})$ between contractual payment dates.

The par value of a CDO tranche spread is one that makes the present value of premium leg cash flows and the present value of contingent leg cash flows equal each other.

The present value of the premium leg is

$$\text{PremiumLeg PV} = s(\underline{L}, \overline{H}) \sum_{i=1}^{N} \Delta(t_i - 1, t_i) D(t_i)[\overline{H} - \underline{L} - EL_i(\underline{L}, \overline{H})] \tag{1.1.2}$$

The present value of the protection leg is

$$\text{ContingentLeg PV} = \sum_{i=1}^{N} D(\overline{t_i})\big([EL_i(\underline{L}, \overline{H}) - EL_{i-1}(\underline{L}, \overline{H})]\big) \tag{1.1.3}$$

where $\Delta(t_{i-1}, t_i)$ is the time increment for period $[t_{i-1}, t_i]$, $D(t_i)$ is the discount factor at time t_i and $\overline{t_i}$ is the mid point in period $[t_{i-1}, t_i]$.

Therefore the model spread for tranche $[\underline{L}, \overline{H}]$ can be derived as the following at par, given that there's no upfront payment

$$s(\underline{L},\overline{H}) = \frac{\sum_{i=1}^{N} D(\overline{t_i})\big([EL_i(\underline{L},\overline{H}) - EL_{i-1}(\underline{L},\overline{H})]\big)}{\sum_{i=1}^{N} \Delta(t_{i-1},t_i)D(t_i)[\overline{H} - \underline{L} - EL_i(\underline{L},\overline{H})]} \tag{1.1.4}$$

Adequate pricing and hedging of bespoke CDO tranches are crucial to an institution that aims to minimise the risks of its trades books. However, bespoke CDOs have the following specifications, in increasing order of complexity and difficulty to evaluate.

1. The underlying portfolio and maturities are the same as the referenced names in a standard CDS index portfolio such as iTraxx or CDX but the tranche subordinations are different from standard index tranches.

2. The underlying portfolio is the same but the maturities of the portfolio are different from those of observed referenced names in a CDS index.

3. The constituents of the underlying portfolio are different from the referenced names.

1.1.1.6. *More exotic credit derivatives.* The CDOs are more and more like vanilla derivatives and other more exotic credit derivatives pose even more difficulty in terms of modelling and pricing. They include CDO-squared, tranchelets, credit CPPI, forward-starting tranches (FDTr), options on tranches and leveraged super-senior (LSS).

A CDO-squared is a CDO backed by a pool of CDO tranches.

Tranchelets is the name given to very thin tranches on the standard indices. Their width is typically 1% of the capital structure.

A credit CPPI (Constant Proportion Portfolio Insurance) is a dynamic trading strategy in which all the payments to the investor are replicated exactly by the issuer who follows the CPPI rebalancing strategy to allocate assets between a risky credit portfolio and safe assets.

An FDTr allows the investor to sell protection on a single-tranche CDO starting at a forward date t_F at a spread agreed upon today. There are two variations. In the first variation any loss incurred before t_F decreases tranche attachment and detachment levels. In the second variation loss that happens before the option starts does not reduce tranche levels.

An *option on a tranche* gives the option holder the right, but not the obligation, to enter into a single-tranche CDO at a future expiry date t_E. The underlying tranche matures

at a later time T and pays a tranche spread agreed today. The tranche attachment and detachment levels are also defined and fixed at option initiation.

An LSS tranche is a product in which the investor assumes a leveraged exposure to the super-senior part of the capital structure.

A sound pricing model is crucial for development in the portfolio credit derivatives market. Earlier credit models which find applications in areas such as economic capital measurement are typically static in nature. With the development of a liquid-credit indices market and the invention of more exotic credit derivatives that are increasingly dynamic and path dependent in nature, such as those mentioned above, the requirements on pricing models are even more stringent. Ideally, portfolio credit models should be able to describe the frequency of default of the referenced entities and their joint default distribution, default intensity evolution, recoveries given default and CDS spread dynamics for various maturities. However, it is extremely difficult to satisfy all these requirements in one model. We review a wide range of portfolio credit models, which all have advantages that are suitable for specific applications. We start with industry copula models.

1.1.2. Copula models. The copula models are the first-generation pricing models for CDOs. The first Gaussian latent variable model can be found in Vasicek (1987). Its multi-state form is the underlying model for CreditMetrics. It was later reintroduced as the Gaussian copula model in Li (2000) for pricing synthetic CDOs.

Copula models typically use copula functions to describe the joint distribution of credit events assuming marginal default probabilities bootstrapped from single-name CDS contracts.

Given a n-dimensional Gaussian copula function

$$C(u_1, \cdots, u_n) = \Phi_n(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \cdots, \Phi^{-1}(u_n))$$

$u_i, 1 \leq i \leq n$ is a uniform random variable, Φ_n is the n dimensional Gaussian cumulative distribution function with correlation coefficient matrix $\sum$.

The joint default probabilities of a credit portfolio with n names in Li (2000) is given by

$$P\{\tau_1 < t_1, \cdots, \tau_n < t_n\} = \Phi_n(\Phi^{-1}(F_1(t_1)), \Phi^{-1}(F_2(t_2)), \cdots, \Phi^{-1}(F_n(t_n))) \quad (1.1.5)$$

by matching uniform variable u_i with the cumulative default density function $F_i(t_i), 1 \leq i \leq n$.

To generate the correlated default times, we must first generate correlated random variables $[X_i, \cdots, X_n]$ from an n-dimension Gaussian distribution. And as there is a one-to-one correspondence between X_i and t_i, then default times can be derived by mapping X_i with default time t_i using

$$t_i = F_i^{-1}(\Phi(X_i)), i = 1, 2, \cdots, n.$$

1.1.2.1. *MC or analytical method.* The method of generating correlated default times in the last section is suggested in Li (2000), where it introduced the Monte Carlo (MC) method for pricing credit derivatives. However, this method is time consuming when applied to a standard credit portfolio of 125 names. Subsequent researchers have investigated the factor form of the copula model and developed semi-analytical pricing techniques. In the literature, the factor models based on the original Vasicek specification are typically called"copula models", with the "copula" given by the distribution of the common factor.

In Laurent & Gregory (2005) the Gaussian copula model 1.1.5 is given by

$$X_i = \rho_i M + \sqrt{1 - \rho_i^2} Z_i \tag{1.1.6}$$

where M, Z_i and $X_i \sim N(0, 1)$ and $\rho_i, -1 < \rho_i < 1$ is the correlation between variable X_is.

This representation reduces the problem with a copula whose dimension equals the number of names in the underlying credit portfolio to the dimension of two factors - a common factor M and an idiosyncratic factor Z_i. Conditional on the common factor M defaults are independent among obligors.

Therefore we have

$$P\{\tau_1 < t_1, \cdots, \tau_n < t_n \mid M\} = \prod_{i=1}^{n} P\{\tau_i < t_i \mid M\} = \prod_{i=1}^{n} P(t_i \mid M)$$

where τ_i is the default time of obligor i.

For the Gaussian copula, the conditional default probabilities $\prod_{i=1}^{n} P(t_i \mid M)$ can be derived from 1.1.6.

$$\begin{aligned} P(t_i \mid M) &= P\{X_i < k \mid M\} \\ &= P\{\rho_i M + \sqrt{1-\rho_i} Z_i < k \mid M\} \\ &= \Phi\Big(\frac{k - \rho_i M}{\sqrt{1-\rho_i^2}}\Big) \end{aligned}$$

where k is a time-dependent threshold which can be matched to marginal default probability at t_i by equating $k = \Phi^{-1}(F_i(t_i))$.

Equation 1.1.5 therefore becomes

$$P\{\tau_1 < t_1, \cdots, \tau_N < t_N\} = \int \prod_{i=1}^{N} P(t_i \mid M) f(M = m) dm \quad (1.1.7)$$

$$= \int \prod_{i=1}^{N} \Phi\Big(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i m}{\sqrt{1-\rho_i^2}}\Big) f(m) dm \quad (1.1.8)$$

$f(m) = \frac{1}{\sqrt{2\pi}} e^{\frac{-m^2}{2}}$ is a standard Gaussian density function.

In Laurent & Gregory (2005) a fast Fourier transform method was adopted to derive the loss distribution $p(\ell, t)$

The characteristic function of the loss process $L(t)$ (equation 1.1.1) can be derived as

$$\begin{aligned} \varphi_{L(t)}(u) &= \mathbb{E}[\exp(iuL(t))] \\ &= \mathbb{E}\big[\mathbb{E}[\exp(iuL(t)) \mid M]\big] \\ &= \int \prod_{i=1}^{N} ((1 - P(t \mid m)) + P(t \mid m) \varphi_{1-R_i}(uI_i)) f(m) dm \end{aligned}$$

where R_i can be stochastic and follows a Beta distribution and φ_{1-R_i} is the characteristic function of R_i.

Then loss distribution $p(\ell, t)$ by certain time t can be solved by a Fourier inversion technique.

In Andersen, Sidenius & Basu (2003), a *probability basket* approach or *recursive method*[4] is used to build up loss distributions $p(\ell, t)$. This method is faster than the fast Fourier transform and has become the most popular among practitioners. The authors also show how to use the recursive method together with the factor-reduction technique in building correlation matrix to calculate the *Greeks*.

The recursive method is used throughout this thesis to build up the loss distribution. In a two-state form it works out as the following.

Suppose we know the loss distribution $p^k(\ell, t), \ell = 0, ..., l_{max,k}$ for a reference portfolio with $k \geq 0$ referenced names where ℓ is the loss number and $l_{max,k}$ is the maximum number of losses which can be reached by a portfolio with k names. Now we add another name to the portfolio with loss weight w_{k+1} and known default probability $p_{k+1}(t)$. We get the loss distribution of the larger basket by

$$p^{k+1}(\ell, t) = p^k(\ell, t)(1-p_{k+1}(t))+p^k(\ell-w_{k+1})p_{k+1}(t), \ell = 0, \cdots, l_{max,k}+w_{k+1} \quad (1.1.9)$$

Then the loss distribution $p^N(\ell, t)$ for a N-obligor portfolio can be built from the boundary case of the empty basket $p^0(\ell, t) = \delta_{0,\ell}$ where $\delta_{i,j}$ is one for $i = j$ and zero otherwise.

Here the non-negative integer w_i is the loss weight assigned to the k-th credit in the reference pool. It is calculated by rounding $(1 - R_k)I_k/u$ to the nearest integer where u is the loss unit.

There are some variations of the recursive method in the literature. In Hull & White (2004), two alternative approaches to fast Fourier transform were developed. The *probability basket* approach, independently developed by the authors, claimed to be more accurate than that of Andersen et al. (2003) in some situations. The other alternative is the recurrence method, which is more suitable for pricing basket default swaps.

1.1.2.2. *Correlation skew.* As default correlation is the driving factor for credit portfolio valuations, in practice the standard Gaussian model reaches the status of the Black-Scholes model in the equities market to quote market-implied correlations. In the *compound correlation* approach, a tranche spread is fixed and the pricing formula is inverted so that a correlation parameter can be chosen to yield the quoted spread. However, the

[4]For abuse of terminology, *recursive method* or *probability basket approach* are equivalent here.

Gaussian copula model fails to reproduce market quotes of different CDO tranches in the capital structure with a single correlation number. Different tranches *imply* different correlations in the form of a *correlation smile* similar to the concept of *volatility smile* of the Black-Scholes model.

To price bespoke tranches, one has to interpolate or extrapolate correlation parameters from standard tranches in the Gaussian copula model. However, the tranche spreads along the capital structure are not monotonic with correlation and therefore are very difficult to fit with a satisfying correlation curve.

The more serious underlying problem is that some no-arbitrage conditions can be violated in the curve fitting.

One of the no-arbitrage conditions is that the sum of the tranche's expected loss across the capital structure should equal the sum of the expected loss of all the credits in the underlying reference portfolio (O'Kane 2008).

Suppose we buy protection on nTr continuous tranches with the i-th tranche having strikes $[\underline{L}_{i-1}, \overline{H}_i]$ where $\underline{L}_{i-1}$ represents the attachment point for tranche $i-1$, $\overline{H}_i]$ represents the detachment point for tranche i, $0 \leq i \leq K$, $\underline{L}_0 = 0$ and $\overline{H}_{nTr} = 1$.

The expected loss of tranche $[\underline{L}_{i-1}, \overline{H}_i]$ is

$$EL(\underline{L}_{i-1}, \overline{H}_i) = \mathbb{E}_{\rho(\underline{L}_{i-1}, \overline{H}_i)}[\min(L(t), \overline{H}_i) - \min(L(t), \underline{L}_{i-1})] \tag{1.1.10}$$

where $\rho(\underline{L}_{i-1}, \overline{H}_i)$ is the correlation for tranche $[\underline{L}_{i-1}, \overline{H}_i]$ in the Gaussian copula.

The sum of expected loss of all tranches is

$$\sum_{i=1}^{nTr} EL(\underline{L}_{i-1}, \overline{H}_i) \quad = \quad \sum_{i=1}^{nTr} \mathbb{E}_{\rho(\underline{L}_{i-1}, \overline{H}_i)}[\min(L(t), \overline{H}_i) - \min(L(t), \underline{L}_{i-1})]$$

The above terms will cancel out if we use the same correlation across all tranches.

$$\begin{aligned}
\sum_{i=1}^{nTr} EL(L_{i-1}, H_i) &= \mathbb{E}_\rho\Big[\sum_{i=1}^{nTr}\big(\min(L(t), \overline{H}_i) - \min(L(t), \underline{L}_{i-1})\big)\Big] \\
\text{(Cancelling terms)} &= \mathbb{E}_\rho\Big[\min(L(t), \overline{H}_{nTr}) - \min(L(t), \underline{L}_0)\Big] \\
&= \mathbb{E}_\rho[L(t)] \\
&= \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[L_i(t)]
\end{aligned}$$

where $\mathbb{E}[L_i(t)]$ is the expected loss of issuer i to time t which is independent of correlation and therefore the sum is actually the sum of the expected loss of all names in a credit portfolio. However, if we use a different correlation value for each tranche, the sum of the expected loss of all tranches is very likely to be different from the expected loss of the sum of all credits $\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[L_i(t)]$ and creates arbitrage opportunities.

The interpolation problem with compound correlation is lessened by the introduction of the base correlation technique by JP Morgan in 2004 (McGinty & Ahluwalia 2004). Base correlations are developed on the insight that any tranche can be represented by two base or equity tranches whose detachment points coincide with the attachment and detachment points of this tranche. To understand this, it is easy to see that the expected loss of any tranche can be represented by the difference of that of two equity tranches.

$$\begin{aligned}
EL(\underline{L}_{i-1}, \overline{H}_i) &= \mathbb{E}[\min(L(t), \overline{H}_i)] - \mathbb{E}[\min(L(t), \underline{L}_{i-1})] \\
&= EL(0, \overline{H}_i) - EL(0, \overline{H}_{i-1})
\end{aligned}$$

where $\underline{L}_{i-1} = \overline{H}_{i-1}$.

Therefore, given a set of market-quoted tranches on the capital structure of a CDO, we can bootstrap the spreads of a series of equity tranches with corresponding seniorities. To do that, we must start with the lowest tranche on the capital structure - the equity tranche.

Notice that for the equity tranche $[0, \overline{H}_1]$ the compound correlation is the same as the base correlation. Expected loss of the first two tranches $[0, \overline{H}_1]$ and $[\underline{L}_1, \overline{H}_2]$ can be calculated by equating the model spreads to the market quotes. Then the expected loss of the equity tranche $[0, \overline{H}_2]$ is calculated as the sum of the expected loss of $[0, \overline{H}_1]$ and $[\underline{L}_1, \overline{H}_2]$. Then

the model spread can be calculated and base correlation for $[0, \overline{H}_2]$ can be backed out. Proceeding inductively, from the market CDS index tranches, all the correlations for the corresponding equity tranches can be built up.

Typically, the base correlation curve forms a monotonically increasing skew rather than a smile. Therefore, to price any bespoke tranche one can interpolate on the base correlation curve for the corresponding equity tranches and then reverse the bootstrapping procedure described in the last paragraph to calculate the expected loss of the bespoke tranche and then the model spread.

One improvement from the compound correlation approach is that the base correlation approach preserves the sum of expected loss across the capital structure that the compound correlation approach violates.

Proof. The expected loss of a tranche $[\underline{L}_{i-1}, \overline{H}_i]$ is,

$$\begin{aligned} EL(\underline{L}_{i-1}, \overline{H}_i) &= EL(0, \overline{H}_i) - EL(0, \underline{L}_{i-1}) \\ &= \mathbb{E}_{\rho(0,\overline{H}_i)}[\min(L(t), \overline{H}_i)] - \mathbb{E}_{\rho(0,\underline{L}_{i-1})}[\min(L(t), \underline{L}_{i-1})] \end{aligned}$$

Therefore, since

$$\mathbb{E}_{\rho(0,\overline{H}_i)}[\min(L(t), \overline{H}_i)] = \mathbb{E}_{\rho(0,\underline{L}_i)}[\min(L(t), \underline{L}_i)]$$

because $\overline{H}_i = \underline{L}_i$, then

$$\begin{aligned} \sum_{i=1}^{N} EL(\underline{L}_{i-1}, \overline{H}_i) &= \sum_{i=1}^{N} \left(\mathbb{E}_{\rho(0,\overline{H}_i)}[\min(L(t), \overline{H}_i)] - \mathbb{E}_{\rho(0,\underline{L}_{i-1})}[\min(L(t), \underline{L}_{i-1})] \right) \\ &= \mathbb{E}_{\rho(0,1)}[\min(L(t), 1)] \\ &= \mathbb{E}[L(t)] \end{aligned}$$

However, the base correlation method still failed to convince academics that the bespoke tranche spreads that are calculated from the interpolated base correlation curves are not necessarily arbitrage free, which can violate the following no-arbitrage condition.

CONDITION 1.1.1. *The higher up in the capital structure of a CDO, the less likely that a tranche is hit by defaults in the underlying portfolio. Therefore, the model spreads should be monotonically decreasing while remaining positive with regarding to the seniority of the tranches on the capital structure*

O'Kane (2008) shows that using linear interpolation the spreads for tranchelets[5] are shown to go negative for subordination from 25% to 30% percent on the capital structure of CDX. And using cubic spline interpolation the spreads of tranchelets are not monotone for subordinations of above 20% on the capital structure of iTraxx.[6] Therefore interpolation of base correlation (be it with splines or otherwise) does not guarantee monotonicity of tranchelet spread, but this monotonicity is required for the interpolation to be financially plausible.

1.1.2.3. *Wiping off the smile.* The cause of the correlation skew generated by Gaussian copula is that the model cannot produce a single correlation parameter that makes all tranche spreads equal to the market quotes. The root cause of this problem was firstly identified in the copula function literature as the lacking of tail dependence in the Gaussian copula. Tail dependence in a copula can describe the increased joint probability of defaults.

An introduction to tail dependency can be found in Joe (1997), and Umberto Cherubini & Vecchiato (2004) present a few tail- dependent copula with finance applications.

Tail dependence can be measured by tail index. For example, the lower tail index is given by

$$\lambda_L = \lim_{v \to 0^+} \frac{C(v, v)}{v}$$

where $C(v, v)$ is a two-dimensional copula function.

Gaussian copula has a lower tail index of zero, whereas some other copulas, such as Clayton or Student t, have positive tail indices.

[5]Using tranchelets the spreads for different subordinations on the capital structure can be *zoomed in* and closely examined.

[6]For interested readers, another paper which deals with interpolation in building curves in finance is Hagan & West (2006)

Models adopting copula functions with tail dependence have shown to improve the fit to market quotes with varied degrees of success.

Hull & White (2004) tested the double-t copula and claimed to have achieved a better fit over the Gaussian copula model. The double-t copula is constructed by replacing the factors M and Z_i in equation 1.1.6 with Student-t distributions. However, the sum of two student-t distributions does not have a known functional form and therefore the distribution of X_i cannot be calculated analytically, which greatly increases the computational expense.

One class of copula models, however, which is built on generalised hyperbolic (GH) distributions, is more promising and has been subject to active research.

Normal Inverse Gaussian (NIG) distribution, a member of the GH family, was used to construct a NIG copula model in Kalemanova, Schmid & Werner (2007). In this paper the factor form of a NIG copula is given by

$$X_i(t) = \rho M(t) + \sqrt{1-\rho^2} Z_i(t) \tag{1.1.11}$$

with mutually independent random variables $M(t)$ and $Z_i(t)$ where

$$M(t) \sim F_{NIG}\Big(\alpha, \beta, -\frac{\beta\gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}\Big)$$

$$Z_i(t) \sim F_{NIG}\Big(\frac{\sqrt{1-\rho^2}}{\rho}\alpha, \frac{\sqrt{1-\rho^2}}{\rho}\beta, -\frac{\sqrt{1-\rho^2}}{\rho}\frac{\beta\gamma^2}{\alpha^2}, \frac{\sqrt{1-\rho^2}}{\rho}\frac{\gamma^3}{\alpha^2}\Big)$$

with $\gamma = \sqrt{\alpha^2 - \beta^2}$

Kalemanova et al. (2007) shows that NIG copula can achieve a similar but enhanced fit over double-*t* copula. The model also achieves faster computational speed than the double-t copula, since the NIG distribution, like the Gaussian distribution, is stable under convolution.

Therefore, the NIG copula serves as an ideal example to study copula tail behavior as compared to Gaussian copula. The first chapter of the thesis develops a new NIG copula simpler than that of equation 1.1.11 and studies its expected loss and spreads sensitivities to model parameters.

A whole class of the GH-based copula models has been shown to generally achieve better fit (Albrecher, Ladoucette & Schoutens 2006) than the standard Gaussian copula. However, while the fittings were close to market quoted tranche spreads, they were not perfect. Interpolation would still be needed if they were to be used in practice. A base correlation method for these GH models was developed in Garcia, Goossens, Masol & Schoutens (2007). The paper shows that the correlation skew with a GH copula is significantly flatter than that of the Gaussian copula, which indicates the correlations generated by the GH copulas are closer to reality. It also concludes that no single GH-derived copula is better than the others and recommends taking into consideration the whole family for modelling purposes.

However, the problem with the base correlation approach, as shown in the last section, cannot be reconciled with a particular choice of copula model in general. In Parcell & Wood (2007), the authors argued that the choice of the interpolation method in the base correlation approach may permit model arbitrage in expected loss and extrapolation is even more problematic. They suggested replacing the *base correlation* approach with an alternative *base expected loss* approach, which avoids the arbitrage problem.

Rather than inventing new copulas, another way to improve on existing copula models is to transform a copula function with distortion functions to increase tail dependency (Crane & van der Hoek 2008). The paper shows how a transformed Gaussian copula is able to produce realistic synthetic CDO tranche prices, given a single dependency parameter for an entire portfolio.

However, copula functions with tail dependence do not always guarantee a close match with the correlation skew observed from market. In a comparison study done by Burtschell, Gregory & Laurent (2008) on Gaussian copula, stochastic correlation to Gaussian copula, Student-*t* copula, double-*t* copula, Clayton and Marshall-Olkin copula models the authors show that Student-*t* copula and Clayton copula, both exhibiting tail dependency, behave similarly to the Gaussian copula model, Marshall-Olkin copula provides an overly skewed smile and double-*t* copula and stochastic correlation model lie in between and provide a better fit.

Difficulties in achieving a good fit to the market with a copula model parametrically also give rise to non-parametric ways of creating copulas. In Hull & White (2006) a non-parametric model was constructed such that a copula can be implied, in the form of hazard rate scenarios, from observed market quotes. The rationale behind this is the following.

Notice that coupled with marginal default probability $P(t)$, the factor form of equation 1.1.6 can be transformed to the conditional default probability as,

$$P(t \mid M) = \Phi\left(\frac{\Phi^{-1}(F_i(t)) - \rho_i M}{\sqrt{1-\rho_i}}\right)$$

From equation 1.1.7 one can see that a copula model is completely defined by the conditional independent default probability $P(t \mid M)$ and the distribution of the common factor M. If we define $\lambda_i(t \mid M)$ as the conditional hazard rate, the conditional default probability can also be written as

$$P(t \mid M) = 1 - \exp\Big[-\int_0^t \lambda_i(s \mid M)ds\Big]$$

Therefore, Hull & White (2008) argue that the copula functions are defined by hazard rate scenarios together with their distributions, which can be implied from market quotes exactly. This implied copula approach is to back out these hazard rates and their distributions by non-parametric methods.

Specifically, the cumulative default probabilities are given by

$$d_k = P(t \mid \lambda_k) = 1 - \exp(-\lambda_k t), k = 1 \cdots, nL$$

where there are nL point distributions for hazard rate scenarios and the probability for a particular hazard rate scenario λ_k is π_k.

By choosing a set of λ_ks and π_ks, the model is calibrated to the marginal default probability d

$$d = \sum_{k=1}^{nL} \pi_k d_k$$

and for the CDO market standard tranches plus the index, the following equations hold

$$\begin{aligned}\sum_{k=1}^{nL}\pi_k V_m(\lambda_k) &= 0, m = 1, \cdots, 6\\ \sum_{k=1}^{nL}\pi_k &= 1\\ \pi_k \geq 0 &\qquad k = 1, \cdots, nL\end{aligned}$$

where $V_m(\lambda_k)$ are the differences of the present values of premium leg and contingent leg of the five tranches plus the index tranche given a particular hazard rate scenario.

They find that when assuming constant recovery rates there are generally no solutions to the above equations. By assuming a negative relationship between default probability and recovery rate, exact solutions can be found.

In a broader context, Rosen & Saunders (2007) introduced the multi-factor Generalised Linear Mixed Models (GLMMs), which includes the implied copula approach as a special case. The multi-factor model incorporates extra default correlation within sector or geographical concentrations.[7] However, the paper only developed a static version of the model and extension to dynamic modelling increases computational effort substantially.

1.1.2.4. *The search for a dynamic model.* Even if a copula model can match market quotes perfectly, it is still a static model and not suitable for pricing CDOs with customised maturities or products dynamic in nature such as FDTrs or options on tranches. Their applicability is therefore inherently limited.

With market development for more exotic and sophisticated products, the need for a theoretically sound pricing model is even more important. Given the popularity of copula models one line of research is to extend copula models into dynamic ones.

In Schönbucher & Schubert (2001) the dynamics of survival probabilities and credit spreads of an obligor in a credit portfolio are derived by conditioning on the filtration generated by the default information of other obligors. It is shown that in situations with positive dependence, the default of one obligor causes the credit spreads of the other obligors to jump upwards, which is observed empirically in credit contagion.

[7]The contagion effects, which will be discussed later in the chapter, are normally observed at the sector level. The multi-factor model therefore incorporates the contagion causes of defaults.

There are two basic filtrations generated by the factors that affect defaults.

The first filtration is $(\mathcal{G}_t)_{t\in[0,T]}$ which contains the background information about the process of general market variables until time t such as share prices, default-free interest rates or exchange rates, and also all default-relevant information except explicit information on the occurrence or non-occurrence of defaults.

The second filtration $(\mathcal{F}_t^i)_{t\in[0,T]}$ contains only information on default and survival of obligor i up to time t.

In this model, default happens when the default countdown process $\gamma_i(t)$ reaches the level of the trigger variable U_i:

$$\tau_i := \inf\{t : \gamma_i(t) \leq U_i\}$$

where the default trigger variables $U_i, i = 1, \cdots, I$ are correlated uniform random variables on the unit interval $[0, 1]$ which can be generated by a copula function.

The default countdown process $\gamma_i(t)$ is defined as

$$\gamma_i(t) := \exp\Big\{ -\int_0^t \lambda_i(u)du\Big\}$$

where $\lambda_i(u)$ is the stochastic default intensity of obligor i.

Different combinations of the two filtrations generate different information sets. Based on these different information sets, the survival probabilities dynamics of an obligor can be derived. Further, the spread dynamics of an obligor given these economic scenarios generated by particular copula models can also be derived from survival probabilities.

It shows that a copula model can produce the observed contagion effects that the credit spread of one obligor jumps given default information of other obligors.

In particular, it is found that for the Clayton copula these jumps are proportional to the pre-default intensity.

However, in this approach the dependency structure is static and exogenously given in the form of a copula.

In Totouom & Armstrong (2005) a dynamic Clayton copula class is developed and compared with the Gaussian base correlation approach. However, the calibration results are shown to be similar.

These dynamic copula models do not improve model fit due to the fact that the dependency structures are exogenously given by the the copula functions as in a static copula model.

However, dynamic models do not have to be derived from copula functions. By modelling the stochastic process of the factors that drive default, copula functions can be implied from the factor dynamics at a particular point in time. Enough flexibility can be achieved by giving a variety of specifications for the factor dynamics and a more realistic and flexible dependency structure can be generated. This class of models belongs to the bottom-up dynamic models introduced next. Also, the no-arbitrage conditions along the time dimension can be built into the models in construction.

1.1.3. Dynamic models. Dynamic models are characterised by explicit or inferred modelling of dynamic evolution of default intensities and point or asset processes. One group of dynamic credit portfolio modelling stems from modelling single obligor default. When generalised to modelling credit portfolios, the driving factors in these models are further identified as common and idiosyncratic factors. Conditioning on the common factors, individual default can then be lifted up to the portfolio level by building up the loss distribution from conditional default probabilities. Therefore, this type of model is called a *bottom-up* model. Bottom-up models can be further classified as structural and intensity models, like their single-credit counterparts.

Copula models also belong to bottom-up models and can be seen as the static reduced-form version of dynamic models. The dynamic models class preceded copula models in development but lost their favour among practitioners for the computational complexities. A major hurdle is that pricing has to be done via Monte Carlo simulation. However, some recent developments in dynamic models has successfully overcome this problem.

1.1.3.1. *Structural models.* The structural approach, pioneered in the single-obligor case by Black & Scholes (1973) and Merton (1974), models the evolution of firm value explicitly and default happens if, at maturity, the market value of the company falls below an exogenously given threshold or its debt value. The model was later extended by Black

& Cox (1976) to incorporate the case that default occurs in a first-passage-time manner. However, the model produces unrealistic short-term spreads for the reason that asset price follows a diffusion process and therefore default time is predictable. A major contribution recently comes from Zhou (2001b) that a few CDO pricing models are based on.

Zhou (2001b) extended Merton (1974) by introducing a jump factor Y into the asset price dynamics V_t as the following.

$$\frac{dV_t}{V_t} = (\mu - \lambda v)dt + \sigma dZ_1 + (\Pi - 1)dY \tag{1.1.12}$$

where μ represents the expected return of a firm's assets; v, λ and σ are positive constants and λ is the compensator for the Poisson process dY; Z_1 is a standard Brownian motion; dY is a Poisson process with intensity parameter λ; $\Pi > 0$ is the jump amplitude with expected value equal to $v + 1$; dZ_1, dY and Π are mutually independent.

In this model, the asset prices have unpredictable jumps and the company can default whenever its asset price falls below the threshold. In this way a short-term company can still default and thus realistic short-term spread is achievable.

Zhou (2001a) was the first to extend the first-passage-time diffusion model in Black & Cox (1976) to a two-firm default correlation model and an analytical formula was derived for calculating joint default probabilities.

Hull, Predescu & White (2006) also extended the Black & Cox (1976) structural model to CDO pricing. In Hull et al. (2006) the asset price of a company i is defined by $V_i(t)$ and $V_i(t)$ follows a diffusion process as in Black & Cox (1976).

$$\frac{dV_i}{V_i} = \mu_i dt + \sigma_i dZ_i$$

where Z_i is a Wiener process.

To generate asset correlations, Z_i is modelled by

$$dZ_t^i = a_i dW_t^c + \sqrt{1 - a_i^2} dW_t^i \tag{1.1.13}$$

where W_t^c and W_t^i are independent Wiener processes.

The pricing is done in two steps.

First, an iterative procedure is used to find the barrier parameters so that the default probabilities between two payment dates match those implied by the index tranche spreads.

Second, the Monte Carlo method is used to simulate the asset price process and thus the loss process can be generated and expected loss can be calculated.

This model is similar to the Gaussian copula model in equation 1.1.6 statically. Therefore, the calibration results were also similar to those of the Gaussian copula model.

One can infer that the pure diffusion process cannot generate the observed correlation skew. By adding a jump term to the diffusion process, however, it turns out that the jump-diffusion process produces realistic spreads in a single obligor case but also generates a more flexible dependency structure.

Willemann (2007) extended the jump-diffusion model in equation 1.1.12 to correlation modelling. To adapt the model to the risk-neutral framework, the author assumes that any jump-risk premium is already absorbed in model parameters. The asset-price dynamic is then given by,

$$d\ln X = \Big(r - \frac{\sigma^2}{2} - \lambda v + \phi\Big)dt + \sigma dZ + \ln \Pi dY$$

where r is the risk-free interest rate.

In contrast to Hull et al. (2006), who value CDOs using Monte Carlo simulation in a structural model, here the author does not assume that default occurs the first time the value of assets is at or below the barrier. Instead, he assumes that default at time t occurs if the value of assets is below the barrier at the time of payment, ignoring the fact that the company's assets may have earlier fallen below its debt threshold. This way the probability bucket approach can be used to build up the loss distributions.

To achieve correlations between assets, the author decomposes the driving Brownian Motion Z_i of each firm i, as in equation 1.1.13. The jump terms are common to all obligors.

Conditional on the integral of W_t^C and the number of jumps Y_t until any payment time t the default probabilities of an obligor can be calculated.

Then the recursive method in equation 1.1.9 can be used for fast semi-analytic computation. The calibration results to iTraxx and CDX data show that good fits are achieved for equity and senior tranches but mezzanine tranches are over valued.

Notice that the common factor that drives default correlation is the common Brownian term W_t^C and the common jump term $(\pi - 1)dY$. The jump term makes the tail of the distribution of the Brownian term fatter.

The author argues that the diffusive correlation achieved through correlating the innovation terms of the driving Brownian motion is symmetric and that it is not possible simply to increase the correlation to fit both the most senior tranches and equity tranches. However, the default correlation achieved through common jumps turns out to revolve around the occurrence of predominantly negative jumps and this causes asymmetric correlation in the sense that the asset values only move together when they move down. This type of correlation can be increased significantly without destroying the fit to junior tranches achieved through the symmetric diffusive correlation.

There are some later developments in the dynamic structural models.

Kiesel & Scherer (2007) generalised the Willemann (2007) model into the first-passage-time default type and introduced an efficient Monte Carlo method for pricing.

Inglis & Lipton (2007) introduces another structural jump diffusion model with multiple diffusion and Poisson terms. However, only the results of a static version of the model were given when calibrated to iTraxx and CDX prices and achieved good fit.

1.1.3.2. *Intensity models.* The intensity-based or reduced-form approach to credit risk modelling can be found in earlier papers by Jarrow & Turnbull (1995), Duffie & Singleton (1999), Lando (1998) and Schönbucher (1998).

The reduced form approach models the *intensity* of default rather than the explicit mechanism of corporate default as in the structural approach. In this way, the literature of interest-rate term-structure modelling can be borrowed. These models can be calibrated to credit spreads directly and became popular in single-credit modelling.

Lando (1998) introduced the intensity-based approach with the Cox or doubly stochastic process[8]. In a Cox process, conditional on the state variable X_s, the survival probability of a company is given by

[8]The name comes from the fact that there are two stochastic processes involved - the first is the stochastic intensity and the second is the inhomogeneous Poisson process when it is conditioned on the realisations of the intensity

$$P\{\tau > t \mid (X_s)_{0<s<t}\} = e^{-\int_0^t \lambda(X_s)ds}$$

where $\lambda(X_s)$ is the conditional default intensity. To extend it to multi-obligor settings the state variable can be further modelled by the sum of common and idiosyncratic factors.

However, the extension of the intensity-based approach to default correlation modelling has been criticised for not being able to produce enough correlation unless jumps are added to the diffusion process (Schönbucher 2003). Therefore, jump-diffusion processes are a favourable class for the default intensities and any Stochastic Deferential Equations(SDEs) admitting closed-form solutions are convenient choices. A notable class of jump diffusion processes is the affine process. For a comprehensive introduction to affine process in financial modelling see Duffie, Filipovic & Schachermayer (2003).

Duffie & Garleanu (2001) is one of first papers to adopt an intensity model for pricing CDOs. In the paper, the stochastic intensity $\lambda(t)$ is modelled by a basic Affine Jump Diffusion (AJD) process

$$d\lambda(t) = \kappa[\theta - \lambda(t)]dt + \sigma\sqrt{\lambda(t)}dW(t) + \Delta J(t)$$

with parameters $(\kappa, \theta, \sigma, \mu, \ell)$ where $W(t)$ is a standard Brownian motion and $\Delta J(t)$ denotes any jump that occurs at time t of a pure-jump process $J(t)$, independent of $W(t)$, whose jump sizes are independent and exponentially distributed with mean μ and whose jump times are those of an independent Poisson process with mean jump arrival rate ℓ. κ is the mean reverting rate, the long run mean is $\overline{m} = \theta + \ell\mu/\kappa$ and the long-run variance of $\lambda(t)$ is given by $\frac{\sigma^2\overline{m}}{2\kappa} + \frac{l\mu^2}{\kappa}$

The basic affine process has the property that if X and Y are two such processes with parameters $(\kappa, \theta_X, \sigma, \mu, \ell_X)$ and $(\kappa, \theta_Y, \sigma, \mu, \ell_Y)$, then $X+Y$ is also a basic affine process with parameters $(\kappa, \theta, \sigma, \mu, \ell)$ with $\theta = \theta_X + \theta_Y$ and $\ell = \ell_X + \ell_Y$.

In Duffie & Garleanu (2001), correlation is introduced by

$$\lambda_i = X_C + X_i \tag{1.1.14}$$

where X_C and X_i are basic affine processes with, respectively, parameters $(\kappa, \theta_C, \sigma, \mu, \ell_C)$, and $(\kappa, \theta_j, \sigma, \mu, \ell_j)$ and $X_1, \cdots, X_N$ and X_C are mutually independent. By the property of basic affine process described above, λ_i is also a basic affine process with parameters $(\kappa, \theta, \sigma, \mu, \ell)$, where $\theta = \theta_C + \theta_i$ and $\ell = \ell_C + \ell_i$

The parameter

$$\rho = \frac{\ell_C}{\ell}$$

is the long-run fraction of jumps to a given obligors intensity that are common to all (surviving) obligors' intensities. If $X_c(0)/\lambda_i(0) = \rho$ then, for any distinct i and j, ρ is also treated as the initial instantaneous correlation between λ_i and λ_j, that is, the limiting correlation between λ_i and λ_j as t goes to zero.

While the model is not calibrated to standard CDS index tranches, the author conducts extensive tests of the model on risk analysis and market valuations on CDOs via Monte Carlo simulation with the effects of correlation for prioritisation schemes.

Based on Duffie & Garleanu (2001), Mortensen (2006) introduced a more flexible AJD model and a semi-analytical pricing method.

He proposes that the intensity λ is given by

$$\lambda_i = a_i X_C + X_i \tag{1.1.15}$$

When $a_i = 1$, equation 1.1.15 reduces to equation 1.1.14. This simple modification introduces a unique parameter for different obligors in the portfolio and the models can be used to price heterogenous portfolios. It can be inferred that an obligor with low default intensity will have low correlation with the common factor.

Notice that the new λ_i is no longer an affine process but marginal default probabilities still have a closed form, given by

$$\begin{aligned} P(\tau_i \leq t) &= 1 - \mathbb{E}[e^{-\int_0^t \lambda_{i,s} ds}] \\ &= 1 - \mathbb{E}[e^{-a_i \int_0^t Y_s ds}] \times \mathbb{E}[e^{-\int_0^t X_{i,s} ds}] \end{aligned}$$

where $\mathbb{E}[e^{-a_i \int_0^t Y_s ds}]$ and $\mathbb{E}[e^{-\int_0^t X_{i,s} ds}]$ can be further derived as AJD processes.

Then the integral of the common factor X_C is defined as

$$Z(t) = \int_0^t X_C(s) ds$$

By conditioning on the factor $Z(t)$, defaults are independent across all the entities.

The conditional default probability for obligor i is,

$$P(\tau_i \leq t \mid Z(t) = z) = 1 - e^{-a_i z} E[e^{-\int_0^t X_{i,s} ds}]$$

Then one can either use the recursive algorithm in equation 1.1.9 in the non-homogenous case or binomial distributions in the homogenous case to derive loss distributions.

The distribution of the common factor can be obtained by inversion of the character function using fast Fourier transform (FFT).

The calibration results show that the jump term is needed to obtain realistic correlation levels, and matching to market-quoted iTraxx and CDX data is at least as good as those reported by good copula models, such as the *t* copula model in Hull & White (2004). However, computation time is slower than the fastest copula models.

Eckner (2007) improved on Mortensen (2006) in terms of increasing computational speed and accuracy. First, the author makes the model identifiable and parsimonious by making parameter restrictions.

$$\frac{1}{m} \sum_{i=1}^{m} a_i = 1$$

where $a_i = \frac{c_{i,t}^{cds}}{Avg_i(c_{i,t}^{cds})}$ and $c_{i,t}^{cds}$ denotes the five-year CDS spread at time t for the i-th reference entity.

Further parameter simplifications are given by

$$\begin{aligned}
\kappa_i &= \kappa_Y = \kappa \\
\sigma_i &= \sqrt{a_i}\sigma_Y \\
\mu_i &= a_i\mu \\
\omega_1 &= \frac{l_Y}{l_i + l_Y} \\
\omega_2 &= \frac{a_i\theta_Y}{a_i\theta_Y + \theta_i}
\end{aligned}$$

These conditions reduce the number of free parameters to only seven and hence λ_i is a basic affine process, as follows.

$$\lambda_i \sim bAJD(\lambda_{i,0}, \kappa, \theta_i + a_i\theta_Y, \sqrt{a_i}\sigma_Y, l_i + l_Y, \mu_i)$$

Second, the author argues that the recursive algorithm 1.1.9 is the most time-consuming aspect of pricing and requires a large amount of time calculating the probability of events that are extremely unlikely to occur and therefore have a negligible impact on credit-tranche spreads. By avoiding calculating these unlikely events, the author modified the algorithm and reduced the time consumption of the recursive method from 65% to 38% of computing time.

In defence of the introduction of a premium to the risk-free discount rate in pricing to improve model fit, the author makes the following arguments.

First, since credit indices and credit tranches recognise only *bankruptcy* and *failure to pay* as credit events, whereas CDS contracts usually also include certain forms of *restructuring*, he expects the risk-neutral default probabilities implied by CDS spreads to be higher than those implied by credit indices and tranche spreads;

Second, the credit indices are among the most liquid contracts in the credit derivatives market, so an investor who wants to buy protection on the underlying portfolio is likely to prefer to trade the index instead of the less liquid individual CDS contracts.

To incorporate these two effects, the author supposes that risk-neutral expected cash flows to the seller of protection for a credit tranche are discounted at rates above the risk-free rate.

Specifically, the value at time s of a payment Z that is received at some future time $t > s$ is given by,

$$V_s^{\text{cds}}(t) = E_s^{\mathbb{Q}}\Big(e^{-\int_s^t (r_u+\eta_u^{\text{cds}})ds} Z\Big)$$

for a CDS contract and

$$V_s^{\text{tr}}(t) = E_s^{\mathbb{Q}}\Big(e^{-\int_s^t (r_u+\eta_u^{\text{tr}})ds} Z\Big)$$

for credit index tranche. Here the liquid discounts are η_u^{cds} and η_u^{tr} which are assumed to be constant.

In Feldhutter (2007) the author did a comprehensive empirical test with a few AJD credit models including that of Mortensen (2006) on an extensive data set of CDSs and CDO tranche spreads on the CDX Investment Grade data. The paper found that the variations over time in actual tranche spreads are matched well for the most risky tranches but did not match the variation in senior tranches. It also found that the hedging performance for equity tranches are better than that of the Gaussian copula model. It suggested that incorporating time-varying jump intensity can in a parsimonious way improve the pricing and hedging performance of senior tranches.

1.1.3.3. *Levy models.* Levy processes have been popular in modelling stock prices in previous years. The GH distributions, which form the basis of a class of Levy processes, have been successfully applied in static copula models. Therefore, some GH distributions-based Levy models have also been developed and shown promising results. There have been some interesting models in both intensity-based and structural models in portfolio credit-risk modelling.

In Joshi & Stacy (2005) corporate defaults are driven by an information arrival process. Each issuer i defaults at a rate of $c_i(t)$ per unit of information arrival. Therefore, the probability that a name survives to time T is given by

$$\exp\Big\{-\int_0^T c_i(t)dI_t\Big\}$$

where I_t is the intensity of business time modelled by a sum of gamma processes. Pricing in the model is done using Monte Carlo simulation. With an efficient algorithm, a typical tranche of a 125-name CDO with a five-year maturity can be priced to three significant

figures within about five seconds on a desktop PC with C++ code. And it is found that the sum of two gamma processes can closely match the correlation skew.

Another paper is by Baxter (2007), where the assets in a structural model follow several gamma processes. Here the author shows, apart from achieving a better fit than the Gaussian model, the model also has stable model parameters when calibrated to data of a number of consecutive days. A more accurate model can be developed in the same framework by including jump terms.

1.1.3.4. *Lattice models.* Lattice models refer to a class of models in which defaults are driven by factors that assume a discrete set of states. They include binomial models as a special case.

Hull & White (2008) directly models the cumulated default intensities $X_t = \int_0^t \lambda(s)ds$ of an obligor with a jump process

$$dX_t = \mu dt + dq$$

where λ is intensity and H is the jump size with $\mu \geq 0$ and $H \geq 0$.

In any short interval of time Δt, $dq = H$ with probability $\lambda \Delta t$ and $dq = 0$ with probability $1 - \lambda \Delta t$

The basic version of the model assumes a homogenous portfolio with the same marginal default probabilities and recovery rate across all names. The probability of n defaults in a pool with N credits follows a binomial distribution given by

$$b(n, N, q) = \frac{N!}{n!(N-n)!} q^n (1-q)^{N-n}$$

where $q = 1 - S(J, t)$. $S(J, t)$ is the survival probability of an obligor by time t conditional on the number of jumps J up to time t given by

$$S(J, t) = \exp\Big\{ - M(t) - \sum_{j=1}^{J} H_j \Big\}$$

where $M(t)$ is the integral of μ up to time t.

The probability of J jumps between time 0 and time t is

$$P(J,t) = \frac{\Lambda(t)^J e^{-\Lambda(t)}}{J!}$$

where $\Lambda(t) = \int_0^t \lambda(s)ds$.

The expected notional on the CDO tranche at time t conditional on J jumps is

$$E(t \mid J) = \sum_{n=0}^{N} b(n, N, 1 - S(J,t))W(n,t)$$

where $W(n,t)$ is the tranche notional at time t when there are n defaults

$$W(n,t) = \begin{cases} 1 & \text{when } n < g(n_{\underline{L}}) \\ \frac{\overline{H} - n(1-R)/N}{\overline{H} - \underline{L}} & \text{when } g(n_{\underline{L}}) \leq n < g(n_{\overline{H}}) \\ 0 & \text{when } n \geq g(n_{\overline{H}}) \end{cases}$$

where R is the recovery rate, $g(n_{\underline{L}})$ is the smallest integer greater than $n_{\underline{L}}$ and $g(n_{\overline{H}})$ is the smallest integer greater than $n_{\overline{H}}$.

$n_{\underline{L}}$ and $n_{\overline{H}}$ are defined as,

$$\begin{aligned} a_{\underline{L}} &= \frac{\underline{L} \cdot N}{1 - R} \\ a_{\overline{H}} &= \frac{\overline{H} \cdot N}{1 - R} \end{aligned}$$

In the basic model, the drift μ is assumed to be zero and jump size as constant. The jump intensity $\lambda(t)$ is chosen to match the CDS spreads. The only free parameter is the jump size, which is, similarly to the correlation parameter in the Gaussian copula model, calibrated to the CDO market quotes.

When modelling a heterogenous portfolio, the drift μ is chosen to match the CDS spread and the intensity λ and jump size H are implied from market CDO quotes.

This model setup is problematic since the drift μ is not guaranteed to be increasing when calibrated to all CDS spreads for different payment dates, which violates the no-arbitrage conditions along the time dimension. For a closer examination see Schlögl (2008).

1.1.3.5. *Markov chain models.* There are both bottom-up and top-down Markov chain models. In this section we introduce bottom-up models and in the next section we will introduce the relevant top-down Markov chain models.

In Graziano & Rogers (2006) the intensity dynamics of each obligor are driven by a common factor represented by a continuous time Markov chain $(\xi_t)_{t\geq 0}$ that determines the probability of the survival of the obligor i to time t via

$$q_t^i = P(\tau^i \geq t \mid F_t^\xi) = \exp(-C_t^i)$$

where τ_i indicates the default time of the i-th reference entity, and C_i is an additive function of the chain of the following form

$$C_t^i = \int_0^t \lambda^i(\xi_u)du + \sum_{j\neq k} w_{jk}^i J_{jk}(t)$$

where λ_i is a deterministic function of the chain, $J_{jk}(t)$ denotes the number of jumps by time t from state j to state k, and the w_{jk}^i is a non-negative weight.

Furthermore, the continuously compounded short rate is assumed to be a deterministic function of ξ given by

$$B(0,t) = \exp\Big\{ \int_0^t r(\xi_u)du \Big\}$$

The process ξ is modelled by a continuous-time finite-state irreducible Markov chain. Thus the dependency between the defaults of different obligors is determined by λ, w_{jk}^i and the generating matrix of the Markov chain. These form the calibration parameters of the model, to be fitted to market quotes for CDSs and CDO tranche spreads. In addition, r as a function of ξ must be calibrated to match the observed term structure of default-free interest rates.

The loss distribution is calculated by Laplace transform of the loss process $L_t = A_i(1 - R_i)I_{\{\tau_i \leq t\}}$.

Pricing can be done by exact analytical method, Poisson simulation or MC method. The exact method using linear algebra routines is too cumbersome for most applications. The

MC method is generally faster but for large factor numbers the Poisson approximation method is better.

This model can be simultaneously fitted to market-quoted CDO tranches at five-, seven- and ten-year maturities - not perfectly, but within a reasonable number of basis points.

In bottom-up models both the static copula models and dynamic models developed in the past years have been improved in both model fit and pricing efficiency. However, if required to fit correlation skew exactly, more factors and parameters have to be added, which greatly decreases tractability and calibration efficiency. To further incorporate contagion effects, it becomes even more difficult to price and calibrate.

Some of the problems faced with bottom-up models can be directly dealt with by top-down models, which are introduced below.

1.1.4. General loss models. General loss models directly deal with the loss process at the portfolio level rather than modelling the default process of single obligors. They can normally achieve fast and accurate market calibrations. These models have spread dynamics that are rich enough to fit the correlation skew or the tranche option volatility that the market may imply. They are suitable to price such exotic derivatives as FDTrs, options on CDO tranches and leveraged super-senior tranches. However, since they model loss at the aggregate level, the hedges generated by these models are best at a systematic risk level. Hedging idiosyncratic risks is difficult because it requires the use of conditioning techniques to extract the information about a specific credit.

The focus of this thesis is on the study of bottom-up modelling. Therefore, only a brief review of the literature on top-down models is conducted.

Top-down models are to a large extent inspired by the interest-rate modelling literature.

There are two classes of interest-rate model. The first class models the instantaneous spot rate r_t with a suitable stochastic process. In these models the value of a zero-coupon bond at time t maturing at time T with face value of 1 is given by

$$B(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) \mid \mathcal{F}_t\right]$$

The second class of model is the Heath-Jarrow-Morton (HJM) framework introduced by Heath, Jarrow & Morton (1992) in which the modelling quantity is the whole forward curve $f(t,T), t \leq T$. In this class of model, a zero-coupon bond at time t maturing at time T with face value of 1 is given by

$$B(t,T) = \exp\left(- \int_t^T f(t,s)ds \right)$$

The HJM-type forward-rate models take the forward curve as input and capture the full dynamics of the entire forward-rate curve movement, whereas the short-rate models only capture a point in the curve. No arbitrage conditions can be derived by forming a two bond portfolio with different maturities. The result of the no-arbitrage condition is that the drifts restrictions can be expressed as functions of volatilities.

Similar to interest-rate modelling, the first class of general loss models picks the forward-loss process as the underlying dynamics, as for example in Bennani (2005), Schönbucher (2005) and Sidenius, Piterbarg & Anderson (2008). This class of model takes the market quotes as initial inputs and derives the no-arbitrage conditions for the loss dynamics. However, the challenge is much greater than interest-rate models since the no-arbitrage evolution of an entire loss surface is of concern rather than only a one-dimensional term structure.

Bennani (2005) introduced the forward-loss process L_T of a given portfolio and defined the no-arbitrage conditions that a model has to satisfy. The outstanding notional of the whole portfolio, $ON(t,T) = \mathbb{E}(1 - L_T \mid \mathcal{F}_t)$, is described by

$$ON(t,T) = \exp\left(- \int_0^t x_s ds \right) \exp\left(- \int_t^T X(t,s)ds \right)$$

where x_s is instantaneous loss rate and $X(t,s)$ is forward loss rate, which is modelled by a sum of Brownian Motions and Poisson processes.

Bennani (2005) implemented a one-dimensional version of the model with a trinomial tree and calibration results to standard iTraxx quotes are reasonable with two free parameters.

Schönbucher (2005) also presents a top-down model inspired by the HJM framework. Instead of modelling the loss-rate dynamics as in Bennani (2005), the paper starts by

modelling the transition rate $p_n(t,T)$ of the loss distributions with an auxiliary time-inhomogeneous Markov chain. Here $p_n(t,T)$ is a vector of probabilities viewed at time t with exactly n obligors having defaulted in the reference portfolio by time T.

The current CDO tranche values are totally determined by the current term structure of portfolio loss distributions. Therefore, the initial term structure $p(0,T)$ is set up to match the market tranche quotes with different strikes and maturities. Next, dynamics are introduced to the transition rates with diffusion and jump processes and no-arbitrage drift restrictions can be derived. This way the pricing of some more exotic products like CDO^2 can be done more directly. Eventually the model can be calibrated to individual obligor defaults by thinning methodology.

This approach allows a straightforward calibration of the model and, as in HJM-type interest-rate models, the arbitrage-free dynamics are then fully determined by the specification of the volatilities of the transition rates.

Ehlers & Schonbucher (2006) extended Schönbucher (2005) by considering non-constant interest rates in the model. These introduce conditional forward interest rates and forward protection rates (spreads) given the realisation of the loss process. An HJM-type specification of the loss-contingent forward interest and loss rates are then proposed and no-arbitrage conditions are derived.

Sidenius et al. (2008) models forward loss distribution as a state variable and assumes a diffusion process for the forward rates, requiring the dynamics to be specified under a background filtration which does not contain information about default times. Then the loss process is built up from the augmented filtration conditional on the paths of forward rates.

der Voort (2006) implements a concrete example under the aforementioned HJM-type framework and conducts model calibration using a lattice approach. The calibration results are used for pricing forward-starting CDOs and are compared with factor copula models.

Under the same framework, Krekel (2006) obtains an implied loss surface of a credit portfolio by means of a simple CDS bootstrapping procedure with sequential quadratic programming.

Filipovic, Overbeck & Schmidt (Forthcoming) provides a unified framework in top-down modelling by introducing a defaultable $(T; x)$-bond. A defaultable $(T; x)$-bond pays $1_{\{L_T \leq x\}}$ at maturity where $x \in [0, 1]$ represents the state where $100x\%$ of the overall nominal has defaulted and L_t represents the ratio of CDO-losses occurring by time t.

Essentially, all contingent claims on the CDO-pool, such as STCDOs, can be written and thus hedged and priced as linear combinations of $(T; x)$-bonds. For example, any European-type contingent claim on the loss process with (regular enough) payoff function $F(L_T)$ at maturity T can be decomposed into a linear combination of $(T; x)$-bonds.

The (T; x)-bond price term-structure movements are described explicitly in the form

$$P(t, T, x) = 1_{\{L_T \leq x\}} e^{-\int_t^T f(t,u,x)du}$$

where the corresponding $(T; x)$-forward rates, $f(t, u, x)$, are modelled as semi-martingales driven by Brownian motion and the jump processes. This setup allows for feedback, or contagion effects, from the loss process on the forward curve. Then absence of arbitrage in terms of a drift condition and a relation between the short end of the spread curve and the prevailing loss intensity are obtained. Mathematical evidence is provided that arbitrage-free $(T; x)$-bond models uniquely exist under general assumptions. Risk-neutral pricing has to be done by a generic Monte-Carlo algorithm.

The second class of general loss models, like their counterpart in interest-rate modelling, model the spot-loss process directly. They include models that either describe the point process for default or the intensity dynamics that drive these point processes. Compared to the first type of model, these models have to be calibrated to market-traded CDS index tranches rather than taken as initial conditions. There is also a rich range of specifications which can be applied to the intensity dynamics. One feature of these models is that they can incorporate different causes of corporate defaults - either factor driven or contagion effects.

The contagion-driven defaults were frequently observed, for example in the Asian crisis of 1997 or the bursting of the Internet and telecommunications bubble in 2001 and the recent sub-prime credit crisis. A research finding by Das, Duffie, Kapadia & Saita (2007) shows that contagion effects explain the extra clustering which exceeded that implied by

doubly stochastic models through testing historical default data of US corporations from 1979 to 2004.

Contagion models were introduced into correlated default modelling by Davis & Lo (2001) and Jarrow & Yu (2001) and have been studied by Schönbucher (2004), Giesecke & Weber (2006) and Kraft & Steffensen (2007).

In an early investigation, Davis & Lo (2001) considers a homogeneous portfolio in which an event ramps up the intensity by a fixed factor for an exponential time. Ordinary differential equations are then derived that govern the distribution of the default process.

In multi-obligor settings, a popular way of modelling contagion is using the Markov process, as in Herbertsson (2008) and Frey & Backhaus (2008).

In Herbertsson (2008), the correlated default is reinterpreted as a Markov jump process and individual default intensity jumps in the event that other obligors default. It allows the authors to use a matrix-analytic approach to derive computationally tractable closed-form expressions for CDO pricing.

Another class of models strives to capture both factors driving defaults and the interactions between defaults, such as contagion, using point processes.

In Giesecke & Tomecek (2005), the authors show that a time change with continuous paths increasing to 1 transforms a standard Poisson process into a general-point process with totally inaccessible arrivals and the compensator is given by the time change. The method can generate both path-dependent or self-exciting processes and the classical Hawkes process and doubly stochastic process are special examples.

In Giesecke & Goldberg (Forthcoming) the authors show that any point process can be thinned down to a sum of individual defaultable obligors. This makes it possible for portfolio credit models to be calibrated to individual default. Random thinning is a method to allocate portfolio intensity to the constituent names and its values represent the conditional probability of a firm which defaults or the next given default is imminent. The method facilitates both calibration to single-name credit swaps and multiname credit derivatives. The results can be used to hedge risk against single obligor defaults.

In Errais, Giesecke & Goldberg (2007), default times are governed by intensity-driven market-wide risk factors that follow an affine jump-diffusion process. The loss process

itself is a risk factor so that both the timing of past defaults and their recovery rates influence the future evolution of the portfolio loss. This affine point-process model leads to computationally tractable valuation, hedging and calibration of credit derivatives.

In Ding, Giesecke & Tomecekz (2009), a time-changed birth process is introduced for modelling correlated credit events. It can also be viewed as a *doubly stochastic birth process*, which is a self-exciting process compared to the normal doubly stochastic process. The pricing is computationally tractable by applying Laplace transform to the time change.

In the Generalised-Poisson Loss (GPL) model introduced in Brigo, Pallavicini & Torresetti (2006), the loss process is modelled by a sum of Poisson processes, with each Poisson process representing the event of one obligor default, two simultaneous obligor defaults, and so on.

Arnsdorf & Halperin (2007) introduces a Bivariate Spread-Loss Portfolio (BSLP) model. In this model the portfolio loss process follows a Markov chain whose generator is driven by a stochastic intensity (so that the generator itself becomes stochastic). The intensity is given by a diffusion process which can incorporate default-induced jumps. The portfolio default intensity is a derived process in this model and is shown to be a jump-diffusion process governed by contagion factors. Efficient lattice implementations are available for this two-dimensional Markovian model. The model can be viewed as a low-dimensional short-rate version of Schönbucher (2005) which requires Monte-Carlo simulations because of its non-Markovian nature.

Lopatin & Misirpashaev (2007), similar to Arnsdorf & Halperin (2007), introduces a two-dimensional intensity-based Markovian model. This model can be easily calibrated to a generic distribution of portfolio loss without sacrificing tractability and robustness. Different to Arnsdorf & Halperin (2007), the calibration procedure consists of two steps. The the first step is to find the intensity of an auxiliary one-step Markov chain model consistent with the CDO tranches. Because the intensity of the Markov chain is a deterministic function of accumulated loss and time, it can be referred to as local intensity, to distinguish it from the stochastic intensity of the full model. In the second step, the full two-dimensional model is calibrated to match the local intensity.

1.2. Motivation

This thesis is devoted to the modeling of defaults in the context of valuation and hedging of tranches of CDOs. As clearly manifested by the current credit crisis, this topic is of great practical importance. Indeed, the fact that the market practice in valuation and hedging of CDOs relied on an over-simplified static market model (Gaussian model) can be seen an important factor which contributed to the severity of the crisis. The main goals of my research are:

(i) to examine some of the existing copula-based approaches to correlated defaults and to assess the model risk associated with a model's choice, (ii) to provide static and dynamic extensions of copula-based models.

1.3. Thesis Structure

The thesis begins with a sensitivities analysis of the Normal Inverse Gaussian (NIG) copula model. It provides a deep understanding of default correlation, which leads to the later development of three binomial lattice models. The main contributions are the three binomial models, with increasing sophistication in succession to capture the dynamics of default while maintaining no-arbitrage conditions.

The thesis consists of four chapters:

1. *Tranche sensitivities for (NIG) copula model*. Copula models are a good vehicle for studying correlations. Chapter 2 investigates the performance of a *NIG* copula in correlation modelling by conducting tranche sensitivities analysis to model parameters. We compare it to the Gaussian model and show its significance for risk management. We conclude by suggesting better modelling parameterisation schemes.

2. *A static lattice model*. This model can be calibrated to all tranches in a capital structure with one set of model parameters and therefore fits the correlation skew. The distributions on a re-combining binomial lattice can fully capture the factor distribution implied by market-quoted tranches. We also experiment with a combined Cross Entropy and Nelder Mead method to increase calibration speed.

3. *A dynamic lattice model* The dynamic binomial model generalises the static model by modifying the model as a Cox process and satisfies no-arbitrage conditions in the time

dimension. However, since the conditional default probabilities are not a Markov process, the pricing has to be done along each path on the lattice. Computing load increases exponentially with increasing steps on the lattice and therefore the calibration performance is greatly compromised.

4. *A Markovian binomial model* By choosing the integral of the intensity process as the modelling quantity in this model, the conditional default process is Markovian. Therefore, the recombining feature of the lattice is exploited so that a recursive method can be used at each node of the lattice to build up loss distributions. The computational load is linear with the number of steps on the lattice. We also show how to incorporate a stochastic recovery rate in this model.

CHAPTER 2

Tranche sensitivities and model design

2.0.1. Introduction. CDOs are asset-backed securities whose tranche spread values are driven by the default correlations of the credits in the underlying portfolio. The Gaussian copula model has been under criticism because it is unable to reproduce the market quotes of CDS index tranches with one correlation parameter. The popular base correlation approach to solving this "correlation smile" problem, however, is not arbitrage free in the capital-structure dimension, as discussed in section 1.1.2.2. Though the recent effort of credit modelling has focused on developing dynamic models, the copula models remain an interesting topic to study.

In this chapter, we study the tranche sensitivities of Gaussian and Normal Inverse Gaussian (NIG) copulas to model parameters. The quantities under study are tranche expected loss and tranche spreads. The ground work is developed for tranche expected loss sensitivities, which have a more direct link to model parameters, and we further derive spreads sensitivities when testing calibration algorithms. The model parameters include the dependency parameters (for the Gaussian copula there is only one correlation parameter) and the default threshold parameter (the parameter that is calibrated to marginal default probabilities).

The gains from studying tranche sensitivities with regard to model parameters are as follows.

1). Some of these relationships can be observed empirically, such as where equity tranche spreads decrease and senior tranche spreads increase when correlation increases. These relationships are qualitative and model independent. However, the quantitative functional relationships between these variables are different for different models. The model risk involved can be disastrous if not treated properly in practice. By studying the tranche sensitivities to model parameters, the limitations of these models can be explored.

2). Some relationships cannot be easily observed and identified, such as tranche behaviour given changes in co-movements of extreme events. This relationship can be quantified in the model as tranche loss sensitivities given the variation in tail dependency. Therefore, we can study tranche behaviour given tail dependency variations independently of correlation.

We select the NIG copula for our study because it has two extra parameters which control tail dependence apart from a correlation parameter as in the Gaussian copula model. Also, a parameter sensitivities analysis on NIG copulas would be helpful in identifying the features that make it a better alternative to the Gaussian copula. In this sense we also go a step further from the comparison study in Burtschell et al. (2008), where the authors only juxtapose the calibration results for different copula models.

Meng & Sengupta (2008) studies the tranche expected loss sensitivities of the Gaussian copula models with regard to the correlation parameter and provides relevant mathematical proof. In a more complicated model such as the NIG copula model, the burden on mathematical proof is out of the scope of the chapter. Instead, our study focuses on deriving the functional relationship and simulating some of the qualitative behaviour of tranche sensitivities to model parameters.

In deriving tranche sensitivities for the NIG copula, we adopt some of the conventions in Meng & Sengupta (2008)[1] in deriving tranche loss sensitivities for the Gaussian copula. We provide simulation results for both copulas. One application of the results in the sensitivities analysis is for model calibration using Newton's method. Another application is in hedging against market risk in CDS spread change and correlation change. We show that Gaussian and NIG copula models can give quite different hedging quantities.

Our study in tranche sensitivities gives us insight into how the parametrisation of copula functions determines dependency in a CDO structure and drives defaults. This understanding helps us to solve the correlation smile problem by designing more flexible models.

The chapter is presented as follows:

[1]This paper was developed independently of Meng & Sengupta (2008). The current version adopted some conventions from that paper on the standard Gaussian copula model.

In the first section we lay out the assumptions behind our tranche sensitivities analysis and also the results obtained from simulation. In the second and third section, we derive the sensitivities of expected loss with regard to model parameters for the Gaussian copula and the NIG copula. In the fourth section we derive the spread sensitivities to model parameters for both copulas. In the fifth section we apply the spreads sensitivities results for model calibration and describe the calibration results. In the sixth section we discuss dependency structure, model design and risk management issues for copula models.

2.1. Assumptions and results

We assume a homogenous portfolio of $N = 125$ names as in a standard CDX or Itraxx portfolio. A *homogenous portfolio* is defined as a portfolio with all underlying names having the same credit default spreads or marginal default probabilities and constant recovery rates. We further assume a recovery rate of zero for all names. Therefore, a default of a credit results in the loss of one unit in the portfolio.

We study two types of tranches - equity tranche and tranchelet.

An *equity tranche* is a tranche which has a 0% attachment point.

A *tranchelet* in this chapter is a tranche which covers exactly one unit of loss. It is the most disaggregated tranche and can therefore more accurately reflect tranche sensitivities to model parameters. A tranchelet higher up on the capital structure is referred, in this chapter, as a senior tranchelet and one lower on the capital structure is referred as a junior tranchelet. All the derivatives with regard to model parameters will be derived on these two type of tranches.

We aim to verify whether the following results are also true for the NIG copula, as for the Gaussian copula.

1. Equity tranches are long correlation and senior tranches are short correlation [2].

2. Equity tranche deltas, given index spread movements, is a monotonically increasing function ranging from 0 to 1.

We also verify the following results for both Gaussian and NIG copulas through simulation.

[2]Here long (or short) correlation means a tranche investor would benefit from a long (or short) position if correlation increases.

1. Junior tranchelets are long correlation, senior tranchelets are short correlation and there's a *neutral tranchelet* that is close to static given correlation change.

2. Tranchelet deltas are positive.

We also derive the following results for the NIG copula only.

1. Equity tranche sensitivities to tail dependency parameters α and β.

2. Tranchelet sensitivities to tail dependency parameters α and β.

2.2. Gaussian copula tranche sensitivities

In this section we first present some results from Meng & Sengupta (2008) on equity tranche sensitivities to correlation for the Gaussian copula and extend it to tranchelet sensitivities to correlation.

2.2.1. Sensitivities to correlation for equity tranches. Consider a homogenous portfolio with N names, each with a notional unit. We rewrite the factor form 1.1.6 as the following, assuming a constant correlation.

$$X_i = \sqrt{\rho}M + \sqrt{1-\rho}Z_i \tag{2.2.1}$$

where $X_i, M, Z_i, 1 \leq i \leq N$ are independent and follow standard normal distributions and N is the number of obligors in a portfolio. The variables $X_1, \cdots, X_n$ are therefore related through the Gaussian copula with correlation ρ [3]

Proof.

$$\begin{aligned}
\mathbb{E}[X_i X_j] &= \mathbb{E}[(\sqrt{\rho}M + \sqrt{1-\rho}Z_i)(\sqrt{\rho}M + \sqrt{1-\rho}Z_j)] \\
&= \mathbb{E}[\rho M^2 + \sqrt{\rho(1-\rho)}Z_i M + \sqrt{\rho(1-\rho)}Z_j M + (1-\rho)Z_i Z_j] \\
&= \mathbb{E}[\rho M^2] \\
&= \rho\mathbb{E}[M^2] \\
&= \rho
\end{aligned}$$

[3] We only consider a time horizon within one payment in the sensitivities analysis. This will be relaxed when we apply it to model calibration.

Therefore

$$cor(X_i, X_j) = \frac{\mathbb{E}[X_i X_j] - \mathbb{E}[X_j]\mathbb{E}[X_j]}{var(X_i)var(X_j)} = \rho$$

We use the notation

$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and $\Phi(x) = \int_{-\infty}^{x} \phi(s)ds$

for the standard Gaussian density and cumulative distribution functions.

The probability that exactly j names default is

$$p_j = \int_{\mathbb{R}} \binom{N}{j} p^j (1-p)^{N-j} \phi(x) dx \tag{2.2.2}$$

where $p = \mathbb{P}[X_i \leq c \mid M = x] = \mathbb{P}\big[Z_i \leq \frac{c-\sqrt{\rho}x}{\sqrt{1-\rho}}\big] = \Phi\big(\frac{c-\sqrt{\rho}x}{\sqrt{1-\rho}}\big)$ is the conditional marginal default probability and c is the default threshold.

We use l_k^e to denote the loss of the $[0, k]$ equity tranche and l_k^{let} to denote the loss of the $[k-1, k]$ tranchelet that exactly covers the k-th default in the capital structure by the chosen time horizon.

Assuming there are v number of defaults, the loss for an equity tranche l_k^e is,

$$l_k^e = \min\{v, k\} = 1_{[v=1]} + 2 \cdot 1_{[v=2]} + (k-1)1_{[v=k-1]} + k1_{[v \geq k]}$$

The loss for the tranchelet l_k^{let} is,

$$l_k^{let} = \max\{\min\{v, k\} - (k-1), 0\} = 1_{[v \geq k]}$$

Let L_k^e be the expected value of l_k^e, it is proven in Meng & Sengupta (2008) that

$$\frac{dL_k^e}{d\rho} = \int_{\mathbb{R}} I_k(p) \frac{\partial p}{\partial \rho} \phi(x) dx < 0$$

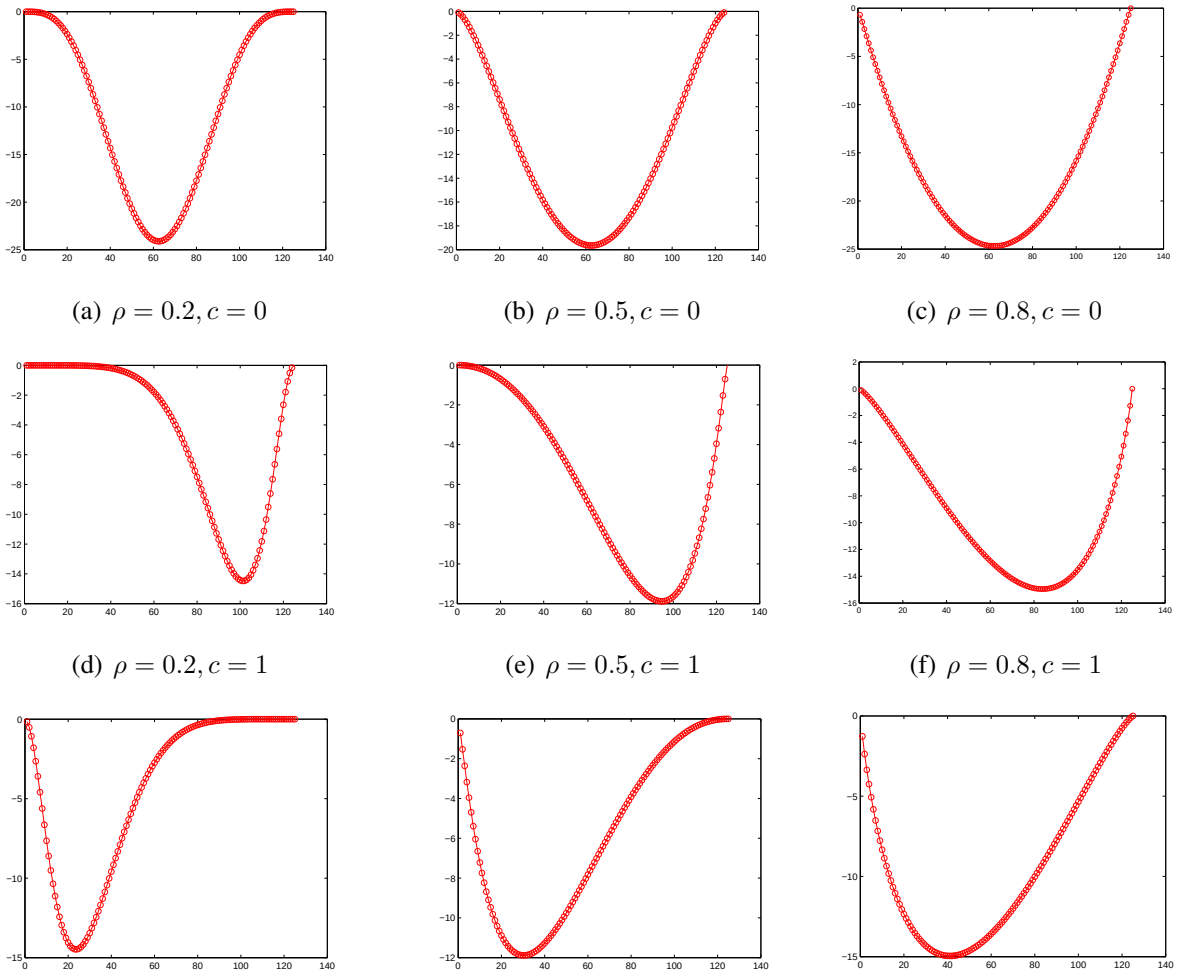
where

$$I_k(p) = \sum_{j=0}^{k-1} \binom{N}{j} (N-j) p^j (1-p)^{N-j-1}$$

$$\frac{\partial p}{\partial \rho} = -\frac{x - c\sqrt{\rho}}{2\sqrt{\rho}(1-\rho)^{3/2}} \phi\Big(\frac{c-\sqrt{\rho}x}{\sqrt{1-\rho}}\Big) \tag{2.2.3}$$

Since $\frac{dL_k^e}{d\rho}$ is always negative, it means that if correlation increases, the expected loss of an equity tranche decreases. Therefore an investor would benefit from such increases. Then we say equity tranches are long correlation.

Figure 2.2.1 illustrates the curve for $\frac{\partial L_k^e}{\partial \rho}$ with different correlation and default threshold levels.



(a) $\rho = 0.2, c = 0$ (b) $\rho = 0.5, c = 0$ (c) $\rho = 0.8, c = 0$

(d) $\rho = 0.2, c = 1$ (e) $\rho = 0.5, c = 1$ (f) $\rho = 0.8, c = 1$

(g) $\rho = 0.2, c = -1$ (h) $\rho = 0.5, c = -1$ (i) $\rho = 0.8, c = -1$

FIGURE 2.2.1. Expected loss sensitivities to correlation for equity tranches in the Gaussian copula. These figures confirm the theoretical results that the correlation is always negative.

2.2.2. Sensitivities to correlation for tranchelets. Let L_k^{let} be the expected loss of tranchelet l_k^{let}. It is given by

$$L_k^{let} = \mathbb{E}[l_k^{let}]$$

The loss of a $[k-1,k]$ tranchelet is the difference of the equity tranches $[0,k]$ and $[0,k-1]$.

$$l_k^{let} = l_k^e - l_{k-1}^e \tag{2.2.4}$$

Therefore take the expectation of both sides of 2.2.4

$$L_k^{let} = L_k^e - L_{k-1}^e \tag{2.2.5}$$

Take the derivative against the correlation parameter ρ of both sides of 2.2.5

$$\frac{\partial L_k^{let}}{\partial \rho} = \frac{\partial L_k^e}{\partial \rho} - \frac{\partial L_{k-1}^e}{\partial \rho} \tag{2.2.6}$$

Therefore

$$\begin{aligned}
\frac{\partial L_k^{let}}{\partial \rho} &= \frac{\partial L_k^e}{\partial \rho} - \frac{\partial L_{k-1}^e}{\partial \rho} \\
&= \int_{\mathbb{R}} I_k(p)\frac{\partial p}{\partial \rho}\phi(x)dx - \int_{\mathbb{R}} I_{k-1}(p)\frac{\partial p}{\partial \rho}\phi(x)dx \\
&= \int_{\mathbb{R}} \Big(I_k(p) - I_{k-1}(p)\Big)\frac{\partial p}{\partial \rho}\phi(x)dx
\end{aligned} \tag{2.2.7}$$

where

$$\begin{aligned}
I_k(p) - I_{k-1}(p) &= \sum_{j=0}^{k-1}\binom{N}{j}(N-j)p^j(1-p)^{N-j-1} - \sum_{j=0}^{k-1}\binom{N}{j}(N-j)p^j(1-p)^{N-j-1} \\
&= \binom{N}{k}(N-k)p^k(1-p)^{N-k-1}
\end{aligned} \tag{2.2.8}$$

and $\frac{\partial p}{\partial \rho}$ is given in equation 2.2.3.

Let $y = x - c\sqrt{\rho}$, we have

$$p = p(y) = \Phi(\frac{c-\sqrt{\rho}x}{\sqrt{1-\rho}}) = \Phi(\frac{c(1-\rho)-\sqrt{\rho}y}{\sqrt{1-\rho}})$$

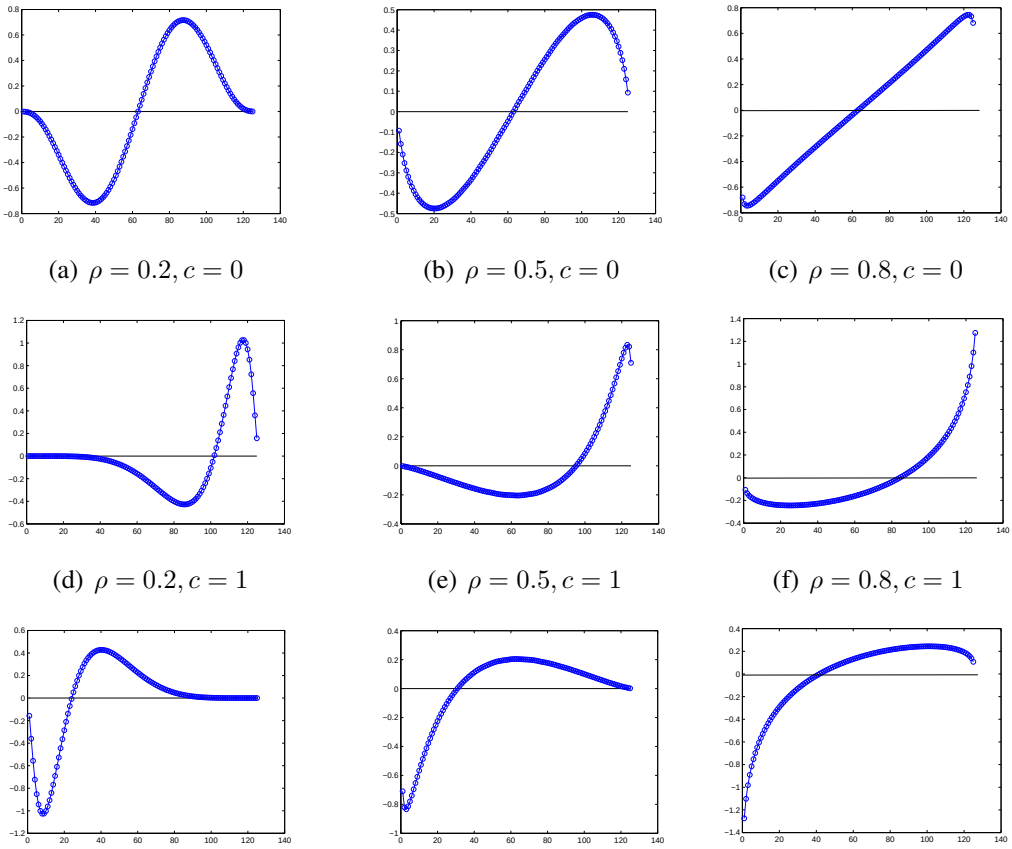
Therefore

$$\begin{aligned}
\frac{\partial L_k^{let}}{\partial \rho} &= -\int_{\mathbb{R}} \Big(I_k(p) - I_{k-1}(p)\Big)\frac{y}{4\pi(1-\rho)^{3/2}\sqrt{\rho}}e^{-\frac{y^2}{2(1-\rho)}-\frac{c^2}{2}}dy \qquad (2.2.9) \\
&= -\int_{\mathbb{R}} \binom{N}{k}(N-k)p^k(1-p)^{N-k-1}\frac{y}{4\pi(1-\rho)^{3/2}\sqrt{\rho}}e^{-\frac{y^2}{2(1-\rho)}-\frac{c^2}{2}}dy
\end{aligned}$$

Figure 2.2.2 plots the curve for $\frac{\partial L_k^{let}}{\partial \rho}$ with different correlation values and default thresholds. Since there are 125 loss units/tranches in a CDO, the curves appear discrete but fine enough to be close to be continuous. Since the curve crosses the x-axle once, if we increase the number of tranches to infinity, there should be one tranche that has a zero derivative value to correlation and thus is insensitive to correlation change. For a 125 name CDO, we call the closest tranche to this tranche that has a zero derivative value the neutral tranche. For junior tranchelets lower in seniority than the neutral tranche, the derivatives are negative and for senior tranchelets above the neutral tranche they are positive. From the figure, we can see that the neutral tranche position is affected by the default threshold c. A realistic threshold in the example is for $c = -1$ where default probability is less than 50% and the neutral tranchelet is closer to the 0% end of the capital structure. Therefore, the practice of hedging a long equity tranche with a short mezzanine tranche can be dangerous. This is because, if they happen to be on different sides of the neutral tranche, the two hedging quantities have different signs given correlation change and such a hedging strategy will increase the risk rather than eliminate it. Even though a Gaussian copula may indicate that the two tranches are both long correlation, the Gaussian copula model is not arbitrage free and the neutral tranche can be wrong. In a different copula model, as we will show in the section on the NIG copula model, the neutral tranche can be different from that of the Gaussian copula. Therefore, such a hedging strategy can be very sensitive to model risk.

2.2.3. Sensitivities to index spread change. The individual CDS spread change, in a homogeneous portfolio, is equivalent to a uniform change in the credit index. Since the default threshold c is calibrated to the marginal default probability and in a homogenous portfolio this threshold is the same across all names, a uniform change in spreads results in a change in this default threshold c for all obligors in a portfolio. Therefore, we are concerned with the tranche variation with regard to the variation in c.

Let Δ_k denote the delta of equity tranche $[0, k]$. It is the factor that makes the portfolio with a long position in the $[0, k]$ tranche and a short position in Δ_k times the full index stationary to first order, against variations of the default threshold c. Thus,



(a) $\rho = 0.2, c = 0$ (b) $\rho = 0.5, c = 0$ (c) $\rho = 0.8, c = 0$

(d) $\rho = 0.2, c = 1$ (e) $\rho = 0.5, c = 1$ (f) $\rho = 0.8, c = 1$

(g) $\rho = 0.2, c = -1$ (h) $\rho = 0.5, c = -1$ (i) $\rho = 0.8, c = -1$

FIGURE 2.2.2. Expected loss sensitivities to correlation for tranchelets in the Gaussian copula

$$\Delta_k = \frac{\frac{\partial L_k^e}{\partial c}}{\frac{\partial L_N}{\partial c}}$$

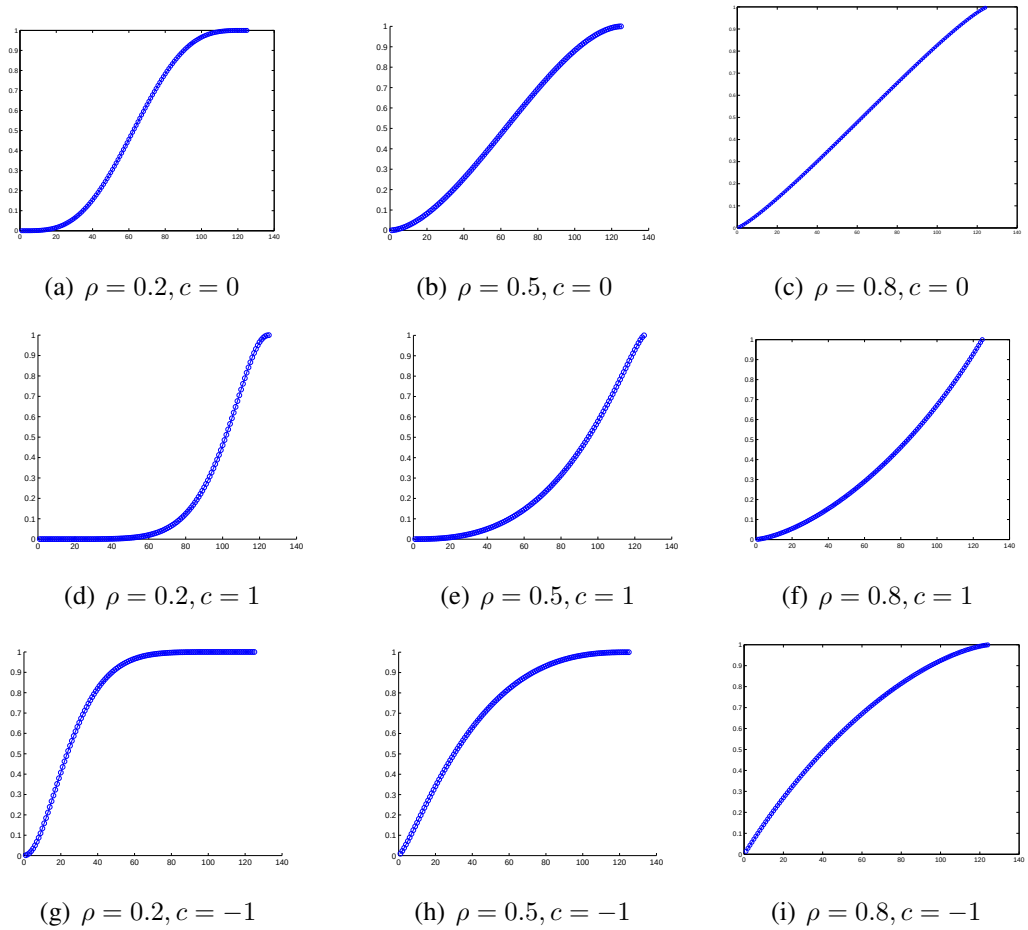
It is proven in Meng & Sengupta (2008) that,

$$\Delta_k = \frac{1}{N} \int_{\mathbb{R}} I_k(p(y)) \frac{1}{\sqrt{2\pi(1-p)}} e^{-\frac{y^2}{2(1-\rho)}} dx$$

Figure 2.2.3 shows the curve of Δ_k with different correlations and default thresholds and similar to the delta of equity derivatives in that it is a monotonically increasing curve between $[0, 1]$.

Let Δ_k^{let} be the delta of tranchelet $[k-1, k]$, which can be derived in the following way.

For the Gaussian copula, the expected loss of the CDS index is



(a) $\rho = 0.2, c = 0$ (b) $\rho = 0.5, c = 0$ (c) $\rho = 0.8, c = 0$

(d) $\rho = 0.2, c = 1$ (e) $\rho = 0.5, c = 1$ (f) $\rho = 0.8, c = 1$

(g) $\rho = 0.2, c = -1$ (h) $\rho = 0.5, c = -1$ (i) $\rho = 0.8, c = -1$

FIGURE 2.2.3. Deltas of equity tranches with a range of different correlations and thresholds given variation in index spreads

$$L_N = \mathbb{E}(l_N) = N\Phi(c)$$

and therefore,

$$\frac{\partial L_N}{\partial c} = N\phi(c) = \frac{N}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}$$

From equation 2.2.5, the expected loss of tranchelet L_k^{let} is

$$L_k^{let} = L_k^e - L_{k-1}^e$$

We have

$$\begin{aligned}\frac{\partial L_k^{let}}{\partial c} &= \frac{\partial L_k^e}{\partial c} - \frac{\partial L_{k-1}^e}{\partial c} \\ &= \int_{\mathbb{R}} \Big(I_k(p) - I_{k-1}(p)\Big)\frac{\partial p}{\partial c}\phi(x)dx\end{aligned}$$

where $I_k(p) - I_{k-1}(p)$ is given in equation 2.2.8.

From Meng & Sengupta (2008), we have

$$\frac{\partial p}{\partial c}\phi(x) = \frac{1}{2\pi\sqrt{1-\rho}} e^{-\frac{(x-c\sqrt{\rho})^2}{2(1-\rho)} - \frac{c^2}{2}}$$

Setting $y = x - c\sqrt{\rho}$, then $\frac{\partial L_k^{let}}{\partial c}$ can be further derived by

$$\begin{aligned}\frac{\partial L_k^{let}}{\partial c} &= \int_{\mathbb{R}} \Big(I_k(p) - I_{k-1}(p)\Big)\frac{\partial p}{\partial c}\phi(x)dx \\ &= \int_{\mathbb{R}} \binom{N}{k}(N-k)p^k(1-p)^{N-k-1}\frac{1}{2\pi\sqrt{1-\rho}} e^{-\frac{(y)^2}{2(1-\rho)} - \frac{c^2}{2}}dx \quad (2.2.10)\end{aligned}$$

Therefore, the delta of tranchelet $[k-1, k]$ is

$$\Delta_k^{let} = \frac{\frac{\partial L_k^e}{\partial c}}{\frac{\partial L_N}{\partial c}} = \frac{1}{N}\int_{\mathbb{R}} \binom{N}{k}(N-k)p^k(1-p)^{N-k-1}\frac{1}{\sqrt{2\pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}}dx$$

Figure 2.2.4 shows the curve of tranchelets delta with different correlation and default thresholds and in all figures the delta are positive.

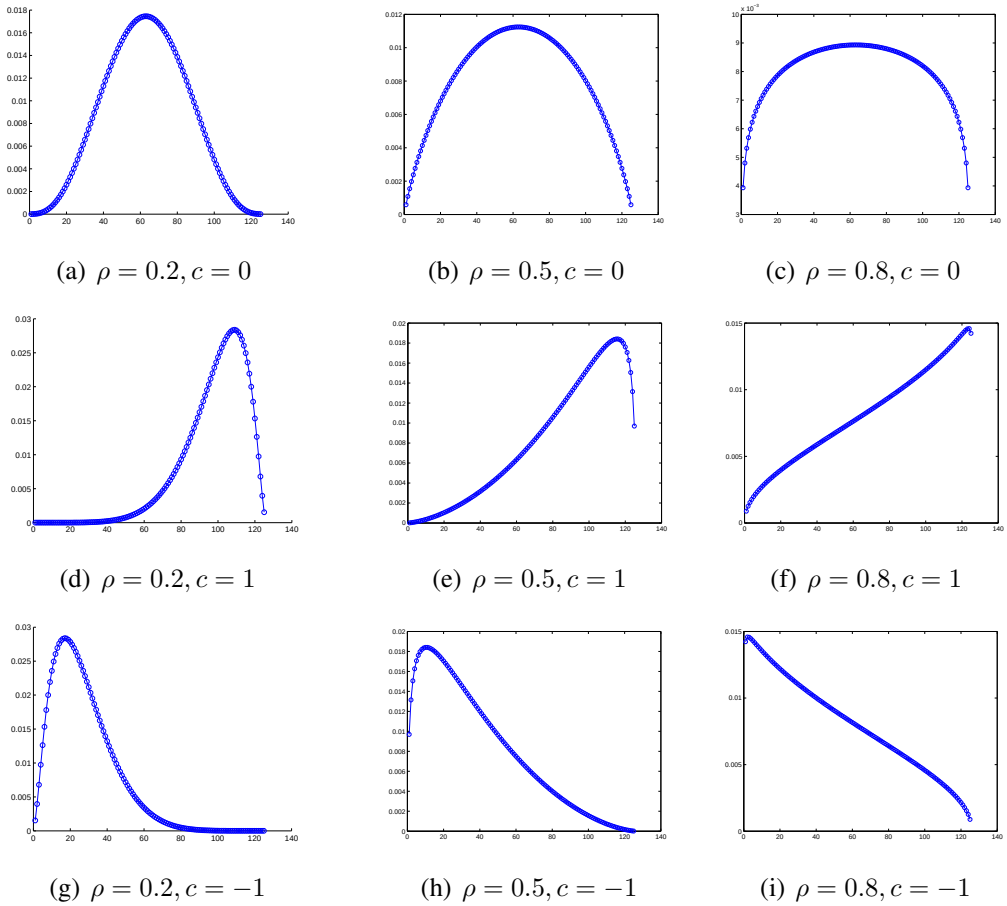
2.3. Normal Inverse Gaussian copula tranche sensitivities

In this section, we derive tranche sensitivities for the NIG copula.

If a random variable X follows a NIG distribution[4], $X \sim NIG(\alpha, \beta, \mu, \delta)$, its density function has the following form,

$$f(x; \alpha, \beta, \mu, \delta) = \frac{\delta\alpha \cdot \exp(\delta\gamma + \beta(x-\mu))}{\pi\sqrt{\delta^2 + (x-\mu)^2}} K_1(\alpha\sqrt{\delta^2 + (x-\mu)^2}) \quad (2.3.1)$$

[4]NIG distribution is a member of the General Hyperbolic (GH) distribution. Here we use NIG as an example and similar results can be obtained with distributions in the GH family.



(a) $\rho = 0.2, c = 0$ (b) $\rho = 0.5, c = 0$ (c) $\rho = 0.8, c = 0$

(d) $\rho = 0.2, c = 1$ (e) $\rho = 0.5, c = 1$ (f) $\rho = 0.8, c = 1$

(g) $\rho = 0.2, c = -1$ (h) $\rho = 0.5, c = -1$ (i) $\rho = 0.8, c = -1$

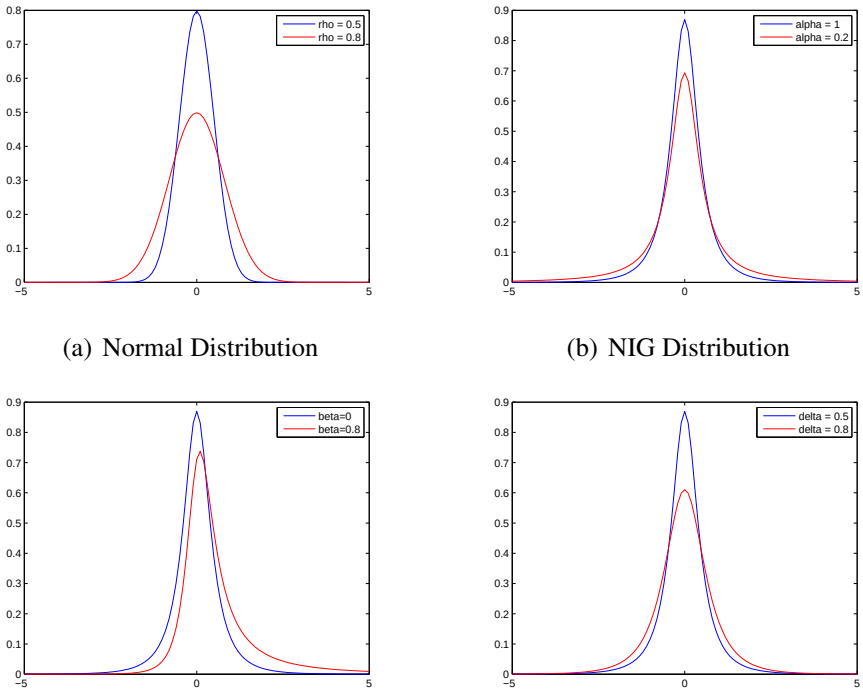
FIGURE 2.2.4. Delta of tranchelets with regard to index spreads. They are positive which confirms empirical observation.

where $\gamma = \sqrt{\alpha^2 - \beta^2}$ and $K_1(\omega) = \frac{1}{2}\int_0^\infty \exp(-\frac{1}{2}\omega(t + \frac{1}{t}))dt$ is the modified Bessel function of the third kind.

There are four parameters in the density function.

- α determines the kurtosis which controls the peakness of the centre and the fatness of the tails.
- β is the skewness parameter which can vary the fatness of the tails by increasing one and decreasing the other.
- μ is the location parameter. However, unlike the Gaussian distribution, μ is not the mean of the distribution. Its mean is $\mu + \delta\beta\gamma$.
- δ is the scale parameter. Unlike the Gaussian distribution, δ is not the variance of the distribution. Its variance is $\delta\alpha^2/\gamma^3$.

Figure (2.3.1) illustrates the distributions of normal and NIG distributions. Sub-figure (a) shows that the NIG distribution has a skinnier peak and fatter tails than the Normal distribution. Sub-figure (b) shows that in the NIG distribution a smaller α generates fatter tails. Sub-figure (c) shows that increasing β increases the fatness of the right tail and decreases the fatness of the left tail. Sub-figure (d) shows that increasing δ increases the variance of the NIG distribution.



(a) Normal Distribution (b) NIG Distribution

(c) NIG Distribution (d) NIG Distribution

FIGURE 2.3.1

The NIG distribution for α and β is stable under convolution.

PROPERTY 2.3.1. *If M and N are independent random variables and follow a NIG distribution*

$$M \sim F_{NIG}(\alpha, \beta, \mu_1, \delta_1), N \sim F_{NIG}(\alpha, \beta, \mu_2, \delta_2)$$

Then the sum $M + N$ follow a NIG distribution[5]

$$M + N \sim F_{NIG}(\alpha, \beta, \mu_1 + \mu_2, \delta_1 + \delta_2)$$

[5]Notice that M and N share a common set of α and β.

Our NIG factor model has the following representation.

$$X_i = M + Z_i \tag{2.3.2}$$

M and Z_i are independent random variables and $M \sim F_{NIG}(\alpha, \beta, 0, \rho)$ and $Z_i \sim F_{NIG}(\alpha, \beta, 0, 1-\rho)$. From Property (2.3.1) $X_i \sim F_{NIG}(\alpha, \beta, 0, 1)$.

The correlation between X_i and X_j can be derived as ρ.

$$\begin{aligned}
cov(X_i X_j) &= \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] \\
&= \mathbb{E}[(M + Z_i)(M + Z_j)] - (\mathbb{E}[M] + \mathbb{E}[Z_i])(\mathbb{E}[M] + \mathbb{E}[Z_j]) \\
&= \mathbb{E}[M^2 + Z_i M + Z_j M + Z_i Z_j] - (\mathbb{E}[M]^2 + \mathbb{E}[M]\mathbb{E}[Z_j] \\
&\quad + \mathbb{E}[M]\mathbb{E}[Z_i] + \mathbb{E}[Z_i]\mathbb{E}[Z_j]) \\
&= \mathbb{E}[M^2] - \mathbb{E}[M]^2 \\
&= var(M) \\
&= \rho\alpha^2/\gamma^3
\end{aligned}$$

Hence,

$$cor(X_i, X_j) = \frac{\mathbb{E}[X_i X_j] - \mathbb{E}[X_j]\mathbb{E}[X_j]}{\sqrt{var(X_i)}\sqrt{var(X_j)}} = \frac{\rho\alpha^2/\gamma^3}{\sqrt{\alpha^2/\gamma^3}\sqrt{\alpha^2/\gamma^3}} = \rho$$

Therefore, ρ is the correlation parameter.

We use the notation

$$f(x; \alpha, \beta, \delta) = \frac{\delta\alpha \cdot \exp(\delta\gamma + \beta x)}{\pi\sqrt{\delta^2 + x^2}} K_1(\alpha\sqrt{\delta^2 + x^2})$$

and $F(x; \alpha, \beta, \rho) = \int_{-\infty}^{x} f(s; \alpha, \beta, \rho) ds$

for the NIG density and distribution functions when $\mu = 0$

The probability that exactly j names default is therefore given by

$$p_j = \int_{\mathbb{R}} \binom{N}{j} p^j (1-p)^{N-j} f(x; \alpha, \beta, \rho) dx$$

where p is the conditional default probability.

$$p = P\{X_i \leq c \mid M = x\} = P\big\{Z_i \leq c - x\big\} = F\big(c - x; \alpha, \beta, 1 - \rho\big) \tag{2.3.3}$$

2.3.1. Equity tranches sensitivities to model parameters. There are three parameters in the NIG copula that are of importance to our study: α, β and ρ. The location parameter μ does not affect the correlation between the variables and is omitted from our analysis. The linear correlation parameter ρ is similar to the one in the Gaussian copula. The extra parameters α and β are of particular interest since they can change the tails of a NIG distribution and hence the tail dependencies in a NIG copula.

2.3.1.1. *Equity tranches sensitivities to ρ, α and β.* The expected loss of an equity tranche is given by

$$\begin{aligned} L_k^e &= k - \sum_{j=0}^{k} (k-j) p_j \\ &= k - \sum_{j=0}^{k} (k-j) \int_{\mathbb{R}} \binom{N}{j} p^j (1-p)^{N-j} f(x; \alpha, \beta, \rho) dx \end{aligned}$$

The equity tranche sensitivities to correlation are derived by taking the derivative with respect to ρ.

$$\begin{aligned}
\frac{\partial L_k^e}{\partial \rho} &= -\sum_{j=0}^{k}(k-j)\int_{\mathbb{R}}\binom{N}{j}\Big\{[jp^{j-1}(1-p)^{N-j}-(N-j)p^j(1-p)^{N-j-1}]\frac{\partial p}{\partial \rho}f(x;\alpha,\beta,\rho) \\
&\quad +p^j(1-p)^{N-j}\frac{\partial f(x;\alpha,\beta,\rho)}{\partial \rho}\Big\}dx \\
&= \int_{\mathbb{R}} I(k)\frac{\partial p}{\partial \rho}f(x;\alpha,\beta,\rho)dx \\
&\quad -\int_{\mathbb{R}}\sum_{j=0}^{k}\binom{N}{j}(k-j)p^j(1-p)^{N-j}\frac{\partial f(x;\alpha,\beta,\rho)}{\partial \rho}dx \\
&= \int_{\mathbb{R}}\sum_{j=0}^{k-1}\binom{N}{j}(N-j)p^j(1-p)^{N-j-1}\frac{\partial p}{\partial \rho}f(x;\alpha,\beta,\rho)dx \\
&\quad -\int_{\mathbb{R}}\sum_{j=0}^{k-1}\binom{N}{j}(k-j)p^j(1-p)^{N-j}\frac{\partial f(x;\alpha,\beta,\rho)}{\partial \rho}dx \\
&= \int_{\mathbb{R}}\Big(I_k\frac{\partial p}{\partial \rho}f(x;\alpha,\beta,\rho)-B_k\frac{\partial f(x;\alpha,\beta,\rho)}{\partial \rho}\Big)dx
\end{aligned}$$

where

$$I_k=\sum_{j=0}^{k-1}\binom{N}{j}(N-j)p^j(1-p)^{N-j-1}$$

and

$$B_k=\sum_{j=0}^{k-1}\binom{N}{j}(k-j)p^j(1-p)^{N-j}$$

The derivative $\frac{\partial p}{\partial \rho}$ can be derived as

$$\begin{aligned}
\frac{\partial p}{\partial \rho} &= \frac{\partial F(c-x;\alpha,\beta,1-\rho)}{\partial \rho} \\
&= -\int_{-\infty}^{c-x}\frac{\partial f(s;\alpha,\beta,1-\rho)}{\partial \delta}ds
\end{aligned}$$

where

$$\frac{\partial f(x;\alpha,\beta,\delta)}{\partial \delta} = \frac{\alpha\text{exp}(\delta\gamma+\beta x)}{\pi(\delta^2+x^2)}\big[\frac{\delta\gamma(\delta^2+x^2)+x^2}{\sqrt{\delta^2+x^2}}K_1(\alpha\sqrt{\delta^2+x^2})-\frac{1}{4}\alpha\delta^2K_2(\alpha\sqrt{\delta^2+x^2})\big]$$

where $K_2(w) = \int_0^\infty (t + 1/t)\exp(-\frac{1}{2}w(t + 1/t))dt$

The simulation results are illustrated in figure 2.3.2 for a variety of different parameters. It shows that for the NIG copula the derivative $\frac{\partial L_k^e}{\partial \rho} < 0$, as in the Gaussian copula. We know in general that the fair spread of the senior tranche(let) is positively related to correlation and the spread of the junior tranche(let) is negatively related to correlation. Therefore, the derivative for the equity tranche curve starts close to 0 and decreases first as its detachment point increases to incorporate more junior tranches. After it starts to incorporate senior tranches, it starts to increase its value until it finally reaches 0, when the detachment point becomes 100%. That is when the equity tranche becomes the full capital structure, which is indifferent to correlation change.

Next we derive the equity tranche sensitivities to parameter α.

It is easy to show that,

$$\frac{\partial L_k^e}{\partial \alpha} = \int_{\mathbb{R}} \Big(I_k \frac{\partial p}{\partial \alpha} f(x; \alpha, \beta, \rho) - B_k \frac{\partial f(x; \alpha, \beta, \rho)}{\partial \alpha} \Big) dx$$

Here calculating $\frac{\partial p}{\partial \alpha}$ is somewhat different from calculating $\frac{\partial p}{\partial \rho}$. We know that the default threshold is calibrated to marginal default probabilities, therefore

$$c = F^{-1}(D; \alpha, \beta, 1) \tag{2.3.4}$$

where D is the default probability of an obligor by the concerned time period. If there is variation in α, the marginal default probability D remains calibrated and constant; thus the default threshold c has to vary in response to the change.

Therefore, equation 2.3.3 becomes

$$p = F(F^{-1}(D; \alpha, \beta, 1) - x; \alpha, \beta, 1 - \rho) \tag{2.3.5}$$

Let $z = F^{-1}(D; \alpha, \beta, 1) - x$, then $p = F(z; \alpha, \beta, 1 - \rho)$

Then $\frac{\partial p}{\partial \alpha}$ is given by,

$$\frac{\partial p}{\partial \alpha} = \frac{\partial F(z; \alpha, \beta, 1 - \rho)}{\partial z} \frac{\partial z}{\partial \alpha} + \frac{\partial F(z; \alpha, \beta, 1 - \rho)}{\partial \alpha} \tag{2.3.6}$$

Further,

$$\frac{\partial z}{\partial \alpha} = \frac{\partial F^{-1}(D;\alpha,\beta,1)}{\partial \alpha} \tag{2.3.7}$$

From equation 2.3.4,

$$D = F(c;\alpha,\beta,1)$$

then we have to use the implicit function differentiation rule to derive $\frac{\partial c}{\partial \alpha}$, given below.

$$\frac{\partial c}{\partial \alpha} = -\frac{\partial F/\partial \alpha}{\partial F/\partial c} \tag{2.3.8}$$

We know $\partial F(c;\alpha,\beta,1)/\partial c = f(c;\alpha,\beta,1)$ and

$$\partial F/\partial \alpha = \int_{-\infty}^{c} \frac{\partial f(s;\alpha,\beta,1)}{\partial \alpha} ds \tag{2.3.9}$$

Therefore, we have,

$$\frac{\partial p}{\partial \alpha} = -\frac{f(c-x;\alpha,\beta,1-\rho)}{f(c;\alpha,\beta,1)} \int_{-\infty}^{c} \frac{\partial f(s;\alpha,\beta,1)}{\partial \alpha} ds + \int_{-\infty}^{c-x} \frac{\partial f(s;\alpha,\beta,1-\rho)}{\partial \alpha} ds \tag{2.3.10}$$

Since mathematically α and β are interchangeable, for β we have

$$\frac{\partial L_k^e}{\partial \beta} = \int_{\mathbb{R}} \Big(I_k \frac{\partial p}{\partial \beta} f(x;\alpha,\beta,\rho) - B_k \frac{\partial f(x;\alpha,\beta,\rho)}{\partial \beta} \Big) dx$$

and

$$\frac{\partial p}{\partial \beta} = -\frac{f(c-x;\alpha,\beta,1-\rho)}{f(c;\alpha,\beta,1)} \int_{-\infty}^{c} \frac{\partial f(s;\alpha,\beta,1)}{\partial \beta} ds + \int_{-\infty}^{c-x} \frac{\partial f(s;\alpha,\beta,1-\rho)}{\partial \beta} ds \tag{2.3.11}$$

It can be proven that

$$\begin{aligned}
\frac{\partial f(x;\alpha,\beta,\delta)}{\partial \alpha} &= \frac{\delta \exp(\delta\gamma + \beta x)}{\pi} \Big[\frac{1 + \frac{\alpha^2 \delta}{\gamma}}{\sqrt{\delta^2 + x^2}} K_1(\alpha\sqrt{\delta^2 + x^2}) - \frac{\alpha}{4} K_2(\alpha\sqrt{\delta^2 + x^2}) \Big] \\
\frac{\partial f(x;\alpha,\beta,\delta)}{\partial \beta} &= (x - \frac{\delta\beta}{\gamma}) f
\end{aligned}$$

Figure 2.3.3 shows the curves of derivative $\frac{\partial L_k^e}{\partial \alpha}$ for different parameter values. Interestingly, it is similar to the curve of derivative $\frac{\partial L_k^e}{\partial \rho}$, except that it is positive. The reason is that for the NIG copula, a bigger α results in smaller tail dependence, whereas, conversely, for the Gaussian copula a bigger ρ results in larger tail co-movements. Similar to our analysis on ρ, in the NIG copula, with increasing α, tail dependency decreases and extreme co-movements of default become unlikely, which decrease senior tranche loss and increase junior tranche loss. Starting close to 0, when the detachment points increase on the capital structure and incorporate more junior tranches, expected loss starts to increase. When senior tranches are incorporated, the expected loss starts to decrease and eventually becomes 0, since the full capital structure is indifferent to dependency change.

How the capital structure is indifferent to dependence parameter change can be seen from the equation 2.3.12 for the expected loss of the full capital structure.

$$L_N^e = N \cdot F(c; \alpha, \beta, 1) = N \cdot D \tag{2.3.12}$$

It is easy to tell that the expected loss is indifferent to correlation change since it does not appear in the formula. However, for α and β, since the model is calibrated to marginal default probability D, a change in α or β will cause a change in c.

2.3.1.2. *Equity tranches sensitivities to index spread change.* To compute the delta for equity tranches in the NIG copula

$$\Delta_k = \frac{\frac{\partial L_k^e}{\partial c}}{\frac{\partial L_N}{\partial c}}$$

We have the expected loss of the index given by

$$L_N = \mathbb{E}(l_N) = N \cdot F(c; \alpha, \beta, 0, 1)$$

and therefore the sensitivity of expected loss to default threshold is

$$\frac{\partial L_N}{\partial c} = N \cdot f(c; \alpha, \beta, 0, 1)$$

The sensitivity of expected loss to the default threshold is given by the derivative,

$$\frac{\partial L_k^e}{\partial c} = \int_{\mathbb{R}} I(k) \frac{\partial p}{\partial c} f(x; \alpha, \beta, \rho) dx$$

$\frac{\partial p}{\partial c}$ can be derived by

$$\begin{aligned} \frac{\partial p}{\partial c} &= \frac{\partial F(c - x; \alpha, \beta, 1 - \rho)}{\partial c} \\ &= f(c - x, \alpha, \beta, 0, \rho) \end{aligned}$$

Therefore, tranche delta Δ_k is given by,

$$\Delta_k = \frac{\int_{\mathbb{R}} I_k f(c - x, \alpha, \beta, 0, \rho) f(x; \alpha, \beta, \rho) dx}{N \cdot f(c; \alpha, \beta, 0, 1)}$$

It can be shown that, in figure 2.3.4, as for the Gaussian copula, the NIG copula has a delta that monotonically increases from 0 to 1.

2.3.2. Tranchelets sensitivities to model parameters. From the expected tranchelet loss in equation 2.2.5, we derive the sensitivity of tranchelet expected loss with regard to correlation given by,

$$\begin{aligned} \frac{\partial L_k^{let}}{\partial \rho} &= \frac{\partial L_k^e}{\partial \rho} - \frac{\partial L_{k-1}^e}{\partial \rho} \\ &= \int_{\mathbb{R}} \Big[(I_k - I_{k-1}) \frac{\partial p}{\partial \rho} f(x; \alpha, \beta, \rho) - (B_k - B_{k-1}) \frac{\partial f(x; \alpha, \beta, \rho)}{\partial \rho} \Big] dx \end{aligned}$$

where $I_k - I_{k-1}$ has been derived to be

$$I_k - I_{k-1} = \binom{N}{k} (N - k) p^k (1 - p)^{N-k-1}$$

and $B_k - B_{k-1}$ can be similarly derived

$$B_k - B_{k-1} = \sum_{j=0}^{k-1} \binom{N}{j} (N - j) p^j (1 - p)^{N-j-1}$$

Figure 2.3.5 shows the curve of $\frac{\partial L_k^{let}}{\partial \rho}$ for different model parameters. Similar to the Gaussian copula, the derivative value with regard to correlation for junior tranchelets is negative,

which indicates that these tranche spreads decrease their values with increasing correlation, and for senior tranchelets the curve values are positive, which indicates that senior tranche spreads increase their values with increasing correlation. As we can see, the neutral tranche position is also affected by β apart from c. Therefore, the results of the hedging strategy of a long equity tranche with a short mezzanine tranche can be very different using the NIG model and the Gaussian model.

Similarly, tranchelet sensitivities to α and β are given by

$$\frac{\partial L_k^{let}}{\partial \alpha} = \int_{\mathbb{R}} \Big[(I_k - I_{k-1}) \frac{\partial p}{\partial \alpha} f(x; \alpha, \beta, \rho) - (B_k - B_{k-1}) \frac{\partial f(x; \alpha, \beta, \rho)}{\partial \alpha} \Big] dx$$

and

$$\frac{\partial L_k^{let}}{\partial \beta} = \int_{\mathbb{R}} \Big[(I_k - I_{k-1}) \frac{\partial p}{\partial \beta} f(x; \alpha, \beta, \rho) - (B_k - B_{k-1}) \frac{\partial f(x; \alpha, \beta, \rho)}{\partial \beta} \Big] dx$$

Figure 2.3.6 shows the sensitivities of tranchelets expected loss with regard to α. It looks like the flip image of the curve for $\frac{\partial L_k^{let}}{\partial \rho}$ along the x-axis. It indicates that increasing α decreases extreme joint defaults or senior tranche spreads and increasing ρ increases them. However, we can see that the maximum value of $\frac{\partial L_k^{let}}{\partial \alpha}$ is much smaller than that of $\frac{\partial L_k^{let}}{\partial \rho}$. It tells us that there's only limited magnitude to what α (or β) can do to vary the expected loss of a copula. The NIG copula allows tail dependence but α and β have a limited capacity to vary it. It can be confirmed later in our calibration test that once ρ is found by the algorithm, α or β can only better the results slightly. However, if we compare sub-figure 2.3.6 (a) with sub-figure 2.3.5 (b), with the same parameter values, we can see that α has more influence on the tranchelet on the tail of the capital structure than ρ does. Therefore, parameter ρ and α together can change both the correlation and tail dependence in the expected loss curve.

2.3.3. Tranchelet sensitivities to index spread change. Following the setup in section 2.2.3, similar to equation 2.2.10, we can derive the following result for the NIG copula.

$$\frac{\partial L_k^{let}}{\partial c} = \int_{\mathbb{R}} (I_k - I_{k-1}) \frac{\partial p}{\partial c} f(x; \alpha, \beta, \rho) dx$$

where

$$\frac{\partial p}{\partial c} = \frac{\partial F(c-x;\alpha,\beta,1-\rho)}{\partial c} = f(c-x;\alpha,\beta,1-\rho)$$

Figure 2.3.7 juxtaposes Gaussian and NIG copula deltas with regard to index spread change. We can see that, if used in hedging, these two models will give very different hedging requirements. Since the NIG copula has more parameters and can achieve a better fit, it can give richer and potentially more accurate hedging quantities.

2.4. Spreads sensitivities to model parameters

Based on the sensitivities of expected loss with regard to model parameters, we can further derive model spreads sensitivities to model parameters and apply the results to model calibration.

We relax the model assumptions to incorporate a heterogenous portfolio and multiple payment periods. Then we replace p_k with $p(\ell, t_i)$ to denote the portfolio loss distribution by time t_i.

Take the derivative of the model spread derived in equation 1.1.4 with regard to a parameter d (i.e. ρ, α and β) as

$$\frac{\partial s}{\partial d} = \sum_{i=1}^{n}\sum_{l=0}^{N} \frac{\partial s}{\partial p(\ell,t_i)} \frac{\partial p(\ell,t_i)}{\partial d} \tag{2.4.1}$$

where n is the number of payments and N is the number of obligors.

Notice that the derivative $\frac{\partial s}{\partial p(\ell,t_i)}$ is model independent.

For $1 \le i \le n-1$

$$\frac{\partial s}{\partial p(\ell,t_i)} = \frac{[(D_i - D_{i+1})\text{PremiumLeg} + D_i\Delta_i\text{ContingentLeg}]}{\text{PremiumLeg}^2} \frac{\partial EL_i}{\partial p(\ell,T_i)} \tag{2.4.2}$$

and for $i = n$

$$\frac{\partial s}{\partial p(\ell,t_n)} = \frac{[\overline{D}_n\text{PremiumLeg} + D_n\Delta_n\text{ContingentLeg}]}{\text{PremiumLeg}^2} \frac{\partial EL_i}{\partial p(\ell,T_i)} \tag{2.4.3}$$

where D_i is discount factor for time t_i and Δ_i is the time increment between $[t_{i-1}, t_i]$.

In a heterogenous portfolio we have to use the recursive method in section 1.1.2.1 to derive $p(\ell, t_i \mid M)$. Since we assume a constant recovery rate, the algorithm 1.1.9 can be simplified as follows.

Assume we know the default distribution for a set of i names.

$$p^i(\ell, t \mid M), \qquad \ell = 0, \cdots, i$$

Add one name, with a conditional default probability $P_{i+1}(t \mid M)$. The default distribution for the new reference portfolio of $i+1$ names is:

$$\begin{aligned} p^{i+1}(0,t \mid M) &= p^i(0,t \mid M)(1 - P_{i+1}(t \mid M)) \qquad (2.4.4)\\ p^{i+1}(\ell,t \mid M) &= p^i(\ell,t \mid M)(1 - P_{i+1}(t \mid M)) + p^i(\ell-1,t \mid M)P_{i+1}(t \mid M) \\ &\quad \text{for} \quad \ell = 1, \cdots, i \\ p^{i+1}(i+1,t \mid M) &= p^i(i,t \mid M)P_{i+1}(t \mid M) \end{aligned}$$

with the initial default distribution for $i = 0$, $p^0(0, t \mid M) = 1$.

And the recursive formula for $\frac{\partial p(\ell, t_i \mid M)}{\partial d}$ can be derived as

$$\begin{aligned} \frac{\partial p^{i+1}(0,t \mid M)}{\partial d} &= (1 - P_{i+1}(t \mid M))\frac{\partial p^i(0,t \mid M)}{\partial d} - p^i(0,t \mid M)\frac{\partial P_{i+1}(t \mid M)}{\partial d} \\ \frac{\partial p^{i+1}(\ell,t \mid M)}{\partial d} &= (1 - P_{i+1}(t \mid M))\frac{\partial p^i(\ell,t \mid M)}{\partial d} + P_{k+1}(t \mid M)\frac{\partial p^i(\ell-1,t \mid M)}{\partial d} \\ &\quad + (p^i(\ell-1,t \mid M) - p^i(\ell,t \mid M))\frac{\partial P_{k+1}(t \mid M)}{\partial d} \qquad \text{for} \quad \ell = 1, \cdots, i \\ \frac{\partial p^{i+1}(i+1,t \mid M)}{\partial d} &= P_{i+1}(t \mid M)\frac{\partial p^i(i,t \mid M)}{\partial d} + p^i(i,t \mid M)\frac{\partial P_{k+1}(t \mid M)}{\partial d} \end{aligned}$$

2.4.1. Tranche spread sensitivities for the Gaussian copula. For the Gaussian copula there is only one correlation parameter; let $d = \rho$, we have

$$\frac{\partial p(\ell, t_i)}{\partial \rho} = \int_{\mathbb{R}} \frac{\partial p(\ell, t_i \mid M = x)}{\partial \rho}\phi(x)dx \qquad (2.4.5)$$

Further for any obligor k, its conditional default probability $P_k(t_i \mid M)$ is given by

$$P_k(t_i \mid M = x) = \Phi\Big(\frac{c_k^i - \sqrt{\rho}x}{\sqrt{1-\rho}}\Big) \tag{2.4.6}$$

where $c_k^i = \Phi^{-1}(D_i)$, D_i is the default probability of obligor k by time t_i and the derivative of $P_k^i(t \mid M)$ with regard to ρ is the same form of equation 2.2.3.

Figure 2.4.1 shows the sensitivities of the tranche spreads percentage change to the correlation parameter $\frac{\partial S/S}{\partial \rho}$ for $\rho = 0.64$.

We can see that the increase of the correlation parameter increases senior tranchelet spreads and lowers junior tranchelet spreads. It is consistent with the sensitivities of expected loss with regard to correlation.

2.4.2. Tranche spreads sensitivities for the NIG copula. For NIG copula, we have the sensitivities of loss distribution to model parameter d given by the following,

$$\frac{\partial p(\ell, t_i)}{\partial \rho} = \int_{\mathbb{R}} \Big(\frac{\partial p(\ell, t_i \mid M)}{\partial \rho} f(x; \alpha, \beta, \rho) + p(\ell, t_i \mid M) \frac{\partial f(x; \alpha, \beta, \rho)}{\partial \rho} \Big) dx \tag{2.4.7}$$

The conditional default probability $P_k(t_i \mid M = x)$ is given by,

$$P_k(t_i \mid M) = F(c_k^i - x; \alpha, \beta, 1-\rho) \tag{2.4.8}$$

where $c_k^i = F^{-1}(D_i; \alpha, \beta, 1)$.

Tranche spread sensitivity to correlation parameter is shown in Figure 2.4.2. Compared to that of the Gaussian copula, for the NIG copula the tranches at the two ends of the capital structure are more sensitive to correlation change, indicating that the NIG copula has a fatter tail and requires more capital protection given correlation change in the market.

2.5. Calibration algorithm for NIG copula and results

Now we can use the Newton-Raphson method in model calibration given the above results.

The objective function to minimise is given by,

$$v = \sqrt{\sum_{j=1}^{J} \Big(\frac{s_j - \text{quote}_j}{\text{quote}_j}\Big)^2} \tag{2.5.1}$$

where $1 \leq j \leq J$ is the tranche number and quote_j is the market quote for tranche j.

Then we can define the derivative with respect to model parameters as

$$\frac{\partial v}{\partial d} = \sum_{j=1}^{J} \frac{\partial v}{\partial s_j} \frac{\partial s_j}{\partial d} = \sum_{j=1}^{J} \frac{(s_j - \text{quote}_j)}{v \cdot \text{quote}_j^2} \frac{\partial s_j}{\partial d} \tag{2.5.2}$$

where the derivative $\frac{\partial s}{\partial d}$ was solved in the last section.

The calibration procedure for the Gaussian model is a direct application of Newton's method to minimise the objective function 2.5.2.

For the NIG copula model, the calibration procedure is as follows.

(1) Start with correlation parameter ρ with initial point $\rho = 0.5$ and use the Newton-Raphson Method to find an optimal value to minimise the objective function.
(2) Continue with the next parameter α and repeat the same procedure.
(3) Next we try parameter β to see if we can further minimise the objective function 2.5.1.
(4) If the error is within the bid/ask spread, then stop. Otherwise continue from step (1) until convergence.

2.5.1. Calibration results for the two models. For the NIG copula, we find that once we optimise correlation parameter ρ, we cannot further optimise the result significantly with parameters α and β. It confirms the results from the sensitivities analysis that α and β only have limited variability to influence expected loss. We test them separately in our calibration. When calibrating α and ρ we fix β to be zero and when calibrating β and ρ we fix α to be 2.

Table 2.5.1 shows the calibration results on CDX investment-grade market quotes from 21-Apr-04.

The results show that, except for the mezzanine tranches, all other tranches can be fit between bid and ask spreads. It confirms other researchers' findings that the NIG copula can fit market quotes better than the Gaussian copula model and the mezzanine tranches are generally overvalued by the model relative to the market.

Tranche(%)	Market mean	ask	bid	Gaussian	NIG(α)	NIG(β)
0-3	39.5(%)	37	42	39.5	39.47	38.4
3-7	305(bp)	280	330	380	320	326
7-10	106(bp)	102	110	112	102	103
10-15	49(bp)	39	59	34	46	47
15-30	11(bp)	6	16	3	12	14
error				2.597	0.1251	0.2383
ρ				0.1989	0.228	0.2497
α					5.9	2
β					0	0.97

TABLE 2.5.1. Calibration results for CDX IG tranches on 21-Apr-04 maturing in five years

2.6. Dependence structure, model design and risk management

2.6.1. Full dependence, independence and tail dependence. For both the Gaussian copula and NIG copula, two extreme forms of dependency can be achieved - full dependency and independence. The full dependency case corresponds to a 100% correlation with $\rho = 1$ and the independence case corresponds to a 0% correlation with $\rho = 0$. [6] Therefore, any correlations that the market implies can be achieved with both copulas.

For $0 < \rho < 1$ there is asymptotically no upper or lower tail dependency for the Gaussian copula, whereas for the NIG copula there are both upper and lower tail dependency for $0 < \rho < 1$. For the Gaussian copula tail dependency cannot be changed. However, for the NIG copula tail dependency can be changed by variation in α or β, while keeping correlation unchanged.

2.6.2. Insights on model design. The NIG copula is built upon NIG distributions such that there is an independent correlation parameter which controls the correlation or general dependency between obligors and tail dependence parameters α and β, which control local dependence. This ensures that one can vary its tail dependency, while maintaining correlation unchanged, to fine tune the tranche spreads to fit better to market quotes. However, α and β have limited capability to further the results.

This feature gives the NIG copula an advantage over some other copulas, such as the Clayton copula. The Clayton copula has only one parameter controlling both correlation

[6] This is not always true with all copula functions. For example, the Student-t copula when $\rho = 0$ still has some tail dependency.

and tail dependency. Therefore, the Clayton copula is structurally not as flexible compared to the NIG copula. However, we should see that for the NIG copula, the variability of model spreads through varying its tail dependency parameters is very limited. Thus, it is very likely that the better fit for the NIG copula comes from the functional form of the copula structure rather than the flexibility of the tail dependency parameters. Also from the calibration results of some other papers such as Burtschell et al. (2008), as the Clayton copula does not fit to market data well. Therefore, it seems that even though the Clayton copula has asymptotic tail dependency, its particular functional form is not suitable for the dependency modelling in CDOs. Since the tail dependency parameters only have limited variability, even though all GH-based copulas have tail dependency, we have to test each individual for different applications.

To gain further understanding of model dependency, we have to know how model parameters affect the tail dependency in a portfolio. In the context of CDO pricing, the dependency in a portfolio is fully reflected in its loss distribution $p(\ell, t)$, which is a direct observer for dependency structure in a CDO. For example, a model with tail dependency means that the loss distribution $p(\ell, t)$ has a fatter tail.

We know that the loss distribution $p(\ell, t)$ is the weighted average of conditional loss distribution $p(\ell, t \mid M)$

$$p(\ell, t) = \mathbb{E}[p(\ell, t \mid M)g(M)]$$

The conditional loss distribution $p(\ell, t \mid M)$ is a binomial distribution which is close to standard Normal distribution when N is large. Therefore, it is close to being model independent. Hence, the loss distribution is the weighted sum of these binomial distributions. Therefore, dependency structure is almost completely determined by the weights $g(M)$ - the density function of M. Therefore the tail dependency parameters α and β in the common factor M also determine the tail dependency of the loss distribution.

Having understood this, if we replace the NIG distribution of the idiosyncratic factor with other distributions, such as the Gaussian distribution, the new model can still achieve results that fit well.

2.6.3. Significance for risk management. The sensitivities analysis indicates that the model risk can be substantial in copula models.

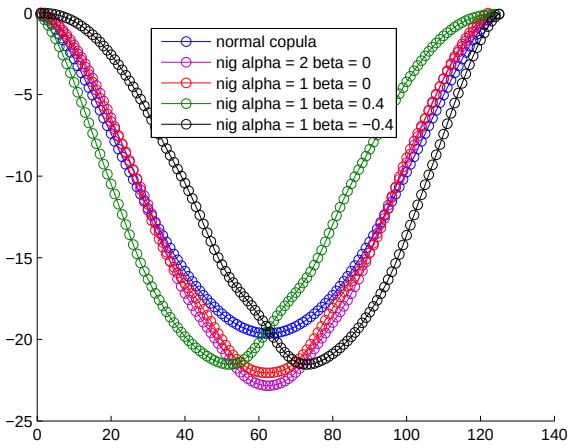
For example, in hedging against individual obligor spread change, the delta for Gaussian and NIG copulas as shown in figure 2.3.7 gives us very different hedging requirements.

In hedging against market correlation the strategy of hedging a long equity tranche with a short mezzanine tranche can be especially dangerous since mezzanine tranche sensitivities can go in opposite directions with different models. In general, the NIG copula should gives us a better protection than Gaussian copula since NIG model has a better fit than the Gaussian model.

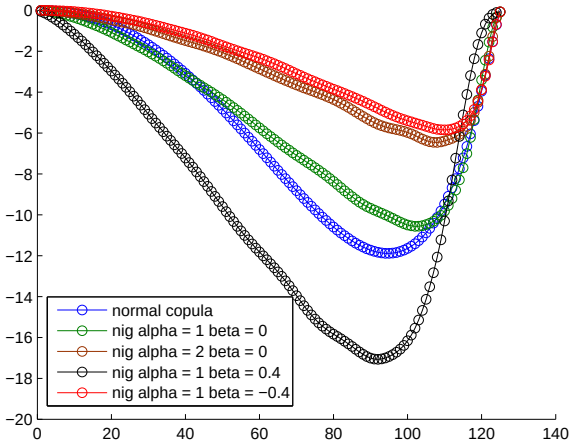
2.7. Conclusion

In this chapter we constructed parameters sensitivities analysis on the Gaussian and NIG copula models and designed a calibration algorithm based on the results. Our analysis shows that the NIG copula is structurally more flexible than the Gaussian copula. However, its tail dependency parameters α and β have limited capability in influencing tranche spreads. The results show that the two copulas can give quite different hedging quantities and this difference can be disastrous in practice. It is evidenced in the aforementioned Ford downgrade-triggered credit-market crash, where some hedge funds lost big money when the market correlation change caused the long and short positions in equity and mezzanine tranches to go in different directions.

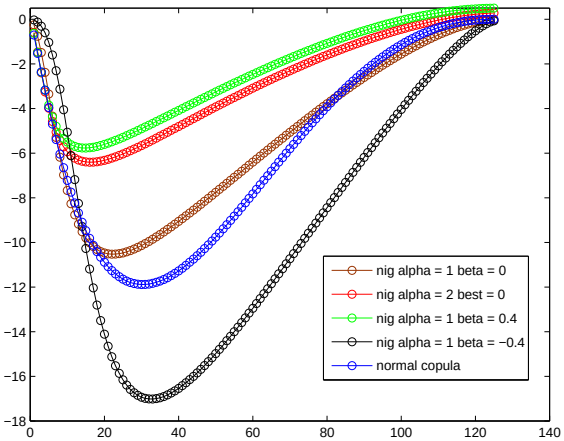
Inspired by the NIG copula structure, we want to construct a model with similar structural flexibility but much more parameter variability so that the resulting model can be calibrated to a variety of market scenarios. A non-parametric model is one such choice.



(a) $\rho = 0.5, c = 0$

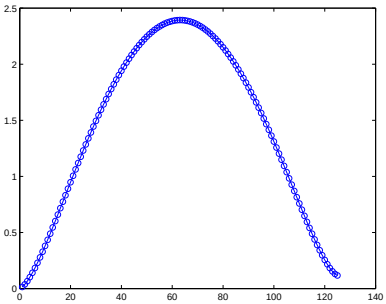


(b) $\rho = 0.5, c = 1$

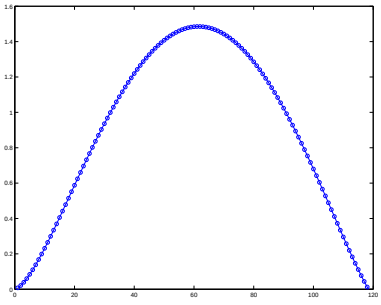


(c) $\rho = 0.5, c = -1$

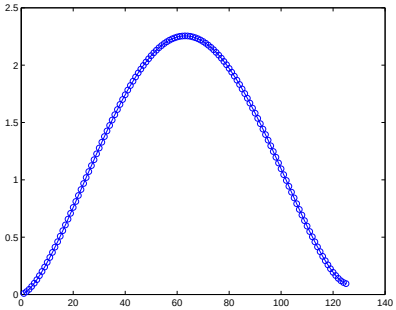
FIGURE 2.3.2. Equity tranche sensitivities to correlation for NIG copula. The same as the Gaussian copula, for the NIG copula the delta is also negative since this property is model independent.



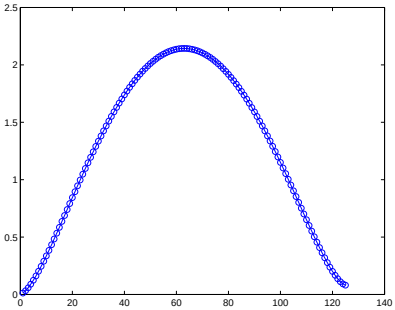
(a) $\alpha = 1, \beta = 0, \rho = 0.5, c = 0$



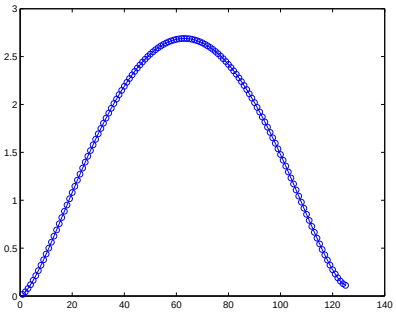
(b) $\alpha = 2, \beta = 0, \rho = 0.5, c = 0$



(c) $\alpha = 1, \beta = 0, \rho = 0.4, c = 0$

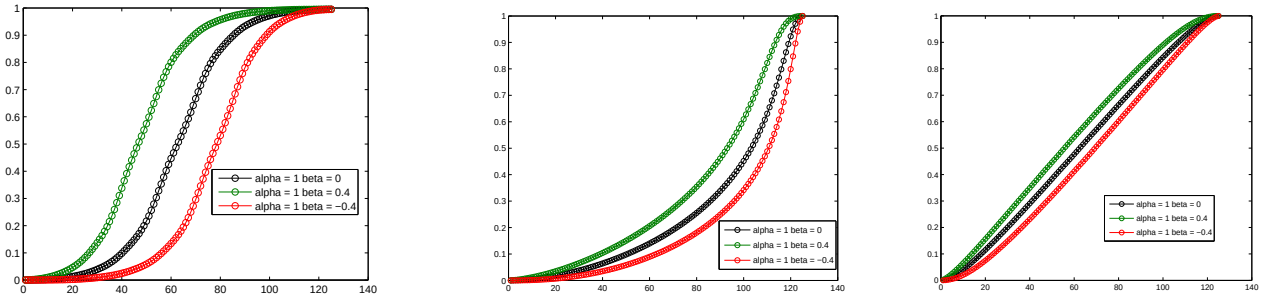


(d) $\alpha = 1.2, \beta = 0, \rho = 0.5, c = 0$



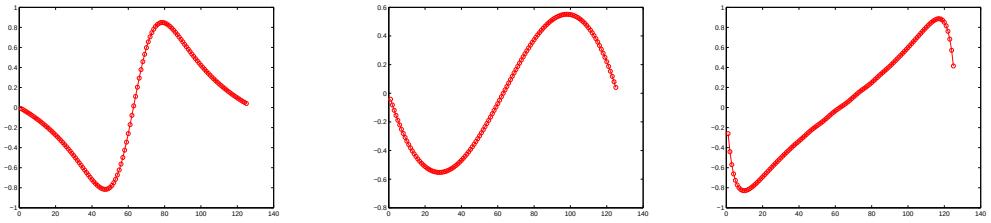
(e) $\alpha = 0.8, \beta = 0, \rho = 0.5, c = 0$

FIGURE 2.3.3. Equity tranche sensitivities to α for NIG copula. We can see that the curve values don't change much for different α.

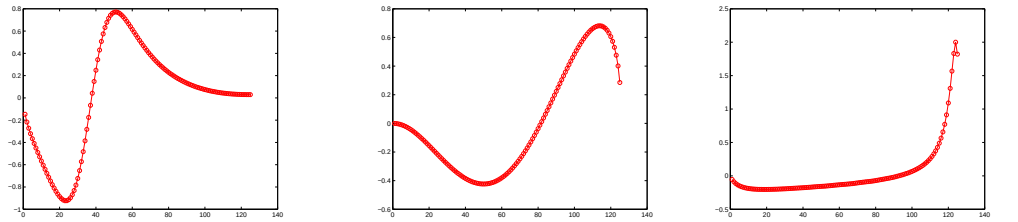


(a) $\rho = 0.2, c = 0$ (b) $\rho = 0.5, c = 1$ (c) $\rho = 0.8, c = 0$

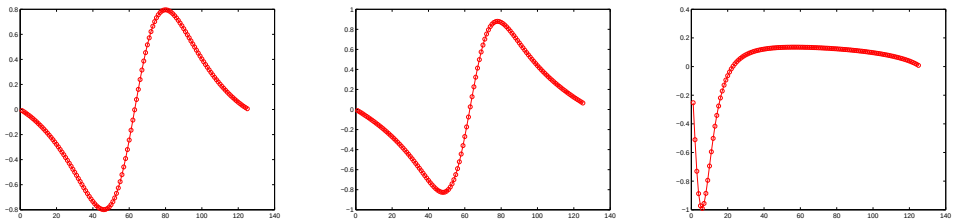
FIGURE 2.3.4. The delta of equity tranches for the NIG copula given a change in the index spread. The three panels show different curvatures for different parameter values.



(a) $\alpha = 1, \beta = 0, \rho = 0.2, c = 0$ (b) $\alpha = 1, \beta = 0, \rho = 0.5, c = 0$ (c) $\alpha = 1, \beta = 0, \rho = 0.8, c = 0$

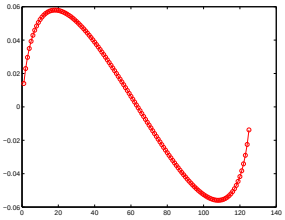


(d) $\alpha = 1, \beta = 0.6, \rho = 0.2, c = 0$ (e) $\alpha = 1, \beta = -0.6, \rho = 0.5, c = 0$ (f) $\alpha = 1, \beta = 0, \rho = 0.8, c = 1$

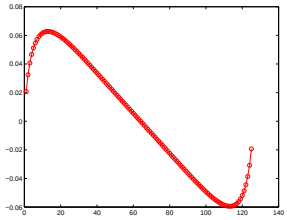


(g) $\alpha = 2, \beta = 0, \rho = 0.2, c = 0$ (h) $\alpha = 0.6, \beta = 0, \rho = 0.2, c = 0$ (i) $\alpha = 1, \beta = 0, \rho = 0.5, c = -1$

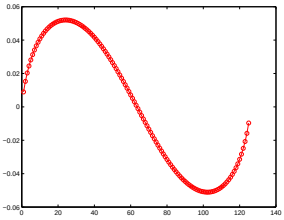
FIGURE 2.3.5. Sensitivities to correlation for tranchelets. We can see that different β gives different delta neutral tranchelets.



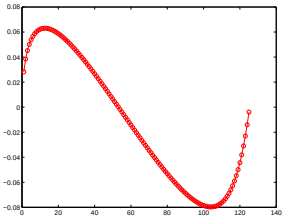
(a) $\alpha = 1, \beta = 0, \rho = 0.5, c = 0$



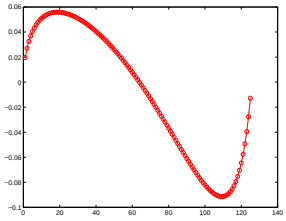
(b) $\alpha = 1, \beta = 0, \rho = 0.6, c = 0$



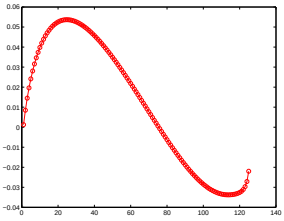
(c) $\alpha = 1, \beta = 0, \rho = 0.4, c = 0$



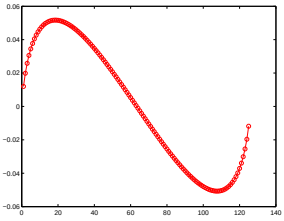
(d) $\alpha = 1, \beta = 0.1, \rho = 0.5, c = 0$



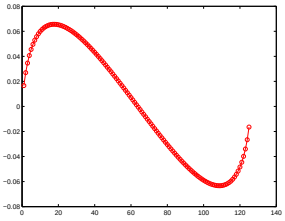
(e) $\alpha = 1, \beta = 0, \rho = 0.5, c = 0.1$



(f) $\alpha = 1, \beta = -0.1, \rho = 0.5, c = 0$

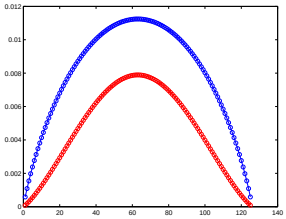


(g) $\alpha = 1.2, \beta = 0, \rho = 0.5, c = 0$

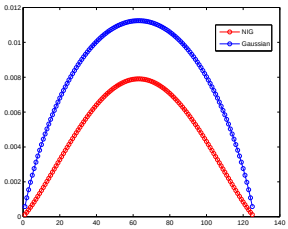


(h) $\alpha = 0.8, \beta = 0, \rho = 0.5, c = 0$

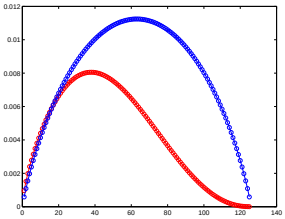
FIGURE 2.3.6. Sensitivities to α for tranchelets for the NIG copula. Similar to the equity tranche sensitivities to α, these curve values don't vary much for different values of α.



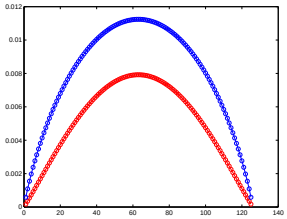
(a) $\alpha = 0.5, \beta = 0\, \rho = 0.5, c = 0$



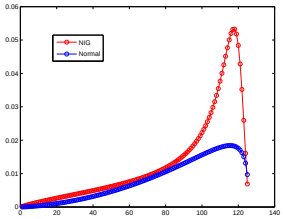
(b) $\alpha = 1, \beta = 0, \rho = 0.5, c = 0$



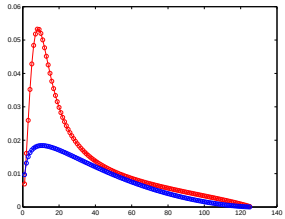
(c) $\alpha = 1, \beta = 0.6, \rho = 0.5, c = 0$



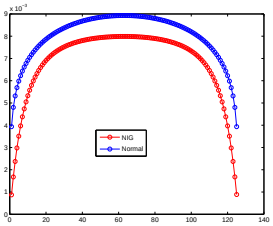
(d) $\alpha = 2, \beta = 0, \rho = 0.5, c = 0$



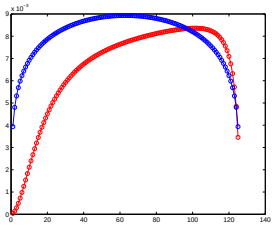
(e) $\alpha = 1, \beta = 0, \rho = 0.5, c = 1$



(f) $\alpha = 1, \beta = 0, \rho = 0.5, c = -1$



(g) $\alpha = 1, \beta = 0, \rho = 0.8, c = 0$



(h) $\alpha = 1, \beta = -0.6, \rho = 0.8, c = 0$

FIGURE 2.3.7. Deltas of the Gaussian and NIG copulas with regard to index spreads change for tranchelets

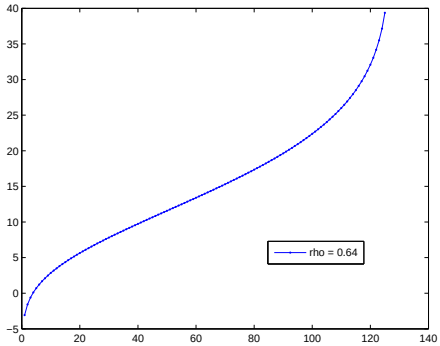


FIGURE 2.4.1. Spread sensitivities to the correlation parameter variation for the Gaussian copula

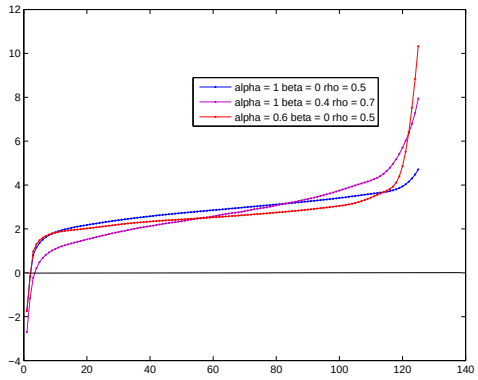


FIGURE 2.4.2. Spread sensitivities to correlation parameter for NIG copula

CHAPTER 3

Semi-parametric lattice models - a static binomial model

The tranche expected loss and tranche spreads sensitivities analysis with regard to model parameters help us gain a significant understanding of default correlation. Combined with model parametrisation analysis, it shows that the NIG factor model is structurally superior to the Gaussian factor model but has limited capability in terms of parameter flexibility to fit a wide range of market quotes and completely remove the correlation skew.

Therefore, together with similar research findings from the literature, we conclude that it is difficult, if not impossible, to provide any parametric form flexible enough to accommodate the diverse dependency structures that the market reality may imply. Hence, in the next chapters of the thesis we adopt a semi-parametric approach to modelling and develop flexible arbitrage-free models in a binomial lattice. The main result of this thesis is the design of three binomial lattice models with increasing sophistication, aiming to satisfy three no-arbitrage conditions in succession. The first, static, version of the model endeavours to fit the correlation skew in the capital-structure dimension. Next we develop the model into a dynamic model, which is automatically arbitrage free in the time dimension. However, the dynamic model loses some of the flexibility and the computational efficiency is compromised, as the conditional survival probabilities are path dependent. The third model is a Markovian model that improves pricing speed and calibration accuracy while maintaining an absence of arbitrage in both dimensions. Also, when we incorporate a stochastic recovery rate in the Markovian model, we will demonstrate that there is a third no-arbitrage condition to be observed.

3.1. The features of the three models

For all three models, the dynamics of the common market factor follow a recombining binomial lattice. We don't specify the dynamics of idiosyncratic factors, as they will be integrated out in pricing. This approach reduces the complexity of the model by dropping the dynamics of the credit spreads, which is unimportant for most applications.

Our semi-parametric method for designing the lattice is conducted by specifying transition probabilities and changing the size of each step in the lattice as model parameters. The transition probabilities are state and time dependent, and can be calibrated to market tranche quotes. The parameters that control the size of the change of the common factor control the variances of the common factor distribution, as we discussed in the last chapter, and thus control the default correlation between different obligors. Since the distribution of the common factor reflects the market-dependency structure, a flexible model should, if perfectly calibrated, imply the exact distribution of the market dynamics.

Further, all three models have parameters that control an individual obligor's default intensity level, which responds to changes in the market factor, and these parameters can be calibrated to marginal default probabilities bootstrapped from CDS spreads.

3.1.1. The choice of calibration methods. Our chosen calibration method - the Cross Entropy (CE) method - is a novel application of this method in finance. Since we have a multi-dimensional non-linear optimisation problem, subject to constraints (e.g. the transition probabilities must reside between $(0, 1)$) and with an objective function that is likely to have multiple local optimal values, some traditional optimisation algorithms fail completely.

The CE method has previously been employed in finance in Edelman (2004), who uses the local CE method to derive an *optimality criterion* for fitting a risk-neutral distribution to observed option prices. In our case the optimality criterion is given solely in terms of fitting *the parameters of the random distribution of model parameters* to market prices, and we are using the CE method as a global optimisation method, as described in Rubinstein & Kroese (2004)(not specific to finance). We also experimented with the Genetic Algorithm (GA). However, the GA does not converge within a reasonable timeframe. This indicates that when dealing with a large-scale non-linear optimisation problem of the type encountered in our context, the CE method is superior.

To further improve calibration efficiency, we also experiment using the CE *global* optimisation method combined with the Nelder Mead (NM) *local* optimisation method to construct a hybrid algorithm and achieve interesting results.

3.2. A static binomial model

The first binomial model is a static model in the sense that the dynamics of the default process are not modelled explicitly. By specifying the factor distribution non-parametrically in different steps of a binomial lattice, we intend to fit the correlation skew exactly and therefore satisfy the first no-arbitrage condition 1.1.1. Non-parametric CDO pricing models can be found in Hull & White (2006) where the author tries to imply the copula structure from market data in the form of hazard rates. Here our model implies the market correlation in the form of market factor distribution.

The condition 1.1.1 is automatically satisfied if the tranches in the same underlying portfolio can be calibrated with one set of model parameters.

3.3. Model setup

We make the following assumptions for the basic versions of all three models.

1). All models are constructed under the risk-neutral measure.

2). All notionals are assumed to be 1 and equal recovery rates of 40% are assumed for all names in the underlying portfolio.

3). Default and premium payments are assumed to occur only at the discrete time points t_n in arrears.

Let the remaining coupon payment dates in a CDO structure be $t_1, t_2, \cdots, t_N$ and $M_{j,k}, 1 \leq j \leq k$ be the value of a node on a recombining binomial lattice in state j and step k.

The survival probability of an obligor i by a payment time $t_n, 1 \leq n \leq N$ conditional on a common factor $M_{j,k}$ takes the following form.

$$P\{\tau_i > t_n \mid M_{j,k}\} = e^{-\lambda_i(t_n)M_{j,k}}$$

where τ_i is the default time of obligor i, and $\lambda_i(t_n)$ determines the sensitivity with which each obligor responds to economic conditions.

The distribution of $M_{j,k}, 1 \leq j \leq k$, is the distribution of the k-th-step on a K-step recombining binomial lattice with $1 \leq k \leq K$. Instead of using one step (final step) in a binomial lattice to compute loss distributions for all the payment dates, we use each step

in the lattice for one or more payment dates. This increases the flexibility of the model without increasing the number of calibration parameters [1].

The number of steps needed on the lattice is generally fewer than the number of payment dates of a CDO to achieve an accurate calibration. Thus we have to allocate the payment dates to the steps on the lattice.

The payment dates are allocated as follows.

First we pick a payment date as the *key* payment date that corresponds to a step on the lattice. This is generally done to ensure that there is an equal number of payment dates between these key payment dates. The payment date for step k is denoted by T_k.

Then the payment dates between these key payment dates are allocated in this way. For any payment time t_n which satisfies $T_{k-1} < t_n \leq T_k$, they belong to step k. Hence, we use the function form $k = \eta(t_n), 1$ for $T_{k-1} < t_n \leq T_k$ to obtain the step number k given a payment date t_n.

Figure 3.3.1 gives a graphical example.

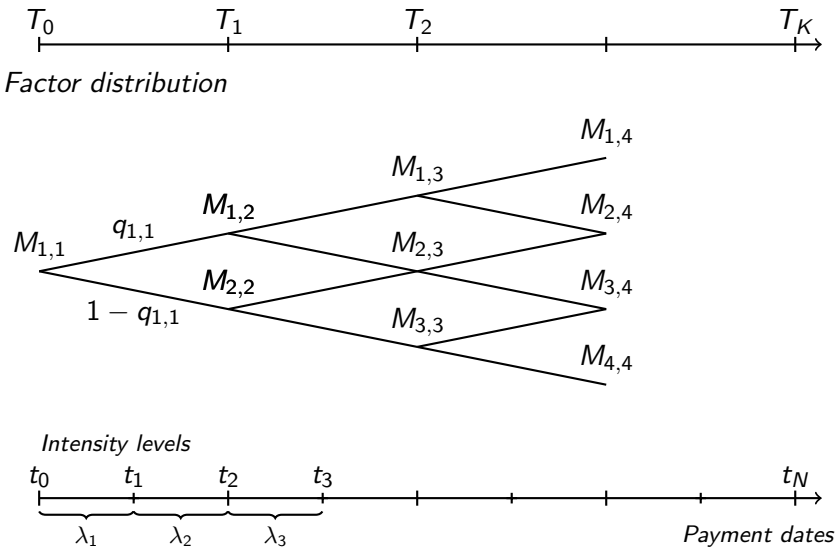


FIGURE 3.3.1. The common factor distribution on the lattice

The $\lambda_i(t_n)$s can be calibrated to marginal risk-neutral default probabilities using the following equation.

[1] As will be explained in model calibration, the calibration parameters are determined by the number of transition probabilities on the lattice; a K step lattice requires $\frac{K(K-1)}{2}$ number of parameters regardless of whether we only use the distribution of the last step or the distributions of all the steps.

$$P\{\tau_i > t_n\} = \sum_{j=1}^{\eta(t_n)} e^{-\lambda_i(t_n)M_{j,\eta(t_n)}} Q(M_{j,\eta(t_n)})$$

where $Q(M_{j,\eta(t_n)})$ is the probability of factor $M_{j,\eta(t_n)}$ at step $\eta(t_n)$ and state j.

To simplify notation, let $Q_{j,\eta(t_n)} = Q(M_{j,\eta(t_n)})$.

From section 1.1.1.5, to compute model spreads we need to build up the loss distribution $p(\ell, t_n)$, which can be written in terms of the loss probabilities *conditional* on the nodes of the lattice as

$$p(\ell, t_n) = \sum_{k=0}^{\eta(t_n)} p(\ell, t_n | M_{j,\eta(t_n)}) Q_{j,\eta(t_n)} \tag{3.3.1}$$

These conditional loss distributions $p(\ell, t_n|M_{j,\eta(t_n)})$ can be built up by the recursive method once the conditional default probabilities $e^{-\lambda_i(t_n)M_{j,\eta(t_n)}}$ are calibrated from marginal default probabilities.

3.4. Algorithmic construction of the model

From the above setup, $M_{j,k}$ denotes the value of the common factor for payment at future T_k in state j for $1 \leq j \leq k$. Let $M_{1,1} = 1$. Assume that $M_{j,k}$ moves up to $M_{j,k+1} = (1 + a)M_{j,k}$ with probability $q_{j,k}$ and down to $M_{j+1,k+1} = (1-a)M_{j,k}$ with probability $1-q_{j,k}$, for some constant a. The distribution of $M_{j,k}$ on each node, $Q_{j,k}$, can be constructed by specifying the transition probabilities $q_{j,k}$ and $M_{j,k}$ can be constructed once a with $0 < a < 1$ is given.

Note that while the transition probabilities can be time- and state-dependent, the proportional sizes of the 'up' and 'down' moves are constant through time. The latter ensures that M evolves according to a recombining lattice.

Therefore the variable a and the transition probabilities $q_{j,k}$ on the lattice form the parameter inputs of the model, which can be calibrated to CDS index tranches.

In summary, we use the following induction algorithm to compute the value of M on each node and their probability distributions.

3.4.0.1. *Initialisation.* We have

$$\begin{cases} Q_{1,1} = 1 \\ M_{1,1} = 1 \end{cases}$$

3.4.0.2. *Forward induction for the factor distribution.* The probability distribution for $M_{j,k}$ is given by

$$Q_{j,k} = \begin{cases} (1 - q_{1,k-1})Q_{1,k-1} & j = 1 \\ q_{j-1,k-1}Q_{j-1,k-1} + (1 - q_{j,k-1})Q_{j,k-1} & 1 < j < k \\ q_{j-1,k-1}Q_{j-1,k-1} & j = k \end{cases}$$

3.4.0.3. *Forward induction for the factor values.*

$$M_{j,k} = \begin{cases} (1 + a)M_{j,k-1} & 1 \leq j < k \\ (1 - a)M_{j-1,k-1} & j = k \end{cases}$$

3.4.1. Calibration to marginal default probabilities. At each step in the induction, the survival probability conditional on factor M by a payment date t_n, $S_i(t_n|M_{j,\eta(t_n)})$, can be calculated using the following formula if $\lambda_i(t_n)$ is known.

$$S_i(t_n|M_{j,\eta(t_n)}) = P\{\tau_i > t_n \mid M_{j,\eta(t_n)}\} = \exp\left\{-\lambda_i(t_n)M_{j,\eta(t_n)}\right\}$$

The marginal survival probabilities are the weighted average of conditional survival probabilities given by

$$\begin{aligned} S_i(t_n) &= \sum_{j=1}^{\eta(t_n)} S_i(t_n|M_{j,\eta(t_n)})Q_{j,\eta(t_n)} \\ &= \sum_{j=1}^{\eta(t_n)} \exp\{-\lambda_i(t_n)M_{j,\eta(t_n)}\}Q_{j,\eta(t_n)} \end{aligned} \tag{3.4.1}$$

$\lambda_i(t_n)$ can be solved, since $Q_{j,\eta(t_n)}$, $M_{j,\eta(t_n)}$ and $S_i(t_n)$ are known.

Notice the right-hand side of the equation 3.4.1 is a continuous monotonically decreasing function of variable $\lambda_i(t_n)$, which has its first derivative.

The derivative of $S_i(t_n)$ with regarding to $\lambda_i(t_n)$ is given by

$$\frac{\partial S_i(t_n)}{\lambda_i(t_n)} = -\sum_{j=1}^{\eta(t_n)} M_{j,\eta(t_n)} \exp\{-\lambda_i(t_n) M_{j,\eta(t_n)}\} Q_{j,\eta(t_n)} \tag{3.4.2}$$

Therefore, we can use Newton's method to find the solution for equation 3.4.1. Let the initial value for $\lambda(t_n)$ start from 0. When $\lambda_i(t_n) = 0$, the right-hand side of equation 3.4.1 is 1. The survival probability on the left-hand side of equation 3.4.1 is always between 0 and 1. If we increase $\lambda(t_n)$ from 0 to ∞, the right-hand side of equation 3.4.1 will eventually go to zero. Therefore, a positive $\lambda(t_n)$ satisfying the equation 3.4.1 always exists.

Once the $\lambda_i(t_n)$ which satisfies equation 3.4.1 is found, the conditional survival probability $S_i(t_n|M_{j,\eta(t_n)})$ can be calculated. Then the recursive method algorithm 2.4.4 can be used to derive the conditional loss distribution $p(\ell, t_n \mid M_{j,\eta(t_n)})$ with the conditional default probability for the i-th name by time t_n as $P_i(t_n \mid M_{j,\eta(t_n)}) = 1 - S_i(t_n \mid M_{j,\eta(t_n)})$.

3.5. Model calibration with CE method

The binomial model has the flexibility to calibrate to market CDO index tranches as closely as possible. However, the non-linear nature of the problem and the large number of calibration parameters involved requires an algorithm that can cope with large-scale continuous multi-extremal optimisation. Stochastic optimisation methods like genetic algorithm (GA) and simulated annealing (SA) have traditionally been employed to handle these challenges. The CE algorithm is relatively new to the finance literature. However, it is especially powerful in combinatorial and continuous multi-extremal optimisation. With a calibration algorithm like CE, some mathematical models that are previously difficult to calibrate now become plausible for applications.

The first objective function we try to minimise in the model calibration is given by

$$\text{Objective function one} = \sqrt{\sum_{m=1}^{nTr} \frac{(\text{Market quote}_m - \text{Model spread}_m)^2}{\text{Bid/ask spread}_m}} \tag{3.5.1}$$

where nTr is the number of tranches and m is the tranche number.

We use bid/ask spreads as weights for spread differences so that different tranche spreads can converge to within the bid/ask spreads at similar speed.

We also use a 'fully adaptive' CE algorithm, which in broad terms proceeds as follows[2]:

(1) Randomise the deterministic optimisation problem by defining a *probability distribution* for all model parameters, which include transitional probabilities $q_{j,k}$ and the relative change a for the common factor $M_{j,k}$. Denote by $\mathbf{v}$ a vector of parameters of the probability distribution of the model parameters.
(2) Choose an initial parameter vector (starting point) $\mathbf{v}_0$. Set the iteration counter $t = 1$.
(3) Generate a sample of N sets of model variables using the density $f(\cdot, \mathbf{v}_{t-1})$.
(4) Calculate the value of the objective function for each of these sets of model variables. Denote by $\hat{\gamma}_t$ the ρ–quantile of these values (for an exogenously chosen level ρ).
(5) Find the parameter vector $\tilde{\mathbf{v}}_t$ which generates distributions for the chosen parameter vectors $\hat{\gamma}_t$ with maximum likelihood.
(6) 'Smooth out' $\tilde{\mathbf{v}}_t$ by weighted averaging with $\mathbf{v}_{t-1}$ to give $\mathbf{v}_t$.
(7) If $\hat{\gamma}$ did not change in the last d steps, say $d = 5$, then stop, otherwise increment the iteration counter and loop back to step (3).

Some further explanations of the algorithm.

In step (1), we choose these variables to follow the beta distribution. The variable a, which is the relative change for M, does not necessarily have to fall between $(0, 1)$ for the model to calibrate. However, we find that the range $(0, 1)$ is enough to generate sufficient correlation to fit all the data we have tested so far.

In step (2), for a new calibration the initial value for beta distribution we choose is $(\alpha = 1, \beta = 1)$, which is equivalent to a uniform distribution with support $[0, 1]$. However, calibration can start with any values as input depending, on the context.

In step (4), the choosing of ρ as the quantile of 'good' parameters selected for the next iteration is more of an art than a science. In our application, we find 10% is a reasonable

[2]For a detailed discussion of the algorithm, see Rubinstein & Kroese (2004).

choice. Factors that need to be taken into consideration when determining ρ are the number of model parameters, the speed of convergence or calibration accuracy[3] and the nature of the problem[4].

In step (5), there are several ways of fitting a distribution to the selected samples. The way we adopt in our model is first to calculate the mean and the variance of the selected sample $\hat{\gamma}_t$. For the beta distribution, they are given by the following formula,

$$\text{mean} = \mu = \frac{\alpha}{\alpha + \beta}$$

and

$$\text{variance} = \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Once we obtain the mean and variance of the 'good parameters', the new distribution parameters are given by,

$$\alpha = \frac{\mu^2(1-\mu)}{\sigma^2}$$

and

$$\beta = (1-\mu)(\frac{\mu(1-\mu)}{\sigma^2} - 1)$$

The reason for doing it this way is in the next step of the calibration. We found that it is more likely to achieve quick convergence by smoothing out mean and variance rather than by smoothing out the distribution parameters.

In step (6), the choice of the weights to smooth out 'old' parameters with 'new' parameters is to avoid premature convergence to a local minimum rather than global minimum. In our case, we take the weighted average of the 'old' and 'new' means and variances of the beta distribution in order to determine the distribution parameters for the next iterations, where 80% of the weight is given to the old means and variances and 20% to the new ones.

We test the model on iTraxx IG data for a series of 30 days from 21 March 2005. Figure 3.5.1 to figure 3.5.5 show the calibration results. All model spreads are fit within bid

[3]The convergence speed and calibration accuracy are trade-offs between each other.

[4]We find that the more 'non-linear' the dependency structure implied in the market data, the longer it takes for the calibration; thus, ρ needs to be smaller for more iterations

and ask spreads and CDS calibrations are accurate to the second significant figure of the marginal default probabilities.

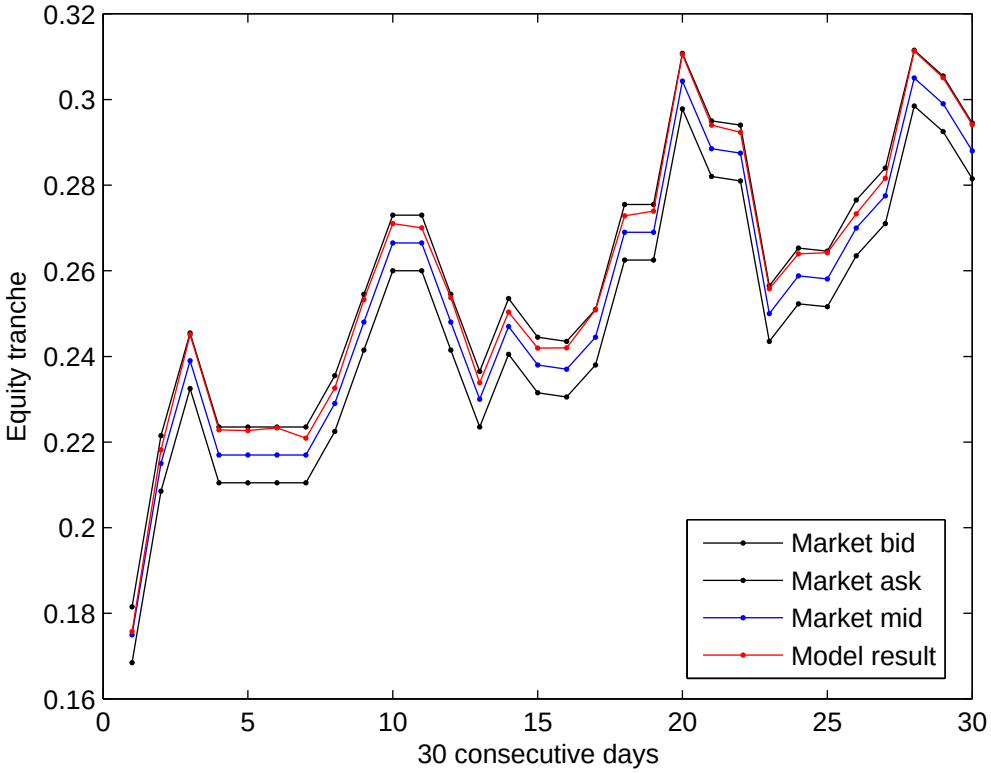


FIGURE 3.5.1. Calibration results of one-month serial iTraxx IG data for the equity tranche. The calibration results for most of the days indicate that the mid market quotes are generally over-estimated but still within the bid and ask spreads.

3.5.1. Combined CE and Nelder Mead algorithm. The CE method is suitable for global optimisation problems where the search space [5] includes a number of local optimal solutions. However, if the search space has only one optimal solution, which is the global solution itself, the CE method can be inefficient since it has to search for the whole parameter space. It can be complemented by an optimisation method that can "go straight to the point".

[5]The definition of a global solution here is relative depending on the calibration accuracy required.

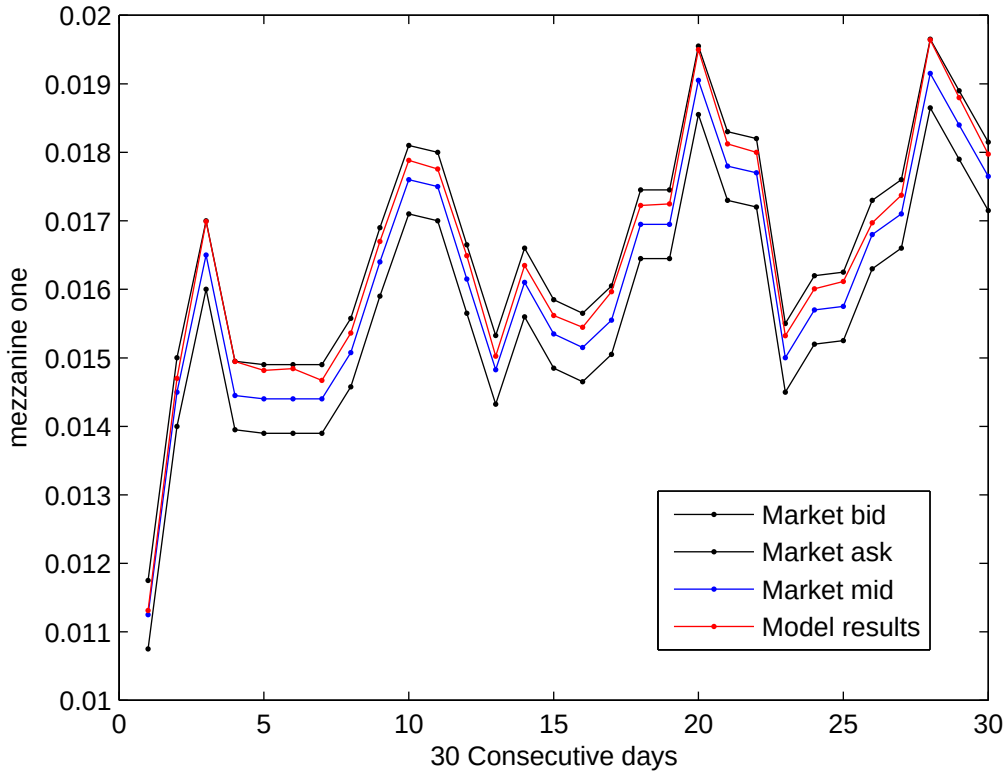


FIGURE 3.5.2. Calibration results of one-month serial iTraxx IG data for the first mezzanine tranche. The calibration results for most of the days indicate that the mid market quotes are generally over-estimated but still within the bid and ask spreads.

In this section, we combine the CE method with a local search stochastic optimisation algorithm, the NM method (Nelder & Mead 1965)[6], to see if a combined global-local optimisation algorithm can speed up calibration.

For the new experiment we design the second objective function 3.5.2 to minimise as,

$$\text{Objective function two} = \sqrt{\sum_{m=1}^{nTr} \max\left(\frac{|\text{Market quote}_m - \text{Model spread}_m|}{4 \times \text{Bid/ask spread}_m} - 2^{1-\text{stage}}, 0\right)^2} \tag{3.5.2}$$

where $stage$ indicates the different levels of threshold towards the mid-market quotes. The function is designed such that once the model spreads fall below the desired threshold,

[6]The Nelder Mead algorithm or downhill simplex method is a traditional optimization method and there are plenty of resources on its usage. For inexperienced reader, a good starting point is the Wikipedia site.

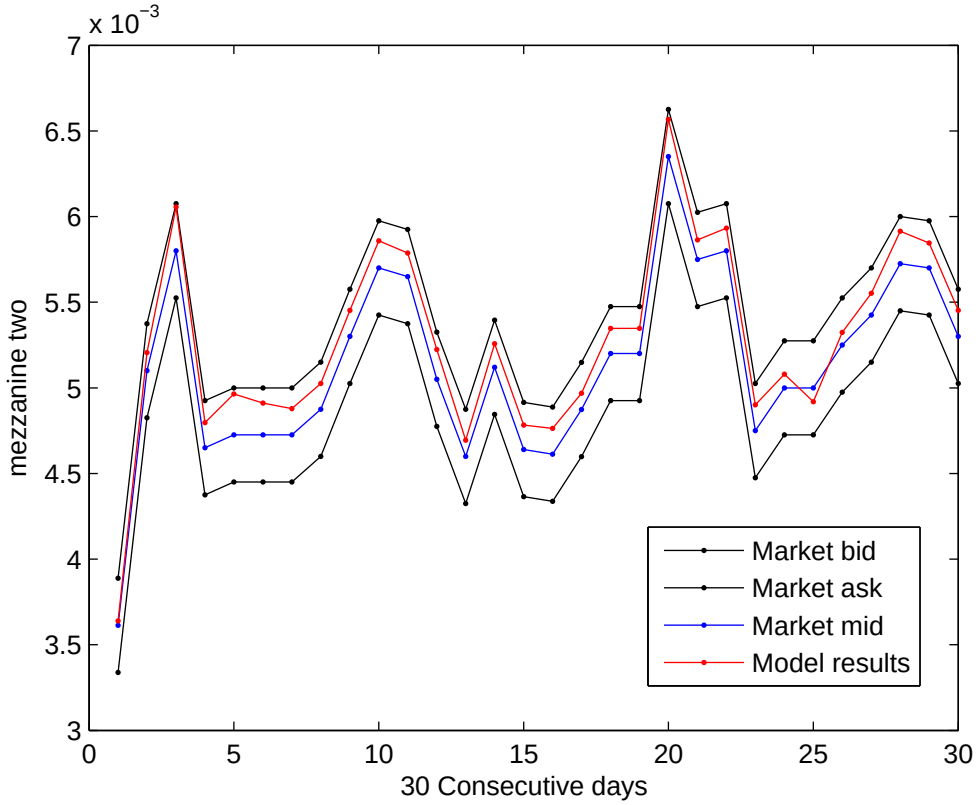


FIGURE 3.5.3. Calibration results of one-month serial iTraxx IG data for the second mezzanine tranche. The calibration results for most of the days indicate that the mid market quotes are generally over-estimated but still within the bid and ask spreads.

objective function 3.5.2 becomes zero and then we record the solutions and the number of iterations it takes to pass that threshold. It starts with 4 times the bid/ask spreads around the mid-market quotes, which we found is a reasonable starting point.

Table 3.5.1 shows the calibration results of five-year iTraxx IG data on 25 March 2005. The results are intermediate results using the CE method only when the algorithm passes different stages of the thresholds. The errors are evaluations from the first objective function 3.5.1.

Then, to initiate the NM algorithm, we take each intermediate best solution when the CE algorithm passes each stage, as shown in table 3.5.1, as input parameters to the NM algorithm, and record how many more stages the NM algorithm can pass through. Table 3.5.2 shows the test results of using the combined algorithm.

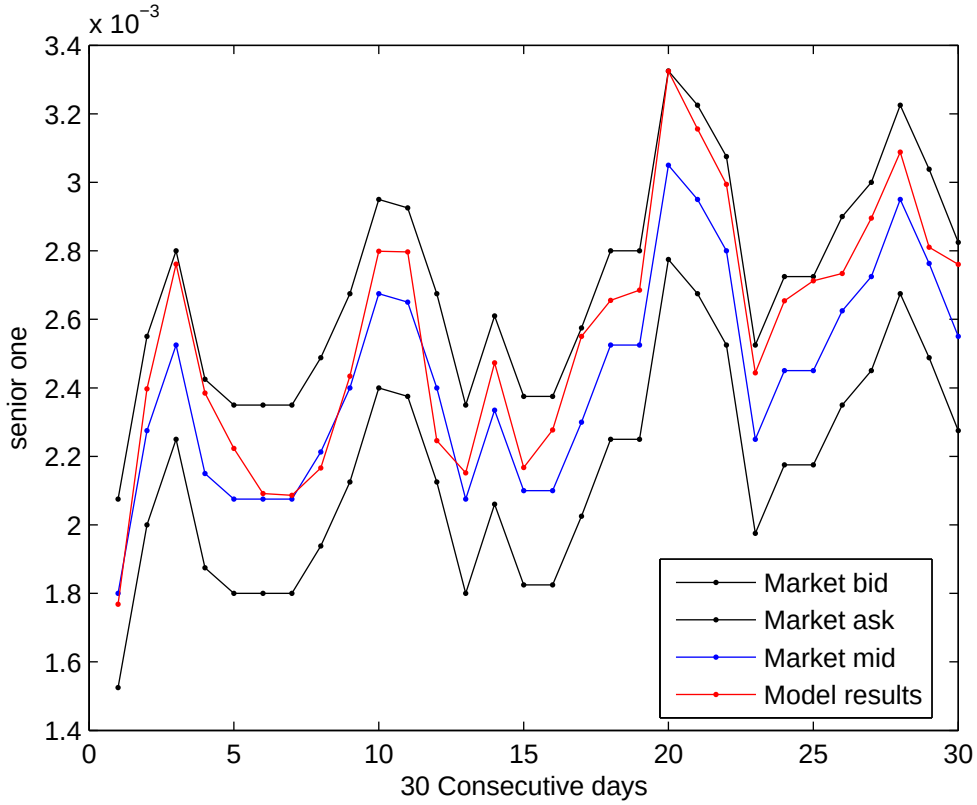


FIGURE 3.5.4. Calibration results of one-month serial iTraxx IG data for the first senior tranche. The calibration results for most of the days indicate that the mid market quotes are generally over-estimated but still within the bid and ask spreads.

Tranche	Mid	Bid	Ask	Stage 1	Stage 2	Stage 3	Stage4	Stage 5	Stage 6
(%)				Iteration 2	10	40	46	195	254
0–3	17.5(%)	16.85	18.15	19.11	18.64	17.93	17.43	17.6	17.51
3–6	112.5(bp)	107.5	117.5	108.7	106.9	116.9	112.6	112.7	112.5
6–9	36.13(bp)	33.38	38.88	36.09	38.06	35.5	36.99	36.57	36.17
9–12	18(bp)	15.25	20.75	21.64	23.02	17.86	17.26	17.91	18.04
12–22	10(bp)	8.25	11.75	10.45	13.1	10.23	10.70	10.15	10.2
Error				0.7311	0.8398	0.2859	0.1473	0.0627	0.0301

TABLE 3.5.1. Calibration results for iTraxx IG tranches maturing in five years on 21 March 2005. Errors are evaluated from the first objective function 3.5.1

The second column is the number of tranche pricing evaluations it takes for the CE method to pass a certain stage. The rest of the columns are stages that the NM algorithm can pass through, from a particular stage of the CE algorithm. Note that in the example each iteration involves 100 tranche pricing evaluations.

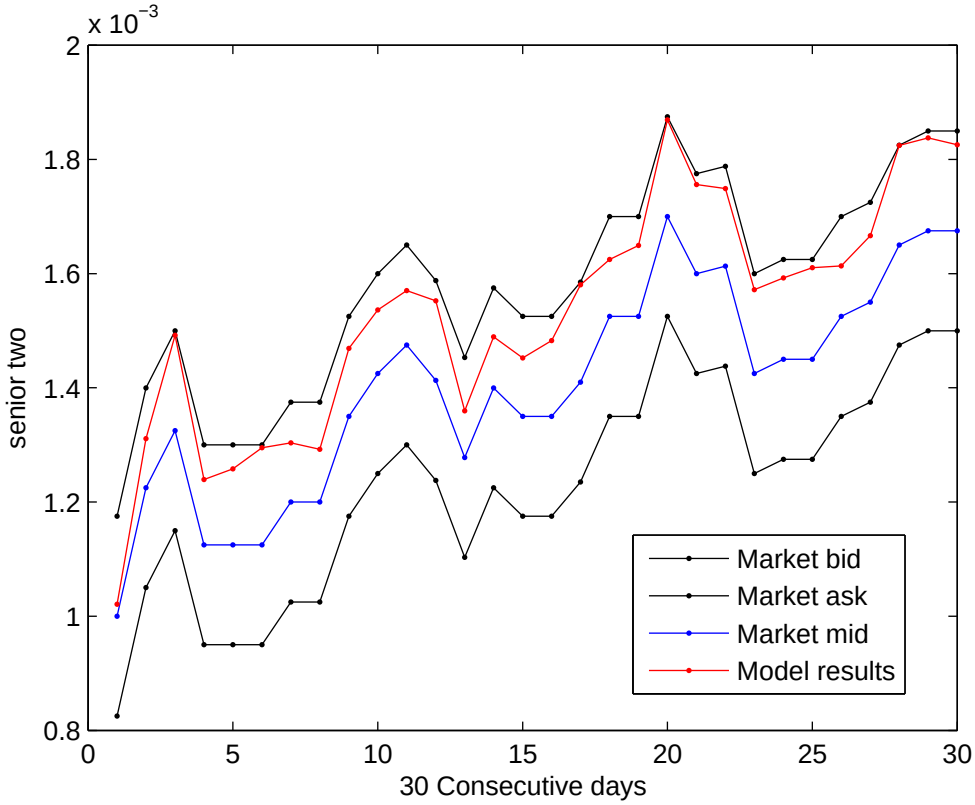


FIGURE 3.5.5. Calibration results of one-month serial iTraxx IG data for supersenior tranche. The calibration results for most of the days indicate that the mid market quotes are generally over-estimated but still within the bid and ask spreads.

From table 3.5.2 we can tell that if both CE and NM algorithms can achieve the desired accuracy by passing a certain threshold, generally the NM algorithm is faster. For example, to reach stage 6 (within one-eighth of the bid/ask spread) the CE algorithm alone requires 25400 tranche pricing evaluations, whereas if we stop the CE algorithm after stage 3 and pass the intermediate parameters on to the NM algorithm, it takes only $4000 + 2045 = 6045$ tranche pricing evaluations.

However, the NM algorithm alone does not always converge to the desired level of accuracy. Starting at stage 1 from the CE algorithm, the NM algorithm only takes it to stage 3. Generally, the closer to global optimum it is, the better the NM algorithm performs, but it is not guaranteed.

	CE	NM	stage					
Stage	# of evaluations	1	2	3	4	5	6	7	8
1	200	1	108	421					
2	1000		1	2141					
3	4000			1	216	1053	2045		
4	4600				1	793	1248	3039	3628
5	19500					1	84	1158	1607
6	25400						1	1244	1735
7	41000							1	1225
4	4600	9	10	11	12	13	14	15	16
5	19500	1921	2084	2219	2280	2467	2552	2629	2713
6	25400	2285							
7	41000	1530	1849	2017	2124	2193	2309	2420	2511

TABLE 3.5.2. CE and NM combined calibration results

Maturity	5 year		7 year		10 year	
Tranche (%)	Market	Model	Market	Model	Market	Model
$0-3$	13.01	12.93	27.87	28.47	42.2895	43.22
$3-6$	67.1	68	147.4	145	396	405
$6-9$	17.93	18	40.4	39	120.2581	120
$9-12$	8.34	9	18.3	21	55.35817	59
$12-22$	3.28	4	8.1	9	18.08	23

TABLE 3.5.3. Calibration to iTraxx index tranches of three maturities on 02 July 2007

3.5.2. No-arbitrage conditions for a dynamic model. The static model can be calibrated to the market index tranches almost perfectly with one set of parameters and therefore satisfies no-arbitrage Condition 1.1.1. It is efficient in calibration and can be used in practice to price products that are static in nature, such as bespoke tranches.

However, one has to be careful when using it to price products that are generally dynamic. For example, we can calibrate the model to CDOs with maturities of 5, 7 and 10 years simultaneously, as in table 3.5.3.

Figure 3.5.6 (a) shows the loss distributions implied by the CDS index tranches. We need to be careful with the interpretation because the time evolution of the loss distribution does not exist for a static model. There are no-arbitrage conditions to be satisfied in the time dimension for a dynamic model. One such condition for dynamic bottom-up models is condition 3.5.1 given below.

CONDITION 3.5.1. *The term structure of default probabilities of an obligor conditional on a realisation of state variables of any given point in time is a monotone increasing function of maturity.*

Another more general and model-independent no-arbitrage condition is the following.

CONDITION 3.5.2. *In a portfolio of N names, starting with any point in time, the probability of less than l number of defaults decreases with time or the probability of more than l number of defaults increases with time.*

This no-arbitrage condition can be formulated mathematically, as given below.

$$\sum_{\ell=0}^{l} P(\ell, t_{n+1}) < \sum_{\ell=0}^{l} P(\ell, t_n)$$

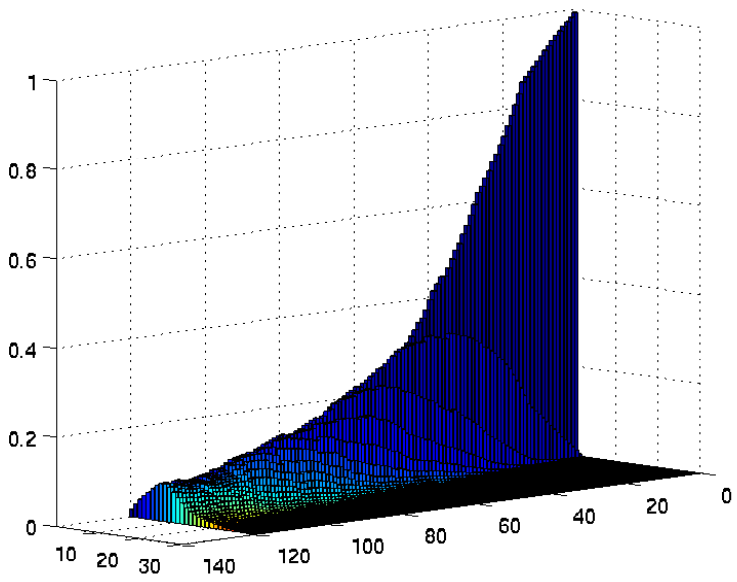
where t_{n+1} is a time step after t_n and l is the number of defaults under consideration.

Condition 3.5.2 can be applied to both bottom-up and top-down models. It is important to know that bottom-up models, such as copula models, satisfy condition 3.5.2 but since the default dynamics are not modelled, copula models are not dynamic models. Copula models are a special case of static models because they use one distribution for all payment dates. By setting up our model using all the steps on the lattice, we want to show that the above condition can be violated by static bottom-up models.

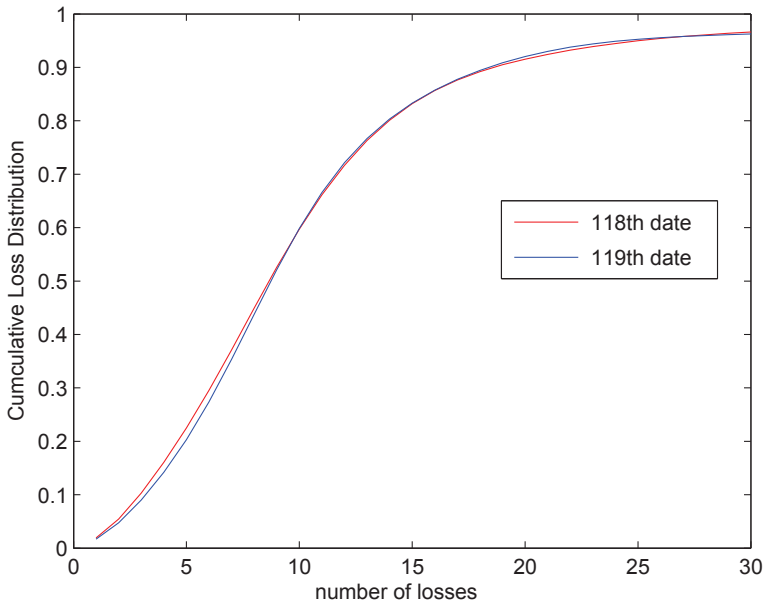
Figure 3.5.6 (b) shows the violation of the no-arbitrage Condition 3.5.2 between the loss distributions by the payment date 118 and payment date 119. From the graph we can see that the two curves intersect each other. To maintain the no-arbitrage Condition 3.5.2, the loss distribution curve for the payment date 118 should be above the payment date 119.

For bottom-up models to be qualified as dynamic models, they have to satisfy condition 3.5.1. Condition 3.5.1 is a stronger condition since condition 3.5.2 is model independent and can be derived from 3.5.1. Condition 3.5.1 is for dynamic bottom-up models with more restrictive assumptions. And since condition 3.5.1 is a stronger condition, it is harder to fit bottom-up models to market quotes than top-down models, in which the loss process is modelled directly.

Figure 3.5.7 gives the conditional default probabilities in the lattice for one of the obligors in the portfolio from our calibrated model. In the example, each step in the lattice covers



(a) Market Implied Loss distribution



(b) Arbitrage violation. The red curve should be above the blue curve rather than intertwined with it.

FIGURE 3.5.6. Calibration statistics for iTraxx data on 21 March 2005

three payment dates; within each step in the lattice, since there is no dynamics for the common factor, the default probabilities are increasing with time. If we treat the transition

on the lattice as the dynamics of the market factor, the resulting default process violates the no-arbitrage condition 3.5.1. The term structure of conditional default probability can actually decrease. For example, when the common factor changes from $M_{1,2}$ to $M_{3,2}$, the conditional default probability decreases from 0.001198 to 0.001082.

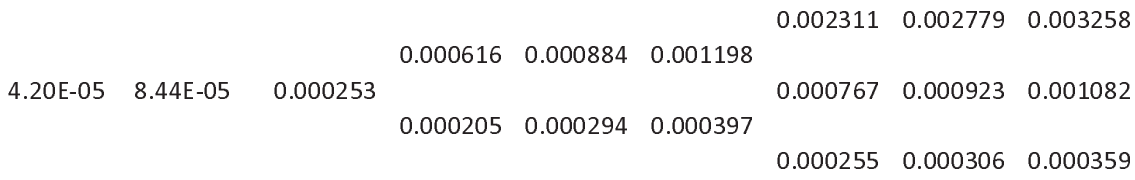


FIGURE 3.5.7. The conditional default probabilities of an obligor in the lattice

3.6. Comparison with implied copula model by Hull and White

There are similarities between the static model and the model in Hull & White (2006). Both are non-parametric static models that are not arbitrage free in the time dimension. The static model can be seen as implying the copula in the form of common factor distribution, whereas Hull & White (2006) implies the copula in the form of conditional hazard rates. Since it does not have a functional form between hazard rate and factor distribution, and the static binomial model does, their model is 'more' non-parametric.

The advantages of the static binomial model are as follows.

Firstly, in the implied copula model the dispersion of hazard rates is hard to determine. In the static binomial model the parameter a controls the variance of the lattice and the dispersion between hazard rates is well placed and controlled.

Secondly, in the implied copula model it is difficult to price a heterogenous portfolio, as the market distribution and the individual correlations to market are all contained in the conditional hazard rates. However, pricing a heterogenous portfolio is natural for the static binomial model. The λ is the response of each obligor to the common market distribution, which controls individual correlation to market and is separate from the common factor.

Once the market factor distribution is implied from the calibration, any bespoke portfolio which has similar composition to the calibration portfolio can be priced by simply using the implied factor distribution to compute for the λs of the new obligors.

3.7. Conclusion

In this chapter we showed how a static binomial model can be calibrated to market-quoted tranches exactly, with one set of model parameters. We further examined that, if viewed dynamically, the model is not arbitrage free along the time dimension. Therefore, it is not suitable for pricing products that are dynamic in nature. Based on this model, in the next two chapters we will develop two more sophisticated dynamic models for a wider applicability.

However, relative to the two dynamic models, the static model enjoys much faster pricing and calibration.

CHAPTER 4

A dynamic binomial model

A dynamic bottom-up model describes the stochastic default process of an obligor. This process can be seen as having two layers. The first layer is the dynamics of the underlying economic variables, which affect defaults such as interest rates, and the second layer is the stochastic default time conditional on the realisations of the underlying variables. This is the so-called *doubly stochastic* or *Cox* process (Lando 1998). The credit-risk models involving this process belong to the type of model known as intensity models. In this type of model the default intensity is chosen as the modelling quantity whose dynamics are generated by the filtration of the underlying variables. This default process in the time dimension has to satisfy the no-arbitrage condition 3.5.1 for a model to qualify as a dynamic model.

4.1. The stochastic intensity process

In a multi-obligor setting, to model default correlations we work in a simplified world where the underlying variables are classified and represented by two factors - a common factor, which affects the default of all obligors under consideration, and an idiosyncratic factor, which only affects individual obligor default.

Specifically, default intensity is modelled by the sum of a common market component and an idiosyncratic component. Since the idiosyncratic factor can be integrated out in building the loss distributions, we only model the dynamics of the common factor in our binomial model. In a general form the conditional survival probability until time t of an obligor i is represented by

$$P\{\tau_i > t \mid \mathcal{M}_t\} = e^{-\int_0^t h(M_s)ds}$$

where M_s is the common market factor at time s, $\{\mathcal{M}_t\}$ is the filtration generated by $\{M_s, 0 \leq s \leq t\}$ until time t and $h(M_s)$ is default intensity at time s.

Conditional on the filtration $\{\mathcal{M}_t\}$, defaults are independent among obligors. For each realisation of the market process $M_s, 0 \leq s \leq t$ the default probability is an increasing function of time as long as the intensity $h(M_s)$ is a positive process.

Therefore, the second no-arbitrage condition 3.5.1 is automatically satisfied by specifying a positive stochastic process for the intensities.

Typically, processes that exhibit fat tails in distributions, such as a jump-diffusion process, can be a good choice for the dynamics of intensities, as in Duffie & Garleanu (2001). However, in general, most of these models require Monte Carlo simulation of the paths of the factor dynamics in pricing and are therefore too slow to be practical.

In our approach the representation of the common factor dynamics with a binomial lattice is a discrete process and thus the number of paths is finite. The probability distribution of each path can be explicitly calculated for pricing purposes.

4.2. The setup of a dynamic binomial lattice model

With the same model assumptions as in section 3.3, we further assume that the default-free discount curve is independent of the default intensity dynamics. Consider a discrete set of time points T_k, $0 \leq k \leq K$, where $T_0 = 0$ denotes the initial time point ('today'). The dynamics of the common factor are represented by the binomial lattice we set up in section 3.3. Assume that the default hazard rate for the i-th obligor in the time interval $(T_{k-1}, T_k]$ is $\lambda_i(t)M_{j,k}$, with $\lambda_i(t)$ a deterministic function of time and $M_{j,k}$ is piecewise constant between $(T_{k-1}, T_k]$ with j as the state variable and k as the time step variable. The probability of default on $(s, t] \subseteq (T_{k-1}, T_k]$ given $M_{j,k}$ is

$$1 - \exp\left\{-\int_s^t \lambda_i(u)M_{j,k}du\right\}$$

Normally, the number of steps in the lattice is fewer than that of the number of payments remaining in a CDO. Therefore, each step on the lattice will cover at least one payment date and $\lambda_i(t)$s are piecewise constant for each payment period.

We discretise the timeline using a set of time points t_n, $0 \leq n \leq N$ when payments occur, with $t_0 = T_0 = 0$ and $t_N = T_K$. The time for state change on the lattice coincides with the payment dates in the CDO. Therefore,

$$\{T_0, \ldots, T_K\} \subseteq \{t_0, \ldots, t_N\}$$

and the step number on the lattice is given by the function $k = \eta(t_n)$. As in the static model, the allocation of payments between time $[T_{k-1}, T_k]$ is to ensure roughly equal numbers between payment dates.

Figure 4.2.1 illustrates these specifications on the lattice.

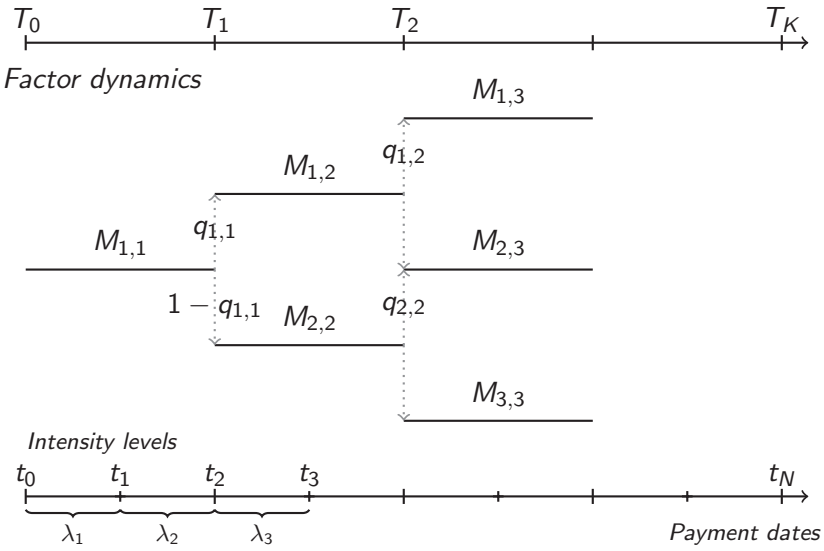


FIGURE 4.2.1. A lattice representation of the factor dynamics

To price a CDO tranche, first we have to generate model parameters a and $q_{j,k}$ to build up the lattice dynamics. For details of the algorithmic construction of the factor values and probability distributions on the lattice, see section 3.4.

4.3. Algorithmic construction of conditional survival probabilities

Once the factor dynamics are generated, the default dynamics in the form of conditional default probabilities can be generated by finding parameter $\lambda_i(t)$s to match marginal default probabilities. Since $\lambda_i(t)$s are time-step dependent, we derive the conditional survival probabilities step-by-step on the lattice.

We know that, given an obligor has survived to the current payment date t_{n-1}, the survival probability until the next payment date t_n is

$$\exp\left\{-\int_{t_{n-1}}^{t_n} \lambda_i(u) M_{j,\eta(u)} du\right\}$$

Notice that the conditional survival probability until time t_N, $S_i(t_N \mid \mathcal{M}_{t_N})$ is the product of conditional survival probabilities of each segment along the path of the market factor given by

$$S_i(t_N \mid \mathcal{M}_{t_N}) = \exp\left\{-\int_0^{t_N} \lambda_i(u) M_{j_n,\eta(u)} du\right\} = \prod_{n=1}^{N} \exp\left\{-\int_{t_{n-1}}^{t_n} \lambda_i(u) M_{j_n,\eta(u)} du\right\}$$

In the construction, we work with the survival probability $S_i(t_n|M_{j,\eta(t_n)})$ that is conditional on the current state of the factor $M_{j,\eta(t_n)}$. Due to the recombining feature of the lattice, $S_i(t_n|M_{j,\eta(t_n)})$ is generated by the overlapping paths that lead to the current node $M_{j,\eta(t_n)}$. In general, along the path on the lattice there are two types of changes that affect the conditional default process - the change subject to time progression when the default intensities are accumulated and the change subject to state change of the common factor when the conditional default probabilities branch out or are combined. Next we derive the induction algorithm for the conditional survival probabilities subject to the change in the two dimensions.

4.3.1. Initialisation. For the first step on the lattice, we have

$$\begin{aligned} S_i(T_1) &= S_i(T_1|M_{1,1}) \\ S_i(T_1|M_{1,1}) &= \exp\left\{-\int_{T_0}^{T_1} \lambda_i(t) M_{1,1} dt\right\} \end{aligned}$$

4.3.2. Forward induction for survival probability on the lattice subject to time progression.

$$S_i(t_{n+1}|M_{j,\eta(t_{n+1})}) = \exp\left\{-\lambda_i(t_{n+1})(t_{n+1}-t_n) M_{j,\eta(t_n)}\right\} S_i(t_n|M_{j,\eta(t_n)}) \qquad (4.3.1)$$

Here $\eta(t_{n+1}) = \eta(t_n)$ since there is only time progression and no state change in this step.

4.3.3. Forward induction for survival probability on the lattice subject to state change.

$$S_i(T_k|M_{j,k+1}) =$$

$$\begin{cases} S_i(T_k|M_{j,k}) & j=1 \\ S_i(T_k|M_{j,k})q(M_{j,k}|M_{j,k+1}) + \ S_i(T_k|M_{j-1,k})q(M_{j-1,k}|M_{j,k+1}) & 1<j<k \\ S_i(T_k|M_{j-1,k}) & j=k \end{cases}$$

where $q(M_{j,k-1}|M_{j,k})$ is the conditional probability that the process M was in state j at time T_{k-1} given that it is in state j at time T_k, which can be calculated from known probabilities using Bayes' Theorem, i.e.

$$\begin{aligned} q(M_{j,k}|M_{j,k+1}) &= \frac{q(M_{j,k+1}|M_{j,k})Q_{j,k}}{Q_{j,k+1}} \\ &= \frac{q_{j,k}Q_{j,k}}{Q_{j,k+1}} \end{aligned}$$

and similarly

$$\begin{aligned} q(M_{j-1,k}|M_{j,k+1}) &= \frac{q(M_{j,k+1}|M_{j-1,k})Q_{j-1,k}}{Q_{j,k+1}} \\ &= \frac{(1-q_{j-1,k})Q_{j-1,k}}{Q_{j,k+1}} \end{aligned}$$

where $q(M_{j,k+1} \mid M_{j,k}) = q_{j,k}, q(M_{j,k+1} \mid M_{j-1,k}) = 1 - q_{j-1,k}$ and $q_{j,k}$s are transition probabilities.

Figure 4.3.1 gives a graphical representation as an example.

Note that by construction, all future term structures of conditional survival probabilities will obey the no-arbitrage requirement of monotonicity, i.e.

$$\mathbb{E}[S_i(s \mid M_{.,\eta(s)}) \mid M_{j,k}] > \mathbb{E}[S_i(t \mid M_{.,\eta(t)}) \mid M_{j,k}] \quad \forall s<t, 1 \le j \le k \tag{4.3.2}$$

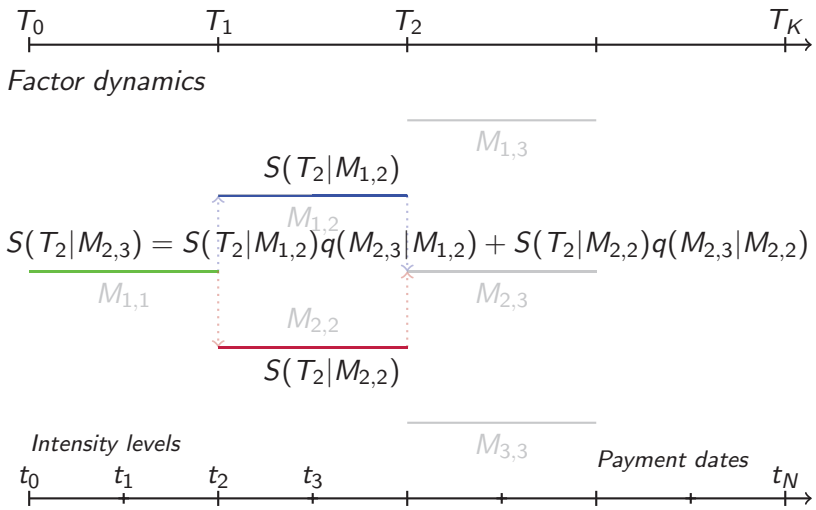


FIGURE 4.3.1. Forward induction on the lattice for conditional survival probabilities

4.4. Calibration to marginal default probabilities

The intensity levels $\lambda_i(t)$s are the response of each obligor to the common market factor at time t. They are piecewise constant between payment dates for each obligor.

From equation 4.3.1 we can derive,

$$\begin{aligned} S_i(t_{n+1}) &= \sum_{j=1}^{n} S_i(t_{n+1}|M_{j,\eta(t_n)})Q_{j,\eta(t_n)} \\ &= \sum_{j=1}^{n} \exp\left\{-\lambda_i(t_{n+1})(t_{n+1}-t_n)M_{j,\eta(t_n)}\right\} S_i(t_n|M_{j,\eta(t_n)})Q_{j,\eta(t_n)} \end{aligned} \tag{4.4.1}$$

where $M_{j,\eta(t_n)}$ and $Q_{j,\eta(t_n)}$ have been derived from model parameters. Therefore, once we know $S_i(t_n|M_{j,\eta(t_n)})$ at time step t_n, we can solve for $\lambda_i(t_{n+1})$ for time step t_{n+1}.

To solve for $\lambda_i(t_{n+1})$, it must first be acknowledged that the solution exists. Notice that when $\lambda_i(t_{n+1}) = 0$ the right-hand side of equation 4.4.1 becomes $S_i(t_n)$. Since function $\exp\left\{-\lambda_i(t_{n+1})(t_{n+1}-t_n)M_{j,\eta(t_n)}\right\}$ is a continuous monotone decreasing function of variable $\lambda_i(t_{n+1})$, if we increase $\lambda_i(t_{n+1})$ starting from 0 to infinity ∞, the right-hand side will go from $S(t_n)$ to 0. Since $0 < S_i(t_{n+1}) < S_i(t_n)$, $\lambda_i(t_{n+1})$ will reach a value that satisfies the equation 4.4.1.

Since the derivative exists for function $\exp\left\{-\lambda_i(t_{n+1})(t_{n+1}-t_n)M_{j,\eta(t_n)}\right\}$ with regard to $\lambda_i(t_{n+1})$, we employ Newton's method in the root finding algorithm.

Set the initial value for $\lambda_i(t_{k+1})$ to be 0. The derivative with regarding to $\lambda_i(t_{k+1})$is given by

$$\frac{\partial S_i(t_{n+1})}{\partial \lambda_i(t_{n+1})} = -(t_{n+1}-t_n)M_{j,\eta(t_n)} \sum_{j=1}^{n} \exp\left\{-\lambda_i(t_{n+1})(t_{n+1}-t_n)M_{j,\eta(t_n)}\right\} S_i(t_n|M_{j,\eta(t_n)})Q_{j,\eta(t_n)}$$

In our experience, it generally takes two steps of Newton's method to reach the desired accuracy of 10^{-5}.

4.5. Pricing on the path

In order to price tranches of CDOs, we require the portfolio loss distribution $P(\ell, t_n)$ to compute the expected loss of a portfolio of obligors up to each payment date.

Here in the dynamic model, the default probabilities are independent conditional on the path generated by factor M and therefore the recursive method can only be applied to this conditional default probabilities to build up the loss distributions conditional on the path.

Then, to compute the loss distribution $p(\ell, t)$, we do not have to use the MC method[1], as the probability distribution $Q(M_{1,1}, \cdots, M_{j,\eta(t_n)})$ for each path $M_{1,1}, \cdots, M_{j,\eta(t_n)}$ until time t_n can be calculated.

Therefore, $p(\ell, t)$ is the weighted average of the conditional loss distributions given by

$$P(\ell, t_n) = \sum_{k=0}^{\eta(t_n)} P(\ell, t_n|M_{1,1}, \cdots, M_{j,\eta(t_n)})Q(M_{1,1}, \cdots, M_{j,\eta(t_n)}) \tag{4.5.1}$$

where the conditional loss probabilities $P(\ell, t_n|M_{1,1}, \cdots, M_{j,\eta(t_n)})$ can be built up by the recursive method which will be implemented in section 4.5.2.

However, the number of paths on the lattice until a certain step has an exponential relationship with regarding to the number of steps on the lattice. A K step lattice has 2^K paths. The computational burden is still of concern if the number of steps becomes large. However, due to the computational speed of most desktop computers, for practical purposes generally less than 12 steps on the lattice are reasonable.

[1]Another reason for not using the MC method is that the CE method requires accurate pricing to be effective. If the pricing result is not stable, as with the MC method, the parameter distributions in the CE algorithm are based on inaccurate evaluation results and the method will be difficult to converge.

4.5.1. Path generation. To compute the conditional default probabilities, we need to generate all the paths on the lattice. The first step is to generate the positions of the nodes on each path on the lattice. If we traverse forward (from left to right) on the lattice, we only need the row positions, which corresponds to the states of the common factor, for the nodes on the lattice. We proceed in such a way, as shown in figure 4.5.1. Starting with the first node with a row position of 1, if in the next step the factor goes up, its row position remains 1, and if it goes down, its row position increases by 1 to 2. Therefore, for the first path, which involves the factor going up in all the steps, its row vector is $[1, 1, \cdots, 1]_{K+1}$ for a K step lattice. If for the next path, the factor goes up in the first $K-1$ step as in the first path but on the last step it goes down, then the row vector for the path is $[1, 1, \cdots, 1, 2]_{K+1}$. If we run through all the paths on the lattice accordingly, the last path is $[1, 2, \cdots, K+1]_{K+1}$.

To generate these vectors algorithmically, a simple way is to use loops. However, there is a better way, as there are 2^K paths for a K step lattice and a K-digit binary number also has 2^K possible $(0, 1)$ combinations. In fact, since there are $K+1$ elements in the path vectors, starting from the second element, if we subtract each element by its left element and form a new vector, we get exactly all the 2^K number of K-digit binary numbers.

Therefore, the algorithm to generate all the paths are as follows.

1). Generate a sequence of decimal numbers from 1 to 2^K.

2). Convert them into binary numbers.[2]

3). Put a 1 in front of each of the binary numbers to form vectors.

4). Calculate the cumulative sum of all the vectors in decimal arithmetics and we get the row vectors.

4.5.2. Implementing the recursive method. Once we generate all the paths on the lattice, we can calculate the conditional survival probabilities and probability distributions.

[2]To convert from a decimal integer numeral to its binary equivalent, the number is divided by two, and the remainder is the least significant bit. The (integer) result is again divided by two, its remainder is the next most significant bit. This process repeats until the result of further division becomes zero.

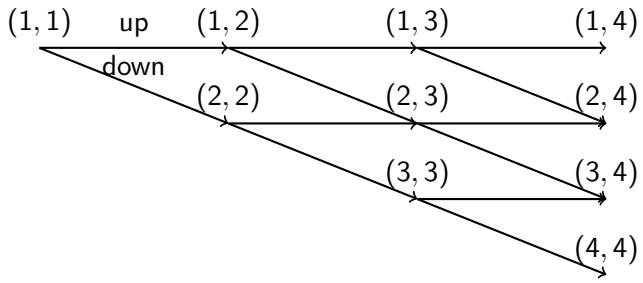


FIGURE 4.5.1. Traverse on the lattice

For the j-th path on a K-step lattice, once we add the column number to the vector we have a position vector $[(j_1, 1), (j_2, 2), \cdots, (j_K, K)]$ which generates the path

$$[M_{j_1,1}, M_{j_2,2}, \cdots, M_{j_K,K}]$$

.

The cumulative survival probabilities conditional on this path until T_K are therefore

$$S(T_K \mid M_{j_1,1}, M_{j_2,2}, \cdots, M_{j_K,K}) = e^{-\int_{T_0}^{T_K} \lambda(s) M_{j_k,\eta(s)} ds}$$

The probability distribution until T_K for the path is given by

$$Q(M_{j_1,1}, M_{j_2,2}, \cdots, M_{j_K,K}) = \prod_{k=1}^{K-1} q_{(M_{j_k,k}, M_{j_{k+1},k+1})}$$

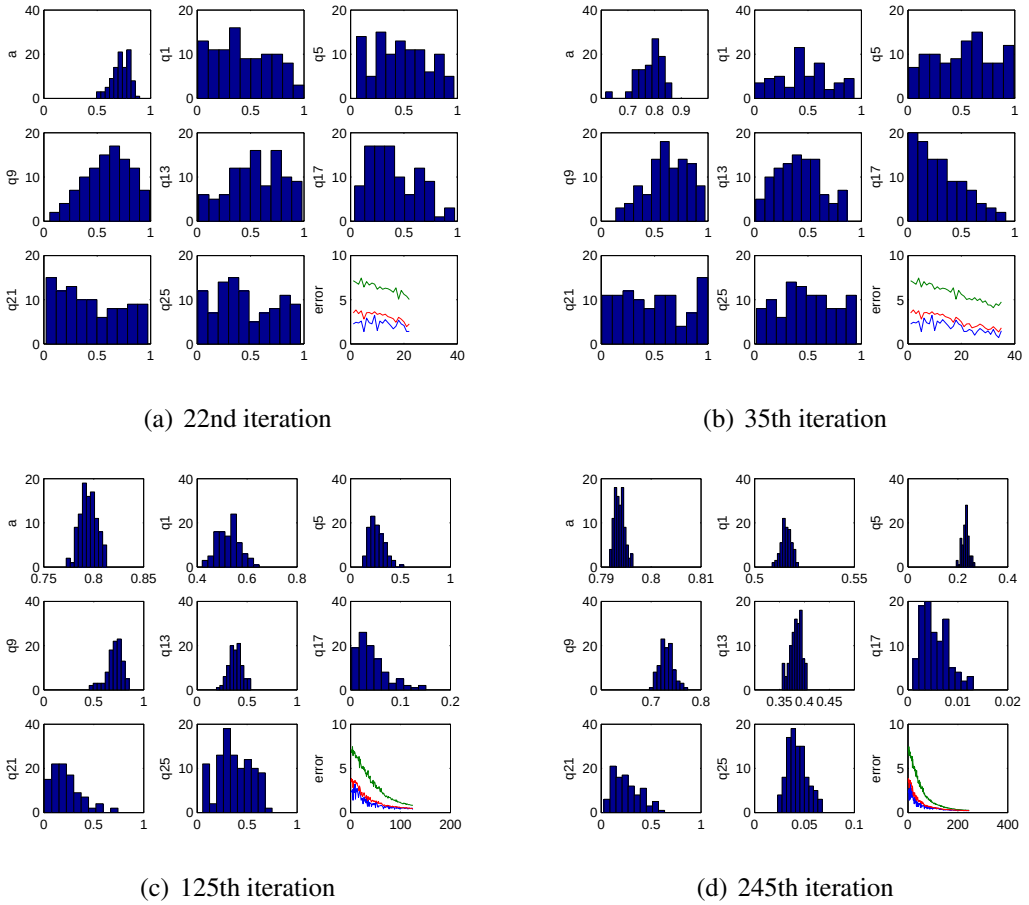
where

$$q_{(M_{j_k,k}, M_{j_{k+1},k+1})} = \begin{cases} q_{j_k,k} & j_k = j_{k+1} \\ 1 - q_{j_k,k} & j_{k+1} = j_k + 1 \end{cases}$$

Then the recursive algorithm 2.4.4 can be applied to derive the conditional loss distribution $P(\ell, t_n | M_{1,1}, \cdots, M_{j,\eta(t_n)})$ with conditional default probability $1 - S_i(t_n \mid M_{j_1,1}, \cdots, M_{j_{\eta(t_n)},\eta(t_n)})$ for each obligor i.

4.6. Calibration performance with the CE method

Figure 4.6.1 presents some 'snapshots' of the parameter distributions during calibration.



(a) 22nd iteration

(b) 35th iteration

(c) 125th iteration

(d) 245th iteration

FIGURE 4.6.1. Parameter distributions in calibration for iTraxx IG data on 21 March 2005

The four sub-figures are for the 22nd, 35th, 125th and 245th iterations, respectively. In each sub-figure, there are nine panels. The $(1,1)$ panel is the distribution for the size change parameter a, the $(1,2)$ to $(3,2)$ panels are selected distributions for transition probabilities $q_{j,k}$ and the last panel $(3,3)$ are the errors for the best solution, the average error of chosen ρ-quantile solutions and the average error of all the solutions in each iteration.

It can be observed that a converges very quickly from the beginning, indicating that the general correlation between obligors is determined first and foremost. The transition probabilities converge slower than a but the speed of convergence differs among them. The errors between model spreads and market quotes fluctuate but in general decrease until

convergence as iteration number increases. It is an interesting to compare this to the NIG copula model calibration analysis from the last chapter. In the NIG copula, once correlation parameter ρ is determined, the results can only be bettered slightly by varying the local dependency parameters α and β. The binomial model, however, shows much greater flexibility by achieving much greater calibration accuracy.

4.7. Calibration results

We tested the model on CDX IG data of 21 April 2003 and iTraxx IG data of 21 March 2005. The calibration results are given in Table 4.7.1 and Table 4.7.2, respectively. The first three columns are mid-market quotes and bid/ask spreads. The last columns are calibration results and are all fitted within bid/ask spreads.

Tranche (%)	Bid	Ask	Mid	Model spread
0–3	37(%)	42	39.5	41.39
3–7	28 (bp)	33	30.5	31.7
7–10	100 (bp)	112	106	112
10–15	39 (bp)	59	49	55
15–30	6(bp)	16	11	14

TABLE 4.7.1. Calibration results for CDX tranches on 21 April 2003 maturing in five years

Tranche (%)	Bid	Ask	Mid	Model spread
0–3	16.85(%)	18.15	17.5	17.81
3–6	107.5(bp)	117.5	112.5	114.3
6–9	33.38 (bp)	38.88	36.13	36.38
9–12	15.25 (bp)	20.75	18	17.48
12–22	8.25 (bp)	11.75	10	10.86

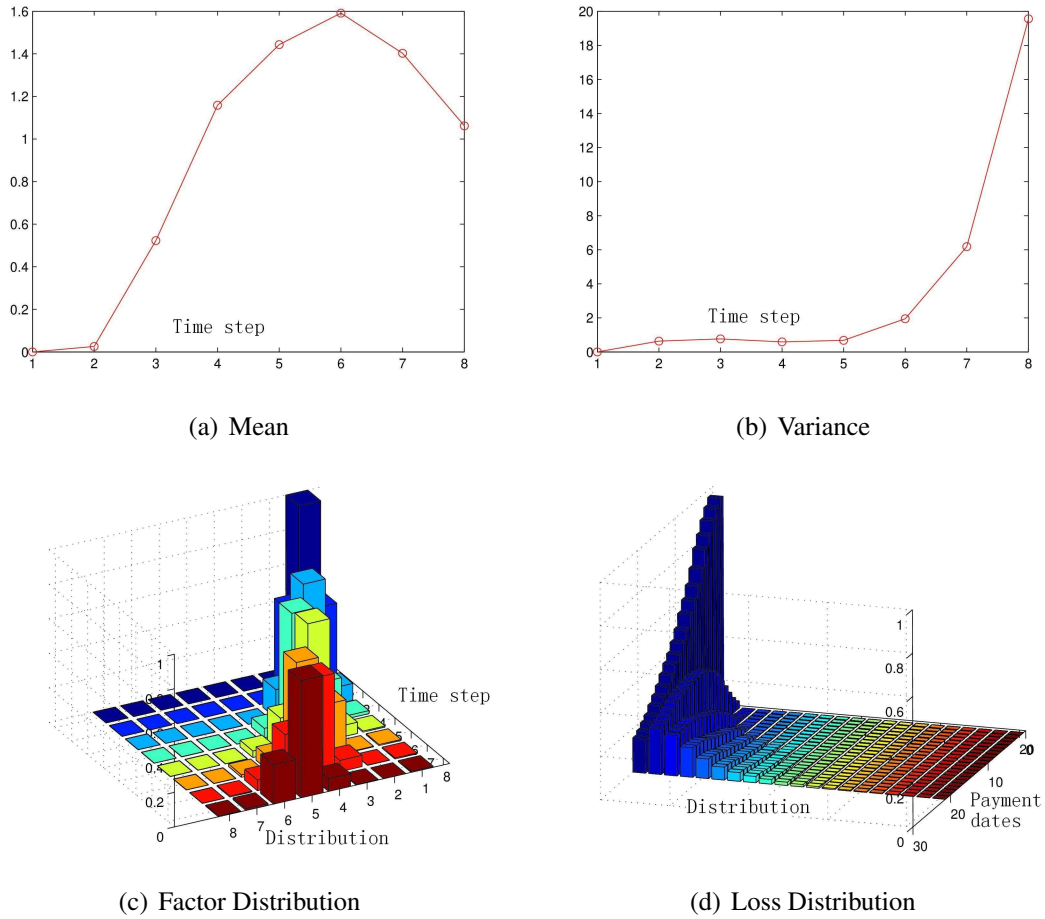
TABLE 4.7.2. Calibration results for iTraxx tranches on 21 March 2005

An important finding is that compared to the static model, calibration with the dynamic binomial model is not as accurate using the same number of steps on the lattice. A probable reason for this is that the integral of the intensity dynamics reduces model flexibility. Further, the flexibility cannot be increased significantly by increasing the number of steps on the lattice.

We will show that the approach to modelling the intensity integral directly, to be introduced in the Markov model in the next chapter, can resolve this problem.

4.8. Implied market process

Compared to top-down models, the bottom-up approach explicitly models the dynamics of the systematic factor and idiosyncratic factor. Therefore, by fitting the model to market quotes, people's view of the market can be implied through systematic factor dynamics. Figure 4.8.1 presents some statistics that we can derive from the calibrated results of table 4.7.2.



(a) Mean

(b) Variance

(c) Factor Distribution

(d) Loss Distribution

FIGURE 4.8.1. Calibration statistics of the common factor M implied by the iTraxx IG data on 21 March 2005

Figure 4.8.1(a) plots the mean of $\log(1/M)$. Therefore, the curve can be either positive or negative, which represents the market's up or down movement, similar to a market index. Figure 4.8.1(b) is the variance of M, which reflects people's view of future market volatility. Figure 4.8.1(c)(only the right triangle) is the distribution of the common factor. And figure 4.8.1 (d) are the loss distributions of the portfolio by the first 20 payment dates.

4.8.1. No-arbitrage condition satisfied. The dynamic model automatically satisfies the no-arbitrage condition 3.5.1. By satisfying this condition, the no-arbitrage condition 3.5.2 is also satisfied, as shown in figure 4.8.2.

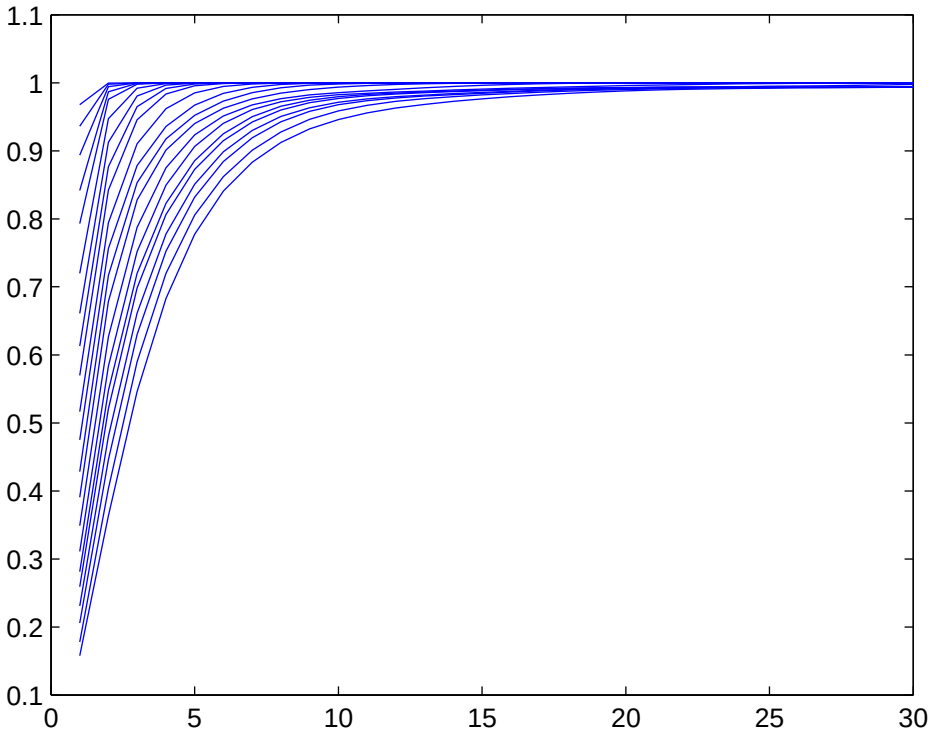


FIGURE 4.8.2. The cumulative loss distribution by different payment dates

4.9. Stochastic recovery rate extension for the binomial model

4.9.1. Recovery rate and probability of default. So far we have assumed that recovery rate is constant throughout different economic states. However, there is a wealth of empirical evidence suggesting that recovery rate has a negative relationship with economic conditions. The underlying logic for this is quite straightforward: if a borrower defaults on a loan, the recovery from its loan collateral may depend on its market value. The market value of the collateral, like any other asset, depends on economic conditions. If the economy is booming, a company's default probability is low and if default occurs, the demand for the company's collateral is high and it is likely to sell for a good price. On the other hand, if the economy is experiencing a recession, recovery rates may decrease just as default rates tend to increase. Therefore, there is a negative correlation

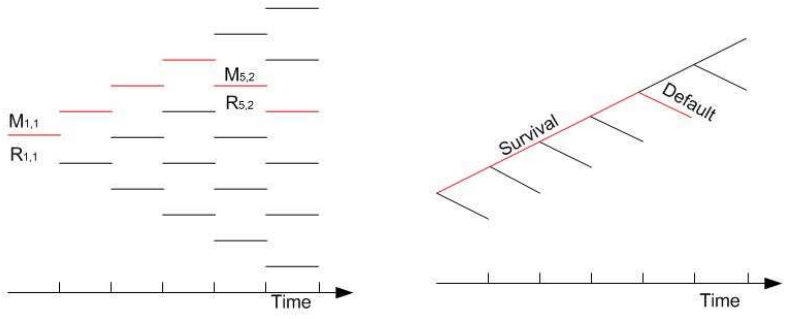
between default probabilities and recovery rates. See Jokivuolle & Peura (2003), Amato & Gyntelberg (2005) and Frye (2000) for a detailed explanation.

The benefit gained through incorporating a stochastic recovery rate in modelling can be seen in achieving improved model fit to market data, as evidenced by several researchers, such as in Hull et al. (2006).

To maintain consistency in pricing, we also relax the constant recovery assumption in bootstrapping default hazard rates from CDS prices. Therefore, the intensity levels λ_is for each obligor are not calibrated to marginal default probabilities but to the CDS spreads in the bootstrapping procedure. Based on each set of market scenarios given by the dynamics of the market factor M, the bootstrapping algorithm output is λ_is rather than marginal default probabilities.

Figure 4.9.1 (a) shows a set of economic scenarios generated by a binomial lattice and (b) shows the default process for an obligor for one realisation on the lattice.

This section is organised as follows. First we derive the standard CDS bootstrapping convention for the model. Then, based on that, we show how to bootstrap the intensity parameter λ_is under the dynamic binomial model from CDS spreads. Then we set up algorithms to correlate the recovery rate with default probabilities. Finally, we review the generalised recursive method from Jackson, Kreinin & Ma (2007) and demonstrate how it is used for generating loss distributions.



(a) Economic scenarios on the lattice (b) One realisation

FIGURE 4.9.1. Stochastic recovery rate mapping diagram

4.9.2. CDS pricing convention in our model. The survival probability of an obligor until time u is given by

$$S(u) = e^{-\int_0^u h(s)ds} \tag{4.9.1}$$

where $h(s)$ is the instantaneous forward hazard rate at time s.

Similarly, the price of a risk-free discount bond maturing at time u is given by

$$B(0,u) = e^{-\int_0^u r(s)ds} \tag{4.9.2}$$

where $r(s)$ is the instantaneous forward rate at time s.

Discretise the period ending in time t into K time blocks and each block is $\Delta u_k, 1 \leq k \leq K$

Assume that default can only occur at the end of each time block. Then the PV (present value) of the premium leg of a CDS contract is given by

$$a_u P(t) = \sum_{k=1}^{K} e^{-\int_0^{u_k}(h(s)+r(s))ds} a_u \Delta u_k \tag{4.9.3}$$

where a_u is the annualised spread.

If we take $\max(u_k) \to 0$, we have the premium leg PV in the continuous time as

$$aP(t) = a\int_0^t e^{-\int_0^u (h(s)+r(s))ds} du \tag{4.9.4}$$

Similarly, the PV of the default leg of the CDS is computed as the following in the discrete case.

$$\begin{aligned} D(t) &= \sum_{k=1}^{K} e^{-\int_0^{u_{k-1}} h(s)ds}(1 - e^{-\int_{u_{k-1}}^{u_k} h(s)ds}) e^{-\int_0^{u_k} r(s)ds}(1 - R_k) && (4.9.5)\\ &= \sum_{k=1}^{K} e^{-\int_0^{u_k}(h(s)+r(s))ds}(e^{\int_{u_{k-1}}^{u_k} h(s)ds} - 1)(1 - R_k) && (4.9.6) \end{aligned}$$

Let $\max(\Delta u_k)$ goes zero, $e^{\int_{u_{k-1}}^{u_k} h(s)ds} \to 1 + h_u du$

Therefore, the PV of the default leg in continuous time is given by

$$D(t) = \int_0^t e^{-\int_0^u (h(s)+r(s))ds} h(u)(1 - R_u) du \tag{4.9.7}$$

The spread a_t for any CDS contract expiring at time t can be derived by equating equation 4.9.7 with equation 4.9.4

$$a_t = \frac{D(t)}{P(t)} = \frac{\int_0^t e^{-\int_0^u (h(s)+r(s))ds} h(u)(1-R_u)du}{\int_0^t e^{-\int_0^u (h(s)+r(s))ds} du} \tag{4.9.8}$$

For a CDS contract that expires in the immediate future. Assume the hazard rate h_1 and recovery rate R_1 are constant, h_1 can then be derived given the market spread a_1.

$$h_1 = \frac{a_1}{1-R_1} \tag{4.9.9}$$

4.9.3. Hazard rate bootstrapping from CDS spreads. Assume that there are a number of K CDS contracts outstanding in the market that we can use for bootstrapping and assume the hazard rates $h(s)$ to be constant between the expiry dates of two adjacent contracts.

For a CDS contract that matures in time t_k, we know the PVs of the two legs are equal

$$a_k P(t_k) = D(t_k) \tag{4.9.10}$$

where a_k is the spread for a CDS contract maturing at t_k.

The increments between the PVs of the two legs of contracts maturing at time t_k and time t_{k+1} are given by

$$a_k P(t_k) - a_{k-1} P(t_{k-1}) = D(t_k) - D(t_{k-1}) \tag{4.9.11}$$

Expansion on the left

$$a_k(P(t_k) - P(t_{k-1})) + (a_k - a_{k-1})P(t_{k-1}) = D(t_k) - D(t_{k-1}) \tag{4.9.12}$$

From equation (4.9.4),

$$\begin{aligned}
P(t_k) - P(t_{k-1}) &= \int_{t_{k-1}}^{t_k} e^{-\int_0^u (h(s)+r(s))ds} du \\
&= e^{-\int_0^{t_{k-1}} (h(s)+r(s))ds} \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^u (h(s)+r(s))ds} du \\
&= e^{-\int_0^{t_{k-1}} (h(s)+r(s))ds} \int_{t_{k-1}}^{t_k} e^{-(h_k+r_k)(u-t_{k-1})} du \\
&= e^{-\int_0^{t_{k-1}} (h(s)+r(s))ds} \int_0^{\Delta t_k} e^{-(h_k+r_k)u} du \\
&= e^{-\int_0^{t_{k-1}} (h(s)+r(s))ds} \frac{1 - e^{-(h_k+r_k)\Delta t_k}}{h_k + r_k} \\
&= S(t_{k-1})B(0,t_{k-1}) \frac{1 - e^{-(h_k+r_k)\Delta t_k}}{h_k + r_k}
\end{aligned}$$

From equation (4.9.7),

$$\begin{aligned}
D(t_k) - D(t_{k-1}) &= h_k(1 - R_k) \int_{t_{k-1}}^{t_k} e^{-\int_0^u (h(s)+r(s))ds} du \\
&= h_k(1 - R_k)(P(t_k) - P(t_{k-1}))
\end{aligned}$$

Therefore equation (4.9.12) becomes

$$\begin{aligned}
a_k(P(t_k) - P(t_{k-1})) + (a_k - a_{k-1})P(t_{k-1}) &= h_k(1 - R_k)(P(t_k) - P(t_{k-1})) \\
(a_k - a_{k-1})P(t_{k-1}) &= (h_k(1 - R_k) - a_k)(P(t_k) - P(t_{k-1})) \\
\frac{(a_k - a_{k-1})P(t_{k-1})}{(h_k(1 - R_k) - a_k)S(t_{k-1})B(0,t_{k-1})} &= \frac{1 - e^{-(h_k+r_k)\Delta t_k}}{h_k + r_k}
\end{aligned}$$

If h_{k-1} is known $P(t_{k-1})$ can be calculated. Then we can solve for h_k. Since h_1 has been derived for the first contract, we can use the above induction formulas to bootstrap all the hazard rates from the rest of the CDS contracts.

4.9.4. Extension to the stochastic recovery rate in the binomial setting. The previous bootstrapping procedure is based on a deterministic model. Since the recovery rate is dependent on economic conditions, to price CDS in a stochastic recovery setting we need to introduce the dynamics of the default process to the bootstrapping procedure. In our dynamic binomial lattice model, recovery rate $R_{j,k}$ is state and time dependent so that

it has a negative relationship with the state variable $M_{j,k}$. If we generate the market scenario on the binomial lattice given by $a_k, q_{j,k}$ and $R_{j,k}$, the bootstrapping procedure gives the intensity levels λ_k for period (t_{k-1}, t_k).

Assume the number of steps on the lattice is the same as the number of CDS contracts outstanding. The pricing model for CDS in a deterministic model can be adapted for the dynamic case and is given below.

The probability of an obligor surviving to time u is the expected value of the survival probability as a function of the path of the state variable $M_s, 0 < s < u$ given by

$$S(u) = \mathbb{E}\big[e^{-\int_0^u \lambda(M_s)ds}\big] \tag{4.9.13}$$

The PV of the default leg is given by

$$a_t P(t) = a_t \mathbb{E}\Big[\int_0^t e^{-\int_0^u (\lambda(M_s)+r(s))ds} du\Big] \tag{4.9.14}$$

and the PV of the premium leg is given by

$$D(t) = \mathbb{E}\Big[\int_0^t e^{-\int_0^u (\lambda(M_s)+r(s))ds} \lambda(M_u)(1 - R(M_u))du\Big] \tag{4.9.15}$$

Now we develop the bootstrapping procedure to compute for the intensity parameter λ. Assume M follows a K step binomial lattice, which corresponds to time periods $t_1, \cdots, t_K$.

Set the end of the first time period as t_1 and the intensity level $\lambda(M_s)$ as $\lambda_1 M_{1,1}$ where $M_{1,1} = 1, Q_{1,1} = 1$

The PV of the premium leg of the CDS becomes

$$a_1 P(t_1) = a_1 \mathbb{E}\Big[\int_0^{t_1} e^{-(\lambda_1 M_{1,1}+r_1)u} du\Big] = a_1 \int_0^{t_1} e^{-(\lambda_1 M_{1,1}+r_1)u} du \tag{4.9.16}$$

and the PV of the default leg is

$$\begin{aligned} D(t_1) &= \mathbb{E}\Big[\int_0^{t_1} e^{-(\lambda_1 M_{1,1}+r_1)u} \lambda_1 M_{1,1}(1 - R_{1,1})du\Big] \\ &= \lambda_1 M_{1,1}(1 - R_{1,1}) \int_0^{t_1} e^{-(\lambda_1 M_{1,1}+r_1)u} du \\ &= \lambda_1 M_{1,1}(1 - R_{1,1}) P(t_1) \end{aligned}$$

Since $a_1 P(t_1) = D(t_1)$,

$$\lambda_1 = \frac{a_1}{1 - R_{1,1}} \tag{4.9.17}$$

The induction can be derived as the following, assuming a deterministic interest rate.

$$P(t_k) - P(t_{k-1}) \;=\; \mathbb{E}\Big[\int_{t_{k-1}}^{t_k} e^{-\int_0^u (\lambda(M_s)+r(s))ds} du\Big]$$

and

$$\begin{aligned} D(t_k) - D(t_{k-1}) &= \mathbb{E}\Big[\int_{t_{k-1}}^{t_k} e^{-\int_0^u (\lambda(M_s)+r(s))ds} \lambda(M_u)(1 - R(M_u)) du\Big] \\ &= \mathbb{E}\Big[\lambda_k M_{j,k}(1 - R_{j,k}) \int_{t_{k-1}}^{t_k} e^{-\int_0^u (\lambda(M_s)+r(s))ds} du\Big] \end{aligned}$$

From equation (4.9.12),

$$\begin{aligned} & (a_k - a_{k-1}) P(t_{k-1}) \\ =\;& D(t_k) - D(t_{k-1}) - a_k(P(t_k) - P(t_{k-1})) \\ =\;& \mathbb{E}\Big[\big(\lambda_k M_{j,k}(1 - R_{j,k}) - a_k\big) \int_{t_{k-1}}^{t_k} e^{-\int_0^u (\lambda(M_s)+r(s))ds} du\Big] \\ =\;& \mathbb{E}\Big[\big(\lambda_k M_{j,k}(1 - R_{j,k}) - a_k\big) e^{-\int_0^{t_{k-1}} (\lambda(M_s)+r(s))ds} \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^u (\lambda(M_s)+r(s))ds} du\Big] \\ =\;& B(0, t_{k-1}) \mathbb{E}\Big[\big(\lambda_k M_{j,k}(1 - R_{j,k}) - a_k\big) S(t_{k-1} \mid M_{j,k}) \int_{t_{k-1}}^{t_k} e^{-(\lambda_k M_{j,k} + r_k)(u - t_{k-1})} du\Big] \\ =\;& B(0, t_{k-1}) \mathbb{E}\Big[\big(\lambda_k M_{j,k}(1 - R_{j,k}) - a_k\big) S(t_{k-1} \mid M_{j,k}) \int_0^{\Delta t_k} e^{-(\lambda_k M_{j,k} + r_k)u} du\Big] \\ =\;& B(0, t_{k-1}) \mathbb{E}\Big[\big(\lambda_k M_{j,k}(1 - R_{j,k}) - a_k\big) S(t_{k-1} \mid M_{j,k}) \frac{1 - e^{-(\lambda_k M_{j,k} + r_k)\Delta t_k}}{\lambda_k M_{j,k} + r_k}\Big] \\ =\;& B(0, t_{k-1}) \sum_{j=1}^{k} \big(\lambda_k M_{j,k}(1 - R_{j,k}) - a_k\big) S(t_{k-1} \mid M_{j,k}) \frac{1 - e^{-(\lambda_k M_{j,k} + r_k)\Delta t_k}}{\lambda_k M_{j,k} + r_k} Q_{j,k} \end{aligned}$$

Since the lattice is recombining, at every step we condition the process in the last step of the filtration $M_{j,k}$. Therefore, to solve for λ_k all the above quantities are known, except that $S(t_{k-1} \mid M_{j,k})$ needs to be derived.

$S(t_{k-1} \mid M_{j,k})$ is the survival probability by time t_{k-1} conditional on $M_{j,k}$, which indicates the economy is in state j at time t_k.

Notice that during time period $[t_{k-2}, t_{k-1})$ the economy has been in state $M_{\cdot k-1}$ and at time t_{k-1} a state change occurs when it jumps to $M_{\cdot k}$ on the binomial lattice.

If given $S(t_{k-1} \mid M_{j,k-1})$ with $1 \leq j \leq k-1$, $S(t_{k-1} \mid M_{j,k})$ is derived using the following formula.

$$S(t_{k-1} \mid M_{j,k}) = \begin{cases} S(t_{k-1} \mid M_{1,k-1}) & j = 1 \\ S(t_{k-1} \mid M_{j-1,k-1})(1 - q_{j-1,k-1})\frac{Q_{j-1,k-1}}{Q_{j,k}} & \\ \quad + S(t_{k-1} \mid M_{j,k-1})q_{j,k-1}\frac{Q_{j,k-1}}{Q_{j,k}} & 1 < j < k \\ S(t_{k-1} \mid M_{k-1,k-1}) & j = k \end{cases}$$

with initial condition $S(t_1 \mid M_{1,1}) = \exp\{-\int_{t_0}^{t_1} \lambda(t) M_{1,1} dt\}$.

4.9.5. Stochastic recovery rate generation. We generate the stochastic recovery rates on the lattice in the following way.

First we generate a pseudo recovery factor $\overline{R}_{j,k}$ for all obligors on the lattice. Let the initial pseudo recovery factor value $\overline{R}_{1,1}$ be 1. Let the percentage change on the lattice be a random variable b and add it as part of the calibration parameters.

On the lattice, if factor $M_{j,k}$ goes up to $M_{j,k+1} = (1+a)M_{j,k}$, then let the corresponding pseudo recovery factor $\overline{R}_{j,k}$ go down to

$$\overline{R}_{j,k+1} = (1-b)\overline{R}_{j,k}$$

And when $M_{j,k}$ goes down to $M_{j+1,k+1} = (1-a)M_{j,k}$, recovery factor $\overline{R}_{j,k}$ goes up to

$$\overline{R}_{j+1,k+1} = 1/(1-b)\overline{R}_{j,k}$$

In this way, $\overline{R}_{j,k}$ has a negative relationship with $M_{j,k}$.[3] but without an explicit functional relationship between them. Since the percentage changes a and b for M and R are calibrated via the CE method, the recovery rate for each obligor is negatively correlated

[3]Note that in our model with the specification $P\{\tau < t\} = 1 - \exp\{-\int_0^t \lambda_s M_s ds\}$ default probability has a positive relationship with M. Therefore, recovery rate has a negative relationship with default probability.

with M in a stochastic way. Once $\overline{R}_{j,k}$s are generated they are multiplied by an average recovery rate of $\overline{r}_i$ for each obligor, which can be estimated using historical data.

Therefore, the real recovery rate for an obligor i is defined as $R_{j,k} = \max(\min(\overline{r}_i \overline{R}_{j,k}, 1), 0)$ to ensure recovery rates are between 0 and 1.

4.9.6. Generalised recursive method. The recursive method introduced in Andersen et al. (2003) can be generalised in the stochastic recovery rate setting. Instead of the constant recovery rate case, where there are two states - default and survival, now there are more than two states - survival state and states for different losses. Assuming the possible loss states for an obligor are finite, building the loss distribution is equivalent to computing the distribution of a sum of mutually independent finite state random variables, which was introduced as the generalised recursive method in Jackson et al. (2007), given below.

Let variable $1_{\{m;N\}}$ be the m-th variable which takes integer value from $\{0, 1, \cdots, N-1\}$ with probability $Q_{m,n}$ where $\sum_{n=0}^{N-1} Q_{m,n} = 1$ and $1_{\{m;N\}}, 1 \leq m \leq M$ are mutually independent. The distribution for the sum of M mutually independent variables $1_{\{m;N\}}$, $\sum_{m=1}^{M} 1_{\{m;N\}}$ can be built up using the following recursive method,

$$\mathbf{p}_{m+1} = \begin{pmatrix} p_{m+1,(N-1)(m+1)} \\ \vdots \\ p_{m+1,1} \\ p_{m+1,0} \end{pmatrix} = \begin{pmatrix} \mathbf{p}_m & 0 & \cdots & 0 \\ 0 & \mathbf{p}_m & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \mathbf{p}_m \end{pmatrix} \begin{pmatrix} Q_{m+1,N-1} \\ \vdots \\ Q_{m+1,1} \\ Q_{m+1,0} \end{pmatrix}$$

where $p_{m,j}$ denotes the probability of distribution that the sum of m variables is in state j, $\mathbf{p}_m = (p_{m,(N-1)m}. \cdots, p_{m,1}, p_{m,0})^T$ and initial condition is $\mathbf{p}_1 = (Q_{1,N-1}, \cdots, Q_{1,1}, Q_{1,0})^T$

4.9.7. Applying the generalised recursive method in the model. Assume the notional for each obligor is 1 and each loss unit has a width of 0.1. There are 11 possible loss states for each obligor - $[0, 0.1, 0.2, \cdots, 1]$ where 0 loss means survival.

For each node $M_{j,k}$ on the lattice its loss state variable $L_{j,k} \in [0, 0.1, \cdots, 1]$ can be computed by round$((1 - R_{j,k})/0.1) \times 0.1$. [4]

[4]The rounding is to the nearest integer.

For each path on the lattice $[M_{j_1,1}, M_{j_2,2}, \cdots, M_{j_K,K}]$ until time T_K the loss probability distribution $Q_{m,j}$ can be generated by summing up probabilities of loss on each node of the path with the same loss units n.

$$Q_{m,j} = \sum_{L_{j,k}=n} P\{T_{k-1} < \tau < T_k \mid L_{j,k}, k = 1, \cdots, K\}$$

where n is the loss units, $n = [0, 0.1, 0.2, \cdots, 1]$.

Then the number of states for the loss distribution $p(\ell, t)$ of a portfolio of N names is $N \times 10 + 1$.

4.10. Conclusion

In this chapter, we constructed a dynamic binomial model that can be calibrated within bid and ask spreads of CDO tranches while maintaining a no-arbitrage condition in the time dimension. The stochastic recovery rate extension to the model shows how recovery rate dynamics can be built into a binomial model of CDS and how to build up a loss distribution for CDO pricing. Compared to the static model, however, the dynamic model is slower to price and calibrate. In addition, the complexity decreases its flexibility. These problems inspired us to develop the Markov model, which is introduced in the next chapter.

CHAPTER 5

The Markovian binomial model

5.1. In search of a Markovian binomial model

The dynamic binomial model prices credit derivatives that are dynamic in nature. However, in the model described in the previous chapter the pricing time increases dramatically if it needs to increase the number of steps on the lattice to increase flexibility. Let the number of steps on the lattice be K. Since the tranche evaluation involves computing default probabilities conditional on the path, the computation time for the last step of the lattice is of the order of $O(2^K)$ and the computation time for all the steps is of the order of $O(2^{K+1})$. For a typical five-year CDO, which normally requires roughly eight steps on the lattice to achieve a good fit, computational time is still acceptable. If one wants to calibrate the model to CDOs of two maturities simultaneously, the number of steps on the lattice needed to achieve a good fit increases to about 20 and the number of paths to be evaluated increases to $2^{20} = 1048576 >$ one million, which is unacceptable in practice.

The cause of this problem is that the default probabilities conditional on the state of the factor dynamics are not Markovian. Since the lattice is a recombining lattice, if there is more than one path leading to a particular node, the default probabilities integrated from each path up to that node are different. Therefore, the default probabilities conditional on the different nodes of a certain step are not independent of each other and are the average of overlapping paths.

To reduce the computational time in tranche evaluation we desire default probabilities that are independent conditional on the nodes rather than the path. This means the computational time has a linear relationship with the number of steps on the lattice.

Therefore, we are looking for a binomial lattice model in that the conditional default/survival probability dynamics are Markovian.

Mortensen (2006) defines the integral of the default intensities generated from the common factor as a conditioning variable. The default probabilities conditional on this integral

by a certain point in time are independent of each other. Inspired by this approach, we directly model the integral of the default intensities with a binomial lattice and specify the dynamics in such a way that the no-arbitrage condition 3.5.1 is maintained.

5.2. Model setup

Recall, in the dynamic binomial model, the conditional survival probabilities process satisfies

$$P\{\tau > t \mid \mathcal{M}_t\} = e^{-\int_0^t h(M_s)ds}$$

where $\mathcal{M}_t$ is the filtration generated by $M_s, 0 < s < t$.

The survival probabilities dynamics dependent on $M_t, 0 < s < t$ is not a Markovian process. To avoid simulating the path of M in tranche evaluations, one possible solution is to use the integral $\int_0^t h(M_s)ds$ as a conditioning variable as in Mortensen (2006). In this way, the default probability process becomes a Markovian process.

In our previous binomial setting, however, because of the non-parametric nature of M_s, the integral $H(t) = \int_0^t h(M_s)ds$ is intractable. We get around this problem by modelling the integral of the intensity $H(t)$ directly.

The survival probability of obligor i given the state of the economy M_t is given by

$$S_i(t \mid M_t) = P\{\tau_i > t \mid M_t\} = e^{-H_i(M_t)}$$

Here M_t follows a recombining binomial lattice. Note that the integral of the intensity $H_i(M_t)$ also follows a recombining lattice where each path is a monotonically increasing process. In this way the no-arbitrage condition 3.5.1 is automatically satisfied.

Since the conditional survival probability $S_i(t \mid M_t)$ is a function of the current state of the economy M_t only, conditional on M_t, the default probabilities between obligors are independent.

5.2.1. Algorithmic induction on the lattice. Under the same model assumptions of the dynamic binomial model in the previous chapter, we consider a discrete set of time

points T_k, $0 \leq k \leq K$, where $T_0 = 0$ denotes the initial time point ('today'). The dynamics of the economic factor are represented by a recombining binomial lattice indicated by $M_{j,k}$, where j is the state variable and k is the time-step variable, as previously defined.

However, we do not specify the values of the factor $M_{j,k}$. Nor do we need to derive the functional relationship between $H^i(M_{j,k})$ and $M_{j,k}$. Instead we work on the dynamics $H^i(M_{j,k})$ directly. And the distribution $Q(M_{j,k})$ of the factor $M_{j,k}$ follows the same setup in section 3.3.

To derive the dynamics of the conditional survival probabilities, we start by specifying the initial conditions for $H_i(M_{j,k})$ and give a step-by-step induction on a binomial lattice so that they satisfy the no-arbitrage condition 3.5.1.

To simplify notation, let

$$H^i_{j,k} = H^i(M_{j,k})$$

to represent the integral of the intensity for obligor i at time step k and state j .

Initialisation. In the initial state $j = k = 1$, $H^i_{1,1}$ can be calibrated to the default probability by the initial time T_0 of obligor i.

$$D_i(T_0) = 1 - e^{-H^i_{1,1}}$$

where $D_i(T_0)$ is the default probability of obligor i by time T_0.

If we set the current time $T_0 = 0$, then $H_{1,1} = 0$

Induction on the lattice. On the second step, $H^i_{1,1}$ can jump to two states $H^i_{1,2}$ and $H^i_{2,2}$, which are given by

$$\begin{cases} H^i_{1,2} = H^i_{1,1}(\lambda_{i,1}a_1 + 1) \\ H^i_{2,2} = H^i_{1,1}(\lambda_{i,1} + 1) \end{cases}$$

Where $a_1 > 1$ and $\lambda_{i,1} \geq 0$. $\lambda_{i,1}$ can be calibrated to the marginal default probability for obligor i by the payment date T_1 of the first step. a_1 is part of the model parameters which can be calibrated to CDO market quotes. This setup ensures $H^i_{1,2}$ greater than $H^i_{2,2}$, and $H^i_{1,2}$ and $H^i_{2,2}$ both greater than the initial state $H_{1,1}$.

On the third step, the induction follows the formula below.

$$\begin{cases} H^i_{1,3} = H^i_{1,2}(\lambda_{i,2}a_2 + 1) \\ H^i_{2,3} = H^i_{1,2}(\lambda_{i,2} + 1) \\ H^i_{3,3} = H^i_{2,2}(\lambda_{i,2} + 1) \end{cases}$$

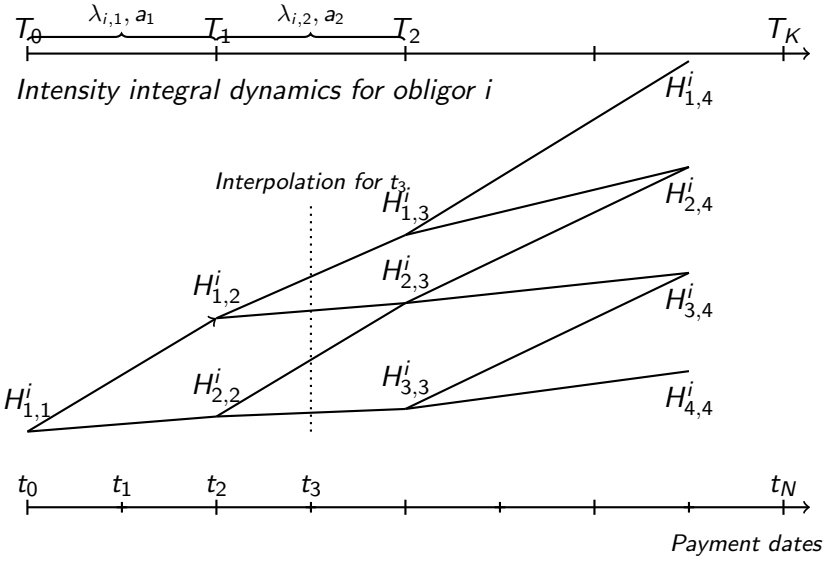


FIGURE 5.2.1. Intensity Integral Dynamics

Note that nodes $H^i_{1,2}$ and $H^i_{2,2}$ recombine at node $H^i_{2,3}$. From the above formula we have $H^i_{2,3} > H^i_{1,2}$. We already know from the second step that $H^i_{1,2} > H^i_{2,2}$. Hence $H^i_{2,3} > H^i_{2,2}$. Therefore, both the paths leading to $H^i_{2,3}$ is a monotonically increasing process.

In general, the derivation on the k-th step from the $k-1$-th step is given by the following formula.

$$H^i_{j,k} = \begin{cases} H^i_{1,k-1}(\lambda_{i,k-1}a_{k-1} + 1) & \text{for } j = 1 \\ H^i_{j-1,k-1}(\lambda_{i,k-1} + 1) & \text{for } 2 \leq j \leq k \end{cases}$$

The induction guarantees the conditional survival probabilities are monotonically decreasing along any path on the lattice. The parameter a_k controls the variances of the common factor and hence the correlation between obligors. To maximise the flexibility in model calibration, they are chosen to be time dependent and are different for each step on the lattice.

Figure 5.2.1 illustrates the dynamics of the intensity integral on the lattice.

5.3. Calibration to marginal default probabilities on the nodes

$\lambda_{i,k}$s are the responses of obligor i to the economic state change and can be chosen to match the marginal default probabilities given by the following formula.

$$S_i(T_k) = P\{\tau_i > T_k\} = \sum_{j=1}^{k} e^{-H^i_{j,k}} Q(M_{j,k}) \tag{5.3.1}$$

To see how $\lambda_{i,k}$ can be calibrated to marginal default probabilities step by step on the lattice, equation 5.3.1 can be further derived as,

$$\begin{aligned} S_i(T_k) &= \sum_{j=1}^{k} e^{-H^i_{j,k}} Q(M_{j,k}) \\ &= e^{-H^i_{1,k}} Q(M_{1,k}) + \sum_{j=2}^{k} e^{-H^i_{j,k}} Q(M_{j,k}) \\ &= e^{-H^i_{1,k-1}(\lambda_{i,k-1}a_{k-1}+1)} Q(M_{1,k}) + \sum_{j=2}^{k} e^{-H^i_{j-1,k-1}(\lambda_{i,k-1}+1)} Q(M_{j,k}) \end{aligned}$$

where $e^{-H^i_{j,k-1}}, 1 \le j \le k-1$ have already been derived from induction.

To solve for $\lambda_{i,k-1}$ numerical methods have to be used. Notice that the derivative of $S_i(T_k)$ with regard to $\lambda_{i,k}$, $\frac{\partial S_i(T_k)}{\partial \lambda_{j,k}}$ exists. It can be derived as follows,

$$\begin{aligned} \frac{\partial S_i(T_k)}{\partial \lambda_{i,k-1}} &= -a_{k-1} H^i_{1,k-1} e^{-H^i_{1,k-1}(\lambda_{i,k-1}a_{k-1}+1)} Q(M_{1,k}) \\ &\quad - \sum_{j=2}^{k} H^i_{j-1,k-1} e^{-H^i_{j-1,k-1}(\lambda_{i,k-1}+1)} Q(M_{j,k}) \end{aligned}$$

Then we can use Newton's method to find a $\lambda_{j,k}$ to match marginal survival probabilities. We start with the smallest possible value for $\lambda_{i,k}$, $\lambda_{i,k} = 0$ and increase its value. In practice, if the solution exists it normally takes two steps to reach a desired accuracy of $\epsilon < e^{-5}$.

However, unlike in the previous two models, we may not find solutions for $\lambda_{i,k}$ in this model when we wish to calibrate $\lambda_{j,k}$ to the survival probabilities by payment date T_k. We know that for an obligor i, the survival probability is a decreasing process - $S_i(T_{k+1}) < S_i(T_k)$. If at time step $k-1$ we have found a $\lambda_{i,k-1}$ that is calibrated to survival probability $S_i(T_k)$, at time step T_{k+1}, to maintain the absence of arbitrage, $\lambda_{i,k}$ has to be greater than 0. Let $S'(T_{k+1})$ be the survival probability when $\lambda_{i,k} = 0$. Then we probably encounter $S'(T_{k+1}) < S(T_{k+1})$. In this case if we increase $\lambda_{i,k}$ from 0, then $S'(T_{k+1})$ will decrease and never reach $S(T_{k+1})$.

To see why it is the case that when $\lambda_{i,k} = 0$ the new $S'_i(T_{k+1})$ does not start from $S_i(T_k)$ but less than $S_i(T_k)$, Figure 5.3.1 shows that when $\lambda_{i,k} = 0$, $H^i_{1,k+1}$ coincides with $H^i_{2,k+1}$ and they equal $H^i_{1,k}$, and also $H^i_{j,k} = H^i_{j+1,k+1}$ for $j \geq 2$. That is

$$S'_i(T_{k+1}) = \sum_{j=1}^{k+1} e^{-H_{j,k+1}} \cdot Q(M_{j,k+1}) = e^{-H_{1,k}} \big(Q(M_{1,k+1}) + Q(M_{2,k+1})\big) + \sum_{j=2}^{k} e^{-H_{j,k}} Q(M_{j+1,k+1})$$

when $\lambda_{i,k} = 0$. It can be easily proved that $S'(T_{k+1}) < S(T_{k+1})$.

Proof.

If we let $Q(M_{j,k}, M_{j,k+1})$ be the probability distribution when the economy is in state j at time k and state j at time $k+1$ and $Q(M_{j,k}, M_{j+1,k+1})$ is the probability distribution when the economy is in state j at time k and state $j+1$ at time $k+1$, then

$$Q(M_{j,k}) = Q(M_{j,k}, M_{j,k+1}) + Q(M_{j,k}, M_{j+1,k+1})$$

Also,

$$Q(M_{j,k+1}) = \begin{cases} Q(M_{j,k}, M_{j,k+1}) & j = 1 \\ Q(M_{j-1,k}, M_{j,k+1}) + Q(M_{j,k}, M_{j,k+1}) & 1 < j < k+1 \\ Q(M_{j-1,k}, M_{j,k+1}) & j = k+1 \end{cases}$$

Therefore,

$$
\begin{aligned}
S(T_k) &= \sum_{j=1}^{k} e^{-H_{j,k}} Q(M_{j,k}) \\
&= e^{-H_{1,k}} Q(M_{1,k}) + \sum_{j=2}^{k} e^{-H_{j,k}} \big(Q(M_{j,k} M_{j,k+1}) + Q(M_{j,k} M_{j+1,k+1}) \big) \\
&= e^{-H_{1,k}} Q(M_{1,k}) + \sum_{j=2}^{k} \big(e^{-H_{j,k}} Q(M_{j,k} M_{j,k+1}) + e^{-H_{j,k}} Q(M_{j,k} M_{j+1,k+1}) \big) \\
&< e^{-H_{1,k}} Q(M_{1,k}) + \sum_{j=2}^{k} \big(e^{-H_{j-1,k}} Q(M_{j,k} M_{j,k+1}) + e^{-H_{j,k}} Q(M_{j,k} M_{j+1,k+1}) \big) \\
&= e^{-H_{1,k}} \big(Q(M_{1,k}) + Q(M_{2,k} M_{2,k+1}) \big) + \sum_{j=2}^{k-1} e^{-H_{j,k}} \big(Q(M_{j,k} M_{j+1,k+1}) \\
&\quad + Q(M_{j+1,k} M_{j+1,k+1}) \big) + e^{-H_{k,k}} Q(M_{k,k}, M_{k+1,k+1}) \\
&= e^{-H_{1,k}} \big(Q(M_{1,k+1}) + Q(M_{2,k+1}) \big) + \sum_{j=2}^{k-1} e^{-H_{j,k}} Q(M_{j+1,k+1}) + e^{-H_{k,k}} Q(M_{k+1,k+1}) \\
&= e^{-H_{1,k}} \big(Q(M_{1,k+1}) + Q(M_{2,k+1}) \big) + \sum_{j=2}^{k} e^{-H_{j,k}} Q(M_{j+1,k+1}) \\
&= S'(T_{k+1})
\end{aligned}
$$

Therefore, it is possible that if there is a payment date t_n after T_k, which satisfies $S'(T_{k+1}) < S(t_n) < S(T_k)$, then $S'(T_{k+1})$ will never reach $S(t_n)$. That is $\lambda_{i,k}$ cannot be calibrated to the default probability by next payment date t_n. If this happens in calibration, we can skip to the payment date after the next one and repeat, if necessary, until we find a payment date T_{k+1} that satisfies $S'(T_{k+1}) \geq S(T_{k+1})$.

Note that the likelihood of this 'skip of payment dates' occurring in calibration depends on model parameter values. The CE algorithm is able to generate a wide range of data distribution in calibration to increase model flexibility. Some 'extreme values' of model parameters produce extreme values in factor values and distributions. Hence, the extreme values in conditional default probabilities cause a skipping of payment dates in calibration. And the skipping of payment dates reduces the number of steps and hence the number of parameters in the model, which reduces flexibility.

Therefore, a good calibration should achieve a balance of the two influencing factors - adequate number of steps on the lattice and adequate factor values and distributions.

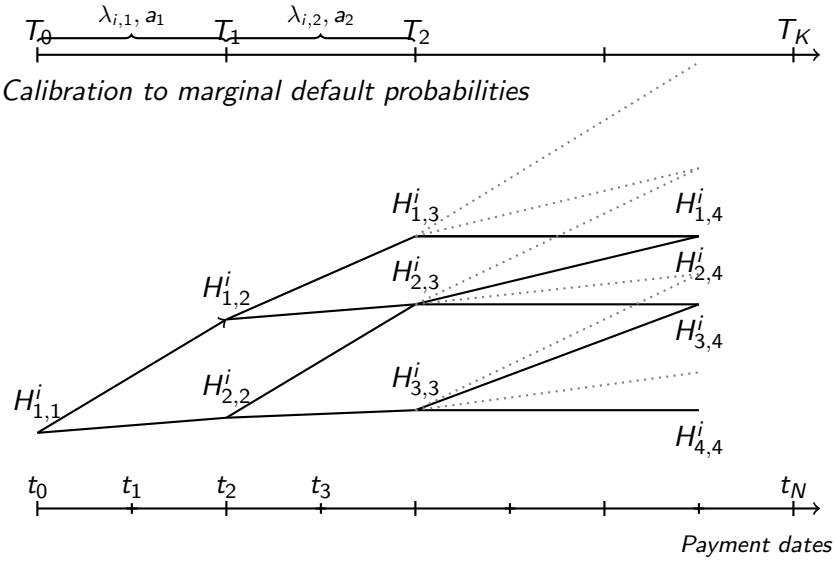


FIGURE 5.3.1. When $\lambda_{i,3} = 0$ the dotted lines coincide with the solid lines for $H^i_{1,4}, \cdots, H^i_{4,4}$

It should be noted that in practice we skip the payment dates on purpose when calibrating the nodes to marginal default probabilities, even without the above-mentioned problem, as the number of steps on the lattice is normally chosen to be smaller than the number of payment dates remaining in the life of the CDO to reduce calibration parameters. The goal is such that we want the number of steps needed on the lattice for the model to be minimal, but still to have enough flexibility in calibration. We will show below how to assign the payment dates to different nodes on the lattice. Once they are assigned, a further skipping of payment dates resulting from the above-mentioned problem does not occur very often. In addition, for the payment dates that are missed in the first stage of calibration on the nodes, we will show how they can be calibrated on the branches between the nodes, which completes the calibration to marginal default probabilities.

5.3.1. Dynamic payment dates allocation and marginal default probabilities calibration on the nodes. As in the previous models, a general rule of thumb to allocate payment dates to steps on the lattice is to ensure an equal number of payment dates between the nodes. With this model, the preallocation does not guarantee all steps will

be calibrated to marginal default probabilities. Therefore, we dynamically allocate payment dates at run time. First we divide the number of payments by the intended number of steps. For example, if there are 20 payments and the number of steps is 5, each step covers $20/5 = 4$ payments. Then, in calibration to marginal default probabilities at each iteration the algorithm increases the payments dates by 4. The algorithm then tests if the marginal default probabilities for that step can be calibrated. If not, the algorithm increases the payment date by 1 and tests again until finding the one that can be calibrated.

In practice, when we test the 30-day series data with 21 payment dates for each set of tranche data, we find that a 7-step lattice with each step covering 3 or 4 payments seems to be optimal to achieve accurate calibration. That is, if the algorithm allocates 3 payments in each step, we only encounter 1 skipped payment date in general. However, there are occasions, for example, when the calibration can also be achieved by more skipping payments and fewer steps, say 3 steps. Since the 3-step lattice is not as flexible as the 7-step lattice, the calibration is not perfect. In general, by running the same calibration a few times, an optimal set of model parameters should be achieved, if they exist.

Once we dynamically allocate the payment dates and finished calibration for all steps the loss distributions conditional on the node $M_{j,k}$, $p(\ell, T_k \mid M_{j,k})$ can be generated using a recursive method given the conditional default probability $D_i(T_k \mid M_{j,k}) = 1 - S_i(T_k \mid M_{j,k})$.

5.3.2. Marginal default probabilities calibration between the nodes. For payment dates that are left out in marginal default probabilities calibration on the nodes, we interpolate them on the branches between the nodes.

Assume that the default intensity integral H increases linearly between each time step, as in figure 5.2.1. At step k, $M_{j,k}$ branches to two states $M_{j,k+1}$ and $M_{j+1,k+1}$. After step k and before the economy states recombine at step $k+1$, there are $2k$ number of states for H on the branches.

The survival probabilities by t_n with $T_k < t_n < T_{k+1}$, conditional on the branch, e.g. $[M_{j,k}, M_{j+1,k+1}]$ can be interpolated on the branches. For any payment date t_n, the interpolation method we use is to find a straight line perpendicular to the time axis that intersects the H lines such that the intersection points on H satisfy the following equation.

$$S(t_n) = \sum_{j=1}^{k} e^{-[(H_{j,k+1}-H_{j,k})b+H_{j,k}]}Q(M_{j,k}M_{j,k+1})+e^{-[(H_{j+1,k+1}-H_{j,k})b+H_{j,k}]}Q(M_{j,k}M_{j+1,k+1})$$

where b is the interpolation parameter and has to be solved for. Since the function is smooth and monotonic, b can also be found using Newton's method by finding the derivative $\frac{\partial S(t_n)}{\partial b}$.

The probabilities distribution for each branch can be easily calculated using the transition probabilities on the lattice.

$$\begin{cases} Q(M_{j,k}M_{j,k+1}) = Q(M_{j,k}) \times Q(M_{j,k+1}/M_{j,k}) \\ Q(M_{j,k}M_{j+1,k+1}) = Q(M_{j,k}) \times Q(M_{j+1,k+1}/M_{j,k}) \end{cases}$$

5.4. Modified calibration algorithm

The calibration is done using the Cross Entropy (CE) method. The calibration parameters are a_ks and all the transition probabilities $q_{j,k}$ on the lattice.

We have modified the CE method slightly from the two previous models.

Let nP be the number of payment dates in a CDO structure. In step one, the algorithm generates $\frac{nP(nP+1)}{2} + nP$ number of model parameters. They are the maximum number of parameters that might be needed in calibration. The first $\frac{nP(nP+1)}{2}$ parameters are allocated to transition probabilities $q_{j,k}$ and the last nP parameters are allocated to the parameter a_is.

Then for parameter $a_i, a_i > 1$, we find that $1 < a_i < 6$ is a good range for most applications. Therefore, we transform a variable $x \subset (0, 1)$ to a_i by $a_i = 5 \times x + 1$.

It should be noted that in each iteration we generate the maximum number of parameters, that is, when the number of steps equal the number of payment dates. However, only the number of parameters stipulated by the number of steps on the lattice, which are dynamically determined by the algorithm at run time, are needed.

5.5. Calibration Results

Our model can be calibrated perfectly to most of the market data we have tested.

The non-parametric feature of the Markov model implies that for one set of market data we can get a range of arbitrage-free prices implied by the market. Because our model is non-parametric and thus makes fewer structural assumptions about the factor distribution, the 'model risk' is arguably less than that embodied in more parametric models.

5.5.1. Multiple solutions to one calibration date. Table 5.5.1 shows four perfect calibration results to mid-market quotes of the same degree of accuracy.

Tranche (%)	Mid	Model spread 1	Model spread 2	Model spread 3	Model spread 4
0–3	39.5 (%)	39.574	39.66	39.45	39.65
3–7	305(bp)	305.1	305.4	304	306.1
6–10	106(bp)	106.0	106.2	105.4	105.6
10–15	49(bp)	48.64	49.55	49.2	49.43
15–30	11(bp)	11.42	11.39	11.36	11.26

TABLE 5.5.1. Four calibration results for 5y CDX IG tranche data on 21 April 2003

The model parameters from the calibration algorithm are given in table 5.5.2.

Table 5.5.3 are the three calibration results of a perfect match on iTraxx index tranche data. Figure 5.5.1 contains the loss distributions implied by these three calibration results, which display slightly different curvatures. This indicates that even perfect calibration results might imply different distributions. One reason for this is that the market tranche widths are not thin enough (not like a tranchelet) to have a close match with loss distributions. Another possible reason is that the non-parametric nature of the model permits greater freedom. This flexibility makes it possible for a perfect calibration but potentially it might allow instability in pricing.

Figure 5.5.2 shows, when fully calibrated to market data, the conditional default probabilities of two obligors on the lattice. We can see that they are monotonically increasing, which satisfies the no-arbitrage conditions in the time dimension.

5.5.2. Calibration to two maturities. The model can be calibrated to CDO index tranches of multi-maturities simultaneously.

Table 5.5.4 shows the calibration results to iTraxx IG data of two maturities on 21 March 2005 and Figure 5.5.3(a) is the loss distribution implied from the market data.

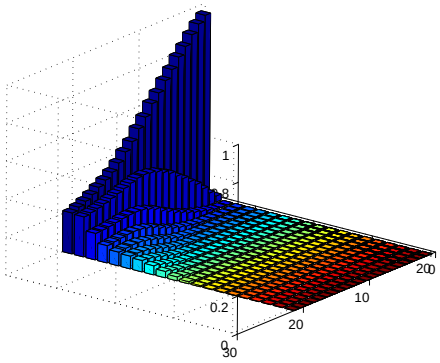
	solution 1	solution 2	solution 3	solution 4
transitional probabilities	0.0705	0.0692	0.0392	0.0669
	0.3440	0.2303	0.4762	0.3261
	0.1444	0.0924	0.0149	0.0499
	0.0887	0.8013	0.7133	0.4239
	0.0741	0.0670	0.1943	0.0549
	0.0326	0.0371	0.0281	0.0465
	0.6359	0.7825	0.3642	0.7213
	0.5981	0.5022	0.6463	0.6511
	0.4432	0.3792	0.2570	0.5181
	0.0552	0.1557	0.1429	0.1496
	0.8253	0.7991	0.7082	0.6693
	0.3693	0.6889	0.6027	0.5542
	0.2265	0.3581	0.2873	0.2182
	0.2729	0.3103	0.2871	0.3266
	0.2409	0.2267	0.2331	0.2192
	0.6320	0.2332	0.3354	0.4037
	0.7329	0.7895	0.8636	0.8690
	0.1485	0.5717	0.3177	0.6265
	0.4964	0.5160	0.2387	0.2640
	0.0698	0.8762	0.4849	0.3212
	0.8008	0.5272	0.5023	0.3802
size change for a	0.5120	0.8707	0.7193	0.5251
	0.7230	0.7330	0.4108	0.5982
	0.5403	0.5455	0.8436	0.4775
	0.0574	0.7242	0.8375	0.5198
	0.9334	0.8895	0.7808	0.9202
	0.4308	0.3471	0.3629	0.3703

TABLE 5.5.2. Model parameters from the four calibration results. The table shows that the four runs of the CE algorithm find different sets of model parameters matching the same set of market data. The transition probabilities $q_{j,k}$ and time dependent a_is are for a lattice with 6 steps.

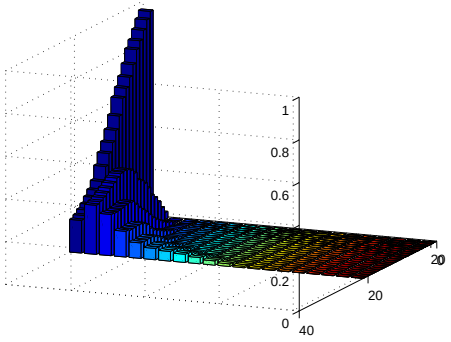
Tranche (%)	Market Mid	Model spread 1	Model spread 2	Model spread 3
0–3	17.5 (%)	17.5000	17.4999	17.5000
3–6	112.5 (bp)	112.5023	112.4983	112.5012
6–9	36.13 (bp)	36.1309	36.1294	36.1297
9–12	18 (bp)	18.0010	18.0005	17.9989
12–22	10 (bp)	9.9993	9.9999	10.0005

TABLE 5.5.3. Three calibration results for iTraxx IG tranches maturing in five years on 21 March 2005

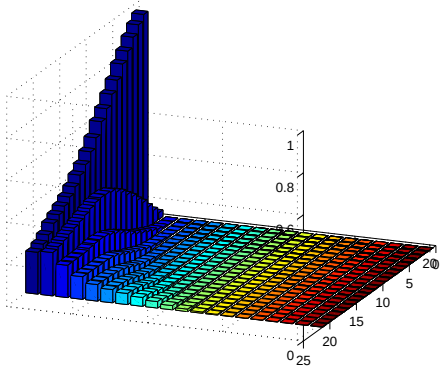
One application of the simultaneous calibration is to calibrate the model to tranches of maturities which are more liquid and use the results to price tranches of maturities that are illiquid.



(a) Loss distribution of calibration 1



(b) Loss distribution of calibration 2



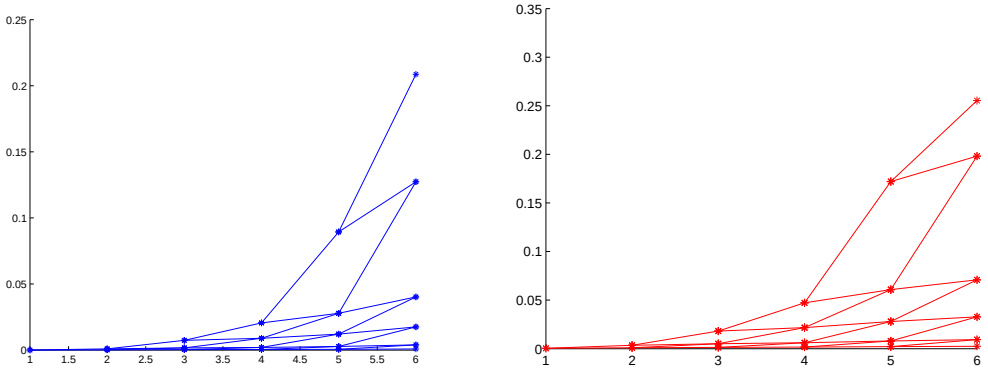
(c) Loss distribution of calibration 3

FIGURE 5.5.1. The loss distributions implied by the three calibration results on iTraxx IG tranches on 21 March 2005

Tranche (%)	Mid	Maturity	Bid	Ask	Model spread
0–3	17.5 (%)	5y	16.85	18.15	17.4999
3–6	112.5(bp)	5y	10.75	11.75	112.5006
6–9	36.13(bp)	5y	33.38	38.88	36.1293
9–12	18(bp)	5y	15.25	20.75	17.9991
12–22	10(bp)	5y	8.25	11.75	10.0000
0–3	45.75(%)	10y	44.06	47.44	45.7492
3–6	347.5(bp)	10y	332.5	362.5	347.5068
6–9	127.5(bp)	10y	117.875	137.125	127.4978
9–12	59.75(bp)	10y	50.675	68.825	59.7492
12–22	37(bp)	10y	30.525	43.475	36.9994

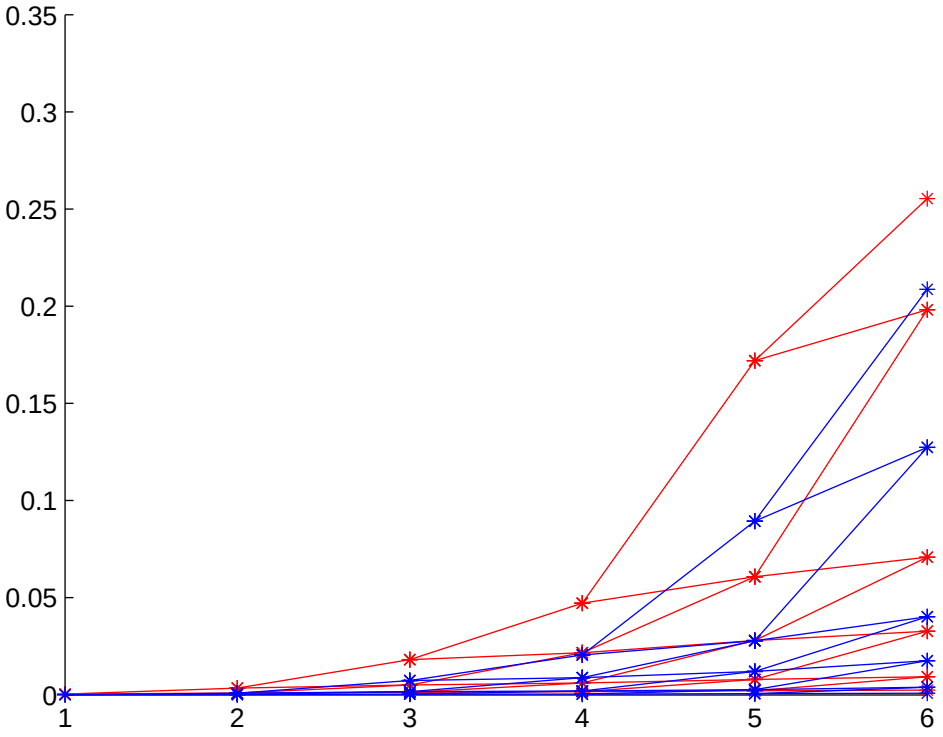
TABLE 5.5.4. Simultaneous calibration to iTraxx tranches of 5- and 10-year maturities on 21 March 2005

Table 5.5.5 shows the calibration results to iTraxx IG data of 5- and 10-year maturities on 02 July 2007 and the 7-year model spread is priced with the calibrated parameters and Figure 5.5.3(b) is the loss distribution implied from the results.



(a) Conditional default probabilities of obligor one

(b) Conditional default probabilities of obligor two



(c) Conditional default probabilities of obligors one and two

FIGURE 5.5.2. default dynamics on the lattice

The errors are the squared sum of the differences between market and model spreads calculated for the tranches of three maturities, respectively. We can see that the 7-year errors are higher than the other two maturities, as expected, but still within reasonable range; that is, the 'model-implied interpolation' of the 7-year tranches from the 5-year tranches

and 10-year tranches yields 7-year tranche spreads that are close to those observed in the market.

Tranches(%)	Mid	Maturity	payments	Model spread	Error
0–3	13.01(%)	5y	60	13.15	
3–6	67.1(bp)	5y	60	67.6	
6–9	17.93(bp)	5y	60	18.27	
9–12	8.34(bp)	5y	60	8.36	
12–22	3.28(bp)	5y	60	3.69	0.93
0–3	27.87(%)	7y	84	28.38	
3–6	147.39(bp)	7y	84	169.39	
6–9	40.41(bp)	7y	84	49.67	
9–12	18.29(bp)	7y	84	24.32	
12–22	8.11(bp)	7y	84	9.78	24.68
0–3	42.28 (%)	10y	120	42.08	
3–6	396.01(bp)	10y	120	398.88	
6–9	120.25(bp)	10y	120	121.59	
9–12	55.35(bp)	10y	120	55.69	
12–22	18.08(bp)	10y	120	18.56	3.22

TABLE 5.5.5. Simultaneous calibration for iTraxx tranches of 5- and 10-year maturities and pricing of 7-year tranches on 02 July 2007

5.5.3. Calibration to 30 days' serial data. Figure 5.5.4 to figure 5.5.8 are the plots of calibration results to iTraxx data for 30 consecutive days. For each day we run three calibrations and take the best results and plot the bid and ask spreads, mid-market quote and model spreads. The figures show that for most of the days the model can achieve exact 'hits' and all are within bid and ask spreads. Therefore, our model is flexible enough to be calibrated to implied distribution for most market realities.

5.5.4. Stability test. In this section we conduct the model stability test on model parameters and tranche spreads. A stable model gives users the confidence to price bespoke tranches and in hedging against market movements.

5.5.4.1. *Parameter stability.* We have four calibration results on the iTraxx IG data of 28 March 2005. Figure 5.5.9 (a) and (b) show the transitional probabilities $q_{j,k}$s and parameters a_ks of the four results. Even though they vary from one another for each solution, qualitatively they still share a resemblance. In particular, the first few points seem to move up and down in the same direction but with different magnitudes.

Figure 5.5.10 and figure 5.5.11 show the factor distribution of the 6-th and 8-th step on the lattice. Interestingly, while quantitatively different, there is a qualitative resemblance in the factor distributions of the four results.

In general, we can say that when the model is perfectly calibrated the parameters are qualitatively stable. However, it is more important to test the stability of the model when it is applied to pricing bespoke tranches, since that is one of the applications of the model.

5.5.4.2. *Model spreads stability.* In market CDO index tranches, the super-senior tranche is generally either not quoted, not traded or not liquid. Therefore, we can use the calibration results to evaluate the super-senior tranches. A good model should give stable price of the supersenior tranche spread.

Table 5.5.6 shows the pricing results of the tranche $[22 - 100]$ when the other quoted tranches are perfectly calibrated and the four super-senior tranche results are within 1 basis point of each other.

Tranches(%)	Mid Market	Solution 1	Solution 2	Solution 3	Solution 4
0–3	22.9(%)	22.9	22.9	22.9	22.9
3–6	150.75(bp)	150.73	150.76	150.74	150.76
6–9	48.75(bp)	48.76	48.75	48.74	48.76
9–12	22.13(bp)	22.12	22.13	22.13	22.13
12–22	12(bp)	12	12	12	12
22 – 100	not quoted	1.2	1.4	1.2	1.2

TABLE 5.5.6. Super-senior tranche spreads on 28 March 2005

Table 5.5.7 shows the pricing results of bespoke tranches with implied model parameters from the four calibration results above. We can see that the differences between the model spreads of different solutions are from within a few basis points for an equity tranche to one basis point for a senior tranche.

The test for pricing bespoke tranches indicates that the loss distributions implied by different calibration results are quite stable and smooth.

5.5.5. Model spreads sensitivities stability. The model spreads sensitivities tests are the model spreads sensitivities to a parallel shift in individual CDS spreads and changes in market correlation once a set of market quotes is perfectly calibrated.

Table 5.5.8 shows the new model spreads if the underlying default probabilities shift up 1%. We substitute a_k for market correlation. To increase market correlation we increase

Tranches(%)	Solution 1	Solution 2	Solution 3	Solution 4
0–4	14.05(%)	14.04	13.98	14.02
4–7	92.48(bp)	92.11	94.3	92.78
7–10	37.04(bp)	38.2	37.56	38.06
10–15	16.67(bp)	15.86	16.42	15.86
15–30	9.6(bp)	10.45	10.04	10.41

TABLE 5.5.7. Bespoke tranche model spreads on 28 March 2005

the parameters a_i by 1%. The new pricing results are shown in table 5.5.9. Figure 5.5.12 (a) and (b) show the plot of these sensitivity values given by the difference between the market spreads and new model spreads for different solutions. We can see that these sensitivities are very close, which implies the model parameters are quite stable.

Tranches	Mid Market	Solution 1	Solution 2	Solution 3	Solution 4
0–3	22.9(%)	23.255	23.2549	23.27	23.257
3–6	150.75(bp)	153.77	153.82	153.769	153.769
6–9	48.75(bp)	49.692	49.573	49.6588	49.5999
9–12	22.13(bp)	22.5329	22.6621	22.61562	22.65
12–22	12(bp)	12.083	12.0658	12.08185	12.05278
22 – 100	1.25	1.216	1.4265	1.1902	1.199

TABLE 5.5.8. New model spreads if CDS spreads shifted up 1% on 28 March 2005

Tranches	Mid Market	Solution 1	Solution 2	Solution 3	Solution 4
0–3	22.9(%)	22.803	22.809	22.806	22.8
3–6	150.75(bp)	151.19	151.34	151.365	151.195
6–9	48.75(bp)	49.22	49.1632	49.182	49.157
9–12	22.13(bp)	22.35	22.44	22.4	22.423
12–22	12(bp)	12.0585	12.05	12.059	12.0334
22 – 100	1.25	1.211	1.42323	1.188	1.197

TABLE 5.5.9. New model spreads if a_is shifted up 1% on 28 March 2005

5.6. Model simplification

To further investigate the model, we try to speed up computation by reducing model parameters. The model has parameters a_ks that are time dependent. In a simplified version we can have a constant correlation parameter a to replace all the a_ks. The resulting model does not have sufficient flexibility to calibrate perfectly to market data but is still accurate enough for applications. Table 5.6.1 shows 4 calibration results of iTraxx IG data on 28 March 2005.

Tranches(%)	Mid Market	Solution 1	Solution 2	Solution 3	Solution 4
0–3	22.9(%)	23.22	23.20	24.11	23.5
3–6	151(bp)	153	153	143	156
6–9	49(bp)	50	48	48	51
9–12	22(bp)	23	22	27	25
12–22	12(bp)	13	13	14	14

TABLE 5.6.1. Calibration results of 5y iTraxx IG data on 28 March 2005

5.7. Extension to stochastic recovery rate

We have assumed the recovery rate to be constant throughout time steps and different states of economy. Now we generalise the constant recovery rate to the stochastic recovery rate.

Let $R_{j,k}$ be the recovery rate that is dependent on economic state $M_{j,k}$. It is consistent with the way in which the default intensity integral $H_{j,k}$ is dependent on $M_{j,k}$. The recovery rate term structure $R_{j,k}$ represents the average recovery rate of all the segments of a path on the lattice from time T_0 to time T_k in state j regardless of the path. That is, all the paths have the same average recovery rate. Therefore, it is also a Markov process.

To describe the stochastic recovery-rate term-structure dynamics on the lattice, we have to develop a few key relationships. We know empirical evidence suggests that recovery rate is in a reverse relationship with default probability. We also know that the no-arbitrage condition in the time dimension governs the conditional default dynamics. But what governs the recovery-rate term-structure dynamics on the lattice? We show that the recovery-rate term-structure dynamics are governed by the following no-arbitrage condition for the expected loss process.

CONDITION 5.7.1. *The expected loss of an obligor conditional on a realisation of its default intensities is a monotonically increasing function of time.*

It is important to know that the expected loss dynamics on the binomial lattice is also a Markov process.

Formulated mathematically, we want to prove that if $M_{j-1,k}$ and $M_{j,k}$ recombine at $M_{j,k+1}$ and let $L(T_{k+1})$ be the accumulated loss for an obligor from time 0 to time T_{k+1}, then the following equation holds.

$$\mathbb{E}[L(T_{k+1}) \mid M_{j,k+1}] = \mathbb{E}[L(T_{k+1}) \mid M_{j-1,k}, M_{j,k+1}] = \mathbb{E}[L(T_{k+1}) \mid M_{j,k}, M_{j,k+1}] \tag{5.7.1}$$

First, the expected loss at node $M_{j,k+1}$ satisfies

$$\mathbb{E}[L(T_{k+1}) \mid M_{j,k+1}] = \mathbb{E}[1_{\{\tau > T_{k+1}\}} \cdot (1 - R(T_{k+1})) \mid M_{j,k+1}] = (1 - R_{j,k+1})P_{j,k+1}$$

where $P_{j,k}$ is the default probability of an obligor by time T_k conditional on $M_{j,k}$

Then, since the recovery rate R is a Markov process,

$$\mathbb{E}[1_{\{\tau > T_{k+1}\}} \cdot (1 - R(T_{k+1})) \mid M_{j-1,k}, M_{j,k+1}] = \mathbb{E}[1_{\{\tau > T_{k+1}\}} \cdot (1 - R(T_{k+1})) \mid M_{j,k+1}] = (1 - R_{j,k+1})P_{j,k+1}$$

Hence $\mathbb{E}[L(T_{k+1}) \mid M_{j-1,k}, M_{j,k+1}] = \mathbb{E}[1_{\{\tau > T_{k+1}\}} \cdot (1 - R(T_{k+1})) \mid M_{j-1,k}, M_{j,k+1}] = (1 - R_{j,k+1})P_{j,k+1}$.

In the same way, we can prove that $\mathbb{E}[L(T_{k+1}) \mid M_{j,k}, M_{j,k+1}] = (1 - R_{j,k+1})P_{j,k+1}$.

Therefore, equation 5.7.1 holds and we can conclude that the expected loss $\mathbb{E}[L(T_k) \mid M_{j,k}]$ is also a Markov process given both the conditional default probabilities and recovery-rate dynamics are Markov processes.

From here we can confirm that from node $M_{j,k}$ to node $M_{j,k+1}$, the expected loss process satisfies,

$$\mathbb{E}[L(T_k) \mid M_{j,k}] + \mathbb{E}[L(T_k, T_{k+1}) \mid M_{j,k}M_{j,k+1}] = \mathbb{E}[L(T_{k+1}) \mid M_{j,k}, M_{j,k+1}]$$

where $L(T_k, T_{k+1})$ is the loss accumulated between time T_k and time T_{k+1}.

From equation 5.7.1, the above equation can be further derived as

$$\mathbb{E}[L(T_k) \mid M_{j,k}] + \mathbb{E}[L(T_k, T_{k+1}) \mid M_{j,k}M_{j,k+1}] = \mathbb{E}[L(T_{k+1}) \mid M_{j,k+1}] \tag{5.7.2}$$

Equation 5.7.2 states that the expected loss of an obligor up to time T_k conditional on state $M_{j,k}$ plus the expected loss that occurs between state $M_{j,k}$ and state $M_{j,k+1}$ equals the expected loss of the obligor up to time T_{k+1} conditional on state $M_{j,k+1}$.

Let x be the recovery rate that applies when default occurs between state $M_{j,k}$ and state $M_{j,k+1}$ and let $P_{j,k}$ be the conditional default probability for an obligor at $M_{j,k}$.

Equation 5.7.2 can be further derived as

$$(1 - R_{j,k})P_{j,k} + (P_{j,k+1} - P_{j,k})(1 - x) = (1 - R_{j,k+1})P_{j,k+1} \tag{5.7.3}$$

x has to be between $(0, 1)$ to maintain the no-arbitrage condition 5.7.1.

Then, if $R_{j,k}$, $P_{j,k}$, x and $P_{j,k+1}$ are known, $R_{j,k+1}$ must satisfy the following condition.

$$0 \leq 1 - x = \frac{(1 - R_{j,k+1})P_{j,k+1} - (1 - R_{j,k})P_{j,k}}{P_{j,k+1} - P_{j,k}} \leq 1$$

The right inequality becomes

$$\frac{P_{j,k+1}}{P_{j,k}} \geq \frac{R_{j,k}}{R_{j,k+1}} \Rightarrow R_{j,k+1} \geq \frac{P_{j,k}}{P_{j,k+1}} R_{j,k} \tag{5.7.4}$$

and the left inequality becomes

$$\frac{P_{j,k+1}}{P_{j,k}} \geq \frac{1 - R_{j,k}}{1 - R_{j,k+1}} \Rightarrow R_{j,k+1} \leq 1 - \frac{(1 - R_{j,k})P_{j,k}}{P_{j,k+1}} R_{j,k} \tag{5.7.5}$$

Note that $\frac{P_{j,k+1}}{P_{j,k}} > 1$ is always true from the second no-arbitrage condition 3.5.1.

These are the two boundaries for $R_{j,k+1}$. The same derivation can be done for $R_{j+1,k+1}$.

Now we can start to derive the conditions for stochastic recovery-rate dynamics on the lattice step by step. Assuming that the initial state $R_{1,1}$ is given, the first condition we want to enforce on the lattice is that at each node the recovery rate can either go up or down in response to an economic state change.

This condition automatically ensures that all the nodes on the boundaries of the lattice have one of the two inequalities (5.7.4 and 5.7.5) satisfied.

Specifically, when $R_{j,k}$ goes up to $R_{j,k+1}$, then $R_{j,k+1} > R_{j,k}$ or $\frac{R_{j,k}}{R_{j,k+1}} < 1$. Therefore, inequality 5.7.4 holds and only inequality 5.7.5 has to be satisfied.

On the other hand, when $R_{j,k}$ goes up from $R_{j,k+1}$, then $R_{j,k+1} < R_{j,k}$ or $\frac{R_{j,k}}{R_{j,k+1}} > 1$ and $\frac{L_{j,k}}{L_{j,k+1}} < 1$. Therefore, inequality 5.7.5 holds and only inequality 5.7.4 has to be satisfied.

In general, if the stochastic recovery rate follows the binomial lattice with up-and-down jumps, for all the boundary nodes, there is one inequality to be satisfied.

For all the recombining nodes, since there are two nodes that lead to each recombining node, each path has to satisfy the two inequalities separately, even though the expected losses at the recombining node are the same from both paths. Further, since one of the conditions for each path is always satisfied, there are two inequalities to be satisfied for the recombining node in total.

Now we show how to implement the stochastic recovery rate dynamics to satisfy the conditions above.

Let the initial value $R_{1,1} = 0.4$. Then let the CE algorithm generate a sequence of random numbers $b_i, 1 \leq i \leq nP$ between $(0, 1)$, where nP is the number of payments.

We then work out the boundary conditions for $R_{1,2}$ and $R_{2,2}$.

For $R_{1,2}$ the upper boundary is $R_{1,1}$ and lower boundary is $LBoundary = \frac{P_{1,1}}{P_{1,2}} R_{1,1}$. For $R_{2,2}$ the lower boundary is $R_{1,1}$ and upper boundary is $UBoundary = 1 - \frac{P_{1,1}}{P_{1,2}}(1 - R_{1,1})$. Therefore, let

$$\begin{cases} R_{1,2} = R_{1,1} - b_1(R_{1,1} - LBoundary) \\ R_{2,2} = R_{1,1} + b_1(UBoundary - R_{1,1}) \end{cases}$$

From $R_{1,2}$ and $R_{2,2}$, we can proceed further to calculate $R_{1,3}$ and $R_{3,3}$.

The recombining node $R_{2,3}$ has to satisfy the following inequalities.

$$\frac{P_{1,2}}{P_{2,3}} R_{1,2} \leq R_{2,3} \leq 1 - \frac{P_{2,2}}{P_{2,3}}(1 - R_{2,2}) \tag{5.7.6}$$

If the above inequalities 5.7.6 are satisfied, we take the average of the two boundaries to be $R_{2,3}$

$$R_{2,3} = \frac{1}{2}\Big(\frac{P_{1,2}}{P_{2,3}} R_{1,2} + 1 - \frac{P_{2,2}}{P_{2,3}}(1 - R_{2,2})\Big)$$

However, inequalities 5.7.6 are not guaranteed to be satisfied in the induction, as we may encounter $\frac{P_{1,2}}{P_{2,3}} R_{1,2} > 1 - \frac{P_{2,2}}{P_{2,3}}(1 - R_{2,2})$. We overcome this problem in the CE algorithm by adding the following step.

Whenever a violation of the inequalities 5.7.6 for recombining nodes occurs, the algorithm automatically marks this set of pricing parameters and gives zero values to the

pricing results. When all the pricing results are passed on to the CE algorithm, the sets of parameters that violate the above conditions will have the maximum values in evaluating the objective functions and be placed in the last of the queue of solutions. Therefore, these parameters won't be used in generating a 'better' set of parameters. Gradually these 'bad' parameters will be eliminated.

Parameter b can be added to the calibration parameters and can be implied from market tranche prices by the CE algorithm.

In calculating loss distributions, we first make R time and state dependent and make the following changes to expected loss calculation to accommodate the stochastic recovery rate.

$$\begin{aligned} EL_i &= \sum_{l=0}^{N} p(l, T_k)\text{max}(\text{min}(lA(1-R), H) - L, 0) \\ &= \sum_{l=0}^{N} \mathbb{E}\big[p(l, T_k \mid M_{j,k})\text{max}(\text{min}(lA(1-R_{j,k}), H) - L, 0)\big] \\ &= \mathbb{E}\big[\sum_{l=0}^{N} p(l, T_k \mid M_{j,k})\text{max}(\text{min}(lA(1-R_{j,k}), H) - L, 0)\big] \end{aligned}$$

Table 5.7.1 presents the calibration results of the iTraxx IG tranche data on 21 March 2005. The stochastic recovery-rate model can reduce the number of steps needed on the lattice from 6 to 5 and only use a constant parameter a but still achieves perfect calibration.

Tranches(%)	Mid Market	5 year	Mid Market	10 year
0–3	17.5(%)	17.49	45.75	45.72
3–6	112(bp)	113	348	347
6–9	36(bp)	36	128	127
9–12	18(bp)	18	60	60
12–22	10(bp)	10	37	37

TABLE 5.7.1. Simultaneous calibration to iTraxx data on 21 March 2005 with stochastic recovery

Figure 5.7.2 (a) shows the implied stochastic recovery-rate dynamics on the lattice and (b) shows its mean value for 10 years. The mean curve indicates that the drift of the implied recovery rate actually has gone up for longer maturities in the example.

Table 5.7.2 shows the calibrated results sub-prime crisis data after July 2007, which cannot be perfectly calibrated with the basic model but can now be perfectly calibrated without increasing the number of steps(5 steps).

Tranches (%)	5 year Mid Market	Bid	Ask	Basic model	Stochastic model
0–3	29.27(%)	28.62	29.92	30.57	29.27
3–6	225(bp)	220	230	234	225
6–9	100(bp)	97	103	105	100
9–12	56(bp)	53	59	58	56
12–22	31(bp)	29	32	34	31

TABLE 5.7.2. Comparison calibration results to iTraxx data on 01 Aug 2007

The findings demonstrate that the stochastic recovery rate can be a realistic assumption in credit modelling.

However, it is still difficult to achieve perfect calibration to multiple maturities for the post-crisis data, as shown in table 5.7.3.

Tranches (%)	5 year Mid Market	Model	10 year Mid Market	Model
0–3	29.27(%)	28.85	46.10	48.12
3–6	225(bp)	220	539	522
6–9	100(bp)	97	299	308
9–12	56(bp)	59	172	184
12–22	31(bp)	30	80	87

TABLE 5.7.3. Calibration results of 5 and 10 years of iTraxx data on 01 Aug 2007

5.8. Comparison to the simplified approach by Hull and White

Due to a number of similarities of the Markov model to Hull & White (2008), in order to better illustrate the advantages of our model, in this section we compare the two models. Each model starts by modelling the same quantity - the integral of the hazard-rate process - and both models use a lattice to describe the factor dynamics.

However, the differences between the models are apparent.

1). Hull's model in the homogenous case is a pure jump process. At every time step, there is an infinite number of states and the jump size is constant. Our model is based on a recombining binomial lattice with time-dependent 'jump' sizes in each step.

2). Our modelling philosophy is as follows.

First, the a_ks control the general variances of the market factor and therefore the correlation between obligors, and $\lambda_{i,k}$ controls each obligor i's response to the market condition change. Hence, they customise each company's correlation to the market and are calibrated to marginal default probabilities.

Second, the aim of our model is to imply the market 'dependency structure' in the form of the distribution of the common factor. Therefore, the transition probabilities $q_{j,k}$s and the parameter a_ks are parameters that are calibrated to market tranche prices. In Hull's model the distribution on the lattice is basically a Taylor series, which is a discretised proximation to Gaussian distribution with only one free intensity parameter. Therefore, it has no flexibility to cope with variable market dependencies.

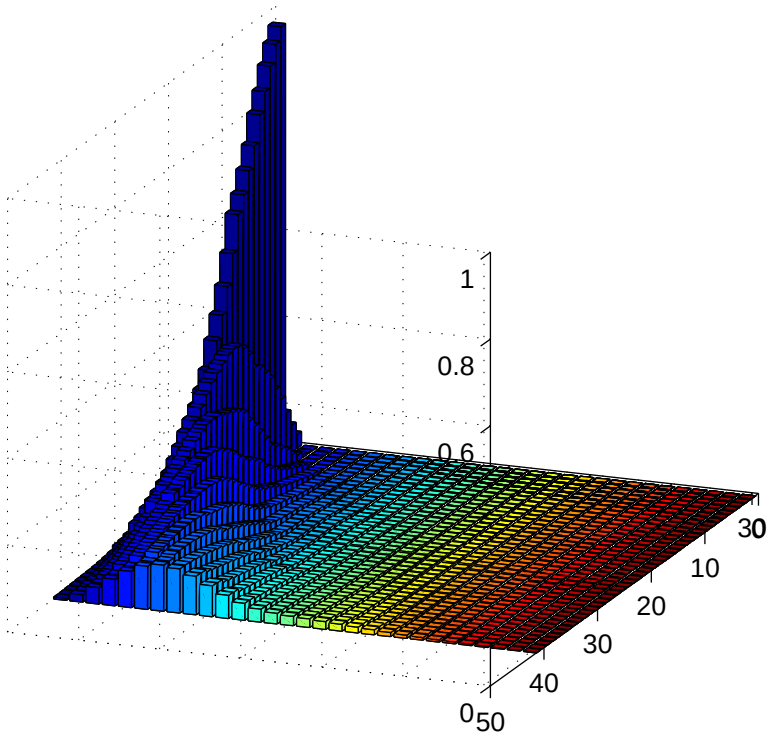
The aim of Hull's model is to serve as an alternative to the Gaussian copula model. It cannot achieve the accuracy achieved by our model and it does not have the ability to match the market correlation skew.

3). Hull's model is not consistent when pricing homogenous and heterogenous portfolios. In the homogenous case, it calibrates the marginal default probabilities by the intensity parameter and assumes a zero drift. In the heterogenous case, it calibrates to marginal default probabilities with a drift parameter. Further, in the heterogenous case no-arbitrage conditions may be violated when calibrating to marginal default probabilities since the drift parameters are not guaranteed to be positive.

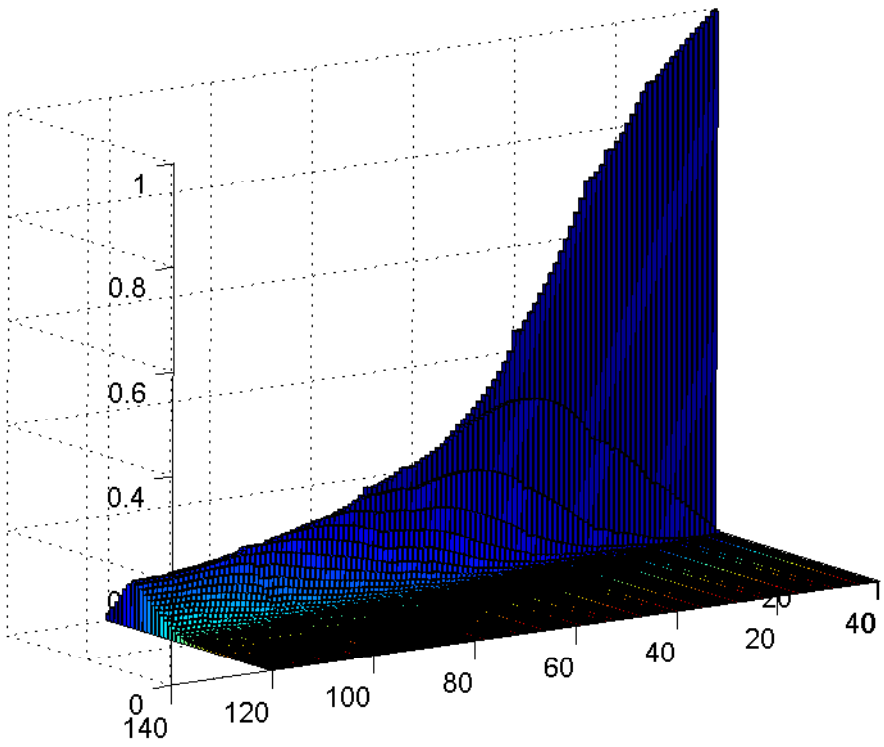
5.9. Conclusion

In this chapter, we developed a Markovian binomial model which addresses the path-dependent problem embedded in the dynamic binomial model. Like the dynamic model, market dependency can be implied in the form of the distribution of the common factor. The new model makes it computationally economical to price CDO index tranches with multiple maturities simultaneously. The calibration results indicate a dramatically improved match compared with the dynamic binomial model. Stability tests show that these non-parametric models are qualitatively stable for applications. The stochastic recovery-rate extension to the model further reduces the number of steps and thus the number of model parameters in calibration with increased accuracy. Our non-parametric approach

makes it possible to calibrate to market quotes exactly, so it is ideal to be applied in pricing exotic credit derivatives and hedging.



(a) Loss distribution to 10 years of iTraxx IG on 21 March 2005



(b) Loss distribution to 10 year of iTraxx IG on 07 July 2007

FIGURE 5.5.3

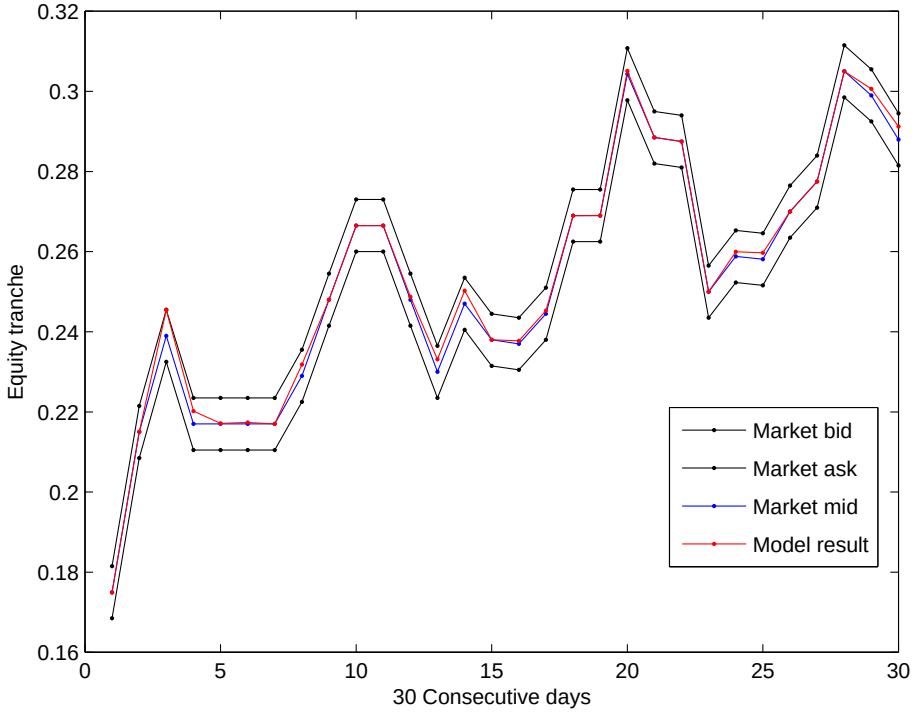


FIGURE 5.5.4. Calibration results for equity tranche to one-month serial data on iTraxx IG

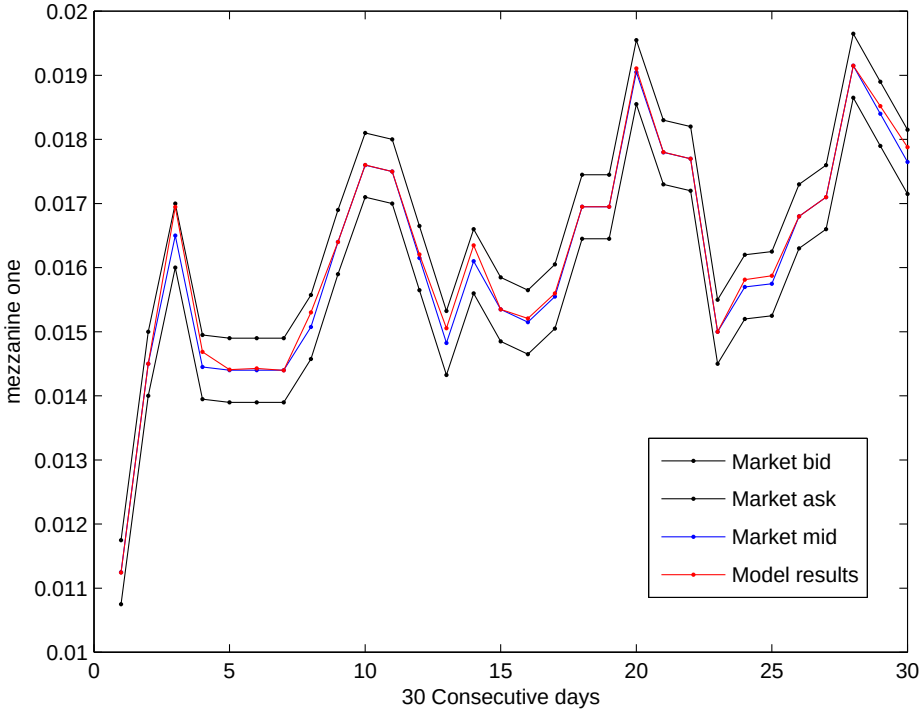


FIGURE 5.5.5. Calibration results for first mezzanine tranche to one-month serial data on iTraxx IG

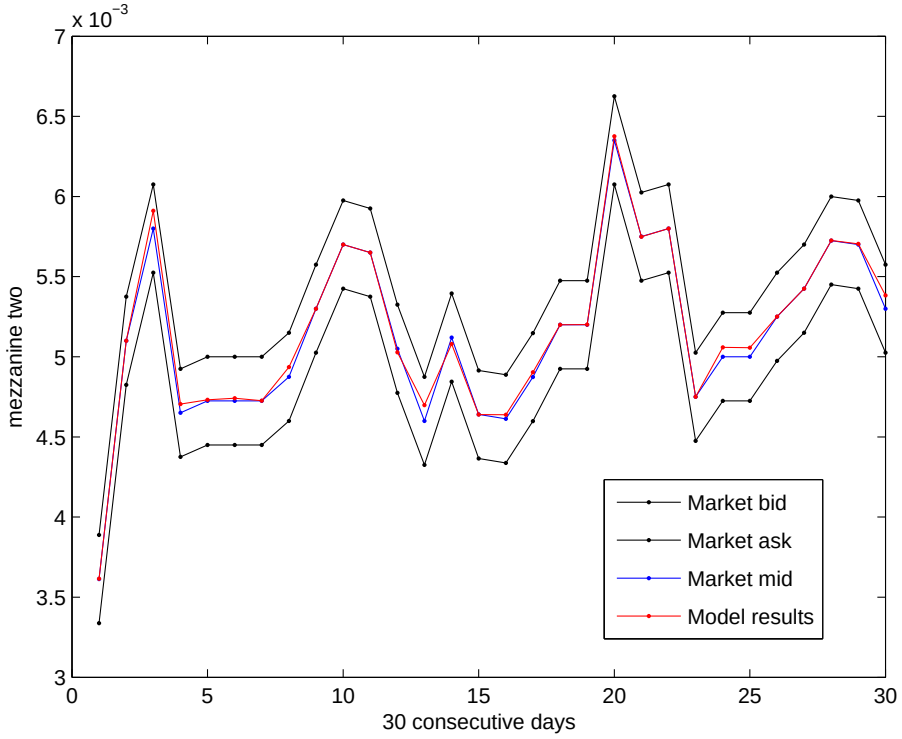


FIGURE 5.5.6. Calibration results for second mezzanine tranche to one-month serial data on iTraxx IG

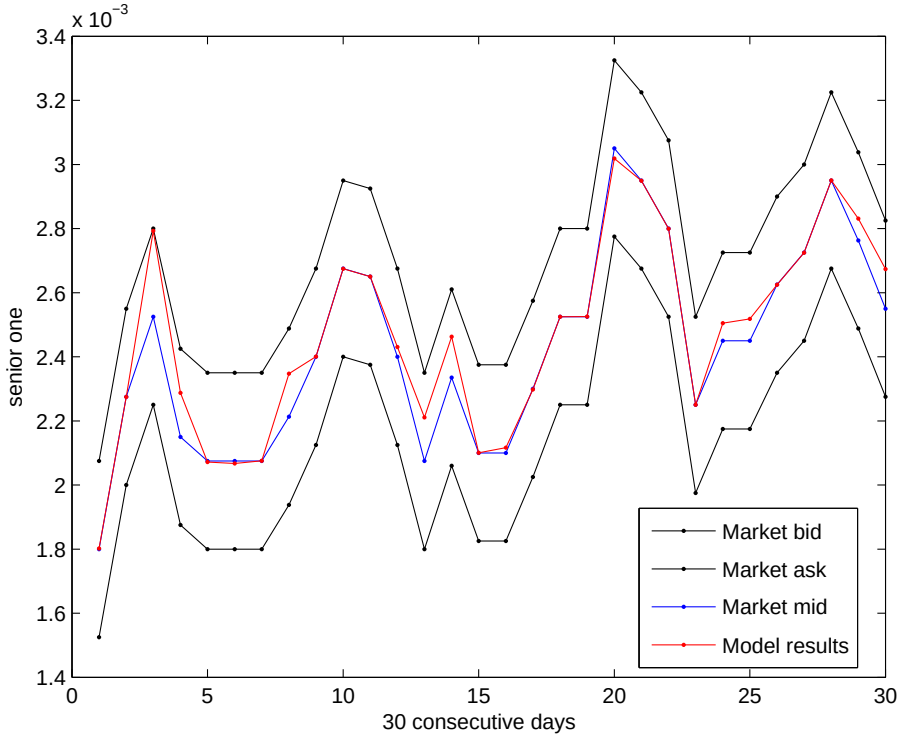


FIGURE 5.5.7. Calibration results for first senior tranche to one-month serial data on iTraxx IG

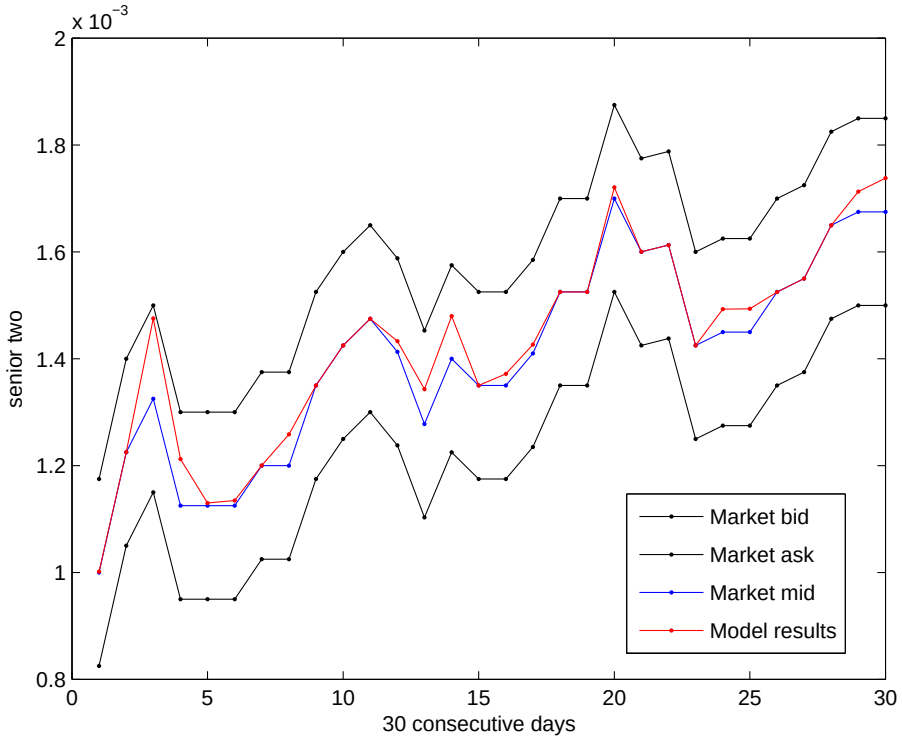
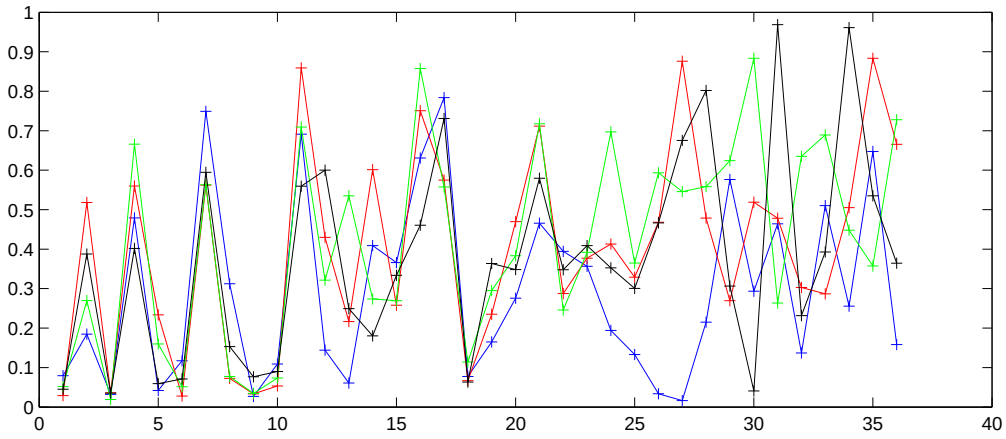
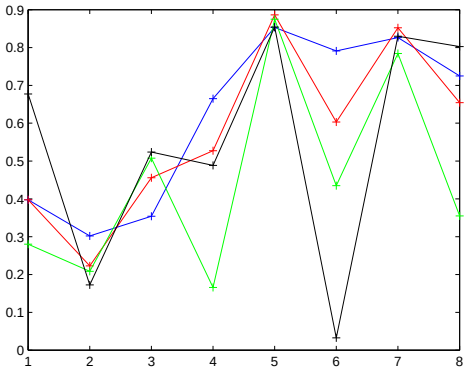


FIGURE 5.5.8. Calibration results for second senior tranche to one-month serial data on iTraxx IG



(a) Transitional probabilities



(b) a_is

FIGURE 5.5.9. Model parameters of 4 different calibration results of CDX IG data on 28 March 2005

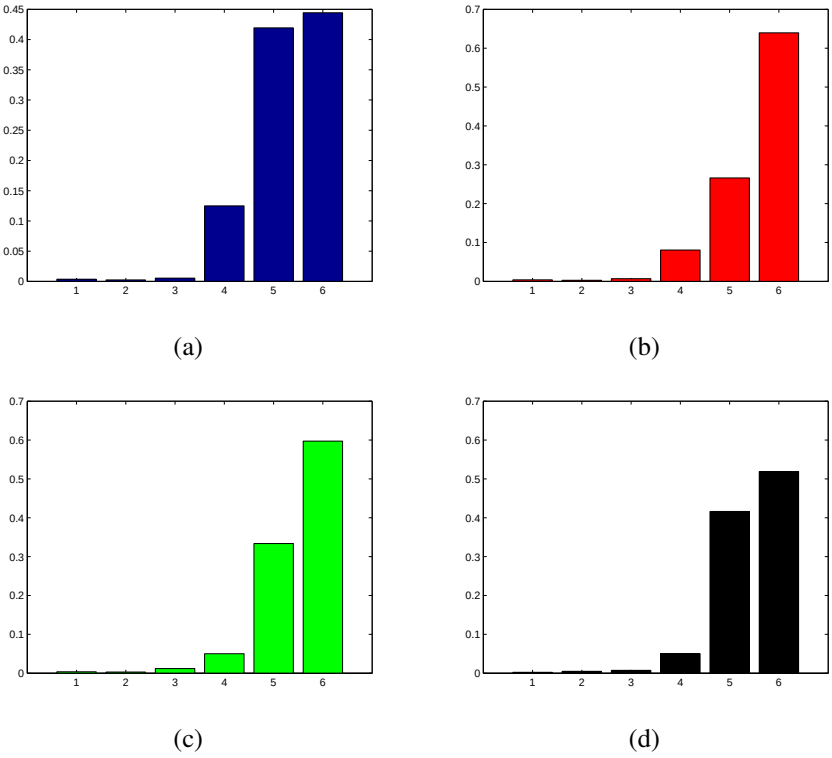


FIGURE 5.5.10. Factor distributions of 4 different calibration results of CDX IG data on 28 March 2005 on the 6th step

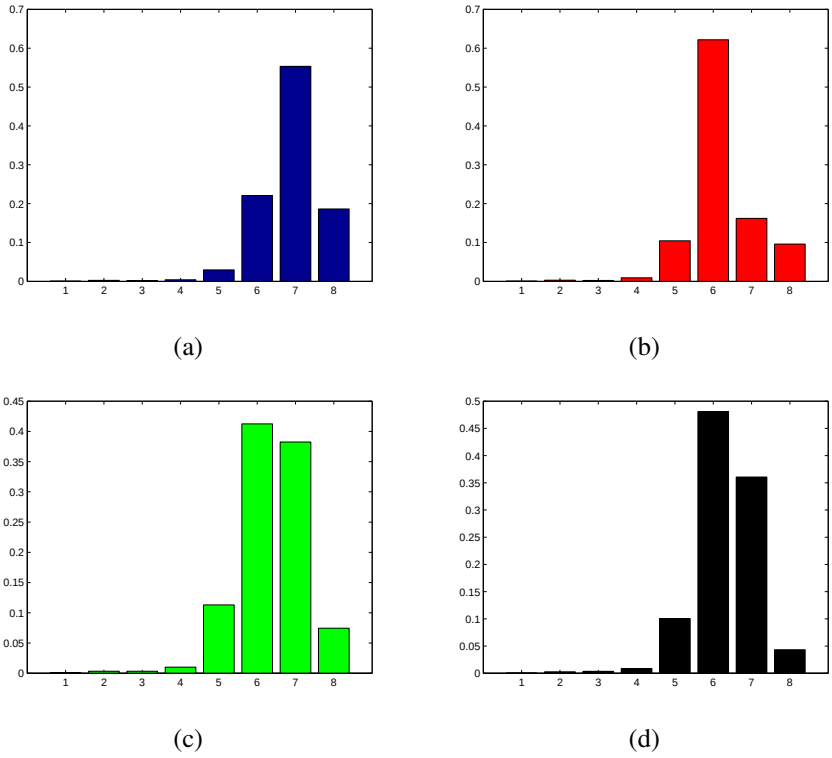
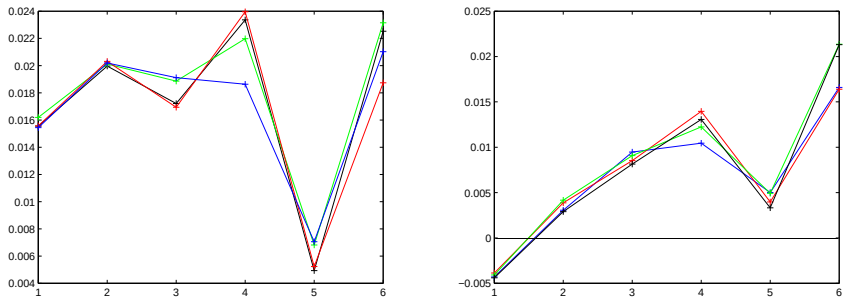


FIGURE 5.5.11. Factor distributions of 4 different solutions of CDX IG data on 28 March 2005 on the 8th step



(a) Model sensitivities to CDS spreads (b) Model sensitivities to correlation

FIGURE 5.5.12. Model sensitivities to spread and correlation change of CDX IG data on 28 March 2005

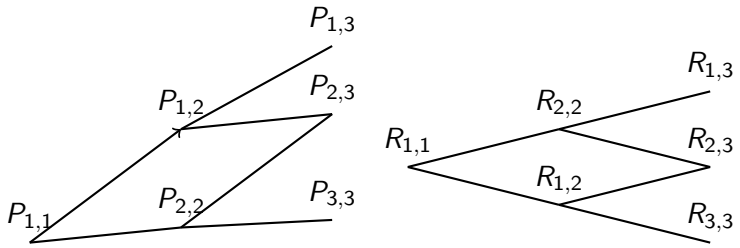
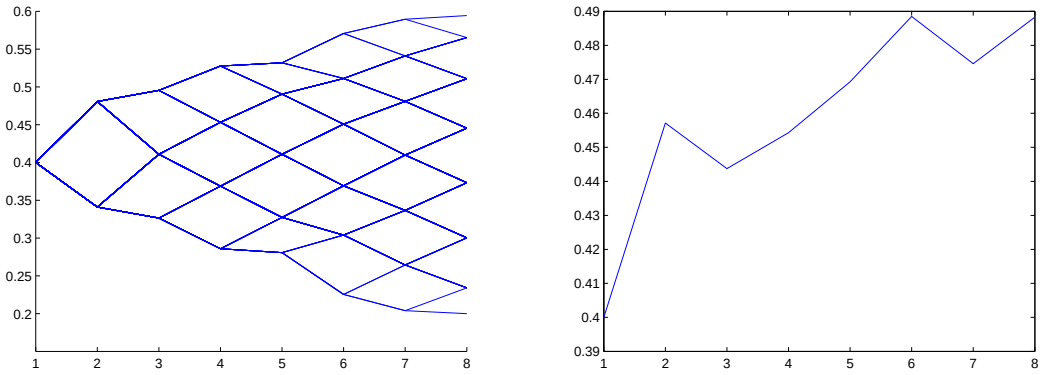


FIGURE 5.7.1. Stochastic recovery rate on the lattice



(a) Stochastic Recovery Rate dynamics on 21 March 2005

(b) The mean of recovery rate on 21 March 2005

FIGURE 5.7.2

CHAPTER 6

Thesis Conclusion

In this thesis, we investigated the bottom up approach to credit risk modeling. This type of models in general has a few inherent problems which serve as the basis of a series of no-arbitrage conditions that we formulate. The violation of these no-arbitrage arguments can cause potential losses in trading of CDOs in different contexts when using these models. The first is that the 'correlation smile' exists because a model cannot be calibrated to all the tranches on a CDO structure by one set of model parameters. We call this no-arbitrage condition in the capital structure dimension. Through the NIG copula model parameters sensitivities analysis, we identified that the cause of this problem is the lack of flexibility in the parameterization of a model. To reduce model risk to the greatest extent, we introduce a semi-parametric way of model design with the construction of binomial lattices to represent market evolution. The static binomial model shows that it is possible to be fully calibrated to all CDO tranche prices simultaneously and thus overcome this issue. Another advantage of this model is that it can achieve fast calibration because of its simplicity. Therefore it can be used to price products that are also static in nature. However, the static model only describes the default probability of an obligor given market distribution at specific points in time and therefore has the potential of violating the no-arbitrage condition of the default process given the time evolution. The second no arbitrage condition is that the default probability process should be an increasing function of time in the realization of an economic scenario. The choosing the right factor to model - the default intensity, we apply the Cox process into the binomial model and the no-arbitrage condition in the time dimension can be satisfied automatically given a realization of the common market factor. Here some of the deficiencies of the non-parametric approach start to surface. Compared to models that are governed by parametric processes such as SDEs, a non-parametric specification can lose its flexibility after numerical manipulation such as integration if the quantity being modeled is in the integrand. Parametric models might not suffer this problem since their flexibility is governed by the number of model parameters

and they don't necessarily lose the degrees of freedom after numerical manipulation. Another interesting point of this model is how to deal with the non-Markovian nature of the integrated intensity. One an advantage of the binomial approach is that we can directly calculate the default probabilities on each path rather than do Monte Carlo simulation in a parametric model. However, in a parametric model the MC simulation can be avoided by conditioning on the integral and if the subsequent numerical challenges are not challenging it is easier than the binomial model. In our binomial approach, to overcome these two problems at once is to model the integrated default intensity directly. The Markovian Binomial Model directly models the integrated intensity process and avoids the problems faced by the Dynamic Binomial Model. This model can achieve relative fast and also accurate calibration as in the Static Model while satisfying the first two no-arbitrage conditions. It was then extended to incorporate stochastic recovery rate. However, here we identified a third no-arbitrage argument that is normally automatically satisfied in the dynamic model and thus has been overlooked by other researchers. The third no-arbitrage condition is that the expected loss process should be an increasing function of time given the realization of an economic scenario. After we explicitly impose this condition on the Markovian Model and apply numerical techniques, Markovian Model works perfectly under stochastic recovery rate. There is potentially substantial work which can be done with this approach to credit risk modeling. One direct application is to test the hedging performance of these binomial models in trading CDOs and more exotic derivatives. Another interesting topic can be a comparison performance testing of the binomial models with other parametric models such as copula models and jump-diffusion models. Some further investigation can be carried out on how to introduce more parametric structure on the binomial lattice to further increase flexibility with minimal steps.

Bibliography

Albrecher, H., Ladoucette, S. A. & Schoutens, W. (2006), 'A generic one factor levy model for pricing synthetic cdos', *http://www.schoutens.be/LevyCDO.pdf* .

Amato, J. D. & Gyntelberg, J. (2005), 'Cds index tranches and the pricing of credit risk correlations', *BIS Quarterly Review, March* .

Andersen, L., Sidenius, J. & Basu, S. (2003), 'All your hedges in one basket', *RISK magazine, November* .

Arnsdorf, M. & Halperin, I. (2007), 'Bslp: Markovian bivariate spread-loss model for portfolio credit derivatives', *http://www.defaultrisk.com* .

Baxter, M. (2007), 'Gamma process dynamic modelling of credit', *Risk magazine, October* pp. 98–101.

Bennani, N. (2005), 'The forward loss model: a dynamic term structure approach for the pricing of portfolio credit derivatives', *http://www.defaultrisk.com/* .

Black, F. & Cox, J. C. (1976), 'Valuing corporate securities: Some effects of bond indenture provisions', *Journal of Finance* **31**(2), 351–367.

Black, F. & Scholes, M. (1973), 'The pricing of options and corporate liabilities', *Journal of Political Economy* **81**, 637–654.

Brigo, D., Pallavicini, A. & Torresetti, R. (2006), 'Calibration of cdo tranches with the dynamical generalized-poisson loss model', *http://www.damianobrigo.it* .

Burtschell, X., Gregory, J. & Laurent, J.-P. (2008), 'A comparative analysis of cdo pricing models', *http://www.defaultrisk.com* .

Crane, G. & van der Hoek, J. (2008), 'Using distortions of copulas to price synthetic cdos', *Insurance: Mathematics and Economics* **42**(3), 903–908.

Das, S. R., Duffie, D., Kapadia, N. & Saita, L. (2007), 'Common failings: How corporate defaults are correlated', *The Journal Of Finance* **62**(1), 93–117.

Davis, M. & Lo, V. (2001), 'Modelling default correlation in bond portfolios', *Mastering Risk Volume 2: Applications, Chapter 8* pp. 141–151.

der Voort, M. V. (2006), 'An implied loss model', *www.DefaultRisk.com* .

Ding, X., Giesecke, K. & Tomecekz, P. I. (2009), 'Time-changed birth processes and multi-name credit derivatives', *working paper, Stanford University* .

Duffie, D. (2007), 'Innovations in credit risk transfer: Implications for financial stability', *working paper, Stanford University* .

Duffie, D., Filipovic, D. & Schachermayer, W. (2003), 'Affine processes and applications in finance', *Annals of Applied Probability* **13**(3), 984–1053.

Duffie, D. & Garleanu, N. (2001), 'Risk and valuation of collateralized debt obligations', *Financial Analysts Journal, Jan/Feb* **57**(1), 41–59.

Duffie, D. & Singleton, K. J. (1999), 'Modeling term structures of defaultable bonds', *Review of Financial Studies,* pp. 687–720.

Eckner, A. (2007), 'Computational techniques for basic affine models of portfolio credit risk', *http://www.eckner.com/papers/* .

Edelman, D. (2004), 'The minimum local cross-entropy criterion in options pricing', *Risk* **7**(7).

Ehlers, P. & Schonbucher, P. J. (2006), 'Pricing interest rate-sensitive credit portfolio derivatives', *Working Paper, Swiss Federal Institute of Technology Zurich* (Swiss Finance Institute Research Paper No. 06-39).

Errais, E., Giesecke, K. & Goldberg, L. R. (2007), 'Pricing credit from the top down with affine point processes', *http://www.stanford.edu/dept/MSandE/people/faculty/giesecke/indexes.pdf* .

Feldhutter, P. (2007), 'An empirical investigation of an intensity-based model for pricing cdo tranches', *http://staff.cbs.dk/pef/CDOpaper.pdf* .

Filipovic, D., Overbeck, L. & Schmidt, T. (Forthcoming), 'Dynamic cdo term structure modelling', *Mathematical Finance* .

Frey, R. & Backhaus, J. (2008), 'Portfolio credit risk models with interacting default intensities: a markovian approach', *http://www.defaultrisk.com/* .

Frye, J. (2000), 'Collateral damage: A source of systematic credit risk', *Risk magazine, April* pp. 91–94.

Garcia, J., Goossens, S., Masol, V. & Schoutens, W. (2007), 'Levy base correlation', *http://www.schoutens.be/* .

Giesecke, K. & Goldberg, L. (Forthcoming), 'A top down approach to multi-name credit', *Operations Research* .

Giesecke, K. & Tomecek, P. (2005), 'Dependent events and changes of time', *http://www.defaultrisk.com/* .

Giesecke, K. & Weber, S. (2006), 'Credit contagion and aggregate losses', *Journal of Economic Dynamics and Control* **30**(5), 741–767.

Graziano, G. D. & Rogers, C. (2006), 'A dynamic approach to the modelling of correlation credit derivatives using markov chains', *http://www.defaultrisk.com/* .

Hagan, P. S. & West, G. (2006), 'Interpolation methods for curve construction', *Applied Mathematical Finance* **13**(2), 89 – 129.

Heath, D., Jarrow, R. & Morton, A. (1992), 'Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation.', *Econometrica* (60), 77–105.

Herbertsson, A. (2008), 'Pricing synthetic cdo tranches in a model with default contagion using the matrix-analytic approach', *http://www.defaultrisk.com/* .

Hull, J., Predescu, M. & White, A. (2006), 'The valuation of correlation-dependent credit derivatives using a structural model', *http://www.rotman.utoronto.ca/ hull/DownloadablePublications/StructuralModel.pdf* .

Hull, J. & White, A. (2004), 'Valuation of a cdo and an nth to default cds without monte carlo simulation', *Journal of Derivatives* **12**(2), 8–23.

Hull, J. & White, A. (2006), 'Valuing credit derivatives using an implied copula approach', *Journal of Derivatives* **14**(2), 8–18.

Hull, J. & White, A. (2008), 'Dynamic models of portfolio credit risk: A simplified approach', *Journal of Derivatives* **15**(4), 9–28.

Inglis, S. & Lipton, A. (2007), 'Factor models for credit correlation', *http://www.defaultrisk.com/* .

Jackson, K., Kreinin, A. & Ma, X. (2007), 'Loss distribution evaluation for synthetic cdos', *www.defaultrisk.com* .

Jarrow, R. A. & Turnbull, S. M. (1995), 'Pricing derivatives on financial securities subject to credit risk', *Journal of Finance* **50**, 53–85.

Jarrow, R. A. & Yu, F. (2001), 'Counterparty risk and the pricing of defaultable securities', *Journal of Finance* **56**(5), 1765–1799.

Joe, H. (1997), *Multivariate Models and Dependence Concepts*, Chapman and Hall, London.

Jokivuolle, E. & Peura, S. (2003), 'Incorporating collateral value uncertainty in loss given default estimates and loan-to-value ratios', *European Financial Management* **9**(9), 299–314.

Joshi, M. & Stacy, A. (2005), 'Intensity gamma: a new approach to pricing cdos', *http://www.defaultrisk.com* .

Kalemanova, A., Schmid, B. & Werner, R. (2007), 'The normal inverse gaussian distribution for synthetic cdo pricing', *Journal of Derivatives* **14**(3), 80.

Kiesel, R. & Scherer, M. (2007), 'Dynamic credit portfolio modelling in structural models with jumps', *http://www.mathematik.uni-ulm.de/finmath/people/kiesel/publications/* .

Kraft, H. & Steffensen, M. (2007), 'Bankruptcy, counterparty risk, and contagion', *Review of Finance* **11**(2), 209–252.

Krekel, M. (2006), 'The implied loss surface of cdos', *www.DefaultRisk.com* .

Lando, D. (1998), 'On cox processes and credit risky securities', *Review of Derivatives Research* **2**, 99–120.

Laurent, J.-P. & Gregory, J. (2005), 'Basket default swaps, cdos and factor copulas', *Journal of Risk* **7**(4), 103–122.

Li, D. X. (2000), 'On default correlation: A copula function approach', *Journal of Fixed Income* **9**(4), 43–54.

Lopatin, A. & Misirpashaev, T. (2007), 'Two-dimensional markovian model for dynamics of aggregate credit loss', *http://www.defaultrisk.com/* .

McGinty, L. & Ahluwalia, R. (2004), 'Introducing base correlations', *JP Morgan, Credit Derivatives Strategy, March* .

Meng, C. & Sengupta, A. N. (2008), 'Cdo tranche sensitivities in the gaussian copula model', *www.defaultrisk.com* .

Merton, R. C. (1974), 'On the pricing of corporate debt: The risk structure of interest rates', *Journal of Finance* **29**(2), 449–470.

Mortensen, A. (2006), 'Semi-analytical valuation of basket credit derivatives in intensity-based models', *Journal of Derivatives* **13**(4), 8–26.

Nelder, J. A. & Mead, R. (1965), 'A simplex method for function minimization', *Computer Journal* **7**, 308C313.

O'Kane, D. (2008), *Modelling Single-name and Multi-name Credit Derivatives*, John Wiley and Sons, Ltd.

Parcell, E. & Wood, J. (2007), 'Wiping the smile off your base (correlation curve)', *http://www.edparcell.com/* .

Rosen, D. & Saunders, D. (2007), 'Valuing cdos of bespoke portfolios with implied multi-factor models', *http://www.stats.uwaterloo.ca/Faculty/Saunders/CDO.pdf* .

Rubinstein, R. Y. & Kroese, D. P. (2004), *The Cross–Entropy Method*, Springer Science+Business Media, Inc.

Schlögl, E. (2008), *Markov Models for CDOs*, The Definitive Guide to CDOs - Market, Application, Valuation, and Hedging, RISK Books.

Schönbucher, P. (1998), 'Term structure modelling of defaultable bonds', *Review of Derivatives Research* **2**, 161–192.

Schönbucher, P. (2003), *Credit Derivatives Pricing Models: Models, Pricing and Implementation*, John Wiley and Sons Ltd.

Schönbucher, P. (2005), 'Portfolio losses and the term structure of loss transition rates: A new methodology for pricing of portfolio of credit derivatives', *Working Paper, Department of Mathematics, ETH Zurich* .

Schönbucher, P. J. (2004), 'Information-driven default contagion', *http://www.defaultrisk.com/* .

Schönbucher, P. & Schubert, D. (2001), 'Copula-dependent default risk in intensity models', *http://www.defaultrisk.com/* .

Sidenius, J., Piterbarg, V. & Anderson, L. (2008), 'A new framework for dynamic credit portfolio loss modelling', *International Journal of Theoretical and Applied Finance* **11**(2), 163–197.

Totouom, D. & Armstrong, M. (2005), 'Dynamic copula processes: A new way of modelling cdo tranches', *http://www.cerna.ensmp.fr/Documents/DTT-MA-DynamicCopula.pdf* .

Umberto Cherubini, E. L. & Vecchiato, W. (2004), *Copula methods in finance*, John Wiley and Sons, Ltd.

Vasicek, O. (1987), Probability of loss on loan portfolio, Technical report, KMV Corporation.

Willemann, S. (2007), 'Fitting the cdo correlation skew: A tractable structural jump-diffusion model', *Journal of Credit Risk, Spring* **3**(1), 63–90.

Zhou, C. (2001*a*), 'An analysis of default correlations and multiple defaults', *The Review of Financial Studies, January* **14**(2), 555–576.

Zhou, C. (2001*b*), 'The term structure of credit spreads with jump risk', *Journal of Banking and Finance* **25**(11), 2015–2040.